SIA for Model 2 using the Taylor series approach

ODE system

$$\text{Glucose}'[t] = D_r * (GF - \text{Glucose}[t]) - r_{glu} * X[t]$$
$$\text{Lactate}'[t] = -D_r * \text{Lactate} + r_{lac} * X[t]$$
$$X'[t] = \left(\left(\mu_{max} * \frac{\text{Glucose}[t]}{km_{glu} + \text{Glucose}[t]} * \frac{ki_{lac}}{ki_{lac} + (\text{Lactate}[t])^2}\right) - D_r\right) * X[t]$$

Initial conditions: Glucose[0]=1.01, Lactate[0]=3.98, X[0]=0.46

All states are observed and the only known parameters are D_r and GF

First two derivatives for each state for when t = 0

Glucose'[0] = $(-1.01 + GF)\, D_r - 0.46\, r_{glu}$

Glucose''[0] =

$$-D_r\, ((-1.01 + GF)\, D_r - 0.46\, r_{glu}) - 0.46\, r_{glu} \left(-D_r + \frac{1.01\, ki_{lac}\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})}\right)$$

Lactate'[0] = $-\text{Lactate}\, D_r + 0.46\, r_{lac}$

Lactate''[0] = $0.46\, r_{lac} \left(-D_r + \frac{1.01\, ki_{lac}\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})}\right)$

X'[0] = $0.46 \left(-D_r + \frac{1.01\, ki_{lac}\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})}\right)$

X''[0] = $0.46 \left(-D_r + \frac{1.01\, ki_{lac}\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})}\right)^2 +$

$$0.46 \left(-\frac{1.01\, ki_{lac}\, ((-1.01 + GF)\, D_r - 0.46\, r_{glu})\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})^2} + \frac{ki_{lac}\, ((-1.01 + GF)\, D_r - 0.46\, r_{glu})\, \mu_{max}}{(15.8404 + ki_{lac})\,(1.01 + km_{glu})} - \frac{8.0396\, ki_{lac}\, (-\text{Lactate}\, D_r + 0.46\, r_{lac})\, \mu_{max}}{(15.8404 + ki_{lac})^2\,(1.01 + km_{glu})}\right)$$

Analysis of Taylor series coefficients

1. r_{glu} can be identified from Glucose'[0]
2. r_{lac} can be identified from Lactate'[0]
3. Use Glucose''[0] to solve for ki_{lac}:

$$ki_{lac} \to \big(3.39273\times10^{15}\,\text{Glucose}''[0] - 3.42665\times10^{15}\,D_r^2 + 3.39273\times10^{15}\,GF\,D_r^2 + 3.35913\times10^{15}\,\text{Glucose}''[0]\,km_{glu} - 3.39273\times10^{15}\,D_r^2\,km_{glu} + 3.35913\times10^{15}\,GF\,D_r^2\,km_{glu} - 3.12131\times10^{15}\,D_r\,r_{glu} - 3.0904\times10^{15}\,D_r\,km_{glu}\,r_{glu}\big) \big/ \big(-2.14182\times10^{14}\,\text{Glucose}''[0] + 2.16324\times10^{14}\,D_r^2 - 2.14182\times10^{14}\,GF\,D_r^2 - 2.12061\times10^{14}\,\text{Glucose}''[0]\,km_{glu} + 2.14182\times10^{14}\,D_r^2\,km_{glu} - 2.12061\times10^{14}\,GF\,D_r^2\,km_{glu} + 1.97047\times10^{14}\,D_r\,r_{glu} + 1.95096\times10^{14}\,D_r\,km_{glu}\,r_{glu} - 9.85237\times10^{13}\,r_{glu}\,\mu_{max}\big)$$

4. Substitute new ki_{lac} solution into X’[0] which leaves km_{glu} as the only unknown parameter and thus km_{glu} is identifiable
5. Use X’[0] to solve for μ_{max}:

$$\mu_{max} \to \frac{1}{ki_{lac}}\,1.72191\times10^{-7}\big(1.99985\times10^{8}\,X'[0] + 9.19931\times10^{7}\,D_r + 1.2625\times10^{7}\,X'[0]\,ki_{lac} + 5.8075\times10^{6}\,D_r\,ki_{lac} + 1.98005\times10^{8}\,X'[0]\,km_{glu} + 9.10823\times10^{7}\,D_r\,km_{glu} + 1.25\times10^{7}\,X'[0]\,ki_{lac}\,km_{glu} + 5.75\times10^{6}\,D_r\,ki_{lac}\,km_{glu}\big)$$

6. Substitute new μ_{max} solution into Glucose’’[0] which leaves ki_{lac} as the only unknown parameter and thus ki_{lac} is now identifiable
7. μ_{max} can now be identified from X’[0] which means that all parameters have been identified and Model 2 is structurally identifiable