

Optics

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Part I

Rays and beams

Part II

Fourier optics

Part III

Light in matter

Part IV

Interference and Coherence e

Chapter 6

Interference

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By the end of this chapter you should be able to explain and experimentally demonstrate the operation of an interferometer.

Overview

Interference describes phenomena resulting from the superposition of two or more waves. We have already seen such phenomena in previous chapters: diffraction of light at apertures and Fourier optics in general, or rotation of polarization states in optically active media. In this chapter we will discuss interference phenomena in more detail. A prerequisite for interference is coherence, which is the topic of the next chapter. Here we assume that all waves are monochromatic and coherent, and only in the next chapter we discuss what this means.

This chapter consists of two parts: first we discuss the most common types of interferometers and their applications, then we move on to interference in planar stacks of dielectric materials, as used in dielectric filters.

Interference in general

In chapter 4 we have defined the intensity I of a wave as the temporal average of the Poynting vector projected on a reference surface:

$$I = \langle \mathbf{S} \cdot \mathbf{n} \rangle_T = \frac{c n \epsilon_0}{2} |E_0|^2 \quad (6.1)$$

where E_0 is the amplitude of the electrical field in one of the two orthogonal polarization eigen-states¹ $\hat{\mathbf{p}}$, i.e.

$$\mathbf{E}(\mathbf{r}, t) = E_0 \hat{\mathbf{p}} e^{i(\mathbf{k}\mathbf{r} - \omega t)} \quad (6.2)$$

¹ horizontal and vertical linear or RCP and LCP

Interference occurs when E_0 results from the superposition of several waves, i.e. $E_0 = \sum E_i$. These waves are summed *coherently*

$$I \propto \left| \sum_i E_i \right|^2 \quad (6.3)$$



in contrast to an incoherent sum

$$I \propto \sum_i |E_i|^2 \quad . \quad (6.4)$$

Waves in the same polarization eigen-state are summed coherently, waves in different polarization eigen-states are summed incoherently. In the latter case, no interference occurs. When only two (complex-valued) wave amplitudes contribute, we can write

$$I = \frac{c n \epsilon_0}{2} |E_1 + E_2|^2 \quad (6.5)$$

$$= \frac{c n \epsilon_0}{2} \{ |E_1|^2 + |E_2|^2 + 2|E_1| |E_2| \cos \Delta\phi \} \quad (6.6)$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \Delta\phi \quad (6.7)$$

where $\Delta\phi$ is the phase difference $\phi_2 - \phi_1$ between the waves with $E_i = |E_i|e^{i\phi_i}$. The detected intensity changes periodically with the phase difference $\Delta\phi$. When both waves have the same intensity, then

$$I = 2I_0(1 + \cos \Delta\phi) = 4I_0 \cos^2 \Delta\phi/2 \quad (6.8)$$

Michelson interferometer

In a Michelson interferometer one and the same beam splitter splits and recombines the waves that travel along the two arms. After recombination, they have acquired a phase difference $\Delta\phi$ due to a path length difference or a difference in index of refraction of the medium in the interferometer arm.

The Michelson interferometer has two outputs: one through the fourth facet of the beam splitter, and another one in reverse direction of the incoming beam. The beam-splitter is reciprocal and loss-free, i.e. its (complex valued) reflection and transmission coefficients r and t do not depend on the direction of travel, and their squares add up to one: $|r|^2 + |t|^2 = 1$. The fields at the outputs are

$$E_a = E_0 (r t + t r e^{i\Delta\phi}) \quad (6.9)$$

$$E_b = E_0 (r r + t t e^{i\Delta\phi}) \quad (6.10)$$

Energy conservation requires that

$$|E_a|^2 + |E_b|^2 = |E_0|^2 \quad (6.11)$$

independent of $\Delta\phi$. This is fulfilled when the phases of reflection and transmission coefficients differ by 90 degrees, i.e.

$$r = |r|e^{i\alpha} \quad \text{and} \quad t = |t|e^{i(\alpha \pm \pi/2)} \quad (6.12)$$

which is equivalent with

$$\frac{r}{t} = -\frac{r^*}{t^*} \quad . \quad (6.13)$$

The Fresnel coefficients of a single interface fulfill this requirement, as do all combinations of interfaces and materials. When one output of the interferometer is at its maximum, the other is at its minimum. Total power is conserved.

Mach-Zehnder interferometer

In a Mach-Zehnder interferometer, one beam splitter separates the waves and a second beam splitter recombines them. Again, we have two output ports of the interferometer. In a Mach-Zehnder interferometer, they are more easily accessible than in a Michelson interferometer because the incoming beam takes a different path. When the path length in both arms is the same ($\Delta\phi = 0$) the symmetric output takes all power, i.e., each beam is transmitted and reflected once.

Mach-Zehnder interferometers are used in optical telecommunications to modulate the intensity of a beam in an optical fiber. By slightly changing the refractive index in one arm of the interferometer, a phase difference is built up that switches one output from bright to dark. The power of the second output is then dumped.

Sagnac interferometer

The Sagnac interferometer is an example of a common-path interferometer. Both beams travel the same path, but in reverse direction. When both beams have seen the same index of refraction, the total power is reflected back through the input port, as in this direction, each beam is reflected and transmitted once by the only beam splitter of the interferometer.

Only a few effects are capable of causing an effective path length difference for the counterpropagating beams: The interferometer could rotate, magneto-optical effects of non-reciprocal media could be involved, or short laser pulses could interact with a rapidly changing medium. The rotating Sagnac interferometer is the basis of a laser gyroscope. The phase difference between the beams depends on the angular velocity of rotation ω , the area A of the interferometer, and the dot product between their directions:

$$\phi = \frac{8\pi}{\lambda c} \omega \cdot \mathbf{A} \quad (6.14)$$

A fiber-loop increases the effect.

Fabry-Perot interferometer

All of the examples above were two-wave interference. The Fabry-Perot interferometer is an example of multiwave interference. It consists of two parallel, partially reflecting mirrors. For simplicity, we will assume that both mirrors are identical. The transmitted beam is composed of several waves: the directly transmitted wave, a wave with one round trip, a wave with two round trips, and so on. Compare to the directly transmitted wave E_1 , all other waves acquire for each round trip one complex factor h that contains two reflections at the mirrors and a phase lag due to the distance between the mirrors. As equation, this reads

$$E_{\text{out}} = E_1 (1 + h + h^2 + \dots) = \frac{E_1}{1 - h} \quad (6.15)$$

The intensity is thus

$$I = \frac{I_0}{|1 - h|^2} = \frac{I_0}{(1 - |h|)^2 + 4|h| \sin^2 \phi/2} \quad (6.16)$$

with $h = |h|e^{i\phi}$. This can be written as

$$I = \frac{I_{\max}}{1 + (2\mathcal{F}/\pi)^2 \sin^2 \phi/2} \quad (6.17)$$

with the *finesse* \mathcal{F}

$$\mathcal{F} = \frac{\pi \sqrt{|h|}}{1 - |h|} \quad (6.18)$$

The higher the quality of the mirrors, the better their reflective, and the closer comes $|h|$ to one, and the higher the finesse becomes.

The Fabry-Perot interferometer is a spectral filter. Let us assume that the reflectivity r of the mirrors does not depend on the wavelength. The distance d between the mirrors leads to a wavelength-dependent phase shift ϕ

$$\phi = 2dk = \frac{4\pi d}{\lambda} = \frac{4\pi d\nu}{c} \quad (6.19)$$

with the optical frequency ν . We get a peak in transmission when $\phi/2 = n\pi$ or when the frequency of the optical wave changes by

$$\Delta\nu_{\text{FSR}} = \frac{c}{2d} \quad (6.20)$$

which is called the *free spectral range*. The FWHM of a peak is as phase $\delta\phi$ or frequency $\delta\nu$

$$\delta\phi = \frac{2\pi}{\mathcal{F}} \quad \delta\nu = \frac{c}{2d\mathcal{F}} \quad (6.21)$$

The finesse \mathcal{F} can be written as

$$\mathcal{F} = \frac{\delta\nu}{\Delta\nu_{\text{FSR}}} \quad (6.22)$$

and defines the usefulness of the interferometer by peak width per free span between the peaks. The Fabry-Perot interferometer is a narrow spectral filter, but its transmission function has repeated maxima. One must therefore know in advance that the expected signal is spectrally narrower than the free spectral range. Or you can cascade several interferometers of different fineness, with the first one acting as a pre-filter for the others. By tuning the distance between the mirrors, the Fabry-Perot interferometer can be used as a spectrometer.

Optics of layered media: Transmission and scattering matrix

We present here the T-matrix method, a versatile technique for studying the optical properties of layered media, i.e. stacks of unstructured films of materials with different dielectric functions. These can be dielectric materials leading to e.g. Bragg reflections and dielectric filters, or metal films leading to surface plasmons. We will discuss transmission and reflection of these stacks.

In a layered medium, a wave traveling through the stack of layers is partially reflected and partially transmitted at each interface. The multiple reflections interfere with each other. To keep track of this, we use in each layer

a combined wave traveling in the $+z$ direction and one traveling in the $-z$ direction. These waves mix at interfaces. This formalism is described in chapter 7 of Saleh and Teich, 1991 and Yeh, 2005. A similar formalism with an E and B field traveling in the same direction is described in Pedrotti et al., 2008 and Macleod, 2001.

Let us assume that we have left of the interface a wave traveling to the right ($+z$ direction) of amplitude U_1^+ , and one wave traveling to the left of amplitude U_1^- . On the right side of the interface, we get the amplitudes U_2^\pm by multiplication with a *transmission* or *transfer* matrix \mathbf{M}

$$\begin{pmatrix} U_2^+ \\ U_2^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} U_1^+ \\ U_1^- \end{pmatrix} = \mathbf{M} \begin{pmatrix} U_1^+ \\ U_1^- \end{pmatrix} \quad (6.23)$$

Below we will derive transmission matrices \mathbf{M}_i for every interface and the homogeneous space in between. The full stack can then be described by a product matrix, multiplying together all partial matrices \mathbf{M}_i along the stack

$$\mathbf{M}_{\text{total}} = \mathbf{M}_n \cdot \mathbf{M}_{n-1} \cdots \mathbf{M}_2 \cdot \mathbf{M}_1 \quad (6.24)$$

This is a very convenient feature of the transmission matrix. Note that we label the interactions from left to right with 1 to n , but the matrices are multiplied from right to left, as mathematics has its origin in Arabic culture.

An inconvenient feature of the transmission matrix is that its matrix elements have no direct physical meaning. The problem is that we multiply on the matrix a vector that is half an input, half an output of this interface. We know what comes out (travels to the left), and the matrix should tell us what comes in from the other side. In this sense, the related *scattering* matrix \mathbf{S} is closer to physical meaning:

$$\begin{pmatrix} U_2^+ \\ U_1^- \end{pmatrix} = \begin{pmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{pmatrix} \cdot \begin{pmatrix} U_1^+ \\ U_2^- \end{pmatrix} = \mathbf{S} \begin{pmatrix} U_1^+ \\ U_2^- \end{pmatrix} \quad (6.25)$$

The scattering matrix connects waves traveling towards the interface with those traveling away from the interface. The entries t_{ij} and r_{ij} are the transmission and reflection coefficients for the amplitudes of the waves traveling from i to j (i.e. 12 is traveling towards the right, $+z$ direction). However, for the scattering matrix \mathbf{S} , the full stack can not be calculated by multiplying together all partial matrices.

It is therefore convenient to switch between both representations, derive the scattering matrix \mathbf{S} for each situation, and then convert into a transmission matrix \mathbf{M} . The relations are²

$$\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{t_{21}} \begin{pmatrix} t_{12}t_{21} - r_{12}r_{21} & r_{21} \\ -r_{12} & 1 \end{pmatrix} \quad (6.26)$$

$$\mathbf{S} = \begin{pmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} AD - BC & B \\ -C & 1 \end{pmatrix} \quad (6.27)$$

as long as D or t_{21} are not zero.

The transmission in backward direction t_{21} is thus the reciprocal of the D -element of $\mathbf{M}_{\text{total}}$. The transmission in forward direction is

$$t_{12} = \frac{\det \mathbf{M}_{\text{total}}}{D} \quad (6.28)$$

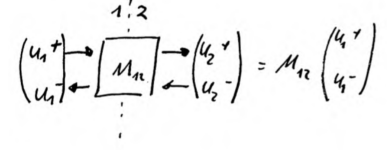


Figure 6.1: The operation of the transmission matrix

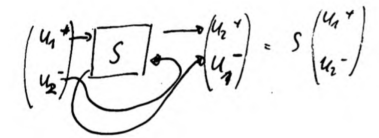


Figure 6.2: The operation of the scattering matrix

² Saleh and Teich, 1991 eq. 7.7

and similar for the reflection from the front side

$$r_{12} = -\frac{C}{D} \quad . \quad (6.29)$$

Electrical fields

We need to define the physical meaning of the amplitudes U_i^\pm to be able to calculate the reflection (r_{ij}) and transmission (t_{ij}) coefficients. We assume plane waves

$$\mathbf{E} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = \hat{\mathbf{E}} U e^{i k_z z} e^{i k_x x} e^{-i \omega t} \quad (6.30)$$

where the wave vector \mathbf{k} lies in the xz -plane, U defines the amplitude of the wave and $\hat{\mathbf{E}}$ the polarization direction. With the full length of the wave vector in vacuum $k_0 = 2\pi/\lambda$ and the refractive index n of the medium we get

$$k_z^2 + k_x^2 = n^2 k_0^2 \quad . \quad (6.31)$$

The polarization directions are

$$\hat{\mathbf{E}}^{(s)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{E}}^{(p)} = \frac{1}{n k_0} \begin{pmatrix} \pm k_z \\ 0 \\ k_x \end{pmatrix} \quad . \quad (6.32)$$

The \pm -sign takes the sign of the direction of travel, see Fig. 2.2 in Novotny and Hecht, 2012. Note that with this definition we have $|\hat{\mathbf{E}}| = 1$, which differs from problem 12.4 in Novotny and Hecht, 2012.

The left and right traveling waves are thus

$$\mathbf{E}^+ = \hat{\mathbf{E}} U^+ e^{+i k_z z} \quad \text{and} \quad \mathbf{E}^- = \hat{\mathbf{E}} U^- e^{-i k_z z} \quad (6.33)$$

where we have split off the global term $e^{i(k_x x - \omega t)}$.

Propagation matrix

Before we come to interfaces, let us discuss the transmission matrix of a homogeneous material layer j of thickness d_j and (complex) refractive index n_j . Relevant is the z -component of the (complex) wave vector $k_{z,j}$. Note that we do *not* use the sign of $k_{z,j}$ to describe the direction of travel. Independent of the propagation direction, each wave sees a reflection coefficient $r = 0$ and a (complex) transmission coefficient t

$$t = t_{12} = t_{21} = e^{+i k_{z,j} d_j} \quad . \quad (6.34)$$

The transmission matrix of a homogeneous medium is thus

$$\mathbf{M} = \begin{pmatrix} e^{+i k_{z,j} d_j} & 0 \\ 0 & e^{-i k_{z,j} d_j} \end{pmatrix} \quad . \quad (6.35)$$

Interface matrix

The transmission and reflection coefficients of an interface are the Fresnel coefficients r and t for s and p polarization, as defined in chapter 4. We assume non-magnetic materials ($\mu = 1$).

With eq. 6.26 we get for both polarization directions the transmission matrix

$$\mathbf{M}_{12} = \frac{1}{t_{21}} \begin{pmatrix} 1 & r_{21} \\ r_{21} & 1 \end{pmatrix}, \quad (6.36)$$

as

$$t_{12}t_{21} - r_{12}r_{21} = t_{21}^2 \frac{k_{z,1}}{k_{z,2}} + r_{21}^2 = 1. \quad (6.37)$$

Note that the transmission matrix from medium 1 to medium 2 uses the Fresnel coefficients of the backwards direction! We can abbreviate this to³ (see also appendix at the end of this chapter)

³ In problem 12.4 in Novotny and Hecht, 2012 the leading $1/\eta$ seems to be missing!

$$\mathbf{M}_{12} = \frac{1}{2\eta} \begin{pmatrix} 1 + \kappa & 1 - \kappa \\ 1 - \kappa & 1 + \kappa \end{pmatrix} \quad (6.38)$$

with

$$\kappa = \eta^2 \frac{k_{z,1}}{k_{z,2}} \quad \text{and} \quad \eta^s = 1 \quad \text{or} \quad \eta^p = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \quad (6.39)$$

The factors η in front of the transmission matrix \mathbf{M}_{12} can be collected in front of the total transmission matrix $\mathbf{M}_{\text{total}}$, in case one is not interested in the distribution of the fields inside the stack. Then, all η^p collapse into $\sqrt{\epsilon_{\text{first}}/\epsilon_{\text{last}}}$, which is equal to one in case the terminating half-spaces of the layered medium have both the same dielectric constant.

Appendix: derivation of eq. 6.38

We start from

$$\mathbf{M}_{12} = \frac{1}{t_{21}} \begin{pmatrix} 1 & r_{21} \\ r_{21} & 1 \end{pmatrix} \quad (6.40)$$

and abbreviate the Fresnel coefficients as

$$r_{21}^s = \frac{k_{z,2} - k_{z,1}}{k_{z,1} + k_{z,2}} = \frac{b - a}{a + b} \quad (6.41)$$

$$t_{21}^s = \frac{2k_{z,2}}{k_{z,1} + k_{z,2}} = \frac{2b\eta}{a + b} \quad (6.42)$$

$$r_{21}^p = \frac{\epsilon_1 k_{z,2} - \epsilon_2 k_{z,1}}{\epsilon_2 k_{z,1} + \epsilon_1 k_{z,2}} = \frac{b - a}{a + b} \quad (6.43)$$

$$t_{21}^p = \frac{2\sqrt{\epsilon_1 \epsilon_2} k_{z,2}}{\epsilon_2 k_{z,1} + \epsilon_1 k_{z,2}} = \frac{2b\eta}{a + b} \quad (6.44)$$

with $a = \epsilon_2 k_{z,1}$, $b = \epsilon_1 k_{z,2}$ and $\eta = \sqrt{\epsilon_2/\epsilon_1}$. In the case of s-polarization, the ϵ_i are ignored / set to one. With this we get

$$\mathbf{M}_{12} = \frac{a + b}{2b\eta} \begin{pmatrix} 1 & (b - a)/(a + b) \\ (b - a)/(a + b) & 1 \end{pmatrix} = \frac{1}{2b\eta} \begin{pmatrix} b + a & b - a \\ b - a & b + a \end{pmatrix} \quad (6.45)$$

$$= \frac{1}{2\eta} \begin{pmatrix} 1 + \frac{a}{b} & 1 - \frac{a}{b} \\ 1 - \frac{a}{b} & 1 + \frac{a}{b} \end{pmatrix} = \frac{1}{2\eta} \begin{pmatrix} 1 + \kappa & 1 - \kappa \\ 1 - \kappa & 1 + \kappa \end{pmatrix} \quad (6.46)$$

with

$$\kappa = \frac{a}{b} = \eta^2 \frac{k_{z,1}}{k_{z,2}} \quad \text{and} \quad \eta^s = 1 \quad \text{or} \quad \eta^p = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \quad (6.47)$$

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Appendices

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