ENGINEERING MATH II

Topic 6.4 (Ch.12.4)

D'Alembert's Solution of Wave Equation

- **Motivation:**
  - Our previous solution for $\dot{u}(x,0)=0$ case:

$$\left[\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[\alpha_n \cos\left(\frac{cn\pi}{L}t\right)\right] \\ \alpha_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}\right]$$

    is not very simple.
  - With some transformation:

$$u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right] \quad ; \text{ for odd func } f^*$$

  - D'Alembert observed: a solution can be obtained by introducing new variables: $v \equiv x+ct$, $w \equiv x-ct$.
  - Let's follow this observation and see where it goes:

- **D'Alembert's Solution:**
  - Let $u(x,t) \rightarrow u(v,w)$

$$v = x+ct$$
$$w = x-ct$$

  - The derivatives:

$$\frac{\partial u}{\partial x} = u_x = u_v v_x + u_w w_x = u_v + u_w \quad ; \quad v_x = 1,\ w_x = 1$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = (u_v+u_w)_x = (u_v+u_w)_v v_x + (u_v+u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

$$\frac{\partial}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial}{\partial w} \qquad u_{vw} = u_{wv}$$

$$\frac{\partial^2 u}{\partial t^2} = u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

  - The new wave eqn:

$$c^2(u_{vv} - 2u_{vw} + u_{ww}) = (u_{vv} + 2u_{vw} + u_{ww})c^2$$

$$\boxed{u_{vw} \equiv \frac{\partial^2 u}{\partial w \partial v} = 0}$$

- The nice thing about $\frac{\partial^2 u}{\partial w \partial v} = 0$ is that it only has one term. Can solve via direct integration:

$$\frac{\partial u}{\partial v} = h(v)$$

$$u = \int h(v)\,dv + \psi(w)$$

$$\equiv \phi(v) + \psi(w)$$

$$= \underbrace{\phi(x+ct) + \psi(x-ct)}_{\text{d'Alembert's Solution.}}$$

⇐ we haven't actually found what's ϕ and ψ.

- **Finding ϕ and ψ using initial conditions**

  - Initial conditions we'll consider:

$$\begin{bmatrix} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{bmatrix}$$

  - Plug in the IC. to the eqn:

$$u(x,0) = \phi(x) + \psi(x) = f(x)$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$$

Divide by c and integrate.

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

$$\begin{bmatrix} \phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + k(x_0) \\ k(x_0) = \phi(x_0) - \psi(x_0) \end{bmatrix}$$

Note: $\frac{\partial u}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial t}$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial t} = \frac{\partial \phi}{\partial v} \cdot c = \frac{\partial \phi}{\partial (x+ct)} \cdot c$$

At: $t=0$:

$$= \frac{\partial \phi}{\partial x} \cdot c = c\phi'$$

(+)

$$2\phi(x) = f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + k(x_0)$$

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}k(x_0) \quad (1)$$

(−)

$$2\psi(x) = f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)\,ds - k(x_0)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2}k(x_0) \quad (2)$$

- Final solution:

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

This looks like ① + ② but not quite.
① and ② are function of x.
We need to replace x with $(x+ct)$ and $(x-ct)$, respectively:

$$\phi(x+ct) = \frac{f}{2}(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \frac{1}{2}k(x_0)$$

$$\psi(x-ct) = \frac{f}{2}(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \frac{1}{2}k(x_0)$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

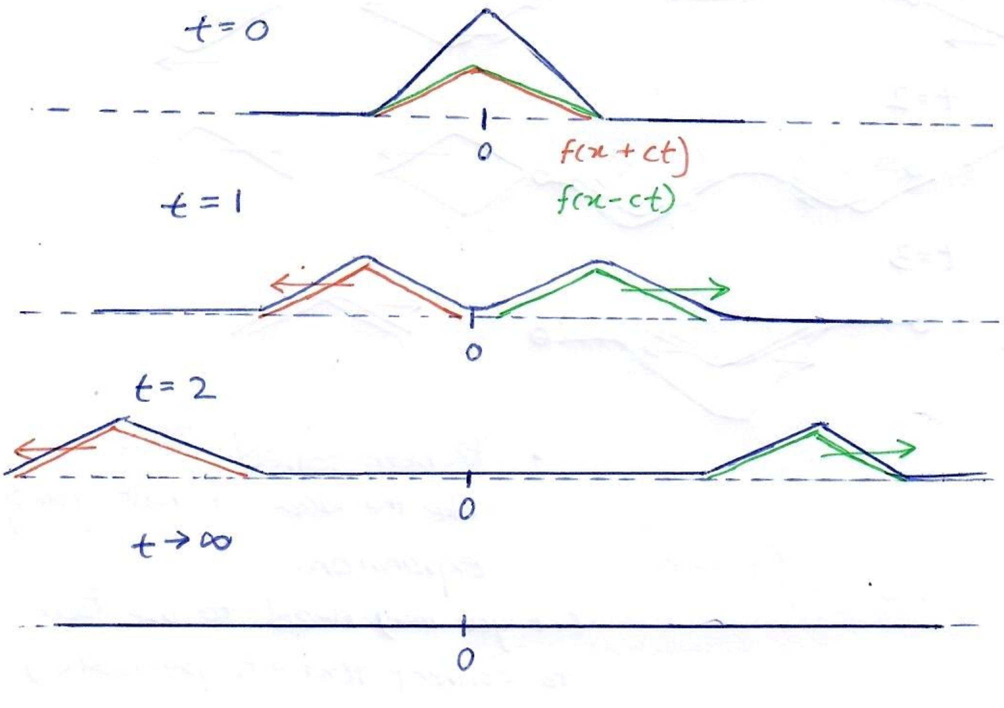
- If the initial velocity $\dot{u}(x,0) = g(x) = 0$:

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]}$$

- Boundary Conditions:

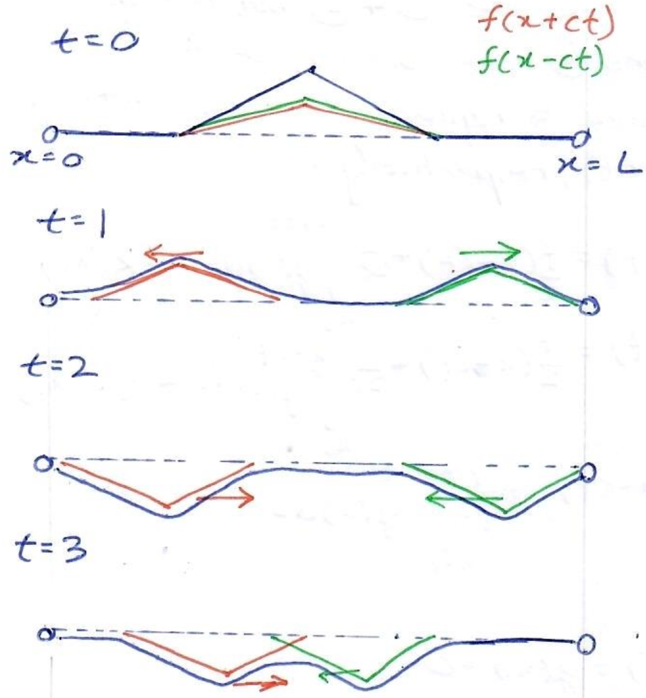
The solution above did not consider boundary conditions.
(i.e. it could also work for a free / infinitely long string).

E.g. Infinite string:



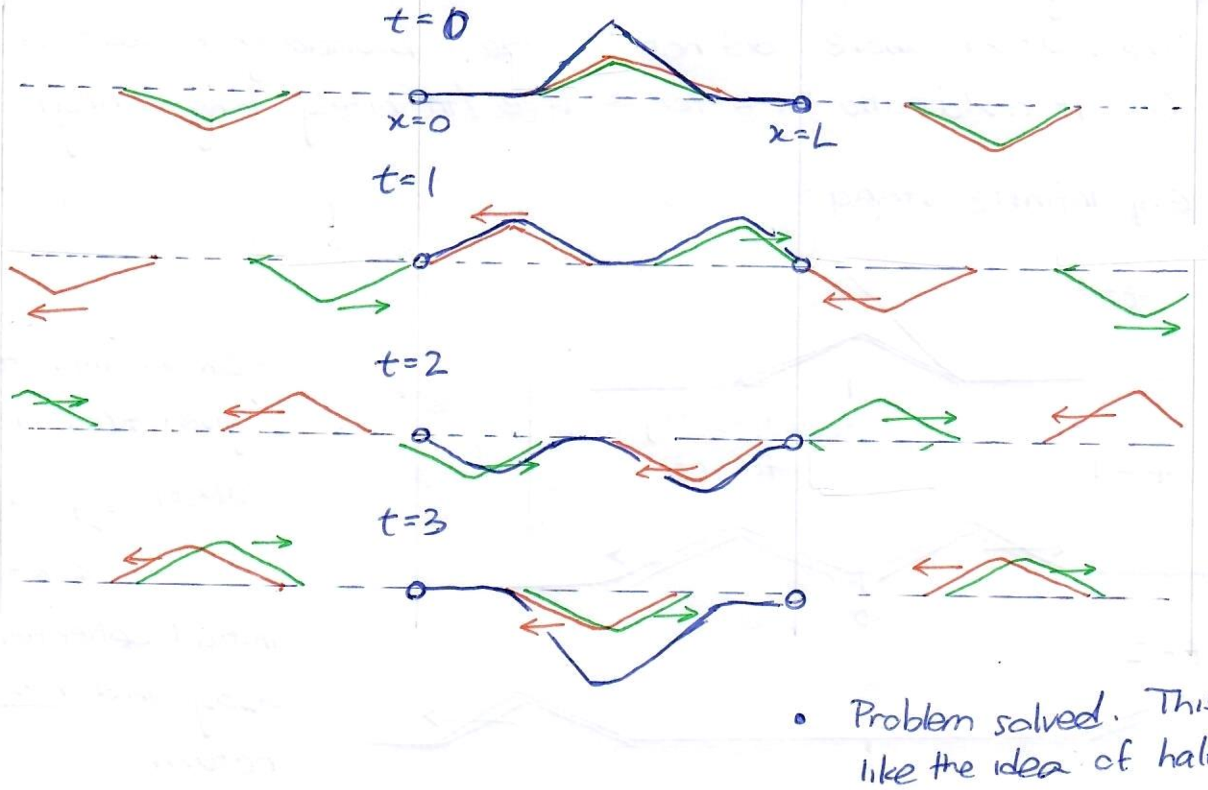
- On an infinite string, given $f(x)$ and $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$, the initial deflections move away and never return.
- That's OK because the string is infinite.

E.g. String tied at two ends:



- When the string is tied at the ends, whatever wave traveling to the end must reflect back and in the reversed amplitude.
- This is basic physics, you can try it using ropes in the gym.
- Obviously our previous eqn don't describe this real behavior.
  How do we modify the eqn?

- Method: Make $f(x)$ an odd periodic function with $P=2L$:



- Problem solved. This is like the idea of half-range expansion.

(i.e. you may need to use Fourier to convert $f(x)$ into periodic)