ENGINEERING MATH II

Topic 6.1 (Ch. 12.1)

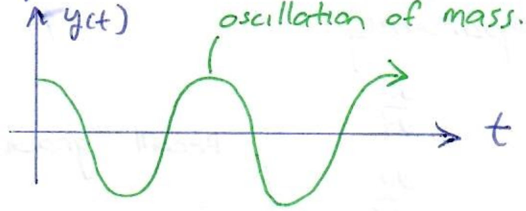
Basic Concepts of PDE's.

• Motivation:

- When your problem is zero-dimensional, such as a point-mass suspended on spring:

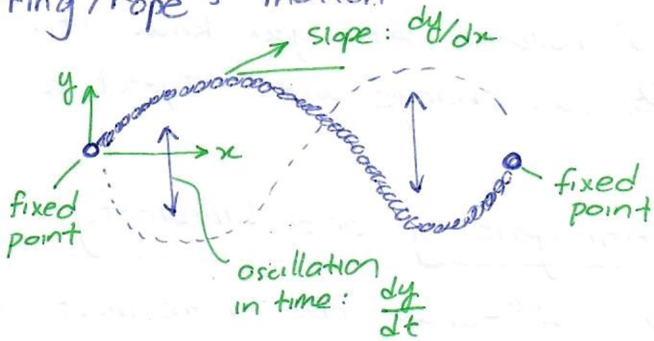
$$y'' + ky = 0$$

The result can only depend on one variable such as time: $y(t)$



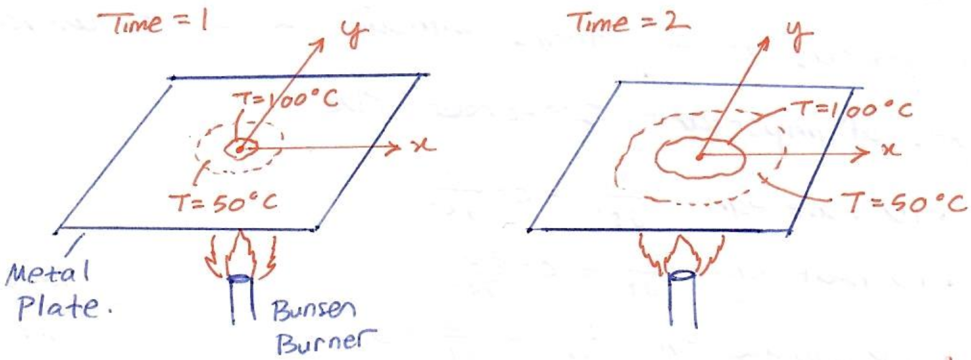
- But many real-life problems have higher dimensions.

E.g. A string/rope's motion:



Here, the answer (the rope's height y) is a function of space x, and time t.

E.g. Heat diffusion:



Here, the temperate of the metal plate can change according to 2 spatial directions and time: $T(x, y, t)$

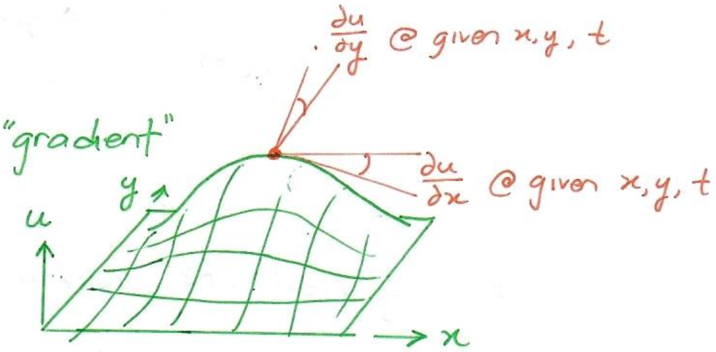
• What is PDE

- Partial Differential Equation is basically a DE where the $\underbrace{\text{variable of interest}}_{\text{or the "unknown function"}}$ depends on ≥ 2 variables.

- Specifically, we take the derivative of unknown function $u(x, y, t, \ldots)$ in terms of each independent variable separately:

$$\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}$$

Recall "gradient"



- The "∂" notation let's you know it's a partial derivative (i.e. u can change with respect to other variables.)

• Order, Homogeneity and Linearity:

- Like ODE, 2nd-order PDE is the most useful for engineering.
- It is linear if we don't have multiplication of u or its derivative to each other.
- Homogeneous if any terms without u or its derivative $= 0$.

Examples of important 2nd-order PDE:

- 1D wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
- 1D heat eqn: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- 2D Laplace eqn: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- 2D wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
- 3D Laplace eqn: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Discuss:
What is meant by "important".

• Concept of Solution in PDE

- Fundamentally: any u that satisfies the PDE is a solution.

E.g. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Solution 1

$u = x^2 - y^2$

$\dfrac{\partial u}{\partial x} = 2x \quad \dfrac{\partial u}{\partial y} = -2y$

$\dfrac{\partial^2 u}{\partial x^2} = 2 \quad \dfrac{\partial^2 u}{\partial y^2} = -2$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ ✓

Solution 2

$u = e^x \cos y$

$\dfrac{\partial u}{\partial x} = e^x \cos y \quad \dfrac{\partial u}{\partial y} = -e^x \sin y$

$\dfrac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \dfrac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$ ✓

Solution 3

$u = \ln(x^2 + y^2)$

$\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2+y^2} \quad \dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2+y^2}$

$\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{2(x^2-y^2)}{(x^2+y^2)^2} \quad \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2(y^2-x^2)}{(x^2+y^2)^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2(x^2-y^2)}{(x^2+y^2)^2} + \dfrac{2(x^2-y^2)}{(x^2+y^2)^2} = 0$ ✓

- Key message:
  - If you think ODE had a lot of solutions, PDE have even more!
  - PDE solutions can come from very different functions. (e.g. poly, trig. log).

  * In general, more complex equation = bigger solution space.

- For homog. linear PDE, if u_1, u_2 are solutions in region R, then:

  $u_3 = c_1 u_1 + c_2 u_2$ is also a solution.

- Bounding conditions:
  - In ODE, we have IVP that uses initial conditions $(y(t=0) = \ldots)$ to find constants $c_1, c_2 \ldots$ etc.
  - In PDE, since $u(t, x, y \ldots)$ is (or can be) a function of time and space, we need initial condition $(u(t=0))$ and boundary conditions $(u(x_0, y_0))$ to find/bound the problem.

    ↳ * Discuss concept of B.C.

• How to Solve PDE :

① Some PDE can be solve like ODE ... <u>if</u> you're lucky:

Eg. $\frac{\partial^2 u(x,y)}{\partial x^2} - u(x,y) \equiv u_{xx} - u = 0$

Although $u = u(x,y)$, the PDE only involves $\frac{\partial^2}{\partial x^2}$:

- Solve like $u'' - u = 0 \rightarrow u = Ae^{x} + Be^{-x}$
- Don't forget A, B can be function of y :

$$u(x,y) = A(y)e^{x} + B(y)e^{-x}$$

i.e. If $A = A(x,y)$ it wouldn't behave like const. in $\frac{\partial}{\partial x}$.
But $A = A(y)$ is like a const. towards $\frac{\partial}{\partial x}$.

Eg.

$u_{xy} = -u_x$.

Here the key is that we can play substitution of variable $u_x \equiv P$:

$P_y = -P$

$\frac{P_y}{P} = -1$

$\downarrow$ integrate in y

$\ln|P| = -y + C(x)$ ← $C(x)$ behaves like a const. towards $\frac{\partial}{\partial y}$.

$\therefore P = c(x)e^{-y}$

$\downarrow$ since $P = \frac{\partial u}{\partial x}$, integrate P to get u:

$u(x,y) = f(x)e^{-y} + g(y)$; $f(x) = \int c(x)\,dx$

② In general, it is <u>not</u> so easy to solve PDE.
In fact, mathematicians / scientists devote careers to solving <u>specific</u> important PDE (eg. wave eqn, heat eqn, Navier-Stokes).

This course will focus on PDE related to <u>wave equations</u> only (i.e. model of vibrating string).