ENGINEERING MATH II

Topic 6.2 (Ch 12.2)

Intro to Wave Equations

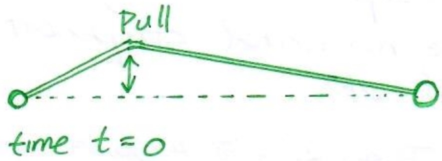
- When we say "wave", we refer to the waves traveling on a vibrating string. (not ocean wave...).
- Ch. 12.2, 3, 4 are all dedicated to this wave eqn.
- We begin by $\boxed{\text{finding/deriving the wave eqn}}$ from modeling an elastic piece of string (e.g. rubber band or guitar):

① String:

②

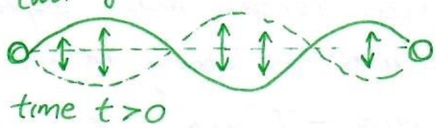
- stretch it and tie to two fixed points.

③

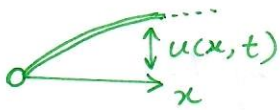


- Pull the string to some initial shape.

④ Twang~!!



- Release and watch it twang.
- I want to find $u(x,t)$., the vertical displacement of string at any time and position.



* Assumptions:

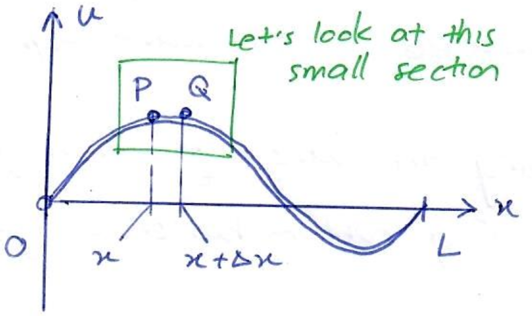
① String has uniform density ρ.

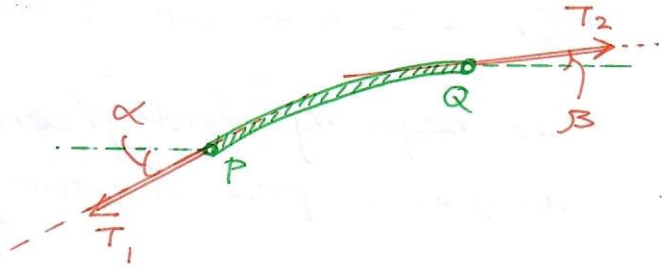
② The tension from stretching (T) is much larger than force of gravity (so ignore gravity).

③ The problem is 1-D. Also, assume $u(x,t)$ is small, so that any point on the string is displaced only vertically.

④ The string has no bending stiffness.

- Engineering Modeling:
  - In modeling (especially using DE), we focus on a very, very small section of the overall problem.
  - Eg. let us model only a small section of the whole string:





- T_1, T_2 are tension forces at P, Q, respectively.
- Since the string is free to bend, T_1, T_2 are tangential to the local slope (angles α, β).
- Eqn of Motion (horizontal): By assumption the string has no horizontal motion, so the horizontal components of T_1, T_2 must cancel out:

$$T_1 \cos\alpha = T_2 \cos\beta = \text{const.} = T$$

- Eqn of Motion (vertical): The vertical forces must equal to acceleration of the mass of string between P-Q:

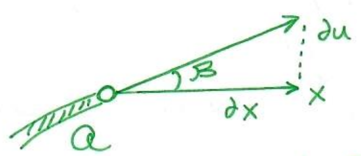
$$T_2 \sin\beta - T_1 \sin\alpha = \rho\,\Delta x \frac{\partial^2 u}{\partial t^2} \quad ; \quad \rho = \left[\text{kg}/\text{m}\right]$$

lengthwise density.

- Algebra: divide the vertical forces by T:

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha$$

$$= \frac{\rho\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$



Since tangent = slope.
$\tan\beta = \frac{\partial u}{\partial x}$, then: $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_x$

$\tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}$

- In combination: $\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_x\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$

As $\Delta x \to 0$: $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$; $c^2 = \frac{T}{\rho}$ ⇐ 1-D WAVE EQUATION!

always ≥ 0