ENGINEERING MATH II

Topic 6.5 (Ch. 12.5) + 12.6

Heat Equation.

⇒ We spent a lot of time on wave eqn.
Here we'll show the heat eqn.

- **Fundamental Physics:**
  - For 1D problem, heat flow (v) from high to low temperature (u) in proportion to the temperature gradient:

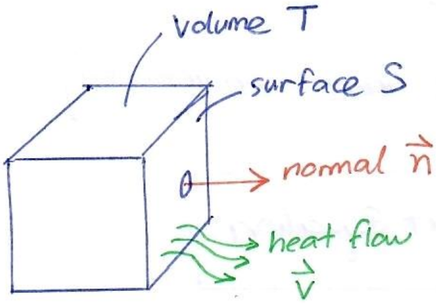
$$v = -K \frac{du}{dx}$$

  - For 3D, use vector form:

$$\vec{v} = -K \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial u/\partial z \end{bmatrix} = -K \operatorname{grad} u. \quad ; \; u = u(x, y, z, t)$$

  - Assume in your object: $\begin{bmatrix} \text{thermal conductivity } K \\ \text{specific heat } \sigma \\ \text{density } \rho \end{bmatrix}$ are all constant.

- **3D Model:**



$\vec{v} \cdot \vec{n} > 0$ heat flow out of T through S
< 0 heat flow in

The amount of heat leaving through a surface S:

$$\iint_S \vec{v} \cdot \vec{n} \, dA$$

- Using Gauss's Divergence Theorem:

Heat leaving S = Time-rate change of heat energy in T.

$$\iint_S \vec{v}\cdot\vec{n}\,dA = -K\iint_S (\text{grad}\,u)\cdot\vec{n}\,dA = -K\iiint_T \text{div}\,(\text{grad}\,u)\,dx\,dy\,dz$$

$$= -K\iiint_T \nabla^2 u\,dx\,dy\,dz$$

where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ is a Laplacian of u.

- The total heat energy in T is also:

$$H = \iiint_T \delta\rho u\,dx\,dy\,dz$$ using the material properties δ and ρ.

$$-\frac{\partial H}{\partial t} = -\iiint_T \delta\rho\frac{\partial u}{\partial t}\,dx\,dy\,dz$$

- Universal Law: Heat leaving = time-rate change of energy in T:

$$-\iiint_T \delta\rho\frac{\partial u}{\partial t}\,dx\,dy\,dz = -K\iiint_T \nabla^2 u\,dx\,dy\,dz$$

$$-\iiint_T \frac{\partial u}{\partial t}\,dx\,dy\,dz = -c^2\iiint_T \nabla^2 u\,dx\,dy\,dz \quad ; \quad c^2 = \frac{K}{\delta\rho}$$

(Thermal Diffusivity)

$$\iiint_T \left(\frac{\partial u}{\partial t} - c^2\nabla^2 u\right)dx\,dy\,dz = 0$$

- Since the eqn hold for any T, assume there is an infinitely small T. Then:

$$\boxed{\frac{\partial u}{\partial t} = c^2\underbrace{\nabla^2 u}_{\nabla^2 u \text{ also models other diffusion process.}} \quad : \text{Heat Equation.}}$$

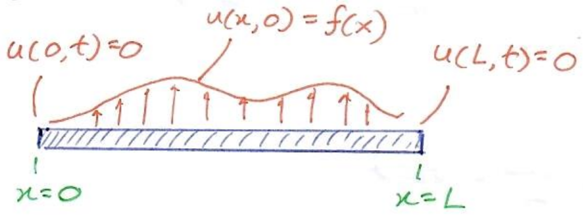
• 1D Heat Equation. with Isothermal BC.

• Our wave equation : $u_{tt} - c^2 u_{xx} = 0$

heat equation : $u_t - c^2 u_{xx} = 0$

They are very similar. Solution will employ almost same steps.

• Model:



• Assume narrow long bar ($\approx$ 1 dimension).

• The ends ($x=0, L$) are "isothermal".
i.e. kept at $u=0$ no matter what.

• Given initial temperature $f(x)$

$\Rightarrow$ Find $u(x,t)$

• Separation of variable technique:

① Assume: $u(x,t) = F(x)\,G(t)$

② Heat Eqn becomes: $F\dot{G} = c^2 F'' G$; $\dot{G} \equiv \frac{dG}{dt}$, $F'' \equiv d^2F/dx^2$

$\downarrow$

③ Separate : $\underbrace{\frac{\dot{G}}{c^2 G}}_{t} = \underbrace{\frac{F''}{F}}_{x \text{ function.}} = k$

④ For $k=0$, and $k>0$, only $u \equiv 0$ works. $\leftarrow$ not interesting

For $k<0$: let $k = -p^2$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2$$

⑤ Two ODE's:

$\dot{G} + c^2 p^2 G = 0$

i) $\dot{G} + \lambda_n^2 G = 0$

$\lambda_n \equiv \frac{cn\pi}{L}$ $\qquad p = \frac{n\pi}{L}$

ii) Solution: $G_n(t) = B_n e^{-\lambda_n^2 t}$

$F'' + p^2 F = 0$

i) General solution : $F(x) = A\cos px + B\sin px$

ii) BC : $u(0,t) = F(0)\,G(t) = 0$

$u(L,t) = F(L)\,G(t) = 0$

since $G(t)=0 \rightarrow u=0$, not interesting,

$\therefore$ let: $\underset{=A}{F(0)=0}$, $\underset{=B\sin pL}{F(L)=0}$.

$\therefore$ If $B \neq 0$ ($u \neq 0$) : $\sin pL = 0 \rightarrow p = \frac{n\pi}{L}$, $n = 1, 2, \cdots$

iii) $F_n(x) = \sin\frac{n\pi x}{L}$; $n = 1, 2, \cdots$

6 Thus: $u_n(x,t) = G(t)\,F(x)$

$$= B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

Note: This is just one term out of many : $n = 1, 2, \dots, \infty$

7 The full solution (a Fourier Series):

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \;;\quad \lambda_n = \frac{cn\pi}{L}}$$

8 Bring in the IC:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

Fourier coefficient: $\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \;;\; n = 1, 2, \dots}$

- 1D Heat Eqn with "Adiabatic" (Insulated) BC.
  - The key difference from isothermal case, is that the end-point can be any temperature now, as long as heat flows across the ends are zero.
  - Heat flow $\propto$ temperature gradient

$$\therefore\quad u_x(0,t) = 0 \;\rightarrow\; F'(0)\,G(t) = 0$$

$$u_x(L,t) = 0 \;\rightarrow\; F'(L)\,G(t) = 0$$

If $F(x)$'s general solution was $F(x) = A\cos px + B\sin px$

$$F'(x) = -Ap\sin px + Bp\cos px.$$

$$\Downarrow$$

$$A = 1,\; B = 0,\; p = p_n = \frac{n\pi}{L} \;;\; n = 0, 1, 2, 3, \dots$$

$$\therefore\; u_n(x,t) = A_n \cos\frac{n\pi x}{L}\, e^{-\lambda_n^2 t} \;;\; \lambda_n = \frac{cn\pi}{L} \;;\; n = 0, 1, 2, \dots$$

using the IC: $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \;;\; n = 1, 2, \dots$$

- **2D Heat Equation.**

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

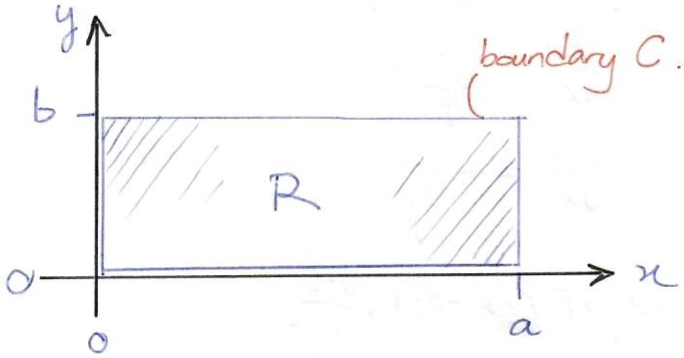
$$\Downarrow$$

If we assume steady : i.e. if u doesn't change with t.

$\frac{\partial u}{\partial t} = 0$ (usually when $t \to \infty$)

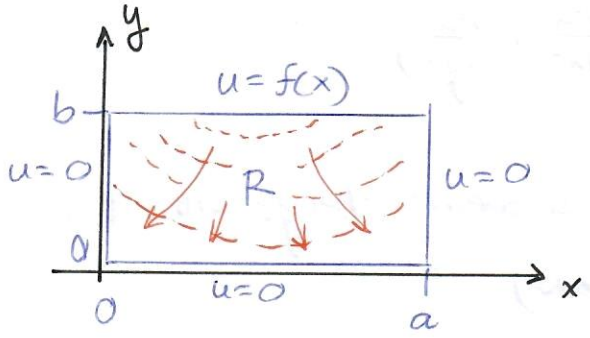
$\boxed{\therefore \nabla^2 u = 0}$ ← This is also Laplace's Eqn.

To make life simple, only consider a rectangular region R :



- This is called a boundary value problem (BVP) since time does not affect the solution.
- 3 kinds of boundary conditions :
  - (i) u is given on C (Dirichlet Problem)
  - (ii) Normal derivative $\partial u / \partial n$ given on C (Neumann Problem).
  - (iii) Mixed type (Robin Problem).

E.g. of Dirichlet Problem.



• Let's model the spread of heat from the top-edge ($u=f(x)$) through region R.

① The heat eqn $\nabla^2 u = 0$ is linear, homog, so we can use separation-of-variables:

$$u(x,y) = F(x)\,G(y)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$G(y)\frac{\partial^2 F}{\partial x^2} = -F(x)\frac{\partial^2 G}{\partial y^2}$$

$$\frac{1}{F}\cdot\frac{\partial^2 F}{\partial x^2} = -\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -k$$

2a $\quad \frac{\partial^2 F}{\partial x^2} + kF = 0$

BC of x: $\left\{ \begin{matrix} F(0)=0 \\ F(a)=0 \end{matrix} \right\}$

Using Fourier Series:

$$F_n(x) = \sin\frac{n\pi}{a}x \; ; n=1,2,\dots$$

where $k = \left(\frac{n\pi}{a}\right)^2$

i.e. Use FS to approx the ODE solution. And from there, get k.

2b $\quad \frac{d^2 G}{dy^2} - kG = \frac{\partial^2 G}{\partial y^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$

ODE solutions:

$$G_n(y) = A_n e^{n\pi\frac{y}{a}} + B_n e^{-n\pi\frac{y}{a}}$$

BC of y: $G_n(0) = 0 = A_n + B_n$

$\therefore B_n = -A_n$.

$$G_n(y) = A_n\left(e^{n\pi\frac{y}{a}} - e^{-n\pi\frac{y}{a}}\right)$$

$\downarrow$ $\sinh x = \frac{e^x - e^{-x}}{2}$

$$= \underbrace{A_n 2}\,\sinh\left(\frac{n\pi y}{a}\right)$$

$$\equiv A^* \sinh\left(\frac{n\pi y}{a}\right)$$

③ Combine:

$$u_n(x,y) = F_n G_n = A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

$\Downarrow$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}$$

④ Use the final BC : $u(x, y=b) = f(x)$:

$$u(x,b) = \sum_{n=1}^{\infty} \underbrace{A_n^*}\ \underbrace{\sin\left(\frac{n\pi x}{a}\right)}_{\text{If this is the Fourier basis}}\ \underbrace{\sinh\left(\frac{n\pi b}{a}\right)} = f(x)$$

$A_n^* \sinh\left(\frac{n\pi b}{a}\right)$ must be the Fourier coeff.

$$\boxed{\therefore\ A_n^* = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx}$$