Topic 6.4+: Method of Characteristics

To explore the "Method of Characteristics (MoC)" fully deserves a few weeks of classes.

But unfortunately we don't have that.

Here's just a quick "taste" of MoC that's just enough to let you know what it is, and… hopefully a bit more in depth and meaningful than the textbook.

Motivating Observation

- The solution of wave-equation in (x, t) space is quite complex:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} F_n(x)G_n(t) \;= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[\alpha_n \cos\left(\frac{cn\pi}{L}t\right)\right]$$

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

For every position and time, the local solution u must evaluate an infinite sum of $F_n(x)G_n(t)$.

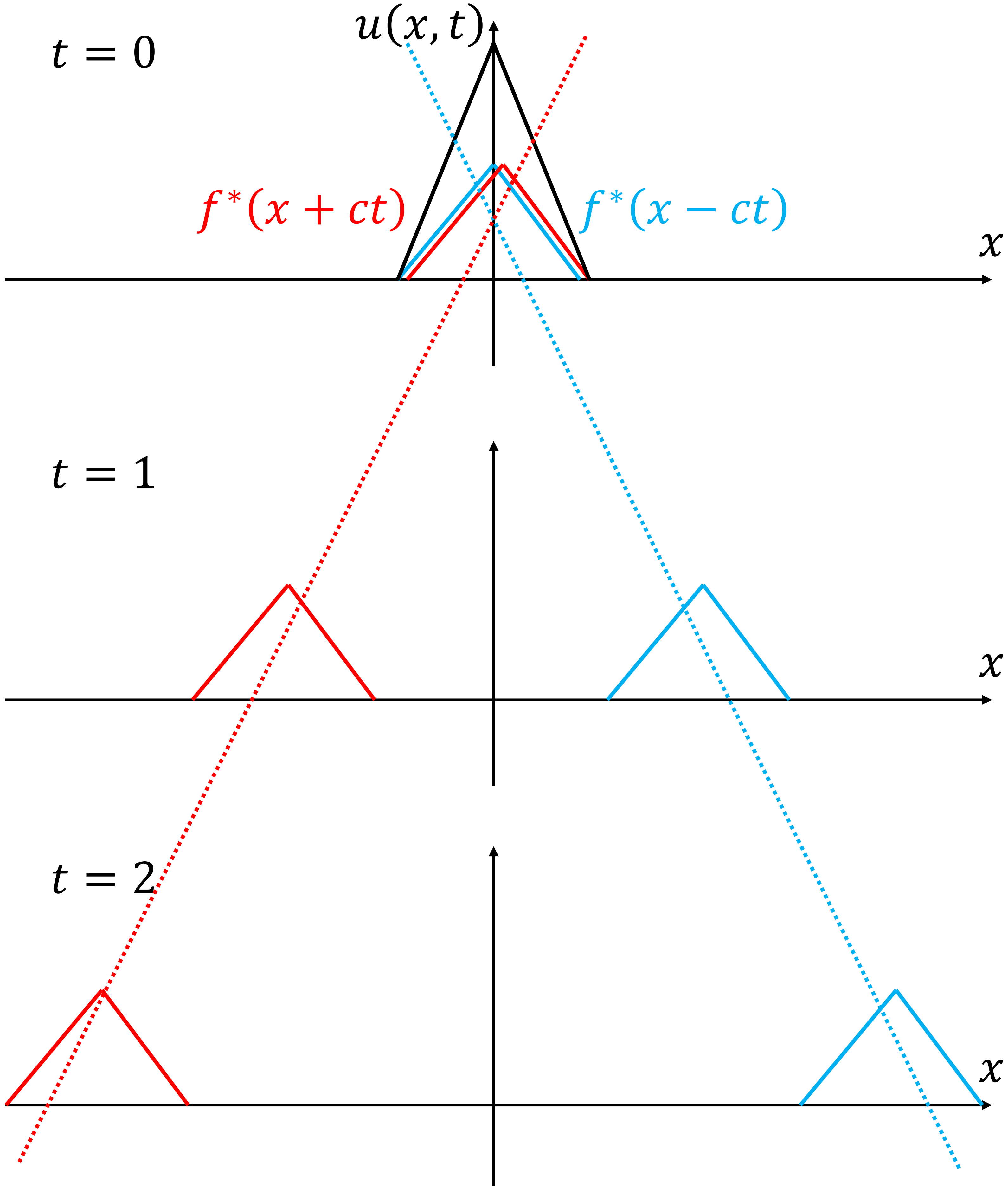
- But if we transform $u(x,t) \to u(v,w)$, where $v = \phi(x,t) = (x + ct)$ and $w = \psi(x,t) = (x - ct)$, the solution suddenly becomes very simple:

$$u(u,v) = f^*(v) + f^*(w)$$

The solution is the same function f^* moving along the lines v and w.
We can v and w the characteristic line of the wave equation.

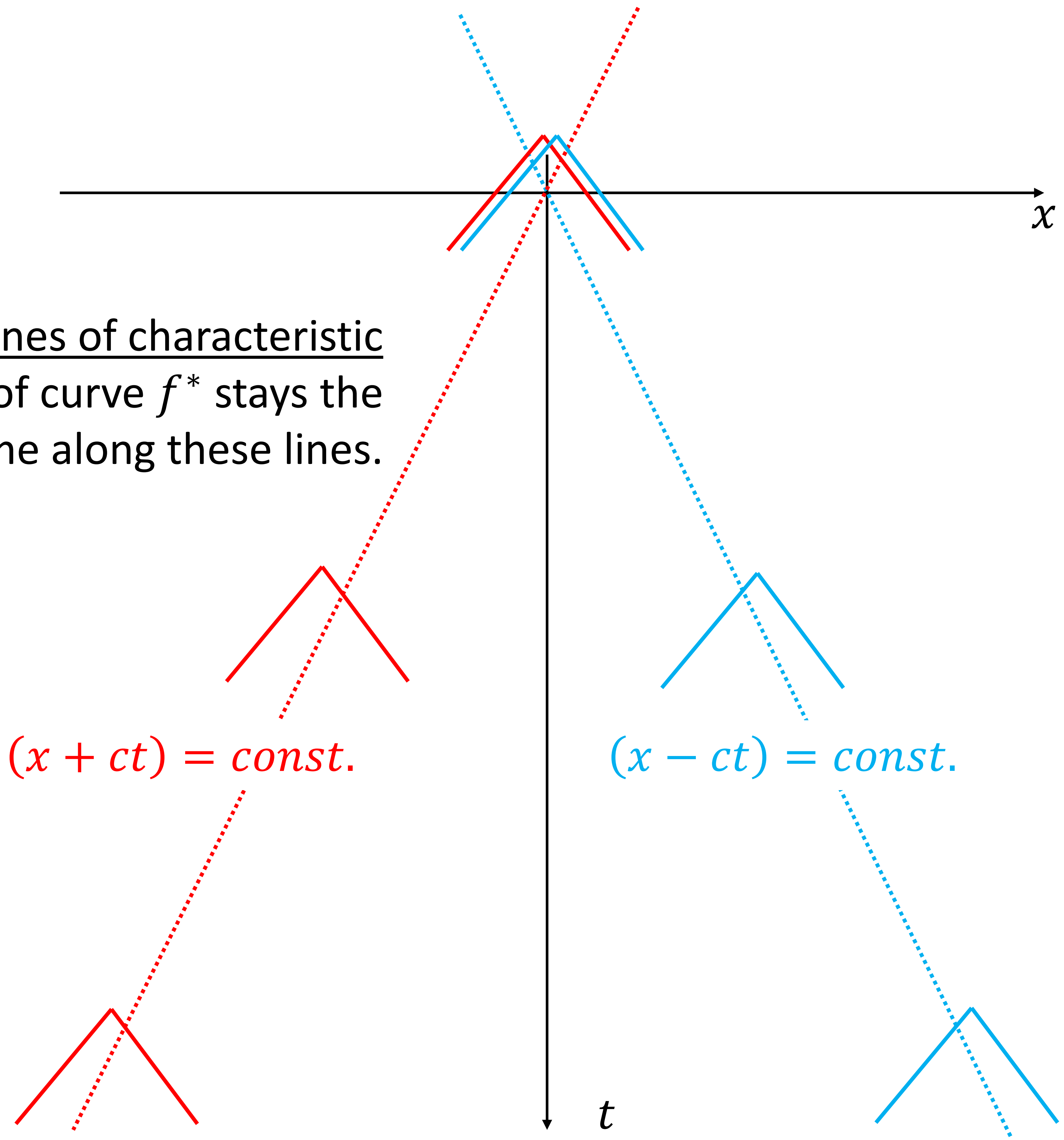
Of course… still need to find what f^* is.

Another Way to Look at It



<u>Lines of characteristic</u>

Shape of curve f^* stays the same along these lines.



Method of Characteristics

1. Given a PDE.

2. What type of PDE is it?

3. What are the characteristics ϕ, ψ

4. What are the transformed variables u, v in terms of ϕ, ψ (i.e. $u = u(\phi, \psi) = u\big(\phi(x,t), \psi(x,t)\big) = u(x,t)$.

5. Transform the PDE in (x,t) into a NEW PDE in (u,v) [called the PDE's "Normal/Canonical Form"].

6. Use BC and IC to find the function(s) f^* that stays constant along (u,v).

7. Transform solution back to (x,t)… and it should be simpler than Fourier Series or other types of solution.

A Simple Version of MoC

Reference: https://youtu.be/rO-f3zh3_kw

Method of Characteristics

- Consider the PDE:

$$Au_{tt} + Bu_{tx} + Cu_{xx} = 0$$

E.g. Our wave eqn: $-\frac{1}{c^2}y_{tt} + y_{xx} = 0$

$$A = -\frac{1}{c^2}$$
$$B = 0$$
$$C = 1$$

- Assume the MoC solution has the form:

$$u(t,x) = f^*(mt + x)$$

- This leaves m to be determined. As usual, differentiate and substitute into the PDE:

$$\frac{\partial f^*}{\partial t} = \frac{\partial f^*}{\partial(mt+x)}\frac{\partial(mt+x)}{\partial t} = mf^{*\prime}$$
$$\frac{\partial^2 f^*}{\partial t^2} = \frac{\partial mf^{*\prime}}{\partial(mt+x)}\frac{\partial(mt+x)}{\partial t} = m^2 f^{*\prime\prime}$$

$$\frac{\partial f^*}{\partial x} = f^{*\prime}$$
$$\frac{\partial^2 f^*}{\partial x^2} = f^{*\prime\prime}$$

$$\frac{\partial^2 f^*}{\partial t \partial x} = mf^{*\prime\prime}$$

- New equation:

$$Am^2 f^{*\prime\prime} + Bmf^{*\prime\prime} + Cf^{*\prime\prime} = 0$$
$$Am^2 + Bm + C = 0$$
$$m = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Types of Solutions

Hyperbolic ($B^2 - 4AC > 0$)

- A typical quadratic solution with 2 real and unequal roots (m_1, m_2)
- This creates two families of characteristic lines, and a solution of the form:

$$u(t,x) = f^*(m_1 t + x) + f^*(m_2 t + x)$$

E.g. Our wave eqn: $-\frac{1}{c^2} y_{tt} + y_{xx} = 0$

$A = -\frac{1}{c^2} \;\; ; \;\; B = 0 \;\; ; \;\; C = 1$

$B^2 - 4AC = 0 - 4(+)(1) > 0$

In this case because $B = 0$, $m_2 = -m_1$

Elliptic ($B^2 - 4AC < 0$)

- Complex conjugates (m_1, m_2)
- This creates two families of characteristic lines, but in complex space.

$$u(t,x) = f^*(m_1 t + x) + f^*(m_2 t + x)$$

Parabolic ($B^2 - 4AC = 0$)

- $m_1 = m_2$
- One family of overlapping characteristic lines:

$$u(t,x) = f^*(m_1 t + x) + t f^*(m_1 t + x)$$

A similar derivation is obtained if
$u(t,x) = f^*(t + mx)$

Expanding a bit further…

- Technically PDE that fits MoC is:

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = R$$

- The coefficients A, B, C, D, E, F … can all be functions of x, t
- Hence, a single PDE can be parabolic, hyperbolic or elliptic in different regions of x, t