Exercise 1

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{pmatrix} \qquad x = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

1) $$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

$$= \frac{\sqrt{2} \cdot 1 + 1 \cdot 1 + 1 \cdot 0}{\sqrt{(\sqrt{2})^2 + 1^2 + 0^2} \times \sqrt{1+1+1}}$$

$$= 0.8047$$

2) Normalise vectors -> find cross product

$$x_n = \frac{x}{\|x\|} = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix} / \sqrt{(\sqrt{2})^2 + 1^2} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 1/\sqrt{3} \\ 0 \end{pmatrix}$$

$$y_n = \frac{y}{\|y\|} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} / \sqrt{3} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\Rightarrow z_n = x_n \times y_n$$

$$z_1 = \begin{vmatrix} 1 & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{vmatrix} = 1/\sqrt{3}$$

$$z_2 = -\begin{vmatrix} \sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{vmatrix} = -\frac{\sqrt{2}}{\sqrt{3}}$$

$$z_3 = \begin{vmatrix} \sqrt{2} & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \end{vmatrix} = \frac{\sqrt{2}-1}{\sqrt{3}}$$

$$\Rightarrow z = \begin{pmatrix} 1/\sqrt{3} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}-1}{\sqrt{3}} \end{pmatrix}$$

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3) $$u = Az = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}-1}{\sqrt{3}} \end{pmatrix}$$

$$\Rightarrow u_1 = 1/\sqrt{3} + \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}-1}{\sqrt{3}} = \frac{2\sqrt{2}}{3}$$

$$u_2 = 2/\sqrt{3} + \frac{2\sqrt{2}}{\sqrt{3}} + \frac{2\sqrt{2}-2}{\sqrt{3}} = \frac{4\sqrt{2}}{\sqrt{3}}$$

$$u_3 = -\frac{1}{\sqrt{3}} - \frac{3\sqrt{2}}{\sqrt{3}} - \frac{3\sqrt{2}+3}{\sqrt{3}} = \frac{-6\sqrt{2}-4}{\sqrt{3}}$$

Exercise 2

2) $F(x, y, z) = \underbrace{(x-c_x)^2 + (y-c_y)^2 + (z-c_z)^2}_{\|(x,y,z)^T - c\|^2 = \|p-c\|^2} - r^2 = 0$

$\Rightarrow F(p) = \|p-c\|^2 - r^2 = 0$

(by properties of vector norm)

3) Given $F(p) = \|p-c\|^2 - r^2 = 0$,

where p is a point within a ~~cir~~ sphere and $\gamma(t) = td$ where d is the direction of the ray, the intersections will be given at points where $td = p$, i.e.

$$\|td - c\|^2 - r^2 = 0$$

4) $\|td - c\|^2 - r^2 = 0$

$$= (td_x)^2 - 2td_x c_x + c_x^2 +$$
$$(td_y)^2 - 2td_y c_y + c_y^2 +$$
$$(td_z)^2 - 2td_z c_z + c_z^2 - r^2 = 0$$

$$= (td_x)^2 + (td_y)^2 + (td_z)^2 - 2t\langle c,d\rangle + \overbrace{c_x^2 + c_y^2 + c_z^2}^{\|c\|^2} - r^2$$

$$= t^2(\underbrace{d_x^2 + d_y^2 + d_z^2}_{\|d\|^2 = 1}) - 2\langle d,c\rangle t + \|c\|^2 - r^2 = 0$$

$$= \underline{t^2 - 2\langle d,c\rangle t + \|c\|^2 - r^2 = 0}$$

5) $t^2 - 2\langle d,c\rangle t + \|c\|^2 - r^2 = 0$

$a = 1$; $b = -2\langle d,c\rangle$; $c = \|c\|^2 - r^2$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\langle d,c\rangle \pm \sqrt{4\langle d,c\rangle^2 - 4\|c\|^2 + 4r^2}}{2}$$

$$= \langle c,d\rangle \pm \sqrt{\underbrace{\langle c,d\rangle^2 - \|c\|^2}_{-D} + r^2} = \underline{\langle c,d\rangle \pm \sqrt{r^2 - D^2}}$$