Quantum Field Theory for Educated Dummies

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Version 1.0

Contents

Chapter 1

Foundations

1.1 Natural Units and Dimensions

Convenient systems of units start with arbitrary definitions for units of certain fundamental. quantities and derive the remaining units from laws of nature. To see how this works, assume we know three basic laws of nature and we want to devise a system of units from scratch. We will do this first for the cgs system and then for natural units. The three lawas are:

- The distance L traveled by a photon is the speed of light multiplied by its time of travel. $L = ct$
- The energy of a massive particle is equal to its mass (at rest) m times the speed of light squared. $E = mc^2$
- The energy of a photon is proportional to its frequency f. The constant of proportionality is Planck's constant $h.E = hf$ or re-expressed as $E = \hbar\omega$

In natural units:the c and $\hbar$ are dimensionless and equal to 1. The unit for energy is MeV.

1.2 Notation

We shall use a notation defining **contravariant components x^μ of the** 4D **position vector** as 3D Cartesian coordinates X_i plus ct (see Appendix $\bar{A}$ if you are not comfortable with this), i.e.,

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} ct, X_i \end{bmatrix}^T \tag{1.2.1}$$

From special relativity, we know the differential proper time passed on an object (with $c = 1$) is

$$(d\tau)^2 = (dt)^2 - dX_1 dX_1 - dX_2 dX_2 - dX_3 dX_3 \tag{1.2.2}$$

If we define **covariant components** of the 4D position vector as

$$x_\mu = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -X_1 \\ -X_2 \\ -X_3 \end{bmatrix} = [t, -X_i]^T \tag{1.2.3}$$

then

$$(d\tau)^2 = dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3 = dx^\mu dx_\mu \tag{1.2.4}$$

Using metric tensor $g_{\mu\nu}$, we have the following relation:

$$x_\mu = g_{\mu\nu} x^\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.2.5}$$

The inverse of $g_{\mu\nu}$, $g^{\mu\nu}$, has the exact same form.Thus,

$$(d\tau)^2 = g_{\mu v} dx^\mu dx^v = g^{\mu v} dx_\mu dx_v \tag{1.2.6}$$

Partial derivative w.r.t. x^μ and x_μ are:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right)^T = \left(\frac{\partial}{\partial t} \cdot \frac{\partial}{\partial X_i}\right)^T \tag{1.2.7}$$

and

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}\right)^T = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial X_i}\right)^T \tag{1.2.8}$$

For a matrix, we can raise the index as

$$M^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} M_{\alpha\beta} \tag{1.2.9}$$

1.3 Review of Variational Methods

Recall also, that given the Lagrangian, we could find the Hamiltonian H, via the Legendre transformation (employing a Cartesian system where $x^i = x_i$ and $p^i = p_i$),

$$H = p_i \dot{x}^i - L, \quad \text{where } p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \ (= p' \text{ for Cartesian system }) \tag{1.3.1}$$

p_i is the conjugate, or canonical, momentum of x '. (Note that a contravariant component in the denominator is effectively equivalent to a covariant component in the entire entity, and vice versa.) Hence, we can define:

First quantization

Keeping the classical Hamiltonian and, changing Poisson brackets to commutators, we could just as readily have used the Lagrangian L, or the equations of motion instead.

1.3.1 Classical Field Theory

From particle theory to field theory, we have the following things changed:

$$L, H, etc \to \mathcal{L}, \mathcal{H},$$

$$x^i(t) \to \phi^r\left(x^\mu\right)$$

$$t \to x^\mu$$

Classical field theory is analogous in many ways to classical particle Lagrangian L, we have the Lagrangian density $\mathcal{L}$. Instead of time t as an independent variable, we have $x^\mu = x^0, x^1, x^2, x^3 = t, x^i$ as independent variables. Instead of a particle described by $x^i(t)$, **we have a field value described by $\phi(x^\mu)$, where r designates different field types, or possibly, different spatial components of the same vector field.**

The Euler-Lagrange equation for fields becomes

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^r, \mu}\right) - \frac{\partial \mathcal{L}}{\partial \phi^r} = 0 \tag{1.3.2}$$

and

$$\mathcal{H} = \pi_r \dot{\phi}^r - \mathcal{L}, \qquad \text{where } \pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} \tag{1.3.3}$$

with π_r being the conjugate momentum density of the field ϕ^r. And the action is

$$S = \int_T \int_V \mathcal{L}\left(\phi, \phi_{,\mu}\right) d^3\mathbf{x} dt = \int_\Omega \mathcal{L}\left(\phi, \phi_{,\mu}\right) d^4x \tag{1.3.4}$$

1.3.2 Key concepts in field theory

For fields

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{dt} = \dot{\phi} \tag{1.3.5}$$

This is generally not true for other quantities. For fields,

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x'}\frac{dx'}{dt} + \frac{\partial \phi}{\partial t}\frac{dt}{dt} \tag{1.3.6}$$

where $\frac{dx^i}{dt}$ are equal to zero.

For a single particle, particle position coordinates are the generalized coordinates and particle momentum components are its conjugate momenta. For fields, each field is itself a gen-

eralized coordinate and each field has its own conjugate momentum (density). As noted, this field conjugate momentum (density) is different from the physical momentum (density) that the field possesses.

For conjugate and physical momentum densities, we have the following density relation

$$p_t = \frac{\partial L}{\partial \dot{x}^t} \quad \frac{\text{for small particle in medium,}}{\text{divide by particle volume}}, \quad \mathcal{R}_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i}$$

We note carefully that our x^i here is the position coordinate of a point fixed relative to the field and thus is time dependent. Further, the total derivative $\dot{x}^i$ equals the partial derivative w.r.t. time, since x^i in the present case is only a function of time. Now take the conjugate momentum density relation for relativistic fields,

$$\pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r}$$

and

$$\frac{\mathcal{R}_i}{\pi_r} = \frac{\partial \mathcal{L}/\partial \dot{x}^i}{\partial \mathcal{L}/\partial \dot{\phi}^r} = \frac{\partial \dot{\phi}^r}{\partial \dot{x}'} = \frac{\partial \phi^r/\partial t}{\partial x^i/\partial t} = \frac{\partial \phi^r}{\partial x^i} \quad \rightarrow \quad \mathcal{R}_i = \pi_r \frac{\partial \phi^r}{\partial x^i} \quad \rightarrow \quad \mathcal{R}^i = -\pi_r \frac{\partial \phi^r}{\partial x^i}$$

The partial derivative of ϕ^r with respect to either of our definitions of x^i (time dependent as the moving position of a point fixed to the field, or time independent as coordinates fixed in space) is the same because by definition, partial derivative means we hold everything else here constant, Thus, the above relation holds in field theory when we consider the x^i as independent variables (coordinates fixed in space).

The field equation (equations of motion) for relativistic fields keep the exact same form in any inertial frame of reference, i.e., they are Lorentz invariant. Four scalars(world scalars) are invariant under a Lorentz transformation and look exactly the same to any observer.

The mass m in relativity of a free particle is four scalar, where $m^2 = p^\mu p_\mu$.
Demanding that the Euler-Lagrange equation be Lorentz invariant, we know that, within that equation, x^μ,ϕ^r, and derivatives of x^μ are Lorentz covariant or invariant. So, in order for the whole equation to be Lorentz invariant, the Lagrangian density $\mathcal{L}$ must be invariant.
Further, L, H, and $\mathcal{H}$ are not Lorentz scalars.

1.4 Schrodinger vs. Heisenberg Pictures

In Schrodinger picture, the expectation value is calculated as:

$$\overline{\mathcal{O}} = \int \psi^\dagger \mathcal{O} \psi d^3x = \langle\psi|\mathcal{O}|\psi\rangle$$

The time derivative of the expectation value is

$$\frac{d\bar{\mathcal{O}}}{dt} = \frac{d}{dt}\langle\psi|\mathcal{O}|\psi\rangle = \left\langle \frac{\partial\psi}{\partial t}|\mathcal{O}|\psi \right\rangle + \left\langle \psi \left| \frac{\partial\mathcal{O}}{\partial t} \right| \psi \right\rangle + \left\langle \psi|\mathcal{O}|\frac{\partial\psi}{\partial t} \right\rangle \tag{1.4.1}$$

where we use partial derivative in the brackets because our states and operators are functions of x^i and t.

In the Schródinger picture (S.P.), the solutions to the Schrödinger equation

$$i\frac{\partial\psi_S}{\partial t} = H\psi_S \quad \text{or} \quad i\frac{\partial}{\partial t}|\psi\rangle_S = H|\psi\rangle_S \tag{1.4.2}$$

In the Schrodinger picture, the operators are usually not time dependent. For example, using the familiar momentum operator $p_1^S = i\partial/\partial x^1$ for the S.P. in the x^1 direction, with

$$\psi_S = Ae^{-i(Et-p\cdot x)} = |\psi\rangle_S \quad A^\dagger A = \frac{1}{V}$$

we have

$$\bar{p}_1 = \int A^+ e^{i(Et-p'x')} \left(i\tfrac{\partial}{\partial x^1}\right) Ae^{-i(Et-p'x')} d^3x = {}_s\langle\psi|p_1^S|\psi\rangle_s$$

Since there is no t in the operator, we have

$$\frac{dp_1^S}{dt} = \frac{\partial p_1^S}{\partial t} = 0$$

and

$$\frac{d\bar{p}_1}{dt} = \frac{d}{dt}\langle\psi\,|p_1^s|\,\psi\rangle = \left\langle \frac{\partial\psi}{\partial t}\,|p_1^s|\,\psi \right\rangle_s + s\left\langle \psi \left| \frac{\partial p_1^s}{\partial t} \right| \psi \right\rangle_s + s\left\langle \psi\,|p_1^s|\,\frac{\partial\psi}{\partial t} \right\rangle_s \tag{1.4.3}$$

Using (1.4.2), and its complex conjugate, we have

$$\frac{d\bar{p}_1}{dt} =_S \left\langle \psi \left| \left(iHp_1^S + \frac{\partial p_1^S}{\partial t} - ip_1^S H \right) \right| \psi \right\rangle_S =_S \langle\psi\,|-i\left[p_1^S, H\right]|\,\psi\rangle_S +_S \left\langle \psi \left| \frac{\partial p_1^S}{\partial t} \right| \psi \right\rangle_S \tag{1.4.4}$$

where the last term is equal to zero.Recall the old NRQM adage that the expectation value of any operator without explicit time dependence that commutes with the Hamiltonian is conserved (its time derivative is zero.)

The Schrodinger picture states and operators can be transformed to states and operators having different form via what is known as a unitary transformation. The particular unitary transformation (where U is a unitary operator) for this is

$$U = e^{-iAt} \quad \left(= e^{-iHt/\hbar} \text{ in non-natural units } \right) \tag{1.4.5}$$

where states and operators transform as

$$\begin{aligned} U^\dagger|\psi\rangle_S &= |\psi\rangle_H \quad U^\dagger \mathcal{O}^S U = \mathcal{O}^H \\ U|\psi\rangle_H &= |\psi\rangle_S \quad U O^H U^\dagger = \mathcal{O}^S \end{aligned} \tag{1.4.6}$$

We find that in the first relation the state is now time independent in H.P.

Taking the time derivative in the second relation, we have:

$$\begin{aligned} \frac{d}{dt}\left(U^\dagger \mathcal{O}^S U\right) &= (iH)\underbrace{e^{iA'}\mathcal{O}^S}_{\mathcal{O}^H} + \underbrace{e^{iH_t}\left(\frac{\partial \mathcal{O}^S}{\partial t}\right)e^{-iH_l}}_{\text{defined as } \hat{\partial} O^H/\partial t} + \underbrace{e^{iHt}\mathcal{O}^S e^{-tHt}}_{\mathcal{O}^H}(-iH) \\ &= \frac{d\mathcal{O}^H}{dt} = -i\left[\mathcal{O}^H, H\right] + \frac{\hat{\partial}\mathcal{O}^H}{\partial t} \end{aligned} \tag{1.4.7}$$

In this note, $\frac{\hat{\partial}\mathcal{O}^H}{\partial t}$ will always be zero. Nonetheless, we see that in the H.P., an operator time derivative can be **non-zero**,and the operator is time dependent.

Box

A unitary transformation is called unitary because its operation on (transformation of) a state vector leaves the magnitude of the state vector unchanged, i.e., the state vector magnitude is multiplied by unity.

A unitary transformation can be thought of as "rotating" a (complex number) state vector in Hilbert space (the complex space where each coordinate axis is an eigenvector) without changing the "length".

Recall, from classical mechanics, that an orthogonal transformation represented by a real matrix **A** has an inverse equal to the transpose of that matrix, i.e., $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{T}}$. In the complex space of state vectors, a unitary transformation U has an analogous form for its inverse, the complex conjugate transpose, i.e., $\underline{U}^{-1} = U^{+}$ and so $U^{\dagger}U = 1$. The following example may make this clearer.

Do a Taylor expansion of $U = e^{-iHt}$ above about t, when U is operating on an energy eigenstate., i.e.,

$$U\left|\psi_E\right\rangle = e^{-iHt}\left|\psi_E\right\rangle = \left(1 - itH - \frac{1}{2}t^2H^2 + \ldots\right)\left|\psi_E\right\rangle = e^{-iEt}\left|\psi_E\right\rangle \tag{1.4.8}$$

Note: Although it is common to write $U = e^{-iHt}$,it is implied that H (if you think of it as $i\partial/\partial t$) does not act on t. To be proper, the t should be placed before the H, as we did in the expansion above, but it usually is not done that way.

Because $H = H^S$ commutes with itself, U and U^{+} commute with H, so using $\mathcal{O}^S = H^S = H$, we have

$$H = H^S = H^H \tag{1.4.9}$$

and after inserting $UU^{+} = 1$, we find

$$\begin{aligned}\frac{d\overline{\mathcal{O}}}{dt} &= s\left(\psi\left|UU^{\dagger}\left(-i\left[\mathcal{O}^S, H\right]\right)UU^{\dagger}\right|\psi\right\rangle_s + {}_S\left\langle\psi\left|UU^{+}\frac{\partial\mathcal{O}^S}{\partial t}UU^{\dagger}\right|\psi\right\rangle_S \\ &= {}_H\left\langle\psi\left|\left(-i\left[\mathcal{O}^H, H\right]\right)\right|\psi\right\rangle_H + {}_H\left\langle\psi\left|\frac{\hat{\partial}\mathcal{O}^{\mu}}{\partial t}\right|\psi\right\rangle_H\end{aligned} \tag{1.4.10}$$

1.5 Extrapolation to Field Theory

According to the correspondence principle, in the macroscopic limit, our quantum relations must reduce to the usual classical relations. So the principle provides us with a key part of our method for quantization. That is, in going from classical theory to NRQM, we must take

$$\{x', p_j\} = \delta^i_j \quad \overrightarrow{\text{1st quantization}} \quad \left[x^i, p_j\right] = i\hbar\delta^i_j$$

For field theory, we have

$$\{\phi^r(\mathrm{x},t), \pi, (\mathrm{y},t)\} = \delta^r{}_s\delta(\mathrm{x}-\mathrm{y}) \overrightarrow{\text{2nd quantization}} \left[\phi^r, \pi_s\right] = i\hbar\delta^r{}_s\delta(\mathbf{x}-\mathbf{y}) \tag{1.5.1}$$

and

$$\left[\phi^r, \phi^s\right] = \left[\pi_r, \pi_s\right] = 0 \tag{1.5.2}$$

Again, Quantization is a means for deducing quantum theory from classical theory.

1.6 Appendix: some useful math relations

First, we have four-velocity of relativity u^μ for an object as:

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} = \begin{bmatrix} u^0 & u^1 & u^2 & u^3 \end{bmatrix} \tag{1.6.1}$$

where

$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{\sqrt{1-v^2/c^2}dt} = \frac{v^i}{\sqrt{1-v^2/c^2}} = \gamma v^i \tag{1.6.2}$$

$$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-v^2/c^2}} = \gamma c \tag{1.6.3}$$

Thus,

$$(u)^2 = |u^\mu|^2 = u^\mu u_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \gamma^2\left(c^2 - \left(v^1\right)^2 - \left(v^2\right)^2 - \left(v^3\right)^2\right) = c^2 \tag{1.6.4}$$

The magnitude of the 4-momentum $p^\mu = mu^\mu$ is then found from:

$$(p)^2 = |p^\mu|^2 = p^\mu p_\mu = m^2 u^\mu u_\mu = m^2c^2 \quad \left(= m^2 \text{ in natural units }\right) \tag{1.6.5}$$

Note that $p^0 = \gamma mc = E/c$, where E is relativistic energy, and p^i is relativistic 3-momentum.

Or

$$p^\mu p_\mu = m^2c^2 = g_{\mu\nu}p^\mu p^\nu = \begin{bmatrix} E/c & p^1 & p^2 & p^3 \end{bmatrix} \begin{bmatrix} E/c \\ -p^1 \\ -p^2 \\ -p^3 \end{bmatrix} \tag{1.6.6}$$

and

$$\frac{E^2}{c^2} = p^2 + m^2c^2 \tag{1.6.7}$$

Chapter 2

Scalars: Spin 0 Fields

2.1 Deducing Klein-Gordon Equation

If we squared the operators in the original Schrodinger equation, we obtain

$$\left(i\hbar\frac{\partial}{\partial t}\right)\left(i\hbar\frac{\partial}{\partial t}\right)\phi = H^2\phi = \left(\mathbf{p}_{oper}^2c^2 + m^2c^4\right)\phi \tag{2.1.1}$$

which becomes

$$-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\phi = \left(-\hbar^2\frac{\partial}{\partial X_i}\frac{\partial}{\partial X_i} + m^2c^2\right)\phi \tag{2.1.2}$$

In a compact form, we have

$$-\frac{\partial}{\partial x^0}\frac{\partial}{\partial x_0}\phi = \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_i} + \frac{m^2c^2}{\hbar^2}\right)\phi \tag{2.1.3}$$

where we define $\mu^2 = \frac{m^2c^2}{\hbar^2}$. Re-arranging, we have the Klein-Gordon equation

$$\left(\partial_\mu\partial^\mu + \mu^2\right)\phi = 0 \tag{2.1.4}$$

The operation $\partial_\mu\partial^\mu = \partial^\mu\partial_\mu$ is called the **d'Alembertian** operator, and is the 4D Minkowski coordinates analogue of the 3D Laplacian operator of Cartesian coordinates.

2.1.1 The solutions to the Klein-Gordon Equation

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2VE_n/\hbar}} \left(A_n e^{-\frac{i}{\hbar}(E_n t - \mathbf{p}_n \cdot \mathbf{x})} + \underbrace{B_n^\dagger e^{\frac{i}{\hbar}(E_n t - \mathbf{p}_n \cdot \mathbf{x})}}_{\text{absent in NRQM}} \right) \tag{2.1.5}$$

Because we are using the square of the relativistic Hamiltonian in RQM, we get additional solutions of exponential form $+i(E_n t - \mathbf{p}_n \cdot \S)/\hbar$ that also solve the relativistic Klein-Gordon equation.

With an aim towards using natural units, we note the following relations, where wave number $k_i = 2\pi/\lambda_i$ and we use the deBroglie relation $p^i = \hbar k^i$

$$p_\mu = \begin{bmatrix} E/c \\ p_i \end{bmatrix} = \begin{bmatrix} E/c \\ -p^i \end{bmatrix} = \hbar k_\mu = \begin{bmatrix} \hbar\omega/c \\ -\hbar k' \end{bmatrix} \tag{2.1.6}$$

in natural units

$$p_\mu = \begin{bmatrix} E \\ -p^i \end{bmatrix} = k_\mu = \begin{bmatrix} \omega \\ -k^i \end{bmatrix} \tag{2.1.7}$$

Recall the notation introduced in the previous chapter, we have

$$px = p_\mu x^\mu = Et - p^i x^i = p^\mu x_\mu \tag{2.1.8}$$

$$kx = k_\mu x^\mu = \omega t - k^i x^i = \frac{Et}{\hbar} - \frac{p^i x^i}{\hbar} = \frac{p_\mu}{\hbar} x^\mu \tag{2.1.9}$$

In natural unit

$$E = \omega, \quad p_i = k_i, \quad p_\mu = k_\mu, \quad px = kx \tag{2.1.10}$$

For free fields, a given wave with wave number vector $\mathbf{k}$ has a particular energy, and we can designate that energy via either $E_\mathbf{k}$ or $\omega_\mathbf{k}$. It is common practice for scalars to use $\mathbf{k}$ (rather than $\mathbf{p}$) and $\mathbf{k}$ (rather than $E_\mathbf{p}$ or $E_\mathbf{k}$.)

The Klein-Gordon equation solutions then become, in natural units

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V_{el}}} \left(A_k e^{-ikx} + B_k^\dagger e^{ikx} \right) \tag{2.1.11}$$

Box

In RQM, the solution ϕ is that of a general (sum of eigenstates) single particle state. Each eigenstate has mathematical form of

$$\phi_{k,A} = \frac{e^{-ikx}}{\sqrt{V}}$$

$$\phi_{\mathrm{k},B^\dagger} = \frac{e^{ikx}}{\sqrt{V}}$$

Each of these forms has what is called **unit norm**. That is, all such eigenstates are **orthonormal**:

$$\int \phi^\dagger_{k,A}\phi_{k',A}d^3x = \frac{1}{V}\int e^{ikx}e^{-ik'x}d^3x = \delta_{kk'} \tag{2.1.12}$$

Similar relations exist for $\phi_{k,B^\dagger}$.

2.1.2 Deducing probability density in RQM

Starting from the Klein-Gordon equation, first post-multiply it by $\phi^\dagger$, then subtract the complex conjugate equation post-multiplied by ϕ,

$$\begin{array}{c} \left\{\frac{\partial^2}{\partial t^2}\phi = \left(\nabla^2 - \mu^2\right)\phi\right\}\phi^\dagger \\ -\left\{\frac{\partial^2}{\partial t^2}\phi^\dagger = \left(\nabla^2 - \mu^2\right)\phi^\dagger\right\}\phi \end{array} \tag{2.1.13}$$

Since $\mu^2\phi^\dagger\phi - \mu^2\phi\phi^\dagger = 0$, we obtain

$$i\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\phi^\dagger - \frac{\partial\phi^\dagger}{\partial t}\phi\right) = i\nabla\cdot\left((\nabla\phi)\phi^\dagger - \left(\nabla\phi^\dagger\right)\phi\right) \tag{2.1.14}$$

where probability density and the probability current for a Klein-Gordon particle are

$$\rho = j^0 = i\left(\frac{\partial\phi}{\partial t}\phi^\dagger - \frac{\partial\phi^\dagger}{\partial t}\phi\right) \tag{2.1.15}$$

and

$$\mathbf{j} = -i\left((\nabla\phi)\phi^\dagger - \left(\nabla\phi^\dagger\right)\phi\right) \quad j^i = -i\left(\phi_{,i}\phi^\dagger - \phi^\dagger_{,i}\phi\right) = i\left(\phi^{,i}\phi^\dagger - \phi^{\dagger,i}\phi\right) \tag{2.1.16}$$

Now we can define the 4-currents as

$$j^\mu = \begin{bmatrix} \rho \\ \mathbf{j} \end{bmatrix} = \begin{bmatrix} \rho \\ j^i \end{bmatrix} = \begin{bmatrix} j^0 \\ j^i \end{bmatrix} = i\left(\phi^\mu \phi^\dagger - \phi^{\dagger,\mu}\phi\right) \tag{2.1.17}$$

The 4D continuity equation is then

$$\frac{\partial j^\mu}{\partial x^\mu} = \partial_\mu j^\mu = j^\mu{}_{,\mu} = 0 \tag{2.1.18}$$

(2.1.18) tells us the important fact that the 4-divergence of the 4-current of any conserved quantity is zero.

The total probability of unity is a relativistic invariant. Further A_k here are constants that do not vary with frame.So the probability of finding any particular state is also independent of what frame the measurements are taken in.

2.1.3 Negative Energies in RQM

If we apply traditional Hamiltonian operator H as $i\partial/\partial t$ to $\phi_{\mathbf{k},B^\dagger}$:

$$i\frac{\partial \phi_{\mathbf{k},B^\dagger}}{\partial t} = i\frac{\partial}{\partial t}\frac{e^{ikx}}{\sqrt{V}} = -\omega_{\mathbf{k}}\frac{e^{ikx}}{\sqrt{V}} = -\omega_{\mathbf{k}}\phi_{\mathbf{k},B^\dagger} = E_{\mathbf{k},B'}\phi_{\mathbf{k},B^+} \tag{2.1.19}$$

Since $\omega_{\mathbf{k}}$ is always a positive number, we have states with negative energies in RQM, and we need QFT to solve this dilemma.

2.2 Klein-Gordon Equation in Quantum Field Theory

The fundamental scalar wave equation of RQM, the Klein-Gordon equation, is also the fundamental scalar wave equation of QFT,except that ϕ therein is considered a field, instead of a state.

The word "field" in quantum theory refers to a quantity that is spread out in space, but also, importantly, as we will soon see, is an **operator** in QFT. More properly, it is called a **quantum field or an operator field**, though the short term field is far more common.

Box

In QFT,

$|\phi\rangle$ symbolizes a state (particle)

ϕ symbolizes a field (operator),

Klein-Gordon equation is called a **field equation**.

2.2.1 From classical relativistic fields to QFT

The classical **Lagrangian density for a free, real, relativisitc scalar field** ϕ has form

$$\mathcal{L}_0^0 = K\left(\partial_\alpha\phi\partial^\alpha\phi - \mu^2\phi\phi\right) = K\left(\dot{\phi}\dot{\phi} - \underbrace{\partial_i\phi\partial_i\phi}_{\nabla\phi\cdot\nabla\phi} - \mu^2\phi\phi\right) \tag{2.2.1}$$

where ϕ, since it is a classical field, is real (not complex), μ is a constant to be determined by experiment, K is an arbitrary constant, the superscript 0 on $\mathcal{L}$ stands for scalar (with spin 0), and the subscript "0" means free.

Using the Legendre transformation, we can readily use (2.2.1) to find the Hamiltonian density, where π_0^0 is the field conjugate momentum:

$$\mathcal{H}_0^0 = \pi_0^0\dot{\phi} - \mathcal{L}_0^0 = \underbrace{\frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}}}_{2K\dot{\phi}}\dot{\phi} - \mathcal{L}_0^0 = K\left(\dot{\phi}\dot{\phi} + \nabla\phi\cdot\nabla\phi + \mu^2\phi\phi\right) \tag{2.2.2}$$

We know that in QM most meaningful things are complex, not real. So now we take **classical field as complex**: ϕ is complex. Choosing $K = 1$, we obtain the **free, complex scalar field Lagrangian and Hamiltonian densities**:

$$\mathcal{L}_0^0 = \left(\partial_\alpha\phi^\dagger\partial^\alpha\phi - \mu^2\phi^\dagger\phi\right) = \left(\dot{\phi}^\dagger\dot{\phi} - \nabla\phi^\dagger\cdot\nabla\phi - \mu^2\phi^\dagger\phi\right) \tag{2.2.3}$$

$$\mathcal{H}_0^0 = \frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}^r}\dot{\phi}^r - \mathcal{L}_0^0 = \underbrace{\frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}}}_{\pi_0^0=\dot{\phi}^\dagger} + \underbrace{\frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}^\dagger}}_{\pi_0^0=\dot{\phi}}\dot{\phi}^\dagger - \mathcal{L}_0^0 = \dot{\phi}\dot{\phi}^\dagger + \nabla\phi^\dagger\cdot\nabla\phi + \mu^2\phi^\dagger\phi \tag{2.2.4}$$

where ϕ and $\phi^\dagger$ are considered separate fields in the summation over field types r. Now

we are ready to **derive the Klein-Gordon field equation.** from **Euler-Lagrange field equation**:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^r_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \phi^r} = 0 \tag{2.2.5}$$

Substituting the Lagrangian density (2.2.3), we obtain the Klein-Gordon equation for fields

$$\begin{aligned} \left(\partial_\mu \partial^\mu + \mu^2\right)\phi &= \left(\Box^2 + \mu^2\right)\phi = 0 \\ \left(\partial_\mu \partial^\mu + \mu^2\right)\phi^\dagger &= \left(\Box^2 + \mu^2\right)\phi^\dagger = 0 \end{aligned} \tag{2.2.6}$$

The discrete palne wave solutions to (2.2.6) are

$$\begin{aligned} \phi(x) &= \underbrace{\sum_k \frac{1}{\sqrt{2Va_k}} a(k) e^{-ikx}}_{} + \underbrace{\sum_k \frac{1}{\sqrt{2V\omega_k}} b^\dagger(k) e^{ikx}}_{} \\ &= \phi^+ \quad + \quad \phi^- \end{aligned} \tag{2.2.7}$$

$$\begin{aligned} \phi(x) &= \underbrace{\sum_k \frac{1}{\sqrt{2Va_k}} b(k) e^{-ikx}}_{} + \underbrace{\sum_k \frac{1}{\sqrt{2V\omega_k}} a^\dagger(k) e^{ikx}}_{} \\ &= \phi^{\dagger +} \quad + \quad \phi^{\dagger -} \end{aligned} \tag{2.2.8}$$

As we will see later, the lower case coefficients are operators. The continuous plane wave solutions are

$$\phi(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}} a(\mathbf{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}} b^\dagger(\mathbf{k}) e^{ikx} \tag{2.2.9}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathrm{k}}}} b(\mathrm{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathrm{k}}}} a^\dagger(\mathrm{k}) e^{ikx} \tag{2.2.10}$$

2.3 Commutation Relations: The Crux of QFT

Of overriding importance in the theory, as we will see, are the following coefficient commutation relations, which we will derive below from the 2^{nd} quantization postulate of (1.5.1).

$$\left[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')\right] = \left[b(\mathbf{k}), b^{\dagger}(\mathbf{k}')\right] = \delta_{\mathbf{k}\mathbf{k}'}(\text{ discrete }); = \delta(\mathbf{k}-\mathbf{k}') \text{ (continuous)} \tag{2.3.1}$$

The form of this relation tells us that $a(\mathbf{k})$ and $b(\mathbf{k})$ are operators.

Proof of coefficient commutation relations

Starting from (1.5.1), we take different spatial coordiantes, but the same time coordinate t for ϕ and ${\pi^0}_0$:

$$\left[\phi(\mathbf{x},t)\pi_0^0(\mathbf{y},t) - \pi_0^0(\mathbf{y},t)\phi(\mathbf{x},t)\right] = \left[\phi(x,t)\dot{\phi}^{\dagger}(y,t) - \dot{\phi}^{\dagger}(y,t)\phi(x,t)\right] = i\delta(x-y) \tag{2.3.2}$$

Plugging the discrete solutions into the middle part, we get

$$\sum_{\mathbf{k}}\sum_{\mathbf{k}'} \frac{i\omega_{\mathbf{k}}}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}}}} \begin{pmatrix} -a_{\mathbf{k}}b_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})} + a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})} \\ -b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}}-\alpha_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})} + b^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})} \\ +b_{\mathbf{k}'}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})} + b_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})} \\ -a^{\dagger}_{\mathbf{k}'}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})} - a^{\dagger}_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})} \end{pmatrix}$$

Using the math identity for the 3D Dirac delta function:

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{V}\sum_{n=-\infty}^{+\infty} e^{-i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y})} \tag{2.3.3}$$

and matching terms, we see that all terms where $\mathbf{k\prime} \neq \pm\mathbf{k}$ must equal zero since the identity (2.3.3) has no terms in $\mathbf{k}$ and $\mathbf{k}'$. The remaining terms all have $k' = \pm k$ which means $\omega_k = \omega_{k'}$.Some of these have an exponential form $i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}})t.$, and those terms give us a summation of terms over $\mathbf{k}$having form, for each possible $\mathbf{k}'$, of

$$\underbrace{(b_k a_k - a_k b_k)}_{\text{must } =0}\underbrace{e^{-i2\omega_k t}}_{\neq 0}e^{ik\cdot(x+y)}$$

$$\underbrace{\left(b^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}\right)}_{\text{must } =0}e^{i2\omega_{\mathbf{k}}t}e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}$$

$$\underbrace{(b_{-k}a_k - a_k b_{-k})}_{\text{must } =0}e^{-i2\omega_k t}e^{ik\cdot(x-y)}$$

$$\underbrace{\left(b^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}-a^\dagger_{-\mathbf{k}}b^\dagger_{\mathbf{k}}\right)}_{\text{must }=0}e^{i2\alpha_{\mathbf{k}}t}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

i.e., all time dependent terms eqaul 0, as no time dependence on RHS of (2.3.2). The remaining relevant terms are in the following forms

$$\underbrace{\left(a_{\mathbf{k}}a^\dagger_{\mathbf{k}}-a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right)}_{\text{must }=1}\underbrace{e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}})t}}_{=1}e^{i\mathbf{k}\cdot(x-y)}$$

$$\underbrace{\left(a_{\mathbf{k}}a^\dagger_{-\mathbf{k}}-a^\dagger_{-\mathbf{k}}a_{\mathbf{k}}\right)}_{\text{must }=0}e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}$$

$$\underbrace{\left(b_{-k}b^\dagger_{\mathbf{k}}-b^\dagger_{\mathbf{k}}b_{-\mathbf{k}}\right)}_{\text{must }=0}e^{-i\mathbf{k}\cdot(x+y)}$$

2.4 The Hamiltonian in QFT

For a free scalar field $\mathcal{H}=\mathcal{H}^0_0$, we have

$$H^0_0=\int\mathcal{H}^0_0d^3x=\int\left(\dot\phi\dot\phi^\dagger+\nabla\phi^\dagger\cdot\nabla\phi+\mu^2\phi^\dagger\phi\right)d^3x=$$
$$\int\left(\sum_{\mathbf{k}}\frac{\partial}{\partial t}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx}+b^\dagger(\mathbf{k})e^{ikx}\right)\right)\left(\sum_{\mathbf{k}'}\frac{\partial}{\partial t}\frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(b\left(\mathbf{k}'\right)e^{-ik'x}+a^\dagger\left(\mathbf{k}'\right)e^{ikx}\right)\right)d^3x$$
$$+\int\left(-\partial_i\phi^\dagger\partial^i\phi+\mu^2\phi^\dagger\phi\right)d^3x$$

Expand the middle line here. All terms in the integration result in zero except when $\mathbf{k}'=\mathbf{k}$ or $\mathbf{k}'=-\mathbf{k}$, because we are integrating orthogonal functions between their boundaries. Also, because of (2.1.12), the integral over $a(n)b(k)e^{-2ikx}$, $b^\dagger(k)a^\dagger(k)e^{2ikx}$, $-a(x)a^\dagger(-k)e^{-ikx}e^{i(-k)x}$, and $-b^\dagger(k)b(-k)e^{ikx}e^{-i(-k)x}$ are zeros too. Since the volume of integration above equals V, we end up with

$$\begin{aligned}\int\dot\phi\dot\phi^\dagger d^3x&=\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{2}\left(-a(\mathbf{k})b(-\mathbf{k})e^{-2i\omega_{\mathbf{k}}t}+a(\mathbf{k})a^\dagger(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})-b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{u}}t}\right)\\&=\sum_{\mathbf{k}}\frac{(\omega_{\mathbf{k}})^2}{2\omega_{\mathbf{k}}}\left(-a(-\mathbf{k})b(\mathbf{k})e^{-2i\omega_{\mathbf{k}}t}+a(\mathbf{k})a^\dagger(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})-b^\dagger(-\mathbf{k})a^\dagger(\mathbf{k})e^{2i\omega_{\mathbf{k}}t}\right)\end{aligned}\tag{2.4.1}$$

Following similar steps we get

$$-\int\partial_i\phi^\dagger\partial^i\phi d^3x=\int\partial_i\phi^\dagger\partial_i\phi d^3x$$
$$=\textstyle\sum_k\frac{k^2}{2\omega_k}\left(b(k)a(-k)e^{-2i\omega_kt}+a^\dagger(k)a(k)+b(k)b^\dagger(k)+a^\dagger(k)b^\dagger(-k)e^{2i\omega_kt}\right)$$

and

$$\int \mu^2 \phi^\dagger \phi d^3x = \sum_{\mathbf{k}} \frac{\mu^2}{2\omega_{\mathbf{k}}} \left(b(\mathbf{k})a(-\mathbf{k})e^{-2i\omega_k t} + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}} t} \right)$$

Adding the final parts of the equations above, and using $\mathbf{k}^2 + \mu^2 = (\omega_{\mathbf{k}})^2$ along with the coefficient commutation relations, we end up with

$$\begin{aligned} H_0^0 &= \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2} \left(\underbrace{a(\mathbf{k})a^\dagger(\mathbf{k})}_{\text{use commutator}} + a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + \underbrace{b(\mathbf{k})b^\dagger(\mathbf{k})}_{\text{use commutator}} \right) \\ &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2} + b^\dagger(\mathbf{k})b(\mathbf{k}) + \frac{1}{2} \right) \end{aligned} \tag{2.4.2}$$

or simply

$$H_0^0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2} \right) \tag{2.4.3}$$

where

$$N_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \quad N_b(\mathbf{k}) = b^\dagger(\mathbf{k})b(\mathbf{k}) \tag{2.4.4}$$

$\underline{N_a}(\mathbf{k})$ = number operator with eigenvalue $n_a(\mathbf{k})$ = number of a particles with 3− mom $\mathbf{k}$ in the ket,

$\underline{N_b}(\mathbf{k})$ = number operator with eigenvalue $n_b(\mathbf{k})$ = number of b particles with 3 -mom $\mathbf{k}$ in the ket,

and, the vacuum has 1/2 quantum of energy for each $\mathbf{k}$ for a particles, and also for b particles.

We also anticipate that the b type particles will be antiparticles, and the a types, normal particles.

· **Example**

10 particle state

$$H_0^0 | 5\phi_{k_1}, 2\phi_{k_2}, \underbrace{3\bar{\phi}_{k_3}}_{b \text{ type particles}} \rangle$$

$$= \sum_k \omega_k \left(N_a(k) + \frac{1}{2} + N_b(k) + \frac{1}{2} \right) |5\phi_{k_1}, 2\phi_{k_2}, 3\bar{\phi}_{k_3}\rangle$$

$$= \left(5\omega_{\mathrm{k}_1} + 2\omega_{\mathrm{k}_2} + 3\omega_{\mathrm{k}_3} + \sum_{\mathrm{k}} \omega_{\mathrm{k}} \left(\frac{1}{2} + \frac{1}{2}\right)\right) \left(5\phi_{\mathrm{k}_1}, 2\phi_{\mathrm{k}_2}, 3\bar{\phi}_{\mathrm{k}_3}\right)$$

Vaccum state

$$H_0^0 \underbrace{|0\rangle}_{\text{vacuum state}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2}\right) |0\rangle$$

$$= \underbrace{\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{1}{2} + \frac{1}{2}\right)}_{\text{infinite energy}} |0\rangle$$

2.4.1 Unit Nors and Orthogonality for Multiparticle States

$$\langle \phi_k || \phi_k \rangle = \int_V \underbrace{\phi_k^\dagger(x,t)\phi_k(x,t)}_{\text{slates}} d^3x = 1 \tag{2.4.5}$$

$$\langle \phi_{\mathbf{k}} \| \phi_{\mathbf{k}'} \rangle = \int_V \underbrace{\phi_{\mathbf{k}}^\dagger(\mathbf{x},t)\phi_{\mathbf{k}'}(\mathbf{x},t)}_{\text{states}} d^3x = 0, \mathbf{k} \neq \mathbf{k}' \tag{2.4.6}$$

Note: It is assumed that the ket is expressed in the position basis. The ket symbol $|\phi\rangle$ in general represents a particle state, but the form of the ket when we write it out mathematically changes with the basis we care to use for eyample. For example, we could express the ket in the momentum basis (in momentim space) instead of the position basis x; or in a number of other ways. Mathematically,

$$|\phi_{\mathbf{k}}\rangle = Ae^{-i(Et - \mathbf{k}\cdot\mathbf{x})} \quad (|\phi_{\mathbf{k}}\rangle \text{ here is expressed in the position basis })$$

is really $|\phi\rangle_{x \text{ basis}} = \langle x|\phi\rangle = Ae^{-i(Et - k\cdot x)}$

2.5 Creation and Destruction Operators

Proof that $a(\mathrm{k})$ is a Particle Destruction Operator

What can we say about the state

$$a(\mathbf{k})\,|n_{\mathbf{k}}\rangle = |m_{\mathbf{k}}\rangle$$

To see this, first operate on this state with our number operator

$$N_a(\mathbf{k})\left|m_{\mathbf{k}}\right\rangle = N_a(\mathbf{k})a(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle = \underbrace{a^\dagger(\mathbf{k})a(\mathbf{k})}_{\text{use commuttor}}\ a(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle$$

Then we use the commutation relations:

$$\left(a(k)a^\dagger(k)-1\right)a(k)\left|n_k\right\rangle = a(k)a^\dagger(k)a(k)\left|n_k\right\rangle - a(k)\left|n_k\right\rangle = a(k)N_a(k)\left|n_k\right\rangle - a(k)\left|n_k\right\rangle =$$

$$= a(k)n_k\left|n_k\right\rangle - a(k)\left|n_k\right\rangle = n_k a(k)\left|n_k\right\rangle - a(k)\left|n_k\right\rangle = (n_k-1)\,a(k)\left|n_k\right\rangle = (n_k-1)\left|m_k\right\rangle \text{ So}$$

$$N_a(\mathrm{k})\left|m_{\mathrm{k}}\right\rangle = (n_{\mathrm{k}}-1)\left|m_{\mathrm{k}}\right\rangle = m_{\mathrm{k}}\left|m_{\mathrm{k}}\right\rangle$$

$$m_k = n_k - 1$$

Similarly, we have

$$N_a(\mathrm{k})\left(a(\mathrm{k})|n_{\mathrm{k}}\right)\rangle = (n_{\mathrm{k}}-1)\left(a(\mathrm{k})\left|n_{\mathrm{k}}\right\rangle\right)$$

$$N_a(k)\left(a^\dagger(k)\left|n_k\right\rangle\right) = (n_k+1)\left(a^\dagger(k)\left|n_k\right\rangle\right.$$

$$N_b(k)\left(b(k)\left|\bar{n}_k\right\rangle\right) = (\bar{n}_k-1)\left(b(k)\left|\bar{n}_k\right\rangle\right)$$

$$N_b(k)\left(b^+(k)\left|\bar{n}_k\right\rangle\right) = (\bar{n}_k+1)\left(b^+(k)\left|\bar{n}_k\right\rangle\right)$$

$a(k), b(k)$ are called lowered operators and $a^\dagger(k), b^\dagger(k)$ are called raised operators.

2.5.1 Normalization factors for raised and lowered states

Consider

$$a^\dagger(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle = A\left|n_{\mathbf{k}}+1\right\rangle$$

$$\left(A\left|n_{\mathrm{k}}+1\right\rangle\right)^\dagger = \left(a^\dagger(\mathbf{k})\left|n_{\mathbf{k}}\right\rangle\right)^\dagger = \left\langle n_{\mathbf{k}}\right|a(\mathbf{k}) = \left\langle n_{\mathbf{k}}+1\right|A^+$$

Note that

$$\left\langle n_{\mathbf{k}}\left|a(\mathbf{k})a^\dagger(\mathbf{k})\right|n_{\mathbf{k}}\right\rangle = \left\langle n_{\mathbf{k}}+1\left|A^\dagger A\right|n_{\mathbf{k}}+1\right\rangle = A^\dagger A\left\langle n_{\mathbf{k}}+1\|n_{\mathbf{k}}+1\right\rangle = A^\dagger A$$

Also

$$\left\langle n_{\mathbf{k}}\left|\underbrace{a(\mathbf{k})a^\dagger(\mathbf{k})}_{\text{use commutator}}\right|n_{\mathbf{k}}\right\rangle = \left\langle n_{\mathbf{k}}|\underbrace{a^\dagger(\mathbf{k})a(\mathbf{k})}_{N_a(\mathbf{k})}+1|n_{\mathbf{k}}\right\rangle = \left\langle n_{\mathbf{k}}\left|n_{\mathbf{k}}+1\right|n_{\mathbf{k}}\right\rangle = n_{\mathbf{k}}+1$$

Thus $A = \sqrt{n_k + 1}$.

$$\begin{aligned} a^\dagger(\mathbf{k})\,|n_\mathbf{k}\rangle &= \sqrt{n_\mathbf{k}+1}\,|n_\mathbf{k}+1\rangle \\ a(\mathbf{k})\,|n_\mathbf{k}\rangle &= \sqrt{n_\mathbf{k}}|n_\mathbf{k}-1\rangle \\ b^\dagger(\mathbf{k})\,|\bar{n}_\mathbf{k}\rangle &= \sqrt{\bar{n}_\mathbf{k}+1}\,|\bar{n}_\mathbf{k}+1\rangle \\ b(\mathbf{k})\,|\bar{n}_\mathbf{k}\rangle &= \sqrt{\bar{n}_\mathbf{k}}\,|\bar{n}_\mathbf{k}-1\rangle \end{aligned} \tag{2.5.1}$$

Note that a lowering operator destroys the vacuum state. In summary, second quantization turned the solution coefficient in RQM, which were merely constants, into creation and destruction operators in QFT.

2.5.2 Total Particle Number

For future use, we define the **total particle number as the number of particles (i.e. a types) minus the number of antiparticles (b types)**. For scalars, the total particle number operator is

$$N(\phi) = \sum_k \left(N_a(\mathbf{k}) - N_b(\mathbf{k})\right) \tag{2.5.2}$$

2.5.3 Operator Fields

The total particle lowering operator field is

$$\phi = \underbrace{\phi^+}_{\text{destroys particles}} + \underbrace{\phi^-}_{\text{creates anti-particles}} \tag{2.5.3}$$

The total particle raising operator field is

$$\phi^\dagger = \underbrace{\phi^{\dagger +}}_{\text{destroys anti-particles}} + \underbrace{\phi^{\dagger -}}_{\text{Creates particles}} \tag{2.5.4}$$

2.5.4 Normal Ordering

One approach to solving "infinite vacuum energy" problem is called normal ordering. Normal ordering, in any term, consists of moving all destruction operators to the right-hand side of that term. With this ordering, we have the Hamiltonian as:

$$H_0^0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right) \tag{2.5.5}$$

Note that normal ordering is often justified because particle/field behavior in our theories of classical mechanics, electromagnetism, and special relativity depends on energy difference, so we can take our reference as the vacuum energy level and all energies of interest are relative to that. In those theories, it is ΔE that is important, not E. So, why not assume our Hamiltonian represents ΔE instead of E? The answer is because in general relativity, the theory depends on E. To be consistent with all of physics, we need H representing E, not ΔE.

2.6 Probability, Four Currents, and Charge Density

Since our governing equation is the Klein-Gordon equation, and tha is the same as in RQM, we can follow the similar steps to derive the same 4-current relations, except that ϕ and $\phi^\dagger$ are now operator fields:

$$j^\mu{}_{,\mu} = 0 \text{ with } j^\mu = i\left(\phi^\mu \phi^\dagger - \phi^{\dagger\mu}\phi\right) \quad j^\mu \text{ is an operator} \tag{2.6.1}$$

$$\rho = j^0 = i\left(\frac{\partial \phi}{\partial t}\phi^\dagger - \frac{\partial \phi^\dagger}{\partial t}\phi\right) \tag{2.6.2}$$

Since (2.6.2) is an operator, we **need its expectation value to find measurable probability density**,

$$\bar{\rho} = \langle \phi, \phi_2, \phi_3, \ldots | \rho | \phi, \phi_2, \phi_3, \ldots \rangle$$

and

$$\rho = \frac{1}{V}\sum_k \left(a^\dagger(\mathrm{k})a(\mathrm{k}) - b^\dagger(\mathrm{k})b(\mathrm{k})\right) = \frac{1}{V}\sum_{\mathrm{k}} \left(N_a(\mathrm{k}) - N_b(\mathrm{k})\right) \tag{2.6.3}$$

For single particle state

$$\bar{\rho} = \langle\phi_k|\rho|\phi_k\rangle = \left\langle \phi_k \left| \frac{1}{V}\sum_k \left(N_a(k) - N_b(k)\right) \right| \phi_k \right\rangle = \left\langle \phi_k \left| \frac{1}{V} \right| \phi_k \right\rangle = \frac{1}{V}$$

For a plane wave, this is exactly our probability density, a flat distribution over the volume.

For antiparticles(type b particles), we obtain negative probability:

$$\bar{\rho} = \langle\bar{\phi}_{\mathrm{k}}|\rho|\bar{\phi}_{\mathrm{k}}\rangle = \left\langle \bar{\phi}_{\mathbf{k}'} \left| \frac{-1}{V} \right| \bar{\phi}_{\mathbf{k}'} \right\rangle = -\frac{1}{V}$$

This was another tip to early researchers that the density here was more readily related to charge density, and the b particles were antiparticles, with opposite charge and charge density from particles.

2.6.1 Charge density not probability density

If we multiply our four current operator (2.6.1) by the charge of a scalar particle q it behaves like a charge density operator, which we will designate by s^μ

$$s^\mu_\mu = 0 \quad \text{with} \quad s^\mu = qj^\mu = iq\left(\phi^{,\mu}\phi^\dagger - \phi^{\dagger,\mu}\phi\right) \tag{2.6.4}$$

So

$$\rho_{\text{charge}} = q^0 = iq\left(\frac{\partial\phi}{\partial t}\phi^\dagger - \frac{\partial\phi^\dagger}{\partial t}\phi\right) \tag{2.6.5}$$

2.6.2 Caution in evaluating expectation values of density operators

For operators with a spatial dependence such as ρ often has, the spatial dependence in the operator is not included in the integration. That is. writing out the expectation values as an

integral, we integrate over the $\mathbf{x}'$ of the state, but not the $\mathbf{x}$ of the operator:

$$\langle \rho(\mathbf{x},t) \rangle = \langle \phi_{\mathbf{k}}(\mathbf{x}',t) \, | \rho(\mathbf{x},t) | \phi_{\mathbf{k}}(\mathbf{x}',t) \rangle = \int \phi^{\dagger}_{\mathbf{k},\text{ state}}(\mathbf{x}',t) \, \rho(\mathbf{x},t) \phi_{\mathbf{k},\text{state}}(\mathbf{x}',t) \, d^3x' \tag{2.6.6}$$

2.6.3 Charge operator

Regarding charge, we need merely to integrate our charge density operator qj^0 over the entire volume, to get the **charge operator**:

$$Q = \int s^0 d^3x = q \int j^0 d^3x = q \sum_{\mathbf{k}} (N_a(\mathbf{k}) - N_b(\mathbf{k})) \tag{2.6.7}$$

2.6.4 Three momentum operator

$$p' = \int \mathcal{R}^i d^3x = -\int \pi_r \frac{\partial \phi^r}{\partial x^i} d^3x = -\int \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \phi}{\partial x^i} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}^{\dagger}} \frac{\partial \phi^{\dagger}}{\partial x^i} \right) d^3x \tag{2.6.8}$$

Substituting the Klein-Gordon solutions, we find

$$P = \sum_k k \, (N_a(k) + N_b(k)) \tag{2.6.9}$$

Similar to what happened to charge, we have 1/2 quanta in the vacuum with 3-momentum, but the total for the vacuum sums to zero. Unlike energy, this is a vector summation, and for every 3-momentum $\mathbf{k}$ in the sum, there is a 3-momentum $-\mathbf{k}$, as well. The net is nil 3-momentum for the vacuum, which again, is a welcome result.

So far in our theory, only energy has proved problematic in having a non-zero vacuum expectation value (VEV).

2.6.5 Four momentum operator

$$\underbrace{P^{\mu} = K^{\mu}}_{\text{operators}} = \underbrace{\begin{pmatrix} H \\ \mathbf{p} \end{pmatrix}}_{for free scalars} = \sum_{\mathbf{k}} \begin{pmatrix} \omega_{\mathbf{k}} \\ \mathbf{k} \end{pmatrix} (N_a(\mathbf{k}) + N_b(\mathbf{k})) \tag{2.6.10}$$

where we note that k^{μ} usually refers to the numeric (not operator) 4 vector $(\omega_{\mathbf{k}}, \mathbf{k})$

2.6.6 Klein-Gordon states are Bosons

Using the same scalar creation operator repeatedly on a state results in a raised state containing a number of the same particle with the same **k** (and thus the same energy and identical in all regards.) For example,

$$a(k)^\dagger|0\rangle = |\phi_k\rangle \rightarrow a(\mathrm{k})^\dagger\,|\phi_k\rangle = \sqrt{2}\,|2\phi_k\rangle \rightarrow a(\mathrm{k})^+\,|2\phi_\mathrm{k}\rangle = \sqrt{3}\,|3\phi_\mathrm{k}\rangle \rightarrow \ldots$$

This means Klein-Gordon states must be bosons.

2.6.7 Other useful operators

To **create a general single particle state**, we use the following creation operator:

$$C = \sum_k A_k a_k^\dagger \tag{2.6.11}$$

For example,

$$C|0\rangle = \sum_\mathbf{k} A_\mathbf{k}\mathbf{a}_\mathbf{k}^\dagger|0\rangle = A_1\,|\phi_1\rangle + A_2\,|\phi_2\rangle + A_3\,|\phi_3\rangle + \ldots = |\phi\rangle$$

and

$$\sum_\mathbf{k} |A_\mathbf{k}|^2 = 1$$

Similarly, we can destroy a general single particle state using

$$D = \sum_\mathbf{k} \mathbf{a}_\mathbf{k} \tag{2.6.12}$$

Box

What $\phi(x)$ and $\phi^\dagger(x)$ Create When Acting on the Vacuum ?

$\phi(x)$ acting on the vacuum will create a single general antiparticle state comprising superposition of an infinite number of eigenstates, each with a constant coefficient in front of it, i.e.,

$$\phi(x)|0\rangle = \sum_\mathbf{k} \frac{1}{\sqrt{2V\omega_\mathbf{k}}} e^{-ikx} \underbrace{a(\mathbf{k})|0\rangle}_{0} + \sum_\mathbf{k} \frac{1}{\sqrt{2Va_\mathbf{k}}} e^{ikx} \underbrace{b^\dagger(\mathbf{k})|0\rangle}_{|\bar{\phi}_k\rangle} = \sum_\mathbf{k} \frac{1}{\sqrt{2V\omega_\mathbf{k}}} \underbrace{e^{ikx}\,|\bar{\phi}_\mathbf{k}\rangle}$$

2.6.8 Fock space and Hilbert space

In NROM and RQM, the states are single particle states and the abstract space they inhabit is called **Hilbert space**, which has a different single particle eigenstate as the basis vector of each "axis'. The dimension of the Hilbert space for a given system is simply the number of linearly independent eigenstates in that system. This can, for many systems, be infinite.

The multiparticle abstract state space of QFT is called Fock space, which is simply an extension of Hilbert space to multiparticle states.

2.6.9 a(k) Destroys Any State without Single a Type Particle in k Eigenstate

· **Example**

For a multi-particle state

$$a_2 \left|\phi_1, 4\phi_3, 7\bar{\phi}_2\right\rangle = 0$$

In general, a particular type particle destruction operator of given **k** acting on a state that has no particles of that type of the same 3 -momentum k results in zero.

2.7 Derive Scalar Feynman Propagator

In the course of deriving the interaction theory, a mathematical relationship arises that is called the Feynman propagator. Physically, it can be visualized as repesenting a virtual particle that exists fleetingly and carries energy, momentum, and in some cases, charge from one real particle to another. **Thus, it is the carrier, or mediator, of force (interaction).**

From a heuristic perspective the operator field $\phi^\dagger(y)$ can be considered to create a virtual scalar particle at event y, and the field operator $\phi(x)$ destroys that virtual particle at event x.

The entire derivation is for continuous (not discrete) eigenstate solutions of the field equation

(Klein-Gordon here), since the propagator represents a virtual particle in the vacuum and the vacuum is not confined to a volume V. We represent the scalar Feynman propagator with the symbol $i\Delta_F(x-y)$. (Including the imaginary factor i is common practice.)

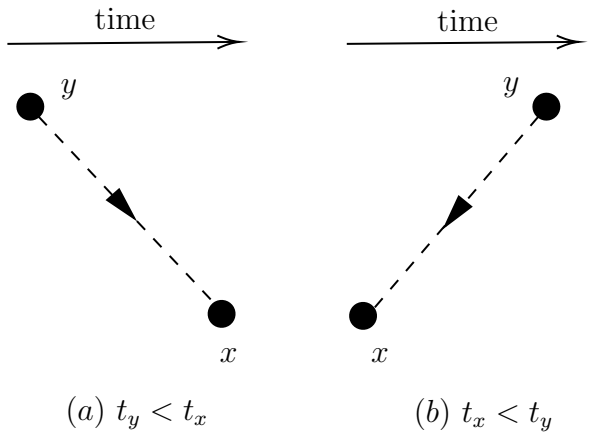


Figure 2.1: Creation and Destruction of Virtual Particle/Antiparticle

Fig 2.1(a) represents creation of a particle, which will be virtual, at y and destruction of it at x. Fig.2.1 (b) represents creation of an antiparticle at x and destruction of it at y. Virtual particles are never detected when real particles interact, so the same effect on the real particles could be realized by either of the processes in the figure above. That is,a virtual particle carrying charge from y to x would represent the same charge exchanges as a virtual antiparticle carrying opposite charge from x to v. Thus, we need a relationship for the propagator that includes both scenarios as possibilities.

That is, we need an operator that will create a particle first if $t_y < t_x$, but create an antiparticle first if $t_x < t_y$. Our Klein-Gordon solutions provide the means for the desired creation and destruction operations. But these have to be arranged to provide us with the time ordering dependence of Fig.2.1. To this end, consider time ordering operator T:

$$\text{for } t_y < t_x \text{ (particle) } T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi(x)\phi^\dagger(y) \tag{2.7.1}$$

Of course. $\phi(x)$ also creates an antiparticle and $\phi^\dagger(x)$ also destroys an anti-particle, but **we will see this effect ultimately drops out and does not play a role in propagator.**

$$\text{for } t_x < t_y \text{ (anti-particle) } T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi^\dagger(y)\phi(x) \tag{2.7.2}$$

We now defined the transition amplitude as

$$\left\langle 0 \left| T\left\{\phi(x)\phi^{\dagger}(y)\right\} \right| 0 \right\rangle \tag{2.7.3}$$

Now, consider

$$T\left\{\phi(x)\phi^{\dagger}(y)\right\}|0\rangle = \phi(x)\phi^{\dagger}(y)|0\rangle \tag{2.7.4}$$

plugging the solutions into the equation above, we have

$$\begin{aligned} T\left\{\phi(x)\phi^{\dagger}(y)\right\}|0\rangle &= \left(\phi^{+}(x)+\phi^{-}(x)\right)F(y)|\phi\rangle \\ &= G(x)F(y)|0\rangle + H(x)F(y)|\bar{\phi}\phi\rangle \end{aligned} \tag{2.7.5}$$

where G, F, and H are numeric factors that result from the creation and destruction operations. Thus $(GF)^{\dagger}(GF)$ represents the probability of observing the vacuum state.To find the amplitude GF, we need only form an inner product,i.e.

$$\langle 0|T\left\{\phi(x)\phi^{+}(y)\right\}|0\rangle = \langle 0|G(x)F(y)|0\rangle + \langle 0|H(x)F(y)|\bar{\phi}\phi\rangle = G(x)F(y)$$

So, the VEV of the time ordering operator is an amplitude, the square of whose magnitude is the probability of the transition from the vacuum initially (represented by $|0\rangle$) to the vacuum finally (represented by $\langle 0|$). Actually, $|G(x)F(y)|^2$ is a probability density (to be precise, a double density), because it is a function of $\mathbf{x}$ and $\mathbf{y}$. That is, the location $\mathbf{y}$ where the virtual particle is created could be anywhere, and so could the location $\mathbf{x}$ where it is destroyed. We would need to integrate the probability density over all possible $\mathbf{x}$ and all possible $\mathbf{y}$ to get the actual probability, and this is what one does in interaction theory to calculate probabilities and cross sections.

Given all of this, we now define the **Feynman propagator** $i\Delta_F$,

$$i\Delta_F(x-y) = \left\langle 0 \left| T\left\{\phi(x)\phi^{\dagger}(y)\right\} \right| 0 \right\rangle \tag{2.7.6}$$

Now we express $i\Delta_F$ in terms of commutators. For $t_y < t_x$, the case for a virtual particle, the Feynman propagator equals

$$i\Delta_F(x-y) = \left\langle 0 \left| \phi(x)\phi^{\dagger}(y) \right| 0 \right\rangle$$

$$= \left\langle 0 \left| \phi^{+}(x)\phi^{\dagger+}(y) \right| 0 \right\rangle + \left\langle 0 \left| \phi^{+}(x)\phi^{\dagger-}(y) \right| 0 \right\rangle + \left\langle 0 \left| \phi^{-}(x)\phi^{\dagger+}(y) \right| 0 \right\rangle + \left\langle 0 \left| \phi^{-}(x)\phi^{\dagger-}(y) \right| 0 \right\rangle$$

$$= \langle 0 \left| \phi^+(x)\phi^{\dagger -}(y) \right| 0 \rangle + (\text{factor } F) \langle 0 \left| \phi^-(x) \right| \phi \rangle = \langle 0 \left| \phi^+(x)\phi^{\dagger -}(y) \right| 0 \rangle$$

where "factor" represents the non-operator quantities in each field operator term that are left. unchanged when the creation and destruction coefficient operators act on a ket.

Since

$$0 = \langle 0| - \phi^{\dagger -}(y) \underbrace{\phi^+(x)|0\rangle}_{=0}$$

we now have

$$i\Delta_F(x-y) = \langle 0 \left| \phi^+(x)\phi^{\dagger -}(y) - \phi^{\dagger -}(y)\phi^+(x) \right| 0 \rangle = \langle 0 \left| \left[\phi^+(x), \phi^{\dagger -}(y) \right] \right| 0 \rangle \tag{2.7.7}$$

In similar fashion, for $t_x < t_y$, we have

$$i\Delta_F(x-y) = \langle 0 \left| \left[\phi^{\dagger +}(y), \phi^-(x) \right] \right| 0 \rangle \tag{2.7.8}$$

Now we express commutator forms of $i\Delta_F$ as integrals. First we define:

$$i\Delta^+(x-y) = \left[\phi^+(x), \phi^{\dagger -}(y) \right] \tag{2.7.9}$$

Using the continuous solutions to Klein-Gordon equation, we have

$$\begin{aligned} i\Delta^+(x-y) &= \frac{1}{2(2\pi)^3} \iint \left[a(\mathrm{k}), a^+(\mathrm{k}') \right] \frac{e^{-ikx} e^{ik'y}}{\sqrt{\omega_{\mathrm{k}} \omega_{\mathrm{k}'}}} d^3\mathrm{k} d^3\mathrm{k}' \\ &= \frac{1}{2(2\pi)^3} \int \left(\int \frac{e^{iky}}{\sqrt{\omega_{\mathbf{k}} \omega_{\mathrm{k}'}}} \delta(\mathrm{k} - \mathrm{k}') \, d^3\mathrm{k}' \right) e^{-ikx} d^3\mathrm{k} \end{aligned}$$

so

$$i\Delta^+(x-y) = \frac{1}{2(2\pi)^3} \int \frac{e^{-ik(x-y)}}{a_{\mathbf{k}}} d^3\mathbf{k} \tag{2.7.10}$$

Similarly,

$$i\Delta^-(x-y) = \left[\phi^{\dagger +}(y), \phi^-(x) \right] = \frac{1}{2(2\pi)^3} \iint \left[b(\mathrm{k}), b^\dagger(\mathrm{k}') \right] \frac{e^{ikx} e^{-iky}}{\sqrt{\omega_{\mathrm{k}} \omega_{\mathrm{k}'}}} d^3\mathrm{k} d^3\mathrm{k}'$$

Thus,

$$i\Delta^\pm(x-y) = \frac{1}{2(2\pi)^3} \int \frac{e^{\mp ik(x-y)}}{\omega_k} d^3k \tag{2.7.11}$$

Because the commutator of these operators is a number. $i\Delta^{\pm}(x-y)$ are simply numbers(to be more precise it is a numeric function, not an operator function), not operators.**The bottom line is: we don't have to worry about operators, their effects, or VEV brackets any more, but can simply evaluatate the Feynman propagator as a numeric mathematical relation.**

Next, **we express the two real integrals $i\Delta^{\pm}$ as contour integrals**. From **Cauchy integral formula**, we have

$$f\left(\omega_{\mathbf{k}}\right)=\frac{1}{i2\pi}\int_{C^{+}}\frac{f\left(k_{0}\right)}{k_{0}-\omega_{\mathbf{k}}}dk_{0} \tag{2.7.12}$$

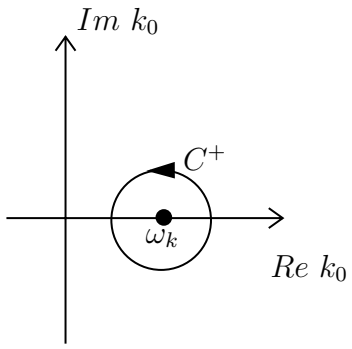


Figure 2.2: Contour Integral for real, positive frequency

Now, rewrite $i\Delta^{+}(x-y)$ as

$$i\Delta^{+}(x-y)=\frac{1}{(2\pi)^{3}}\int e^{i\mathbf{k}\cdot(x-y)}\underbrace{\left\{\frac{e^{-i\omega_{k}(t_{x}-t_{y})}}{2\omega_{\mathbf{k}}}\right\}}_{f(\omega_{\mathbf{k}})}d^{3}\mathbf{k}$$

where we take the bracketed quantity as equal to $f\left(\omega_{\mathbf{k}}\right)$, and where

$$f\left(k_{0}\right)=\frac{e^{-ik_{0}(t_{x}-t_{y})}}{k_{0}+\omega_{\mathbf{k}}}$$

We can then use $f(k_0)$ to find

$$\begin{aligned}i\Delta^{+}(x-y)&=\frac{1}{(2\pi)^{3}}\int e^{i\mathbf{k}\cdot(x-y)}\left\{\frac{1}{i2\pi}\int_{C^{+}}\frac{f\left(k_{0}\right)}{k_{0}-\omega_{\mathbf{k}}}dk_{0}\right\}d^{3}\mathbf{k}\\&=\frac{1}{(2\pi)^{3}}\int e^{i\mathbf{k}\cdot(x-y)}\left\{\frac{1}{i2\pi}\int_{C^{+}}\frac{e^{-ik_{0}(t_{x}-t_{y})}}{\left(k_{0}-\omega_{\mathbf{k}}\right)\left(k_{0}+\omega_{\mathbf{k}}\right)}dk_{0}\right\}d^{3}\mathbf{k}\\&=\frac{-i}{(2\pi)^{4}}\int_{C^{+}}\frac{e^{-ik(x-y)}}{\left(k_{0}\right)^{2}-\left(\omega_{\mathbf{k}}\right)^{2}}d^{4}k\end{aligned} \tag{2.7.13}$$

where the integral notation now implies integration over four dimensions of the 4 -momentum, with the 3 -momentum part from $-\infty$ to $+\infty$ in real space and the energy part a contour integral in complex space. Note that the integral does not "blow up" because $k_0 \neq \omega_{\mathrm{k}}$ over the contour integral. k_0 **has at this point become, for us, a variable that generally does not equal energy** $\omega_{\mathbf{k}}$.

Because the following relations are always true for any four vector,

$$k^2 = (k_0)^2 - (\mathbf{k})^2 \rightarrow (k_0)^2 = k^2 + (\mathbf{k})^2 \tag{2.7.14}$$

we have

$$a_k^2 - (k)^2 = \mu^2 \quad \rightarrow \quad \omega_k^2 = \mu^2 + (k)^2 \tag{2.7.15}$$

and thus,

$$i\Delta^+(x-y) = \frac{-i}{(2\pi)^4}\int_{C^+} \frac{e^{-ik(x-y)}}{k^2-\mu^2} d^4k \tag{2.7.16}$$

For $i\Delta^-(x-y)$, we carry out similar steps except that the contour integral (still counter clockwise [ccw], as in Fig. 2.2) is now about $-\omega_{\mathbf{k}}$. When all is said and done, we find the only differences to be the sign and the contour, which is now about the negative frequency value and designated by C^-:

$$i\Delta^-(x-y) = \frac{i}{(2\pi)^4}\int_{C^-} \frac{e^{-ik(x-y)}}{k^2-\mu^2} d^4k \tag{2.7.17}$$

We then re-express $i\Delta_F$ in most convenient form using the contour shown below:

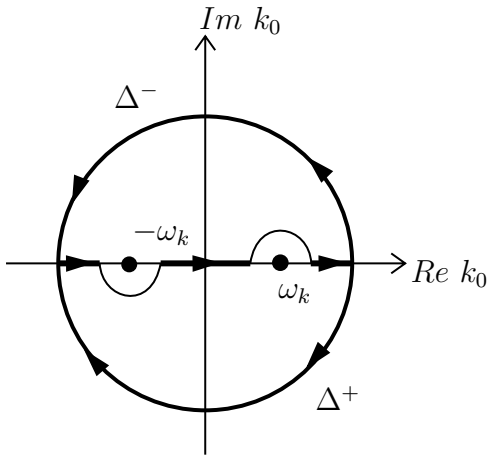


Figure 2.3: Contour integrals for $\Delta^{\pm}$

The lower loop shown above encloses $+\omega_{\mathrm{k}}$, but since it has a cw integration path, the result will have a sign change, and hence equals the ccw integration. Thus, the lower loop represents Δ^+. This means we can define the Feynman propagator as **proportional to the same integral over the two different loops.** We say proportional because we also have to include the concomitant integration over the 3D space of **k**. So we can then re-write the Feynman propagator as:

$$i\Delta_F(x-y) = \frac{i}{(2\pi)^4}\int_{C_F}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \tag{2.7.18}$$

where C_F is the contour shown in Fig.2.3.

Now, consider enlarging the outer hemispheric parts of the two loops, so they extend essentially to infinity. The value of the contour integrals over them will remain unchanged. But the k^2 value in the denominator of (2.7.18) will become so large that any contribution to the integral over those parts of the path will become negligible.

We can further simplify by **moving the poles an infinitesimal distance η of the real axis** as shown in the figure below and deform the contour so that it is all along the real axis.In the limit as $\eta \to 0$, the integral will have the same value, though we must now include this slight pole shift in the propagator expression (2.7.18).

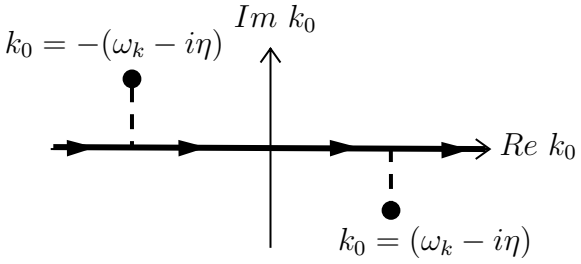


Figure 2.4: Contour and displaced poles for Δ_F

Again, with $k^2-\mu^2 = (k_0)^2 - (\omega_{\mathbf{k}})^2$, we have

$$i\Delta_F(x-y) = \frac{i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{(k_0)^2-(\omega_{\mathbf{k}}-i\eta)^2}d^4k$$

If we use $k^2-\mu^2 = (k_0)^2 - (\omega_{\mathbf{k}})^2$ again, and take $\epsilon = 2\eta\omega_{\mathbf{k}}$, we have our **final result for the Feynman scalar propagator**:

$$i\Delta_F(x-y) = \frac{i}{(2\pi)^4}\int_{-\infty}^{+\infty} \frac{e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon} d^4k \tag{2.7.19}$$

From (2.7.19), we can readily write down **the 4-momentum space form of the propagator**,**the Fourier transform of (2.7.19)**, which is

$$\Delta_F(k) = \frac{1}{k^2-\mu^2+i\varepsilon} \tag{2.7.20}$$

Chapter 3

Spinors: Spin 1/2 Fields

3.1 Dirac's Approach to RQM:

Dirac's primary goal was a 1st order relativistic Schrodinger equation, and he postulated that if it existed, it must have the general form of

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle = (a\cdot\mathbf{p}+\beta m)|\psi\rangle \tag{3.1.1}$$

In 3.1.1, $\mathbf{p}$ is particle three momentum (an operator in quantum theories), and the vector α and the scalar β would have to be determined. Thus, the equation would be first order in the time derivative. To find α and β, Dirac reasoned that H^2 and $|\psi\rangle$ must also satisfy the usual relativistic energy momentum relation (and therefore the Klein-Gordon equation)

$$-\frac{\partial^2}{\partial t^2}|\psi\rangle = H^2|\psi\rangle = \left(\mathbf{p}^2+m^2\right)|\psi\rangle$$

Thus

$$-\frac{\partial^2}{\partial t^2}|\psi\rangle = H^2|\psi\rangle = (\alpha_i p_i+\beta m)(\alpha_j p_j+\beta m)|\psi\rangle$$

$$= \left(\alpha_i^2 p_i^2 + \underbrace{(\alpha_i\alpha_j+\alpha_j\alpha_i)}_{\text{must }=0} p_i p_j + \underbrace{(\alpha_i\beta+\beta\alpha_i)}_{\text{must }=0} p_i m + \beta^2 m^2\right)|\psi\rangle$$

Therefore, $\alpha_i^2 = 1$ and $\beta^2 = 1$. In summary, we have anti-commutators relationship:

$$[\alpha_i, \alpha_j]_+ = [\alpha_i, \beta]_+ = 0 \quad i \neq j \quad \alpha_1, \alpha_2, \alpha_3, \beta \text{ all anti-commute with each other,}$$
$$(\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = (\beta)^2 = 1 \text{ (the identity matrix)} \tag{3.1.2}$$

If α_i and β were numbers they would have to commute and could not possibly anti-commute. Hence, they can only be matrices. since these matrices are operators operating on $|\psi\rangle$, then $|\psi\rangle$ itself must be a multicomponent object (i.e., a column matrix, at least.). Use the relations above, one can show that the α and β matrices are traceless, hermitian, have $\pm$ eigenvalues, and must have an even dimension of at least four.

Square matrices in a 4D space must be 4X4, and thus if $|\psi\rangle$ is a column matrix (a vector), it must have four components (a 4D vector). Take care to note the 4D space we are talking about here is not the four-dimensional physical space of relativity theory, but an abstract pace, often called **spinor space**.

3.1.1 Standard presentation

Choosing the minimum dimension case (four), Dirac and Pauli came up with a set of matrices which solve all of the conditions in (3.1.2):

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix}$$

which are commonly written using Pauli matrices

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}$$

If we define **gamma matrices** or **Dirac matrices** as:

$$\gamma^0 = \beta \quad \gamma^1 = \beta\alpha_1 \quad \gamma^2 = \beta\alpha_2 \quad \gamma^3 = \beta\alpha_3 \tag{3.1.3}$$

we find the **Hermiticity conditions** as

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{3.1.4}$$

3.1.2 Dirac equation expressed with Dirac matrices

Pre-multiplied β, Dirac's original 1^{st} order equation takes on the form

$$i\beta\frac{\partial}{\partial t}|\psi\rangle = \left(\beta\alpha_i p_i + \beta^2 m\right)|\psi\rangle = \left(-i\gamma^i\frac{\partial}{\partial x^i} + m\right)|\psi\rangle \tag{3.1.5}$$

or rearranged as what is formally called the **Dirac equation**:

$$\sum_{\eta=1}^{4}\left(\sum_{\mu=0}^{3} i\left(\gamma^\mu\right)_{\kappa\eta}\partial_\mu - m\delta_{\kappa\eta}\right)|\psi\rangle_\eta = 0 \quad \kappa = 1,2,3,4 \tag{3.1.6}$$

where we have written out the 4X4 spinor space indices in κ and η. Note the Dirac equation is actually **four separate non-matrix equations**, one for each value of the index κ. And each of these equations entails a sum of matrix components (sum over μ), each post multiplied by one of the four components (in η index) of the column vector $|\psi\rangle$.

The common way to write the Dirac equation is to hide the spinor space indices in κ and η,

$$\left(i\gamma^\mu\partial_\mu - m\right)|\psi\rangle = 0 \tag{3.1.7}$$

Another notation commonly used, which is the most streamlined of all, is

$$\not{\partial} = \gamma^\mu\partial_\mu \quad \text{so, the Dirac equation } \rightarrow (i\not{\partial} - m)|\psi\rangle = 0$$

Also, $m \rightarrow \frac{mc}{\hbar}$ in non-natural units in the Dirac equation.

3.1.3 Solutions to the Dirac equation

Write out (3.1.7) fully as:

$$i\gamma^\mu\partial_\mu|\psi\rangle = i\left(\gamma^0\partial_0+\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)|\psi\rangle = m|\psi\rangle =$$

$$= i\left(\begin{bmatrix} \partial_0 & 0 & \partial_3 & \partial_1-i\partial_2 \\ 0 & \partial_0 & \partial_1+i\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1+i\partial_2 & -\partial_0 & 0 \\ -\partial_1-i\partial_2 & \partial_3 & 0 & -\partial_0 \end{bmatrix}\right)\left|\begin{matrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{matrix}\right\rangle = m\left|\begin{matrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{matrix}\right\rangle$$

The Dirac equation solutions in the Dirac-Pauli (standard) representation are

$$\left|\psi^{(1)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{pmatrix}}_{u_1}\underbrace{e^{-ipx}}_{\text{4D physical space part}} = u_1e^{-ipx}$$

$$\left|\psi^{(2)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\\frac{-p^3}{E+m}\end{pmatrix}}_{u_2}e^{-ipx} = u_2e^{-ipx}$$

$$\left|\psi^{(3)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{pmatrix}}_{v_2}e^{ipx} = v_2e^{ipx}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\frac{p^1-ip^2}{E+m}\\\frac{-p^3}{E+m}\\0\\1\end{pmatrix}e^{ipx} = v_1e^{ipx} \tag{3.1.8}$$

We have defined new symbols $u_r(\mathbf{p})$ and $v_r(\mathbf{p})(r = 1, 2)$, which are the column vectors multiplied by the constant shown, are functions only of $\mathbf{p}$ for a given m(since $E = \sqrt{\mathbf{p}^2 + m^2}$), and go by the name **spinors**, or **four-spinors**. Note that the particles represented by $|\psi^{(n)}\rangle$ are also often called spinors.

u_1 represents spin up, and u_2 represents spin down in the particle at-rest system. As you might expect, we will find the solutions containing $v_r(\mathbf{p})$ are associated with antiparticles; and those with $u_r(\mathbf{p})$, with particles. Take care to note the reverse order numbering on $v_{2,1}$ from $u_{1,2}$, which is customary.

If we take **inner products of four spinors**, we have

$$u_1^\dagger(\mathbf{p})u_1(\mathbf{p}) = \frac{E}{m}$$

More generally

$$u_{\underline{r}L}^\dagger(\mathbf{p})u_{\underline{r}}(\mathbf{p}) = v_{\underline{r}}^\dagger(\mathbf{p})v_{\underline{r}}(\mathbf{p}) = \frac{E}{m} \tag{3.1.9}$$

Where underline means no summation. Also, spinors are **orthogonal**

$$\begin{aligned} u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = \frac{E}{m}\delta_{rs} \\ u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = 0 \end{aligned} \tag{3.1.10}$$

Therefore, the eigensolutions are also orthogonal

$$\langle\psi^{(m)}|\psi^{(n)}\rangle = 0 \text{ for } m \neq n \tag{3.1.11}$$

For example

$$\langle\psi^{(1)}|\psi^{(3)}\rangle = \int u_1^\dagger(\mathbf{p})e^{+ipx}v_2(\mathbf{p})e^{+ipx}d^3x = \underbrace{u_1^\dagger(\mathbf{p})v_2(\mathbf{p})}_{=0 \text{ for } \mathbf{p}=0}\underbrace{\int e^{+ipx}e^{+ipx}d^3x}_{=0 \text{ for } \mathbf{p}\neq 0} = 0$$

where we follow **Cesaro integration**:$\int_0^\infty \sin x dx = 1$ and $\int_0^\infty \cos x dx = 0$.

The most general solution to the Dirac equation is:

$$\psi_{\text{state}} = |\psi\rangle = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\text{p}}}} \left(C_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + D_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right) \tag{3.1.12}$$

Unlike the Klein-Gordon equation, the Dirac equation is a matrix equation. So, rather than complex conjugate form of the wave equation, we need to consider **taking a complex conjugate pose of that equation**, we define and use the **adjoint**

$$\bar{\psi}_{\text{state}} = \psi^\dagger_{\text{state}} \gamma^0 = |\psi\rangle^\dagger \gamma^0 = \langle \psi | \gamma^0 = \langle \bar{\psi}| \tag{3.1.13}$$

where an inner product between the row vector $|\psi\rangle^\dagger = \langle\psi| = \psi^+$ state and the gamma matrix are implied. The adjoint Dirac equation is

$$i\partial_\mu \langle \bar{\psi} | \gamma^\mu + m \langle \bar{\psi}| = 0 \tag{3.1.14}$$

Adjoint spinors are defined as the row vectors

$$\bar{u}_r = u_r^\dagger \gamma^0 \quad \bar{v}_r = v_r^\dagger \gamma^0 \tag{3.1.15}$$

which, gives us the **discrete plane wave adjoint general solution form**:

$$\bar{\psi}_{\text{state}} = \langle \bar{\psi}| = \sum_{r,\text{p}} \sqrt{\frac{m}{VE_{\text{p}}}} \left(D_r(\text{p}) \bar{v}_{\text{r}}(\text{p}) e^{-ipx} + C_r^\dagger(\text{p}) \bar{u}_r(\text{p}) e^{ipx} \right) \tag{3.1.16}$$

3.1.4 Probability density for Dirac Fermions

The **four-current** using the Dirac equation can be found to be

$$\partial_\mu j^\mu = 0 \quad j^\mu = (\rho, \mathbf{j}) = \bar{\psi}_{\text{state}} \, \gamma^\mu \psi_{\text{state}} \, = \langle \bar{\psi} | \gamma^\mu | \, \psi \rangle_{\text{not integ}} \tag{3.1.17}$$

where the subscript "not intg" means we are not integrating over space in the bracket shown.

The (3.1.17) means the total quantity

$$\begin{aligned}\int_V j^0 d^3x &= \int_V \rho d^3x = \int_V \bar{\psi}_{\text{state}}\gamma^0\psi_{\text{state}}d^3x = \left\langle \bar{\psi}\left|\gamma^0\right|\psi\right\rangle \\ &= \int_V \psi^\dagger_{\text{state}}\gamma^0\gamma^0\psi_{\text{state}}d^3x = \langle\psi|\psi\rangle = Q'\end{aligned} \tag{3.1.18}$$

is conserved for $V=$ all space. **For a single particle state in RQM, if we assume the solution onlu terms with coefficients** C_r(i.e., only has spinors of form u_r), our ρ becomes

$$\begin{aligned}\rho &= \left(\sum_{r,p}\sqrt{\frac{m}{VE_p}}C_r^\dagger(p)\underbrace{\bar{u}_r(p)}_{u_r^\dagger(p)\gamma^o}\right)e^{ipx}\Bigg)\gamma^0\left(\sum_{r',p'}\sqrt{\frac{m}{VE_{p^p}}}C_{r'}\left(p'\right)u'_{r'}\left(p'\right)e^{-ip'x}\right) \\ &= \left(\sum_{r,p}\sqrt{\frac{m}{VE_p}}C_r^\dagger(p)u_r^\dagger(p)e^{ipx}\right)\left(\sum_{r',p'}\sqrt{\frac{m}{VE_p}}C_{r'}\left(p'\right)u_{r'}\left(p'\right)e^{-ip'x}\right)\end{aligned} \tag{3.1.19}$$

If this is probability density, we have

$$\int \rho d^3x = \sum_{r,p}\frac{m}{V}\left(C_r^\dagger(\mathrm{p})C_r(\mathrm{p})\frac{u_r^\dagger(\mathbf{p})u_r(\mathrm{p})}{E_p/m}\underbrace{\int e^{-ipx}e^{ipx}d^3x}_{V}\right) == \sum_{r,\mathrm{p}}\left|C_r(\mathrm{p})\right|^2 = 1$$

Notice that a Dirac fermion represented by a solution with exponential form $-ipx$(that has spinor u_r and coefficient $C_r(\mathbf{p})$), has positive energy; and one represented by solution form ipx(spinor v_r and coefficient $D_r^\dagger(\mathrm{p})$), has negative energy.

3.1.5 Dirac approach to Spin

The RQM spin operator are defined as:

$$\Sigma_1 = \frac{\hbar}{2}\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}\quad \Sigma_2 = \frac{\hbar}{2}\begin{bmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{bmatrix}\quad \Sigma_3 = \frac{\hbar}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \tag{3.1.20}$$

Note Σ_i (3.1.20) is a 3D object in physical space (3 components of the spin angular momentum), but **each of the components in that space is itself a** 4X4 **matrix in relativistic**

4D **spinor space, rather than non-relativistic 2 D spinor space of NRQM.** In relativity. spin has three spatial components. But, relativistically, each component must act on a 4D column vector in spinor space.

Consider a stationary Dirac particle which has p = 0 in our frame, and the solutions (3.1.8) become much simplified. What then, are their respective spins? For the first such solution, we have

$$\Sigma_3 \left|\psi^{(1)}\right\rangle = \frac{\hbar}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \sqrt{\frac{m+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} = \frac{\hbar}{2}\left|\psi^{(1)}\right\rangle$$

Therefore $\left|\psi^{(1)}\right\rangle$ **is for spin up.**

As soon as we have a moving particle, things get more complicated, as we have to include nonzero p^i values in our solutions (3.1.8) This complication, which we didn't have in NRQM, is due to relativistic effects.

Box

Classical Macroscopic Spinning Object Translating at Relativistic Speed

In 4D relativistic theory, angular momentum is a 2nd order tensor, but it can be treated simply as a vector formed from the integral over a rotating body of $dm\,(\mathbf{r}\times\mathbf{v}_t) = dm(\mathbf{r}\times(\omega\times\mathbf{r}))$, where symbols should be obvious. When a macroscopic object like a spinning disk, as shown below, moves close to the speed of light, distances contract in the direction of the velocity, and this makes the plane of the disk appear to turn. (See figure below.)

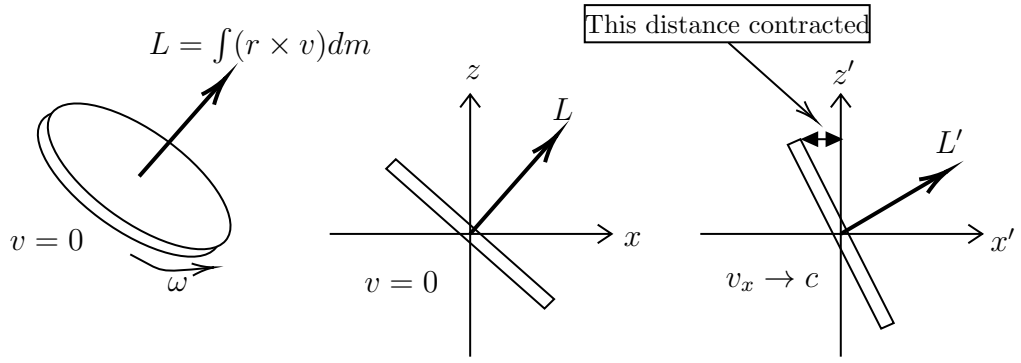


Figure 3.1: Spinning disk

The closer the disk gets to the speed of light, the more the disk surface appears in the observer's frame to align normal to the velocity direction. In the rest frame translating with the disk itself, the disk still appears aligned in the original way. In the observer's frame, though, the angular momentum L appears to turn toward the direction of the velocity becoming L'. The greater the speed, the greater this turning. At light speed, L' and v become parallel.

Quantum mechanically, then, at high speed, a particle's angular momentum (spin) magnitude remains unchanged, but its direction appears to us in our frame to realign itself closer to that of the translational velocity vector.

Mathematically, these kinds of relativistic complications are incorporated into the form of the spinors $u_r(\mathrm{p})$ and $v_r(\mathrm{p})$ (by their dependence on 3 -momentum and thus ultimately, on velocity) and by how they are combined to form more general spin states.

Note that the spinor components are actually dependent on particle velocity, rather than momentum, by the following logic. Energy and momentum are expressed (in non-natural units to make it easier to understand)

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad p^i = \frac{mv^i}{\sqrt{1 - v^2/c^2}}$$

so in the coefficient and spinor components of the Dirac spinor (3.1.8) the mass m drops out. This leaves them a function solely of velocity.

What happens when the particle is not stationary

Note what happens to the spin as seen by us, for an electron whose spin is represented solely by u_1, but has $p^1 \neq 0$, with $p^2 = p^3 = 0$ in our frame (the lab.)

$$\Sigma_3 \left|\psi^{(1)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \frac{E+m}{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p^1}{E+m} \end{pmatrix} e^{-fpx} = \frac{1}{2}\sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{-p^1}{E+m} \end{pmatrix} e^{-ipx} \neq \frac{1}{2}\left|\psi^{(1)}\right\rangle$$

u_1 for a non-translating electron has spin up, but u_1 for an electron with high trnasverse velocity is not an up eigenstate.

Now consider u_1 representing an electron traveling in the z direction instead of the x direction

$$\Sigma_3 \left|\psi^{(1)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ p^3 \\ \frac{p^3}{E+m} \\ 0 \end{pmatrix} e^{-tpx} = \frac{1}{2}\sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{pmatrix} e^{-ipx} = \frac{1}{2}\left|\psi^{(1)}\right\rangle$$

This electron, represented by u_1, is an up eigenstate as it moves, just as it was when it was at rest. Relativistically, this makes sense, as the plane of a spinning disk with $\mathbf{L}$ aligned in the direction of $\mathbf{p}$ would not appear to turn as $\mathbf{p}$ increased from zero to a relativistic value.

In general, boosts in the spin axis direction leave u_1, u_2, v_2 and v_1 in the same spin eigenstates as they would be at rest. Boosts in other directions take them out of these spin eigenstates.

Box

The four-spinors span the 4D spinor space

By analogy, we can surmise that the four Dirac spinors u_1, u_2, v_2 and v_1 of (3.1.8) span the RQM 4D spinor space of all possible spins and momenta, and thus, are basis vectors for that space. Our RQM general solution (3.1.12) contains within it all possible relativistic spin states.

More mathematically, we should know that a 4D space is spanned by four column vectors, where these vectors are all independent of one another. Generally. the vector solutions of an eigenvalue problem, which is what the Dirac equation solutions are, are independent and complete, and thus we can conclude, span the space. They can be used as basis vectors.

General RQM solution contains all possible spin directions

In (3.1.12), different coefficients $C_1(\mathbf{p})$ and $\mathcal{C}_2(\mathbf{p})$ will yield different spin states for C type particles. And different coefficients $D^\dagger_{1(\mathbf{p})}$ and $D^\dagger_{2(\mathbf{p})}$ will yield different spin states for D type particles.

To see how this works, we consider how each of the four states shown below can be represented by their respective terms in the general particle state solution (3.1.12).

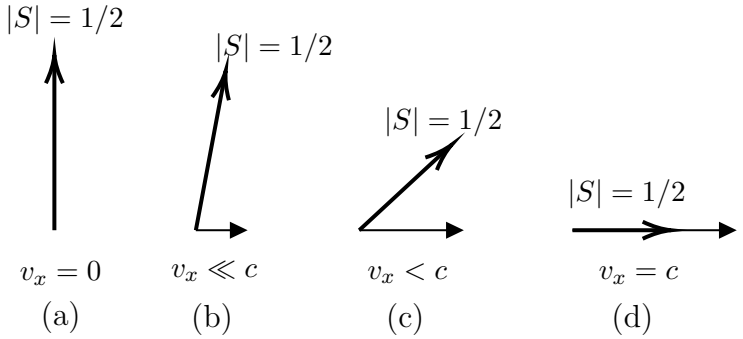


Figure 3.2: Effect of Transverse Velocity on Dirac Particle Spin

In general, for $j = a, b, c, d$, the four states shown (for a C type particle) in the figure are

$$\left|\psi_{(j)}\right\rangle = \sqrt{\frac{m}{VE_{p_j}}}\left(C_1\left(p_j\right)u_1\left(p_j\right) + C_2\left(p_j\right)u_2\left(p_j\right)\right)e^{-ip_j x} \tag{3.1.21}$$

Not that we have a particle here (so no D type terms), and $\mathbf{p}_j$ is known. State (a) there is effectively spin up with $\mathrm{p_a} = 0$, so

$$\left|\psi_{(a)}\right\rangle = \sqrt{\frac{m}{VE_{p_a}}}C_1(0)u_1(0)e^{-ip_a x} = \sqrt{\frac{m}{VE_{\mathrm{p}_a}}}\sqrt{\frac{E_{\mathrm{p}_a}+m}{2m}}\begin{pmatrix}\mathrm{I}\\0\\0\\0\end{pmatrix}e^{-ip_a x} \tag{3.1.22}$$

which is an eigenstate of Σ_3. So for state (a), $|\psi_a\rangle$ has $C_1 = 1$ and $C_2 = 0$.

For the last state (d), where the particle is traveling at the speed of light, (3.1.21) becomes an eigenstate of Σ_1 with eigenvalue 1/2,

$$\left|\psi_{(d)}\right\rangle = \sqrt{\frac{m}{VE_{\mathrm{p}_d}}}C_1(\infty)u_1(\infty)e^{-ip_d x} + \sqrt{\frac{m}{VE_{\mathrm{p}d}}}C_2(\infty)u_2(\infty)e^{-ip_d x}$$

$$= \sqrt{\frac{m}{VE_{\mathrm{p}_d}}}\sqrt{\frac{E_{\mathrm{p}_d}+m}{2m}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}e^{-ip_dx}$$

where here, we must have $C_1 = C_2 = 1$. (in the normalized version, $C_1 = C_2 = 1/\sqrt{2}$).For "in between" states (b) and (c), C_1 and C_2 would have other values. The bottumline is:**p** determines $u_{1,2}$ and then spin is represented by correct linear combination of u_1 and u_2. Note that we can never have a relativistic state where the spin vector and **p** are at right angles.

Note that u_1 and u_2 actually exist in spinor space (they are spinor space basis vectors in that space), but they correspond to directions in physical space. For example, in the at-rest system, u_1 represents spin up and so can be visualized as a spatial vector that points in the $+z$ direction. Similarly, in the at-rest system, u_2 represents spin down, so can be visualized as a vector pointing in the $-z$ direction.

Summary

1. $u_1(\mathrm{p})e^{-ipx}$ and $u_2(\mathrm{p})e^{-ipx}$ is each always an eigenstate of the Dirac equation (for any **p**)
2. $u_1(\mathrm{p})e^{-ipx}$ and $u_2(\mathrm{p})e^{-ipx}$ is each sometimes an eigenstate of z spin, i.e. of Σ_3 (for $\mathbf{p} = 0$ or $= p^3\mathbf{i}_3$)
3. $u_1(\mathbf{p})e^{-ipx}$ and $u_2(\mathbf{p})e^{-ipx}$ are always basis vectors for any general state $|\psi\rangle$ (for any **p**)
4. $u_1(p)$ and $u_2(p)$ is each sometimes an eigenstate of z spin, i.e. of Σ_3 for $\mathbf{p} = 0$ or$= p^3\mathbf{i}_3$)
5. $u_1(\mathbf{p})$ and $u_2(\mathbf{p})$ are always basis vectors in 4D spinor space (for any **p**)
6. $u_1(\mathrm{p})$ **and** $u_2(\mathrm{p})$ **change orientation, as visualized in physical space, as** p **changes.**
7. Spin S (often in relativity as Σ) changes direction with p, but differently than u_1 and u_2

Any general spin state u can be represented as a linear combination of u_1 and u_2 (for any **p**)

$$u(\mathrm{p}) = C_1(\mathrm{p})u_1(\mathrm{p}) + C_2(\mathrm{p})u_2(\mathrm{p})$$

Any general particle state includes a spin part plus a spacetime part(for any given **p**):

$$|\psi_{\mathrm{p}}\rangle = \sqrt{\frac{m}{2VE_{\mathrm{p}}}}u(\mathbf{p})e^{-ipx} = \sqrt{\frac{m}{2VE_{\mathrm{p}}}}\left(C_1(\mathrm{p})u_1(\mathrm{p})e^{-ipx} + C_2(\mathrm{p})u_2(\mathrm{p})e^{-ipx}\right)$$

3.1.6 RQM Helicity operator

For massless particles ($v = c$), the velocity vector must perfectly align with the spin vector. This alignment is called **perfect helicity**. In general, if the spin axis (using the right-hand rule), of a particle is in the direction of v one says the particle has **positive helicity**. If spin points in the direction of $-v$, the particle has **negative helicitv**.

The degree of helicity a particle has be define in terms of the angle between the spin vector and the velocity vector. It is maximum if that angle is zero. **The dot product of the spin vector with a unit vector in the p (or equivalently, the v) direction has come to be the mathematical definition of helicity**.

Our spin operator Σ in RQM plays the role of a 3-vector in physical space that points in the direction of spin. The inner product in physical space of the spin operator Σ and the unit vector in the **p** direction would then be **helicity operator**:

$$\Sigma_p = \Sigma \cdot i_p = \Sigma \cdot \frac{p}{|p|} = \Sigma_1\frac{p^1}{|p|} + \Sigma_2\frac{p^2}{|p|} + \Sigma_3\frac{p^3}{|p|} \tag{3.1.23}$$

(3.1.23) is a 4×4 matrix in spinor space because each Σ_i is a matrix. (3.1.23) is a scalar in physical space because it is the inner product of two vectors.

· **Example**

Consider a case where a particle is in the first eigenstate of (3.1.8), and if $p^3 \neq 0$, we have

$$\Sigma \cdot \frac{\mathbf{p}}{|\mathbf{p}|}\left|\psi^{(1)}\right\rangle = \Sigma_3 \underbrace{\frac{p^3}{|\mathbf{p}|}}_{=1}\left|\psi^{(1)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{pmatrix}e^{-ipx} = \frac{1}{2}\left|\psi^{(1)}\right\rangle$$

· **Example**

Note that if p^3 were negative (-z direction)

$$\Sigma_3 \left|\psi^{(1)}\right\rangle = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ 0 \end{pmatrix} e^{-ipx} = \frac{1}{2}\left|\psi^{(1)}\right\rangle$$

but

$$\Sigma \cdot \frac{\mathbf{p}}{|\mathbf{p}|}\left|\psi^{(1)}\right\rangle = \Sigma_3 \underbrace{\frac{p^3}{|\mathbf{p}|}}_{=-1}\left|\psi^{(1)}\right\rangle = -\frac{1}{2}\left|\psi^{(1)}\right\rangle$$

In general, a $+1/2$ helicity state for spinors means the spin is in the direction of p; a $-1/2$ helicity eigenvalue means spin is in the direction of $-p$.

3.2 The Dirac Equation in QFT

The Dirac equation for fields (where we will, as with scalar fields, work in the Heisenberg picture), is

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{3.2.1}$$

Its eigensolutions

$$\psi^{(1)} = u_1 e^{-ipx} \quad \psi^{(2)} = u_2 e^{-ipx} \quad \psi^{(3)} = v_2 e^{ipx} \quad \psi^{(4)} = v_1 e^{ipx} \tag{3.2.2}$$

The adjoint Dirac equation for fields is

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{3.2.3}$$

with adjoint eigensolutions

$$\bar{\psi} = \psi^\dagger \gamma^0 \rightarrow \bar{\psi}^{(1)} = u_1^\dagger \gamma_e^0 e^{ipx} = \bar{u}_1 e^{ipx} \quad \bar{\psi}^{(2)} = \bar{u}_2 e^{i\phi x} \quad \bar{\psi}^{(3)} = \bar{v}_2 e^{-i\varphi x} \quad \bar{\psi}^{(4)} = \bar{v}_1 e^{-\dot{\varphi} x} \tag{3.2.4}$$

the general discrete plane wave solutions are

$$\begin{aligned}\psi &= \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathrm{p}}}} \left(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right) \\ &= \quad \psi^+ \quad + \quad \psi^-\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}\bar{\psi} &= \sum_{r,p} \sqrt{\frac{m}{VE_p}} \left(d_r(p)\bar{v}_r(p)e^{-ipx} + c_r^\dagger(p)\bar{u}_r(p)e^{ipx}\right) \\ &= \quad \bar{\psi}^+ \quad + \quad \bar{\psi}^-\end{aligned} \tag{3.2.6}$$

The general continuous plane wave solutions are

$$\begin{aligned}\psi &= \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_p}} \left(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right) \\ \bar{\psi} &= \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_p}} \left(d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx}\right)\end{aligned} \tag{3.2.7}$$

the Lagrangian (density) for free spinor fields to be

$$\mathcal{L}_0^{1/2} = \bar{\psi}\left(i\gamma^\alpha\partial_\alpha - m\right)\psi \tag{3.2.8}$$

Conjugate momenta for ψ and $\bar{\psi}$ are

$$\pi^{1/2} = \frac{\partial\mathcal{L}_0^{1/2}}{\partial\psi_0} = i\psi\gamma^0 = i\psi^\dagger\gamma^\dagger\gamma^0 = i\psi^\dagger \tag{3.2.9}$$

$$\bar{\pi}^{1/2} = \frac{\partial\mathcal{L}_0^{1/2}}{\partial\bar{\psi}_{,0}} = 0 \tag{3.2.10}$$

The Dirac Hamiltonian density can be found from the Legendre transformation as

$$\begin{aligned}\mathcal{H}_0^{1/2} &= \pi^{1/2}\dot{\psi} + \bar{\pi}^{1/2}\dot{\bar{\psi}} - \mathcal{L}_0^{1/2} = i\psi^\dagger\dot{\psi} - \mathcal{L}_0^{1/2} = i\underbrace{\psi^\dagger\gamma^0}_{\bar{\psi}}\gamma^0\dot{\psi} - \mathcal{L}_0^{1/2} \\ &= i\bar{\psi}\gamma^0\dot{\psi}\underbrace{-i\bar{\psi}\gamma^0\dot{\psi} - i\bar{\psi}\gamma^i\partial_i\psi}_{-i\bar{\psi}\gamma^\alpha\partial_\alpha\psi} + m\bar{\psi}\psi = -i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi\end{aligned} \tag{3.2.11}$$

3.3 Anti-commutation Relations for Dirac Fields

$$\left[c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')\right]_+ = \left[d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')\right]_+ = \delta_{rS}\delta_{\mathbf{pp}'}(\text{ discrete });= \delta_{rs}\delta(\mathbf{p}-\mathbf{p}')(\text{ continuous }) \tag{3.3.1}$$

All other anti-commutators between coefficients equal zero.

3.4 The Dirac Hamiltonian in QFT

Similar to what we did for scalar fields, we find the Dirac Hamiltonian by integrating the Dirac Hamiltonian density over all space (a volume V containing the discrete solutions, which we can make as large as we like), i.e.,

$$H_0^{1/2} = \int \mathcal{H}_0^{1/2} d^3x = \int \left(-i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi\right) d^3x \tag{3.4.1}$$

we substitute the Dirac general solution into (3.4.1) to give

$$H_0^{1/2} = \int \left(-i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi\right) d^3x =$$

$$\int \left(\sum_{r,p}\sqrt{\frac{m}{VE_p}}\left(d_r(p)\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx}\right)\right) \times$$

$$\left(-i\gamma^i\partial_i\right)\left(\sum_{s,p'}\sqrt{\frac{m}{VE_{p'}}}\left(c_s(\mathbf{p}')\,u_s(\mathbf{p}')\,e^{-ip'x} + d_s^\dagger(\mathbf{p}')\,v_s(\mathbf{p}')\,e^{ip'x}\right)\right) d^3x$$

$$+\int m\left(\sum_{r,p}\sqrt{\frac{m}{VE_p}}\left(d_r(p)\bar{v}_r(p)e^{-ipx} + c_r^\dagger(p)\bar{u}_r(p)e^{ipx}\right)\right) \times$$

$$\left(\sum_{s,\mathbf{p}'}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\left(c_s(\mathbf{p}')\,u_s(\mathbf{p}')\,e^{-ip'x} + d_s^\dagger(\mathbf{p}')\,v_s(\mathbf{p}')\,e^{i\mathbf{p}'\cdot x}\right)\right) d^3x$$

The first of the two integrals above becomes

$$\int \left(\sum_{r,\mathbf{p}}\sqrt{\frac{m}{VE_p}}d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-i\left(E_pt - p^ix^i\right)}\right)\left(\sum_{s,p'}\sqrt{\frac{m}{VE_{p'}}}c_s(\mathbf{p}')\,\gamma^i\underbrace{p'^i}_{\text{from }\partial_i}u_s(\mathbf{p}')\,e^{-i\left(E_{p'}-p'^ix^i\right)}\right) d^3x$$

$$+\int\left(\sum_{r,\mathrm{p}}\sqrt{\frac{m}{VE_{\mathrm{p}}}}d_r(\mathrm{p})\bar{v}_r(\mathbf{p})e^{-i\left(E_{\mathrm{p}}t-p^ix^i\right)}\right)\left(\sum_{s,p'}\sqrt{\frac{m}{VE_{p'}}}d_s^\dagger\left(p'\right)\gamma^i\left(-p'^i\right)v_s\left(p'\right)e^{i\left(E_{p'}t-p'^ix^i\right)}\right)d^3x$$

$$+\int\left(\sum_{r,\mathrm{p}}\sqrt{\frac{m}{VE_{\mathrm{p}}}}c_r^\dagger(\mathrm{p})\bar{u}_r(\mathbf{p})e^{-i\left(E_{\mathrm{p}}t-p^ix^i\right)}\right)\left(\sum_{s,p'}\sqrt{\frac{m}{VE_{p'}}}c_s\left(p'\right)\gamma^i\left(p'^i\right)u_s\left(p'\right)e^{i\left(E_{p'}t-p'^ix^i\right)}\right)d^3x$$

$$+\int\left(\sum_{r,\mathrm{p}}\sqrt{\frac{m}{VE_{\mathrm{p}}}}c_r^\dagger(\mathrm{p})\bar{u}_r(\mathbf{p})e^{-i\left(E_{\mathrm{p}}t-p^ix^i\right)}\right)\left(\sum_{s,p'}\sqrt{\frac{m}{VE_{p'}}}d_s^\dagger\left(p'\right)\gamma^i\left(-p'^i\right)v_s\left(p'\right)e^{i\left(E_{p'}t-p'^ix^i\right)}\right)d^3x$$

because an integral over all space of the oscillating function $e^{if(\mathbf{x})}$, where $f(\mathbf{x}\neq 0)$ is zero. So,in the first and last lines, only terms with $\mathbf{p}=-\mathbf{p}'$ will survive. And in the 2nd and 3rd lines, only terms in $\mathbf{p}'=\mathbf{p}$ will. We assume, as in RQM, that **the order of spinors and coefficients (such as c_r and d_r) can be interchanged at will, but we must preserve the order of spinor entities as it represent matrix/vector multiplication in spinor space.**

$$+\int\left[\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}d_r(-\mathbf{p})\bar{v}_r(-\mathbf{p})\gamma^ip^iu_s(\mathbf{p})c_s(\mathbf{p})e^{-i2E_pt}\right]d^j$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathrm{p}}}d_r(\mathbf{p})\bar{v}_r(\mathbf{p})\gamma^i\left(-p^i\right)v_s(\mathbf{p})d_s^\dagger(\mathbf{p})\right)d^3x$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_p}c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})\gamma^ip^iu_s(\mathbf{p})c_s(\mathbf{p})\right)d^3x$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_p}c_r^\dagger(-\mathbf{p})\bar{u}_r(-\mathbf{p})\gamma^i\left(-p^i\right)v_s(\mathbf{p})d_s^\dagger(\mathbf{p})e^{i2E_pt}\right)d^3x$$

In similar fashion. the last two lines of the expansion of (3.4.1), representing the mass term in $H_0^{1/2}$, become

$$\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_p}d_r(-\mathbf{p})\bar{v}_r(-\mathbf{p})mu_s(\mathbf{p})c_s(\mathbf{p})e^{-i2E_pt}\right)d^3x$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}d_r(\mathbf{p})\bar{v}_r(\mathbf{p})mv_s(\mathbf{p})d_s^\dagger(\mathbf{p})\right)d^3x$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathrm{p}}}c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})mu_s(\mathbf{p})c_s(\mathbf{p})\right)d^3x$$

$$+\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}c_r^\dagger(-\mathbf{p})\bar{u}_r(-\mathbf{p})mv_s(\mathbf{p})d_s^\dagger(\mathbf{p})e^{i2E_{\mathbf{p}}t}\right)d^3x$$

Box

Relationship for $u_s(\mathbf{p})$

Consider the Dirac equation and a single eigensolution to it having 3 -momentum p and spin s

$$(i\gamma^\mu\partial_\mu - m)\,\psi = (i\ \not{\partial} - m)\psi = 0 \quad \text{with} \quad \psi = c_{\underline{s}}(\mathbf{p})u_{\underline{s}}(\mathbf{p})e^{-ipx}$$

$$(\gamma^\mu p_\mu - m)\,c_{\underline{s}}(\mathbf{p})u_{\underline{s}}(\mathbf{p})e^{-ipx} = (\not{p} - m)c_{\underline{s}}(\mathbf{p})u_{\underline{s}}(\mathbf{p})e^{-ipx} = 0$$

Neither $c_s(p)$ nor the exponential equal zero, so the remaining factors must equal zero, thus

$$(\gamma^\mu p_\mu - m)\,u_s(\mathbf{p}) = (\not{p} - m)u_s(\mathbf{p}) = 0$$

Now from the complex conjugate transpose of (3.1.10), where r and s are dummy variables and thus interchangeable and the relation holds for any $\mathbf{p}$, including $-\mathbf{p}$,

$$u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) = 0 \rightarrow v_s^\dagger(-\mathbf{p})u_r(\mathbf{p}) = 0 \rightarrow v_r^\dagger(-\mathbf{p})u_s(\mathbf{p}) = 0$$

$$v_r^\dagger(-\mathbf{p})\gamma^0\gamma^0u_s(\mathbf{p}) = 0 \rightarrow \bar{v}_r(-\mathbf{p})\gamma^0u_s(\mathbf{p}) = 0 \rightarrow \bar{v}_r(-\mathbf{p})\gamma^0p_0u_s(\mathbf{p}) = 0$$

And thus,

$$\bar{v}_r(-\mathrm{p})\gamma^ip^iu_s(\mathrm{p}) = \bar{v}_r(-\mathrm{p})\gamma^i\,(-p_i)\,u_s(\mathrm{p}) - \bar{v}_r(-\mathrm{p})\gamma^0p_0u_s(\mathrm{p}) = -\bar{v}_r(-\mathrm{p})\underbrace{\gamma^\mu p_\mu}_{\not{p}}u_s(\mathrm{p})$$

When we then use the RHS instead of the LHS of the equation above, we get

$$-\int\left(\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}d_r(-\mathbf{p})(\bar{v}_r(-\mathbf{p})\underbrace{(\not{p} - m)u_s(\mathbf{p})}_{=0}c_s(\mathbf{p})e^{-i2E_pt}\right)d^3x = 0$$

Recall that

$$v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p})\gamma^0v_s(\mathbf{p}) = \frac{E_p}{m}\delta_{rs} = \frac{p_0}{m}\delta_{rs}$$

we have

$$\bar{v}_r(\mathbf{p})\gamma^0p_0v_s(\mathbf{p}) = \frac{(p_0)^2}{m}\delta_{rs} = \frac{E_p^2}{m}\delta_{rs}$$

Adding the integrals above, and using the relationship in the box, we find

$$\begin{aligned}
&\int \left(\sum_{r,s,\mathbf{p}} \frac{m}{VE_p} d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) \left(-\gamma^i p^i + m \right) v_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \right) d^3x \\
&= \int \left(\sum_{r \le p} \frac{m}{VE_p} d_r(p) \left(\bar{v}_r(p) \left(\gamma^i p_i + m + \gamma^0 p_0 \right) v_s(p) - \frac{E_p^2}{m} \delta_{rs} \right) d_s^\dagger(p) \right) d^3x \qquad (3.4.2) \\
&= \frac{1}{V} \int_{=1}^{1} d^3x \left(\sum_{r,s,p} \frac{m}{E_p} d_r(p) \left(\gamma^\mu p_\mu + m \right) v_s(p) - \frac{E_p^2}{m} \delta_{rs} \right) d_s^\dagger(p)
\end{aligned}$$

Note that $(\not{p} + m) v_s(p) = 0$, the equation above reduces to

$$\sum_{r,\mathrm{p}} \frac{m}{E_\mathrm{p}} d_r(\mathbf{p}) \left(-\frac{E_\mathrm{p}^2}{m} \right) d_r^\dagger(\mathbf{p}) = -\sum_{r,\mathbf{p}} E_\mathbf{p} \underbrace{d_r(\mathbf{p}) d_r^\dagger(\mathbf{p})}_{\text{use anti-commutator}} \qquad (3.4.3)$$

Finally,

$$H_0^{1/2} = \sum_{r,p} E_p \left(N_r(p) - \frac{1}{2} + \bar{N}_r(p) - \frac{1}{2} \right) \qquad (3.4.4)$$

$$N_r(\mathbf{p}) = c_{\underline{r}}^\dagger(\mathbf{p}) c_{\underline{r}}(\mathbf{p}) \quad \bar{N}_r(\mathbf{p}) = d_{\underline{r}}^\dagger(\mathbf{p}) d_{\underline{r}}(\mathbf{p}) \quad \text{(underbars mean no summation)} \qquad (3.4.5)$$

where

$N_r(\mathbf{p})$ = number operator with eigenvalue $n_r(\mathbf{p})$ = number of c particles of 3 -mom $\mathbf{p}$, spin r in the ket,

$\bar{N}_r(\mathbf{p})$ = number operator with eigenvalue $\bar{n}_r(\mathbf{p})$ = number of d particles with $\mathbf{p}$ and spin r in the ket,

and, the vacuum has $-1/2$ quantum of energy for each p, r for c particles, and also for d particles

Note that

$$H_0^{1/2}|0\rangle = \sum_{r,p} E_p \left(N_r(p) - \frac{1}{2} + \bar{N}_r(p) - \frac{1}{2} \right) |0\rangle = \sum_{r,p} E_p \left(-\frac{1}{2} - \frac{1}{2} \right) |0\rangle \qquad (3.4.6)$$

This infinite negative energy indicates that there is still something missing from the extant theory.

3.5 Creation and Destruction Operators

It will probably not come as a big surprise that the $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ operators destroy Dirac particles, and their complex conjugates create Dirac particles. We prove this below.

From the anti-commutation relations,

$$\left[c_r^\dagger(\mathbf{p}), c_r^\dagger(\mathbf{p})\right]_+ = \left[c_r(\mathbf{p}), c_r(\mathbf{p})\right]_+ = 0 \tag{3.5.1}$$

Thus,

$$c_r^\dagger(\mathbf{p})c_r^\dagger(\mathbf{p}) + c_r^\dagger(\mathbf{p})c_r^\dagger(\mathbf{p}) = 0 \rightarrow \left(c_r^\dagger(\mathbf{p})\right)^2 = 0 \tag{3.5.2}$$

Similarly

$$\left(c_r(\mathbf{p})\right)^2 = 0 \quad \left(d_r^\dagger(\mathbf{p})\right)^2 = 0 \quad \left(d_r(p)\right)^2 = 0$$

Proof that $c_r(\mathbf{p})$ is a Destruction operator

$$c_r(\mathbf{p})\left|\psi_{r.p}\right\rangle = |?\rangle$$

Use the number operator, we have

$$N_r(\mathbf{p})|?\rangle = n_?|?\rangle = n_? c_r(\mathbf{p})\left|\psi_{r,\mathbf{p}}\right\rangle = \left(1 - c_r(\mathbf{p})c_r^\dagger(\mathbf{p})\right) c_r(\mathbf{p})\left|\psi_{r,\mathbf{p}}\right\rangle$$

$$= c_I(\mathbf{p})\left|\psi_{r,\mathbf{p}}\right\rangle - c_r(\mathbf{p})\underbrace{n_r(\mathbf{p})}_{=1}\left|\psi_{r,\mathbf{p}}\right\rangle = (1-1)\underbrace{c_r(\mathbf{p})\left|\psi_{r,\mathbf{p}}\right\rangle}_{|?\rangle}$$

When $c_r^\dagger(\mathbf{p})$ acts on a single particle state, we find

$$c_r^\dagger(\mathbf{p})\underbrace{\left|\psi_{r,\mathbf{p}}\right\rangle}_{c_r^\dagger(\mathbf{p})|0\rangle} = \left(c_r^\dagger(\mathbf{p})\right)^2|0\rangle = 0 \tag{3.5.3}$$

So, the theory we've developed tells us that we cannot create (we cannot have) multiparticle with more than one Dirac particle in a given single particle state.

General rule

Coefficient commutation relations work for bosons and allow more than one identical single particle state to co -exist in the same multiparticle state.

Coefficient anti-commutation relations work for fermions and do not allow more than one identical single particle state to co-exist in the same multiparticle state.

3.5.1 Total Particle number

As with scalars, total particle rtl.llilber is defined as the number of particles (i.e. c types) minus the number of antiparticles (d types). For spinors, the total particle number operator is

$$N(\psi) = \sum_{r,p} \left(N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}) \right) \tag{3.5.4}$$

Again, note the subtle difference in phraseology. "Number of particles" (which is different from "total particle number") equals the number of particles plus the number of antiparticles.

3.6 QFT Spinor Charge Operator and Four Current

From what we know about the number operators, and parallel to what we found for scalar fields, we can simply define our Dirac charge operator as

$$Q = -e \sum_{r,p} \left(N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}) \right) \tag{3.6.1}$$

Where - e is the charge on the electron. Note that, with this definition, d type particles will ave a charge of+ e, which would qualify them as antiparticles of the electron. Note the operation of (3.6.1)on a typical state

$$-e \sum_{r,\mathbf{p}} \left(N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}) \right) \left| \psi_{n,\mathbf{p}_1}, \psi_{n,\mathbf{p}_2}, \bar{\psi}_{n,\mathbf{p}_1} \right\rangle = \underbrace{-e(1+1-1)}_{\text{to charge } = -e} \left| \psi_{n,\mathbf{p}_1}, \psi_{n,\mathbf{p}_2}, \bar{\psi}_{n,\mathbf{p}_1} \right\rangle$$

A state With two electrons and one positron has a total charge Of -e.

3.6.1 The Dirac charge operator from the four current

$$\text{spinor 4-current operator } j^\mu = (\rho, \mathbf{j}) = \bar{\psi}\gamma^\mu\psi \quad \text{with} \quad \partial_\mu j^\mu = 0 \tag{3.6.2}$$

3.7 Dirac Three Momentum Operator

we can simply define our Dirac 3-momentum operator as

$$\mathbf{P} = \sum_{r,\mathbf{p}} \mathbf{p} \left(N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p}) \right) \tag{3.7.1}$$

3.8 Dirac Spin Operator in QFT

We can define the **QFT Dirac spin operator** as

$$_{\text{QFT}}\Sigma_i = \int_V \psi^\dagger \Sigma_i \psi d^3x \rightarrow \quad _{\text{QFT}}\Sigma_3 = \int_V \psi^\dagger \Sigma_3 \psi d^3x \tag{3.8.1}$$

For type c particles, we have

$$_{QFT}{}^c\Sigma_3 = \int_V \left(\sum_{r,p} \sqrt{\frac{m}{VE_\mathbf{p}}} c_r^\dagger(\mathbf{p}) u_r^\dagger(\mathbf{p}) e^{ipx} \right) \Sigma_3 \left(\sum_{s,\mathbf{p}'} \sqrt{\frac{m}{VE_{\mathbf{p}'}}} c_s\left(\mathbf{p}'\right) u_s\left(\mathbf{p}'\right) e^{-ip'x} \right) d^3x \tag{3.8.2}$$

As we should be getting used to by now, all terms where $\mathbf{p} \neq \mathbf{p}'$ will go to zero in the integration. giving us

$$_{QFT}{}^c\Sigma_3 = \left(\sum_{r,s,\text{p}} c_r^\dagger(\mathbf{p}) c_s(\mathbf{p}) u_r^\dagger(\mathbf{p}) \Sigma_3 u_s(\mathbf{p}) \right) \frac{1}{V} \int d^3x = \left(\sum_{r,s,\mathbf{p}} u_r^\dagger(\mathbf{p}) \Sigma_3 u_s(\mathbf{p}) c_r^\dagger(\mathbf{p}) c_s(\mathbf{p}) \right) \tag{3.8.3}$$

For a single particle state of spin s, the c operators will destroy that state, then create ones of spins r, i.e.,

$$\left(\sum_{r,s',p'} \frac{m}{E_{p'}} u_r^\dagger\left(\mathbf{p}'\right) \Sigma_3 u_{s'}\left(\mathbf{p}'\right) c_r^\dagger\left(\mathbf{p}'\right) c_{s'}\left(\mathbf{p}'\right) \right) |\psi_{s,\mathbf{p}}\rangle = \sum_r \frac{m}{E_\mathbf{p}} \underbrace{\left(u_r^\dagger(\mathbf{p}) \Sigma_3 u_s(\mathbf{p}) \right)}_{\text{a number}} |\psi_{r,\mathbf{p}}\rangle \tag{3.8.4}$$

So, the expectation value of what we would measure for spin in the z direction for the given state with s spin would be

$$\langle \psi_{s,\text{p}} |_{QFT}{}^c\Sigma_3| \, \psi_{s,\mathbf{p}} \rangle = \sum_r \frac{m}{E_p} \langle \psi_{s,p}| \left(\text{ a number } \right) |\psi_{r,\mathbf{p}}\rangle$$

$$= 0 \text{ for } r \neq s; \quad = \frac{m}{E_\text{p}} u_r^\dagger(\mathbf{p}) \Sigma_3 u_r(\mathbf{p}) \text{ for } r = s$$

All of the above steps can be repeated analogously for Σ_1 and Σ_2 to yield the general result

$$ {}_{QFT}{}^{c}\Sigma_i = \sum_{r,\mathrm{p}} \frac{m}{E_{\mathrm{p}}} u_r^\dagger(\mathbf{p})\Sigma_i u_r(\mathbf{p}) N_r(\mathbf{p}) \tag{3.8.5} $$

For both type c and d particles

$$ {}_{QFT}{}^{d}\Sigma_i = \left(\sum_{r,\mathrm{p}} \frac{m}{E_{\mathrm{p}}} v_r^\dagger(\mathbf{p})\Sigma_i v_r(\mathbf{p}) \bar{N}_r(\mathbf{p}) \right) \tag{3.8.6} $$

Thus, **QFT spin operator in terms of number operators is**

$$ {}_{QFT}\Sigma_i \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} \left(u_r^\dagger(\mathbf{p})\Sigma_i u_r(\mathbf{p}) N_r(\mathbf{p}) + v_r^\dagger(\mathbf{p})\Sigma_i v_r(\mathbf{p}) \bar{N}_r(\mathbf{p}) \right) \tag{3.8.7} $$

3.9 QFT Helicity Operator

$$ {}_{QFT}\Sigma_{\mathbf{p}} = \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} \left(u_r^\dagger(\mathbf{p})\Sigma_i \frac{p^i}{p} u_r(\mathbf{p}) N_r(\mathbf{p}) + v_r^\dagger(\mathbf{p})\Sigma_i \frac{p'}{p} v_r(\mathbf{p}) \bar{N}_r(\mathbf{p}) \right) \tag{3.9.1} $$

3.10 Anti-commutators are Outer Products

For products of two fields, when the adjoint field is on the left and spinor indices are suppressed, an inner product is implied. Thus, where, as always, repeated indices mean summation,

$$ \bar{\psi}\psi = \bar{\psi}_\beta \psi_\beta = \psi_\alpha^\dagger \gamma^0_{\alpha\beta} \psi_\beta = \text{ a scalar quantity} \tag{3.10.1} $$

When the adjoint field is on the right, an outer product (a tensor/matrix) is implied. For example,

$$ \psi\bar{\psi} = \psi_\alpha \bar{\psi}_\beta = \psi_\alpha \psi_\delta^\dagger \gamma^0_{\delta\beta} = X_{\alpha\beta} = \text{ a matrix quantity in spinor space} \tag{3.10.2} $$

For spinor field anti-commutators, which for us, are almost always outer products, we mean

$$ [\psi, \bar{\psi}]_+ = [\psi, \bar{\psi}]_{+\alpha\beta} = \psi_\alpha \bar{\psi}_\beta + \bar{\psi}_\beta \psi_\alpha = [\bar{\psi}, \psi]_+ = [\bar{\psi}, \psi]_{+\alpha\beta} \tag{3.10.3} $$

3.11 Subtleties Regarding Spin

3.11.1 First issue: general(Non-pure)states

In QFT, one almost invariably deals with kets such as for which, $|\psi_{r,\mathrm{p}}\rangle$ or $|\psi_{r,\mathbf{p}},\psi_{r',\mathbf{p}'}\psi_{r^*,\mathbf{p}''}\rangle$ **it is seemingly implied, the spin values** r, r'**,and** r'' **equal 1 or 2.** These are spin basis states, so it seems these kets cannot represent more general (non-basis) states. The theory still seems a little lacking, if, as it might seem, it only handles pure spin basis states.

3.11.2 Resolution of the first issue

Consider the non spin basis state

$$|\psi_{\mathrm{n.s.b,p}}\rangle = \left| C_1\sqrt{\frac{m}{VE_{\mathrm{p}}}}u_1(\mathrm{p})e^{-ipx} + C_2\sqrt{\frac{m}{VE_{\mathrm{p}}}}u_2(\mathrm{p})e^{-ipx}\right\rangle \tag{3.11.1}$$

where $|\mathrm{C}_1|^2$ is the probability of measuring the u_1 state, and $|\mathrm{C}_2|^2$ is that of the u_2 state. The expectation value of any operator for this state is

$$\overline{\mathcal{O}} = \langle\psi_{\mathrm{n.s.b,p}}|\mathcal{O}|\psi_{\mathrm{n.s.b,p}}\rangle =$$

$$\left\langle C_1\sqrt{\frac{m}{VE_p}}u_1(\mathbf{p})e^{-ipx} + C_2\sqrt{\frac{m}{VE_p}}u_2(\mathbf{p})e^{-ipx}\right| \mathcal{O} \left|C_1\sqrt{\frac{m}{VE_p}}u_1(p)e^{-ipx} + C_2\sqrt{\frac{m}{VE_p}}u_2(p)e^{-ipx}\right\rangle$$

Or, for each spin basis state being an eigenstate of $\mathcal{O}$, with $o_{r=1,p}$ and $o_{r=2,p}$ representing operator eigenvalues for each spin basis state,

$$\begin{aligned}\overline{\mathcal{O}} = &\left\langle C_1\sqrt{\frac{m}{VE_p}}u_1(\mathbf{p})e^{-ipx}|\ \underbrace{o_{r=1,\mathbf{p}}}_{\mathcal{O}eigenval}\ |C_1\sqrt{\frac{m}{VE_p}}u_1(\mathbf{p})e^{-ipx}\right\rangle \\ &+\left\langle C_1\sqrt{\frac{m}{VE_p}}u_1(\mathbf{p})e^{-ipx}\,|o_{r=2,\mathrm{p}}|\,C_2\sqrt{\frac{m}{VE_p}}u_2(\mathbf{p})e^{-i\varphi x}\right\rangle \\ &+\left\langle C_2\sqrt{\frac{m}{VE_p}}u_2(\mathbf{p})e^{-ipx}\,|o_{r=1,p}|\,C_1\sqrt{\frac{m}{VE_p}}u_1(p)e^{-ipx}\right\rangle \\ &+\left(C_2\sqrt{\frac{m}{VE_p}}u_2(\mathbf{p})e^{-ipx}\,|o_{r=2,\mathrm{p}}|\,C_2\sqrt{\frac{m}{VE_p}}u_2(\mathbf{p})e^{-ipx}\right\rangle\end{aligned}$$

The first term in the expression above is

$$o_{r=1,\mathrm{p}}\int_V C_1^\dagger\sqrt{\frac{m}{VE_{\mathrm{p}}}}u_1^\dagger(\mathrm{p})e^{ipx}C_1\sqrt{\frac{m}{VE_{\mathrm{p}}}}u_1(\mathrm{p})e^{-ipx}d^3x$$

$$= \mathcal{O}_{r=1,\mathrm{p}} C_1^\dagger C_1 \underbrace{u_1^\dagger(\mathrm{p}) u_1(\mathrm{p})}_{E_\mathrm{p}/m} \frac{m}{V E_\mathrm{p}} \int_V d^3x = \mathcal{O}_{r=1,\mathrm{p}} \left|C_1\right|^2$$

The second term is zero as $u_1^\dagger(\mathrm{p}) u_2(\mathrm{p}) = 0$. Similarly, we have the third term =0, and the fourth term= $|C_2|^2$. Then,

$$\overline{\mathcal{O}} = o_{r=1,\mathrm{p}} \left|C_1\right|^2 + o_{r=2,\mathrm{p}} \left|C_2\right|^2$$

where $|\mathrm{C}_1|^2 + |\mathrm{C}_2|^2 = 1$. If $\mathcal{O} = H$, the Hamiltonian, then our energy expectation value would be

$$\bar{E} = E_\mathrm{p} \left|C_1\right|^2 + E_\mathrm{p} \left|C_2\right|^2 = E_\mathrm{p} \left(|C_1|^2 + |C_2|^2\right) = E_\mathrm{p}$$

as it must be since the two different spin basis states have the same energy. Similar effects would be seen for other operators like $\mathbf{p}$ or Σ_i.

3.11.3 Second issue: multiparticle states

1. The usual standard representation forms of spinors u_1 and u_2 only represent actual spin if the particle at-rest coordinate system has spin parallel to the z axis direction. That is, p^i used in the usual forms of u_1 and u_2 must be measured m the at-rest system.

2. If we have two (or more) Particles. in a multiparticle state, each typically has a different at-rest coordinate system. So, we can :either i) represent each particle with usual forms for u_1 and u_2 using p^i values for each measured in its own at-rest system, and thereby make computation very difficult, or ii) **represent all particles by the usual forms of u_1 and u_2 using p^i values for all measured in the same coordinate system**, but then, at best, only one particle would have its actual spin represented correctly.

3.11.4 Solution of the second issue

Suppose you had a particle in the particle at-rest coordinate system. Now you wished to express the state in a rotated, primed coordinate system where the x'^3 axis is aligned with spin direction. But how would we express it in the primed system?

In the primed system, the spin is up, and so we figure our state must be an eigenstate of Σ_3 in that system. We might then consider that we need to find C_1 and C_2 in the state ket such that this is true for 3-momentum aligned as p'^3 and $p'^1 \neq 0$,$p'^2 = 0$. Thus,

$$\left|\psi'_{\uparrow_{\text{spin,p'}}}\right\rangle = \left|C_1 u_1\left(\mathbf{p}'\right) e^{-ipx'} + C_2 u_2\left(\mathbf{p}'\right) e^{-ip'x'}\right\rangle =$$

$$\left| C_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p'^3}{E+m} \\ \frac{p'^1}{E+m} \end{pmatrix} e^{-ip'x'} + C_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p'^1}{E+m} \\ \frac{-p'^3}{E+m} \end{pmatrix} e^{-ip'x'} \right\rangle$$

Then operate on the state above with Σ_3,

$$\Sigma_3 \left|\psi'_{\uparrow spin,\mathbf{p}'}\right\rangle = \frac{1}{2} \left| \begin{pmatrix} C_1 \\ -C_2 \\ \frac{C_1 p'^3 + C_2 p'^1}{E+m} \\ \frac{-C_1 p'^1 + C_2 p'^3}{E+m} \end{pmatrix} e^{-ip'\chi} \right\rangle \stackrel{?}{=} \frac{1}{2} \left|\psi'_{\uparrow spin,\mathbf{p}'}\right\rangle$$

If the equal sign with the question mark over is truly an equal sign, then the state is an eigenstate of spin up. But for that to be true, $C_2 = 0$, and $p'^1 = 0$. But we already demand that $p'^1 \neq 0$.

Conclusion: The traditional forms for $u_1, u_2,$ and Σ_3 (more generally, Σ_i) can only be used in a coordinate system for which the particle at-rest coordinate system spin is aligned with the z direction (positive or negative direction.)

Correcting a mistake in the derivation above: In the above example, we should have transformed u_1, u_2, Σ_3 into their equivalent forms in the primed system, i.e., where T is the transformation in spinor space for a coordinate rotation transformation for Cartesian coordinates in physical space such as that

$$u'_1 = Tu_1 \quad u'_2 = Tu_2 \quad \Sigma'_i = T\Sigma_i T^{-1} \qquad \left|\psi'_{\uparrow\text{spin,p'}}\right\rangle = T\left|\psi_{\nearrow\text{spin,p}}\right\rangle$$

where T is a 4X4 matrix in spinor space. Now we find

$$\Sigma'_3 \left|\psi'_{\uparrow_{\text{spin,p'}}}\right\rangle = \frac{1}{2}\left|\psi'_{\uparrow_{\text{spin}},\text{p'}}\right\rangle =$$

$$\underbrace{\Sigma_3'}_{T\Sigma_3T^{-1}}\left|C_1\underbrace{u_1'\left(\mathbf{p}'\right)}_{Tu_1}e^{-ip'x'}+C_2\underbrace{u_2'\left(\mathbf{p}'\right)}_{Tu_2}e^{-ip'x'}\right\rangle=\frac{1}{2}\left|C_1u_1'\left(\mathbf{p}'\right)e^{-ip'x'}+C_2u_2'\left(\mathbf{p}'\right)e^{-ip'x'}\right\rangle$$

We did all of the above to emphatically demonstrate conclusion stressed above.

The issue then, in QFT, is that in every case with two or more particles in different spin state in the same multiparticle ket, we have to use a different coordinate system for each particle, if we want to employ the usual forms for u_1, u_2 and Σ_i. For example, consider two electrons in the same multiparticle state as them having different velocity and different at-rest system spin direction. We would represent the state as:

$$|\psi_A,\psi_B\rangle=|\left(C_{A1}u_1\left(\mathbf{p}_A\right)e^{-ip_Ax_A}+C_{A2}u_2\left(\mathbf{p}_A\right)e^{-ip_Ax_A}\right),\left(C_{B1}u_1\left(\mathbf{p}_B\right)e^{-ip_Bx_B}+C_{B2}u_2\left(\mathbf{p}_B\right)e^{-ip_Bx_B}\right)\rangle \tag{3.11.2}$$

where for given $\mathbf{p}_A$ we would determine C_{A1} and C_{A2} to yield the correct spin for electron A when it is moving as we would see in the lab, and carry out similar steps for the constants for electron B. u_1 and u_2 here have the standard form we are familiar with. Component p_A^i values would be for those of electron A in the at-rest system direction coordinate axes for electron A p_{B}^i would be for components of 3 -momentum of electron B in the coordinate at-rest system of electron B.

Now, we transform ψ_B and represent it in the A at-rest coordinate system:

$$\left|\psi_A,\left(C_{B1}\underbrace{T_{AB}u_1\left(\mathbf{p}_{B/A}\right)}_{u_{1B/A}\left(\mathbf{p}_{B/A}\right)}e^{-ip_{B/A}x_{B/A}}+C_{B2}\underbrace{T_{AB}u_2\left(\mathbf{p}_{B/A}\right)}_{u_{2B/A}\left(\mathbf{p}_{B/A}\right)}e^{-ip_{B/A}x_{B/A}}\right)\right\rangle \tag{3.11.3}$$

where $u_{rB/A}$ is the r th spinor column matrix for the B electron expressed in the A electron at-rest system.

As we should have learned well by now, the critical element in quantum theories is what we can measure when we do experiments, and that is reflected in the expectation value of the operator corresponding to the particular dynamical variable measured. We want to demonstrate that t**he expectation value of any dynamical variable, corresponding to operator $\mathcal{O}$, is the same no matter what system we express our particle state in.**

To keep things simple, consider measuring the expectation value of $\mathcal{O}$ for only the single B

particle of (3.11.3) in the at-rest coordinate system A, $\mathcal{O}_{B/A}$:

$$\overline{\mathcal{O}}_{B/A} = \langle \psi_{B/A} | \mathcal{O}_{B/A} | \psi_{B/A} \rangle = \langle \psi_{B/A} | T_{AB} \mathcal{O}_{B/B} T_{AB}^{-1} | \psi_{B/A} \rangle \tag{3.11.4}$$

In integral form, the (3.11.4) above is:

$$\int_V \begin{pmatrix} C_{B1}^\dagger u_1^\dagger \left(\mathbf{p}_{B/A}\right) T_{AB}^\dagger e^{ip_{B/A}x_{B/A}} \\ +C_{B1}^\dagger u_2^\dagger \left(\mathbf{p}_{B/A}\right) T_{AB}^\dagger e^{ip_{B/A}x_{B/A}} \end{pmatrix} T_{AB} \mathcal{O}_{B/B} T_{AB}^{-1} \begin{pmatrix} C_{B1} T_{AB} u_1 \left(\mathbf{p}_{B/A}\right) e^{-ip_{B/A}x_{B/A}} \\ +C_{B2} T_{AB} u_2 \left(\mathbf{p}_{B/A}\right) e^{-ip_{BA}x_{B/A}} \end{pmatrix} d^3x$$

Where $T_{AB}^\dagger = T_{AB}^{-1}$. Every T_{AB} in (3.11.4) is pre-multiplied by its inverse and drops out. We thus get

$$\overline{\mathcal{O}}_{B/A} = \underbrace{\left\langle \begin{matrix} C_{B1} u_1 \left(\mathbf{p}_{B/A}\right) e^{-ip_{B/A}x_{BiA}} \\ +C_{B2} u_2 \left(\mathbf{p}_{B/A}\right) e^{-ip_{B/A}x_{B/A}} \end{matrix} \right|}_{\text{usual form of spinors with } \mathbf{p}_B \text{components in A coord sys}} \mathcal{O}_{B/B} \left| \begin{matrix} C_{B1} u_1 \left(\mathbf{p}_{B/A}\right) e^{-ip_{B/A}x_{BiA}} \\ +C_{B2} u_2 \left(\mathbf{p}_{B/A}\right) e^{-ip_{B/A}x_{B/A}} \end{matrix} \right\rangle \tag{3.11.5}$$

So, the expectation value of any operator expressed in the A coordinate system, even if acting on a particle with an at-rest coordinate system different from A, can be found using (3.11.5). That is, **it can be found using kets expressed in terms of the standard relations for $u_{1,2}$ and components of 3 momentum expressed in the A system.**

3.12 The Spinor Feynman Propagator

Note, that as we showed with scalars while this "creation/destruction at a point" perspective helps in understanding the derivation of the propagator, the propagator really corresponds to a kind of probability density function in y and x. It represents the probability density (actually, the square its magnitude represents probability density, though it is a bit more complicated as other factors are eventually involved) of a Dirac particle appearing at y and disappearing at x. **It is a double density tn that it is a function of both y and x, two independent variables, rather than one.**

By following the steps similar to the ones in the previous chapter, we have

Spinor Feynman propagator S_F

$$iS_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle \tag{3.12.1}$$

Note that the RHS is an outer product in spinor space and thus the LHS is a matrix:

$$iS_{F\alpha\beta}(x-y) = \left\langle 0 \left| T\left\{\psi_\alpha(x)\bar{\psi}_\beta(y)\right\}\right| 0\right\rangle \tag{3.12.2}$$

When we express anti-commutato forms of iS_F as integrals, we have

$$iS^+_{\alpha\beta}(x-y) = \left[\psi^+_\alpha(x), \bar{\psi}^-_\beta(y)\right]_+ = \left[\psi^+(x), \bar{\psi}(y)\right]_{+\alpha\beta} \tag{3.12.3}$$

Using the continuous solution to the Dirac equation, we find

$$iS^+_{\alpha\beta}(x-y) =$$

$$\frac{m}{(2\pi)^3}\sum_r\sum_r\iint\left(c_r(\mathbf{p})c^\dagger_s(\mathbf{p}')\,u_{r\alpha}(\mathbf{p})\bar{u}_{s\beta}(\mathbf{p}') + c^\dagger_s(\mathbf{p}')\,c_r(\mathbf{p})\underbrace{\bar{u}_{s\beta}(\mathbf{p}')\,u_{r\alpha}(\mathbf{p})}_{=u_{r\alpha}(\mathbf{p})\bar{u}_{s\beta}(\mathbf{p}')}\right)\frac{e^{-ipx}e^{ip'y}}{\sqrt{E_\mathbf{p}E_{\mathbf{p}'}}}d^3\mathbf{p}d^3\mathbf{p}'$$

$$= \frac{1}{2(2\pi)^3}\int \underbrace{(p+m)}_{\alpha,\beta \text{ indices suppressed}}\frac{e^{-ip(x-y)}}{E_p}d^3\mathbf{p}$$

where $\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{\not{p}+m}{2m}$. In the second line, since the index labeling means we take α first (row index) and β second (column index), we take $\bar{u}_{s\beta}u_{r\alpha} = u_{r\alpha}\bar{u}_{s\beta} = (u_r\bar{u}_s)_{\alpha\beta}\cdot$.

By using contour integral, we have

$$S_F(x-y) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ip(x-y)}(\not{p}+m)}{p^2-m^2+i\varepsilon}d^4p \tag{3.12.4}$$

The 4-momentum space form of the propagator as:

$$S_F(p) = \frac{\not{p}+m}{p^2-m^2+i\varepsilon} = (\not{p}+m)\Delta_F(p) \tag{3.12.5}$$

3.13 Dirac Matrices and Spinor Relations

Hermiticity conditions:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{3.13.1}$$

Spinor relations from Dirac equation

$$\begin{aligned} (\not{p} - m)u_r(p) = 0 \quad (\not{p} + m)v_r(p) = 0 \\ \bar{u}_r(\mathrm{p})(\not{p} - m) = 0 \quad \bar{v}_r(\mathrm{p})(\not{p} + m) = 0 \end{aligned} \tag{3.13.2}$$

Most General Form of Orthonormality Conditions

$$\bar{u}_r(\mathrm{p})\gamma^\mu u_s(\mathrm{p}) = \bar{v}_r(\mathrm{p})\gamma^\mu v_s(\mathrm{p}) = \frac{p^\mu}{m}\delta_{rs} \tag{3.13.3}$$

Spinor outer product relations

$$u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{\not{p} + m}{2m} = \frac{\gamma^\mu p_\mu + m}{2m} = \frac{\gamma^\mu_{\alpha\beta} p_\mu + m I_{\alpha\beta}}{2m} = u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) \tag{3.13.4}$$

$$v_r(\mathbf{p})\bar{v}_r(\mathbf{p}) = \frac{\not{p} - m}{2m} = \frac{\gamma^\mu p_\mu - m}{2m} = \frac{\gamma^\mu_{\alpha\beta} p_\mu - m I_{\alpha\beta}}{2m} = v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p}) \tag{3.13.5}$$

Dirac matrices sort of like 4-vector components

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \tag{3.13.6}$$

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu \tag{3.13.7}$$

Contraction identities

$$\begin{aligned} &\gamma_\lambda\gamma^\lambda = 4, && \gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha \\ &\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta} && \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\lambda = -2\gamma^\gamma\gamma^\beta\gamma^\alpha \\ &\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\lambda = 2\left(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma + \gamma^\gamma\gamma^\beta\gamma^\alpha\gamma^\delta\right) \end{aligned} \tag{3.13.8}$$

$$\not{p}\ \not{p} = p^2 \tag{3.13.9}$$

For a four vector A_μ, where "A slash" is defined by $\not{A} = \gamma^\alpha A_\alpha$, the following contraction relations hold.

$$\begin{aligned} &\gamma_\lambda\ \not{A}\gamma^\lambda = -2\ \not{A} \\ &\gamma_\lambda\ \not{A}\ \not{B}\gamma^\lambda = -2\ \not{C}\ \not{B}\ \not{A} \quad \gamma_\lambda\ \not{A}\ \not{B}\ \not{C}\ \not{D}\gamma^\lambda = 2(\not{D}\ \not{A}\ \not{B}\ \not{C} + \not{C}\ \not{B}\ \not{A}\ \not{D}) \end{aligned} \tag{3.13.10}$$

Completeness relations

$$(u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) - v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p})) = \delta_{\alpha\beta} \tag{3.13.11}$$

Trace for a product of an odd number of γ-matrices

$$\mathrm{Tr}\left(y^{\alpha}\gamma^{\beta}\ldots,\gamma^{\mu}\gamma^{\nu}\right) = 0 \tag{3.13.12}$$

For products of an even number of γ -matrices,

$$\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\right) = 4g^{\alpha\beta}$$

$$\mathrm{Tr}\left[\gamma^{\alpha},\gamma^{\beta}\right] = \mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha}\right] = 0$$

$$\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}\right) = 4\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

3.14 Review of Spin in NRQM

In NRQM our spin operator S_i was the Pauli matrices σ_i times the factor $\hbar/2$,

$$S_1 = \frac{\hbar}{2}\sigma_1 = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S_2 = \frac{\hbar}{2}\sigma_2 = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} S_3 = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.14.1}$$

S_i in physical space acted like a vector with three components. In spinor space, however, was represented by a 2×2 matrix, one for each 3D component.

The spin up and spin down states are eigenstates of S_3, the z direction component of the spin operator S_i.Thus, a wave function (ket) with a column matrix representing spin up in the z direction $(1,0)^T$ had spin $\hbar/2$, the down direction $(0,1)^{\mathrm{T}}$ had spin $-\hbar/2$. Up and Down (z direction) eigenstates span the 2D spinor space and are Basis vectors of that space.

General Solution Includes All Possible Spin States:

$$|\psi\rangle = \sum_{\mathbf{p}}\left(C_{+}(\mathbf{p})e^{-ipx}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_{-}(\mathbf{p})e^{-ipx}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \tag{3.14.2}$$

Thus, **if we develop our theory for up and down states, it will be applicable to all possible spin states,too.**

Chapter 4

Vectors: Spin 1 Fields

4.1 Review of Classical Electromagnetism

4.1.1 Maxwell's Equations in 3D plus time formulation

In the formulation conceived by Oliver Heaviside, we have the sourceless Maxwell's equations as:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \vec{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{4.1.1}$$

Now, if we define a scalar potential $\Phi(\mathbf{x}, t)$ and a vector potential $\mathbf{A}(\mathbf{x}, t)$ so they solve

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{4.1.2}$$

Substitution gives

$$-\nabla^2\Phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0 \tag{4.1.3}$$

$$\underbrace{\nabla \times \nabla \times \mathbf{A}}_{\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}} = -\nabla\frac{\partial\Phi}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2} \Rightarrow \frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = -\nabla\frac{\partial\Phi}{\partial t} - \nabla(\nabla \cdot \mathbf{A}) \tag{4.1.4}$$

which Φ and $\mathbf{A}$ must solve. If we can solve for Φ and $\mathbf{A}$, then we can find the fields $\mathbf{E}$ and $\mathbf{B}$.

Φ and $\mathbf{A}$ are not unique. Note we can define other quantities by

$$\Phi' = \Phi + \frac{\partial f}{\partial t}, \quad \mathbf{A}' = \mathbf{A} - \nabla f \tag{4.1.5}$$

and the new quantities still solve the Maxwell's equation, regardless of the form of f.

The formal name for any theory formulated in terms of one or more potentials (two potentials, Φ and A here), where different potentials result in the same observable quantities (E and B here), is **gauge theory**. A gauge-invariant transformation changes the potential(s), also called gauge(s), from one form to another, but leaves the observables unchanged (invariant).

Picking a Useful Gauge

Let's pick f such that the following Coulomb gauge is true:

$$\nabla \cdot \mathbf{A} = 0 \tag{4.1.6}$$

When we do this,(4.1.3) and (4.1.4) become

$$\begin{aligned} \nabla^2 \Phi &= 0 \\ \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= -\nabla \frac{\partial \Phi}{\partial t} \end{aligned} \tag{4.1.7}$$

One solution to the equations above is $\Phi = 0$. Using that, we have

$$\partial_\mu \partial^\mu \mathbf{A} = \Box^2 \mathbf{A} = 0 \tag{4.1.8}$$

i.e., the wave equation. This has the simple plane wave solution

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0 e^{\pm i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{4.1.9}$$

Leading to

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \mp i\omega \mathbf{A}_0 e^{\pm i(\omega t - \mathbf{k}\cdot\mathbf{x})} = \mp\omega \mathbf{A} \tag{4.1.10}$$

From the eqn. above, we can see field E is parallel to A.

$$\mathbf{B} = \nabla \times \mathbf{A} = \mp i \left(\mathbf{k} \times \mathbf{A}_0\right) e^{\pm i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{4.1.11}$$

From the eqn. above, we can see field B is perpendicular to A.

Since we can always readily find **E** and **B** from **A** whenever we want, it is simplest to work with a single equation and the single field **A**, rather than multiple equations in **E** and **B**. Thus, it is common practice to represent, and refer to, electromagnetic fields as **A**.

If we pick our potential **A** such that it satisfies the Coulomb gauge (4.1.6), then solving Maxwell's equations becomes greatly simplified. That gauge lets us take $\Phi = 0$ and results in the single, well know, and easily solvable eave equation in **A**:

$$\partial_\mu \partial^\mu \mathbf{A} = 0 \tag{4.1.12}$$

4.1.2 Maxwell's equation in 4D(covariant) formualtion

The formulation in the previous section **is not relativisitcally covariant**. For that, let's define a 4D potential using Φ and **A** as:

$$A^\mu(x) = \begin{pmatrix} \Phi(x) \\ A^1(x) \\ A^2(x) \\ A^3(x) \end{pmatrix} \tag{4.1.13}$$

Then, let's define a field $F^{\mu\nu}(x)$ (which is a tensor field since it has two 4D indices μ and ν) that we can construct from (4.1.13) as

$$F^{\mu v}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x) \tag{4.1.14}$$

Consider (4.1.14), where $\mu = 1$ and $v = 2$ and we refer to (4.1.3,4.1.4), we have

$$F^{12}(x) = \underbrace{\partial^2}_{\partial_2} A^1(x) - \underbrace{\partial^1}_{\partial_1} A^2(x) = \partial_1 A^2(x) - \partial_2 A^1(x)$$

$$= \frac{\partial}{\partial x^1} A^2(x) - \frac{\partial}{\partial x^2} A^1(x) = \underbrace{(\nabla \times \mathbf{A}(x))^3}_{x^3 \text{ direction component}} = B^3(x)$$

For $\mu = 0$ and $v = 1$, we have instead

$$F^{01} = \partial^1 A^0 - \partial^0 A^1 = \frac{\partial \Phi}{\partial x_1} - \frac{\partial A^1}{\partial t} = \underbrace{\left(-(\nabla \Phi) - \frac{\partial \mathbf{A}}{\partial t} \right)^1}_{x^1 \text{ direction component}} = E^1$$

If can be shown that

$$F^{\mu\nu}(x) = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix} \tag{4.1.15}$$

where E^1 and B^1 represent what we designated by E_x and B_x in Cartesian coordinates before we worked with contravariant and covariant components, just as x^1 represents X_1 in Cartesian coordinates. Ditto for the other E^i and B^i.

With the aid of (4.1.3,4.1.4), we can show that **Maxwell's equations for $A^\mu(x)$ are**

$$\partial^\alpha \partial_\alpha A^\mu(x) - \partial^\mu \left(\partial_\nu A^\nu(x)\right) = 0 \tag{4.1.16}$$

Again, if we have a solution to Maxwell's equations $A^\mu(x)$, then we can transform that solution to another solution $A''(x)$, using the same function $f(x)$, i.e.,

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu f$$

Picking a useful 4D gauge

If we have a gauge like the following, called the **Lorenz gauge**, then (4.1.16) would be greatly simplified

$$\partial_v A^v(x) = 0 \tag{4.1.17}$$

Let's assume we have a valid solution $A''^\mu(x)$ for Maxwell's equation. Then

$$A^\mu(x) = A'^\mu(x) + \partial^\mu f(x) \tag{4.1.18}$$

is also a solution to Maxwell's equations. But to make those solutions easier to solve we also want A^μ also solve (4.1.17). *Can we choose $A^\mu(x)$ so this is so?*

Plugging $A^\mu(x)$ of (4.1.18) into (4.1.17) yields

$$\partial_\mu A^\mu(x) + \partial_\mu \partial^\mu f(x) = 0 \quad \rightarrow \partial_\mu A^\mu(x) = -\partial_\mu \partial^\mu f(x) \tag{4.1.19}$$

So, knowing $A''^{\mu}(x)$ we can, in principle, solve (4.1.19) for $f(x)$, and for that particular $f(x), A^{\mu}(x)$ will, in addition to solving Maxwell's equations, also solve (4.1.17). By doing the latter it will make our Maxwell equations in terms of the four-potential easier to solve.

We never need to actually solve for $f(x)$. We just need to know that we could solve for it, and so doing would give us a four potential $A^{\mu}(x)$ that solves the Lorenz gauge. We also never need to know what $A''^{\mu}(x)$ is. We just know that such a solution must exist. Knowing that $f(x)$ and $A''(x)$ exist if we wanted to find them 1s all that is necessary.

So, with (4.1.17), Maxwell's equation become

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}(x) = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^i \partial x_i}\right) A^{\mu}(x) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i \partial x^i}\right) A^{\mu}(x) = 0 \tag{4.1.20}$$

The bottom line is that the Maxwell's equations in Lorenz gauge is

$$\partial_{\alpha}\partial^{\alpha}A^{\mu}(x) = 0 \tag{4.1.21}$$

Solutions to the 4D wave equation

The wave equation (4.1.21) is virtually identical to the Klein-Gordon equation except for three thins:i) photons are massless ($\mu = 0$ in Klein-Gordon equation),ii) an electromagnetic wave is a classical world, measurable entity and thus is real, not complex, and in the solution $A^{\mu}(x)$ is a four-vector, not a scalar like ϕ Hence, we can show that if the photon field is real,$A^{\mu}(x) = A^{\dagger\mu}(x)$, then its **plane wave discrete solution has form**

$$\begin{aligned} A^{\mu}(x) &= \underbrace{\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}} \varepsilon_r^{\mu} A_r(\mathbf{k}) e^{-ikx}}_{A^{\mu+}} + \underbrace{\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}} \varepsilon_r^{\mu} A_r^{\dagger}(\mathbf{k}) e^{ikx}}_{A^{\mu-}} \\ &= A^{\mu+} \quad + \quad A^{\mu-} \end{aligned} \tag{4.1.22}$$

where $A_r(\mathbf{k})$ is a number, generally complex, and **for each r, ϵ_r^{μ} is a four dimensional vector**, which We can take, without loss of generality, to be unit length.For each r, μ, ϵ_r^{μ} is a real number.

The four polarization vectors

The ε_r^μ, called polarization vectors, **must have four components. Second, to span a 4D space, we need four independent vectors.** The μ superscript stands for the four components ($\mu = 0, 1, 2, 3$). The subscript r stands for the four independent vectors ($r = 0, 1, 2, 3$), which we will take to be orthogonal. In general, each independent vector ϵ_r^μ has components along each of the four axes in 4D.

In general, the orthogonality conditions for ϵ_r^μ are

$$\varepsilon_{\mu r}\varepsilon_s^\mu = g_{rs} = -\zeta_{\underline{r}}\delta_{\underline{r}s} \quad \text{where} \quad \zeta_0 = -1 \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \tag{4.1.23}$$

Note we can choose to align our ε_3^μ vector with the $\mathbf{k}$, the direction of travel of the photon. We call this the **photon polarization vector alignment**, and for it, ϵ_3^μ is called the **longitudinal polarization**(vector), i.e.,

$$\varepsilon_3 = \mathbf{k}/|\mathbf{k}|$$

ε_1^μ and ε_2^μ are orthogonal to ε_3^μ, and for this alignment, are called the **transverse polarizations**

$$\mathbf{k} \cdot \varepsilon_1 = \mathbf{k} \cdot \varepsilon_2 = 0$$

Thus, we will expect ε_1^μ and ε_2^μ here to be in the same plane as the $\mathbf{E}$ and $\mathbf{B}$ vectors, since they are transverse to the direction of travel of an electromagnetic wave.

ε_0^μ, points in the time (4^{th} dimension) direction and in such systems is called **the time-like or scalar polarization**.

4.1.3 The classical electromagnetic Lagrangian

The simplest form of the Lagrangian is

$$\mathcal{L}_0^{e/m} = -\frac{1}{2}\left(\partial_v A_\mu(x)\right)\left(\partial^v A^\mu(x)\right) \tag{4.1.24}$$

which can be verified by inserting it into Euler-Lagrange field equation

$$\frac{\partial}{\partial x^v}\left(\frac{\partial \mathcal{L}}{\partial \phi^n, v}\right) - \frac{\partial \mathcal{L}}{\partial \phi^n} = 0, \quad \text{with} \quad \phi^n = A_\mu; \mathcal{L} = \mathcal{L}_0^{e/m} \tag{4.1.25}$$

4.2 RQM for Photons

4.2.1 First quantization

The first step in 1st quantization comprises taking the same e/m wave equation we had classically, and thus, the same solution form for the state $|A^\mu(x)\rangle$ as for the classical $A^\mu(x)$ which we repeat below.

$$\begin{gathered}\partial_\alpha\partial^\alpha A^\mu_{\text{state}}(x) = \partial_\alpha\partial^\alpha |A^\mu\rangle = 0 \\ |A^\mu\rangle = \sum_{r,\mathbf{k}} \tfrac{1}{\sqrt{2V\omega_k}}\varepsilon^\mu_r(\mathbf{k})A_r(\mathbf{k})e^{-ikx} + \sum_{r,\mathbf{k}} \tfrac{1}{\sqrt{2V\omega_k}}\varepsilon^\mu_r(\mathbf{k})A^\dagger_r(\mathbf{k})e^{ikx}\end{gathered} \tag{4.2.1}$$

Box

Polarization, B, E Unrelated to Spin

When people learn of circular polarization states, where the transverse **E** and **B** states rotate around the **k** vector direction as the e/m wave propagates, they often confuse that rotation with photon spin. The two are unrelated. Classical angular momentum increases with rotation rate, and the rate of the rotation of A (and thus of E and B) increases with ω_k, i.e., with the energy of the photon. But every photon, regardless of energy, has the same spin 1 value.

In electromagnetism and the quantum theories derived from it, we formulate the theory with polarization states rather than spin states. Photon spin is always in the + or $-\mathbf{k}$ direction and comprises only two possible states for given **k**. Polarization vectors have four possible states, mutually orthogonal in 4D space, and thus can serve as basis states, wheres, whereas spin (for photons) cannot.

RQM comutation relations for photons

The second step in 1st quantization is taking Poisson brackets over into commutators (with the factor of i):

$$\left[x^1, p^1\right] = i \quad \rightarrow \quad \left[x^1, p^1\right]|A^\mu\rangle = i\,|A^\mu\rangle = i\frac{1}{\sqrt{2V\omega_k}}\varepsilon^\mu_r A_r(\mathrm{k})e^{-ikx}$$

$$\underbrace{p^1}_{\text{operator}} = -i\frac{\partial}{\partial x^1}$$

The commutation relations-mean the dynamical variables in classical theory become operators in quantum theory, as we have seen before. Note now, why the term "vector" is used for spin 1 bosons. Because $A^\mu(x)$, which represents that boson, is a four vector.

4.3 The Maxwell Equation in QFT

The classical Lagrangian density (4.1.24) equals the QFT Lagrangian density. We repeat it here, but change the superscript from "e/m" to "1", indicating a spin 1 boson.

$$\mathcal{L}_0^1 = -\frac{1}{2}\left(\partial_v A_\mu(x)\right)\left(\partial^v A^\mu(x)\right) \tag{4.3.1}$$

And from Euler-Lagrange equation, the QFT field equation is

$$\partial_\alpha \partial^\alpha A^\mu(x) = 0 \tag{4.3.2}$$

where $A^\mu(x)$ is a quantum field, not a quantum state.

The discrete plane wave solution is

$$A^\mu(x) = \underbrace{\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e_t^{-ikx}}_{A^{\mu+}} + \underbrace{\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx}}_{A^{\mu-}} \tag{4.3.3}$$

The continuous plane wave solutions.

$$A^\mu(x) = \underbrace{\sum_r \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}}\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx}}_{A^{\mu+}} + \underbrace{\sum_r \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}}\varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx}}_{A^{\mu-}} \tag{4.3.4}$$

4.3.1 Conjugate momentum and Hamiltonian density

$$\begin{aligned}\pi^1_\mu &= \tfrac{\partial \mathcal{L}^1_0}{\partial \dot{A}^\mu} = \tfrac{\partial}{\partial \dot{A}^\mu}\left(-\tfrac{1}{2}\underbrace{(\partial_0 A_v)}_{\dot{A}_v}\underbrace{\left(\partial^0 A^v\right)}_{\dot{A}^v}\ \underbrace{\frac{1}{2}\left(\partial_i A_v\right)\left(\partial^i A^v\right)}_{\text{no } \dot{A}^v,\dot{A}_v, \text{ so drops out}}\right) \\ &= -\tfrac{1}{2}(\underbrace{\frac{\partial}{\partial \dot{A}^\mu}\dot{A}_v}_{g_{\mu v}})\dot{A}^v + \dot{A}_v(\underbrace{\frac{\partial}{\partial \dot{A}^\mu}\dot{A}^v}_{\delta_{\mu v}}) = -\dot{A}_\mu\end{aligned} \tag{4.3.5}$$

$$\begin{aligned}\mathcal{H}^1_0 &= \pi^1_\mu \dot{A}^\mu - \mathcal{L}^1_0 = -\dot{A}_\mu\dot{A}^\mu + \frac{1}{2}\dot{A}_\mu\dot{A}^\mu + \frac{1}{2}\left(\partial_i A_\mu\right)\left(\partial^i A^\mu\right) \\ &= -\frac{1}{2}\dot{A}_\mu\dot{A}^\mu + \frac{1}{2}\left(\partial_i A_\mu\right)\left(\partial^i A^\mu\right) \\ &= -\frac{1}{2}\dot{A}_\mu\dot{A}^\mu - \frac{1}{2}\left(\partial^i A_\mu\right)\left(\partial^i A^\mu\right) = -\frac{1}{2}\left(\partial^\nu A_\mu\right)\left(\partial^\nu A^\mu\right)\end{aligned} \tag{4.3.6}$$

4.4 Commutation Relations for Photon Fields

$$\left[A^\mu(\mathbf{x},t), \pi^1_v(\mathbf{y},t)\right] = i\delta^\mu_v\delta(\mathbf{x}-\mathbf{y}) \to \left[A^\mu(x), \pi^{v1}(y)\right] = ig^{\mu v}\delta(\mathbf{x}-\mathbf{y}) \tag{4.4.1}$$

$$\begin{aligned}\left[a_r(\mathbf{k}), a^\dagger_s(\mathbf{k}')\right] &= \zeta_{\underline{r}}\delta_{\underline{r}s}\delta_{\mathbf{k}\mathbf{k}'} \qquad\qquad \zeta_0 = -1, \zeta_{1,2,3} = 1 \\ &= \zeta_{\underline{r}}\delta_{\underline{r}s}\delta\left(\mathbf{k}-\mathbf{k}'\right) \quad \text{(continuous)}\end{aligned} \tag{4.4.2}$$

All other commutators, such as $[a_r(\mathbf{k}), a_s(\mathbf{k})]$, equal zero for any r and s, as with scalar.

4.5 The QFT Hamiltonian for Photons

$$H^1_0 = \sum_{\mathbf{k},r}\omega_{\mathbf{k}}\left(N_r(\mathbf{k}) + \frac{1}{2}\right) \tag{4.5.1}$$

$$N_r(\mathbf{k}) = \zeta_{\underline{r}}a^\dagger_{\underline{r}}(\mathbf{k})a_{\underline{r}}(\mathbf{k}) \text{ the number operator for photons} \tag{4.5.2}$$

4.6 Other Photon Operators in QFT

Photon creation and destruction operators

$$\begin{aligned} a_r^\dagger(\mathbf{k})\,|n_{\mathbf{k},r}\rangle &= \sqrt{n_{\mathbf{k},r}+1}\,|n_{\mathbf{k},r}+1\rangle \\ a_r(\mathbf{k})\,|n_{\mathbf{k},r}\rangle &= \sqrt{n_{\mathbf{k},r}}\,|n_{\mathbf{k},r}-1\rangle \end{aligned} \tag{4.6.1}$$

Total photon particle number

$$N\left(A^\mu\right) = \sum_{\mathbf{k},r} N_r(\mathbf{k}) \tag{4.6.2}$$

Total particle number lowering and raising

$A^{\mu+}$ particle lowering loperator field (contains $a_r(\mathbf{k})$)

$A^{\mu-}$ particle raising operator field (contains $a_r^\dagger(\mathbf{k})\big)$

Four-current operator

$$j^\mu = -i\left(A_\alpha^{,\mu\dagger}A^\alpha - A_\alpha^{,\mu}A^{\alpha\dagger}\right) = 0 \tag{4.6.3}$$

If we were to take j^0 of the eqn. above as our probability operator, as early researchers expected it to be, then we would have zero probability of ever finding a photon.

Three-momentum operator

$$\mathbf{P} = \sum_{\mathbf{k},r} \mathbf{k} N_r(\mathbf{k}) \tag{4.6.4}$$

4.7 Photon Propagator

$$D_F^{\mu\nu}(x-y) = \frac{-g^{\mu\nu}}{(2\pi)^4}\int \frac{e^{-ik(x-y)}}{k^2+i\varepsilon}d^4k \quad \text{in physical space} \tag{4.7.1}$$

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu v}}{k^2+i\varepsilon} \quad \text{in four-momentum space} \tag{4.7.2}$$

4.8 Weak Lorenz Condition

$\partial_\mu A^\mu$ of the Lorenz gauge would be considered an operator in QFT that is identically equal to zero. Unfortunately, that is not strictly true, because, the Lorenz condition, employed in the direct way, is incompatible with the commutation relations.

Consider the commutator

$$[\underbrace{\partial_\mu A^\mu(x)}_{=0 \text{ in Lorenz gauge}}, A^v(y)] = \sum_{r,\mathbf{k}} \frac{-ik_\mu}{2V\omega_\mathbf{k}} \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}) \left(e^{-ik(x-y)} + e^{ik(x-y)}\right)$$

$$= \sum_{r,\mathbf{k}} \frac{-ik_\mu}{V\omega_\mathbf{k}} \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^\nu(\mathbf{k}) \cos(k(x-y)) \neq 0$$

Now, **we replace the Lorenz condition with the weaker condition**:

$$\partial_\mu A^{\mu+}(x)|\Psi\rangle = 0 \tag{4.8.1}$$

and its adjoint is

$$\langle\Psi|\partial_\mu A^{\mu-}(x) = 0 \tag{4.8.2}$$

So, the expectation value of the Lorenz condition equals zero,

$$\overline{\partial_\mu A^\mu(x)} = \langle\Psi\,|\partial_\mu A^\mu(x)|\,\Psi\rangle = \underbrace{\langle\Psi|\left(\partial_\mu A^{\mu-}(x)\right.}_{=0} + \underbrace{\partial_\mu A^{\mu+}(x)\big)\,|\Psi\rangle}_{=0} = 0$$

4.8.1 Meaning of the weak Lorenz condition

To understand (4.8.1), we substitute $A^{\mu+}$ and consider the photon aligned coordinate system, where

$$\varepsilon_0^\mu = (1,0,0,0)^{\mathrm{T}} \quad \varepsilon_1^\mu = (0,1,0,0)^{\mathrm{T}} \quad \varepsilon_2^\mu = (0,0,1,0)^{\mathrm{T}} \quad \varepsilon_3^\mu = (0,0,0,1)^{\mathrm{T}}$$

Thus,

$$\partial_\mu A^{\mu+}(x) = \sum_\mu \partial_\mu \left(\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathrm{k}}}} \varepsilon_r^\mu a_r(\mathbf{k}) e^{-ikx} \right) = \sum_{r,\mathbf{k},\mu} \frac{1}{\sqrt{2V\omega_{\mathrm{k}}}} \varepsilon_r^\mu a_r(\mathbf{k}) \partial_\mu e^{-ikx}$$

$$= \sum_{\mathbf{k}} \frac{-i}{\sqrt{2V\omega_\mathbf{k}}} \left[\underbrace{\varepsilon_0^0}_{=1} a_0(\mathbf{k})\omega_\mathbf{k} + \underbrace{\varepsilon_0^1}_{=0} a_0(\mathbf{k}) \underbrace{k_1}_{=0} + \underbrace{\varepsilon_0^2}_{=0} a_0(\mathbf{k}) \underbrace{k_2}_{=0} + \underbrace{\varepsilon_0^3}_{=0} a_0(\mathbf{k}) k_3 + \ldots \right.$$

$$+\underbrace{\varepsilon_3^0}_{=0}a_3(\mathbf{k})\omega_{\mathbf{k}}+\underbrace{\varepsilon_3^1}_{=0}a_3(\mathbf{k})\underbrace{k_1}_{=0}+\underbrace{\varepsilon_3^2}_{=0}a_3(\mathbf{k})\underbrace{k_2}_{=0}+\underbrace{\varepsilon_3^3}_{=1}a_3(\mathbf{k})k_3\Bigg]e^{-ikx}$$

In the very last term we invoke the relativistic relation $m^2=E^2-\mathbf{p}^2=(ak)^2-p^2=(ak)^2-(k^3)^2$, where $m=0$. So $k^3=\omega_k=-k_3$. Then, for every $\mathbf{k}$, $\partial_\mu A^{\mu+}(x)$ acting on $|\psi\rangle$ becomes

$$\partial_\mu A^{\mu+}(x)|\Psi\rangle=0\rightarrow\left(a_3(\mathbf{k})-a_0(\mathbf{k})\right)|\Psi\rangle=0,\qquad\text{all }\mathbf{k}\tag{4.8.3}$$

Its adjoint is

$$a_3(\mathbf{k})|\Psi\rangle=a_0(\mathbf{k})|\Psi\rangle\quad\rightarrow\quad\left\langle\Psi\right|a_3^\dagger(\mathbf{k})=\langle\Psi|a_0^\dagger(\mathbf{k})\tag{4.8.4}$$

Thus

$$\bar{H_0^1}=\left\langle\Psi\left|H_0^1\right|\Psi\right\rangle=\sum_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{r=1}^{2}\left\langle\Psi\left|a_r^\dagger(\mathbf{k})a_r(\mathbf{k})\right|\Psi\right\rangle\tag{4.8.5}$$

which means the only contribution to the energy expectation value is from transverse photons. The scalar energy expectation value is negative, but it is always cancelled by a positive longitudinal energy expectation value of the same magnitude.

This jibes with classical e/m theory, since we know that **E** and **b** in a classical electromagnetic wave are always perpendicular to **k**. And from the definition of our potential A^μ, we know that **E** points in the same direction as **A**. Thus, classically, **A** is always othogonal to the wave propagation direction $\mathbf{k}/|\mathbf{k}|$.

4.9 Appendix: Completeness Relations

$$\sum_{r=0}^{3}\zeta_r\varepsilon_r^\mu\varepsilon_r^\nu=-g^{\mu\nu}\qquad\text{where}\qquad\zeta_0=-1\quad\zeta_1=\zeta_2=\zeta_3=1\tag{4.9.1}$$

Since the equation above is a tensor equation,transformation to another Lorentz coordinate system (such as by rotating the spatial axes) will leave the relation unchanged. (i.e., it is valid no matter how we choose to align the axes.)

Chapter 5

Symmetry, Invariance, and Conservation for Free Fields

5.1 Symmetry Mathematically

The transformation depicted in Fig. (5.1) can be can be understood either as a rotation of the cylinder in one direction while we remain fixed (an **active transformation**, by name), or alternatively, as a rotation of our viewing frame of reference in the other direction while the cylinder remains fixed (a **passive transformation**).

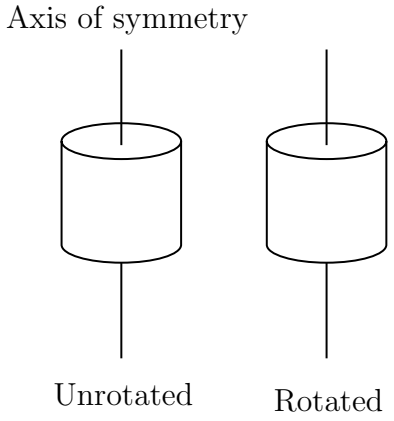


Figure 5.1: Symmetry of a cylinder

Mathematically, when we change our position of observation, it is equivalent to using a new, different reference frame and coordinate system, oriented differently from, and/or displaced relative to, the original. So a transformation can be viewed simply as a change of coordinate system, and this is often represented as a shifting from unprimed to primed coordinates. In QFT, **we will focus on passive transformation interpretation.**

· **Example**

Consider the function

$$g\left(x^1, x^2\right) = \left(x^2\right)^2 \tag{5.1.1}$$

if we have the following rorational transformation as

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{T} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \quad \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{T^{-1}=T^T} \begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix}$$

we can express the function in the primed coordinate as:

$$g = \left(x^2\right)^2 = \left(x'^1 \sin\theta + x'^2 \cos\theta\right)^2 = \left(x'^1\right)^2 \sin^2\theta + \left(x'^2\right)^2 \cos^2\theta + 2x'^1 x'^2 \sin\theta\cos\theta \neq \left(x'^2\right)^2$$

The transformed form of g, represented by g', has the same value at the same physical point, but it is not the same form in terms of the primed coordinates as g was in terms of the unprimed coordinates. But f', the transformed form of f, did have the same form in terms of both sets of coordinates, and thus, we dropped the prime on f on the RHS.

In spite of its non-symmetry under rotation, g is symmetric under a different kind of Fransfort of the firc-symment ander founding f the bynaced relative to the first along transformation, the translation to a coordinate system which is displaced relative to the first along the x^1 axis, i.e., $x^1 \to x^1 = x^1 +$ constant , or

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} K \\ 0 \end{bmatrix} \quad K = \text{ constant}$$

Substitution yields g' have the same form.

From the example above, we can deduce the general rule that if a coordinate is missing in a given function, that function is invariant under a transformation solely in the direction of that coordinate (and also under multiplication of the coordiante by a constant).

So under any transformation of coordinate axes, the value at a physical point of every possible scalar function is invariant.

5.1.1 Scalar are invariant, vectors are covariant

The scalar value at the point (equal to the length of the position vector at that point) is the same in both systems, but the coordinate values are not. For a 2D position vector in physical space, we have

$$|x| = \left|x^i\right| = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} = \sqrt{\left(x'^1\right)^2 + \left(x'^2\right)^2} = \left|x'^i\right|$$

It is generally true of every vector **v**, not just the position vector shown here, that its physical, measurable length (a scalar value) remains unchanged under any coordinate transformation, but its component values change. This is called covariance. Scalar values are invariant under coordinate transformation; vector components are **covariant**.

General rule: if a function h is not a function of the jth coordinate x^j, then h is symmetric under the transformation $x^j \to x^j +$ constant.

5.2 Symmetry in Classical Mechanics

Recall that a Galilean transformation is

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \to \begin{bmatrix} x^{11} \\ x'^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} x^1 - v^1 t \\ x^2 - v^2 t \\ x^3 - v^3 t \end{bmatrix}$$

Newtonian mechanics is invariant under the Galilean transformation. But **Maxwell's eqs are not invariant under Galilean transformation. Instead, it is invariant under Lorentz**

transformation

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma\left(x^0 - \frac{v}{c}x^1\right) \\ \gamma\left(x^1 - \frac{v}{c}x^0\right) \\ x^2 \\ x^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Lambda} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{5.2.1}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad v = v^1$$

The index notation for Lorentz transformations is

$$x'^\mu = \Lambda^\mu{}_v x^v \quad V'^\mu\left(x'^\alpha\right) = \Lambda^\mu{}_v V^v\left(x^\alpha\right) \quad T'^{\mu\nu}\left(x'^\alpha\right) = \Lambda^\mu{}_\delta \Lambda^\nu{}_\gamma T^{\delta\gamma}\left(x^\alpha\right) \tag{5.2.2}$$

Note that Λ^{-1}, the inverse of Λ, can be obtained by taking $\mathbf{v} \rightarrow -\mathbf{v}$ since each coordinate system seems to be going in the opposite direction with respect to the other. Λ^{-1} will transform x'^μ back into x^μ.

Recall that the length of a vector in 3D is unchanged under a coordinate system transformation, 1.e., the length 1s a scalar and thus invariant. The same thing is true in 4D for four-vectors.

$$w_\mu w^\mu = w_0 w^0 + w_1 w^1 + w_2 w^2 + w_3 w^3 = w^0 w^0 - w^1 w^1 - w^2 w^2 - w^3 w^3 = \text{ scalar invariant}$$

and that this is the same for any observer in any inertial coordinate system. This applies to any vector, be it a position vector like x^μ, the differential of position dx^μ, the four-velocity u^μ, the four potential A^μ, the partial derivative ∂^μ, or any other.For instance,

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu \partial^\mu = \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3} = \text{ scalar invariant derivative,}$$

So if X represents a quantity, we have

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'_\mu} \rightarrow \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}X = \frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'_\mu}X$$

5.2.1 Poincare transformation

The most general transformation we could have in spacetime would comprise 1) a 4D translation(translating our coordinate axes in space, time, or both), 2) a rotation in space, and 3) a Lorentz transformation to a frame with different relative velocity. (We ignore reflection.)

The rotation in 3D is the same. It does allow us to rotate our 3D axes, however, so that the relative velocity between our original and transformed systems is along the x^1 axes of both. **This lets us use the Lorentz transformation in its simplest form (5.2.1)**. With this form we state the general transformation between coordinate systems, know as **Poincare transformation** as

$$x^\mu = \Lambda^\mu_\nu \left(x^\nu + a^\nu\right) \quad a^\nu = \text{constant four vector} \tag{5.2.3}$$

5.2.2 Other kinds of symmetry

Consider the Euler-Lagrange equation for a particle in Newtonian mechanics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad L = T - V \quad p_i = \frac{\partial L}{\partial \dot{x}^i}$$

If the Lagrangian L is not an explicit function of the spatial coordinate x^i, then $\partial L/\partial x^i = 0$ on the LHS above. Thus, the time derivative of p_i is zero.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{dp_i}{dt} = 0 \quad \text{with} \quad \frac{\partial L}{\partial x^i} = 0 \quad \text{when} \quad L \neq L\left(x^i\right)$$

Hence p^i is constant and thus, conserved. This makes sense since the only source for spatial dependence in L is the potential energy V. If we have no V dependence on x^i, then there is no force in the x^i direction, and momentum p_i is constant.**Note this means the Lagrangian is symmtric.**

If the Lagrangian is symmetric in a coordinate, then the conjugate momentum for that coordiante is conserved.

5.3 Transformations in QFT

5.3.1 Spinor transformation

For spinor, we seek a matrix which is four by four in spinor space and which represents what ha pens to a spmor under a Lorentz transformation and/or a rotation of coordinates. That is, we seek D in

$$\psi'\left(x'^{\mu}\right)=D\psi\left(x^{\mu}\right) \quad \frac{\text{with spinor indices}}{\text{written out}} \quad \psi'_{\alpha}\left(x'^{\mu}\right)=D_{\alpha\beta}\psi_{\beta}\left(x^{\mu}\right) \tag{5.3.1}$$

The spinor transformation under Lorentz and rotation transformation is

$$D=e^{-i(\mathbf{L\Theta}+\mathbf{MQ})} \quad L^{k}=-\frac{i}{2}\varepsilon_{ij}^{k}\gamma^{i}\gamma^{j}, \quad \Theta^{k}=\left(\theta^{1},\theta^{2},\theta^{3}\right), M^{k}=\frac{i}{2}\gamma^{0}\gamma^{k}, Q^{k}=\left(v^{1},v^{2},v^{3}\right) \tag{5.3.2}$$

where Θ^{k} represents rotation angles of the primed system with respect to the unprimed system; Q^{k} is a three vector of the boost velocities; and $\epsilon_{ij}{}^{k}$ is zero unless i,j,k are all different, 1 if $ijk=123,231,312$, and -1 for the others.

Note that in formal mathematical language, the set of all possible Lorentz transformations (all possible v) is known as the Lorentz group. When the Lorentz group acts on the coordinate system, it changes what our spinors look like in the new system and this change is represented by D. So, D **is called a representation of the Lorentz group.** It "represents" that group in spinor space.

$\bar{\psi}\psi=$ world scalar $\quad \bar{\psi}\gamma^{\mu}\psi=$ transforms like four vector.

5.4 Lorentz Symmetry of the Lagrangian Density

Lagrangian Density is symmetric under Lorentz transformation. We conclude this because of Einstein's postulate that the laws of nature (the field equation here) is invariant in form under Lorentz transformation. The Euler-Lagrange equation for fields, which is another form of the field equation, is a law of nature and must, therefore be invariant in form as well.

5.5 Noether's Theorem

There are other ways the Lagrangian density can be symmetric, for example

$$\phi \to \phi' = \phi e^{-i\alpha} \tag{5.5.1}$$

and for free scalar field, we have

$$\mathcal{L}_0^0 = \partial_v \underbrace{\phi^{r+} e^{-i\alpha}}_{\phi^\dagger} \partial^v \underbrace{\phi' e^{ia}}_{\phi} - \mu^2 \underbrace{\phi' e^{-i\alpha}}_{\phi^\dagger} \underbrace{\phi' e^{i\alpha}}_{\phi} = \mathcal{L}_0^0 \left(\phi'^\dagger, \phi'\right)$$

5.5.1 Internal and external symmetries

Poincaré transformations (Lorentz plus 4D translation) and 3D rotations involve changes t to physical coordinates x^μ of our external world and are called **external transformations**.

Transformation like (5.5.1) have nothing to do with x^μ, but instead function in hidden spaces, behind the scene, like Hilbert or Fock space. They are called **internal transformation**.

Noether's theorem in words: **A symmetry in the Lagrangian density implies an associated quantity is conserved.**
Mathematically:If the Lagrangian density $\mathcal{L}\left(\phi^r, \phi^r, \mu\right)$ is symmetric in form with respect to a transformation in ϕ^r which is a function of parameter α,i.e., $\phi^r\left(x^\mu\right) \to \phi^r\left(x^\mu, \alpha\right)$ then the four current (using $\phi^r\left(x^\mu, \alpha\right)$)

$$j^\mu\left(\phi^r, \phi^r_{,\nu}\right) = \frac{\partial \mathcal{L}}{\partial \phi^r_{,\mu}} \frac{\partial \phi^r}{\partial \alpha} \quad (\text{ sum on } r) \tag{5.5.2}$$

bas zero four-divergence,$\partial_\mu j^\mu = 0$. Thus, its zeroth component j^0 integrated over all the space is conserved, as is qj^0 integrated over all space, where q is a constant.

5.5.2 Apply Noether's theorem to free scalar field

$$\frac{\partial \mathcal{L}_0^0}{\partial \phi_\mu} = \frac{\partial}{\partial \phi_{,\mu}}\left(\phi^\dagger_{,\nu}\phi^\nu - \mu^2\phi^\dagger\phi\right) = \frac{\partial}{\partial \phi_\mu}\left(\phi^\dagger_{,\mu}\phi^\nu\right) = \frac{\partial \phi^\dagger_{,\nu}}{\partial \phi_\mu}\phi^{,\nu} + \phi^\dagger_{,\nu}\frac{\partial \phi^{,\nu}}{\partial \phi_{,\mu}} = \phi^\dagger_{,\nu} g^{\nu\mu} = \phi^{\dagger,\mu}$$

$$\frac{\partial \mathcal{L}_0^0}{\partial \phi^\dagger, \mu} = \frac{\partial}{\partial \phi^\dagger, \mu}\left(\phi^\dagger_{,\nu}\phi^\nu - \mu^2\phi^\dagger\phi\right) = \frac{\partial}{\partial \phi^\dagger}\left(\phi^\dagger, \phi^{,\nu}\right) = \frac{\partial \phi^\dagger_{,\nu}}{\partial \phi^\dagger_{,\mu}}\phi^{,\nu} = \delta^\mu_v \phi^{,v} = \phi^{,\mu}$$

and

$$\frac{\partial \phi(x^\eta,\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\phi\left(x^\eta\right)e^{-i\alpha} = -i\phi\left(x^\eta\right)e^{-i\alpha}$$
$$\frac{\partial \phi^\dagger(x^\eta,\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\phi^\dagger\left(x^\eta\right)e^{i\alpha} = i\phi^\dagger\left(x^\eta\right)e^{i\alpha}$$

Using the relation above, we have

$$j^\mu\left(\phi^r, \phi^r_{,v}\right) = i\left(\phi^{,\mu}\left(x^\eta\right)\phi^\dagger\left(x^\eta\right) - \phi^{\dagger,\mu}\left(x^\eta\right)\phi\left(x^\eta\right)\right)$$

This is identical to the scalar four-current.The expectation value (expected measurement) of a conserved operator is conserved. If the state measured is in an eigenstate, any measurement at any time will yield the same eigenvalue.

5.6 Symmetry, Gauges, and Gauge theory

- **Gauge invariance** (or gauge symmetry) is the property of a field theory in which different configurations of the underlying fundamental, but unobservable, field(s) result in identical observable properties.
- the unobservable field, often a potential field, is called the **gauge field**.
- A gauge transformation changes the gauge field from one configuration to another. Each different configuration of the gauge field is a different gauge.
- A theory have gauge invariance (symmetry) is called a gauge theory.

We can also say that a gauge theory is a type of field theory in which the Lagrangian (density) is invariant under a continuous (not discrete) transformation.

5.7 A Solved Exercise

Problem 14. Show that the total (not density) 3-momentum k^i for free scalars is conserved. Use our knowledge that the conjugate momentum for x^i is k_i, the total (not density)

3 -momentum (expressed in covariant components), and it is conserved if L is symmetric (invariant) under the coordinate translation transformation $x^t \to x^1 = x^t$, where α^t is a constant 3− vector. Then, show the same result via commutation of the three-momentum operator.

Solution The Lagrangian density is $\mathcal{L}_0^0 = \phi^\dagger_{,\mu}\phi^\mu - \mu^2\phi^\dagger\phi$. We must integrate this over all volume to get the total Lagrangian L. If k_i is conserved, then of course, so is k^i. So, we need to show L is invariant under $x^i \to x^i = x^i + \alpha^i$

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2Va_{\mathbf{k}}}} \left(a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right) \quad \phi^\dagger = \sum_{\mathbf{k}} \frac{1}{\sqrt{2Va_k}} \left(b(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi_s, \mu = \textstyle\sum_{\mathbf{k}} \frac{ik_\mu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$
$$\phi^{,\mu} = \textstyle\sum_{\mathbf{k}} \frac{ik^\mu}{\sqrt{2Va_k}} \left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi^\dagger_{z\mu}\phi^z = \sum_{\mathbf{k}}\sum_{\mathbf{k}^*} \frac{-1}{2V} \frac{k_\mu k^{*\mu}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}^*}}} \left(b(\mathbf{k})a\left(\mathbf{k}^*\right)e^{-ik_A x^\mu}e^{-ik_A^* x^\mu} - b(\mathbf{k})b^\dagger\left(\mathbf{k}^*\right)e^{-ik_\mu x^\mu}e^{ik_\mu x^\mu}\right.$$
$$\left. -a^\dagger(\mathbf{k})a\left(\mathbf{k}^*\right)e^{ik_\mu x^\mu}e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})b^\dagger\left(\mathbf{k}^*\right)e^{ik_\mu x^\mu}e^{ik_\mu x^\mu}\right.$$

For $k_i = -k_i^*$, we have

$$k_\mu k^{*\mu} = \omega_{\mathbf{k}}^2 + k_i k^{*i} = \omega_{\mathbf{k}}^2 + k_i\left(-k^i\right) = \omega_{\mathbf{k}}^2 + k_i k_i = k_\mu k_\mu$$

$$\int \phi^\dagger_{,\mu}\phi^{,\mu}dV = \sum_k \frac{-1}{2\omega_k} \begin{pmatrix} k_\mu k_\mu e^{-i2\omega_k t}b(\mathbf{k})a(-\mathbf{k}) - k_\mu k^\mu b(\mathbf{k})b^\dagger(\mathbf{k}) \\ -k_\mu k^\mu a^\dagger(\mathbf{k})a(\mathbf{k}) + k_\mu k_\mu e^{i2\omega_k t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) \end{pmatrix}$$

Similarly,

$$\int \mu^2\phi^\dagger\phi dV = -\sum_{\mathbf{k}} \frac{\mu^2}{2\omega_{\mathbf{k}}} \left(e^{-i2\omega_{\mathbf{k}}t}b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + e^{i2\omega_{\mathbf{k}}t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})\right)$$

It is easy to show that, after the transformation the forms of $\int \mu^2\phi^\dagger\phi dV$ and $\int \phi^\dagger_{,\mu}\phi^{,\mu}dV$ will not change, so L is symmetric in some coordinate, then the conjugate momentum of that coordinate is conserved. k_i, the particle(s) 3 -momentum is the conjugate momentum of x^i. Thus, k_i, is conserved.

Also

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(N_a(\mathbf{k}) + N_b(\mathbf{k})\right) \quad \mathbf{P} = \sum_{\mathbf{k}} \mathbf{k} \left(N_a(\mathbf{k}) + N_b(\mathbf{k})\right) \to [H, \mathbf{P}] = 0 \quad \begin{pmatrix} \text{because all number} \\ \text{operators commute} \end{pmatrix}$$

Thus $\mathbf{P}$ is conserved for the free Hamiltonian.

Chapter 6

Interactions: The Underlying Theory

6.1 Interactions in RQM

6.1.1 Maxwell's equation with sources

If we include the charge density and current density into Maxwell's equation, we have

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho_{charge} \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}_{charge} \\ \nabla \cdot \mathbf{B} &= 0 \\ \vec{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{6.1.1}$$

By introducing the potentials Φ and $\mathbf{A}$, where

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

one gets

$$\begin{aligned} -\nabla\frac{\partial \Phi}{\partial t} - \nabla(\nabla \cdot \mathbf{A}) &= \rho_{charge} \\ \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla\frac{\partial \Phi}{\partial t} + \nabla(\nabla \cdot \mathbf{A}) &= \mathbf{j}_{charge} \end{aligned} \tag{6.1.2}$$

Rewrite the equations above in terms of 4-potential and 4-current $-ej^\mu$=$(\rho_{charge}, \mathbf{j}_{charge})$:

$$\partial^\alpha \partial_\alpha A^\mu(x) - \partial^\mu \left(\partial_\nu A^\nu(x)\right) = -ej^\mu(x) \tag{6.1.3}$$

Using Lorenz gauge condition, we have

$$\partial^\alpha \partial_\alpha A^\mu(x) = -ej^\mu \tag{6.1.4}$$

6.1.2 The classical Lagrangian density for interaction

The full electromagnetic Lagrangian(density), including interactions, must give rise to (6.1.4) when substituted into the Euler-Lagrange field equation:

$$\frac{\partial}{\partial x^v}\left(\frac{\partial \mathcal{L}}{\partial \phi^n_v}\right) - \frac{\partial \mathcal{L}}{\partial \phi^n} = 0, \quad \text{with} \quad \phi^n = A_\mu; \mathcal{L} = \mathcal{L}^{e/m}$$

One can prove that the full electromagnetic field classical Lagrangian is

$$\mathcal{L}^{e/m} = \underbrace{-\frac{1}{2}\left(\partial_v A_\mu(x)\right)\left(\partial^v A^\mu(x)\right)}_{\mathcal{L}_0^{e/m}} + \underbrace{ej^\mu(x)A_\mu(x)}_{\mathcal{L}_1^{e/m}} \tag{6.1.5}$$

where "0" and "1" subscripts denote the free and interaction parts, respectively, of the Lagrangian.

6.1.3 Electromagnetic Interactions in RQM

Relations (6.1.4) and (6.1.5) hold for classical electromagnetism where $-ej^\mu$ is the classical electric charge 4-current. **In quantization, we assume the quantum form of the Lagrangian (density or total) is the same as the classical, and thus, so would be the resulting wave equation.** In RQM, we would then consider A^μ to represent the quantum photon state (ket, wave function). Thus, (6.1.5) represents the RQM electromagnetic Lagrangian. But then, how should one interpret $-ej^\mu$? In Chap.3, we saw that the probability 4-current for an electron in RQM, where ψ_{state} represents the electron wave function state, is

$$j^\mu = (\rho, \mathbf{j}) = \bar{\psi}_{\text{state}}\gamma^\mu\psi_{\text{state}} \quad \text{where } \partial_\mu j^\mu = 0$$

It seems natural to assume charge density varies directly with probability density. Thus, we can assume

$$-ej^{\mu} = -e\bar{\psi}_{\text{state}}\gamma^{\mu}\psi_{\text{state}} \tag{6.1.6}$$

where the total charge of the electron would be

$$-e\int j^{0}d^{3}x = -e$$

Using (6.1.4),(6.1.5), and(6.1.6), we can then represent the **RQM interaction wave equation for a photon** as

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}_{\text{stale}} = -e\bar{\psi}_{\text{state}}\gamma^{\mu}\psi_{\text{state}} \tag{6.1.7}$$

with the corresponding **RQM e/m interaction Lagrangian for a photon** as

$$\mathcal{L}^{e/m} = \underbrace{-\frac{1}{2}\left(\partial_{v}A_{\mu,state}\right)\left(\partial^{v}A^{\mu}_{\text{state}}\right)}_{\mathcal{L}_{0}^{e/m}} + \underbrace{e\bar{\psi}_{\text{state}}\gamma^{\mu}\psi_{\text{state}}A_{\mu,state}}_{\mathcal{L}_{I}^{e/m}} \tag{6.1.8}$$

(6.1.7) governs the behavior of a photon (A^{μ}_{state}) in the presence of an electron (ψ_{state}).

6.1.4 The electromagnetic interaction Dirac equation

We now develop the full Dirac equation describing the electron interacting with a photon. To this end, consider that $\mathcal{L}_{0}^{e/m}$ of (6.1.8) represents the free photon part of the full e/m Lagrangian. If we assume $\mathcal{L}_{1}^{e/m}$ represents the e/m interaction part for both the photon and the electron, then **we need only add the free electron contribution to (6.1.8) to get a Lagrangian containing all terms relevant to photons, electrons, and interactions between them.** Since from Chap 3

$$\mathcal{L}_{0}^{1/2} = \bar{\psi}_{\text{state}}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_{\text{state}}$$

Thus, the **full e/m Lagrangian** is

$$\mathcal{L}^{1/2,1} = \underbrace{-\frac{1}{2}\left(\partial_{v}A_{\mu,state}\right)\left(\partial^{v}A^{\mu}_{\text{state}}\right)}_{\mathcal{L}_{0}^{1}=\mathcal{L}_{0}^{e/m}} + \underbrace{\bar{\psi}_{\text{state}}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_{\text{state}}}_{\mathcal{L}_{0}^{1/2}} + \underbrace{e\bar{\psi}_{\text{state}}\gamma^{\mu}\psi_{\text{state}}A_{\mu,state}}_{\mathcal{L}_{1}^{1/2,1}=\mathcal{L}_{1}^{e/m}} \tag{6.1.9}$$

To find the interaction form of the Dirac equation, we use (6.1.9) in Euler-Lagrange equation with $\phi^{n=1} = \bar{\psi}$. The result is

$$(i\gamma^\mu \partial_\mu - m)\,\psi_{\text{state}} = -e\gamma^\mu \psi_{\text{state}}\, A_{\mu,state} \tag{6.1.10}$$

As an aside, using (6.1.9) with $\phi^{n=2} = \psi$ results in the adjoint full Dirac equation.

Box

A this point, some might be concerned that we have used the Lagrangian density methodology, which is normally reserved for quantum and classical fields, to develop the full Dirac equation for quantum states (corresponding to particles, not fields).One would expect to use the total Lagrangian L (integration of $\mathcal{L}$ over all space) instead of $\mathcal{L}$, since (6.1.10) is a wave equation for interacting states, not fields.

However, even in the context of 1^{st} (particle) and 2^{nd} (field) quantization as we have come to understand them, the issue is not such a big one. This is because we did not employ commutation relations for fields A^μ and ψ, analogous to Poisson bracket relations, in the above development, It is the adth ${}^\mu A^\mu$ and ψ, analogous to Poisson bracket relations, in the above development, **it is the adoption of commutation relations for those fields that turns them into creation and destruction operators quantum mechanically.** We did not do that, so A^μ and ψ remain as states, not quantum fields, in the above treatment. Of course, for RQM, we would still have commutation relations for dynamical variable gnerators, such as p_x and X_1, though we would not have them for A^μ and ψ

6.2 Interactions in QFT

Our interacting spinor field and photon wave equations in the Heisenberg picture should simply be the following for fields,

$$\partial^\alpha \partial_\alpha A^\mu = -e\bar{\psi}\gamma^\mu \psi \tag{6.2.1}$$

$$(i\gamma^\mu \partial_\mu - m)\,\psi = -e\gamma^\mu A_\mu \psi \tag{6.2.2}$$

where the order of ψ and A_μ in the equations above is unimportant, even though they are operators, since different type fields commute. The associated Lagrangian for the ψ and A^μ operator fields is

$$\mathcal{L}^{1/2,1} = \underbrace{-\frac{1}{2}\left(\partial_v A_\mu\right)\left(\partial^v A^\mu\right)}_{\mathcal{L}_0^1} + \underbrace{\bar{\psi}\left(i\gamma^a\partial_\alpha - m\right)\psi}_{\mathcal{L}_0^{1/2}} + \underbrace{e\bar{\psi}\gamma^\mu A_\mu\psi}_{\mathcal{L}_I^{1/2,1}} \tag{6.2.3}$$

Modern day computers can help in providing numerical solutions to these equations, but early researchers in QFT did not have such things. Also, the route those researchers did take provides considerable insight into the inner workings of the theory. That route, for the QFT e/m interaction theory known as **quantum electrodynamics (QED)**, was forged in large part by Richard Feynman, Freeman Dyson, Julian Schwinger, and Sin-Itiro Tomonaga. It involves two things,

- perturbation theory, and
- a trick known as the Interaction picture

The expectation value for any quantum field, including spinor and vector fields, is zero.

6.3 Interaction Picture

It turns out that a third picture, the Interaction Picture (I.P.) is easier to use for interactions in QFT. For one reason, it facilitates use of perturbation theory in place of trying to solve the coupled, non-linear, partial differential equations (6.2.1) and (6.2.2).

Additionally, the LP. allows us to analyze interacting fields using all the results of our free QFT development.

Breaking the Hamiltonian into Free and Interaction Parts The Hamiltonian (total, not density) for e/m interactions in the (Schrödinger Picture) S.P. can be expressed, from the Lagrangian density (6.1.9) and the Legendre transformation, as (with ϕ^r generically representing

any quantum field)

$$\underbrace{H^S}_{H} = \underbrace{H^{1/2,1}}_{\text{just e/m}} = \underbrace{\int \left(\pi_r \dot{\phi}^r - \mathcal{L}_0^1 - \mathcal{L}_0^{1/2}\right) d^3x}_{H_0^S = H_0(\text{ free part })} - \underbrace{\int \mathcal{L}_I^{1/2,1} d^3x}_{H_I^S(\text{ interaction part })} \tag{6.3.1}$$

Thus,$H = H_0 + H_I^S$. Note, for future reference, that for all cases $\mathcal{H}_I = -\mathcal{L}_I$ and $H_I = -L_I$.

Using only the free part of the Hamiltonian to transform to the interaction picture

The transformation from the S.P. to the I.P. is

$$U_0 = e^{-iH_0 t} \tag{6.3.2}$$

where U_0 is a unitary operator, where

$$U_0^\dagger |\Psi\rangle_S = |\Psi\rangle_I \tag{6.3.3}$$

and where subscripts "S" and "I" on generic states $|\Psi\rangle$ indicate the S.P. and I.P, respectively. For operators, where superscripts "S" and "I" represent the S.P. and I.P., respectively,

$$U_0^\dagger \mathcal{O}^S U_0 = \mathcal{O}^I \tag{6.3.4}$$

Parts of the Hamiltonian expressed in the I.P.

For the free part of the Hamiltonian operator $H_0 = H_0^S$, we see that

$$H_0^I = U_0^\dagger H_0^S U_0 = U_0^\dagger H_0 U_0 = e^{iH_0 t} H_0 e^{-iH_0 t} = H_0 e^{iH_0 t} e^{-iH_0 t} = H_0 \tag{6.3.5}$$

because H_0 commutes with itself. Thus,

$$H_0 = H_0^S = H_0^I \tag{6.3.6}$$

this equality generally does not hold for the interaction part:

$$H_I^I = U_0^\dagger H_I^S U_0 \neq H_I^S$$

and thus, we will represent the interaction picture Hamiltonian as

$$H^I = H_0 + H_I^I \tag{6.3.7}$$

6.3.1 Equations of Motion in the I.P.

$$\frac{d\mathcal{O}^l}{dt} = \frac{d}{dt}\left(U_0^+ \mathcal{O}^s U_0\right) = \frac{dU_0^\dagger}{dt}\mathcal{O}^s U_0 + \underbrace{U_0^\dagger \frac{\partial \mathcal{O}^S}{\partial t} U_0}_{\text{defined } = \frac{\partial \mathcal{O}^I}{\partial t}} + U_0^\dagger \mathcal{O}^s \frac{dU_0}{dt} =$$

$$\frac{de^{iH_0t}}{dt}\mathcal{O}^S e^{-iH_0t} + e^{iH_0t}\mathcal{O}^S \frac{de^{-iH_0t}}{dt} + \frac{\partial \mathcal{O}^I}{\partial t} = iH_0 \underbrace{e^{iH_0t}\mathcal{O}^S e^{-iH_0t}}_{\mathcal{O}^I} - \underbrace{e^{iH_0t}\mathcal{O}^S e^{-iH_0t}}_{\mathcal{O}^I} iH_0 + \frac{\partial \mathcal{O}^I}{\partial t}$$

or

$$\boxed{\frac{d\mathcal{O}^I}{dt} = -i\left[\mathcal{O}^I, H_0\right] + \frac{\partial \mathcal{O}^I}{\partial t}} \tag{6.3.8}$$

where the last term is zero in this note, because we only deal with operators for which $\partial \mathcal{O}^S / \partial t = 0$. Thus, the equation of motion for operators in the I.P. depends only on the free part of the Hamiltonian.

The I.P. equation of motion for states is

$$i\frac{d}{dt}|\Psi\rangle_I = H_I^I |\Psi\rangle_I \tag{6.3.9}$$

and hence, the equation of motion for states in the I.P. depends only on the interaction part of the Hamiltonian.

The I.P. equation of motion for expectation values is

$$\frac{d\overline{\mathcal{O}}}{dt} = {}_I\left\langle \Psi \middle| \left(-i\left[\mathcal{O}^I, H^I\right] + \frac{\partial \mathcal{O}^I}{\partial t}\right) \middle| \Psi \right\rangle_I \tag{6.3.10}$$

Box

Note that (6.3.8) has the same form as the operator equation of motion in the H.P., except that we have H_0 in the I.P. and H in the H.P. **Hence, we can teke all results we obtained for operator behavior in the H.P. free field and use them in the I.P. for interacting fields.**

For field operators such as $\mathcal{O}^l = \phi, \psi$, or A^μ), (6.3.8) has identical form to the H.P. equation of motion for fields where $H = H_0$. Thus (6.3.8) reduces to the Klein-Gordon equation for scalars, the free Dirac equation for spinor fields, and the free Maxwell equation for photons. **Quantum fields in the I.P. behave just like the free quantum fields**

6.3.2 Visualizing states in the I.P.

Consider a single particle scalar plane wave state expressed in the coordinate basis. In the S.P., it looks like, where K is a normalization factor and sub/superscript meaning should be obvious,

$$|\phi\rangle_S = \phi^S_{\text{state}} = Ke^{-iEt+i\mathbf{k}\cdot\mathbf{x}} = Ke^{-iE_0t-iE_It+i\mathbf{k}\cdot\mathbf{x}}$$

Note that E_t is a number, and thus is the same in any picture. Transform the state to the I.P.,

$$U_0^\dagger|\phi\rangle_S = e^{iH_0t}\phi_{S,\text{ state}} = Ke^{iH_0t}e^{-iE_0t-iE_It+ik\cdot x} = Ke^{iE_0t}e^{-iE_0t-iE_It+i\mathbf{k}\cdot\mathbf{x}} = Ke^{-iE_It+i\mathbf{k}\cdot\mathbf{x}}$$

So we see that the state in the I.P. varies in time only with the interaction energy. The operator $U_0^\dagger$ takes out the H_0 dependence of the ket.

In the I.P.

- the state equation of motion depends on only the interaction Hamiltonian H_I^I,
- operator equations of motion depend on only the free Hamiltonian H_0, thus, importantly,
- the operator equations of motion in the I.P. are the same as the operator equations of motion in the H.P. for free fields (i.e., for $H^H = H_0$ with $H_I^H = 0$), so all operator relations derived for free field are valid in I.P.

- meaning the free field case Klein-Gordon, Dirac, and Maxwell equations (of motion) from the H.P. are the same as those in the interacting case in the I.P., and so
- quantum fields ϕ, ψ, and A^μ in the L.P. (the solutions to the field equations of motion) are the same as the free quantum fields solutions in the H.P.

So in the I.P. We only need to solve the state equation of motion(6.3.9).

6.4 S matrix

Each S_{fi} of S Matrix is the transition amplitude between an initial eigenstate and a final eigenstate, S_{fi}^2 is probability of transition from initial eigenstate $|i\rangle$ to final eigenstate $|f\rangle$. For example

$$S_{21}^\dagger S_{21} = |S_{21}|^2 = \text{probability of 1 st eigenstate transitioning to 2nd}$$

S_{fi} is a transition amplitude for a particular reaction. That is, for the operator $S_{oper,fi}$

$$S_{fi} = \langle f\,|S_{\text{oper},fi}|\,i\rangle$$

$$S_{\text{oper},fi}|i\rangle = S_{fi}|f\rangle \quad \text{(no sum on } i \text{ or } f) \tag{6.4.1}$$

and

$$\sum_f |S_{fi}|^2 = 1 \tag{6.4.2}$$

in general

$$S_{fi} = \langle f\,|S_{oper}|\,i\rangle \tag{6.4.3}$$

6.5 Finding the S operator

We can find the S_{oper} from the state equation of motion, the only thing we haven't already solved for in the I.P. formulation. In the I.P., our state equation of motion, where $\mathbf{\Psi}$) represents

a generic state (multiparticle typically), is

$$i\frac{d}{dt}|\Psi(t)\rangle_I = H_I^I|\Psi(t)\rangle_I$$

Let

$$|F\rangle = \sum_f S_{fi}|f\rangle = |\Psi\left(t_f\right)\rangle_I = S_{oper}\left(t_f,t_i\right)|\Psi\left(t_i\right)\rangle_I = S_{oper}\left(t_f,t_i\right)|i\rangle \tag{6.5.1}$$

Taking our final time t_f as time t in the eqn. of motion, we have

$$|\Psi(t)\rangle_I = S_{\text{oper}}\left(t,t_i\right)|\Psi\left(t_i\right)\rangle_I \tag{6.5.2}$$

Using the equation above in the equation of motion for state yields

$$i\frac{d}{dt}\left(S_{oper}|\Psi\left(t_i\right)\rangle_I\right) = H_l^I\left(S_{oper}|\Psi\left(t_i\right)\rangle_I\right) \tag{6.5.3}$$

This becomes

$$i\frac{dS_{\text{oper}}}{dt}|\Psi\left(t_i\right)\rangle_I + iS_{\text{oper}}\underbrace{\frac{d}{dt}|\Psi\left(t_i\right)\rangle_I}_{=0 \text{ as indep. of } t_f} = H_I^I S_{\text{oper}}|\Psi\left(t_i\right)\rangle_I \tag{6.5.4}$$

and thus the differential equation for S_{oper}

$$i\frac{dS_{\text{oper}}}{dt} = H_I^I S_{\text{oper}} \tag{6.5.5}$$

This has the solution

$$S_{\text{oper}} = e^{-i\int_{t_i}^{t_f} H_I^I dt} = e^{-i\int_{t_i}^{t_f}\int_V \mathcal{H}_I^I d^4x} \tag{6.5.6}$$

In infinite volume and time frame,

$$S = S_{\text{oper}}\ \left(V\to\infty t_f\to\infty, t_i\to-\infty\right) = e^{-i\int_{-\infty}^{\infty}\mathcal{H}_I^I d^4x} \tag{6.5.7}$$

Box

In Fock space where every eigenstate can be visualized as a separate axis in an infinite dimensional space, the S_{oper} can be visualized as a sort of abstract "rotation" in that space. **The initial state vector $|i\rangle$ is "rotated" by the S_{oper} into a new vector with components along the eigenstate basis axes.**

6.6 Expanding S operator

S_{oper} can be expanded (in a Taylor series like $e^x = 1 + x + x^2/2! + x^3/3! + \dots$) as

$$S_{oper}\left(t_f, t_i\right) = e^{-i\int_{t_i}^{t_f} H_I^I(t)dt} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t_f} \dots \int_{t_i}^{t_f} T\left\{H_I^I\left(t_1\right) H_I^I\left(t_2\right) \dots H_I^I\left(t_n\right)\right\} dt_1 dt_2 \dots dt_n$$

If the H^I above were numeric functions of time, it wouldn't really matter what order, with respect to the t_n, we carry out the integrations above. However, since they are comprised of operators that act on a ket state to their right, we have to be sure that at each point in the integration, the time-wise earliest operators are acting first.

This means that at each point in the n dimensional space each axis is a different t_n, the H_I^I dependent on the earliest time of the t_n should act first, the H_I^I dependent on the next earliest of the t_n should act next. **The e order of the integrand operators is rearranged as we integrate over all t_n dimensions, such taht the operator are time ordered at every point** $(t_1, t_2, t_3, ...)$

Taking our integration limits to infinity in both space and time into the time ordered infinite spacetime S_{oper} , i.e., the **Dyson expansion of the S operator**,

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} T\left\{\mathcal{H}_I^I\left(t_1\right) \mathcal{H}_I^I\left(t_2\right) \dots \mathcal{H}_I^I\left(t_n\right)\right\} d^4x_1 d^4x_2 \dots d^4x_n \tag{6.6.1}$$

Also note the symbols we can use for the terms above as

$$S = \underbrace{I}_{S^{(0)}} \underbrace{-i \int_{-\infty}^{\infty} \mathcal{H}_I^I\left(x_1\right) d^4x_1}_{S^{(1)}} \underbrace{-\frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T\left\{\mathcal{H}_I^I\left(x_1\right) \mathcal{H}_I^I\left(x_2\right)\right\} d^4x_1 d^4x_2}_{S^{(2)}} + \dots = \sum_{n=0}^{\infty} S^{(n)}$$

6.7 Wick's Theorem Applied to Dyson Expansion

Time ordering is cumbersome to handle, because **we can't keep the same order of operators throughout the integration over time**. Fortunately, we can convert S into

non-time ordered form, where the order of operators does not change during integration, via a handy theorem developed by Gian-Carlo Wick.

Wick's theorem converts time ordered products of operators into normal ordered t products of operators and some things called "contractions".

For generic fields A and B (either of which could be ϕ, ψ, A^μ, ϕ, etc) where

$$A = \underbrace{A^+}_{\text{destruc}} + \underbrace{A^-}_{\text{creation}} = A^d + A^c$$

$$B = \underbrace{B^+}_{\text{destruc}} + \underbrace{B^-}_{\text{creation}} = B^d + B^c$$

A contraction is defined as (note the under bracket symbol)

$$\underbrace{A(x_1)\, B}(x_2) = \left[A_1^+(x_1), B^-(x_2)\right]_\mp = \left[A^d(x_1), B^c(x_2)\right]_\mp \quad \text{if } t_2 < t_1 \tag{6.7.1}$$

$$= \pm\left[B^+(x_2), A^-(x_1)\right]_\mp = \pm\left[B^d(x_2), A^c(x_1)\right]_\mp \quad \text{if } t_1 < t_2 \tag{6.7.2}$$

In these two cases, we have two sets of time+location coordinates (t_1, x_1) and (t_2, x_2). The plus sign subscript implies anti-commutation, which is used **is both A and B are fermions.**$\pm$ in front of the relations on the 2 nd row takes a "+" sign for commutation, "-" for anti-commutation. All commutators/anti-commutators are zero unless $A = \phi$ and $B = \phi^\dagger$, or $A = \psi$ and $B = \bar{\psi}$.

Note that (6.7.2) has the same form as the Feynmann propagator. Thus, whenever a contraction of scalar field is non-zero, it is a scalar Feynmann propagator. Similar results hold for Spinor and photon fields. Thus, special cases of contractions (the only non-zero cases) are

$$\underbrace{\phi(x_1)\, \phi^\dagger}(x_2) = \underbrace{\phi^\dagger(x_2)\, \phi}(x_1) = i\Delta_F(x_1 - x_2) \tag{6.7.3}$$

Remember, the propagator is a number because of the commutation relations like $[a(k), a^\dagger(k)] = [b(k), b^\dagger(k)] = 1$.

$$\underbrace{\psi_\alpha(x_1)\, \bar{\psi}}_{\beta}(x_2) = -\underbrace{\bar{\psi}_\beta(x_2)\, \psi}(x_1) = iS_{F\alpha\beta}(x_1 - x_2) \tag{6.7.4}$$

$$\underbrace{A^\mu(x_1)\, \overset{\nu}{A}}(x_2) = iD_F^{\mu\nu}(x_1 - x_2) \tag{6.7.5}$$

6.7.1 Review of normal ordering

Normal ordering consists of placing construction operators to the right-hand side inside a given term.

$$N(ABCD\ldots) = \text{all destruction operators placed to right of all creation operators.}$$

As part of our definition of normal ordering, note that the **switching places of two adjacent fermionic fields gives rise to a sign change**. This can be justified, in part, because fermionic fields obey anti-commutation relations. For example [1], given $[C, D]_+ = 0, CD = -DC$

Normal ordering of terms including contractions:

For B a destruction operator

$$N\left\{\underbrace{A(x_1)\,B^d(x_2)\,C}\,(x_3)\right\} = \pm\underbrace{A(x_1)\,C}\,(x_3)\,B^d(x_2) \tag{6.7.6}$$

where minus sign for C and B^d both fermionic. And for B a creation operator

$$N\left\{\underbrace{A(x_1)\,B^c(x_2)\,C}\,(x_3)\right\} = \pm B^c(x_2)\,\underbrace{A(x_1)\,C}\,(x_3) \tag{6.7.7}$$

Box

We will henceforth use subscripts in place of parenthesis arguments for fields, i.e.,

$$A(x_1) \to A_{x_1} \quad A(x_1)\,B(x_1)\,C(x_1) \to (ABC)_{x_1}$$

$$\psi(x_1)\,\gamma^\mu A_\mu(x_1)\,\bar{\psi}(x_1) \to \left(\psi\gamma^\mu A_\mu\bar{\psi}\right)_{x_1}$$

6.7.2 Wick's theorem

We only state Wick's theorem here and leave the justification to other textbook. The theorem turns a time ordered product into a series of normal ordered terms and contractions, which, as we noted, helps us in calculating interaction probabilities, because we can keep the operators in the same order as we integrate over time.

$$T\{(AB\ldots)_{x_1}\ldots\ldots(AB\ldots)_{x_n}\} = N\{(AB\ldots)_{x_1}\ldots\ldots(AB\ldots)_{x_n}\}$$
$$+N\left\{\underbrace{(AB..}x_1\underbrace{(A}B.)x_2\cdots\right\} + N\left\{\underbrace{(AB..}x_1\underbrace{(AB}.)x_2\cdots\right\} + \ldots$$
$$+N\left\{\overbrace{(\underbrace{ABC...}_{B-B-contraction})x_1(ABC...}^{A-A-contraction})x_2\cdots\right\} + N\left\{\overbrace{(\underbrace{ABC...}_{B-C-contraction})x_1(ABC...}^{A-A-contraction})x_2\cdots\right\} + etc \quad (6.7.8)$$

Note that there are no contractions between operators operating-at the same time. We say there are no "equal times contractions" in Wick's theorem. Equal-time contractions don't play a role in Wick's theorem for QED.

6.8 Wick's Theorem in Words

To get Wick's theorem, we start with a series of operator fields, operating in arbitrary order and set it equal to itself, i.e., $A_1B_2C_3D_4\ldots = A_1B_2C_3D_4\ldots$

On the LHS, we then re-arrange operator fields using commutation/ant-commutation relations such that earlier times are to the right of later times. We herein use the symbol T_c to represent this re-ordering procedure. The final result of the LHS equals the original LHS expression, since at each, we simply substituted equivalent relations for the original pair of adjacent operators.

On the RHS, we re-arrange operator fields using commutation/anti-commutation relations such that destruction are all to the right of creation operators. We herein use the symbol N_c to represent this re-ordering procedure. The final result of the RHS equals the original RHS expression.

The final result of these operations is the same as employing Wick's theorem.

6.9 Comment on Normal Ordering of the Hamiltonian Density

Non-zero commulators (anti-commulators) values are very small, ≈ 0 at macroscopic scales. So from human perspective, all fields effectively commute(anti-commute). So $\mathcal{H}$ could be normal ordered at microscale, but our classical theory formulation would be blind to it & have evolved in non-normal ordered from.

Chapter 7

QED: Quantum Field Interaction Theory Applied to EM

7.1 Dyson-Wick's Expansion/or QED Hamiltonian Density

The Dyson expansion of the S operator is

$$S = I - i\int_{-\infty}^{\infty} \mathcal{H}_I^I(x_1)\, d^4x_1 - \frac{1}{2!}\int_{-\infty}^{\infty} T\left\{\mathcal{H}_I^I(x_1)\,\mathcal{H}_l^I(x_2)\right\} d^4x_1 d^4x_2 + \ldots. \tag{7.1.1}$$

For the interaction Hamiltonian density in (7.1.1) we use the relation discovered to work for RQM, because we have learned that Hamiltonians for RQM expressed in the Schrödinger Picture, as a rule, take the same form for QFT expressed in the Heisenberg Picture. We are working in the Interaction Picture, for which operators (such as the Hamiltonian density) take the same form in the I.P. as in the H.P. So, for electromagnetic interactions between electrons, positrons, and photons, the quantum Hamiltonian density takes the form, where, $A_\mu\gamma^\mu = \not{A}$

$$\mathcal{H}_I^I = -\mathcal{L}_I^I = -e\bar{\psi}A_\mu\gamma^\mu\psi = -e\bar{\psi}\not{A}\psi = -e\left(\bar{\psi}^+ + \bar{\psi}^-\right)\left(A^+ + \bar{A}^-\right)\left(\psi^+ + \psi^-\right) \tag{7.1.2}$$

and

$$S = \underbrace{I}_{S^{(0)}} + \underbrace{ie\int_{-\infty}^{\infty}(\bar{\psi}\not{A}\psi)_{x_1}d^4x_1}_{S^{(1)}} - \underbrace{\frac{1}{2!}e^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\right\} d^4x_1 d^4x_2}_{S^{(2)}} + \ldots \tag{7.1.3}$$

We can approximate (7.1.3) by taking only the first few terms. In this chapter, we only deal with $S^{(0)}, S^{(1)}, S^{(2)}$.

The second term in (7.1.3) has factors operating all at the same time t_1, and so can be considered time ordered. Wick's theorem for this case reduces to

$$T\left\{(AB\ldots)_{x_1}\right\} = N\left\{(AB\ldots)_{x_1}\right\}$$

$$S^{(1)} = ie\int_{-\infty}^{\infty} N\{\bar{\psi}\not{A}\psi\}_{x_1} d^4x_1$$

7.2 Physical Meaning of S(1)

Consider $S^{(1)}$ term on an initial state

$$S^{(1)}|i\rangle = (-i)\int d^4x_1 N\{-e\bar{\psi}\not{A}\psi\}_{x_1}|i\rangle = i\int d^4x_1 N\left\{e\left(\bar{\psi}^+ + \bar{\psi}\right)\left(\not{A}^+ + \not{A}^-\right)\left(\psi^+ + \psi^-\right)\right\}_{x_1}|i\rangle \tag{7.2.1}$$

If we multiply out the factors in (7.2.1) we will have eight different sub-terms contributing to the $S^{(1)}$ term in S. We will label these sub-terms as $S_j^{(1)}$ where $j = 1, 2, \ldots, 8$. For example

$$S_1^{(1)}|e^-_{p_1,q}, e^+_{p_2,r_2}\rangle = ie\int d^4x_1 N\left\{\bar{\psi}^+ A_\mu^- \gamma^\mu \psi^+\right\}_{x_1}\left|e^-_{p_1,r_1}, e^+_{p_2,r_2}\right\rangle = ie\int d^4x_1\left\{A_\mu^- \bar{\psi}^+\gamma^\mu\psi^+\right\}_{x_1}\left|e^-_{p_1,r_1}, e^+_{p_2,r_2}\right\rangle$$

Substituting the expressions for the photon and spinor fields, we have

$$S_1^{(1)}\left|e^-_{p_1,n}, e^+_{p_2,r_2}\right\rangle = ie\int d^4x_1\left(\sum_{s,k}\sqrt{\frac{1}{2V\omega_k}}\varepsilon_{\mu,s}(\mathbf{k})a_s^\dagger(\mathbf{k})e^{ikx_1}\right)\left(\sum_{r',p'}\sqrt{\frac{m}{VE_{p'}}}dr'\left(p'\right)\bar{v}_{r'}\left(p'\right)e^{-ip'x_1}\right)\gamma^\mu\times$$

$$\left(\sum_{r'',p''}\sqrt{\frac{m}{VE_{p''}}}c_{r''}\left(p''\right)u_{r''}\left(p''^e\right)e^{-p''\gamma_1''}\right)\left|e^-_{p_1,r_2}, e^+_{p_2,r_2}\right\rangle$$

Destruction operators $d_{r'}$ and $c_{r''}$ will destroy the ket (i.e., make it equal to zero) for all terms in the sum except when i) $p' = p_2$ and $r' = r_2$, and when in $r'' = r_1$. Those will reduce the ket to the vacuum state by destroying the electron and positron we started out with. Thus, we have

$$S_1^{(1)}|e^-_{p_1,r_1}, e^+_{p_2,r_2}\rangle =$$

$$ie\int d^4x_1\left\{\left(\sum_{s,k}\sqrt{\frac{1}{2V\omega_k}}\varepsilon_{\mu,s}(\mathbf{k})a_s^\dagger(\mathbf{k})e^{ikx_1}\right)\frac{m}{V}\sqrt{\frac{1}{E_{p_1}E_{p2}}}\bar{v}_{r_2}\left(p_2\right)e^{-ip_2x_1}\gamma^\mu u_{r_1}\left(p_1\right)e^{-ip_1x_1}\right\}|0\rangle$$

Each term of the remaining sum above creates a photon with different momentum and polarization states. So

$$S_1^{(1)}|e^-_{\mathrm{p}_1,r_1}, e^+_{\mathrm{p}_2,r_2}\Big) =$$

$$ie\int d^4x_1 \left\{\sum_{s,k}\sqrt{\frac{1}{2V\omega_k}}\varepsilon_{\mu,s}(\mathbf{k})e^{ikx_1}\frac{m}{V}\sqrt{\frac{1}{E_{p_1}E_{p_2}}}\bar{v}_{r_2}\left(\mathbf{p}_2\right)e^{-ip_2x_1}\gamma^\mu u_{r_1}\left(\mathbf{p}_1\right)e^{-ip_1x_1}\right\}|\gamma_{\mathbf{k},s}\rangle$$

Suppose $|\gamma_{\mathbf{k}_1,s_1}\rangle$ is our final state of a single photon. For this final state, note that

$$\left\langle \gamma_{\mathbf{k}_1,s_1}\left|S_1^{(1)}\right|e^-_{\mathbf{p}_1,r_1}, e^+_{\mathrm{p}_2,r_2}\right\rangle = S^{(1)}_{1,fi}$$

which is the transition amplitude for the following Feynman diagram

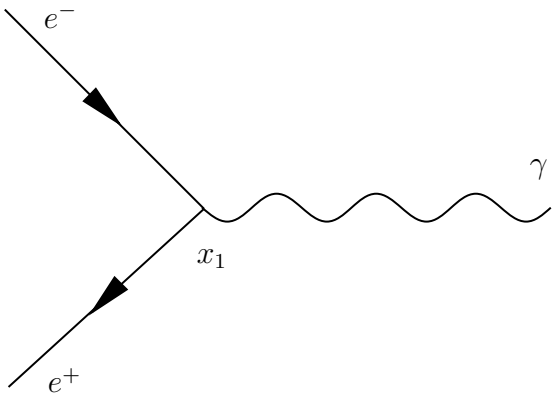


Figure 7.1: Single vertex interaction

From equations above, where all terms having different bra and ket states drop out,

$$S^{(1)}_{1,fi} = ie\int d^4x_1\left\{\sqrt{\frac{1}{2V\omega_{k_1}}}\varepsilon_{\mu,s_1}\left(\mathbf{k}_1\right)e^{ik_1x_1}\frac{m}{V}\sqrt{\frac{1}{E_{p_1}E_{p_2}}}\bar{v}_{r_2}\left(\mathbf{p}_2\right)e^{-ip_2x_1}\gamma^\mu u_{r_1}\left(\mathbf{p}_1\right)e^{-ip_1x_1}\right\}\langle\gamma\|\gamma\rangle$$

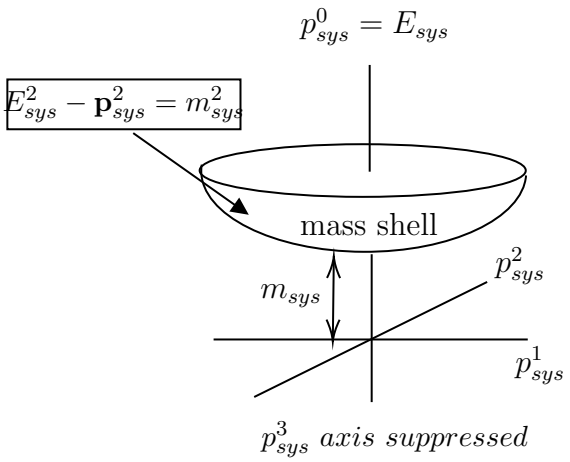
$$= ie\frac{m}{\sqrt{2V^3}}\sqrt{\frac{1}{\omega_{\mathrm{k}_1}E_{\mathrm{p}_1}E_{\mathrm{p}_2}}}\varepsilon_{\mu,s_1}\left(\mathrm{k}_1\right)\bar{v}_{\mathrm{r}_2}\left(\mathrm{p}_2\right)\gamma^\mu u_{r_1}\left(\mathrm{p}_1\right)\underbrace{\int e^{i(k_1-p_2-p_1)x_1}d^4x_1}_{(2\pi)^4\delta^{(4)}(k_1-p_2-p_1)}$$

$$= ie(2\pi)^4\delta^{(4)}\left(k_1-p_2-p_1\right)\sqrt{\frac{1}{2V\omega_1}}\sqrt{\frac{m}{V_{\mathrm{p}}}}\sqrt{\frac{m}{VE_{p_2}}}\varepsilon_{\mu,s_1}\left(\mathbf{k}_1\right)\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^\mu u_{\mathrm{r}_1}\left(\mathbf{p}_1\right) \qquad (7.2.2)$$

The Dirac delta function arising in our calculation ensures that the outgoing 4-momentum of the final state photon equals the incoming total 4-momentum of the two initial state particles. This, we will see, is a general principle that holds for all transition amplitudes, throughout

QFT. Outgoing 4-momentum for any interaction vertex (three particles interacting at a point in a Feynman diagram) equals incoming 4-momentum.

The interaction represented mathematically above and pictorially by Fig. (7.1) is not physically viable and does not occur.



The schematic above shows the following relation:(i.e. a shell upon which real particle energies and 3-momenta values must lie.)

$$E^2_{sys} - \mathbf{p}^2_{sys} = m^2_{sys} \tag{7.2.3}$$

For a system of particles, we determine an invariant mass m_{sys}. The surface in the figure is called the **mass shell**. Real particles must be "on the mass shell". Virtual particles can be, and generally are "off shell". $m_{sys} = 0$ for photons, so the photon mass shell touches the origin.

From the delta function in (7.2.2), we know that the outgoing 4-momentum for the photon of Fig(7.1) equals the incoming total 4-momentum of the electron and positron:

$$p^\mu_i = \begin{pmatrix} E_1 + E_2 \\ \mathbf{p}_1 + \mathbf{p}_2 \end{pmatrix} = p^\mu_f = \begin{pmatrix} \omega_{k_1} \\ \mathbf{k}_1 \end{pmatrix}$$

For initial state

$$p^\mu_i p_{i\mu} = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2) \cdot (\mathbf{p}_1 + \mathbf{p}_2) = m^2_{sys} \neq 0$$

For final state

$$p^\mu_f p_{f\mu} = \underbrace{(E_1 + E_2)^2}_{\omega_{k_1}} - \underbrace{(\mathrm{p}_1 + \mathrm{p}_2)}_{\mathrm{k}_1} \cdot \underbrace{(\mathrm{p}_1 + \mathrm{p}_2)}_{\mathrm{k}_1} = m^2_{\mathrm{sys}} \neq 0$$

which must equal zero for a photon. That it doesn't equal zero means we can't produce a real photon.

$$S_{fi} = \left\langle \gamma \left| S_1^{(1)} + \underbrace{S_2^{(1)} + \ldots + S_8^{(1)}}_{\text{all yield zero}} \right| e^-_{\mathbf{p}_1,n}, e^+_{\mathbf{p}_2,r_2} \right\rangle = \underbrace{\langle\gamma|}_{\text{on-shell}} \underbrace{S_1^{(1)}|e^-_{p_1,n}, e^+_{\mathrm{p}_2,r_2}\rangle}_{\text{off-shell photon}} = 0$$

because the only ket left is an off-shell photon with $k^\mu = p_1^\mu + p_2^\mu$, and that is a different state from, and thus orthogonal to, any real final state photon, which cannot have this value for k^μ. Thus, the transition amplitude for Fig. (7.1) is zero. Similar logic for all single vertex interactions means we can simply ignore $S^{(1)}$ from here on.

7.3 Physical Meaning of S(2)

$S^{(2)}$ **will represent two vertex interactions.** The first term of $S^{(2)}$ is $S_A^{(2)}$:

$$S_A^{(2)} = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\right\}$$

This represents two independent processes like $S^{(1)}$. The two processes do not interact with one another and each behaves as if the other did not exist. There is no virtual particle (Feynman propagator) linking them. Think of two separate single vertex Feynman diagrams. Neither of these can occur, so $S_A^{(2)}$ does not represent a real physical process and is ignored in QFT.

7.3.1 The photon propagator term

Consider the second term of $S^{(2)}$ acting on an initial state

$$S_B^{(2)}|i\rangle = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\right\}|i\rangle \tag{7.3.1}$$

The only initial states (7.3.1) could destroy would be electron/positron states; and the only final states it could create would be electron/positron states. **We call these types of interactions four external lepton interactions.**

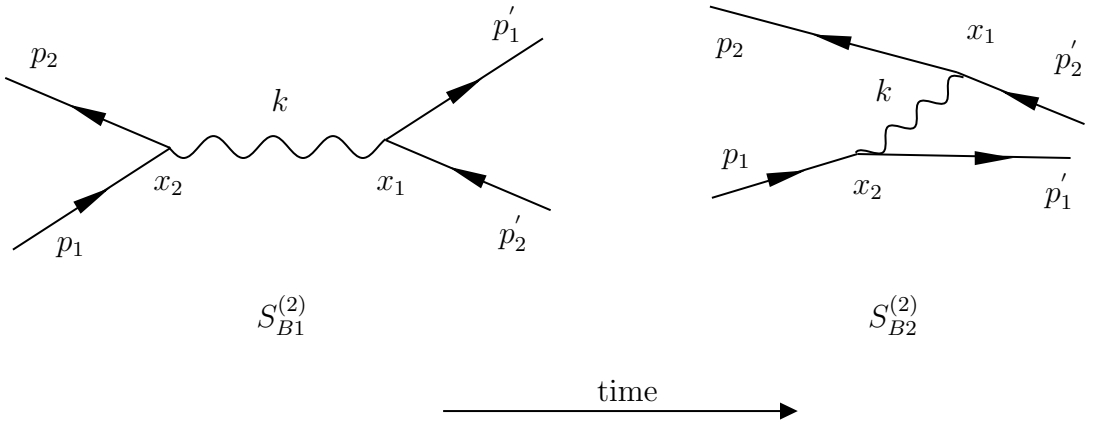


Figure 7.2: Bhabha scattering can occur in two ways

In Fig.(7.2), both scenarios have the same incoming and outgoing particle states, which are all that we can measure. The internal virtual particle interaction is not measurable so there is no way we can tell which of the two interactions gave us the Bhabha scattering.Bhabha scattering actually entails both types of interaction, i) an annihilation of e^- with an e^+ followed by a creation of the same two types of particles, and ii) one of the incoming particles emitting a virtual photon which is then absorbed by the other particle. i.e., the same incoming momenta and spins for both and the same outgoing momenta and spins for both.

First type of Bhabha Scattering

Consider the first type of Bhabha scattering with the S operator acting on the incoming state,

$$S_B^{(2)} \left|e^-_{\mathrm{p_1,r_1}}, e^+_{\mathrm{p_2,r_2}}\right\rangle = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 N \left\{ (\bar{\psi}\underbrace{\not{A}\psi)_{x_1}(\bar{\psi}\not{A}}\psi)_{x_2} \right\} \left|e^-_{\mathrm{p_1,r_1}}, e^+_{\mathrm{p_2,r_2}}\right\rangle \tag{7.3.2}$$

At x_2, the ψ^+ part of ψ_{x_2} would destroy the ket electron; the $\bar{\psi}^+$ part of $\bar{\psi}_{x_2}$ would destroy the ket positron; and we would be left with the vacuum $|0\rangle$. The propagator would then create a virtual photon at x_2 and propagate it to x_1 where it would be destroyed. Then, the ψ^- part of ψ_{x_1} would create a positron at x_1; and the $\bar{\psi}$ part of $\bar{\psi}_{x_1}$ would create an electron there.

$$S_{B1}^{(2)} = \frac{-e^2}{2} \left\langle e^-_{p_1',r_1'}, e^+_{p_2',r_2'}\right| \iint d^4x_1 d^4x_2 \left((\bar{\psi}^-\underbrace{\not{A}\psi^-)_{x_1}(\bar{\psi}^+\not{A}}\psi^+)_{x_2} +\right.$$

$$N\left\{(\bar{\psi}^{+}\not{A}\psi^{+})_{x_1}(\bar{\psi}^{-}\not{A}\psi^{-})_{x_2}\right\}\Big)\left|e^{-}_{\mathfrak{p}_1,r_1},e^{+}_{\mathfrak{p}_2,r_2}\right\rangle$$

in which we can think of the second term above as annihilation at x_1 and creation at x_2 instead of the other way around. When we normal order the second term, $\psi^{+}(x_1)$ is switched once with $\bar{\psi}^{-}$ (x_2) introducing a minus sign, then switch order with $\psi^{-}(x_2)$, introducing a second minus sign and resulting in no total sign change. The propagator is just a number, so it can be moved anywhere without effect (though care has to be taken with keeping the correct spinor multiplication order).

Carrying out similar switching for $\bar{\psi}^{+}(x_1)$ with $\bar{\psi}^{-}(x_2)$ and then $\psi^{-}(x_2)$, we end up with

$$S^{(2)}_{B1}=\frac{-e^2}{2}\left\langle e^{-}_{p'_1,r'_1},e^{+}_{p'_2,r'_2}\right|\iint d^4x_1d^4x_2\left((\bar{\psi}^{-}\not{A}\psi^{-})_{x_1}(\bar{\psi}^{+}\not{A}\psi^{+})_{x_2}+(\bar{\psi}^{-}\not{A}\psi^{-})_{x_2}(\bar{\psi}^{+}\not{A}\psi^{+})_{x_1}\right)\left|\, e^{-}_{p_1,r_1},e^{+}_{p_2,r_2}\right\rangle$$

$$=-e^2\left\langle e^{-}_{p'_1,r'_1},e^{+}_{p'_2,r'_2}\right|\iint d^4x_1d^4x_2(\bar{\psi}^{-}\not{A}\psi^{-})_{x_1}(\bar{\psi}^{+}\not{A}\psi^{+})_{x_2}\left|e^{-}_{\mathfrak{p}_1,r_1},e^{+}_{p_2,r_2}\right\rangle$$

where in second line we simply exchanged dummy variables in the second term of the first line to get the second line. Substituting the field equation solutions for the operators at x_2 above, we have

$$S^{(2)}_{B1}=-e^2\left\langle e^{-}_{\mathrm{p}'_1,r'_1},e^{+}_{\mathrm{p}'_2,r_2}\right|\iint d^4x_1d^4x_2\left(\bar{\psi}^{-}\gamma^{\mu}\psi^{-}\right)_{x_1}iD_{F\mu\nu}\left(x_1-x_2\right)\times$$

$$\left(\sum_{r',p'}\sqrt{\frac{m}{VE_{p'}}}d_{r'}\left(p'\right)\bar{v}_{r'}\left(p'\right)e^{-ip'x_2}\right)\gamma^{v}\left(\sum_{r''p''},\sqrt{\frac{m}{VE_{p''}}}c_{r''}\left(\mathbf{p}''\right)u_{r''}\left(\mathbf{p}''\right)e^{-ip''\mathbf{x}_2}\right)\left|e^{-}_{\mathfrak{p}_1,r_1},e^{+}_{\mathfrak{p}_2,r_2}\right\rangle$$

The terms do match up turn the ket into the vacuum. So the equation above becomes

$$S^{(2)}_{B1}=-e^2\left\langle e^{-}_{\mathrm{p}'_1,r'_1},e^{+}_{\mathrm{p}'_2,r'_2}\right|\iint d^4x_1d^4x_2\left(\bar{\psi}^{-}\gamma^{\mu}\psi^{-}\right)_{x_1}iD_{F\mu\nu}\left(x_1-x_2\right)\times$$

$$\frac{m}{V}\sqrt{\frac{1}{E_{\mathrm{p}_1}E_{\mathrm{p}_2}}}\bar{v}_{r_2}\left(\mathrm{p}_2\right)\gamma^{\nu}u_{r_1}\left(\mathrm{p}_1\right)e^{-ip_2x_2}e^{-ip_1x_2}|0\rangle$$

Substituting the propagator relation we derived at the end of Chap 4 and rearranging,

$$S^{(2)}_{B1}=-e^2\left(e_{\mathrm{p}'\mathrm{i},\mathrm{r}'},e^{+}_{\mathrm{p}'_2,\mathrm{r}'_2}\right|\int\left(\bar{\psi}^{-}\gamma^{\mu}\psi^{-}\right)_{x_1}\times\left(\int\left(\frac{-ig_{\mu\nu}}{(2\pi)^4}\int\frac{e^{-ik(x_1-x_2)}}{k^2+i\varepsilon}d^4k\right)\frac{m}{V}\right.$$

$$\left.\sqrt{\frac{1}{E_{\mathrm{p}_1}E_{\mathrm{p}_2}}}\bar{v}_{r_2}\left(\mathrm{p}_2\right)\gamma^{v}u_{r_1}\left(\mathbf{p}_1\right)e^{-ip_2x_2}e^{-ip_1x_2}d^4x_2\right)d^4x_1|0\rangle$$

$$=e^2\left\langle e^{-}_{\mathrm{p}'_1,r'_1},e^{+}_{\mathrm{p}'_2,r'_2}\right|\int\left(\bar{\psi}^{-}\gamma^{\mu}\psi^{-}\right)_{x_1}ig_{\mu\nu}\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^{\nu}u_{r_1}\left(\mathbf{p}_1\right)\times$$

$$\frac{m}{V}\sqrt{\frac{1}{E_{p_1}E_{p_2}}}\left(\int \frac{1}{k^2+i\varepsilon}e^{-ikx_1}\frac{1}{(2\pi)^4}\underbrace{\left(\int e^{ikx_2}e^{-ip_2x_2}e^{-ip_1x_2}d^4x_2\right)}_{(2\pi)^4\delta^{(4)}(k-p_2-p_1)}d^4k\right)d^4x_1|0\rangle$$

The delta relation that results above tells us that $k = p_1 + p_2$, the virtual photon 4-momentum equals th sum of the incoming particles 4-momenta. Again, we see conservation of energy and 3-momentum at a vertex. **And again, we see that the photon particle is off-shell, i.e.,**$k_\mu k^\mu = m_{sys}^2 \neq 0$**.**

The delta relation picks the single value $k = p_1 + p_2$ out of the integration over k. So now,

$$S_{B1}^{(2)} = e^2\left\langle e^-_{\mathrm{p}_1',r_1'}, e^+_{\mathrm{p}_2',\mathrm{r}_2'}\right|\frac{m}{V}\sqrt{\frac{1}{E_{\mathrm{p}_1}E_{\mathrm{p}_2}}}\left(\int\left(\bar{\psi}^-\gamma^\mu\psi^-\right)_{x_1}\frac{e^{-i(p_2+p_1)x_1}}{(p_2+p_1)^2+i\varepsilon}d^4x_1\right)\times$$

$$ig_{\mu\nu}\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^\nu u_{r_1}\left(\mathbf{p}_1\right)|0\rangle$$

Substituting the relations for the spinor creation field operators at x_1 yields

$$S_{B1}^{(2)} = e^2\left(\frac{m}{V}\right)^2\sqrt{\frac{1}{E_{\mathrm{p}_1}E_{\mathrm{p}_2}}}\sqrt{\frac{1}{E_{\mathrm{p}_1'}E_{\mathrm{p}_2'}}}\left\langle e^-_{\mathrm{p}_1',r_1'}, e^+_{\mathrm{p}_2',\mathrm{r}_2'}\right|\left(\int e^{-i(p_2+p_1)x_1}e^{i\left(p_1'+p_2'\right)x_1}d^4x_1\right)\times$$

$$\bar{u}_{r_1'}\left(\mathbf{p}_1'\right)\gamma^\mu v_{r_2'}\left(\mathbf{p}_2'\right)\frac{ig_{\mu v}}{(p_2+p_1)^2+i\varepsilon}\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^v u_{r_1}\left(\mathbf{p}_1\right)\left|e^-_{p_1',r_1'}, e^+_{p_2',r_2'}\right\rangle$$

The integral over x_1 is another delta function telling us that $p_2' + p_1' = p_2 + p_1$, the total outgoing 4 -momentum equals the total incoming 4 -momentum, which, as we discovered earlier, equals the 4-momentum of the virtual photon propagator. Energy and 3 -momentum are conserved at every vertex. And the final, real particle state is on-shell.

The result of all this, where we note that the fraction factor in the equation above with the $i\varepsilon$ as part of the denominator is simply (up to a sign) the Feynman propagator in momentum space, is

$$\underbrace{S^{(2)}_{B1}}_{\text{transition amplitude}} = \sqrt{\frac{m}{VE_{\mathrm{p}_1}}}\sqrt{\frac{m}{VE_{\mathrm{p}_2}}}\sqrt{\frac{m}{VE_{\mathrm{p}'_1}}}\sqrt{\frac{m}{VE_{\mathrm{p}'_2}}}(2\pi)^4\delta^{(4)}\left(p_1+p_2-(p'_1+p'_2)\right)$$

$$\times \underbrace{\left(-e^2\right)\bar{u}_{r'_1}\left(\mathbf{p}'_1\right)\gamma^\mu v_{r'_2}\left(\mathbf{p}'_2\right) iD_{F\mu v}\left(k=p_1+p_2\right)\bar{v}_{r_2}\left(p_2\right)\gamma^v u_{r_1}\left(p_1\right)}_{\text{Feynman amplitude } \mathcal{M}^{(2)}_{B1}}$$

Note the definition of the Feynman amplitude for this interaction $\mathcal{M}^{(2)}_{B1}$. Using that, we have

$$S^{(2)}_{B1} = \left(\prod_{\mathbf{p}}^{\text{all ext fermions}} \sqrt{\frac{m}{VE_{\mathbf{p}}}}\right)(2\pi)^4\delta^{(4)}\left(p_1+p_2-(p'_1+p'_2)\right)\mathcal{M}^{(2)}_{B1} \tag{7.3.3}$$

The second type of Bhabha scattering

Note in Fig.(7.2), that on the RHS, either x_2 or x_1 could occur first, and our math takes care of both cases automatically. We start with the same fundamental relation we had for the first type of Bhabha scattering,

$$S^{(2)}_B\left|e^-_{\mathrm{p}_1,r_1},e^+_{\mathrm{p}_2,r_2}\right\rangle = -\frac{e^2}{2!}\iint d^4x_1d^4x_2N\left\{(\bar{\psi}\underbrace{A\psi})_{x_1}(\bar{\psi}\underbrace{A\psi})_{x_2}\right\}\left|e^-_{\mathrm{p}_1,r_1},e^+_{\mathrm{p}_2,r_2}\right\rangle \tag{7.3.4}$$

but this time, we pick the creation and destruction operators corresponding to the second type of Bhabha scattering. That is, we have

$$S^{(2)}_{B2} = \frac{-e^2}{2}\left\langle e^-_{p'_1,r'_1},e^+_{p'_2,r'_2}\right|\times$$

$$\iint d^4x_1d^4x_2N\left\{(\bar{\psi}^+\underbrace{A_\mu\gamma^\mu\psi^-})_{x_1}(\bar{\psi}^-\underbrace{A_v}\gamma^v\psi^+)_{x_2}+(\bar{\psi}\underbrace{A_\mu\gamma^\mu\psi^+})_{x_1}(\bar{\psi}^+\underbrace{A_v}\gamma^v\psi^-)_{x_2}\right\}\left|e^-_{\mathbf{p}_1,r_1},e^+_{\mathbf{p}_2,r_2}\right\rangle$$

which, if **we switch dummy integration variables again, the first and second terms above are equivalent.** That eliminates the 1/2 factor and leaves (where we now show spinor indices).

$$S^{(2)}_{B2} = -e^2\langle e^-_{\mathrm{p}'_1,r'_1},e^+_{\mathrm{p}'_2,r'_2}|\iint d^4x_1d^4x_2N\{(\bar{\psi}^+_\alpha\underbrace{A_\mu\gamma^\mu_{\alpha\beta}\psi^-_\beta})_{x_1}(\bar{\psi}^-_\delta\underbrace{A_\nu}\gamma^\nu_{\delta\eta}\psi^+_\eta)_{x_2}\}|e^-_{\mathrm{p}_1,r_1},e^+_{\mathrm{p}_2,r_2}\rangle \tag{7.3.5}$$

This can be visualized as the incoming electron destroyed at x_2 with the outgoing electron created a x_2 and a photon propagator starting at x_2. The incoming positron is destroyed at

x_1 with the outgoing positron positron created at x_1 and the photon propagator ending at x_1. The reverse situation, where x_1 and x_2 are interchanged has been included by the factor of two enveloped.

Normal ordering (7.3.5), where we now see the value of writing out spinor indices, give us

$$S_{B2}^{(2)} = e^2 \langle e^-_{p_1',r_1'}, e^+_{p_2',r_2'}| \iint d^4x_1 d^4x_2 (\bar{\psi}^-_\delta)_{x_2} (A_\mu \gamma^\mu_{\alpha\beta} \psi^-_\beta)_{x_1} (\bar{\psi}^+_\alpha)_{x_1} (A_v \gamma^v_{\delta\eta} \psi^+_\eta)_{x_2}) |e^-_{p_1,r_1}, e^+_{p_2,r_2}\rangle \quad (7.3.6)$$

Note carefully that we have to put (7.3.6) not simply in any normal order, but **in the same normal order as (7.3.5). That is, both transition sub amplitudes must have, in order from the right side moving leftward, operators performing e^- destruction, e^+ destruction, e^- creation, and e^+ creation.** We carry out similar steps to get

$$S_{B2}^{(2)} = e^2 \frac{m^2}{V^2} \sqrt{\frac{1}{E_{\mathrm{p}1} E_{\mathrm{p}_2} E_{\mathrm{p}_1'} E_{\mathrm{p}_2'}}} \bar{v}_{r_2}(\mathrm{p}_2) \gamma^\mu v_{r_2'}(\mathbf{p}_2') \bar{u}_{\mathrm{r}'}(\mathbf{p}_\mathrm{I}') \gamma^v u_{\mathrm{r}_1}(\mathbf{p}_1) \times$$
$$\left(\frac{-ig_{\mu\nu}}{(p_1 - p_1')^2 + i\varepsilon}\right) \underbrace{\left(\int e^{-i(p_1 - p_1')x_1} e^{ip_2' x_1} e^{-ip_2 x_1} d^4x_1\right)}_{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - (p_1' + p_2'))}$$

And thus, finally,

$$S_{B2}^{(2)} = \left(\prod_{\mathbf{p}}^{\text{allext fermions}} \sqrt{\frac{m}{VE_\mathrm{p}}}\right) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - (p_1' + p_2')) \mathcal{M}_{B2}^{(2)} \quad (7.3.7)$$

where

$$\mathcal{M}_{B2}^{(2)} = e^2 \bar{v}_{r_2}(\mathbf{p}_2) \gamma^\mu v_{r_2'}(\mathbf{p}_2') iD_{\mu v}(p_2' - p_2) \bar{u}_{r_1'}(\mathbf{p}_1') \gamma^v u_{r_1}(\mathbf{p}_1)$$

The total transition amplitude for 2nd order (n=2) Bhabha scatering is

$$S_{Bhabba} = S_{B1}^{(2)} + S_{B2}^{(2)} = \left(\prod_{\mathrm{p}}^{\text{all fermions}} \frac{m}{\sqrt{\frac{m}{VE_\mathrm{p}}}}\right) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - (p_1' + p_2')) \left(\mathcal{M}_{B1}^{(2)} + \mathcal{M}_{B2}^{(2)}\right) \quad (7.3.8)$$

7.3.2 Compton Scattering

Consider the third and fourth terms in $S^{(2)}$, acting on an initial state

$$S_C^{(2)}|i\rangle = -\frac{1}{2!} e^2 \iint d^4x_1 d^4x_2 (N\{(\bar{\psi}\not{A}\psi)_{x_1} (\bar{\psi}\not{A}\psi)_{x_2}\} + N\{(\bar{\psi}\not{A}\psi)_{x_1} (\bar{\psi}\not{A}\psi)_{x_2}\})|i\rangle \quad (7.3.9)$$

The first term in (7.3.9) actually equals the second term as we can see from the following Box, and thus we can re-express (7.3.9) as

$$S_C^{(2)}|i\rangle = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 N\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\}|i\rangle \tag{7.3.10}$$

The contraction here is a fermion propagator and the incoming particles that could be destroyed by (7.3.10) could comprise an electron and a photon. Effectively, the electron and photon could scatter off one another as in Fig. (7.3). This is called **Compton scattering**. Note from Fig. (7.3) than occur in two different ways, i.e., have the same real particles in and out.

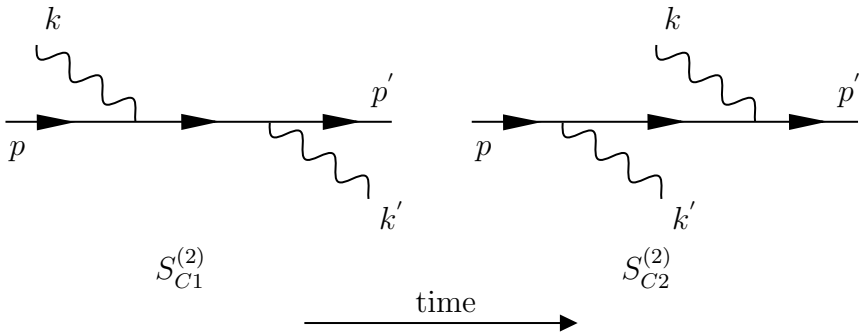


Figure 7.3: Compton Scattering Can Occur in Two Ways

The LH vertex on each side of Fig.(7.3) with x_2, and the RH with x_1, as it would be easier to track the analysis. The interaction could occur in either of the ways we label above as $S_{C1}^{(2)}$ or $S_{C2}^{(2)}$. That is, either i) an electron could absorb a photon (equivalent to an electron and a photon being destroyed and a virtual electron being created at x_2) and later emit a photon(equivalent to the virtual electron being destroyed and both a real electron and a real photon being created at x_1); or ii) an electron could emit a photon (equivalent to an electron being destroyed and a real photon along with a virtual electron being created at x_2), and later absorbed a photo (equivalent to the virtual electron and a real photon being destroyed while a real electron is create at x_1).

Note that only the $S_C^{(2)}$ terms of all the $n = 2$ terms will result in destruction of an initial electron and photon ket. The S matrix transition amplitude for second order Compton

scattering is thus (with incoming particles unprimed, outgoing primed)

$$S_{\text{Compton}} = \langle f|S|i\rangle = \left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}'r'}\right| \left(-e^2\right)\iint d^4x_1d^4x_2N\int\left(\left(\bar{\psi}\underbrace{\not{A}\psi}\right)_{x_1}\left(\bar{\psi}\underbrace{\not{A}\psi}\right)_{x_2}\right)\left|e^-_{\mathbf{p},s},\gamma_{\mathbf{k},r}\right\rangle \tag{7.3.11}$$

$$= -e^2\left\langle e^-_{\mathbf{p}'s'},\gamma_{\mathbf{k}',r'}\right| \iint d^4x_1d^4x_2N\left\{\left(\bar{\psi}^+ + \bar{\psi}^-\right)_{x_1}\left(\not{A}^+ + \not{A}^-\right)_{x_1}\times \right. \tag{7.3.12}$$

$$\left.\left(iS_F\left(x_1-x_2\right)\right)\left(\not{A}^+ + \not{A}^-\right)_{x_2}\left(\psi^+ + \psi^-\right)_{x_2}\right\}\left|e^-_{\mathbf{p},s},\gamma_{\mathbf{k},r}\right\rangle$$

After the operators raise and lower the ket, only two terms in the equation above remain:

$$\begin{aligned} S_{\text{Compton}} = -e^2\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right| \iint d^4x_1d^4x_2 \underbrace{N\left\{\bar{\psi}^-_{x_1}\not{A}^-_{x_1}\left(iS_F\left(x_1-x_2\right)\right)\not{A}^+_{x_2}\psi^+_{x_2}\right.}_{S^{(2)}_{C1}\text{ term}} \\ \left.+\underbrace{\bar{\psi}^-_{x_1}\not{A}^+_{x_1}\left(iS_F\left(x_1-x_2\right)\right)\not{A}^-_{x_2}\psi^+_{x_2}}_{\text{will result in } S^{(2)}_{C_2}\text{ term}}\right\}\left|e^-_{\mathbf{p},s},\gamma_{\mathbf{k},r}\right\rangle \end{aligned} \tag{7.3.13}$$

The full second order Compton transition amplitude, including both cases of Fig.(7.3), is

$$\begin{aligned} S_{\text{Compton}} = &\left(\prod_{\mathbf{p}''}^{\text{all fermions}}\sqrt{\frac{m}{VE_{\mathrm{p}''}}}\right)\left(\prod_{\mathbf{p}''}^{\text{all bosons}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}''}}}\right)(2\pi)^4\times \\ &\delta^{(4)}\left(p'+k'-p-k\right)\left(\mathcal{M}^{(2)}_{C1}+\mathcal{M}^{(2)}_{C2}\right) \end{aligned} \tag{7.3.14}$$

where

$$\mathcal{M}^{(2)}_{C1} = -e^2\bar{u}_{s',\alpha}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu_{\alpha\beta}iS_{F\beta\delta}(q=p+k)\varepsilon_{\nu,r}(\mathbf{k})\gamma^\nu_{\delta\eta}u_{s,\eta}(\mathbf{p})$$

$$\mathcal{M}^{(2)}_{C2} = -e^2\bar{u}_{s',\alpha}\left(\mathbf{p}'\right)\varepsilon_{\mu,r}(\mathbf{k})\gamma^\mu_{\alpha\beta}iS_{F\beta\delta}\left(q=p-k'\right)\varepsilon_{\nu,r'}\left(\mathbf{k}'\right)\gamma^\nu_{\delta\eta}u_{s,\eta}(\mathbf{p})$$

Box

Consider normal ordering the commutator of two boson fields which do not commute,such as

$$N\underbrace{\left[\phi^{+}\left(x_{1}\right),\phi^{\dagger-}\left(x_{2}\right)\right]}_{\neq 0}=N\left\{\phi^{+}\left(x_{1}\right)\phi^{\dagger-}\left(x_{2}\right)-\phi^{\dagger-}\left(x_{2}\right)\phi^{+}\left(x_{1}\right)\right\}=\phi^{\dagger-}\left(x_{2}\right)\phi^{+}\left(x_{1}\right)-\phi^{\dagger-}\left(x_{2}\right)\phi^{+}\left(x_{1}\right)=0$$

Similarly, the normal ordering of an anti-commutator of two fermion fields, which do not anti-commute, equals zero as well. **Thus, we can simply exchange order of adjacent bosons inside any normal ordered product, even if they don't commute and even if they are part of a contraction.** For example

$$N\left\{\phi\left(x_{1}\right)\phi^{\dagger}\left(x_{2}\right)\psi\left(x_{3}\right)\bar{\psi}\left(x_{4}\right)\ldots\right\}$$

$$=N\left\{\left(\phi^{\dagger}\left(x_{2}\right)\phi\left(x_{1}\right)+\underbrace{\left[\phi\left(x_{1}\right),\phi^{\dagger}\left(x_{2}\right)\right]}_{\text{drops out}}\right)\psi\left(x_{3}\right)\bar{\psi}\left(x_{4}\right)\ldots\right\}$$

$$=N\left\{\phi^{\dagger}\left(x_{2}\right)\phi\left(x_{1}\right)\psi\left(x_{3}\right)\bar{\psi}\left(x_{4}\right)\ldots\right\}$$

Similarly, we can exchange the order of adjacent fermion fields, along with a sign change, inside any normal ordered product. Fermion and adjacent bosons always commute.With the above results, we can re-order factors in the first term of $Sc^{(2)}$

$$N\{(\underbrace{\bar{\psi}_{\alpha}A_{\mu}\gamma^{\mu}_{\alpha\beta}\psi_{\beta}}_{)x_1}(\bar{\psi}_{\delta}A_{\nu}\underbrace{\gamma^{\nu}_{\delta\eta}\psi_{\eta}}_{)x_2}\}=N\{(\bar{\psi}_{\delta})_{x_2}(\underbrace{\bar{\psi}_{\alpha}A_{\mu}\gamma^{\mu}_{\alpha\beta}\psi_{\beta}}_{)x_1}(A_{\nu}\underbrace{\gamma^{\nu}_{\delta\eta}\psi_{\eta}}_{)x_2}\}=$$

$$N\left\{\left(\bar{\psi}_{\delta}A_{v}\gamma^{v}_{\delta\eta}\right)_{x_2}(\underbrace{\bar{\psi}_{\alpha}A_{\mu}\gamma^{\mu}_{\alpha\beta}\psi_{\beta}}_{)x_1}\underbrace{(\psi_{\eta})}_{x_2}\right\}=N\left\{\left(\bar{\psi}_{\delta}A_{v}\gamma^{v}_{\delta\eta}\right)_{x_2}\underbrace{(\bar{\psi}_{\alpha})}_{x_1}\underbrace{(\psi_{\eta})}_{x_2}\left(-A_{\mu}\gamma^{\mu}_{\alpha\beta}\psi_{\beta}\right)_{x_1}\right\}$$

$$=N\left\{(\bar{\psi}_{\delta}A_{v}\gamma^{\nu}_{\delta\eta}\underbrace{\psi_{\eta})_{x_2}(\bar{\psi}}_{\alpha}A_{\mu}\gamma^{\mu}_{\alpha\beta}\psi_{\beta})_{x_1}\right\}$$

the x_1 and x_2 are dummy variables so we can make the exchange $x_1 \leftrightarrow x_2$

7.3.3 Returning to the Photon Propagator Term $S_B^{(2)}$

Similar to Compton and Bhabha scattering, there are two ways for Moller scattering (see the figure below) to occur in which the outgoing electrons have the same individual momenta (and spins). And since we can only measure the incoming and outgoing particles, we have no way of knowing which of the two may have occurred. We thus must add the two amplitudes to get the total amplitude.

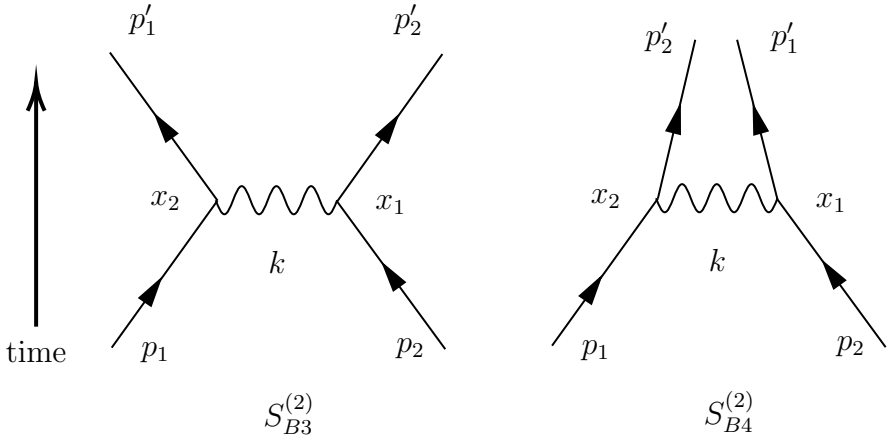


Figure 7.4: Moller Scattering

In finding the result, take care to note that either x_1 or x_2 can occur first and this is already accounted for in the Dyson time ordering converted, via Wick's theorem, to normal ordering.

A separate issue is that in our expression for the transition amplitude, **the p_2 electron could be destroyed at x_2 instead of x_1 (and vice versa for the p_1 electron).** If you work the math you will see these are two different terms, each of which contributes to the amplitude. However, they have the same form if we simply switch dummy integration variables $x_2 \leftrightarrow x_1$ in one of them. Hence, we will get two equal terms for the LHS of Fig. 7.4 and two equal terms for the RHS. This allow us to use only one of them for the LHS and one for the RHS, but multiply the transition amplitude expression by 2. The final results are

$$S_{Moller} = \left(\prod_{\mathbf{p}}^{\text{all fermions}} \right) \sqrt{\frac{m}{VE_p}} \right) (2\pi)^4 \delta^{(4)} \left(p_1' + p_2' - p_1 - p_2 \right) \left(\mathcal{M}_{B3}^{(2)} + \mathcal{M}_{B4}^{(2)} \right) \qquad (7.3.15)$$

$$\begin{aligned}\mathcal{M}^{(2)}_{B3} &= +e^2\bar{u}_{r_1'}\left(\mathbf{p}_1'\right)\gamma^{\mu}u_{r_1}\left(\mathbf{p}_1\right)iD_{F\mu\nu}\left(k=p_1-p_1'\right)\bar{u}_{r_2'}\left(\mathbf{p}_2'\right)\gamma^{\nu}u_{r_2}\left(\mathbf{p}_2\right)\\ \mathcal{M}^{(2)}_{B4} &= -e^2\bar{u}_{r_2'}\left(\mathbf{p}_2'\right)\gamma^{\mu}u_{r_1}\left(\mathbf{p}_1\right)iD_{F\mu\nu}\left(k=p_1-p_2'\right)\bar{u}_{r_1'}\left(\mathbf{p}_1'\right)\gamma^{v}u_{r_2}\left(\mathbf{p}_2\right)\end{aligned} \tag{7.3.16}$$

In the process of finding $\mathcal{M}^{(2)}_{B3}$, we have to normal order. In so doing, we put the destruction operators $\psi^+\left(p_1\right)$ and $\psi^+\left(\mathbf{p}_2\right)$ at the end. But they could be ordered there as $\psi^+\left(p_1\right)\psi^+\left(p_2\right)$, or as $\psi^+\left(p_2\right)\psi^+\left(p_1\right)$. Either would be correct normal order, but the sign of the resulting amplitude would be different. **The key is that in finding $\mathcal{M}^{(2)}_{B4}$, we have to normal order in the same order as we did for. $\mathcal{M}^{(2)}_{B3}$.**

7.3.4 The Electron/Positron Closed Loop Term $S^{(2)}_D$

For the closed loop terms, we have

$$S^{(2)}_D = -\frac{1}{2!}e^2\iint d^4x_1d^4x_2(N\left\{\underbrace{\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}\right\}_{x_2} + N\left\{(\bar{\psi}\underbrace{\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}\right\}_{x_2}) \tag{7.3.17}$$

In the manner detailed in the Box above, we can reorder either of these by switching adjacent fields (and introducing a minus sign each time we switch fermions). Doing this to the first term makes it look like the second term. Thus, (7.3.17) becomes

$$S^{(2)}_D = -e^2\iint d^4x_1d^4x_2N\left\{(\bar{\psi}\underbrace{\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}\right\}_{x_2} \tag{7.3.18}$$

which is represented by the following Feynman diagram

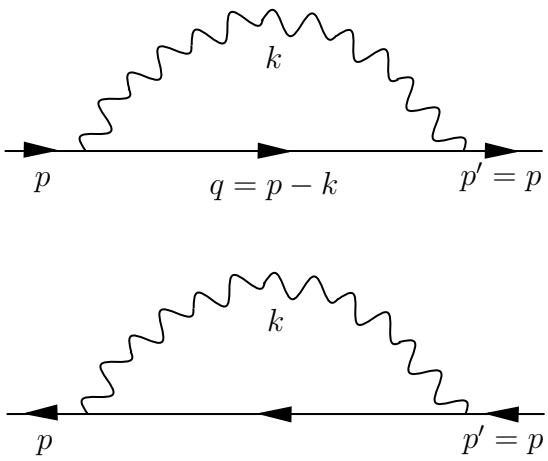


Figure 7.5: Electron Closed Loop(top) and Positron Closed Loop(bottom)

The dosed loop diagrams are also called self-energy diagrams. In Fig.7.5, with closed loops, however, one of the virtual particles' 4-momentum can be anything. That is, for any value of k(the virtual photon 4-momentum),q simply takes on the value $q = p - k$. The sum total of k and q must equal p, but this does not determine k and q separately. Thus, we need to integrate the expression for the transition amplitude of Fig.7.5 over all values of k i.e., over all values of k^0 from $-\infty$ to $+\infty$, and all yalues of $\mathbf{k}$ from $-\infty$ to ∞ along all three spatial directions.

$$S_{\text{eloop}} = \left\langle e_{\mathbf{p}',s'} \left| S_D^{(2)} \right| e_{\mathbf{p},s}^- \right\rangle$$

$$= \left\langle e_{\mathbf{p}'.s'}^- \left| -e^2 \iint d^4x_1 d^4x_2 i D_{F\mu\nu}\left(x_1 - x_2\right) \left(\bar{\psi}^-\right)_{x_1} \gamma^\mu i S_F\left(x_1 - x_2\right) \gamma^\nu \left(\psi^+\right)_{x_2} \right| e_{\mathbf{p}.s}^- \right\rangle$$

$$= -e^2 \left\langle e_{p',s}^- \right| \iint d^4x_1 d^4x_2 i D_{F\mu\nu}\left(x_1 - x_2\right) \left(\sum_{\mathbf{p}',s''} \sqrt{\frac{m}{VE_{p''}}} \bar{u}_{s''}\left(\mathbf{p}''\right) c_{s''}^\dagger\left(\mathbf{p}''\right) e^{ip''x_1} \right) \gamma^\mu \times$$

$$i S_F\left(x_1 - x_2\right) \gamma^\nu \sqrt{\frac{m}{VE_p}} u_s(\mathbf{p}) e^{-ipx_2} |0\rangle$$

Re-arranging factors, **and integrate over spatial coordinates, we have**

$$S_{\text{e loop}} = -e^2 \iint d^4q d^4k \left(\frac{1}{(2\pi)^4}\right) \underbrace{\left(\frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}\right)}_{iD_{\mu\nu}(k)} \underbrace{\left(\int e^{-ikx_1} e^{-iqx_1} e^{ip_1'x_1} d^4x_1\right)}_{(2\pi)^4\delta^{(4)}(q-(p'-k))} \underbrace{\left(\int e^{ikx_2} e^{iqx_2} e^{-ipx_2} d^4x_2\right)}_{(2\pi)^4\delta^{(4)}(q-(p-k))} \times$$

$$\frac{m}{V} \sqrt{\frac{1}{E_{p'}E_p}} \bar{u}_{s'}\left(\mathbf{p}'\right) \gamma^\mu \frac{1}{(2\pi)^4} \underbrace{i\left(\frac{(\not{q} + m)}{q^2 - m^2 + i\varepsilon}\right)}_{iS_F(q)} \gamma^\nu u_s(\mathbf{p})$$

With the Dirac delta functions, integration over q leaves $p' - k = p - k$, meaning $p' = p$, and thus,

$$S_{eloop} = -e^2 \delta^{(4)}\left(p - p' = 0\right) \frac{m}{VE_p} \int d^4k i D_{F\mu\nu}(k) \bar{u}_{s'}(p) \gamma^\mu i S_F(p-k) \gamma^\nu u_s(p)$$

We can show that only the $s = s'$ spin state survives. Thus, the electron loop transition amplitude is

$$S_{eloop} = \prod_{\mathbf{p}'}^{\text{all fermions}} \sqrt{\frac{m}{VE_{p'}}} (2\pi)^4 \delta^{(4)}\left(p - p'\right) \mathcal{M}_{eloop} \tag{7.3.19}$$

where

$$\mathcal{M}_{eloop} = \frac{-e^2}{(2\pi)^4} \int d^4k i D_{F\mu\nu}(k) \bar{u}_s(\mathbf{p}) \gamma^\mu i S_F(p-k) \gamma^\nu u_s(\mathbf{p}) \tag{7.3.20}$$

7.3.5 The Photon Closed Loop Term $S_E^{(2)}$

Like the electron, the photon has a closed loop interaction:

$$S_E^{(2)} = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 N \left\{ (\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2} \right\} \tag{7.3.21}$$

As presented by the Feynman diagram of Fig.7.6, the photon closed loop (or self-energy) diagram is also known as a vacuum polarization loop diagram because, in it, the chargeless photon sitting in the vacuum is polarized, i.e., it split into a particle with plus(pole) charge and a particle with negative (pole) charge.

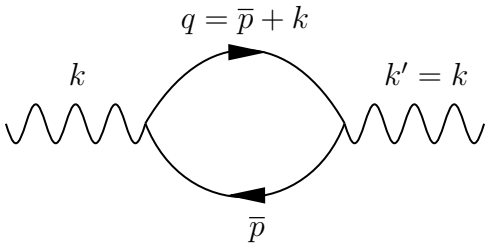


Figure 7.6: Photon Closed Loop

Using the equation (6.7.4), we can re-express (7.3.21) as

$$S_E^{(2)} = -\frac{1}{2!}e^2 \iint d^4x_1 d^4x_2 \left(-iS_{F\eta\alpha}\left(x_2 - x_1\right)\right) \gamma^{\mu}_{\alpha\beta} iS_{F\beta\delta}\left(x_1 - x_2\right) \gamma^{\nu}_{\delta\eta} N \left\{ \left(A^+_{\mu} + A^-_{\mu}\right)_{x_1} \left(A^+_{\nu} + A^-_{\nu}\right)_{x_2} \right\} \tag{7.3.22}$$

Since x_1 and x_2, μ and ν are dummy indeces, we find

$$\underbrace{\left(A^-_{\mu}\right)_{x_1} \left(A^-_{v}\right)_{x_2}}_{\text{will go to zero}} \quad \underbrace{\left(A^+_{\mu}\right)_{x_1} \left(A^+_{v}\right)_{x_2}}_{\text{will go to zero}}$$

and

$$\underset{loop}{S_{\gamma}} = \langle \gamma_{\mathbf{k}',r'}| \, e^2 \iint d^4x_1 d^4x_2 iS_{F\eta\alpha}\left(x_2 - x_1\right) \gamma^{\mu}_{\alpha\beta} iS_{F\beta\delta}\left(x_1 - x_2\right) \gamma^{\nu}_{\delta\eta} \left(A^-_{\mu}\right)_{x_1} \left(A^+_{\nu}\right)_{x_2}$$

$$= \frac{-e^2}{2Va_k}\delta^{(4)}\left(k - k'\right) \int d^4\bar{p} \int d^4\bar{p} \underbrace{S_{F\eta\alpha}(\bar{p})\gamma^{\mu}_{\alpha\beta} S_{F\beta\delta}(\bar{p}+k)\gamma^{v}_{\delta\eta}}_{\text{trace in spinor space of matrix, } M^{\mu v}_{\eta\eta}} \varepsilon_{\mu,r'}(\mathbf{k})\varepsilon_{v,r}(\mathbf{k})$$

$$\mathcal{M}_\gamma = \frac{-e^2}{(2\pi)^4}\left\{\mathrm{Tr}\int d^4\bar{p}S_F(\bar{p})\gamma^\mu S_F(\bar{p}+k)\gamma^\nu\right\}\varepsilon_{\mu,r'}(\mathbf{k})\varepsilon_{v,r}(\mathbf{k}) \tag{7.3.23}$$

$$S_{\gamma\ \mathrm{loop}} = \left(\prod_{\mathbf{k}}^{\mathrm{bosons}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\right)(2\pi)^4\delta^{(4)}\left(k-k'\right)M_{\gamma\ \mathrm{loop}} \tag{7.3.24}$$

Note the real positrons have positive momenta, as in the real world, whereas the virtual positron, for our Feynman diagram and our math, has momentum equal to the negative of its true value.

7.3.6 The Vacuum Bubble Term $S_F^{(2)}$

$$The final non-zero term is S_F^{(2)} = -\frac{1}{2!}e^2\iint d^4x_1d^4x_2(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2} \tag{7.3.25}$$

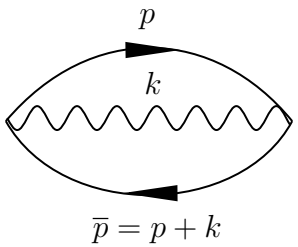


Figure 7.7: Vacuum Bubble Feynman Diagram

$$S_F^{(2)} = -\delta^{(4)}(0)\frac{e^2}{(2\pi)^4}\iint S_{F\eta\alpha}(p+k)\gamma^\mu_{\alpha\beta}S_{F\beta\delta}(p)\gamma^\nu_{\delta\eta}iD_{F\mu\nu}(k)d^4kd^4p \tag{7.3.26}$$

In vacuum bubble means we start with zero 4-momentum in the vacuum, awe end with zero four-momentum in the vacuum, and in between we have a sum total 4 -momentum of zero for all the virtual particles mediating the interaction.

7.4 Feynman Rules

The rules themselves apply to what are termed **topologically different Feynman diagrams.** These are different from one another in ways other than simply changing labeling of vertices:

- The S matrix element for a given interaction is

$$S_{fi} = \delta_{fi} + \left((2\pi)^4 \delta^{(4)}\left(P_f - P_i\right)\right)\left(\prod^{\text{bosons}} \sqrt{\frac{1}{2V\omega_k}}\right)\left(\prod^{fermions} \sqrt{\frac{m}{VE_{\mathbf{P}}}}\right)\mathcal{M} \quad \mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}^{(n)}$$

  where P_f is the total 4-momentum of all final particles, P_i is the total 4-momentum of all initial particles, and the contribution $\mathcal{M}^{(n)}$ comes from the n th order perturbation term of the S operator, $S^{(n)}$

- The Feynman amplitude $\mathcal{M}^{(n)}$ is obtained from all of the topologically distinct, connected (i.e., all lines connected to one another in a given diagram) Feynman diagrams which contain n vertices. The contribution to each $\mathcal{M}^{(n)}$ is obtained by the following:

  1 For each vertex, include a factor $ie\gamma^{\mu}$

  2 For each internal photon line, labeled by 4-momentum k, include a factor $iD_{F\mu v}(k) = i\frac{-g_{\mu v}}{k^2+i\varepsilon}$

  3 For each internal fermion line, labeled by 4-momentum p, write a factor $iS_F(p) = i\frac{\not{p}+m}{p^2-m^2+i\varepsilon}$

  4 For each external line, write one of the following spinor factors, where **p** and **k** indicate basis states of corresponding 3 -momenta, r represents spin state for fermions and polarization state for photons,

  4a) for each initial electron: $u_r(\mathbf{p})$

  4b) for each final electron: $\bar{u}_r(\mathbf{p})$

  4c) for each initial positron: $\bar{v}_r(\mathbf{p})$

  4d) for each final positron: $v_r(\mathbf{p})$

  4e) for each initial photon: $\varepsilon_{r,\mu}(\mathbf{k})$

4f) for each final photon: $\varepsilon_{r,\mu}(\mathbf{k})$ The spinor factors (γ matrices, S_F functions, spinors) for each fermion line are ordered so that, reading from right to left, they occur in the same sequence as following the fermion line in the direction of its arrows through the vertex. (Order is important as it conveys spinor matrix multiplication order when we do not show spinor indices.) The four-momenta at each vertex are conserved (same total after as before).

4f)1. For each closed loop of internal fermions only (without photons inside the loop itself, like what we call a "photon loop" which internally has an electron and a positron), take the trace (in spinor space) of the resulting matrix and multiply by a factor of (-1)

5 For each 4 -momentum q which is not fixed by 4-momentum conservation, carry out the integration $\frac{1}{(2\pi)^4}\int d^4q$ One such integration for each closed loop (fermion/fermion or fermion/photon loop).

6 Multiply the expression by (-1) for each interchange of neighboring fermion operators (each associated with a particular spinor factor) which would be required to place the expression in appropriate normal order. "Appropriate", when we are adding sub amplitudes, means each sub amplitude must be in the same, not just any, normal order of destruction and creation operators.

Box

Each term in $\mathcal{H}_I^I$, or equivalently in $\mathcal{L}_I^I$(since $\mathcal{L}_I^I = -\mathcal{H}_I^I$) results in a vertex interaction comprising the particles generated by the particular fields occurring in that term. Here so far, in QED, the only term in $\mathcal{L}_I^I$ is $\bar{\psi}\not{A}\psi$, so this gives rise to a vertex with two Dirac fermions and a photon.

7.5 Including Other Charged Leptons in QED

So far, we have dealt only with charged leptons of the electron/positron type. As physicists have learned from experiment, there are two more families of leptons, the muon/anti-muon (symbols and $\mu^\dagger$ and the tau/anti-tau particle (symbols τ^- and τ^+) families. Electrons, muons,

and taus all have -1 charge. Positrons, anti-muons and anti-taus all have +1 charge. Electrons and positrons have the same mass. Muons and anti-muons have the same mass, which is about 200 times the electron mass. Taus and anti-taus have the same mass, which is about 170 times the muon mass. **The new interaction Hamiltonian that correctly describes all three families is**

$$\mathcal{H}_I^I = -\mathcal{L}_I^I = -e\sum_{l=1}^{3}\bar{\psi}_l A^\mu \gamma_\mu \psi_l = -e\sum_{l=1}^{3}\left(\bar{\psi}_l^+ + \bar{\psi}_l^-\right)\left(\not{A}^+ + \not{A}^-\right)\left(\psi_l^+ + \psi_l^-\right) \tag{7.5.1}$$

where the summation over l represents a separate Hamiltonian density for electrons/positrons ($l = 1$), muons/anti-muons ($l = 2$), and taus/anti-taus ($l = 3$).

7.5.1 Feynman Rules for Multiple Families

7 Obtain the Feynman amplitude assuming all leptons are electrons/positrons

8 For lines representing other lepton "flavors" replace spinor and/or propagators with those representing the other flavers.

9 The only difference in the form of the result will be the masses.

7.5.2 Elastic vs Inelastic Scattering

The term inelastic scattering (or inelastic interaction) refers to **the particles involved changing into different types of particles with different masses (recall we mean "rest mass" by the term "mass")**, and hence some kinetic energy must be converted into mass (or vice versa depending on the particular interaction).

The total amount of energy (in the form of mass plus kinetic energy) must stay the same. But if the total mass changes, then that change must be compensated for by an equivalent opposite change in kinetic energy.

Elastic scattering (or elastic interaction), on the other hand, implies we have the same type final particles (with the same masses) as the initial particles. No mass is created out of, or

destroyed to yield, kinetic energy. **All energy exchange is purely kinetic, and this is a characteristic of classical elastic interactions, hence the name**.

7.6 Attraction and Repulsion of Particles

For virtual particles, 3-momenturn can actually be in the opposite direction of velocity. In fact, for attraction, this sort of behavior is essential. 3-momentum in other cases can be in other directions not parallel to virtual particle velocity (where we define that velocity in terms of the length vector between emission and absorption events divided by the time between them.)

Of course, in reality, we are looking at particles which are field-like in the sense that they are spreading out in space. An interaction is more like an interaction of one particle field with another, and during this interaction, 3-momentum of.appropriate direction is transferred. Again, the Feynman diagrams tend to make us think of point-like particles, rather than field-like particles.

In all cases, however, we do find total 3-momentum and total energy, including all real and virtual particles, are conserved. These conservation laws still hold.

Now consider two macroscopic. charged objects approaching one another along the same line of action.When they are repulsing each other, they lose kinetic energy, and the total energy

$$E_{\text{total}} = KE_1 + V_{\text{classical}} + KE_2 = \text{ constant}$$

where $V_{\text{classical}} \propto \frac{q_1 q_2}{r} > 0$. As their distance gets smaller, KE of the macro bodies decreases, but the electric field potential energy increases. In QFT, we would say the kinetic energies of the macro bodies plus all the energy of the virtuals is conserved,

$$E_{\text{total}} = KE_1 + E_{\text{virtuals}} + KE_2 = \text{ constant}$$

And so, we can see that what is interpreted classically as classical field energy, is, in the quantum realm, energy of virtual particles,

$$V_{\text{classical}} = E_{\text{virtuals}} > 0$$

Similar to repulsion, the classical energy relation for attraction is

$$E_{\text{total}} = KE_1 + V_{\text{cluascical}} + KE_2 = const.$$

where $V_{\text{classical}} \propto \frac{q_1 q_2}{r} < 0$. Given that we must have the same energy relation, in terms of virtual particles, it must mean that **the virtual particles energy must be negative as two objects approach**.

Table 7.1: Summary of Virtual Photon Properties for 1D Attraction and Repulsion

	Virtual particles in repulsion		virtual particle in attraction	
	3-momentum	Energy	3-momentum	Energy
Approaching and receding objects	**v** direction	positive	$-$**v** direction	negative

7.7 A Solved Exercise

Draw the two Feynman diagram for the interaction $\gamma\gamma \to e^+e^-$. Using Feynman's rules, write out the amplitude for one of the diagrams.

Solution

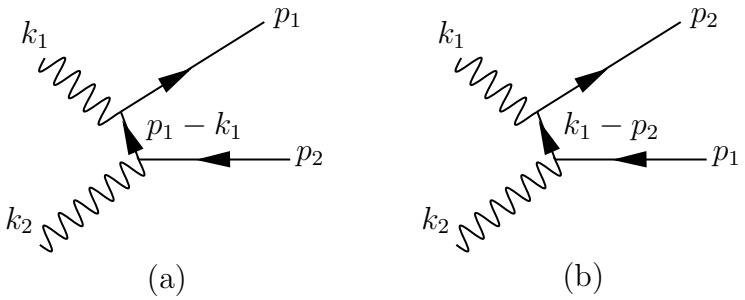


Figure 7.8: Feynman's diagram for $\gamma\gamma \to e^+e^-$

Note the positron propagator relations will carry 4-momentum with opposite sign of what we would find physically,i.e., $k_1 = p_1 + \bar{q}_{phys} \to \bar{q}_{phys} = k_1 - p_1 \to \bar{q} = p_1 - k_1$ in going from left to right time flow.

Follow direction of fermion arrows thru RH vertex: $iS_F(p_1 - k_1)ie\gamma^\nu v_{r_2}(\mathbf{p}_2)$

Including other vertex (LH one) fermion only: $\bar{u}_{r_1}(\mathbf{p}_1)ie\gamma^\mu iS_F(p_1 - k_1)ie\gamma^\nu v_{r_2}(\mathbf{p}_2)$

Including external initial photons: $\bar{u}_{r_1}(\mathbf{p}_1)ie\gamma^\mu iS_F(p_1 - k_1)ie\gamma^\nu v_{r_2}(\mathbf{p}_2)\varepsilon_{\mu,r_1}(\mathbf{k}_1)\varepsilon_{\nu,r_3}(\mathbf{k}_2)$

Thus, $\mathcal{M}^{(2)}_{(a)} = -e^2\bar{u}_{r_1}(\mathbf{p}_1)\gamma^\mu iS_F(p_1 - k_1)\gamma^\nu v_{r_2}(\mathbf{p}_2)\varepsilon_{\mu,r_1}(\mathbf{k}_1)\varepsilon_{\nu,r_3}(\mathbf{k}_2)$, where

$$S^{(2)}_{\gamma\gamma\to e^+e^-} = \frac{1}{2V\sqrt{\omega_1\omega_2}}\frac{m}{2V\sqrt{E_1E_2}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2)(\mathcal{M}^{(2)}_{(a)} + \mathcal{M}^{(2)}_{(b)})$$

Chapter 8

Higher Order Corrections

8.1 Third Order in e Correction Terms

Let us first look at a single contraction term in $S^{(3)}$:

$$N\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}(\bar{\psi}\not{A}\psi)_{x_3}\right\} \tag{8.1.1}$$

One of the Feynmann diagrams for this would look like the following figure (This is an unconnected diagram, not all lines are connected to one another in a single network)

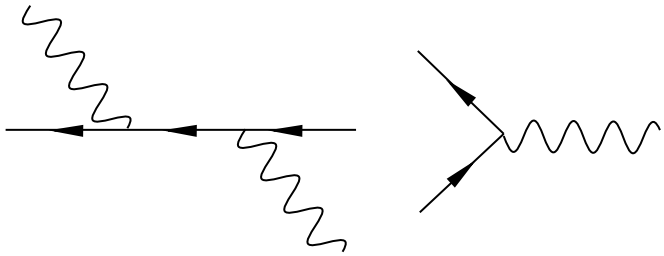


Figure 8.1: Single Contraction term of $S^{(3)}$

As mentioned in the previous chapter, the single vertex interaction is not physically possible as it produce off-shell photon. We conclude here that any possible term in $S^{(3)}$ that has a vertex factor $(\bar{\psi}\not{A}\psi)_{x_i}$ alone, unconnected to a contraction, is **not physical and can be ignored.**

Let's look at one of three contraction term:

$$N\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}(\bar{\psi}\not{A}\psi)_{x_3}\right\} \tag{8.1.2}$$

A typical Feynman diagram for this looks like

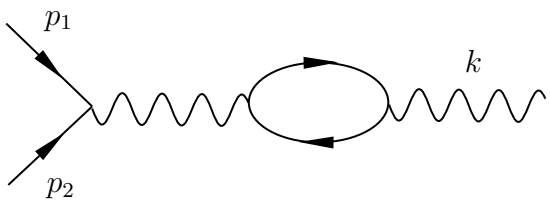
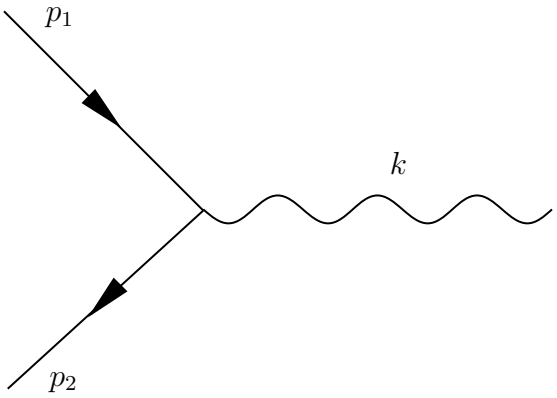


Figure 8.2: A Three Contraction Term of $S^{(3)}$

Note that the net result of Fig.(8.2) is similar to what we saw in the previous chapter, where a real electron and a real positron transmute into a single off-shell photon. Thus, it is unphysical too.



In similar fashion, we can show that every term in $S^{(3)}$ is unphysical and contributes zero to the transmission amplitude.

Feynman's rules ignore the factorial in the denominator of each term because those rules are applied to each topologically distinct diagram.

8.2 Fourth Order in e Correction Terms

The e^4 order contributions to 1st kind of Bhabha scattering is shown in the following figure:

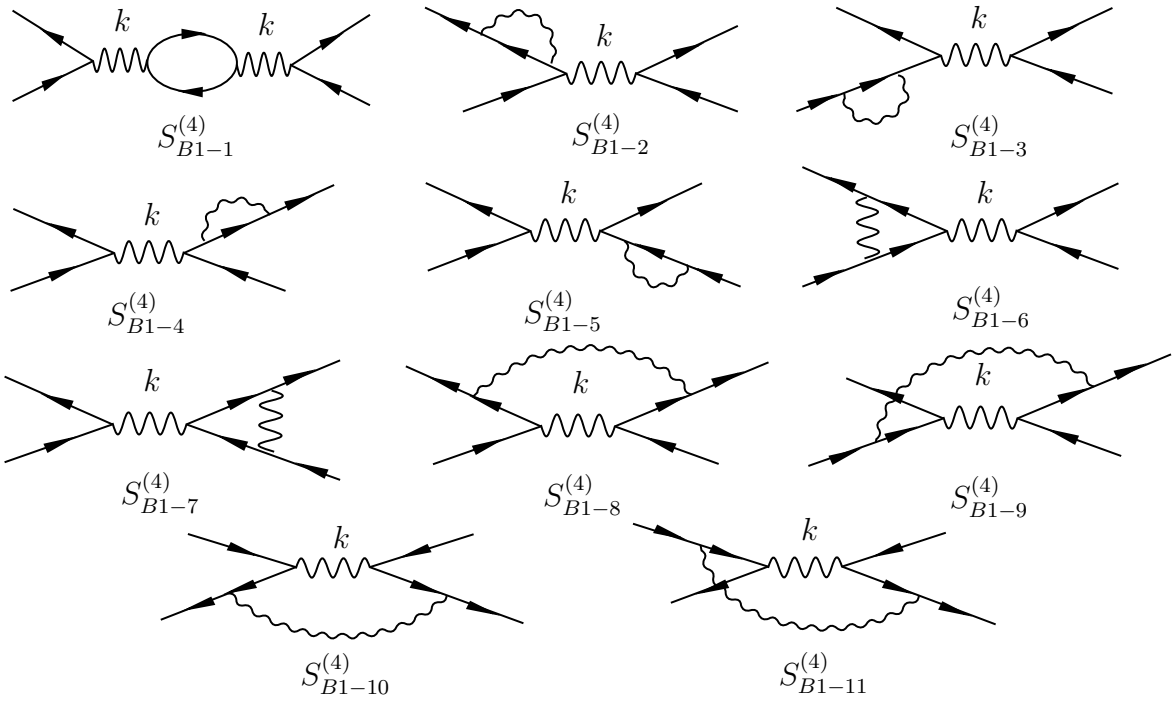


Figure 8.3: e^4 Order Contributions to First Kind of Bhabha Scattering

Unfortunately, some of the terms in Fig.8.3 turn out to be infinite. Let's look at one example to see how the infinity arises.

8.2.1 Photon Loop Diagram

$S^{(4)}_{B1-1}$ has a photon loop (made up of a virtual electron and a virtual positron). We show it in the figure below with the momenta labeled, as well as the spacetime index at each vertex to help keep track when using Feynman's rule)

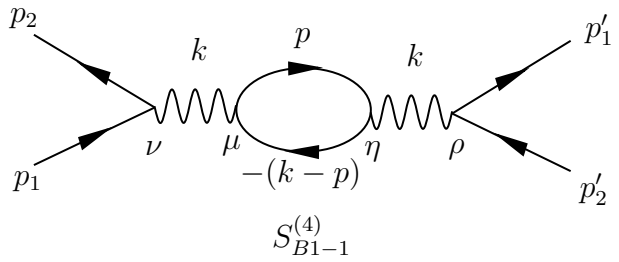


Figure 8.4: $S^{(4)}_{B1-1}$ term Feynman diagram for Bhabha Scattering

The Feynman amplitude for this is

$$\frac{-p^4}{(2x)^4}\bar{u}'_\eta\left(\mathbf{p}'_1\right)\gamma^\rho v_{r'_2}\left(\mathbf{p}'_2\right)D_{F\rho\eta}(k)\left\{\mathrm{Tr}\int S_F(p-k)\gamma^\eta S_F(p)\gamma^\mu d^4p\right\}D_{F\mu\nu}(k)\bar{v}_{\tau_2}\left(\mathbf{p}_2\right)\gamma^v u_{r_1}\left(\mathbf{p}_1\right) \tag{8.2.1}$$

Every momentum value in (8.2.1) is fixed except p, which must be integrated over 4D momentum space from $+\infty$ to $-\infty$ along all four axes. And we know all of the factors are finite except for the integral. Let's estimate the integral value by assuming parts of the integral where any component p^μ is large could be the problematic portions. That is, we'll specifically investigate whether the integral blows up for large values of p^μ.

$$\int S_F(p)\gamma^\mu S_F(p-k)\gamma^\eta d^4p = \int \frac{\not{p}+m}{p^2-m^2+i\varepsilon}\gamma^\mu\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\varepsilon}\gamma^\eta d^4p$$

$$\frac{\text{for contribution}}{\text{from large } p} \to\approx \int \frac{p_v}{p^2}\gamma^v\gamma^\mu\gamma^\sigma\gamma^\eta\frac{p_\sigma}{p^2}d^4p \xrightarrow{\text{ignore } \gamma \text{ matrices}} \int\frac{pp}{p^4}d^4p$$

Since

$$dA = d^2r = 2\pi r dr \quad dV = d^3r = 4\pi r^2 dr \quad dR_{4D} = d^4r = 2\pi^2 r^3 \quad dr$$

Applying the last expression, we have

$$\int_{-\infty}^{\infty}\frac{d^4p}{p^2} = \int_0^\infty 2\pi^2\frac{p^3}{p^2}dp = 2\pi^2\int_0^\infty pdp = \pi^2p^2\big|_0^\infty$$

which diverges with the square of p as p gets large, and is called **quadratically divergent**. This procedure of estimating the degree of divergence is called **power counting, naive power counting, or superficial power counting**. It is called naive/superficial because, as we will see in later chapters, the actual degree of divergence can be less than this estimate. Power counting tells us the maximum degree of divergence an integral may have. For example, the photon loop actually diverges with the log of p at high p. More on this later.

Whenever we have a photon loop, in any scattering case, we will get a factor of infinity in our transition amplitude. Naive power counting indicates the divergence may be proportional to p^2, for large memontum.

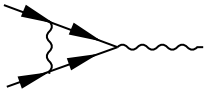
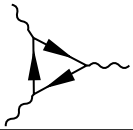
Term	Terminology	Diagram	Divergence $k, p \to \infty$	Comments
photon loop	photon self-energy		p^2	quadratic
fermion loop	fermion self-energy		k	linear
vertex loop	vertex correction		lnk	logarithmic
lepton triangle	triangle graph		p	arises in compton scattering.cancels with another opposite term

Figure 8.5: Wholeness Chart on Loop corrections

8.3 Other terms

Consider the $S^{(4)}_{B1-8}$ diagram in Fig.8.3, which we show below with momenta and spacetime labels. **Note there are no simple loops** in Fig.8.6. The term loop in QFT generally refers only to one the three simple loops shown above.

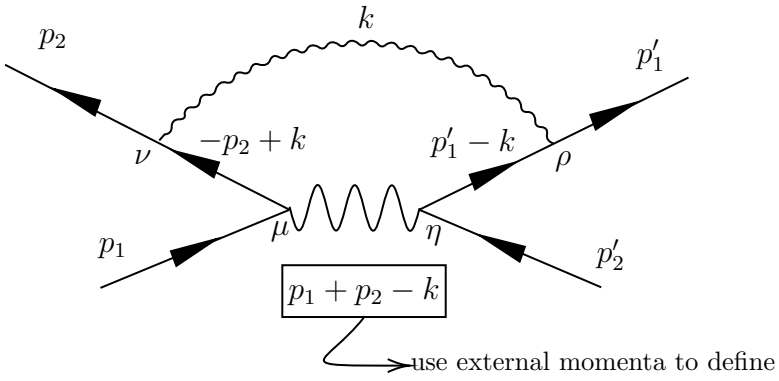


Figure 8.6: $S^{(4)}_{B1-8}$

The Feynman amplitude for the fig above is

$$\frac{1}{(2\pi)^4}e^4\int d^4k\bar{u}_{r'}\left(\mathbf{p}_1'\right)\gamma^\rho S_F\left(p_1'-k\right)\gamma^\eta v_{s'}\left(\mathbf{p}_2'\right)\times$$

$$D_{F\eta\mu}\left(p_1+p_2-k\right)D_{F\rho v}(k)\bar{v}_s\left(\mathbf{p}_2\right)\gamma^v S_F\left(-p_2+k\right)\gamma^\mu u_r\left(\mathbf{p}_1\right)$$

Perform naive power counting, we get

$$\int S_F\left(p_1'-k\right)D_F\left(p_1+p_2-k\right)D_F(k)S_F\left(-p_2+k\right)d^4k$$

$$\xrightarrow{\text{for large}k}\approx\int\frac{1}{k}\frac{1}{k^2}\frac{1}{k^2}\frac{1}{k}d^4k=2\pi^2\int\frac{1}{k^6}k^3dk=-\pi^2\frac{1}{k^2}$$

So this Feynman amplitude integral converges in the large k region. For $k\to 0$, we have

$$\mathcal{M}^{(4)}_{B1-8}\xrightarrow{\text{for small }k}\int\frac{\not{p}_1'+m}{p_1'^2-m^2+i\varepsilon}\frac{1}{\left(p_1+p_2\right)^2+i\varepsilon}\frac{1}{k^2+i\varepsilon}\frac{-\not{p}_2+m}{p_2^2-m^2+i\varepsilon}d^4k$$

$$\approx\int(\text{ constant })\frac{1}{k^2}d^4k\approx 2\pi^2\int\frac{1}{k^2}k^3dk=\pi^2k^2\text{ for small }k$$

Hence, the integral is finite.

The photon,electron, and vertex loops would all lead to factors of infinity in the transition amplitude. All other terms would be finite, only the loop factors are unbounded.
High momentum divergences are referred to as **ultraviolet divergence**, while divergence at small momentum region is called **infrared divergences**.

Tricks for Writing out Feynman Amplitude

1. Before write an amplitude down, label the vertices in Feynman diagram using different greek letters

2. By following the time line, the vertex labels should appear in the expression sequentially

3. For each vertex, always put a $ie\gamma^{\mu}$ between two spinor vectors

4. Always put spinor vector at the left of a gamma matrix, and put its adjoint vector at the right of the matrix

5. If there are multiple virtual particles in a diagram, finish its business at earlier spacetime first then go to the later spacetime.

Chapter 9

The Vacuum Revisited

9.1 Casimir Plates

two plates brought close together experience a small attractive force at very small separation distance. This effect was first predicted by Dutch physicists Hendrik B. G. Casimir and Dirk Polder in 1948. The attractive force has been attributed to ZPE, in heuristic and very simple terms, because the the vacuum quantum waves outside the plates presumably exert greater force than the vacuum quantum waves between the plates. However, there are two key things to note:

1) While the Casimir effect can be calculated by assuming ZPE half quanta, the same result can also be calculated another way without using them at all. It thus does not prove their existence, contrary to what is often claimed.

2) In the Casimir calculation that does employ ZPE, the quanta are assumed to be continuously existing standing waves, not particle pairs popping in and out of existence.

9.2 Lamb Shift

The Lamb shift is a small difference between the two energy levels $^2S_{12}$ and $^2P_{12}$ of the hydrogen atom, which according to RQM, should have the same energies. QFT, in its QED form, predicts this shift, and that prediction was one of the great early successes of the theory.

The Lamb shift calculation is long and difficult. It is often described as taking vacuum fluctuations into account in order to obtain the correct result. However, in actuality, **these "vacuum fluctuations" are really the radiative, or higher order, corrections (extra virtual particles in Feynman diagrams)**. These corrections to the Coulomb potential of the hydrogen atom (in diagrams, extra virtual photons, electrons, and positrons) yield the correct energy levels.

The Lamb shift does not prove vacuum pair production/destruction.

Chapter 10

Symmetry and Conservation for Interaction Fields

10.1 Modify Lagrangian using $F_{\mu\nu}$

Recall the QED full Lagrangian we have been using has form

$$\mathcal{L}^{1/2,1} = -\frac{1}{2}\left(\partial_v A_\mu\right)\left(\partial^v A^\mu\right) + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu \tag{10.1.1}$$

and

$$F^{\mu v}(x) = \partial^v A^\mu(x) - \partial^\mu A^v(x) = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

Now, if we redefine our QED Lagrangian by changing only the first term, we get

$$\mathcal{L}^{1/2,1} = -\frac{1}{4}F^{\mu v}F_{\mu v} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu \tag{10.1.2}$$

or finally

$$\mathcal{L}^{1/2,1} = -\frac{1}{2}\left(\partial^v A^\mu\partial_v A_\mu - \partial^v A^\mu\partial_\mu A_v\right) + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$$

Note that

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\left(-\mathbf{E}^2 + \mathbf{B}^2\right) \tag{10.1.3}$$

10.2 External Symmetry for Interacting Fields

Note that both of the full QED Lagrangian forms above are invariant under a Lorentz transformation. Recall that $\bar{\psi}\gamma^{\mu}\psi$ behaves under Lorentz transformation like a 4-vector. Hence **every term in the both forms is a Lorentz scalar, or world scalar, since a contraction of two 4-vectors is a Lorentz scalar.** Such Scalar are invariant in form under Lorentz transformations. and so

$$\mathcal{L}\left(\bar{\psi}(x),\psi(x),A_{\mu}(x)\right)=\mathcal{L}\left(\bar{\psi}'\left(x'\right),\psi'\left(x'\right)A'_{\mu}\left(x'\right)\right) \tag{10.2.1}$$

$\mathcal{L}$ is symmetric under Lorentz transformation. It has external symmetry.

10.2.1 External symmetry of the interaction probability

In a good theory, the probability of an interaction occurring should not vary with the observer measuring it. That is, the transition amplitude (whose absolute value squared is the probability) should remain invariant under Lorentz transformation.

Note that $\mathcal{L}_I$ is invariant under Lorentz transformation. It takes the same form with fields and coordinates primed, as with them unprimed. Thus, $\mathcal{H}_I=-\mathcal{L}_I$ is also invariant. The S operator is a function of $\mathcal{H}_I$, $S=e^{-i\int_{-\infty}^{\infty}\mathcal{H}_I^I d^4x}$ so **S is also invariant**. If Λ indicates a generic Lorentz transformation representation, we have

$$S_{fi}=\langle f|\underbrace{\Lambda^{-1}\Lambda}_{I}S\underbrace{\Lambda^{-1}\Lambda}_{I}|i\rangle=\left\langle f\left|\Lambda^{-1}\underbrace{\Lambda S\Lambda^{-1}}_{S'=S}\Lambda\right|i\right\rangle=\langle f'|S_{f't'}|f'\rangle=S_{f't'} \tag{10.2.2}$$

So, the transition amplitude S_{fi} for a particular initial multi-particle state scattering particular final multi-particle state is the same as seen from two different Lorentz frames. Thus, $|S_n|^2$ is the same as well. In summary

$$\mathcal{L}_I \text{ sym } \rightarrow \mathcal{H}_l \text{ sym } \rightarrow S \text{ sym } \rightarrow S_{fi} \text{ sym } \rightarrow |S_{fi}|^2 \text{ sym.}$$

10.3 Internal Symmetry and Conservation of Interactions

Now, we apply Noether's theorem to (10.1.2), with $\phi^{r=1} = \psi, \phi^{r=2} = \bar{\psi}$, and $\phi^{r=3} = A^v$, we have our conserved four-current as

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \frac{\partial \psi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{,\mu}} \frac{\partial \bar{\psi}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial A^\nu_{,\mu}} \frac{\partial A^\nu}{\partial \alpha} \tag{10.3.1}$$

where

$$\frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} = \bar{\psi} i \gamma^\mu \quad \frac{\partial \psi}{\partial \alpha} = -i\psi \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}_{,\mu}} = 0 \quad \frac{\partial A^\nu}{\partial \alpha} = 0 \tag{10.3.2}$$

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$
$$Q' = -e \int j^0 d^3x \text{ conserved}$$

If we write the charge operator in terms of number operators, we have

$$Q = -e \sum_{\mathbf{p},\mathbf{r}} \left(N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}) \right)$$

Because it can be proven that Q commutes with S, then

$$Q|f\rangle = q_{ftot}|f\rangle = Qs|i\rangle = sQ|i\rangle = Sq_{itot}|i\rangle = q_{itot}S|i\rangle = q_{itot}|f\rangle$$

and

$$q_{f\text{tot}} = q_{\text{itot}}$$

The total charge remains unchanged during the transition.

10.4 Local Symmetry and Interaction Theory

Up to this point, we have dealt exclusively with global symmetry, which **gives same change everywhere in spacetime**. In contrast, we could have an internal transformation where

$\alpha = \alpha(x^\mu)$, where **the phase angle change is not constant but a function of 3D position and time,**

$$\psi \to \psi' = e^{-i\alpha(x^\mu)}\psi \tag{10.4.1}$$

If the Lagrangian is invarient under such a transformation, we say it has **local symmetry**.

For $\mathcal{L}_I$,

$$\mathcal{L}_0^{1/2} = \bar{\psi}\left(i\gamma^\nu\partial_\nu - m\right)\psi \xrightarrow{\psi \to e^{-ia(x^\mu)}\psi} \bar{\psi}e^{i\alpha(x^\mu)}\left(i\gamma^\nu\partial_\nu - m\right)\psi e^{-ia(x^\mu)}$$

$$= \bar{\psi}\left(i\gamma^\nu\partial_\nu - m\right)\psi + \bar{\psi}\gamma^\nu\psi\partial_\nu\alpha\left(x^\mu\right) \neq \mathcal{L}_0^{1/2}$$

Thus, $\mathcal{L}_Q$ **is not symmetric under this local transformation**. Similarly, The full Lagrangian $\mathcal{L}$ is not symmetric under this transformation.

By carrying out the set of local transformations $\psi \to \psi' = e^{-ia(x^\mu)}\psi$ and $A_\nu \to A'_\nu = A_\nu - (1/e)\partial_\nu\alpha\left(x^\mu\right)$, and using $\mathcal{L}_0^1 = -\frac{1}{4}F_{\nu\beta}F^{\nu\beta}$ for the free photon field Lagrangian (second form above), the full QED Lagrangian $\mathcal{L}$ remains invariant.

If we require our theory to have local symmetry, it must

- use the second form for our Lagrangian with the $F_{\mu\nu}$ term for free photons
- use the particular local transformation set for ψ and A_ν shown above, and
- use the full Lagrangian, including the specific interaction term of form $e\bar{\psi}\gamma^v\psi A_v$

This has led to the following general rule for QFT

If we start with the free Lagrangian and require it to be locally symmetric then it can only be so if we add to it the particular interaction terms that actually describes interactions in the real world.

Note the following

1. Noether's theorem applies locally, as well as globally,
2. Thus, we can derive a conserved current (the same conserved current) from global symmetry or local symmetry using the second form of the full Lagrangian.
3. Symmetry under transformation of one or more fields means the fields are gauge fields (different α in the QED case, means different gauges).
4. You may hear the term Lie group in this context. A Lie group is the set of all possible continuous transformations of the fields in a gauge theory. In our case, for the field ψ, it is the set of $e^{-i\alpha(x^\mu)}$ for all possible $a\left(x^\mu\right)$. This group is called a $U(1)$ group, where the U means unitary and the 1 means it is a 1×1 matrix transformation. (U(n) would entail an $n \times n$ matrix.)

10.5 Minimal Substitution

Note that if we define something called the **gauge covariant derivative** as

$$D_v = \partial_v - ieA_v \tag{10.5.1}$$

then **we can find the interaction Lagrangian by substituting equation above for ∂_ν into the free Lagrangian.** Specifically, This process of substituting the gauge covariant derivative in the free Lagrangian is called minimal substitution.

10.6 Review of Noether Theorem for Free Fields

10.6.1 For Free Scalar Field

The free scalar Lagrangian was invariant under a transformation in phase. That is,

$$\phi \to \phi' = e^{-i\alpha}\phi$$

$$\phi^\dagger \to \phi'^\dagger = \phi^\dagger e^{i\alpha}$$

where α is a real constant, the Lagrangian

$$\mathcal{L}_0^0\left(\phi^\dagger,\phi,\phi^\dagger_{,\mu},\phi^{,\mu}\right)=\phi^\dagger_{,\mu}\phi^{,\mu}-\mu^2\phi^\dagger\phi$$

remains the same function of primed fields as it was of unprimed fields, Applying Noether's theorem with $\phi^{r=1}=\phi$ and $\phi^{r=2}=\phi^\dagger$,we found

$$j^\mu\left(\phi^r,\phi^r_{,\nu}\right)=\frac{\partial\mathcal{L}}{\partial\phi^r_{,\mu}}\frac{\partial\phi^r}{\partial\alpha}\quad(\text{ sum on } r)$$

$$j^\mu_\phi\left(\phi^r,\phi^{r,\mu}\right)=i\left(\phi^{,\mu}\left(x^\eta\right)\phi^\dagger\left(x^\eta\right)-\phi^{\dagger,\mu}\left(x^\eta\right)\phi\left(x^\eta\right)\right)$$

And total charge $Q_\phi=q_\phi\int j^0d^3x$ conserved.

10.6.2 For Free Dirac Fermion Fields

Under the transformation

$$\psi\to\psi'=e^{-i\alpha}\psi\ \left(\text{implies also }\bar\psi\to\bar\psi'=\bar\psi e^{i\alpha}\right)$$

the free Dirac Lagrangian is symmetric. And thus, via Noether's theorem with $\phi^{r=1}=\psi$, $\phi^{r=2}=\bar\psi$, and $-e$ the charge on an electron, we have

$$j^\mu_\psi=(\rho,\mathbf{j})=\bar\psi\gamma^\mu\psi$$

total charge conserved, since $Q_\psi=-e\int j^0d^3x$ conserved.

10.6.3 For free Photon fields

Under the transformation

$$A_\mu\to A'_\mu=e^{-i\alpha}A_\mu$$

where no conjugate transformation is for photon (since its anti-particle field is itself), the free electromagnetic Lagrangian following is not symmetric, and there is no 4-current for photons.

$$\mathcal{L}_0^1=-\frac{1}{2}\left(\partial_vA_\mu\right)\left(\partial^vA^\mu\right)$$

and

$$j^\mu_{A_\mu}=0\to\ \text{total charge } Q_{A_\mu}=0 \text{ conserved}$$

Chapter 11

Overview of Renormalization

11.1 Brief Math Interlude: Regularization

In the process of renormalization, we need to recast infinite integrals in a manageable form (one that is not infinite), at least during part of our renormalization process. During the renormalization process, we can then play some tricks that cause the troublesome integrals to drop out of the final result. We can then, at the end, restore them to their rightful, infinite value, but our final result will no longer diverge. This process of temporarily rendering the infinite integrals as finite is called regularization. We inllustrate the simplest way to regularize with the following example.

Consider the divergent integral

$$\int_{-\infty}^{\infty} x^2 dx = \frac{1}{3}x^3\Big|_{-\infty}^{\infty} \to \infty$$

We can regularize as follows

$$\int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big|_{-\Lambda}^{\Lambda} \to \frac{2}{3}\Lambda^3 \quad \text{later take } A \to \infty$$

Now imagine we express this integral temporarily with Λ, further, we have the eqn. above multiplied by $1/\Lambda^3$. We would find the Λ factors cancel, leaving a finite number result.

11.2 A Renormalization Example: Bhabha Scattering

Go back to Fig. (8.3) and Fig. (7.2). It represents all of the possible first order and second order (in α) Feynman diagrams for the first kind of Bhabha scattering. The first order in α (second order in e) diagram is called a **tree diagram**. (In math the term is used to refer to a diagram in which lines branch out from points without forming any closed loops.)

Box

From now on, k and p sometimes can refer to interaction energy.

For the Feynman amplitude of a given interaction type, $\mathcal{M}^{(n)}$ is the sum of the amplitudes from all diagrams of order n. If n is the order to which e is raised (that is, e^n is found in the amplitude). then all diagrams of order n would have n vertices.

$$\mathcal{M}_{\text{Bhabha total to } e^4} = \underbrace{\mathcal{M}^{(2)}_{B1} + \sum_{i=1}^{11} \mathcal{M}^{(4)}_{B1-i}}_{\text{first type}} + \underbrace{\mathcal{M}^{(2)}_{B2} + \sum_{i=1}^{11} \mathcal{M}^{(4)}_{B2-j}}_{\text{second type}} \tag{11.2.1}$$

It turns out that when we include higher than tree level amplitudes, **the effective coupling e^2 (or equivalently, α) changes. That is, e or α appears to have a different value, according to whether or not we include the higher order parts of the amplitude. A similar thing occurs for the mass m in the lepton propagators.**

Because of this, we will want to take the symbols e and m to represent what we would actually measure in experiment (for which nature would automatically include all the higher order contributions). Thus, we will redefine the symbols e and m that we have been using so far as e_0 and m_0. Those latter symbols will represent charge and mass as they would be found if only tree level diagrams played any role in nature. We will call these the bare charge and bare mass,respectively, since they are not "dressed up" with contributions from additional Feynman diagram beyond the tree level diagram.

- e in all prior work will from henceforth be re-labeled as e_0 (bare charge)
- α in all prior work will from henceforth be re-labeled as $\alpha 0$ (bare coupling constant)
- m in all prior work will from henceforth be re-labeled as m_0 (bare mass)

11.2.1 Result of the Calculation

Adding first type amplitude for Bhabha scattering to second order in α, we have

$$\mathcal{M}_{B1,2\text{ terms}} = -e_0^2 \bar{u}_{r'}\left(\mathbf{p}_1'\right)\gamma^\mu v_{r_2'}\left(\mathbf{p}_2'\right)\left\{\underbrace{iD_{F\mu v}(k)}_{\text{tree diagram}} + \underbrace{iD_{F\mu\eta}(k)e_0^2 X^{\eta\rho}(k,\Lambda)D_{F\rho v}(k)}_{\text{photon self energy diagram}}\right\}\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^v u_{r_1}\left(\mathbf{p}_1\right) \tag{11.2.2}$$

where

$$iX^{\eta\rho}(k,\Lambda) = \frac{1}{(2\pi)^4}\left(\text{Tr}\int \underbrace{S_F(p)\gamma^\rho S_F(p-k)}_{\text{with } m_0 \text{ not } m}\gamma^\eta d^4p\right)$$

with the integration to Λ not ∞. The big job would be to evaluate $X^{\eta\rho}(k,\Lambda)$. When we evaluate that, we find (Chap. 13) the photon self-energy diagram term ends up making two changes to the amplitude as compared to the original tree diagram amplitude. That is, we get first a term that is a function of e_0^2, k, and Λ, and second, a modified form for the photon propagator. This modified propagator (modified to 2^{nd} order in α) is only a function of k, but a different function of k than the original, non-loop, propagator.

$$\mathcal{M}_{B1,2\text{ terms}} = -e_0^2\left\{1 + \underbrace{\left(\text{ function of } e_0^2, k, \Lambda\right)}_{\text{from photon self energy}}\right\}\bar{u}_{r_1}'\left(\mathbf{p}_1'\right)\gamma^\mu v_{r_2'}\left(\mathbf{p}_2'\right)\underbrace{iD_{F\mu\nu}^{2nd,Mod}(k)}_{\text{from photon self energy}}\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^\nu u_{r_1}\left(\mathbf{p}_1\right) \tag{11.2.3}$$

The process of (11.2.2) and (11.2.3) is, however, only the beginning. We need to add in the other 10 diagrams of Fig.(8.3).

$$\mathcal{M}_{B1-e^4} = -e^2(k,\Lambda)\bar{u}_{r'}\left(\mathbf{p}_1'\right)\gamma^\mu_{Mod,2nd}\left(p_1',p_2'\right)v_{r_2'}\left(\mathbf{p}_2'\right)iD_{F\mu v}^{Mod,2nd}(k)\bar{v}_{r_2}\left(\mathbf{p}_2\right)\gamma^\nu_{Mod,2nd}\left(p_1,p_2\right)u_n\left(\mathbf{p}_1\right) \tag{11.2.4}$$

The quantity $e^2(k,\Lambda)$ is dependent on k, the energy level of the interaction. **For e_0 a non-zero constant and $\Lambda = \infty$, e is unbounded.** All other terms are finite. Note that the equation above is identical with our tree level relation, except that

$$e_0 \to e(k,\Lambda) \quad -iD_{F\mu\nu}(k) \to iD^{Mod,2nd}_{F\mu\nu}(k)$$

$$\gamma^\nu \to \gamma^\nu_{Mod,2nd}(p_1,p_2), \quad \gamma^\mu \to \gamma^\mu_{\text{Mod,2nd}}(p_1',p_2')$$

11.2.2 Renormalizing to e^4 Only

Note though that we still haven't resolved the infinity problem, since via

$$e^2(k,\Lambda) = e_0^2\left(1 + e_0^2 2b_n \ln\frac{k}{\Lambda} + \mathcal{O}\left(e_0^4\right)\right) \tag{11.2.5}$$

as $\Lambda \to \infty, e^2(k) \to -\infty$. This is where something radical is required, i.e., renormalization.Instead of $e^2(k)$ going to negative infinity in the limit, we assume that **the limit of $\Lambda \to \infty$ corrsponds to the finite value of the measured charge squared $e^2(k)$ at energy level k**. That is, we assume

$$\lim_{\Lambda\to\infty}\left(e_0^2\left(1 + e_0^2 2b_n \ln\frac{k}{\Lambda} + \mathcal{O}\left(e_0^4\right)\right)\right) = \underbrace{\lim_{\Lambda\to\infty} e_0^2}_{=e_0^2} + \lim_{\Lambda\to\infty} e_0^4 2b_n \ln\frac{k}{\Lambda} + \lim_{\Lambda\to\infty}\mathcal{O}(e_0^6) \tag{11.2.6}$$

Now the only way to obtain a finite value for the measured charge squared $e^2(k)$ is to somehow cancel the minus infinity from the natural logarithm in the second term on the right side. This can only be done by having e_0 equal to zero, but in such a way that when we restore our theory to what it really is by taking $\Lambda \to \infty$.

Taking our bare charge as zero, no doubt, seems strange. Essentially, it means fermions have no inherent charge, but that all charge is a result of higher order interactions.

11.2.3 Total Bhabha Scattering to All Orders

Bottom line: We can use our tree diagrams (both types) and simply substitute the measured charge e for e_0, iD_F^{Mod} for iD_F, and γ^μ_{Mod} for γ^μ to get the finite, correct

total amplitude for Bhabha scattering(to all orders here).

$$\begin{aligned}\mathcal{M}_{Bhabha} &= \mathcal{M}_{B1} + \mathcal{M}_{B2} \\ &= -e^2(k)\bar{u}_\eta\left(\mathbf{p}_1'\right)\gamma^\mu_{Mod}\left(p_1', p_2'\right)v_{r_2'}\left(\mathbf{p}_2'\right)iD^{Mod}_{F\mu\nu}\left(p_1+p_2\right)\bar{v}_{12}\left(\mathbf{p}_2\right)\gamma^v_{Mod}\left(p_1,p_2\right)u_1\left(\mathbf{p}_1\right) \\ &+ e^2(k)\bar{v}_{1/2}\left(\mathbf{p}_2\right)\gamma^\mu_{Mod}\left(p_2,p_2'\right)v_{22}'\left(\mathbf{p}_2'\right)iD^{Mod}_{F\mu v}\left(p_2-p_2'\right)\bar{u}_1'\left(\mathbf{p}_1'\right)\gamma^v_{Mod}\left(p_1',p_1\right)u_\eta\left(\mathbf{p}_1\right)\end{aligned}$$

Where e^2 is normalized to all orders in Bhabha scattering:

$$\lim_{\Lambda\to\infty} e_0^2\left(1+e_0^2 2b_n\ln\frac{k}{\Lambda}+\left(e_0^4\,\text{term}\right)+\left(e_0^6\,\text{term}\right)+\ldots\right)=e^2(k) \tag{11.2.7}$$

11.2.4 Running QED Coupling "Constant"

We can of course, with $e_0^2 = 4\pi\alpha_0$, re-express (11.2.7) as

$$\lim_{\Lambda\to\infty} \alpha_0\left(1+\alpha_0 8\pi b_n\ln\frac{k}{\Lambda}+\left(\alpha_0^2\,\text{term}\right)+\left(\alpha_0^3\,\text{term}\right)+\ldots\right)=\alpha(k) \tag{11.2.8}$$

This is how theorists tend to prefer it, actually. The fine structure "constant" α is more commonly referred in QFT, as the electromagnetic coupling "constant".

Note the word "constant" is now a bit of a misnomer, as we are finding both α and e are not Constants, but functions of k. In fact, since they "run" with k, α is often called the **running constant**.

11.3 Renormalize Mass

There is one more wrinkle to the renormalization business. Just as the bare charge we have been using led to infinite amplitude, so does lepton mass. And just as we had to renormalize the charge. so we will also have to do with mass.

In previous sections, we dealt with Bhabha scattering, where the tree level propagator was photon, having zero mass, and to make things easier in that discussion, we ignored any renormalization effect on lepton mass in the second order fermion loop. But now we need to refine that approach to make it completely correct.

For example, in Compton scattering, we have a tree level fermion propagator, where the mass is non-zero. The presence of fermion propagators in amplitude calculations leads to additional infinite terms. To begin this, recall the momentum space fermion propagator has form

$$S_F(p) = \frac{(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon} = \frac{1}{\not{p} - m_0 + i\varepsilon} \tag{11.3.1}$$

We now extend our trick of using the tree level amplitudes with modified charge, propagators, and vertices as long as we also modify mass. Specifically, when we include all four vertex diagrams in our amplitude calculation for a given interaction like Compton scattering, and we limit loop integrations to a very large energy Λ, rather than infinity, we find the propagator part of the amplitude takes on the from, with the function $H(p)$ a finite function of energy p,

$$S_F(p)\frac{\text{all diagrams}}{\text{to order } e^4} \to \frac{1}{\not{p} - m_0 - \delta m + i\varepsilon}\underbrace{(1 - H(p))}_{\text{finite}} = S_F^{Mod}(p) \tag{11.3.2}$$

and

$$\delta m = e_0^2 \frac{3m}{8\pi^2} \ln \frac{\Lambda}{m} + \mathcal{O}(1) \tag{11.3.3}$$

Note from eqn above, as we take $\Lambda \to \infty$ to restore the actual theory as it is, $\delta m \to \infty$. If m is the finite mass we observe, then we must have the bare mass $m_0 = -\infty$. The difference between the positive infinity of δm and the negative infinity leaves the finite quantity we measure, m.

$$\begin{aligned} &m_0 = -\alpha \\ &\lim_{\Lambda\to\infty} m = \lim_{A\to\infty} m_0 + \lim_{\Lambda\to\infty} \delta m = m_0 + \lim_{\Lambda\to\infty} \delta m = -\infty + \infty = m \end{aligned} \tag{11.3.4}$$

If we were to sum all the higher order diagrams, we have

$$S_F^{Mod}(p) = \frac{1}{\not{p} - m + i\varepsilon}\underbrace{\left(1 - H(P) - \left(\begin{array}{c}\text{Higher}\\ \text{order}\end{array}\right)\right)}_{finite}, \quad m = m_0 + \delta m \tag{11.3.5}$$

$$\delta m = c_0^2 \frac{3m}{8\pi^2} \ln \frac{\Lambda}{m} + \mathcal{O}(1) + \left(\text{ terms in } e_0^4, \Lambda\right) + (\text{ higher order }) \tag{11.3.6}$$

Note that repormalized mass m is not a function of the energy level of the interaction whereas $e(k)$ is a function of energy level, k (symbol p also used for this energy).

11.4 Express $e(k)$ as $e(p)$ or other symbol for energy

Taking $k \to p$, and using the expansion relation $\sqrt{1+2x} = 1 + x + (\text{ higher order in } x)$, we have

$$e(p) = e_0 \left(1 + e_0^2 b_n \ln \frac{p}{\Lambda} + (\text{ higher order in } e_0^2)\right) \tag{11.4.1}$$

The first thing we have to know about $e(p)$ is that it depends only on the photon propagator contribution. Second, **The second term in $e(p)$ has a lot to do with the photon self-energy** (also called the photon loop or the closed fermion loop). Since our fermion loop could be made of different particle pairs (muon/anti-muon, tau/anti-tau,etc), b_n takes these particle pairs into account by enumerating possible pairs at given energy level:

$$b_n = \sum_{a=1}^{n} \frac{1}{12\pi^2} \lambda_a Q_b^2 \tag{11.4.2}$$

where $\lambda_2 =$ number of possible pair types, $Q_a =$ charge in units of e_0.

For $a = 1$, we would have particle/tanti-particle pairs of electron/positron, muon/anti-muon, tawanti-tau, each having associated particle charge $Q_1 = 1$ (1 unit of es per particle), and $\lambda_1 = 3$ (3 families of particles). For quarks of charge magnitude (2/3) e_0, we would take $a = 2$ with $Q_2 = 2/3$ and $\lambda_2 = 9$, because there are three such quarks (up, charmed, and top, each having an associated anti-quark), **each of which comes in three different colors, making a total of nine such pairs**. For quarks of charge magnitude $e_0/3$ (down, strange, and bottom quarks, each having an associated antiquark), we would have $Q_3 = 1/3$ and $\lambda_3 = 9$, each coming in three different possible colors. For W^+ and its anti-particle $W^-, Q_4 = 1$, and $\lambda_4 = 1$.

One thing to be cautious about is that a given particle/anti-particle pair can only arise if the energy of the interaction (incoming particles) is sufficient to create a given pair. For example, the tau has a mass of 1.78 GeV. If our interaction energy k were only 1 GeV, we would not

be able to create tau/anti-tau pairs (needing two times 1.78GeV). So we would have to leave them out of (11.4). Thus, We need an incoming energy equal to, or greater then twice the mass of the particular particle type in the loop, or we will never have that loop. For future reference, see the table below:

Table 11.1: Table of Electrically Charged Particle Masses

Lepton	Q_a	Mass	**Quark**	Q_a	Mass for Virtual Loops	Effective, in proton	**Boson**	Mass
e^-	-1	.511MeV	up	+2/3	3 ± 2 MeV	.336 GeV	W^+, W^-	$80.39 \pm .02$ GeV
μ	-1	.106GeV	down	-1/3	6 ± 2 MeV	.336 GeV		
τ	-1	1.777GeV	charmed	+2/3	$1.25 \pm .1$ GeV			
			strange	-1/3	100 ± 20 MeV			
			top	+2/3	172.5 ± 1 GeV			
			bottom	-1/3	$4.25 \pm .1$ GeV			
$1\text{eV} = 1.60 \times 10^{12} erg = 1.78 \times 10^{-33}\text{gm} \quad M = 10^6 \quad \text{G} = 10^9$								

11.4.1 Deriving e(p), given $e(\mu)$

Evaluating (11.4.1) is problematic if we consider that, in reality, $\Lambda \to \infty$. But what we can do is evaluate $e(p)$ given a reference measurement at another energy level μ. To see this, consider

$$\begin{aligned} e(p) &= e_0 \left(1 + e_0^2 b_n \ln \frac{p}{\Lambda} + \left(\text{ higher order in } e_0^2\right)\right) \\ e(\mu) &= e_0 \left(1 + e_0^2 b_n \ln \frac{\mu}{\Lambda} + \left(\text{ higher order in } e_0^2\right)\right) \end{aligned} \tag{11.4.3}$$

Dividing two equations we have

$$\frac{e(p)}{e(\mu)} = 1 + e_0^2 b_n \ln \frac{p}{\Lambda} - e_0^2 b_n \ln \frac{\mu}{\Lambda} + \begin{pmatrix} \text{higher order} \\ \text{in } e_0^2 \end{pmatrix}$$

$$= 1 + e_0^2 b_n \ln \frac{p}{\mu} + \left(\text{ higher order in } e_0^2\right)$$

Thus, from 11.4.3, we have

$$e_0 = \frac{e(\mu)}{1 + e_0^2 b_n \ln \frac{\mu}{\Lambda} + \left(\text{ higher order in } e_0^2\right)}$$

Squaring the eqn. above we get

$$\frac{e(p)}{e(\mu)} = 1 + \left(e^2(\mu)\left(1 - 2e^2 b_n \ln\frac{\mu}{\Lambda} + (\text{ higher order in } e_0^2)\right)\right) b_n \ln\frac{p}{\mu} + (\text{ higher order in } e_0^2)$$

$$= 1 + e^2(\mu) b_n \ln\frac{p}{\mu} - e^2(\mu) 2e_0^2 b_n \ln\frac{\mu}{\Lambda} b_n \ln\frac{p}{\mu} + (\text{ higher order in } e_0^2)$$

Finally we end up with

$$e(p) = e(\mu)\left(1 + e^2(\mu) b_n \ln(\frac{p}{\mu}) + (\text{ higher order })\right) \tag{11.4.4}$$

So, if we know $e(\mu)$ from experiment, we can calculate (to high accuracy if higher order terms are small, which they are $e(p)$. We never have to take $\Lambda \to \infty$. Of course, we will have to take care with b_n as it might change over the region between energy μ and p. In terms of the running coupling constant, we can rewrite (11.4.4) based on eqn. (11.2.7) as

$$\alpha(p) = \alpha(\mu)\left(1 + \alpha(\mu) 8\pi b_n \ln\frac{p}{\mu} + (\text{ higher order })\right) \tag{11.4.5}$$

Because of the de Broglie relation, where 3 Note that bet have an effective inverse relation between purticle energy (which depends on 3-momentum) and distance (i.e., particle wavelength). Higher energy particles have higher 3-momentum and shorter wavelengths. Thus, $\alpha(p)$ can be plotted against λ.For all distances larger than sub-atomic, we can take $\alpha = \alpha_0 = 1/137$.

11.4.2 Renormalization Group Equation

Using (11.4.5), one can derive a differential equation for the behavior of the running QED coupling constant. First take the partial derivative with respect to mu and ignore negligible contributions from higher order.

$$\underbrace{\frac{\partial}{d\mu}\alpha(p)}_{=0} = \frac{\partial}{\partial\mu}\alpha(\mu) + \left(\frac{\partial}{\partial\mu}\alpha^2(\mu)\right) 8\pi b_n \ln\frac{p}{\mu} - \alpha^2(\mu) 8\pi b_n \frac{\partial}{\partial\mu}\ln\mu + \alpha^2(\mu) 8\pi b_n \underbrace{\frac{\partial}{\partial\mu}\ln p}_{0}$$

Now evaluate at $\mu = p$, this give us

$$\left(\mu\frac{\partial}{\partial\mu}\alpha(\mu)\right)_{\mu=p} = \left(\alpha^2(\mu) 8\pi b_n\right)_{\mu=p}$$

or finally, what is called the **renormalization group equation (RGE)**(11.4.6) for the QED coupling constant, where RHS is called the **beta function.**

$$p\frac{\partial}{\partial p}\alpha(p) = \alpha^2(p)8\pi b_n = \beta\,(p_n b_n) \tag{11.4.6}$$

Beta function is positive. If the beta function were zero, then the QED coupling constant, and the theory itself, would be **energy/scale invariant**. As an aside, the beta function in weak and strong interactions is negative.

11.5 Adiabatic Hypothesis

Prior to interaction, particles are not truly free of interactions, since they can have self-energy interactions like the incoming leptons do for $S^{(4)}_{B1-2}$ to $S^{(4)}_{B1-9}$ of Fig (8.3). That is, there is really **no time that interaction is not occurring.** That is, **there are no "hare" particles.** All real particles are always "dressed with loops like those in the cited diagrams.

However, when we compute amplitudes, we consider the incoming particles (l.e., the ket $|i\rangle$) be "bare", which one can visualize, in a sense that isn't really true, as the incoming lines in the cited diagrams prior to the loops forming. So mathematically, we think in terms of "bare" incoming particles that acquire "dressing" as they enter the interaction arena.

In doing this, we are using what is called the adiabatic hypothesis. "Adiabatic" in thermodynamics means a system does not exchange heat with the environment. Radiation is a form of heat. So by assuming there are no self-energy radiative loops, we assume an adiabatic situation exists prior to the forming of the loops. For this, **we must assume the coupling constant α is turned off, so particles cannot self-interact.** That is, we consider its effect on out interactions modified by a function of time, $e(p) \to f(t)e(p)$, where $f(t)$ is shown in the following figure:

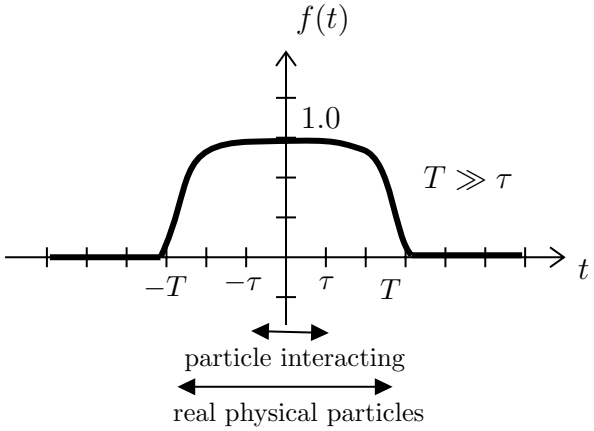


Figure 11.1: The adiabatic hypothesis

Prior to $-T$, the particles are bare (ie, as $|i\rangle$). After $-T$, they become "dressed" due to self interaction, but are not close enough yet to interact with one another. At $t = -\tau$, they become close chough to interact with one another, which continues until τ. The dressed particles then move apart and cease interacting, but remain dressed until T. At T, self-interaction ceases leaving the outgoing particles as bare, once again (i.e., as $|f\rangle$).

This is all fiction, of course, due in part to the fact that bare particle are never seen in the physical world and can never be measured. But it allows us to use our math expressions which take $|i\rangle$ and $|f\rangle$ to be bare particles. After all our calculations are done, we take $T \to \infty$ to restore our fiction to reality.

11.6 Chapter Summary

11.6.1 The Renormalization Procedure

1. Draw all relevant Feynman diagrams with 2 to $2n$ vertices.

2. Add all the related Feynman amplitudes.

3. Evaluate all the propagator integrals in the amplitudes via regularization, yielding quantities e and m dependent on e_0 and m_0 respectively, and a parameter Λ, wherein e and m become

infinite when $\Lambda \to \infty$. Other quantities will be obtained that are finite and not dependent on Λ. These other quantities modify propagators and vertex relations.

4. Redefine e_0 and m_0 such that, as $\Lambda \to \infty, e$ and m equal the finite, physical values for the QED charge constant and fermion particle (rest) mass. e is dependent on interaction energy level p and is expressed as $e(p)$.

11.6.2 Solving Scattering Problems to Order n

1. Write down the tree level amplitude with bare quantities e_0 and m_0

2. Replace e_0 and m_0 with renormalized values $e(p)$ and m, where p is the energy level of the interaction, m is the measured particle (rest) mass, and $e(p)$ is measured charge at energy level p.

3. Replace propagators with modified propagators

4. Replace the vertex relations with modified vertex relations.

11.7 Solved Exercises

1. Does it make sense to you that for our mass renormalization, we can have limit $\lim_{\Lambda\to\infty} \delta m =$ (constant) $\lim_{\Lambda\to\infty} e_0^2 \ln \frac{\Lambda}{m} = \infty$ and $e_0^2 = 0$, whereas in our charge renormalization, we have (different constant) $\lim_{\Lambda\to\infty} e_0^4 \ln \frac{k}{\Lambda} = e^2(k) =$ finite and $e_0^2 = 0$? Why?

Solution:

Note first that $\ln \frac{1}{\Lambda} = -\ln \Lambda$ and the minus sign can be part of a constant. So, consider, as $\Lambda \to \infty$, $e_0^2 ln\Lambda = \infty$ and $e_0^4 ln\Lambda = finite$. It makes sense because $e_0^4 ln\Lambda = finite$ means $e_0^4 \propto \frac{1}{\ln \Lambda}$. So, $e_0^2 \propto \frac{1}{\sqrt{\ln \Lambda}}, e_0^4 \ln \Lambda \propto \frac{1}{\ln \Lambda} \ln \Lambda = 1$, but $e_0^2 \ln \Lambda \propto \frac{1}{\sqrt{\ln \Lambda}} \ln \Lambda = \sqrt{\ln \Lambda}$ goes $\to \infty$ as $\Lambda \to \infty$. For $\Lambda \to \infty$,$e_0^2 \propto \frac{1}{\sqrt{\ln \Lambda}} \to 0, e_0^4 \propto \frac{1}{\ln \Lambda} \to 0$.

2. Determine what the QED coupling constant will be at 6.0MeV. (You need to assume the evaluation is at a virtually immeasurable amount less than 6.0MeV.) Then determine what

the coupling constant is at 12.0MeV (i.e., immeasurably below 12.0).

$$\alpha(6.0) \approx \frac{1}{137}\left(1 + \frac{1}{137} 8\pi \frac{1}{12\pi^2} \ln \frac{6.0}{1.022}\right) = \frac{1}{137}\left(1 + \frac{1}{137}\frac{2}{3\pi}(1.770)\right) \approx \frac{1}{136.63}$$

Now use the above value as the starting point for the next calculation. Up/anti-up loops contribute from 6.0 MeV upward, but down/anti-down loops don't contribute below 12 MeV. So now $\mu = 6.0$ MeV and $p = 12.0$ MeV, with

$$b_n = \frac{1}{12\pi^2}\left((1)^2 + 1 \cdot 3\left(\frac{2}{3}\right)^2 + 0\right) = \frac{1}{12\pi^2}\left(\frac{3}{3} + \frac{4}{3}\right) = \frac{7}{36\pi^2}$$

$$\alpha(p) \approx \alpha(\mu)\left(1 + \alpha(\mu) 8\pi b_n \ln \frac{p}{\mu}\right) = \frac{1}{136.28}$$

Chapter 12

Renormalization Toolkit

12.1 The Three Key Integrals

Applying Feynman's rules in Fig.(12.1), excluding the incoming and outgoing particles, yields the amplitudes under each diagram, where the symbols are defined in (12.1.1) to (12.1.3)

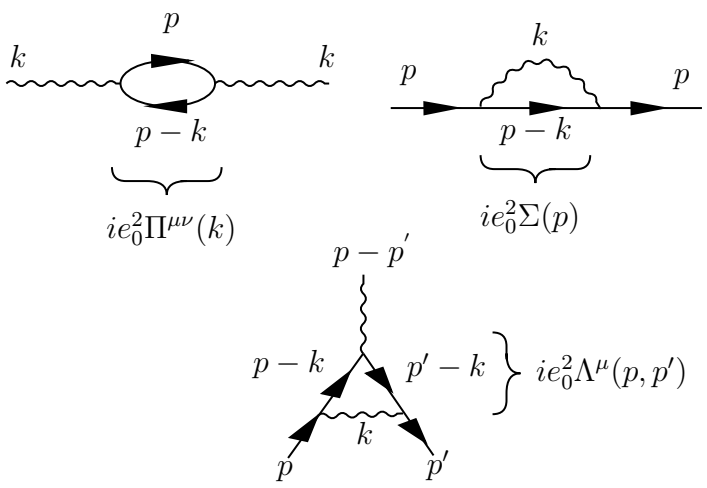


Figure 12.1: Photon Self-energy, Fermion Self-energy, and Vertex Loop Correction

To aid us in the future, we will represent the integrals in these amplitudes, respectively and apart from ie_0^2 factors, by the symbols $\Pi^{\mu\nu}(k)$, $\Sigma(p)$, and $\Lambda^{\mu}(p, p')$.

$$\Pi^{\mu\nu}(k) = \frac{-i}{(2\pi)^4}\,\mathrm{Tr}\int iS_F(p)\gamma^\mu iS_F(p-k)\gamma^\nu d^4p \tag{12.1.1}$$

$$\Sigma(p) = \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha iS_F(p-k)\gamma^\beta d^4k \tag{12.1.2}$$

$$\Lambda^\mu(p,p') = \frac{-1}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha iS_F(p'-k)\,\gamma^\mu iS_F(p-k)\gamma^\beta d^4k \tag{12.1.3}$$

When one carries out the regularization process on the equations above, it allows integrals to be evaluated, with the final result expressed in terms of Λ (not Λ^μ). Then, in the final result at the end, one takes the limit of Λ to obtain the real-world expression of the divergent integral. In each case, **one finds a result with at least one term is finite and at least one other that is finite for finite Λ, but infinite with we take the limit of Λ.** We represent these different terms by the symbols shown below:

$$\Pi^{\mu\nu}(k) = -\underbrace{g^{\mu\nu}k^2A'(k,\Lambda)}_{\infty \text{ for } \Lambda\to\infty} - \underbrace{g^{\mu\nu}k^2\Pi_c\left(k^2\right)}_{\text{finite}} \tag{12.1.4}$$

$$\Sigma(p) = \underbrace{A(\Lambda,m)}_{-\infty \text{ for } \Lambda\to\infty} + \underbrace{(\not{p}-m)B(\Lambda)}_{\infty \text{ for } \Lambda\to\infty} + \underbrace{(\not{p}-m)\Sigma_c(\not{p}-m)}_{\text{finite}} \tag{12.1.5}$$

$$\Lambda^\mu(p,p') = \underbrace{L(\Lambda)\gamma^\mu}_{\infty \text{ for } \Lambda\to\infty} + \underbrace{\Lambda_c^\mu(p,p')}_{\text{finite}} \tag{12.1.6}$$

Note the subscript "c" represents the convergent part of the integral. $\Sigma(p), \Pi^{\mu\nu}(k)$ and $\Lambda^\mu(p,p)$ be considered as Taylor expansions. Their full expressions are

$$\Pi^{\mu\nu}(k) = g^{\mu\nu}k^2\underbrace{2b_n ln\frac{k}{\Lambda}}_{-A'(k,\Lambda)} + g^{\mu\nu}k^2\underbrace{\frac{1}{2\pi^2}\int_0^1 z(1-z)\ln\left(z(1-z)-m^2/k^2\right)dz}_{-\Pi_c(k^2)} \tag{12.1.7}$$

$$\Sigma(p) = \underbrace{-\frac{3m}{8\pi^2}\ln\frac{\Lambda}{m}}_{A(\Lambda,m)} + (\not{p}-m)\underbrace{\frac{1}{8\pi^2}\ln\Lambda}B(\Lambda) + (\not{p}-m)\Sigma_c\left(\not{p}-m\right) \tag{12.1.8}$$

$$\Lambda^\mu(p,p') = \underbrace{\frac{1}{8\pi^2}\gamma^\mu\ln\Lambda}_{L(\Lambda)\gamma^\mu} + \Lambda_c^\mu(p,p') \tag{12.1.9}$$

12.2 Relations We'll Need

12.2.1 Auxiliary Relations

$$(\gamma^v p_v - m)\, u_r(\mathbf{p}) = 0 \quad \bar{u}_r(\mathbf{p})\,(\gamma^v p_v - m) = 0 \tag{12.2.1}$$

$$[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{12.2.2}$$

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\left([\gamma^\mu, \gamma^\nu]_+ + [\gamma^\mu, \gamma^\nu]\right) = g^{\mu v} + \frac{1}{2}[\gamma^\mu, \gamma^v] \tag{12.2.3}$$

For A and B any two operators, which need not commute,

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \ldots \tag{12.2.4}$$

Box

Proof

$$1 = \frac{1}{A-B}(A-B) = \frac{1}{A}(A-B) + \frac{1}{A}B\frac{1}{A}(A-B) + \frac{1}{A}B\frac{1}{A}B\frac{1}{A}(A-B) +$$

$$= 1 \underbrace{-\frac{B}{A} + \frac{B}{A}}_{=0} \underbrace{-\frac{B}{A}\frac{B}{A} + \frac{B}{A}\frac{B}{A}}_{=0} \underbrace{-\frac{B}{A}\frac{B}{A}\frac{B}{A} + \frac{B}{A}\frac{B}{A}\frac{B}{A}}_{=0} + 0 + \ldots$$

12.2.2 Gordon's Identity

Consider relations like the following for a different momentum and spin state

$$(\gamma^\nu p'_\nu - m)\, u_{r'}(\mathbf{p}') = 0 \quad \bar{u}_{r'}(\mathbf{p}')\,(\gamma^\nu p'_v - m) = 0$$

Multiply the the first by $\bar{u}_{r'}(\mathbf{p}')\,\gamma^\mu$ on the left; then multiply the second by $r^\mu u_r(p)$ on the right; then add the two to get

$$2m\bar{u}_{r'}(\mathbf{p}')\,\gamma^\mu u_r(\mathbf{p}) = \bar{u}_{r'}(\mathbf{p}')\,(\gamma^\mu\gamma^\nu p_v + \gamma^\nu\gamma^\mu p'_v)\, u_r(\mathbf{p})$$

Using the relation for $\gamma^\mu\gamma^\nu$ above, we find

$$2m\bar{u}_{r'}(\mathbf{p}')\,\gamma^\mu u_r(\mathbf{p}) = \bar{u}_{r'}(\mathbf{p}')\left(p_v\left(g^{\mu\nu} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) + p'_v\left(g^{v\mu} + \frac{1}{2}[\gamma^v, y^\mu]\right)\right) u_r(\mathbf{p})$$

$$= \bar{u}_{r'}(\mathbf{p}')\left(p_v\left(g^{\mu v} + \frac{1}{2}[\gamma^\mu, \gamma^v]\right) + p'_v\left(g^{\mu v} - \frac{1}{2}[\gamma^\mu, y^v]\right)\right) u_r(\mathbf{p})$$

Re-arranging and dividing by $2m$, we have **Gordon's Identity**:

$$\bar{u}_{r'}(\mathbf{p}')\,\gamma^{\mu}u_r(\mathbf{p}) = \frac{p^{\mu}+p'^{\mu}}{2m}\bar{u}_{r'}(\mathbf{p}')\,u_r(\mathbf{p}) + \frac{p_v - p'_v}{4m}\bar{u}_{r'}(\mathbf{p}')\,[\gamma^{\mu},\gamma^{v}]\,u_r(\mathbf{p}) \tag{12.2.5}$$

12.2.3 Original Ward Identity

When $p = p'$, we have

$$\frac{\partial\Sigma(p)}{\partial p_{\mu}} = \Lambda^{\mu}(p,p) \tag{12.2.6}$$

Proof of the Original Ward Identity

From

$$(S_F(p))^{-1} = \not{p} - m$$

we find

$$0 = \frac{\partial(1)}{\partial p_{\eta}} = \frac{\partial}{\partial p_{\eta}}\left((S_F(p))\,(S_F(p))^{-1}\right) = \frac{\partial}{\partial p_{\eta}}\left((S_F(p))\,(\not{p} - m)\right)$$

or

$$\frac{\partial S_F(p)}{\partial p_{\eta}}\,(S_F(p))^{-1} = -S_F(p)\gamma^{\eta}$$

Thus,

$$\frac{\partial S_F(p)}{\partial p_{\eta}} = -S_F(p)\gamma^{\eta}S_F(p)$$

Taking $p\eta \to p\eta - k\eta$, we have

$$\frac{\partial S_F(p-k)}{\partial\,(p_{\eta}-k_{\eta})} = -S_F(p-k)\gamma^{\eta}S_F(p-k)$$

Then, with this equation used below, we have

$$\frac{\partial\Sigma(p)}{\partial p_{\mu}} = \frac{\partial}{\partial p_{\mu}}\frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^{\alpha}iS_F(p-k)\gamma^{\beta}d^4k$$

$$= \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^{\alpha}i\frac{\partial S_F(p-k)}{\partial\,(p_{\eta}-k_{\eta})}\underbrace{\frac{\partial\,(p_{\eta}-k_{\eta})}{\partial p_{\mu}}}_{=\sigma^{\mu}_{\eta}}\gamma^{\beta}d^4k$$

$$= \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha i\left(-S_F(p-k)\gamma^\mu S_F(p-k)\right)\gamma^\beta d^4k$$
$$= \frac{-1}{(2\pi)^4}\int iD_{Fa\beta}(k)\gamma^a iS_F(p-k)\gamma^\mu iS_F(p-k)\gamma^\beta d^4k = \Lambda^\mu(p,p)$$

Q.E.D

12.2.4 The Ward Identities

Local gauge invariance means our Lagrangian $\mathcal{M}$ is symmetric in form under the transformations:

$$\psi \to \psi' = e^{-i\alpha(x)}\psi \quad A_y \to A'_v = A_v - \frac{1}{e}\partial_v\alpha(x)$$

And thus, our transition amplitude must also be the same in form, as

$\mathcal{L}$ sym $\to$ $\mathcal{L}_I$ unchanged $\to$ $\mathcal{H}_I$ unchanged $\to$ S unchanged $\to$ S_{fi} unchanged $\to$ $|S_{fi}|^2$ unchanged.

Thus, Feynman amplitude $\mathcal{M}$ is gauge invariant if S_{fi} is. **But it must be the total Feynman amplitude from all diagrams (for given order n).** For $2 \leq n$, the point is that $\mathcal{M}^{(n)}$ is gauge invariant, but the individual $\mathcal{M}^{(n)}_{B1}$ and $\mathcal{M}^{(n)}_{B2}$ need not be.

Recognize that if $\mathcal{L}$ is gauge invariant, then $\mathcal{H}_I$ remains the same under any such gauge, and each term in our S operator expansion (each term contains n factors of $\mathcal{H}_i$) does also. Thus, for each order of interaction n, $S^{(n)}$ is effectively gauge invariant. Hence, so are $S^{(n)}_{ft}$ and $\mathcal{M}^{(n)}$.

For any interaction having one or more photons as initial or final particle(s), we can represent the gauge invariant Feynman amplitude for any order n as

$$\mathcal{M}^{(n)}_{fi} = \varepsilon_{r_1\mu}(\mathbf{k}_1)\,\varepsilon_{r_2\nu}(\mathbf{k}_2)\,\varepsilon_{r_3\eta}(\mathbf{k}_3)\ldots\mathcal{M}^{(n)\mu\nu\eta\cdots}_{fi}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\ldots) \tag{12.2.7}$$

Where we again note that (12.2.7) is gauge invariant only when the amplitude includes the sub amplitudes for every diagram having the same incoming and outgoing states. Consider the initial photon of the LHS of Fig.(12.1) to be a real photon, the self-energy Feynman amplitude of the real photon is

$$M^{(2)}_{\gamma\text{self}} = \varepsilon_{r'\mu}(\mathbf{k}')\underbrace{\left\{\frac{1}{(2\pi)^4}\,\mathrm{Tr}\int S_F(p)ie_0\gamma^\mu S_F(p-k)ie_0\gamma^\nu d^4p\right\}}_{\mathcal{M}^{(2)\mu v}_{\gamma\text{ self}} = ie_0^2\Pi^{\mu\nu}(k)}\varepsilon_{rv}(\mathbf{k})$$

The gauge invariance leads to the **Ward identities**:

$$k_{1\mu}\mathcal{M}_{fi}^{(n)\mu}\left(\mathbf{k}_1,\mathbf{k}_2,\ldots\right)=k_{2\nu}\mathcal{M}_{fi}^{(n)\nu}\left(\mathbf{k}_1,\mathbf{k}_2,\ldots\right)=k_{1\mu}k_{2v}\mathcal{M}_{fi}^{(n)\mu\nu}\left(\mathbf{k}_1,\mathbf{k}_2,\ldots\right)=\ldots=0 \tag{12.2.8}$$

where the superscript of $\mathcal{M}^{(n)}$ represents the vertices labels.

For any amplitude relation of the form on the LHS of the equation below, RHS representing the Ward identities, is true. That is, we simply replace the polarization vector by the associated four-momentum and the result equals zero.

$$\mathcal{M}_{fi}^{(n)}\left(\mathbf{k}_1,\ldots,\mathbf{k}_j,\ldots\right)=\varepsilon_{r_j\mu}\mathcal{M}_{fi}^{(n)\mu}\left(\mathbf{k}_1,\ldots,\mathbf{k}_j,\ldots\right)\quad\rightarrow\quad k_{j\mu}\mathcal{M}_{fi}^{(n)\mu}\left(\mathbf{k}_1,\ldots,\mathbf{k}_j,\ldots\right)=0 \tag{12.2.9}$$

Local gauge invariance leads to both charge conservation and the Ward identities. All three are different ways of saying the same thing. Each implies the other two.

charge conservation $\leftrightarrow$ local gauge invariance $\leftrightarrow$ Ward identities

12.3 Ward Identities, Renormalization, and Gauge Invariance

Consider the scattering of light by light shown in Fig. (12.2). Two incoming photons scatter via fermion virtual particles to yield two outgoing photons. This is called photon-photon scattering, or light-by-light scattering, or less commonly, Delbrick scattering. Occasionally, it is referred to as "four photon vertex", but this is misleading as there are really four vertices, not a single one with four photons connected directly to it.

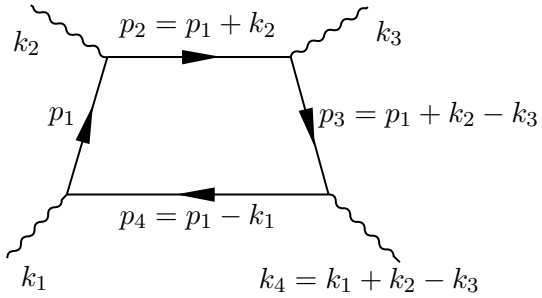


Figure 12.2: One way for Photon-photon scattering

Using Feynman rules, the second order amplitude for the photon-photon scattering of Fig. (12.2), where we distinguish that diagram from its sibling diagrams with the subscript (a), is

$$M_{(a)} = \varepsilon_\mu(\mathbf{k}_4)\,\varepsilon_v(\mathbf{k}_3)\,\varepsilon_\rho(\mathbf{k}_2)\,\varepsilon_\sigma(\mathbf{k}_1)\times$$

$$\underbrace{\frac{-e_0^4}{(2\pi)^4}\,\mathrm{Tr}\int \frac{1}{\not{p}_4 - m + i\varepsilon}\gamma^\mu \frac{1}{\not{p}_3 - m + i\varepsilon}\gamma^\nu \frac{1}{\not{p}_2 - m + i\varepsilon}\gamma^\rho \frac{1}{\not{p}_1 - m + i\varepsilon}\gamma^\sigma d^4p_1}_{\mathcal{M}^{\mu\nu\rho\sigma}_{(a)}}$$

From our Ward identities (12.2.9), where letter subscripts represent different sub-amplitudes contributing to the total second order amplitude, we have

$$k_{1\mu}\mathcal{M}^{\mu v o\sigma}_{\gamma\gamma\to\gamma\gamma} = 0$$

For $k_{1\mu}$ being arbitrary (any possible components), this can only be true if $\mathcal{M}^{\mu\nu\rho\sigma}_{\gamma\gamma\to\gamma\gamma}$ is finite. Therefore, **the amplitude for light-light scattering does not diverge.**

Each of the following being true implies the others are also.

1. The theory is locally gauge invariant(locally symmetric)
2. A quantity (charge) is conserved
3. The theory has the correct interactions
4. The Ward identities hold
5. The theory is renormalizable

If the above are true, then **the gauge boson(photon) is massless.**

12.4 Changes in the Theory with m instead of m_0

12.4.1 Counterterms in Lagrangian and Hamiltonian

If we need to use the measured mass m, we need to investigate what that means for the QED Lagrangian

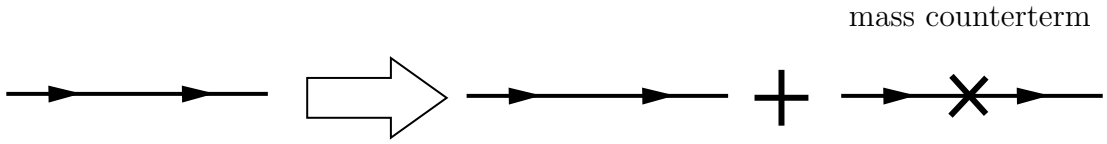
$$\mathcal{L} = -\frac{1}{4}F^{\mu v}F_{\mu v} + \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m_0\right)\psi + e_0\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

We can get the Lagrangian into a form with m by substituting $m_0 = m - \dot{c}m$

$$\mathcal{L} = -\frac{1}{4}F^{\mu v}F_{\mu v} + \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + \underbrace{e_0\bar{\psi}\gamma^{\mu}\psi A_{\mu} + \overbrace{\delta m\bar{\psi}\psi}^{\text{mass counter term}}}_{\text{mass normalized } \mathcal{L}_I} \tag{12.4.1}$$

Recall that $\mathcal{L}_I = -\mathcal{H}_I$ and that **each term in $\mathcal{L}_I$ (or equivalently, $\mathcal{H}_l$) represents an interaction** (a vertex, typically, in the sense that it gives rise to a corresponding vertex in a Feynman diagram).

If we want to use a Lagrangian with the measured mass instead of the bare mass, We must also include a fermion self-interaction diagram as shown in the RHS of Fig.(12.3):



a) Free Fermion for m_0 b) Free Fermion for m

Figure 12.3: Equivalent Free Fermion Feynman Diagrams

Thus, **our Feynman rule for this term is**

10. For each mass counterterm diagram, add a term to the Feynman amplitude with a factor equal to $i\delta m$.

With mass renormalized, the Dirac equation is

$$\left(i\gamma^{\alpha}\partial_{\alpha} - m\right)\psi = 0 \tag{12.4.2}$$

12.5 B in $\Pi(p)$ = L in $\Lambda^\mu(p,p')$

Let's now express $\Lambda^\mu(p,p')$ in its most general possible form. Such a form must contain all possible related entities in the theory having components μ, and these are simply γ^μ,p^μ, and p'^μ. Thus,

$$\Lambda^\mu(p,p') = a\gamma^\mu + b_1 p^\mu + b_2 p'^\mu \tag{12.5.1}$$

$\Lambda^\mu(p,p')$ represents a vertex. If $p = p'$, then $\Lambda^\mu(p,p))$ **represent a free fermion with the vertex loop becoming a fermion self-energy loop.**

Thus, where $p = p'$ and $b = b_1 + b_2$, (12.5.1) becomes

$$\Lambda^\mu(p,p) = a\gamma^\mu + bp^\mu \rightarrow \bar{u}_r(\mathbf{p})\Lambda^\mu(p,p)u_r(\mathbf{p}) = \bar{u}_r(\mathbf{p})a\gamma^\mu u_r(\mathbf{p}) + \bar{u}_r(\mathbf{p})bp^\mu u_r(\mathbf{p})$$

In Gordon's identity (12.2.5) that when $p' = p$, we have

$$\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p}) = \frac{p^\mu + p^\mu}{2m}\bar{u}_r(\mathbf{p})u_r(\mathbf{p}) = \frac{p^\mu}{m}\bar{u}_r(\mathbf{p})u_r(\mathbf{p})$$

and

$$\bar{u}_r(\mathbf{p})\Lambda^\mu(p,p)u_r(\mathbf{p}) = a\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p}) + mb\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p})$$

So,

$$\bar{u}_r(\mathbf{p})\Lambda^\mu(p,p)u_r(\mathbf{p}) = L\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p}) \quad \text{where } L = a + mb$$

Thus,$\Lambda^\mu(p,p) = L(\Lambda)\gamma^\mu$, and

$$\Lambda^\mu(p,p') = \underbrace{L(\Lambda)\gamma^\mu}_{\text{free particle part}} + \underbrace{\Lambda_c^\mu(p,p')}_{=0, for (p=p')}$$

From the original Ward identity,

$$\bar{u}_r(\mathbf{p})\frac{\partial\Sigma(p)}{\partial p_\mu}u_r(\mathbf{p}) = \bar{u}_r(\mathbf{p})\Lambda^\mu(p,p)u_r(\mathbf{p})$$

$$= \bar{u}_r(\mathbf{p})\frac{\partial}{\partial p_\mu}\left(A + (p_v\gamma^v - m)B + (p_v\gamma^v - m)\Sigma_c\right)u_r(\mathbf{p}) = \bar{u}_r(\mathbf{p})L\gamma^\mu u_r(\mathbf{p})$$

$$= \bar{u}_r(\mathbf{p})\gamma^\mu B u_r(\mathbf{p}) + \bar{u}_r(\mathbf{p})\gamma^\mu \underbrace{\Sigma_c(\not{p} - m)u_r(\mathbf{p})}_{=0} + \underbrace{\overline{u_r}(\mathbf{p})(\not{p} - m)}_{=0}\frac{\partial}{\partial p_\mu}\Sigma_c(\not{p} - m)u_r(\mathbf{p})$$

Note that $\Sigma_c(\not{p}-m)$ is part of an expansion in $p-m$, so each term in it has $\not{p}-m$ raised to a power. The factor of $\not{p}-m$ on the right in any such term acts on $u_r(\mathbf{p})$ results in zero. Thus,

$$B\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p}) = L\bar{u}_r(\mathbf{p})\gamma^\mu u_r(\mathbf{p}) \rightarrow B = L \tag{12.5.2}$$

12.6 Re-expressing 2nd Order Corrections

12.6.1 The 2nd Order Photon Propagator

The figure below shows how the **Feynman diagram for the photon propagator at first order becomes two diagrams at second order.**

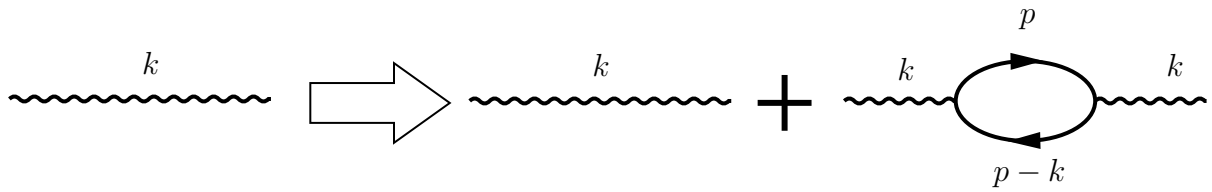


Figure 12.4: Photon Propagator Self-energy Correction to 2nd Order in α

This means the photon propagator becomes

$$iD_{F\alpha\beta}(k) \Rightarrow iD^{2nd}_{F\alpha\beta} = iD_{F\alpha\beta}(k) + iD_{F\alpha\mu}(k)ie_0^2\Pi^{\mu\nu}(k)iD_{F\nu\beta}(k) \tag{12.6.1}$$

The RHS of (12.6.1) is thus

$$iD^{2nd}_{F\alpha\beta}(k) = -\frac{ig_{a\beta}}{k^2+i\varepsilon} + \frac{-ig_{\alpha\mu}}{k^2+i\varepsilon}ie_0^2 g^{\mu\nu}\left(-k^2A'(k,\Lambda) - k^2\Pi_c\left(k^2\right)\right)\underbrace{\frac{-ig_{\upsilon\beta}}{k^2+i\varepsilon}}_{\approx k^2}$$

$$= \frac{-ig_{\alpha\beta}}{k^2+i\varepsilon} + \frac{-ig_{\alpha\mu}}{k^2+i\varepsilon}e_0^2\delta^\mu{}_\beta\left(-A'(k,\Lambda) - \Pi_c\left(k^2\right)\right) = \underbrace{\frac{-ig_{\alpha\beta}}{k^2+i\varepsilon}}_{iD_{F\alpha\beta}(k)}\left(1 - e_0^2A'(k,\Lambda) - e_0^2\Pi_c\left(k^2\right)\right)$$

So

$$iD^{2nd}_{F\alpha\beta}(k) = iD_{F\alpha\beta}\left(1 - e_0^2A'(k,\Lambda) - e_0^2\Pi_c\left(k^2\right)\right) \tag{12.6.2}$$

12.6.2 The 2nd Order Fermion Propagator

The figure below depicts Feynman diagrams for two different ways we can represent the 2nd order fermion propagator, depending on whether we wish to work with m_0 or m.

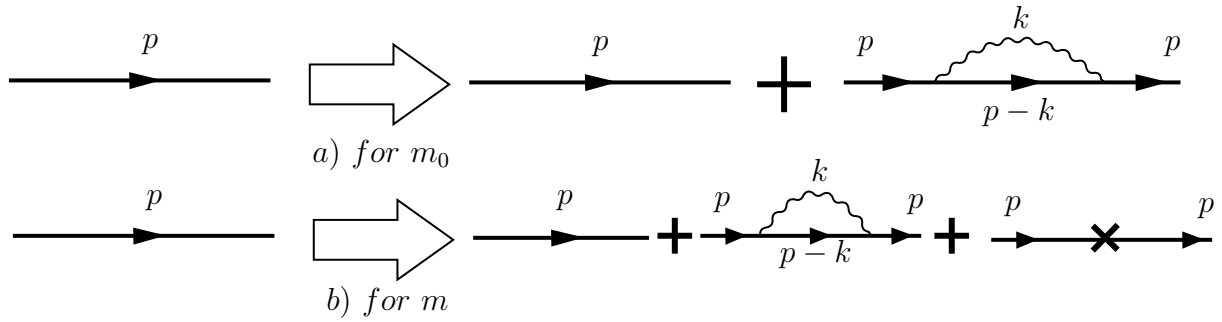


Figure 12.5: Fermion Propagator Self-energy Correction to Diagrams of 2nd Order in α

We will work with a) here first. Thus, the 2nd order fermion propagator is

$$iS_F(p) \Rightarrow iS_F^{2nd}(p) = iS_F(p) + iS_F(p)ie_0^2\Sigma(p)iS_F(p) \tag{12.6.3}$$

and

$$iS_F^{2nd}(p) = i\frac{1}{\underbrace{\not p - m_0 + i\varepsilon}_{A \text{ operator}}} + i\frac{1}{\underbrace{\not p - m_0 + i\varepsilon}_{A \text{ operator}}}\underbrace{\left(-e_0^2\Sigma(p)\right)}_{B \text{ operator}}\frac{1}{\not p - m_0 + i\varepsilon}$$

$$= \frac{i}{A-B} + \begin{pmatrix} \text{higher} \\ \text{order} \end{pmatrix}$$

$$= \frac{i}{\not p \underbrace{-m_0 + e_0^2 A(\Lambda, m)}_{-m} + e_0^2(\not p - m)B(\Lambda) + e_0^2(\not p - m)\Sigma_c(\not p - m) + i\varepsilon} + h.o.$$

where $A(\Lambda, m)$ is unbounded, actually $A(\Lambda, m) \to -\infty$. And h.o. means higher order corrections. Thus, our renormalized fermion propagator to 2nd order, is

$$iS_F^{2nd}(p) = \frac{i\left(1 + e_0^2 B(\Lambda) + e_0^2\Sigma_c(\not p - m)\right)^{-1}}{(\not p - m) + i\varepsilon\left(1 + e_0^2 B(\Lambda) + e_0^2\Sigma_c(\not p - m)\right)^{-1}} + h.o.$$

$$= \frac{i\left(1 + e_0^2 B(\Lambda) + e_0^2\Sigma_c(\not p - m)\right)^{-1}}{(\not p - m) + i\varepsilon\left(1 + e_0^2 B(\Lambda) + e_0^2\Sigma_c(\not p - m)\right)^{-1}} + h.o.$$

Using $\frac{1}{1+x} = 1 - x + \ldots$, we have

$$iS_F^{2nd}(p) = \frac{i\left(1 - e_0^2 B(\Lambda) - e_0^2 \Sigma_c(\not{p} - m)\right)}{(\not{p} - m) + i\varepsilon\left(1 - e_0^2 B(\Lambda) - e_0^2 \Sigma_c(\not{p} - m)\right)} + h.o.$$

$$= \underbrace{\frac{i}{\not{p} - m + i\varepsilon}}_{iS_F(p)}\left(1 - e_0^2 B(\Lambda) - e_0^2 \Sigma_c(\not{p} - m)\right) + h.o.$$

So

$$iS_F^{2nd}(p) \approx iS_F(p)\left(1 - e_0^2 B(\Lambda) - e_0^2 \Sigma_c(\not{p} - m)\right) \tag{12.6.4}$$

12.6.3 2nd Order Incoming and Outgoing Particles

We also have to include higher order corrections to the incoming and outgoing particles, not just the propagators and vertices. The external 2nd fermion corrections, for example, is depicted below

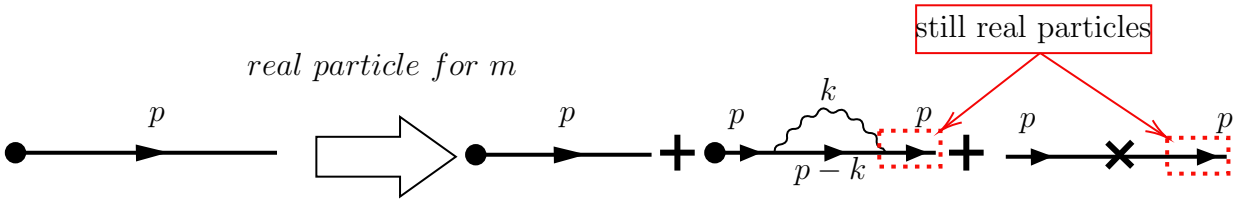


Figure 12.6: Real Fermion Self-energy Correction to Diagrams of 2nd Order in α

The initial fermion contribution to the amplitude, upon renormalization, becomes (where we note that in the last two diagrams of Fig. (12.6) the last line is really an internal line in the overall Feynman diagram for an entire reaction, so must be represented by a propagator)

$$u_r(\mathbf{p}) \Rightarrow u_r^{2nd}(\mathbf{p}) = u_r(\mathbf{p}) + \frac{i}{\not{p} - m + i\varepsilon} ie_0^2 \Sigma(p) u_r(\mathbf{p}) + \frac{i}{\not{p} - m + i\varepsilon}(i\delta m) u_r(\mathbf{p})$$

$$= \left(1 - \frac{e_0^2 A(\Lambda, m) + e_0^2(\not{p} - m)B(\Lambda) + e_0^2(\not{p} - m)\Sigma_c(\not{p} - m) + \delta m}{\not{p} - m + i\varepsilon}\right) u_r(\mathbf{p})$$

since $\delta m = -e_0^2 A(\Lambda, m)$, two terms will cancel above. Also, $e_0^2 \Sigma_c(\not{p} - m)$ is an expansion having terms of $(\not{p} - m)$ to various powers. So each of these terms will lead to a factor of $(\not{p} - m)u_r(\mathbf{p})$

above. Thus,

$$u_r(\mathbf{p}) \Rightarrow \left(1 - \frac{e_0^2(\not{p} - m)B(\Lambda)}{\not{p} - m + i\varepsilon}\right) u_r(\mathbf{p}) \quad \text{indeterminate and naive} \tag{12.6.5}$$

We must be wary with (12.6.5) for two reasons. First the **fermion is real and on shell**, so we might at first consider that $(\not{p} - m)u_r(\mathbf{p}) = 0$. However, we also have a factor of $\not{p} - m$ in the denominator, which would make us think the second term would be $e_0^2 B(\Lambda)$. We thus do not know really what that second term is.

Second, we have not considered that the incoming fermion is initially bare, but becomes dressed via self-interactions. That is, **we would need to incorporate the adiabatic hypothesis, whereby $\mathcal{H}_1$ is turned off initially, but then is turned on well before the particle begins to interact with any other particle.**

Note that

$$iS_{F\alpha\beta}(x - y) = \langle 0 \left| T\left\{\psi_\alpha(x)\bar{\psi}_\beta(y)\right\}\right| 0\rangle$$

where the amplitude has a spinor factor associated with each field. In essence, for the fermion (not anti-fermion) propagator, there is a factor of $u_r(\mathbf{p})\bar{u}_r(\mathbf{p})$. To renormalize the propagator, we multiply it by $1 - e_0^2 B(\Lambda) - e_0^2\Sigma_c(\not{p} - m)$. **So, in effect, each of the two spinors u_r and $\bar{u}_r$ is multiplied by the square root of $1 - e_0^2 B(\Lambda) - e_0^2\Sigma_c(\not{p} - m)$.** So we can surmise that the real particle renormalization is the square root of $1 - e_0^2 B(\Lambda) - e_0^2\Sigma_c(\not{p} - m)$, where we drop the Σ_c term. This lead to

$$\begin{aligned}
\bar{u}_r(\mathbf{p}) &\Rightarrow \bar{u}_r^{2nd}(\mathbf{p}) \approx \left(1 - \frac{1}{2}e_0^2 B(\Lambda)\right)\bar{u}_r(\mathbf{p}) \\
v_r(\mathbf{p}) &\Rightarrow v_r^{2nd}(\mathbf{p}) \approx \left(1 - \frac{1}{2}e_0^2 B(\Lambda)\right) v_r(\mathbf{p}) \\
\bar{v}_r(\mathbf{p}) &\Rightarrow \bar{v}_r^{2nd}(\mathbf{p}) \approx \left(1 - \frac{1}{2}e_0^2 B(\Lambda)\right)\bar{v}_r(\mathbf{p}) \\
\varepsilon_\mu(\mathbf{k}) &\Rightarrow \varepsilon_\mu^{2nd}(\mathbf{k}) \approx \left(1 - \frac{1}{2}e_0^2 A'(\Lambda)\right)\varepsilon_\mu(\mathbf{k})
\end{aligned} \tag{12.6.6}$$

Relations above are often called **external line renormalizations**

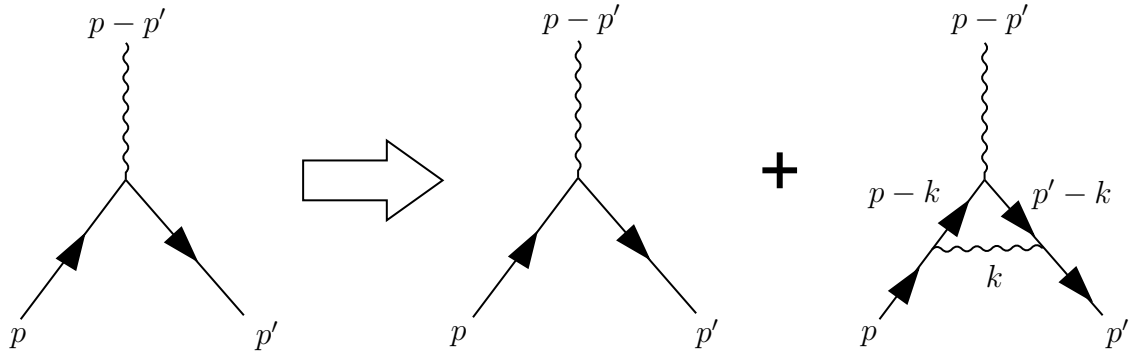


Figure 12.7: Vertex Correction to 2nd Order in α

12.6.4 The 2nd Order Vetex

The vertex modification to second order is depicted in the figure below.

$$i^{\mu}_{0_0\gamma} \Rightarrow ie_0\gamma^{\mu}_{2nd}(p,p') = ie_0\left(\gamma^{\mu} - (ie_0)^2\,\Lambda^{\mu}(p,p')\right)$$

and

$$ie_0\gamma^{\mu} \Rightarrow ie_0\gamma^{\mu}_{2nd}(p,p') = ie_0\left\{\gamma^{\mu}\left(1 + e_0^2 L(\Lambda)\right) + e_0^2\Lambda^{\mu}_c(p,p')\right\} \tag{12.6.7}$$

Chapter 13

Renormalization: Putting It All Together

13.1 Renormalization Example: Compton Scattering

The additional 2nd order in α_0(4th order in e_0) amplitude contributions we need to include. The Feynman diagrams for all of these are shown in Fig. (13.1)

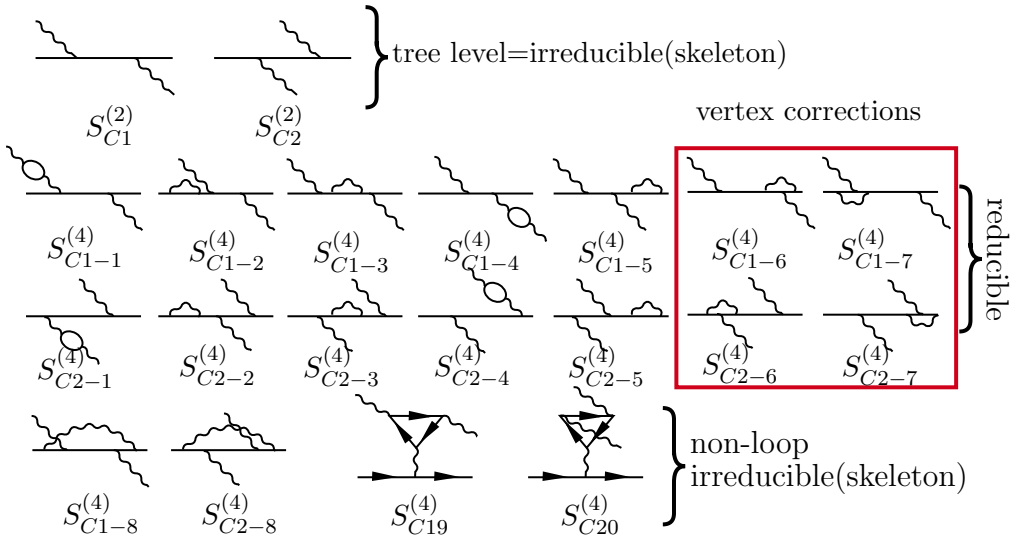


Figure 13.1: Compton Scattering: Feynman Diagram Including 2nd Order Corrections

We will discuss the last four diagrams of Fig.(13.1) first, but before that, we need to define two classes of Feynman diagrams, which are handled differently while renormalizing. The first, represented by the middle two rows in Fig. 13.1 contain propagator/vertex loops and are called reducible diagrams. The second class, represented in the first and last rows, are called irreducible (or skeleton) diagrams and contain no propagator or vertex loops. The word "reducible" is used because reducible diagrams can be reduced to irreducible diagrams by removing the loops. For example, if we take the loop out of any diagram in row two, we get the 1st diagram of row one.

13.1.1 The Incoming to Outgoing Linking Virtual Photon Contribution

The first diagram in the last row,via Feynman rules with k'' as the virtual photon four momentum, has the amplitude

$$\mathcal{M}_{C1-8}^{(4)} = \frac{(ie_0)^4}{(2\pi)^4}\int d^4k''\bar{u}_{r'}(\mathbf{p}')\gamma^\alpha i\delta_F(p-k''+k-k')\varepsilon_\beta(\mathbf{k}')\gamma^\beta\times$$

$$iS_F(p-k''+k)\gamma^\eta\varepsilon_\eta(\mathbf{k})iS_F(p-k'')iD_{F\alpha\delta}(k'')\gamma^\delta u_r(\mathbf{p})$$

Isolate the integral part and power count to determine the maximum possible divergence to get

$$\mathcal{M}_{C1-8}^{(4)} \to \int \frac{1}{(p-k''+k-k')-m+i\varepsilon}\frac{1}{(p-k''+k)-m+i\varepsilon}\frac{1}{(p-k'')-m+i\varepsilon}\frac{1}{k''^2+i\varepsilon}d^4k''$$

$$\text{for large } k'' \to\approx \int \frac{1}{k''}\frac{1}{k''}\frac{1}{k''}\frac{1}{(k'')^2}d^4k'' = 2\pi^2\int\frac{1}{(k'')^5}(k'')^3\,dk^* = -2\pi^2\frac{1}{k''}$$

So this contribution is finite. **The amplitude for the first two diagrams in the last row of Fig (13.1) are finite and negligible at 2nd order compared to the other diagrams**.

13.1.2 The Triangle Diagrams Contribution

Note that the last diagram of Fig. 13.1 can be expressed in a different, but completely equivalent, way. That is, **the order of vertex events can be shown in different time order (time progresses from left to right here) without changing the mathematical**

expression of the associated Feynman amplitude. For the last diagram, switching time ordering of the last to occur vertex with the first to occur vertex is shown in Figure below. The RHS of that figure is identical mathematically to the LHS.

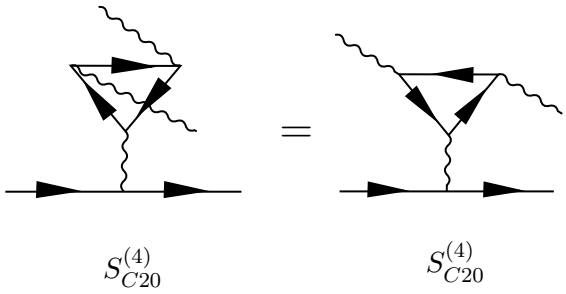


Figure 13.2: Equivalent Triangle Diagrams

Recall from the derivation of the Dirac particle propagator that the propagator for an anti-fermion has the opposite sign of the fermion. So switching fermion and anti-fermion propagators introduces a minus sign into the total amplitude for each such switch. There are three switches from the RHS of Fig.13.2 to the next to last diagram in Fig.13.1. Since $(-1)^3 = -1$, the amplitude for the two diagrams are the same except for sign. So when added, **they cancel exactly, and we don't have to consider them anymore.**

Furry's Theorem: Diagrams containing an all fermion sided polygon having an odd number of sides always occur in pairs, and the contributions of such pairs to the total amplitude cancel out.

Only reducible diagrams yield divergent integrals and thus, divergent amplitudes.

13.2 Renormalizing 2nd Order Divergent Amplitudes

13.2.1 Steps of Renormalization

1.Feynman Diagram Modification

From Fig.(12.5-12.7), we need to first modify the Feynman diagrams at tree level. The propagator, leg, and vertex are modified as

$$\begin{aligned} iD_F\alpha\beta(k) &\Rightarrow iD^{2nd}_{F\alpha\beta} \\ &= iD_{Fa\beta}(k) + iD_{F\alpha\mu}(k)ie_0^2\Pi^{\mu\nu}(k)iD_{F\nu\beta}(k) \end{aligned}$$

$$\begin{aligned} iS_F(p) \overset{(A)\text{ for } m_0}{\Rightarrow} \quad & iS^{2nd}_F(p) \\ &= iS_F(p) + iS_F(p)ie_0^2\Sigma(p)iS_F(p) \end{aligned}$$

$$\begin{aligned} u_r(\mathbf{p}) &\Rightarrow u^{2nd}_r(\mathbf{p}) \\ &= u_r(\mathbf{p}) + iS_F(p)ie_0^2\Sigma(p)u_r(\mathbf{p}) + iS_F(p)(i\delta m)u_r(\mathbf{p}) \end{aligned}$$

analogous for $\bar{u}_r(\mathbf{p}), v_r(\mathbf{p}), \bar{v}_r(\mathbf{p}), \varepsilon_\mu(\mathbf{k})$.

$$\gamma^\mu \Rightarrow \gamma^\mu_{2nd}(p,p') = \gamma^\mu + e_0^2\Lambda^\mu(p,p')$$

2.Expand Loop Integral

From previous chapters we know

$$\Pi^{\mu\nu}(k) = -i(2\pi)^{-4}\,\mathrm{Tr}\int iS_F(p)\times\gamma^\mu iS_F(p-k)\gamma^\nu d^4p$$

$$\Sigma(p) = i(2\pi)^{-4}\int iD_{F\alpha\beta}(k)\times\gamma^\alpha iS_F(p-k)\gamma^\beta d^4k$$

$$\Lambda^\mu(p,p') = -(2\pi)^{-4}\int iD_{Fa\beta}(k)\gamma^\alpha\times iS_F(p'-k)\,\gamma^\mu iS_F(p-k)\gamma^\beta d^4k$$

By performing regularization process, we can write these loop integrals in their most general from as:

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}A\left(k^2\right) + \underbrace{k^\mu k^v B\left(k^2\right)}_{notinthiscase}$$

$$\Sigma(p) = A + (\not{p} - m)B + (\not{p} - m)\Sigma_c(\not{p} - m)$$

where A and B are different in the two equations above. The most general form for vertex modification is

$$\Lambda^\mu(p, p') = a\gamma^\mu + b_1 p^\mu + b_2 p'^\mu$$

Note that the series expansion of $\Pi^{\mu\nu}$ and $\Lambda^\mu(p, p')$ are

$$\Lambda^\mu\left(p, p'\right) = L\gamma^\mu + \Lambda_c^\mu\left(p, p'\right)$$

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}\left(\underbrace{A(0)}_{=0} + k^2 A'(0) + k^2\Pi_c\left(k^2\right)\right)$$

3.Put Expansion of Loop Integral into Modified Leg, Vertex, and Propagator

To second order, we then have

$$iD_{Fa\beta}^{2nd}(k) = \underbrace{\frac{-ig_{a\beta}}{k^2 + i\varepsilon}}_{iD_{F\alpha\beta}(k)}\left(\underbrace{1}_{1storder} - e_0^2 A' - e_0^2\Pi_c\right) \tag{13.2.1}$$

$$iS_F^{2nd}(p) \approx \frac{i}{p\underbrace{-m_0 + e_0^2 A}_{-m} + e_0^2(\not{p} - m)B + e_0^2(\not{p} - m)\Sigma_c + i\varepsilon} = \frac{i}{(\not{p} - m)\left(1 + e_0^2 B + e_0^2\Sigma_c\right) + i\varepsilon}$$

$$\approx \frac{i}{(\not{p} - m + is)\left(1 + e_0^2 B + e_0^2\Sigma_c\right)} \approx \underbrace{\frac{i}{\not{p} - m + i\varepsilon}}_{iS_F(p)}(1 - \underbrace{-e_0^2 B - e_0^2\Sigma_c}_{\text{loop contribution}}) \tag{13.2.2}$$

$$u_r^{2nd}(\mathbf{p}) = \left(1 - e_0^2 B\right)^{1/2} u_r(\mathbf{p}) = \left(Z_f^{2nd}\right)^{1/2} u_r(\mathbf{p}) \approx \left(1 - \frac{1}{2}e_0^2 B\right) u_r(\mathbf{p}) \approx \frac{1}{\left(1 + e_0^2 B\right)^{1/2}} u_r(\mathrm{p}) \tag{13.2.3}$$

same for $\bar{u}_r^{2nd}(\mathbf{p}), v_r^{2nd}(\mathbf{p}), \bar{v}_r^{2nd}(\mathbf{p})$

$$\varepsilon_\mu^{2nd}(\mathbf{k}) = \left(1 - e_0^2 A'\right)^{1/2} \varepsilon_\mu(\mathbf{k}) = \left(Z_\gamma^{2nd}\right)^{1/2} \varepsilon_\mu(\mathbf{k}) \approx \left(1 - \frac{1}{2} e_0^2 A'\right) \varepsilon_\mu(\mathbf{k})$$

and

$$\gamma_{2nd}^\mu(p, p') = \gamma^\mu \left(1 + e_0^2 L\right) + e_0^2 \Lambda_c^\mu = \underbrace{\gamma^\mu}_{\text{1st order}} \underbrace{+\gamma^\mu e_0^2 L + e_0^2 \Lambda_c^\mu}_{\text{2nd order}} \tag{13.2.4}$$

4.Set New Definitions

After we renormalize the mass of fermions as $\not{p} - m = p - m_0 + e_0^2 A(\Lambda) = \not{p} - m_0 - \delta m$, we define shorthand symbols for quantities we will see repeatedly. The abbreviation h.o. stands for "higer order". The subscripts indicate association with photons(γ), fermion(f), or vertices(V).

$$Z_\gamma^{2nd} = 1 - e_0^2 A'$$

$$Z_f^{2nd} = 1 - e_0^2 B = \frac{1}{1 + e_0^2 B} (\text{ ignore h.o. terms on RHS, at 2 nd order })$$

$$Z_V^{2nd} = 1 + e_0^2 L$$

Because $B = L$, then

$$Z_Y^{2nd} = 1/Z_f^{2nd}$$

5.Assembling second order amplitude

Here we use Compton scattering as an example:

$$\underbrace{\mathcal{M}_{C1}^{(2)} + \mathcal{M}_{C2}^{(2)}}_{\text{tree level, finite}} + \underbrace{\sum_j^{i=7} \mathcal{M}_{C1-i}^{(4)} + \sum_i^{i=7} \mathcal{M}_{C2-i}^{(4)}}_{\text{contain divergent integrals}}$$

Let us begin by examining just the firs way for Compton scattering:

$$\mathcal{M}_{C1}^{(2)} = -e_0^2 \bar{u}_{s'}(\mathbf{p}') \varepsilon_{\mu,s'}(\mathbf{k}') \gamma^\mu i S_F(p+k) \varepsilon_{\nu,r}(\mathbf{k}) \gamma^v u_s(\mathbf{p})$$

$$M_{C1-1}^{(4)} = -e_0^2 \bar{u}_{s'}(\mathbf{p}') \varepsilon_{\mu,r'}(\mathbf{k}') \gamma_{iS_F}^\mu (p+k) \underbrace{\left(-\frac{1}{2} e_0^2 A'\right) \varepsilon_{v,r}(\mathbf{k})}_{\text{external photon loop contrib'}} \gamma^v u_s(\mathbf{p})$$

Note that we only have $(-\frac{1}{2}e_0^2A')$ in the eqn. above because we do not want to double counting the 1st contribution.

$$\mathcal{M}^{(4)}_{C1-2} = -e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^v\left(-\frac{1}{2}e_0^2B\right)u_s(\mathbf{p})$$

$$\mathcal{M}^{(4)}_{C1-3} = -e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\left(-e_0^2B - e_0^2\Sigma_c\right)\varepsilon_{\nu,r}|(\mathbf{k})\gamma^v u_s(\mathbf{p})$$

$$\mathcal{M}^{(4)}_{C1-4} = -e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\left(-\frac{1}{2}e_0^2A'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^v u_s(\mathbf{p})$$

$$\mathcal{M}^{(4)}_{C1-5} = -e_0^2\left(-\frac{1}{2}e_0^2B\right)\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu i\delta_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^v u_s(\mathbf{p})$$

$$\mathcal{M}^{(4)}_{\mathrm{Cl}-6} = -e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\left(\gamma^v e_0^2L + e_0^2\Lambda_c^v\right)u_s(\mathrm{p})$$

$$\mathcal{M}^{(4)}_{C|-7} = -e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\left(\gamma^\mu e_0^2L + e_0^2\Lambda_c^\mu\right)\quad iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^v u_s(\mathbf{p})$$

Keep in mind: Σ_c and Λ_c^μ are spinor quantities, A', B, L are scalars.

$$\mathcal{M}^{(2)}_{C1} + \textstyle\sum_i^{i=7}\mathcal{M}^{(4)}_{C1-i} = \left(1 - \frac{1}{2}e_0^2A' - \frac{1}{2}e_0^2B - e_0^2B - \frac{1}{2}e_0^2A' - \frac{1}{2}e_0^2B + e_0^2L + e_0^2L\right)x$$
$$\underbrace{\left(-e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^v u_s(\mathbf{p})\right)}_{M^{(2)}_{C1}}$$

$$-e^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\left(-e_0^2\Sigma_c\right)\varepsilon_{\nu,r}(\mathbf{k})\gamma^\nu u_s(\mathbf{p})$$
$$-e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\gamma^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})e_0^2\Lambda_c^v u_s(\mathbf{p})$$
$$-e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)e_0^2\Lambda_c^\mu iS_F(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^\nu u_s(\mathbf{p})$$

where the last three lines are finite (and so is the factor shown in the second row). Ignoring higher order terms and using our newly defined symbols, we can re-write the equation above as

$$\mathcal{M}^{(2)}_{C1} + \sum_i^{i=7}\mathcal{M}^{(4)}_{C1-i} \approx$$

$$\left(1-\frac{1}{2}e_0^2A'\right)\left(1-\frac{1}{2}e_0^2B\right)\left(1-e_0^2B\right)\left(1-\frac{1}{2}e_0^2A'\right)\left(1-\frac{1}{2}e_0^2B\right)\left(1+e_0^2L\right)\left(1+e_0^2L\right)\times$$
$$-e_0^2\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,r'}\left(\mathbf{k}'\right)\left(\gamma^\mu + e_0^2\Lambda_c^\mu\right)iS_F(p+k)\left(1-e_0^2\Sigma_c\right)\varepsilon_{v,r}(\mathbf{k})\left(\gamma^v + e_0^2\Lambda_c^v\right)u_s(\mathbf{p})$$

where

$$\left(\gamma^\mu + e_0^2\Lambda_c^\mu\right) = \gamma^\mu_{e_0,mod}$$

$$iS_F(p+k)\left(1-e_0^2\Sigma_c\right) = iS_F^{e_0,mod}(p+k)$$

$$\left(\gamma^\nu + e_0^2\Lambda_c^\nu\right) = \gamma^\nu_{e_0,mod}$$

Using these new symbols, we can express the eqn. above more succinctly as

$$\mathcal{M}^{(2)}_{C1}+\sum_{i}^{i=7}\mathcal{M}^{(4)}_{C1-i}\approx\left(Z_{\gamma}^{2nd}\right)\left(Z_{f}^{2nd}\right)^{2}\left(Z_{V}^{2nd}\right)^{2}\times\underbrace{\left(-e_{0}^{2}\bar{u}_{s'}\left(\mathbf{p}'\right)\varepsilon_{\mu,t'}\left(\mathbf{k}'\right)\gamma^{\mu}_{e_0,Mod}iS_{F}^{e_0,mod}(p+k)\varepsilon_{v,r}(\mathbf{k})\gamma^{\nu}_{e_0Mod}u_{s}(\mathbf{p})\right)}_{\mathcal{M}^{(2)}_{C1,e_0,mod}}$$

where the last term is a modified parallel to our tree level amplitude. Because $B=L$, we finally arrive at

$$\mathcal{M}^{(2)}_{C2}+\sum_{i}^{i=7}\mathcal{M}^{(4)}_{C2-i}\approx Z_{\gamma}^{2nd}\mathcal{M}^{(2)}_{C2,e_0,mod} \tag{13.2.5}$$

This is the key step where we cancel out the infinities introduced by higher order corrections.

In the divergent parts of our second order amplitude, for any type of QED scattering, the divergent fermion factors will cancel with the divergent vertex factors, leaving only a photon divergent factor Z_{γ}^{2nd}. **The remainder of the amplitude can be found by substituting finite modifications to the propagator and vertex relations for those in the tree level amplitude.**

6.Charge renormalization

We now renormalize charges, as

$$e_{0}^{2}\left(1-e_{0}^{2}A'\right)=e_{0}^{2}Z_{\gamma}^{2nd}=e^{2}=\text{ measured charge squared} \tag{13.2.6}$$

or

$$e_{0}\left(1-\frac{1}{2}e_{0}^{2}A'\right)\approx e_{0}\left(Z_{\gamma}^{2nd}\right)^{1/2}\approx e=\text{ measured charge} \tag{13.2.7}$$

Bottom line: So, from now on we can simply replace $\left(1-e_{0}^{2}A'\right)e_{0}^{2}$ with e^{2} in every QED amplitude calculation to 2^{nd} order, and our total amplitude to that order will be finite. We need to keep in mind that e depends on energy level (modestly via a log dependence).

For e_0 in $iS_{F}^{e_0,mod}$,$\gamma^{\mu}_{e_0,mod}$, and $iD_{F\mu\nu}^{e_0,mod}$, we simply replace e_0 with e in their expressions. This is because e_0 parts are themselves modified by 3rd and higher order corrections. Thus

$$iS_{F}^{2nd,mod}(p+k)=iS_{F}(p+k)\left(1-e^{2}\Sigma_{e}\right) \tag{13.2.8}$$

$$\gamma^{\mu}_{2nd,mod}=\gamma^{\mu}+e^{2}\Lambda_{c}^{\mu} \tag{13.2.9}$$

$$iD_{F\mu v}^{2nd,mod}(k) = iD_{F\mu v}(k)\left(1 - e^2\Pi_c\right) \tag{13.2.10}$$

To renormalize any interaction amplitude to 2nd order in α, in the tree level amplitude make the substitutions:

- $e_0 \Rightarrow e(k)$
- $iS_F, \gamma^\mu, iD_{F\mu\nu} \Rightarrow$ (13.2.8)-(13.2.9)

13.2.2 The Short Cut Route

To get the final amplitude, we can use **only tree amplitude and substitute relations (13.2.1)-(13.2.4) to tree amplitude.** Then we rearrange amplitude, via approximations to have factors susch as

$$Z_f^{2nd} = \left(1 - e_0^2 B\right) \text{ and } iS_F^{2nd,e_0,Mod} = \left(1 - e_0^2\Sigma_c\right) iS_F$$

$$\text{Cancel } Z_f^{2nd} \text{ and } Z_V^{2nd} \text{ factors leaving } Z_\gamma^{2nd}\mathcal{M}_{\text{Generic}}^{(2)}$$

Finally, renormalize via $Z_\gamma^{2nd} e_0^2 = e^2$.

13.3 The Total Amplitude to 2nd Order

The total Compton scattering amplitude to second order is then the finite value

$$\mathcal{M}_{C,2nd} = \underbrace{\dot{M}_{C1}^{(2)} + \mathcal{M}_{C2}^{(2)}}_{\text{tree level, finite}} + \underbrace{\sum_i^{i=7} M_{C1-i}^{(4)} + \sum_i^{i=7} M_{C2-i}^{(4)}}_{\text{contain divergent integrals}} + \underbrace{M_{C1-8}^{(4)} + \mathcal{M}_{C2-8}^{(4)}}_{\text{finite using } e_0^4} + \underbrace{M_{C19}^{(4)} + M_{C20}^{(4)}}_{\text{cancel out}} \tag{13.3.1}$$

$$= \mathcal{M}_{C1,mod,2nd}^{(2)} + \mathcal{M}_{C2,mod,2nd}^{(2)} + \underbrace{M_{C1-8,mod,2nd}^{(4)} + M_{C2-8,mod,2nd}^{(4)}}_{\text{finite using} e^4, negligible}$$

The sub-amplitudes $\mathcal{M}_{C1-8}^{(4)}$ and $\mathcal{M}_{C2-8}^{(4)}$ originally had factors of e_0^4,but in similar manner as discussed in previous section, higher order corrections modify this to e^4. They are then of

higher order in e than the other terms, which are of order e^2, and can, at 2nd order, be ignored. Thus, for any type of interaction in general, the renormalized amplitude to 2nd order is

$$\mathcal{M}_{\text{Generic}} = \sum_{j}^{\#\text{ ways}} \mathcal{M}^{(2)}_{\text{Gen } j} + \underbrace{\sum_{j,k} \mathcal{M}^{(4)}_{\text{Gen } j-k}}_{\text{finite using } e^4, negligible} \tag{13.3.2}$$

Chapter 14

Regularization

14.1 Ways to Regularization

There are a number of ways to regularize unbounded integrals. The four most common are

1. Cut off regularization

Instead of integrating from $-\infty$ to $+\infty$, we integrate from $-\Lambda$ to $+\Lambda$. When we renormalize using the resulting relation, we take $\Lambda \to \infty$

2. Pauli-Villars regularization

We add in to QED an additional fictitious particle with mass Λ. So in the propagator for this particle, $m^2 = \Lambda^2$ appears as a term in the denominator (as mass does for all propagators). This adds an additional term to the Feynman amplitude (think of an extra Feynman diagram having this extra virtual particle). As it turns out, this causes the amplitude to converge over the $-\infty$ to $+\infty$ integration range. The result is in terms of Λ but turns out to be divergent as $\Lambda \to \infty$. But we can use that result to renormalize. When we take $\Lambda \to \infty$ in the renormalization process, that means our fictitious particle has infinite mass, so it really never shows up anywhere in creation and drops out of the theory. Mathematically, the propagator denominator goes to infinity, so the term with that propagator makes zero contribution to the amplitude, at the end of the day.

3. Dimensional Regularization

Our unbounded loop integrals are over four dimensional spacetime. It turns out that for dimensions D other than $D = 4$ for spacetime, these integrals can be evaluated readily. So we take the same integrals over $D = 4 - \eta$, where $\eta \neq 0$. In the result, we get terms that are unbounded as $\eta \to 0$ (which corresponds to $\Lambda \to \infty$). We use these terms for renormalization in the same way as we do the results from any other method of regularization. Interestingly, η (and thus D) does not have to be an integer in this, method. That is, we can have fractional dimensions. This seems weird, but mathematically it works.

4. Gauge Lattice regularization(Wilson)

This approach approximates continuous spacetime by breaking it into a lattice comprising a large number of small hyper-cubes(4D "cubes") of fixed grid size (cube edge). Thus, **fields with wavelengths approaching, and smaller than the cube grid length cannot be represented since the fields are approximated by their values at the boundaries of the cubes.** This means particle 3-momenta and energies for shorter wavelengths are excluded. The result, for any given grid size, is a finite Feynman amplitude. After doing calculations on lattices with several different size grids, one can extrapolate to zero grid length, i.e., our natural universe with virtually unbounded possible particle energy and 3-momentum. The gauge lattice approach is an advanced topic and will not be covered in this note.

Ideally, every regularization method should give us the same result.

14.2 Relations We need

Recall that for the Γ function used below, we have

$$\Gamma(n) = (n-1)! \quad n\Gamma(n) = n(n-1)! = n! = \Gamma(n+1)$$

The following integrals can simply be accepted. In the following section we derive one of these to illustrate a procedure called Wick rotation that plays a vital role in regularization.

$$\int \frac{d^4p}{(p^2 + s + i\varepsilon)} = i\pi^2 \left(s \log(-s) - s \log\left(\Lambda^2 - s\right) - \Lambda^2\right) \tag{14.2.1}$$

$$\int \frac{d^4p}{\left(p^2+s+i\varepsilon\right)^2} = -i\pi^2\left(\ln(-s) - \ln\left(\Lambda^2-s\right) + \frac{s}{\Lambda^2-s} + 1\right) \tag{14.2.2}$$

$$\int \frac{d^4p}{\left(p^2+s+i\varepsilon\right)^n} = i\pi^2\frac{\Gamma(n-2)}{\Gamma(n)}\frac{1}{s^{n-2}} \quad n \geq 3 \tag{14.2.3}$$

$$\int \frac{p^\mu}{\left(p^2+s+i\varepsilon\right)^n} d^4p = 0 \quad n \geq 3 \tag{14.2.4}$$

$$\int \frac{p^\mu p^v}{\left(p^2+s+i\varepsilon\right)^n} d^4p = i\pi^2\frac{\Gamma(n-3)}{2\Gamma(n)}\frac{g^{\mu v}}{s^{n-3}} \quad n \geq 4 \tag{14.2.5}$$

$$\int \frac{d^4p}{\left(p^2+2pq+t+i\varepsilon\right)^n} = i\pi^2\frac{\Gamma(n-2)}{\Gamma(n)}\frac{1}{\left(t-q^2\right)^{n-2}} \quad n \geq 3 \tag{14.2.6}$$

$$\int \frac{p^\mu}{\left(p^2+2pq+t+i\varepsilon\right)^n} d^4p = -i\pi^2\frac{\Gamma(n-2)}{\Gamma(n)}\frac{q^\mu}{\left(t-q^2\right)^{n-2}} \quad n \geq 3 \tag{14.2.7}$$

$$\int \frac{p^\mu p^v}{\left(p^2+2pq+t+i\varepsilon\right)^n} d^4p = i\pi^2\frac{\Gamma(n-3)}{2\Gamma(n)}\frac{\left(2(n-3)q^\mu q^v + \left(t-q^2\right)g^{\mu v}\right)}{\left(t-q^2\right)^{n-2}} \quad n \geq 4 \tag{14.2.8}$$

$$\int p^\mu p^v d^4p = \frac{1}{4}\int g^{\mu v}p^2 d^4p \tag{14.2.9}$$

$$\int \frac{p^\mu p^v}{p^2+s} d^4p = \frac{1}{4}\int g^{\mu v}\frac{p^2}{p^2+s} d^4p \tag{14.2.10}$$

14.2.1 Deriving Spacetime Integrals Using Wick Rotation

An integral in 4D Euclidean space is "relatively" easy to evaluate since we can convert from 4D Cartesian form of differential element $d^4x = dwdxdydz$ to 4 D spherical coordinates of differential element $dV_{4D} = d^4r = 2\pi^2r^3dr$, where $2\pi^2r^3$ is the 3 D "surface" of a $4D$ hypersphere. Additionally, our radial distance is $r = \sqrt{w^2+x^2+y^2+z^2}$ and that is simply the measured distance from the origin to a point.

In $4D$ spacetime, however, things are not so simple because $r = \sqrt{t^2-x^2-y^2-z^2}$ and there is no simple interpretation of that as a distance in 4D space. Further, defining a suitable differential element, due to the minus signs in the metric, is problematic. To evaluate 4D spacetime integrals we use a trick called **Wick rotation**

Wick rotation is used to convert 4 D spacetime to an associated 4 D Euclidean space, in which a given integral is easier to evaluate. For it, we simply transform

(or, equivalently, make a substitution of variables in the spacetime integral of) $E \to iE$

The differential element d^4p has a factor of dE in it, so it transforms as shown in Wholeness chart below:

Table 14.1: Wick Rotation Summary

	Minkowski		Euclidean
Time component	E	$\to$	iE
Space component	p^i or $\mathbf{p}$	$\to$	p^i or $\mathbf{p}$
Differential element	d^4p	$\to$	id^4p_E
4D vector definition	$p^\mu = (E, \mathbf{p})$		$p^\mu_E = (E, \mathbf{p})$
4D vector transformation	$(E, \mathbf{p})$	$\to$	$(iE, \mathbf{p})$
Square of 4D vector definition	$p^2 = p^\mu p_\mu = E^2 - \mathbf{p}^2$		$p^2 = p^\mu_E p_{E\mu} = E^2 + \mathbf{p}^2$
Square of 4D vector, transformation	p^2	$\to$	$-p^2_E$

Note, for the last line in the chart if $p_E^2 = E^2 + p^2 = E^2 + (p^1)^2 + (p^2)^2 + (p^3)^2$, then it is essentially the length squared in Euclidean energy-momentum space. And thus, the transformation is

$$p^2 = p^\mu p_\mu = E^2 - \sum (p')^2 = E^2 - \mathrm{p}^2 \xrightarrow[\text{Rotation}]{\text{Wick}} -E^2 - \mathrm{p}^2 = p^\mu_E p_{E\mu} = -p^2_E \tag{14.2.11}$$

To derive (14.2.2), we first convert our integral to Euclidean coordinates via the Wick transformation. We can ignore the $i\varepsilon$, or simply think of it as included temporarily in the constant s.

$$I = \int \frac{d^4p}{(p^2+s)^2} = \int \frac{id^4p_E}{(-p_E^2+s)^2} = i\int \frac{d^4p_E}{(p_E^2-s)^2} = i2\pi^2 \int_0^\Lambda \frac{p_E^3 dp_E}{(p_E^2-s)^2} \tag{14.2.12}$$

We then use the following relation from integral tables (or from manipulating the RHS integrands)

$$\int \frac{x^m dx}{(ax^n+c)^r} = \frac{1}{a}\int \frac{x^{m-n}dx}{(ax^n+c)^{r-1}} - \frac{c}{a}\int \frac{x^{m-n}dx}{(ax^n+c)^r} \tag{14.2.13}$$

where $x = p_E, a = 1, c = -s, n = 2, m = 3$, and $r = 2$. We have

$$I = i2\pi^2 \underbrace{\int_0^\Lambda \frac{p_E dp_E}{(p_E^2 - s)}}_{\frac{1}{2}\ln\left(p_E^2 - s\right)\Big|_0^\Lambda} \underbrace{-(-s)i2\pi^2 \int_0^\Lambda \frac{p_E dp_E}{(p_E^2 - s)^2}}_{I'} \tag{14.2.14}$$

From integral tables, we find

$$\int x\left(ax^2 + c\right)^n dx = \frac{1}{2a}\frac{(ax^2 + c)^{n+1}}{n+1} \tag{14.2.15}$$

Taking $x = p_E, a = 1, c = -s, n = -2$, we have

$$I' = -\, i\pi^2 s \frac{1}{(p_E^2 - s)}\Bigg|_0^\Lambda \tag{14.2.16}$$

Thus,

$$I = i\pi^2 \left[\ln\left(p_E^2 - s\right) - s\frac{1}{(p_E^2 - s)}\right]_0^\Lambda = -i\pi^2\left(\ln(-s) - \ln\left(\Lambda^2 - s\right) + \frac{s}{\Lambda^2 - s} + 1\right) \quad QED. \tag{14.2.17}$$

14.2.2 Some Gamma Matrix Relations from Chapter 3

$$\mathrm{Tr}\left(\gamma^a\gamma^\beta\right) = 4g^{\alpha\beta} \tag{14.2.18}$$

$$\mathrm{Tr}\left(\gamma^\sigma\gamma^\delta\gamma^\mu\gamma^\beta\right) = 4\left(g^{\sigma\delta}g^{\mu\beta} - g^{\sigma\mu}g^{\delta\beta} + g^{\sigma\beta}g^{\delta\mu}\right) \tag{14.2.19}$$

$$\begin{aligned} &\gamma_\lambda\gamma^\lambda = 4, \quad \gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha \\ &\gamma_\lambda\gamma^a\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta} \quad \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\lambda = -2\gamma^\gamma\gamma^\beta\gamma^\alpha \\ &\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\lambda = 2\left(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma + \gamma^\gamma\gamma^\beta\gamma^\alpha\gamma^\delta\right) \end{aligned} \tag{14.2.20}$$

14.2.3 Feynman Parameterization

Integrals (14.2.8) and (14.2.3) are not of the same form as the loop integrals. But thanks to a technique developed by Feynman, we can convert the loop integrals to forms for which we can use (14.2.8) and (14.2.3). The loop integrals typically have a product of several different polynomials multiplied in the denominator rather than a single such polynomial (raised to a power typically), as in the integrals (14.2.8) and (14.2.3).

The fist of these useful relations is

$$\frac{1}{ab} = \frac{1}{b-a}\int_a^b \frac{dt}{t^2} \quad = \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{b-a}\left(\frac{b-a}{ab}\right) = \frac{1}{ab} \tag{14.2.21}$$

Define the **Feynman parameter** z via

$$t = b + (a-b)z \rightarrow dt = (a-b)dz$$

where for the integration limits, $t = a$ means $z = 1$, and $t = b$ means $z = 0$. The LHS of (14.2.21) then becomes (where the RHS of (15-26) follows from the symmetry of a and b on the LHS)

$$\frac{1}{ab} = -\int_1^0 \frac{dz}{(b+(a-b)z)^2} = \int_0^1 \frac{dz}{(b+(a-b)z)^2} = \int_0^1 \frac{dz}{(a+(b-a)z)^2} \tag{14.2.22}$$

Relation (14.2.22) readily extends to three factors,

$$\begin{aligned} \frac{1}{abc} &= 2\int_0^1 dx \int_0^x dy \frac{1}{(a+(b-a)x+(c-b)y)^3} \\ &= 2\int_0^1 dx \int_0^{1-x} dz \frac{1}{(a+(b-a)x+(c-a)z)^3} \end{aligned} \tag{14.2.23}$$

These results can be generalized, via induction, to

$$\frac{1}{a_0 a_1 a_2 \ldots a_n} = \Gamma(n+1)\int_0^1 dz_1 \int_0^{z_i} dz_2 \cdots \int_0^{z_{n-1}} dz_n \frac{1}{\left(a_0 + (a_1 - a_0)\, z_1 + \ldots (a_n - a_{n-1})\, z_n\right)^{n+1}} \tag{14.2.24}$$

14.2.4 Leading Log Approximations

Note that for a function of ε, $f(\varepsilon) = \ln(\Lambda' + \varepsilon)$ where $\varepsilon << \Lambda'$

$$\ln(\Lambda' + \varepsilon) =$$

$$\ln(\Lambda' + \varepsilon)|_{\varepsilon=0} + \frac{\varepsilon}{(\Lambda' + \varepsilon)_{\varepsilon=0}} - \frac{\varepsilon^2}{2\,(\Lambda' + \varepsilon)^2\big|_{\varepsilon=0}} + \ldots = \ln\Lambda' + \frac{\varepsilon}{\Lambda'} - \frac{1}{2}\left(\frac{\varepsilon}{h'}\right)^2 + \ldots \tag{14.2.25}$$

$$\approx \ln\Lambda'$$

14.3 Pauli-Villars Regularization

14.3.1 The Concept

Imagine there were an extremely heavy fermion, identical to the electron, muon, and tau in all qualities except mass. In our low energy (far below the mass of this fermion) experiments it could never play a role as a real particle, since all of them would have decayed into lighter particles at the beginning of the universe, and we would not have enough energy in the experiment to create another one. And it could never influence particle collision interactions (such as Compton scattering) as a tree level virtual particle since those interactions would never reach the mass-energy level of this fermion. We would never know it exists.

However, in our higher order correction loop integrals we integrate to infinite energy levels and it could play a role there. If so, the effect would be like adding another Feynman diagram loop with that particular particle in addition to the one for the electron (and muon and tau) coupled with its antiparticle. That is, if the heavy fermion mass were Λ, then all our propagators in our theory would have to be modified as in (14.3.1). But, in order for this to work, we have to change the propagator of the so-called heavy particle a bit. **We give it a negative sign and keep the ” m” in the numerator without changing it to ”Λ”,h e we do m the denominator.** The goal mathematically is first to perturb the propagator by adding a term to it containing Λ, then take Λ to infinity. For this, the form chosen in (14.3.1) works best

$$\frac{\not{p}+m}{p^2-m^2+i\varepsilon} \to \frac{\not{p}+m}{p^2-m^2+i\varepsilon} - \frac{\not{p}+m}{p^2-\Lambda^2+i\varepsilon} = (\not{p}+m)\frac{m^2-\Lambda^2}{(p^2-m^2+i\varepsilon)\,(p^2-\Lambda^2+i\varepsilon)} \quad (14.3.1)$$

The valuable part of all this, from the point of view of regularization, is that **the RHS of (14.3.1) falls off, at high p, with $1/p^3$, whereas the LHS falls off with $1/p$. This allows loop integrals to converge as $p \to \infty$.** They will diverge with Λ, however. But we take care of that with renormalization where, as $\Lambda \to \infty$ (and our heavy fermion cannot then exist in any sense), quantities like e and m take on physical, finite values.

14.3.2 A Simple(Unphysical) Example

Suppose we had an integral of the form

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2+i\varepsilon\right)^2}d^4p \tag{14.3.2}$$

We could use the Pauli-Villars methodology of (14.3.1)(applied just to the denominator of that expression) to turn this into (where we drop the small ε for convenience)

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}d^4p \to \frac{1}{(2\pi)^4}\int \left(\frac{1}{\left(p^2-m^2\right)^2}-\frac{1}{\left(p^2-\Lambda^2\right)^2}\right)d^4p \tag{14.3.3}$$

Using(14.2.2) in the above with the integration limit Λ in (14.2.2) now equal to infinity and $s=-m^2$ for the first term and $-\Lambda^2$(now the heavy fermion mass) for the second, we get

$$\begin{aligned}
&\tfrac{1}{(2\pi)^4}\int \tfrac{1}{(p^2-m^2)^2}d^4p \to \\
&\quad = \tfrac{1}{(2\pi)^4}\left(-i\pi^2\left(\ln\left(m^2\right)-\ln\left(\infty^2+m^2\right)+1\right)+i\pi^2\left(\ln\left(\Lambda^2\right)-\ln\left(\infty^2+\Lambda^2\right)+1\right)\right) \\
&\quad = \tfrac{i}{(4\pi)^2}\Big(\Big(\ln\tfrac{\Lambda^2}{m^2}\Big)+\underbrace{\ln\left(\infty^2+m^2\right)}_{2\ln\infty+\frac{m^2}{\infty^2}+\ldots}-\underbrace{\ln\left(\infty^2+\Lambda^2\right)}_{2\ln\infty+\frac{\Lambda^2}{\infty^2}+\ldots}\Big)
\end{aligned}$$

This gives us

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}d^4p \to \frac{i}{(4\pi)^2}\left(\ln\frac{\Lambda^2}{m^2}\right)-\frac{i\Lambda^2}{(4\pi)^2\infty^2}+\ldots=\frac{i2}{(4\pi)^2}\left(\ln\Lambda-\ln m-\frac{1}{2}\frac{\Lambda^2}{\infty^2}\right)+\ldots$$

and

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}d^4p \overset{\Lambda\to\infty}{\to} \frac{i2}{(4\pi)^2}\ln\Lambda$$

14.4 Dimensional Regularization

14.4.1 The Concept

We take the same integrals over $D=4-\eta$, where $\eta\neq 0$. In the result, we get terms that are unbounded as $\eta\to 0$ (which corresponds to $\Lambda\to\infty$). We then re-express the result in 4D and Λ for use in renormalization, just as we do for any other regularization technique. Note that in $D\neq 4$ spacetime, one dimension is time and all the rest are spatial.

14.4.2 Relations for Arbitrary Dimension Spacetime

For any integer D, $g_{\mu\nu}$ is a $D \times D$ matrix. Parallel to $g_{\mu\nu}g^{\mu\nu} = 4$ for $D = 4$ spacetime, we have

$$g_{\mu v}g^{\mu\nu} = D \tag{14.4.1}$$

In D dimensions, where D is an integer, there are D gamma matrices labeled $\gamma^0, \gamma^1, \ldots \gamma^{D-1}$ These are $f(P) \times f(D)$ matrices, where $f(D)$ is an integer that depends on D. For $D = 4, f(D) = 4$. And we have anti-commutation relations as

$$\gamma^\mu\gamma^v + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{14.4.2}$$

From these, one can derive contraction and trace relation as

$$\begin{aligned} &\gamma_\lambda\gamma^\lambda = D, \quad \gamma_\lambda\gamma^\alpha\gamma^\lambda = -(D-2)\gamma^\alpha \\ &\gamma_\lambda\gamma^a\gamma^\beta\gamma^\lambda = -(D-4)\gamma^\alpha\gamma^\beta + 4g^{a\beta} \quad \text{etc.} \end{aligned} \tag{14.4.3}$$

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = f(D)g^{\alpha\beta} \tag{14.4.4}$$

$$\mathrm{Tr}\left(\gamma^\sigma\gamma^\delta\gamma^\mu\ldots\right) = 0 \quad \text{for any odd number of gamma matrices,} \tag{14.4.5}$$

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\delta\gamma^\nu\gamma^\sigma\right) = f(D)\left(g^{\mu\delta}g^{\nu\sigma} - g^{\mu\nu}g^{\delta\sigma} + g^{\mu\sigma}g^{\delta v}\right) \tag{14.4.6}$$

For a Euclidean space of arbitrary integer dimension D, the mathematicians have provided us with the integral:

$$\int \frac{1}{(2\pi)^D}\frac{d^Dp_E}{\left(p_E^2 - s\right)^2} = \frac{1}{(4\pi)^{D/2}} \cdot \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)}\frac{1}{s^{2-\frac{D}{2}}} \tag{14.4.7}$$

We can use this to find its equivalent in D dimensional spacetime. First perform a inverse Wick rotation transformation to get

$$\int \frac{1}{(2\pi)^D}\frac{1}{\left(p_E^2 - s\right)^2}d^Dp_E = \frac{1}{(2\pi)^D}\int \frac{1}{\left(-p_E^2 + s\right)^2}d^Dp_E = \frac{-i}{(2\pi)^D}\int \frac{1}{\left(p^2 + s\right)^2}d^Dp \tag{14.4.8}$$

From (14.4.7),

$$\frac{1}{(2\pi)^D}\int \frac{1}{\left(p^2 + s\right)^2}d^Dp = \frac{i}{(4\pi)^{D/2}}\frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)}\frac{1}{s^{2-\frac{D}{2}}} \tag{14.4.9}$$

In similar fashion, one can deduce other relations for D dimensional spacetime parallel to (14.2.1) to (14.2.10). We list the most relevant of these below where we use q instead of p to represent the general case.

$$\int \frac{1}{(q^2+s)^n} d^D q = i\pi^{D/2} \frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)} \frac{1}{s^{n-\frac{D}{2}}} \tag{14.4.10}$$

$$\int \frac{q^\mu}{(q^2+s)^n} d^D q = 0 \tag{14.4.11}$$

$$\int \frac{q^\mu q^v}{(q^2+s)^n} d^D q = i\pi^{D/2} \frac{\Gamma\left(n-1-\frac{D}{2}\right)}{2\Gamma(n)} \frac{g^{\mu\nu}}{s^{n-1-D/2}} \tag{14.4.12}$$

$$\int \frac{q^2}{(q^2+s)^n} d^D q = i\pi^{D/2} \frac{\Gamma\left(n-1-\frac{D}{2}\right)}{2\Gamma(n)} \frac{D}{s^{n-1-D/2}} \tag{14.4.13}$$

Note that the gamma function Γ is also defined for non-integer D, so the RHS of integrals above remain valid in that case as well.

14.4.3 The Same Unphysical Example Again

$$\frac{1}{(2\pi)^4} \int \frac{1}{\left(p^2-m^2\right)^2} d^4 p \longrightarrow \frac{1}{(2\pi)^D} \int \frac{1}{\left(p^2-m^2\right)^2} d^D p = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(2)} \left(\frac{1}{-m^2}\right)^{2-\frac{D}{2}}$$

$\Gamma(z)$ has poles (goes to infinity) at $0, -1, -2, \ldots\ldots$ so eqn. above has poles at $D = 4, 6, 8, \ldots$. To examine the behavior around $D = 4$, define $\eta = 4 - D$ and use the approximation

$$\Gamma\left(2-\frac{D}{2}\right) = \Gamma\left(\frac{\eta}{2}\right) \overset{\eta\to 0}{\to} \frac{2}{\eta} - \gamma + \mathcal{O}(\eta) \tag{14.4.14}$$

where γ is the Euler-Mascheroni constant≈ 0.5772, which will always cancel, or be negligible in observable quantities.

We also use the standard relation

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \ldots = e^{\ln a^x} \tag{14.4.15}$$

with $a = 1/m^2$ and $x = \eta/2$ to obtain

$$\left(\frac{1}{-m^2}\right)^{2-\frac{D}{2}} = (-1)^{\frac{\eta}{2}} \left(\frac{1}{m^2}\right)^{\frac{\eta}{2}} \overset{\eta\to 0}{\to} \frac{i}{(4\pi)^2} \left(\frac{2}{\eta} - \gamma + \mathcal{O}(\eta)\right) \left(1 - \frac{\eta}{2} \ln m^2\right) = \frac{i2}{(4\pi)^2} \left(\frac{1}{\eta} - \frac{\gamma}{2} - \ln m\right)$$

Of course, in the full limit $\eta \to 0$ and $D \to 4$, we have

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}d^4p \to \frac{i2}{(4\pi)^2}\frac{1}{\eta} \tag{14.4.16}$$

14.4.4 Important Conclusion

If the dimensional regularization gives the same results as Pauli-Villars regularization, we can conclude that

$$\text{for finite } \Lambda \text{ and } \eta \text{ small, } \ln\Lambda = \frac{1}{\eta} - \frac{\gamma}{2} \tag{14.4.17}$$

$$\text{For } \Lambda \to \infty \text{ and } \eta \to 0, \quad \ln\Lambda = \frac{1}{\eta}$$

14.5 Comparing Various Regularization Approach

- Cut-off method: Simple in concept but violates gauge and Lorentz invariance, and not very useful.

- Pauli-Villars: Works for QED, but not (to date) for weak or strong interactions. Dimensional regularization: Works for QED and weak, but not strong, interactions. (Because strong interaction theory is non-perturbative, so our renormalization scheme does not hold.)

- Gauge lattice: Works for QED, weak, and strong interactions.

14.6 Finding Photon Self Energy Factor Using Dimensional Regularization

We now reproduce the second order photon propagator below

$$iD_{Fa\beta}^{2nd}(k) = iD_{F\alpha\beta}(k) + iD_{Fa\mu}(k)\frac{(-1)}{(2\pi)^4}\left\{\mathrm{Tr}\int ie_0\gamma^\mu iS_F(p-k)ie_0\gamma^\nu iS_F(p)d^4p\right\}iD_{F\nu\beta}(k) \tag{14.6.1}$$

where

$$ie_0^2\Pi^{\mu\nu}(k) = \frac{(-1)}{(2\pi)^4}\left\{\mathrm{Tr}\int ie_0\gamma^\mu i\frac{\not p - \not k + m}{(p-k)^2 - m^2 + i\varepsilon}ie_0\gamma^\nu i\frac{\not p + m}{p^2 - m^2 + i\varepsilon}d^4p\right\} \tag{14.6.2}$$

We introduce new symbol $N^{\mu\nu}(p,k)\,\mathrm{Tr}\left\{\gamma^\mu(\not p - \not k + m)\gamma^\nu(\not p + m)\right\}$. Converting equation above to D diminsional space and dividing by ie_0^2, we have

$$\Pi^{\mu\nu}(k) = \frac{i}{(2\pi)^D}\int\frac{N^{\mu\nu}(p,k)}{\left((p-k)^2 - m^2 + i\varepsilon\right)\left(p^2 - m^2 + i\varepsilon\right)}d^Dp \tag{14.6.3}$$

Now using Feynman parameterization with $a = (p-k)^2 - m^2 + i\varepsilon$ and $b = p^2 - m^2 + i\varepsilon$,

$$\Pi^{\mu\nu}(k) = \frac{i}{(2\pi)^D}\iint_0^1\frac{N^{\mu\nu}(p,k)}{\left(p^2 - m^2 + i\varepsilon + (k^2 - 2pk)\,z\right)^2}dzd^Dp$$

We introduce the new variable

$$q = p - kz \quad dq = dp \to d^Dq = d^Dp \quad p = q + kz$$

and substituting for p gives us

$$\begin{aligned}\Pi^{\mu\nu}(k) &= \frac{i}{(2\pi)^D}\int_0^1\int\frac{N^{\mu\nu}(q+k,k)}{\left((q+kz)^2 - m^2 + k^2z - 2(q+k)k + i\varepsilon\right)^2}d^Dqdz \\ &= \frac{i}{(2\pi)^D}\int_0^1\int\frac{N^{\mu\nu}(q+k,k)}{\left(q^2 + 2qkz + k^2z^2 - m^2 + k^2z - 2qkz - 2k^2z^2 + i\varepsilon\right)^2}d^Dqdz \\ &= \frac{i}{(2\pi)^D}\int_0^1\int\frac{N^{\mu\nu}(q+kz,k)}{\left(q^2 + k^2z(1-z) - m^2 + i\varepsilon\right)^2}d^Dqdz\end{aligned} \tag{14.6.4}$$

Note that the denominator is even in the integration variable q.

Using the trace relations in the numerator of $\Pi^{\mu\nu}$ yields

$$\begin{aligned}
N^{\mu\nu}(p,k) &= \mathrm{Tr}\left\{\gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu(\not{p}+m)\right\} \\
&= \mathrm{Tr}\left\{\gamma^\mu(\not{p}-\not{k})\gamma^\nu p + \gamma^\mu(\not{p}-\not{k})\gamma^\nu m + \gamma^\mu m\gamma^\nu \not{p}^\mu + \gamma^\mu\gamma^\nu m^2\right\} \\
&= \left\{(p_\delta - k_\delta)\, p_\sigma\, \mathrm{Tr}\,\gamma^\mu\gamma^\delta\gamma^v\gamma^\sigma + (p_\delta - k_\delta)\, m \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\delta\gamma^v}_{=0}\right. \\
&\left. + p_\sigma^m \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\sigma}_{=0} + m^2 \underbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu}_{f(D)g^{\mu\nu}}\right\}
\end{aligned}$$

This becomes,

$$\begin{aligned}
N^{\mu\nu}(p,k) &= f(D)\left\{(p^\mu - k^\mu)\, p^\nu - g^{\mu\nu}\,(p_\delta - k_\delta)\, p^\delta + (p^\nu - k^\nu)\, p^\mu + m^2 g^{\mu\nu}\right\} \\
&= f(D)\left\{(p^\mu - k^\mu)\, p^\nu + (p^\nu - k^\nu)\, p^\mu + (m^2 - \underbrace{(p_\delta - k_\delta)\, p^\delta}_{(p-k)p})g^{\mu\nu}\right\}
\end{aligned}$$

Since $q = p + kz$, we have

$$\begin{aligned}
N^{\mu\nu}(q+kz,k) = f(D)\{&(q^\mu + k^\mu z - k^\mu)\,(q^\nu + k^\nu z) + (q^\nu + k^\nu z - k^\nu)\,(q^\mu + k^\mu z) \\
&+ \left(m^2 - (q+kz-k)(q+kz)\right) g^{\mu\nu}\}
\end{aligned}$$

In expanding the eqn. above we can drop all terms linear in q since they make the integrand odd and will therefore yield zero. We can then take $N^{\mu\nu}$ as

$$\begin{aligned}
N^{\mu\nu}(q+kz,k) &= f(D)\left\{q^\mu q^\nu + k^\mu k^\nu z^2 - k^\mu k^\nu z + q^\mu q^\nu + k^\mu k^\nu z^2 - k^\mu k^\nu z\right. \\
&\left. + m^2 g^{\mu\nu} - q^2 g^{\mu\nu} - k^2 z^2 g^{\mu\nu} + k^2 z g^{\mu\nu}\right\} + \ldots \\
&= f(D)\{\underbrace{\left(2q^\mu q^v - q^2 g^{\mu v}\right)}_{N_1^{\mu\nu}} \underbrace{-2k^\mu k^\nu z(1-z)}_{N_2^{\mu v}} + \underbrace{\left(m^2 + k^2 z(1-z)\right) g^{\mu v}}_{N_3^{\mu v}}\} + \ldots
\end{aligned} \tag{14.6.5}$$

Returning to the Whole integral and breaking it into three parts

$$\begin{aligned}
\Pi^{\mu\nu}(k) &= \frac{i}{(2\pi)^D} f(D) \int_0^1 \int \frac{N_1^{\mu\nu} + N_2^{\mu\nu} + N_3^{\mu\nu}}{\left(q^2 + k^2 z(1-z) - m^2 + i\varepsilon\right)^2} d^D q dz \\
&= \frac{i}{(2\pi)^D} f(D) \int_0^1 \left(I_1^{\mu\nu} + I_2^{\mu\nu} + I_3^{\mu\nu}\right) dz
\end{aligned} \tag{14.6.6}$$

For $I_1^{\mu\nu}$

$$l_1^{\mu\nu} = \int \frac{2q^\mu q^\nu - q^2 g^{\mu\nu}}{\left(q^2 + k^2 z(1-z) - m^2 + i\varepsilon\right)^2} d^D q \tag{14.6.7}$$

With (14.4.12) and (14.4.13), $s = k^2z(1-z) - m^2 + i\varepsilon$, and $n = 2$, this becomes

$$I_1^{\mu\nu} = i\pi^{D/2}\frac{\Gamma\left(1-\frac{D}{2}\right)}{2\Gamma(2)}\frac{2g^{\mu\nu}}{\left(k^2z(1-z)-m^2\right)^{1-D/2}} - i\pi^{D/2}\frac{\Gamma\left(1-\frac{D}{2}\right)}{2\Gamma(2)}\frac{Dg^{\mu v}}{\left(k^2z(1-z)-m^2\right)^{1-D/2}}$$

$$= \frac{i\pi^{D/2}g^{\mu\nu}}{\left(k^2z(1-z)-m^2\right)^{1-D/2}}\frac{\Gamma\left(1-\frac{D}{2}\right)}{2}(2-D) = \frac{i\pi^{D/2}g^{\mu v}}{\left(k^2z(1-z)-m^2\right)^{1-D/2}}\underbrace{\left(1-\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}_{\Gamma\left(2-\frac{D}{2}\right)}$$

$$= \frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{1-D/2}}g^{\mu\nu} = \frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}\left(k^2z(1-z)-m^2\right)g^{\mu\nu}$$

For $I_2^{\mu\nu}$

$$I_2^{\mu\nu} = \int \frac{-2k^\mu k^\nu z(1-z)}{\left(q^2 + k^2z(1-z) - m^2 + i\varepsilon\right)^2}d^Dq \tag{14.6.8}$$

From (14.4.10) with $n = 2$ and $s = k^2z(1-z) - m^2 + i\varepsilon$, this becomes

$$-i\pi^{D/2}\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(2)}\frac{2k^\mu k^\nu z(1-z)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}} = -\frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}2z(1-z)k^\mu k^\nu$$

For $I_3^{\mu\nu}$, using (14.4.10) but with a different numerator

$$I_3^{\mu\nu} = \frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}\left(m^2 + k^2z(1-z)\right)g^{\mu\nu} \tag{14.6.9}$$

Adding three integrals yields

$$I_1^{\mu\nu} + I_2^{\mu\nu} + I_3^{\mu\nu} = \frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}2z(1-z)\left(k^2g^{\mu\nu} - k^\mu k^\nu\right) \tag{14.6.10}$$

Now we find

$$\Pi^{\mu\nu}(k) = \frac{i}{(2\pi)^D}f(D)\int_0^1\left(\frac{i\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}2z(1-z)\left(k^2g^{\mu\nu} - k^\mu k^\nu\right)\right)dz$$

$$= \frac{-1}{2^{D-1}\pi^{D-D/2}}f(D)\Gamma\left(2-\frac{D}{2}\right)\left(k^2g^{\mu\nu} - k^\mu k^\nu\right)\int_0^1\frac{z(1-z)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}dz$$

Recall from previous chapter, we have

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}A\left(k^2\right) + \underbrace{k^\mu k^\nu B\left(k^2\right)}_{\text{drops out of amplitude}} \tag{14.6.11}$$

The term with $k^\mu k^\nu$ drops out due to current conservation (gauge invariance). So we can only consider that

$$\Pi^{\mu\nu}(k) = \frac{-1}{2^{D-1}\pi^{D/2}}f(D)\Gamma\left(2-\frac{D}{2}\right)k^2g^{\mu\nu}\int_0^1\frac{z(1-z)}{\left(k^2z(1-z)-m^2\right)^{2-D/2}}dz \tag{14.6.12}$$

Now take the limit of $D \to 4$, and use the expansion of a^x with $a = (k^2 z(1-z) - m^2)$, we have that

$$\begin{aligned} &\lim_{D\to 4} \left(k^2 z(1-z) - m^2\right)^{2-D/2} = \lim_{D\to 4} \left(k^2 z(1-z) - m^2\right)^{\eta/2} \\ &= \lim_{D\to 4} \left(k^2 z(1-z) - m^2\right)^{-\eta/2} = 1 - \tfrac{\eta}{2} \ln\left(k^2 z(1-z) - m^2\right) + \ldots \\ &\eta \to 0 \end{aligned} \tag{14.6.13}$$

Recalling the behavior of Gamma function(14.4.14), and now with $f(D) \to 4$, we have

$$\begin{aligned} \Pi^{\mu\nu}(k) &= \frac{-1}{2^3\pi^2} 4k^2 g^{\mu\nu} \left(\frac{2}{\eta} - \gamma + \mathcal{O}(\eta)\right) \int_0^1 z(1-z) \left\{1 - \frac{\eta}{2} \ln\left(k^2 z(1-z) - m^2\right)\right\} dz \\ &= \frac{-1}{2\pi^2} k^2 g^{\mu\nu} \left(\frac{2}{\eta} - \gamma\right) \int_0^1 z(1-z) \left\{1 - \frac{\eta}{2} \ln\left(k^2 z(1-z) - m^2\right)\right\} dz \\ &= \frac{-1}{2\pi^2} k^2 g^{\mu\nu} \left(\left(\frac{2}{\eta} - \gamma\right) \int_0^1 z(1-z) dz - \int_0^1 z(1-z) \ln\left(k^2 z(1-z) - m^2\right) dz\right) \end{aligned}$$

where we again dropped terms of order η in the last line. Since $(2/\eta - \gamma) = \ln \Lambda$, this becomes

$$\Pi^{\mu\nu}(k) = \frac{g^{\mu\nu}}{6\pi^2} k^2 \ln\frac{k}{\Lambda} + \frac{g^{\mu\nu}}{2\pi^2} k^2 \int_0^1 z(1-z) \ln\left(z(1-z) - m^2/k^2\right) dz \tag{14.6.14}$$

Compare what we have in Chap 12, we now have

$$\Pi^{\mu\nu}(k) = g^{\mu\nu} k^2 \underbrace{2b_n \ln\frac{k}{\Lambda}}_{-A'(k,\Lambda)} + g^{\mu\nu} k^2 \underbrace{\frac{1}{2\pi^2} \int_0^1 z(1-z) \ln\left(z(1-z) - m^2/k^2\right) dz}_{-\Pi_c(k^2)} \tag{14.6.15}$$

where $b_n = 1/(12\pi^2)$ for $n = 1$ (only electron-position loop) as we have here.

14.7 Finding the Vertex Correction Factor Using Dimensional Regularization

From Chapter 14, we have

$$\gamma^\mu \Rightarrow \gamma^\mu_{2nd}(p, p') = \gamma^\mu + e_0^2 \Lambda^\mu(p, p') \tag{14.7.1}$$

and

$$\Lambda^{\mu}\left(p,p'\right)=\frac{1}{(2\pi)^4}\int\frac{-ig_{\alpha\beta}}{k^2+i\varepsilon}i\gamma^{\alpha}\frac{i}{(\not{p}'-\not{k})-m+i\varepsilon}\gamma^{\mu}\frac{i}{(\not{p}-\not{k})-m+i\varepsilon}i\gamma^{\beta}d^4k \tag{14.7.2}$$

Not that we do not have 2nd field spinors u_{2nd} or v_{2nd} here, this is just for a vertex. Since

$$\Lambda^{\mu}(p,p')=\frac{-i}{(2\pi)^4}\int\frac{g_{a\beta}\gamma^{\alpha}\left(\not{p}'-\not{k}+m\right)\gamma^{\mu}(\not{p}-\not{k}+m)\gamma^{\beta}}{k^2\left(\left(p'-k\right)^2-m^2\right)\left((p-k)^2-m^2\right)}d^4k \tag{14.7.3}$$

Doing the contraction in the numerator yields

$$N^{\mu}\left(p,p',k\right)=\gamma_{\beta}\left(\not{p}'-\not{k}+m\right)r^{\mu}\left(\not{p}-\not{k}+m\right)\gamma^{\beta} \tag{14.7.4}$$

Note that if $k\rightarrow 0$, then $p'^2\rightarrow m^2$ and $p^2\rightarrow m^2$. Thus, the denominator of the integrand in (14.7.3) approaches zero proportional to the sixth power, ie., as 0^6, whereas the numerator (with d^4k taken as proportional to k^3dk) as, at best, proportional to 0^5. So **the integrand will diverge for k approaching zero.**

We can correct that by temporarily assuming the virtual photon has a mass λ. That keeps the photon propagator finite and prevents the integrand from diverging in the infrared regime. That is

$$\Lambda^{\mu}\left(p,p'\right)=\frac{-i}{(2\pi)^4}\int\frac{N^{\mu}\left(p,p',k\right)}{\left(k^2-\lambda^2\right)\left(\left(p'-k\right)^2-m^2\right)\left((p-k)^2-m^2\right)}d^4k \tag{14.7.5}$$

and at the end of our calculation we take $\lambda=0$. Ignoring the small ε terms, and using Feynman parameterization with

$$a=k^2-\lambda^2 \qquad b=\left(p'-k\right)^2-m^2 \quad c=(p-k)^2-m^2$$

in the denominator of (14.7.5), we can rewrite (14.7.5) as

$$\Lambda^{\mu}\left(p,p'\right)=\frac{-i}{(2\pi)^D}\iint_0^1\int_0^{1-x}\frac{2N^{\mu}\left(p,p',k\right)}{\left(a+(b-a)x+(c-a)z\right)^3}dzdxd^Dk$$

$$=\frac{-2i}{(2\pi)^D}\iint_0^1\int_0^{1-x}\frac{N^{\mu}\left(p,p',k\right)}{\left(k^2-2k\left(p'x+pz\right)+\underbrace{x\left(p'^2-m^2\right)+z\left(p^2-m^2\right)-\lambda^2(1-x-z)}_{-r}\right)^3}dzdxd^Dk$$

where the symbol r, is used as a shorthand symbol to keep notation streamlined. Recall that in the present case, photon $k = p - p' \to 0$, so the fermions approach on-shell, i.e., $(p')^2 \approx p^2 \approx m^2$. If we further define the following variables:

$$t^\mu = k^\mu - \underbrace{(p'x + pz)^\mu}_{a^\mu} = k^\mu - a^\mu \quad d^4t = d^4k \to d^Dt = d^Dk$$

$$t^2 = k^2 - 2k\,(p'x + pz) + (p'x + pz)^2 = k^2 - 2k\,(p'x + pz) + a^2$$

then

$$\begin{aligned}\Lambda^\mu\,(p,p') &= \frac{-2i}{(2\pi)^D}\iint_0^1\int_0^{1-x}\frac{N^\mu\,(p,p',t+a)}{\left(t^2-r-a^2\right)^3}dzdxd^Dt \\ &= \frac{-2i}{(2\pi)^D}\int_0^1\int_0^{1-x}\int\frac{N^\mu\,(p,p',t+a)}{\left(t^2-r-a^2\right)^3}d^Dtdzdx\end{aligned} \tag{14.7.6}$$

By rearranging the numerator, we have

$$N^\mu\,(p,p',t+a) = \underbrace{\gamma_\beta\,(\not{p}' - \not{a} + m)\,\gamma^\mu(\not{p} - \not{a} + m)\gamma^\beta}_{N_0^\mu(p',p,a)}$$

$$\underbrace{-\gamma_\beta\,(\not{t}\gamma^\mu(\not{p} - \not{a} + m) + (\not{p}' - \not{a} + m)\,\gamma^\mu\not{t})\,\gamma^\beta}_{N_1^\mu(p',p,a,t)}\underbrace{+\gamma_\beta\not{t}\gamma^\mu\not{t}\gamma^\beta}_{N_2^\mu(p',p,a)}$$

For $\Lambda_0^\mu(p,p')$, we have

$$\Lambda_0^\mu\,(p,p') = \frac{-2i}{(2\pi)^D}\int_0^1\int_0^{1-x}\gamma_\beta\,(\not{p}' - \not{a} + m)\,\gamma^\mu(\not{p} - \not{a} + m)\gamma^\beta\left(\int\frac{1}{\left(t^2-r-a^2\right)^3}d^Dt\right)dzdx$$

Using (14.4.10) with $q = t$ and $s = -r - a^2$. The result is

$$\begin{aligned}&\Lambda_0^\mu\,(p,p') = \tfrac{-2i}{(2\pi)^D}\textstyle\int_0^1\int_0^{1-x}\gamma_\beta\,(\not{p}' - \not{a} + m)\,\gamma^\mu(\not{p} - \not{a} + m)\gamma^\beta\times \\ &\left(i\pi^{D/2}\tfrac{\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(3)}\tfrac{1}{(-r-a^2)^{3-\frac{D}{2}}}\right)dzdx\end{aligned}$$

For $D \to 4$, this becomes

$$\Lambda_0^\mu(p,p') = -\frac{1}{2^4\pi^2}\int_0^1\int_0^{1-x}\frac{\gamma_\beta\,(\not{p}' - \not{a} + m)\,\gamma^\mu(\not{p} - \not{a} + m)\gamma^\beta}{(r+a^2)}dzdr \tag{14.7.7}$$

where r and a are functions of x, z, p', and p. We see that the eqn. above has a finite value and can be rewritten, using the gamma matrix relations of (14.4.3) as

$$\begin{aligned}\Lambda_0^\mu\,(p,p') = -\frac{1}{16\pi^2}\int_0^1\int_0^{1-x}\frac{1}{(r+a^2)}\Bigg(&\underbrace{\gamma_\beta\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\beta}_{-2\gamma^\rho\gamma^\mu\gamma^\sigma}\,(p'_\sigma - a_\sigma)\,(p_\rho - a_\rho) + \\ &\underbrace{\gamma_\beta\gamma^\sigma\gamma^\mu\gamma^\beta}_{4g^{\sigma\mu}}\,(p'_\sigma - a_\sigma)^m + \underbrace{\gamma_\beta\gamma^\mu\gamma^\rho\gamma^\beta}_{4g^{\mu\rho}}\,(p_\rho - a_\rho)\,m + \underbrace{\gamma_\beta\gamma^\mu\gamma^\beta}_{-2\gamma^\mu}\,m^2\Bigg)\,dzdx\end{aligned} \tag{14.7.8}$$

Or, switching the dummy variable in the next to last term from ρ to σ,

$$\begin{aligned}\Lambda_0^\mu(p,p') = \int_0^1 \int_0^{1-x} \frac{1}{(r+a^2)} \left(\frac{1}{8\pi^2}\gamma^\rho\gamma^\mu\gamma^\sigma (p'_\sigma - a_\sigma)(p_\rho - a_\rho) \right. \\ \left. -\frac{1}{4\pi^2} g^{\mu\sigma}(p'_\sigma + p_\sigma - 2a_\sigma) m + \frac{1}{8\pi^2}\gamma^\mu m^2 \right) dzdx\end{aligned} \tag{14.7.9}$$

For Λ_1^μ we see that N_1^μ is odd in t, so this term will be 0.

For Λ_2^μ, we have

$$\Lambda_2^\mu(p,p') = \frac{-2i}{(2\pi)^D} \int_0^1 \int_0^{1-x} \int \frac{\overbrace{\gamma_\beta \not{t} \gamma^\rho \gamma^\mu \gamma^\sigma \not{t} \gamma^\beta}^{\gamma_\beta\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\beta t_\rho t_\sigma}}{\left(t^2 - r - a^2\right)^3} d^D t dz dx \tag{14.7.10}$$

Using (14.4.12) with $q = t$ makes this

$$\begin{aligned}\Lambda_2^\mu(p,p') &= \frac{-2i}{(2\pi)^D}\gamma_\beta\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\beta \int_0^1 \int_0^{1-x} \int \frac{t_\rho t_\sigma}{\left(t^2 - r - a^2\right)^3} d^D t dz dx \\ &= \frac{-2i}{(2\pi)^D}\gamma_\beta\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\beta \int_0^1 \int_0^{1-x} \left(i\pi^{D/2} \frac{\Gamma\left(\frac{\eta}{2} = 2 - \frac{D}{2}\right)}{2\Gamma(3)} \frac{g_{\rho\sigma}}{\left(-r-a^2\right)^{\frac{\eta}{2}}} \right) dzdx\end{aligned}$$

Now take $D \to 4$ (i.e., $\eta \to 0$) and use (14.4.14) and (14.4.15) with $a = -r - a^2$ above and $x = -\eta/2$. This, along with two gamma matrix relations of (14.4.3) yields

$$\begin{aligned}\Lambda_2^\mu(p,p') &= \frac{2}{2^4\pi^2} \underbrace{\gamma_\beta\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\beta}_{-2\gamma^\sigma\gamma^\mu\gamma^\rho} g_{\rho\sigma} \int_0^1 \int_0^{1-x} \lim_{\eta\to 0} \left(\frac{\Gamma\left(\frac{\eta}{2}\right)}{2\Gamma(3)} \left(-r-a^2\right)^{-\frac{\eta}{2}} \right) dzdx \\ &= \frac{-1}{4\pi^2 \cdot 2 \cdot 2} \underbrace{\gamma_\rho\gamma^\mu\gamma^\rho}_{-2\gamma^\mu} \int_0^1 \int_0^{1-x} \left(\frac{2}{\eta} - \gamma + \mathcal{O}(\eta) \right) \left(1 - \frac{\eta}{2} \ln\left(-r-a^2\right) + \mathcal{O}\left(\eta^2\right) \right) dzdx\end{aligned}$$

Keeping only lowest order terms gives us

$$\Lambda_2^\mu(p,p') = \frac{1}{8\pi^2}\gamma^\mu \int_0^1 \int_0^{1-x} \left(\frac{2}{\eta} - \gamma - \ln\left(-r-a^2\right) \right) dzdx$$

Using 14.4.17 yields

$$\Lambda_2^\mu(p,p') = \frac{1}{8\pi^2}\gamma^\mu \ln\Lambda - \frac{1}{8\pi^2}\gamma^\mu \int_0^1 \int_0^{1-x} \ln\left(-r-a^2\right) dzdx \tag{14.7.11}$$

Compare with what we used in Chap 13 and 14, we would find:

$$L(\Lambda) = \frac{1}{8\pi^2}\ln\Lambda \tag{14.7.12}$$

and

$$\left.\begin{array}{c} -\frac{1}{8\pi^2}\gamma^\mu \int_0^1 \int_0^{1-x} \ln\left(-r-a^2\right) dzdx + \frac{1}{8\pi^2} \int_0^1 \int_0^{1-x} \frac{1}{(r+a^2)} \times \\ \left(\gamma^\rho\gamma^\mu\gamma^\sigma (p'_\sigma - a_\sigma)(p_\rho - a_\rho) - 2\left(p'^\mu + p^\mu - 2a^\mu\right) m + \gamma^\mu m^2\right) dzdx \end{array}\right\} = \Lambda_c^\mu(p,p') \tag{14.7.13}$$

14.8 Finding Fermion Self Energy Factor Using Dimensional Regularization

$$iS_F^{2nd}(p) = iS_F(p) + iS_F(p)ie_0^2\Sigma(p)iS_F(p) \tag{14.8.1}$$

where

$$\begin{aligned}\Sigma(p) &= \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha iS_F(p-k)\gamma^\beta d^4k \\ &= \frac{i}{(2\pi)^4}\int \frac{-ig_{\alpha\beta}}{k^2+i\varepsilon}\gamma^\alpha i\frac{(\not{p}-\not{k}+m_0)}{(p-k)^2-m_0^2+i\varepsilon}\gamma^\beta d^4k\end{aligned} \tag{14.8.2}$$

By following similar procedures, we find

$$\Sigma(p) = \underbrace{-\frac{3m}{8\pi^2}\ln\frac{\Lambda}{m}}_{A(\Lambda,m)} + (\not{p}-m)\underbrace{\frac{1}{8\pi^2}\ln\Lambda}_{B(\Lambda)}\underbrace{+(p-m)\Sigma_c(p-m)}_{\text{complicated}} \tag{14.8.3}$$

14.9 Additional Notes on Integrals

In the relation

$$I'' = \int p^\mu p^v d^4p = \frac{1}{4}\int g^{\mu v}p^2 d^4p \tag{14.9.1}$$

we can see that for $\mu \neq v$, the RHS is zero. For $\mu \neq v$ in the middle part, we have an odd factor of at least one 3 -momentum component p^1. so the integral from $+\infty$ to $-\infty$ will be zero as well. So, we only have to worry about diagonal terms, i.e., terms with $\mu = \nu$. We can therefore express the middle part of the equation above after Wick rotation, as

$$I'' = \int p^\mu p^v d^4p = i\int \begin{bmatrix} -E^2 & & & \\ & \left(p^1\right)^2 & & \\ & & \left(p^2\right)^2 & \\ & & & \left(p^3\right)^2 \end{bmatrix} d^4p_E$$

So the absolute value of each integral is the same. That is

$$\int (E)^2 idEdp^1dp^2dp^3 = \int \left(p^1\right)^2 idEdp^1dp^2dp^3 = \int \left(p^2\right)^2 idEdp^1dp^2dp^3 = \int \left(p^3\right)^2 idEdp^1dp^2dp^3$$

With this, we find

$$I'' = i\int \begin{bmatrix} -E^2 & & & \\ & E^2 & & \\ & & E^2 & \\ & & & E^2 \end{bmatrix} d^4p_E = -i\int g^{\mu v}E^2 d^4p_E$$

$$= -\frac{i}{4}\int g^{\mu\nu}\left(E^2 + \left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2\right) d^4p_E$$

We now do a reverse Wick rotation for the eqn. above to get back to 4D spacetime (Minkowski) coordinates. That is, take $E \to E/i$ and $d^4p_E \to d^4p/i$ to find

$$I'' = -\frac{i}{4}\int g^{\mu\nu}\left(\left(\frac{E}{i}\right)^2 + \mathbf{p}^2\right)\frac{d^4p}{i} = \frac{1}{4}\int g^{\mu\nu}\left(E^2 - \mathbf{p}^2\right) d^4p = \frac{1}{4}\int g^{\mu\nu}p^2 d^4p \qquad (14.9.2)$$

Chapter 15

Postdiction of Experimental Results

15.1 Coulumb Potential in RQM

Coulomb's potential is different in that it describes a potential field extending radially outward from a source (charge), so using a spherical coordinates would be far simpler. In addition, Maxwell's equation inside the charged region becomes

$$\partial^\alpha \partial_\alpha A^\mu = -e\bar{\psi}\gamma^\mu\psi \tag{15.1.1}$$

For N negatively charged fermions occupying the charged region, we can use a modified form:

$$\partial^\alpha \partial_\alpha A^\mu = -Ne\bar{\psi}\gamma^\mu\psi \tag{15.1.2}$$

However, for the Coulomb potential this becomes simplified because that potential is measured in the region outside the charged region, where no charged fermion field ψ exists. That is, the fermion field carrying the charge extends throughout the source particle/object to its surface, but no futher. Outside the surface, $\psi = 0$. and that is our region of interest.

So $\partial_\alpha \partial^\alpha A^\mu(x) = 0$ governs in that region, and in spherical coordinates, we have

$$\frac{\partial^2}{\partial t^2}A^\mu - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rA^\mu) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}A^\mu\right) - \frac{1}{r^2\sin^2\theta\partial\phi^2}A^\mu = 0 \tag{15.1.3}$$

But since the field is symmetric spherically about the origin, where the charge is located, A^μ

can only be a function of r and t. The Coulomb potential is static (not a function of t), so

$$\frac{\partial^2}{\partial r^2}\left(rA^\mu\right)=0 \tag{15.1.4}$$

The general solution to this equation is $A^\mu=\varepsilon_5^\mu C/r+\varepsilon_s^\mu D$, where C and D are constants. Physically, the potential must vanish at infinity, so $D=0$, and

$$A^\mu\propto\frac{1}{r}\varepsilon_s^\mu \quad \text{or as column matrix,} \quad A^\mu=\begin{bmatrix}A^t\\A^r\\A^\theta\\A^\phi\end{bmatrix}=\frac{1}{r}\begin{bmatrix}A_0^t\\A_0^r\\A_0^\theta\\A_0^\phi\end{bmatrix}=\frac{1}{r}\begin{bmatrix}\Phi_0\\A_0^1\\A_0^2\\A_0^3\end{bmatrix}=\begin{bmatrix}\Phi(r)\\\mathbf{A}(r)\end{bmatrix} \tag{15.1.5}$$

From the physical symmetry, the 3 D yector potential can only have a radial direction, so it cannot bow have any component in the angular directions θ or $A_0^\theta=A_0^\phi=0$. Thus,

$$\mathbf{B}=\nabla\times\mathbf{A}=0 \tag{15.1.6}$$

and no magnetic field is product. To keep things simple, we can therefore just take $\mathcal{A}=0$, (i.e.,$A_0^r=0$ also) without loss of generality. From boundary conditions on the surface of the charged spherical source ($A^t=\Phi$ just on either side of the surface must be equal, though we won't go through the formal mathematics), we obtain the constant A_0^t. We then end up with the following well-known Coulomb potential in Heaviside-Lorentz units:

$$A^\mu=\begin{bmatrix}\Phi\\0\\0\\0\end{bmatrix}=\begin{bmatrix}-eN/(4\pi r)\\0\\0\\0\end{bmatrix} \tag{15.1.7}$$

15.2 Coulomb Potential in QFT

One could simply assume in QFT that the form of Coulomb potential is same as that in RQM, since we found throughout our development of both theories that they paralleled one another in terms of the governing equations and solution forms, and differed only in the interpretation of the solution coefficients as constants or operators. Here we present a derivation of the Coulomb potential from the perspective of QFT.

15.2.1 Repulsive Coulomb Scattering Equivalence to Moller Scattering

Repulsive Coulomb scattering can be represented by Moller scattering as shown in Fig. 15.1, where the source charge particle is spherical (has a radial distribution of its radiation)

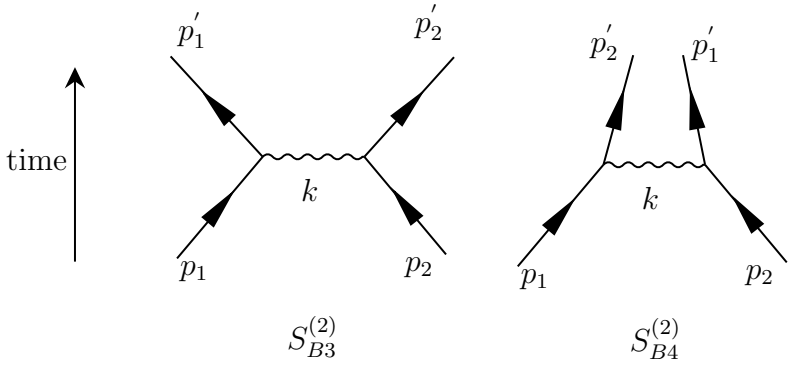


Figure 15.1: Moller Scattering

If the incoming particles in Fig. 15.1 are indistinguishable, such as two electrons, we need to include both diagrams to determine the amplitude. But, if they are distinguishable, such as an electron and a muon, then we only need to consider the LH diagram, (Because there is no indeterminancy in which orlginal particle mutated into which final particle.) Further, the classical Coulomb potential is always between macro (distinguishable) objects So, to make things simpler, we will assume the particles are distinguishable and examine the transition amplitude for only the LH diagram in the figure.

We also assume non-relativistic speeds of our incoming and outgoing particles, as that is typically the case for Coulomb scattering. One can think of the particle labeled 1 as the source, whose radiated virtual particle affects the particle labeled 2.

15.2.2 Relations We Need

Recall that the Fourier transforms of each other, $g(k)$ and $f(x)$, are defined as

$$g(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \quad \Leftrightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(k) e^{ik\pi} dk \tag{15.2.1}$$

From almost any table of Fourier transform pairs, we can get the following relations.

$$\frac{1}{k} \quad \Leftrightarrow \quad \frac{i}{2}\,\mathrm{sgn}\,x \left(\frac{i}{2} \text{ for } x > 0 \quad -\frac{i}{2} \text{ for } x < 0 \right) \tag{15.2.2}$$

$$\frac{1}{k^2} \quad \Leftrightarrow \quad -\frac{1}{2}|x| \tag{15.2.3}$$

In deriving the Coulomb potential using QFT under non-relativisitic conditions, we will need to refer to a result from NRQM, namely the scattering of two charged particles from one another. In NRQM, actually, the charge on one particle is assumed to provide a potential $V(\S)$ to which the other particle wave function responds. So we consider the behavior of one wave function in a given potential.

To get the NRQM transition amplitude, one usually employs the Born approximation, which assumes the incoming wave remains undistorted even within the scattering region. In fact, it is distorted, but modestly so, and the approximation gives results quite close to experiment. Thus, (with V the volume of the interaction, and $\tilde{V}(\mathbf{k})$ the Fourier transform of $V(\mathbf{x})$) the Born approximation scattering amplitude in NRQM in terms of a single particle in a potential $V(x)$ is

$$S_{fi} = \frac{i}{V}\tilde{V}(\mathbf{k})2\pi\delta\left(E_f - E_i\right) \qquad \text{where } \mathbf{k} = \mathbf{p}_i - \mathbf{p}_f \tag{15.2.4}$$

This result extends to the case where two particles interact, such as in Fig. 15.1 where $\tilde{V}(\mathbf{k})$ is the Fourier transform of $V(x)$, the potential field one particle feels due to the other.

$$S_{fi} = \frac{i}{V^2}\tilde{V}(\mathbf{k})(2\pi)^4\delta^{(4)}\left(p_f - p_i\right) \text{ where } \mathbf{k} = \mathbf{p}_i - \mathbf{p}_f \tag{15.2.5}$$

15.2.3 A Detour for the Planar Potential

We have been using plane wave fields and particles to develop our theory. So it will be easier if we investigate plane wave potentials that arise from plane wave fields/particles first. Thus, we now i) derive the transition amplitude for plane waves (meaning a planar charge distribution) in Moller scattering using QFT at non-relativistic speeds, then

ii) compare the result with (15.2.4) (b) to deduce the potential in 3 -momentum space, and finally,

iii) Fourier transform the result to obtain that potential in 3 D physical space

Using Feynman's rule, we have

$$\begin{aligned} S_{\text{difiting}} &= \left(\prod_p^{\text{allexternal}} \sqrt{\tfrac{m}{VE_p}}\right)(2\pi)^4\delta^{(4)}\left(p_1'+p_2'-p_1-p_2\right)\mathcal{M}_{B3}^{(2)} \\ \mathcal{M}_{B3}^{(2)} &= e^2\bar{u}_{r1'}\left(\mathbf{p}_1'\right)\gamma^\mu u_{r1}\left(\mathbf{p}_1\right) iD_{F\mu\nu}\left(k=p_1-p_1'\right)\bar{u}_{r_2'}\left(\mathbf{p}_2'\right)\gamma^\nu u_{r2}\left(\mathbf{p}_2\right) \end{aligned} \tag{15.2.6}$$

where a subscript such as r_2 means the r spin state ($r_2 = 1, 2$) of particle #2. We will be considering non-relativistic speeds, where $E \approx m$ and $|p| << m$ So our spinors will, to good approximation, become (with subscripts here referring to the spin state r)

$$u_{r=1}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u_{r=2}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ \frac{-p^3}{E+m} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For non-relativistic conditions, spinors become effectively independent of $\mathbf{p}$,so

$$u_{r_1=1}\left(\mathbf{p}_1\right) \approx u_{r_2=1}\left(\mathbf{p}_2\right) \quad u_{r_1=2}\left(\mathbf{p}_1\right) \approx u_{r_2=2}\left(\mathbf{p}_2\right)$$

and

$$\begin{aligned} \vec{u}_n &= u\left(\mathbf{p}_1'\right) = u_{n^2=1}^\dagger\left(\mathbf{p}_1'\right)\gamma^0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \\ \vec{u}_n &= 2\left(p_1'\right) = u_{r_1'=2}^\dagger\left(p_1'\right)\gamma^0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \bar{u}_2' &= 1\left(\mathrm{p}_2'\right) = u_2^\dagger + \left(\mathrm{p}_2'\right)\gamma^0 \Rightarrow (1 \quad 0 \quad 0 \quad 0) \\ \bar{u}_{r_2'=2}\left(\mathrm{p}_2'\right) &= u_{r_2'=2}^\dagger\left(\mathrm{p}_2'\right)\gamma^0 \Rightarrow (0,1,0 \quad 0) \end{aligned}$$

So for non-relativisitc conditions, the feynman amplitude becomes,where we assume the case where $r_1 = 1$ and $r_2 = 1$,

$$\mathcal{M}_{B3}^{(2)} = e^2\bar{u}_{r_1'}\left(\mathbf{p}_1'\right)\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{-ig_{00}}{k^2+i\varepsilon}\bar{u}_{r_2'(\mathbf{p}_2')}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is enly non-zero if the adjoint spinors both equal (1 0). As an aside, this means the spins of the incoming and outgoing particles each remain unchanged.

Additionally, for elastic scattering in the center of mass (COM) system for two particles of equal mass, the two particles start with the same speed and end with the same speed, but they exchange velocities (directions). Thus, neither changes its kinetic energy, and thus, **no energy is carried by the virtual particle from one real particle to the another**. However, there is 3 -momentum exchange because the drection of each purticle's velocity clanges. Hence $k^2 = -\mathbf{k}^2(\omega = 0)$ in the propagator. So

$$M_{B3}^{(2)} = -ie^2 \frac{1}{-\mathbf{k}^2 + i\varepsilon} \tag{15.2.7}$$

And the treansition amplitude is

$$S_{\text{Moller}} = ie^2 \frac{1}{V^2}(2\pi)^4 \delta^{(4)}\left(p_1' + p_2' - p_1 - p_2\right) \frac{1}{\mathbf{k}^2} \tag{15.2.8}$$

Comparing the eqn. above with (15.2.4), we conclude that

$$\tilde{V}(\mathrm{k}) = \frac{e^2}{\mathbf{k}^2} \tag{15.2.9}$$

To find the potential in physical space, consider k in the x^1 direction and we have

$$V\left(x^1\right) = -\frac{1}{2}e^2\left|x^1\right| \quad = -\frac{1}{2}e^2 x^1 \tag{15.2.10}$$

15.2.4 Repulsive Coulomb Potential via QFT

We start with plane wave derivation, deduce the 3-momentum space potential $\tilde{V}(\|)$, and in Fourier transforming that, change from Cartesian to spherical 3D coordiantes. To this end, we can use the same low velocity Moller scattering amplitude that we already derived. From that and (15.2.4) we get the same momentum space potential. The only issue remaining is to transform that to position space, expressed in spherical coordinates. Start with

$$V(\mathrm{x}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\mathrm{k}) e^{i\mathrm{k}\cdot\mathrm{x}} d^3k = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{e^2}{\mathrm{k}^2} e^{i\mathrm{k}\cdot\mathrm{x}} d^3k \tag{15.2.11}$$

and note that for spherical coordinates r, θ, ϕ in k space we can align the k^3 axis (the $\theta = 0$ direction) with the direction of the position vector x. With that alignment and with x

representing any point in position space, $\mathbf{k}\cdot\mathbf{x} = |\mathbf{k}||\mathbf{x}|\cos\theta$, where we now use the symbol k as $= |\mathbf{k}|$ (usually it has been shorthand for k'') and $|x| = r$, for the remainder of this section. Thus, $k\cdot x = kr\cos\theta$ and $k^2 = k^2$ for the temporary notation.

The differential volume element d^3k in spherical coordinates in $\mathbf{k}$ space is k^2 sin $\theta d\theta d\phi dk$. Given this,

$$V(r) = \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^2}{k^2}e^{ikr\cos\theta}k^2\underbrace{\sin\theta d\theta}_{-d(\cos\theta)}d\phi dk = \frac{-e^2}{(2\pi)^3}2\pi\int_0^\infty\int_0^\pi e^{ikr\cos\theta}d\underbrace{(\cos\theta)}_{\text{take as } u}dk \tag{15.2.12}$$

$$= \frac{e^2}{(2\pi)^2}\int_0^\infty\left(\frac{e^{ikr}-e^{-ikr}}{ikr}\right)dk$$

The last part can then be re-expressed as an integral from - ∞ to $+\infty$, as follows (where in going to the last term in the first row below, we substitute variable $k \to -k$).

$$V(r) = \frac{e^2}{(2\pi)^2}\left(\int_0^\infty\left(\frac{e^{ikr}}{ikr}\right)dk - \int_0^\infty\left(\frac{e^{-ikr}}{ikr}\right)dk\right) = \frac{e^2}{(2\pi)^2}\left(\int_0^\infty\left(\frac{e^{ikr}}{ikr}\right)(dk) - \int_0^{-\infty}\left(\frac{e^{-i(-k)r}}{i(-k)r}\right)(-dk)\right)$$

$$= \frac{e^2}{(2\pi)^2}\left(\int_0^\infty\left(\frac{e^{it\tau}}{ikr}\right)(dk) + \int_{-\infty}^0\left(\frac{e^{ikr}}{ikr}\right)dk\right)$$

$$= \frac{e^2}{(2\pi)^2}\int_{-\infty}^\infty\frac{e^{ikr}}{ikr}dk = \frac{e^2}{ir2\pi}\underbrace{\frac{1}{2\pi}\int_{-\infty}^\infty\frac{e^{ikr}}{k}dk}_{=i/2 \text{ from table}}$$

thus

$$V(r) = \frac{e^2}{4\pi r} \tag{15.2.13}$$

15.2.5 Attractive Coulomb Potential via QFT

For oppositely charged particle, we have Bhabha scattering (without annihilation) instead of Moller. Its transition amplitude is

$$S^{(2)}_{\text{Bhabha}} = S^{(2)}_{B2} = \left(\prod_{\text{no annih}}^{\text{allextemal}}\sqrt{\frac{m}{VE_{\text{p}}}}\right)(2\pi)^4\delta^{(4)}\left(p'_1+p'_2-(p_1+p_2)\right)\mathcal{M}^{(2)}_{B2} \tag{15.2.14}$$

where

$$\mathcal{M}^{(2)}_{B_2 2} = -e^2\bar{v}_{r1}\left(\mathbf{p}_1\right)\gamma^\mu v_{r'_1}\cdot\left(\mathbf{p}'_1\right)iD_{F\mu\nu}\left(p_1-p'_1\right)\bar{u}'_{r_2}\left(\mathbf{p}'_2\right)\gamma^v u_{r2}\left(\mathbf{p}_2\right) \tag{15.2.15}$$

So, (15.2.14) is similar to Moller except for the sign as we have to maintain normal ordering. At non-relativistic speeds, we have

$$v_{r=1}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_{r=2}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p^1-ip^2}{E+m} \\ \frac{-p^3}{E+m} \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And we can show that

$$\mathcal{M}_{B2}^{(2)} = ie^2 \frac{1}{-\mathbf{k}^2 + i\varepsilon} \tag{15.2.16}$$

at low speed. And

$$\tilde{V}(\mathrm{k}) = -\frac{e^2}{\mathbf{k}^2} \tag{15.2.17}$$

$$V(r) = -\frac{e^2}{4\pi r} \tag{15.2.18}$$

Chapter 16

A Short CheatSheet for Path Integral

Let

$$U\left(x_i, x_f; T\right) = \left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle \tag{16.0.1}$$

and

$$\left|U\left(x_i, x_f; T\right)\right|^2 = \left|\psi\left(x_f, T\right)\right|^2 = \psi^*\left(x_f, T\right)\psi\left(x_f, T\right) = \left\{ \begin{array}{c} \text{probability density of measuring} \\ \text{particle at } x_f \text{ at time } T \end{array} \right. \tag{16.0.2}$$

Approximation for transition amplitude

$$U\left(i, f; T = t_f - t_i\right) \approx C \int_{x_n = x_L}^{x_n = x_R} \ldots \int_{x_2 = x_L}^{x_2 = x_R} \int_{x_1 = x_L}^{x_1 = x_R} e^{i \int_{t_i}^{t_f} \frac{L}{\hbar} dt} dx_1 dx_2 \ldots dx_n \tag{16.0.3}$$

To get an exact, not approximate, relation we have to do two things. 1. Take the x range from i to infinity, i.e., $x_{\mathrm{L}} \to -\infty$ and $x_{\mathrm{R}} \to \infty$, and

2. Take $\Delta t \to dt$ for the saine T (time between events) so the number of time slices $\to \infty$

Doing this, we have

$$U\left(i, f; T = t_f - t_i\right) = C \int_{x = x_i}^{x = x_f} e^{i \int_{t_i}^{t_f} \frac{L}{\hbar} dt} \mathcal{D}_x \tag{16.0.4}$$

For QFT, we have

$$U(i, f; T) = C \int_{\phi_i}^{\phi_f} e^{i \int_0^T \frac{\mathcal{L}}{\hbar} d^4 x} \mathcal{D}\phi\left(x^\mu\right) \tag{16.0.5}$$

Some well-known relations

$$\delta\left(x-x_i\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ik(x-x_i)}dk=\frac{1}{2\pi h}\int_{-\infty}^{+\infty}e^{i\frac{p}{\hbar}(x-x_i)}dp \tag{16.0.6}$$

Borel's theorem

$$\begin{aligned}&\mathcal{F}\left\{\int\ldots\iint_{\zeta}\left(x_f-x_n\right)\ldots f_\gamma\left(x_3-x_2\right)f_\beta\left(x_2-x_1\right)f_\alpha\left(x_1\right)dx_1dx_2\ldots dx_n\right\}\\&=\mathcal{F}\left\{f_\zeta\right\}\ldots\mathcal{F}\left\{f_\gamma\right\}\mathcal{F}\left\{f_\beta\right\}\mathcal{F}\left\{f_\alpha\right\}\end{aligned} \tag{16.0.7}$$