Finite Difference Method In Pricing Options

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Abstract

In this paper we experiment with a finite different method for option pricing of European call. The partial differential equations are derived from the Black-Scholes formula, which is widely used to price options analytically. The Black-Scholes formula is also used to verify the outcome of the model. We take two approaches in solving the finite different method, forward in time and central in space method and the Crank-Nicolson method. After validating the model outputs, we compare the stability of both methods. We test the option pricing capabilities as well as the greek approximating abilities. We find that the Crank-Nicolson method is more stable then the forward in time and central in space method and both are capable of producing a correct option price. In addition we find it very easy to compute greeks with the finite different method that yield an accurate approximation. Some stability issues are mentioned that should be taken into account when using these methdos.

1. Introduction

Firstly traded on the Chicago Board Option Exchange, options are a type of derivative which allow the holder the right to exercise the contract but not the obligation. The contract consist of an agreement between the option's writer and the holder to buy (put option) or to sell (put option) a underlying stock at a certain price. A call option is exercised if the price of the underlying asset is higher than the agreed buying price called strike price K at maturity time T of the contract. A put option is exercised if the price of the underlying asset is lower than the agreed selling price called also strike price K. Thus the holder for a call option gets a payoff which is the difference between the stock price and the strike price. For a holder of a put option, the payoff is the difference between the strike price and the underlying stock price. If no profit can be made at maturity time, the holder doesn't exercise the contract. Due to the freedom of honouring the contract or not, the holder might always draw profit from it. Therefore, the writer imposes a base charge on the contract which allows him to earn the price of the contract no matter if the contract is honoured or not. Thus, the right pricing of the contract is crucial for the writer in order to not attract other market actors who will take profit of the mispricing. Over time various techniques have been developed for pricing options. One famous technique is the Black-Scholes analytical solution which can be applied to European option which can only be exercised at maturity time (Black & Scholes, 1973). For American options, the analytical solution is not possible to the complexity introduced by the possibility of early exercise of the option. One can use the Ross-Rubinstein market model which construct a binomial tree to define the price of the option (Cox et al., 1979). Another method is to use the Monte Carlo method (Boyle, 1977). However, the Monte Carlo method is advantageous for low dimensional problems. For higher dimensions, ≥ 4, the finite-difference method (FDM) yields faster results. Additionally, it can handle cases with early exercise options, which is not the case for the Black-Scholes equation. Furthermore, it is compatible with complex boundaries and barriers. It is also handy for computing greeks such as the delta Δ, gamma Γ or theta Θ.

The finite difference method is a numerical application of the Black-Scholes Partial Differential Equation (BS-PDE) derived by Black & Scholes (1973), which is a diffusion equation of the option price in time and different stock prices it can take. Different methods can be applied such as the FTCS (Forward Time Centred in Space), BTCS (Backward Time Centred in Space). The latest method also called an implicit finite difference is robust, it converges to the solution as ΔS and ΔSt decreases to zero. However, the implicit method needs to solve $M-1$ equation simultaneously with M being the number of discrete stock price value (Hull, 2003). Therefore, one can use the finite difference method, which assumes that $\partial f/\partial S$ and $\partial^2 f/\partial S^2$ are the same for two consecutive time points. The explicit method is computational less expensive but is, therefore, less stable. In this report, the FTCS is derived and its stability is analysis. Furthermore, a combination of the FTCS and BTCS called the Crank-Nicolson method (Crank & Nicolson, 1996) is derived and numerically compared to the FTCS.

2. Background & Theory

Let's assume the dynamic of the stock price is governed by the Geometric Brownian motion such that

$$dS_t = rSdt + \sigma SdW_t \tag{1}$$

with S being the stock price, σ being the implied volatility and r being the risk-free rate. dW_t is equal to $\epsilon\sqrt{dt}$ with *epsilon* being a random number from a normal distribution $\phi(0,1)$ and defines the Brownian motion.

Applying Itô calculus, the change in the option price dV can be expressed as follows

$$dV = \left(rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW_t \tag{2}$$

For a risk-free portfolio, its value Π should to buying $\frac{\partial V}{\partial S}$ shares and selling one option

$$\Pi = -V + \frac{V}{S}S$$

. Thus by adding a risk-free rate r, its value Π

$$d\Pi = -dV + \frac{\partial V}{\partial S}dS \tag{3}$$

Thus using equation 1 and 2 into 3, the Black-Scholes Partial Differential equation can be derived

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV \tag{4}$$

However, for numerical application, the latest equation is not convenient due to the fact that it is not a constant coefficient equation. At each step ∂V is divided by ∂S and multiplied by S. To make it a constant coefficient equation, X=ln(S) is introduced. Since the diffusion equation is solved backwards in time, $\frac{\partial V}{\partial t}$ can be rewritten as

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$$

Thus we have

$$\begin{aligned}
\frac{\partial V}{\partial \tau} =& rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV \\
=& rS\frac{\partial V}{\partial X}\frac{dX}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial}{\partial S}(\frac{\partial^2 V}{\partial X}\frac{dX}{dS}) - rV \\
=& rS\frac{\partial V}{\partial X}\frac{1}{S} + \frac{1}{2}\sigma^2 S^2\frac{\partial}{\partial S}(\frac{\partial^2 V}{\partial X}\frac{1}{S}) - rV \\
=& r\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial V}{\partial X} + \frac{\partial^2 V}{\partial X^2}\frac{dX}{dS}\frac{1}{S}\right) - rV \\
=& r\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial V}{\partial X} + \frac{\partial^2 V}{\partial X^2}\frac{1}{S^2}\right) - rV \\
=& \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2} - rV
\end{aligned} \tag{5}$$

For numerical purposes, the constant coefficient diffusion equation is discretised over time and stock price by using the Forward Time Centred in Space (FTCS) method. The time component is discretised in $\Delta\tau$ increments and the stock price in ΔS. Figure 1 shows a finite-difference mesh grid with time and the stock price being discretized.

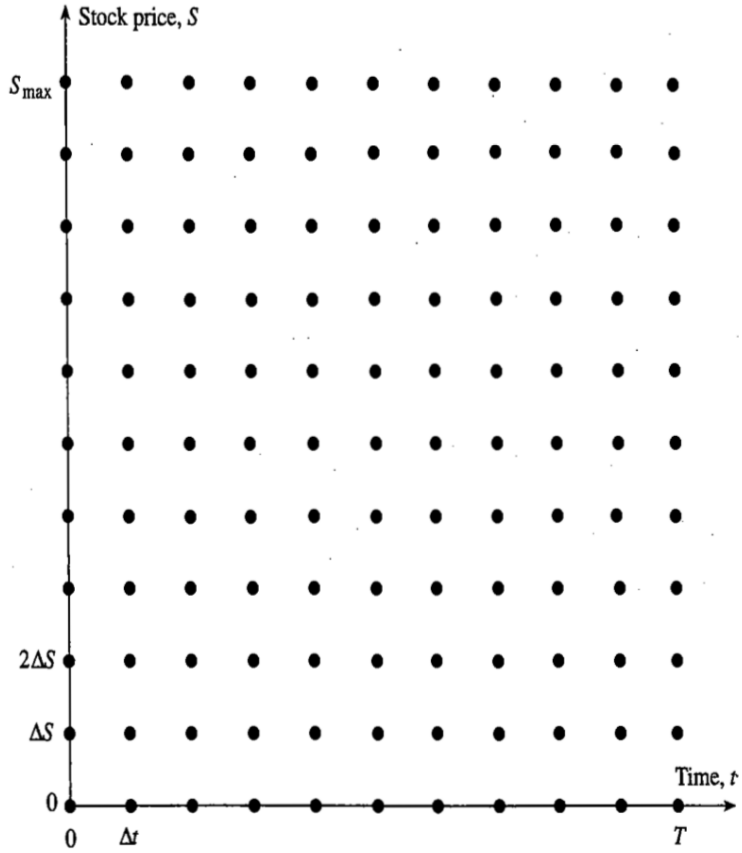

Figure 1: Finite Difference grid (Hull, 2003).

Thus the fundamental theorem of calculus states that a change in x by Δx can be apporximated by the taylor series such that

$$f(x+h) = f(x) + hf'(x) + o(h)$$

Ignoring the higher order terms, $f'(x)$ can be rewritten as follow

$$f'(x) = \lim_{x\to 0} \frac{f(x+h) - f(x)}{h}$$

Same method can be used to approximate $f''(x)$

$$f''(x) = \lim_{x\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Applying the derived relation for $\frac{\partial V}{\partial \tau} \frac{\partial V}{\partial X}, \frac{\partial^2 V}{\partial X^2}$ can be rewritten as a finite difference equation where (n,j) denotes the point in time and stock price respectively.

$$\left(\frac{\partial V}{\partial \tau}\right)_j^n \approx \frac{V_j^{n+1} - V_j^n}{\Delta\tau}$$
$$\left(\frac{\partial V}{\partial X}\right)_j^n \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}$$
$$\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n \approx \frac{V_{j+1}^n - 2V_j^n - V_{j-1}^n}{\Delta S^2}$$

Using these relation in equation 5, we obtain

$$\frac{V_j^{n+1} - V_j^n}{\Delta\tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} + \frac{1}{2}\sigma^2\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} - rV_j^n \tag{6}$$

$$V_j^{n+1} = V_j^n + (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta S}(V_{j+1}^n - V_{j-1}^n) + \frac{1}{2}\sigma^2\frac{\tau}{\Delta S^2}(V_{j+1}^n - 2V_j^n + V_{j-1}^n) - r\Delta\tau V_j^n \tag{7}$$

Another finite difference method which achieves faster convergence by combining the explicit and implicit method is the Crank-Nicolson method (Crank & Nicolson, 1996). The method takes the average between the Forward Time Centred in Space and Backward Time Centred in Space (BTCS).

$$\frac{V_j^n - V_i^{n-1}}{\Delta\tau} = \frac{1}{2}\left[FTCS + BTCS\right] \tag{8}$$

Thus the Backward Time Centred in Space is derived the same way as the Forward Time Centred in Space

$$V_j^{n+1} = V_j^n + (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta S}(V_{j+1}^{n+1} - V_{j-1}^{n+1}) + \frac{1}{2}\sigma^2\frac{\tau}{\Delta S^2}(V_{j+1}^{n+1} - 2V_j^n + V_{j-1}^{n+1}) - r\Delta\tau V_j^{n+1} \tag{9}$$

Puting equation 6 and 9 in 8, one get

$$\begin{aligned}\frac{V_i^{n+1} - V_j^n}{\Delta\tau} =& \frac{1}{2}(\left(r - \frac{1}{2}\sigma^2\right)\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} + \frac{1}{2}\sigma^2\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} - rV_j^n + (r - \frac{1}{2}\sigma^2)\frac{1}{2\Delta S}(V_{j+1}^{n+1} - V_{j-1}^{n+1}) \\ &+ \frac{1}{2}\sigma^2\frac{1}{\Delta S^2}(V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}) - rV_j^{n+1})\end{aligned}$$

$$\begin{aligned}V_j^{n+1} =& V_j^n + (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta S}(V_{j+1}^n - V_{j-1}^n + V_{j+1}^{n+1} - V_{j-1}^{n+1}) \\ &+ \frac{1}{4}\sigma^2\frac{\tau}{\Delta S^2}(V_{j+1}^n - 2V_j^n + V_{j-1}^n + V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}) - \frac{r\Delta\tau}{2}(V_j^n + V_j^{n+1})\end{aligned} \tag{10}$$

Equation 6 and 9 are based on the Taylor series expansion. Thus equation 10 is based on the Taylor series where the space ΔS is second order.

3. Methodology

The numerical analysis was done in Python 3 with the scientific computing package NumPy.

The investigation was focused on the FTCS and Crank-Nicolson method and their stability. For both method the boundary condition was based on the payoff for tree different cases, the column at maturity time, the row for s_{max} and s_{min}. At maturity time, the payoff was set between the strike price and the possible stock prices. In case of an call option as shown in figure 2, the row for s_{max} is based on the the discounting equation $payoff \times e^{-rt}$ with r being the interest rate. The row at s_{min}, the payoff is set to 0. In case of an put option, the discount equation is applied to the row corresponding to s_{min} and s_{max} is set to 0 if the s_{max} is above the strike price.

S \ T	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
100.00	36.19	36.56	36.92	37.30	37.67	38.05	38.43	38.82	39.21	39.60	40.00
95.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	35.00
90.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	30.00
85.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	25.00
80.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	20.00
75.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	15.00
70.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	10.00
65.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	5.000
60.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
55.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
45.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
40.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
35.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
25.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5.0000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.0000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 2: Setting the boundary condition of the grid for a call option with strike price $K = 60$ and a interest rate of $r = 0.3\%$. The grid corresponds to the schematic representation of figure 1.

Thus, knowing the values at maturity time, the grid is solved backwards. Hence $(\frac{\partial V}{\partial \tau})_j^n$ needs to be rewritten

$$\left(\frac{\partial V}{\partial \tau}\right)_j^n \approx \frac{V_j^n - V_j^{n-1}}{\Delta \tau}$$

Plugging $(\frac{\partial V}{\partial \tau})_j^n, (\frac{\partial V}{\partial X})_j^n$ and $(\frac{\partial^2 V}{\partial X^2})_j^n$ in equation 2 with S being $j\Delta S$ where $j = 0, 1, ..., s_{max} - 1$ we get

$$V_j^{n-1} = \alpha_j V_{j-1}^n + \beta_j V_j^n + \gamma_j V_{j+1}^n$$

with

$$\alpha = \frac{1}{2}\Delta\tau(\sigma^2 j^2 - rj)$$
$$\beta = 1 - \Delta\tau(\sigma^2 j^2 + r)$$
$$\gamma = \frac{1}{2}\Delta\tau(\sigma^2 j^2 + rj)$$

For numerical practice, α, β, γ can be stored in matrix A which is multiplied with each time step (each column) starting backwards.

$$A = \begin{pmatrix} \beta_1 & \gamma_1 & & & & \\ \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \alpha_3 & \beta_2 & \gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{j-1} & \beta_{j-1} & \gamma_{j-1} \end{pmatrix}$$

The Crank-Nicolson method is a combination of the FTCS with the BTCS where the BTCS can be rewritten using the same reasoning as before.

$$V_j^n = \alpha_j^\star V_{j-1}^{n-1} + \beta_j^\star V_j^{n-1} + \gamma_j^\star V_{j+1}^{n-1}$$

with

$$\alpha^\star = \frac{1}{2}\Delta\tau(rj - \sigma^2 j^2)$$
$$\beta^\star = 1 + \Delta\tau(\sigma^2 j^2 + r)$$
$$\gamma^\star = \frac{1}{2}\Delta\tau(-rj - \sigma^2 j^2)$$

Thus the Crank-Nicolson states that

$$\frac{1}{2}FTCS = \frac{1}{2}BTCS$$

$$\frac{1}{2}\alpha_j^\star V_{j-1}^{n-1} + \frac{1}{2}(1+\beta_j^\star)V_j^{n-1} + \frac{1}{2}\gamma_j^\star V_{j+1}^{n-1} = \frac{1}{2}\alpha_j V_{j-1}^n + \frac{1}{2}\beta_j V_j^n + \frac{1}{2}(1+\gamma_j)V_{j+1}^n$$

which can be rewritten in from of matrices

$$BV_j^{n-1} = AV_j^n$$

with

$$B = \begin{pmatrix} \beta_1^\star & \gamma_1^\star & & & & & \\ \alpha_2^\star & \beta_2^\star & \gamma_2^\star & & & & \\ & \alpha_3^\star & \beta_2^\star & \gamma_3^\star & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \alpha_{j-1}^\star & \beta_{j-1}^\star & \gamma_{j-1}^\star \end{pmatrix}$$

Thus the matrix equation can be solved to find V_j^{n-1} be using a LU decomposition.

3.1. Greeks

The most common Greeks include the delta, gamma, theta, and Vega which are partial derivatives. If we look at the different part of the PDE in linear hyperbolic we actually see three greeks theta, delta and gamma. Since we calculate these steps every iteration, we can easily extract these greeks from the PDE. To show how well this method works we compare the analytical delta from the black-scholes with the numerical delta from the PDE.

$$\left(\frac{\partial V}{\partial \tau}\right)_j^n = Theta \approx \frac{V_j^{n+1} - V_j^n}{\Delta\tau}$$
$$\left(\frac{\partial V}{\partial X}\right)_j^n = Delta \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}$$
$$\left(\frac{\partial^2 V}{\partial X^2}\right)_j^n = Gamma \approx \frac{V_{j+1}^n - 2V_j^n - V_{j-1}^n}{\Delta S^2}$$

All greeks from the PDE are approximations, so will always have some kind of error. We expect that the error will be small enough to use. Gamma and Delta are both compared with the analytical value to test whether this method is effective enough.

4. Results & Discussion

Table 1: Comparison the determined option price of the forward and Crank-nicolson method to the well-known analytical Black Scholes solution. The grid is defined between $s_{min} = 0$ and $s_{max} = 300$ for a call option with a maturity time $T = 1$, $K = 110$,$r = 0.04\%$,$\sigma = 0.3\%$.

Method	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$	parameters
Forward	9.581	14.984	21.393	ds=1, dt=0.001
Crank-Nicolson	9.624	15.126	21.784	ds=1, dt=0.1
Black-Scholes	9.625	15.129	21.789	

Table 1 shows the calculated call option price using the two methods for a market with a volatility σ of 0.30%, an interest rate $r = 0.04\%$, a strike price $K = 110$. No deviation is shown since all three methods are deterministic. One can see that both methods generate results close to the well-known Black-Scholes solution. However, the Crank-nicolson method yields more accurate results using smaller time increments dt. Furthermore, it produces stable results using a smaller s_{max}. In other words, the Crank-nicolson method yields more accurate results using a smaller finite difference mesh.

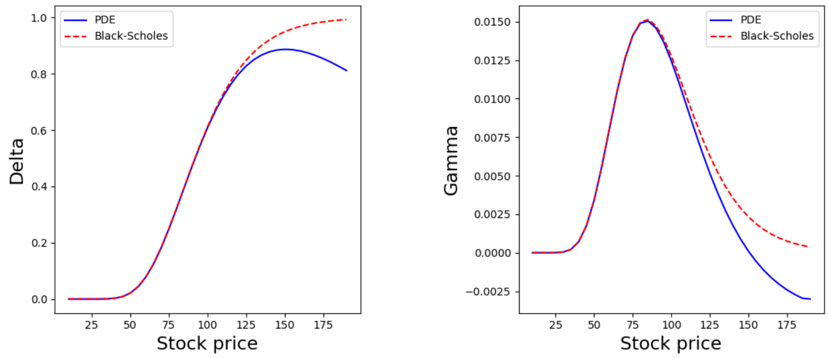

Figure 3: Analysis of Delta left and gamma right. Using the Crank-Nicolson methods for different stock prices. The following parameters are $s_{min} = 0$, $s_{max} = 200$, $t_{max} = 1$ for the grid $K = 100$, $r = 0.04\%$, $sigma = 0.3\%$, $option = call$. The numerical method is compared with the analytical

Figure 3 left shows that the upper boundary is interfering with the delta at higher stock prices, but overall does an excellent job of describing delta. The effect of the boundary can be reduced when the distance from the boundary is enlarged or a more suitable boundary condition is found. The lower boundary doesn't experience this problem since we fine-tuned the boundary on the delta, so at the lower boundary is equal to the analytical solution.

Figure 3 right shows the same characteristics as Delta. This is what we expected since Gamma is a differentiation of Delta. We even see a negative gamma due to the declining of Delta at the end.

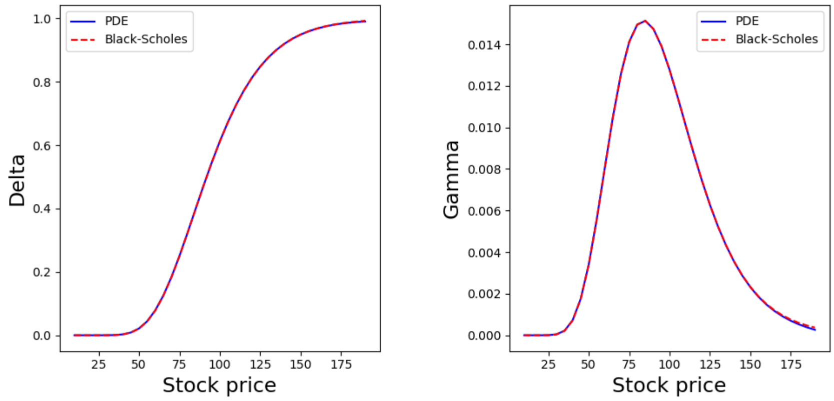

Figure 4: Analysis of Delta left and gamma right with a larger s_{min}. Using the Crank-Nicolson methods for different stock prices. The following parameters are $s_{min} = 0$, $s_{max} = 400$, $t_{max} = 1$ for the grid $K = 100$, $r = 0.04\%$, $sigma = 0.3\%$, $option = call$. The numerical method is compared with the analytical

Figure 4 shows that moving the border from 200 to 400 dramatically increases the accuracy of the approximation of the greeks, which indirectly improves the option price in general. However, it comes with an additional computational cost.

Figure 5 shows different boundaries of the stock price on the y-axis can affect the method (middle). Furthermore, the effect of the time increment is shown (right). The left graph shows the diffusion of the option price for every point in time and space (stock price boundary). One can see a smooth diffusion going from the max payoff of 150 to zero for a call option. If one changes the stock price boundary from 0 to 250 instead, one can see enormously high values at time 0 for around s_{max}. Thus the high values result in erroneous results when the interpolation of all the option values at time point 0 happens. Thus the option is priced to a tremendously high value which can be considered as infinity.

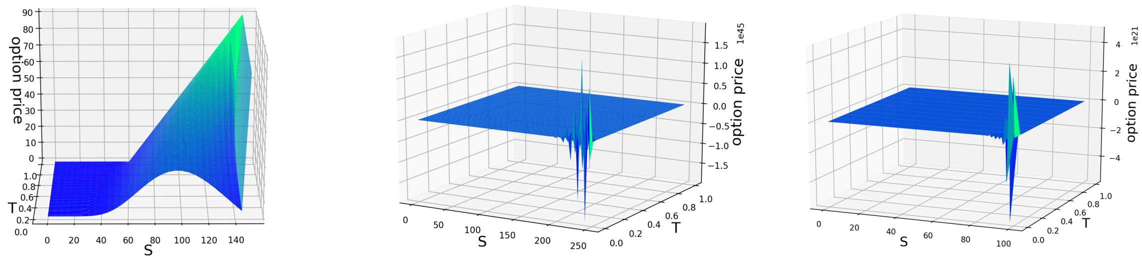

Figure 5: 3D representation of a Finite-difference mesh using the FTCS method with z-axis being the option values for each point in time and stock price. The volatility $\sigma = 0.3\%$, the interest rate $r = 0.1$, the stock price $S0 = 50$ and the strike price $K = 60$, $ds = 5$. The first graph is based on $dt = 0.01$ and $s_{min} = 0, s_{max} = 150$.The resulting value for the option is 4.205. For second graph, $s_{min} = 0, s_{max} = 250$. The resulting value for the option is ∞. The third graph is based on $dt = 0.1$. The resulting value for the option is -1.370.

The same happens when the time increment dt is not adapted to the grid size. By choosing an small high increment (0.1) the diffusion explodes at the same position around s_{max} at time 0 which misprices the option to -1.370. Thus this result is not possible. The 3 graphs shows what a stable and unstable parameters do to the diffusion and how it affects the end pricing.

Hence, to have a stable FD-mesh, it is important to find the right balance between the boundary for the stock price and the time increment. Therefore, an investigation was made into the influence of grid size on the option pricing. Thereby, the space increment ds was changed and the time increment dt. Figure 6 and 7 shows the effect of the grid size on the option pricing. Thus the following graph differs from the previous ones in that it shows the computed option price given the ds and dt used. The previous graphs show the finite grid mesh with the following option price corresponding to the respective point on that gird and not the final computed option price. Using an initial stock price $S0$ of 50, a strike price K of 60, an interest rate r of 0.04% and a volatility σ of 0.27%, the option price according the analytical solution should be 1.5 . Setting $s_{min} = 0, s_{max} = 100$, and $t_{max} = 1$, the FTCS and the Crank-nicolson method are applied using different increments ds for the price S and dt for the time interval.

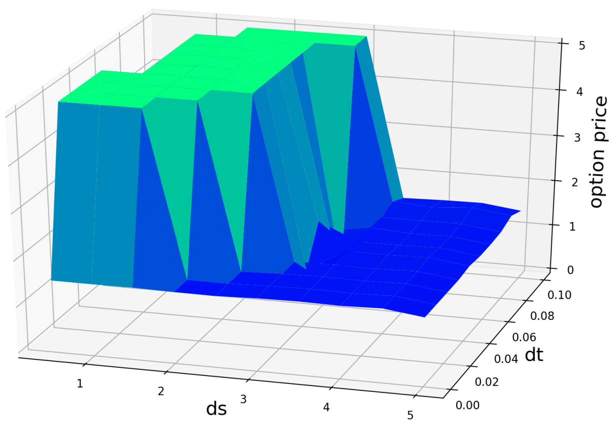

Figure 6: Stability Analysis of the FTCS methods for different increments in dt and dS. The following parameters are $s_{min} = 0$, $s_{max} = 100$, $t_{max} = 1$ for the grid and $S0 = 50$,$K = 60$, $r = 0.04\%$, $sigma = 0.2\%$, $option = call$ concerning the market characteristics. The analytical solution generates an option price of 1.5. For the unstable cases, which produces enormously high numbers were set to 5 for visualisation purposes.

For the FTCS method (figure 6, unstable configuration yielded tremendously high results for the price of an option. For visualisation purposes, results higher than 5 were set to 5. One can see that the FTCS gets rapidly unstable. To get results close to the analytical solution, it seems that with an increasing dt, ds should increase as well.

For the Crank-nicolson method (figure 7), one can see some fluctuation over the whole dt/ds space. From the figure, it is difficult to determine which parameters would improve the method. However, for the whole space, the method seems to be stable in comparison with the FTCS by yielding results close to the known solution of 1.5. We see a small improvement if ds and dt are lower with a small exception for

$ds = 2.5$.

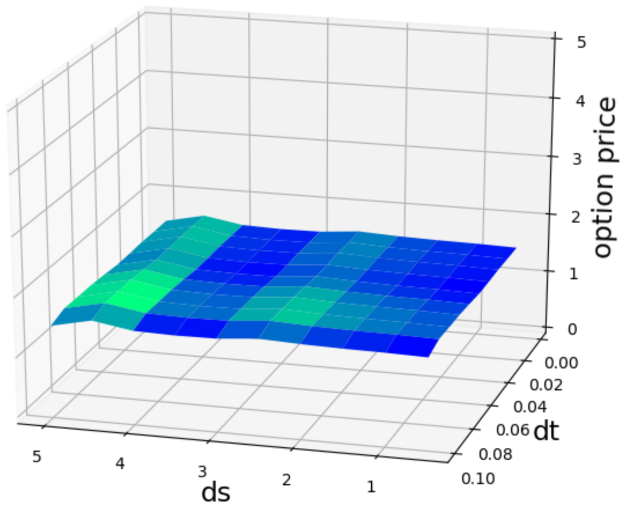

Figure 7: Stability Analysis of the Crank-Nicolson methods for different increments in dt and dS. The following parameters are $s_{min} = 0$, $s_{max} = 100$, $t_{max} = 1$ for the grid and $S0 = 50$,$K = 60$, $r = 0.04\%$, $sigma = 0.2\%$, $option = call$ concerning the market characteristics.The analytical solution generates an option price of 1.5.

5. Conclusion

We succeeded in deriving a partial differential equation from the Black-scholes and showed that it is possible to price a European option equal to the Black-Scholes solution using both methods. Comparing both methods shows that the Crank-Nicolson is more stable than the Forward method, which was also predicted by the literature Crank & Nicolson (1996). Our sensitivity analysis proof the right parameters are crucial to price the option correctly since the PDE can be unstable. Thus, the boundary for the stock price as well as the step size in time is important and needs to be adapted to achieve the desired accuracy. Furthermore, the diffusion equation can be applied to determine some greeks as done for the Delta Gamma. In addition, it was possible to show that the PDE differs from the Analytical solution if the stock price surpassed the strike price, so in an "out of money" situation.

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