# Analysis of Monte Carlo Methods and Variance Reduction Techniques on Option Pricing

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**Abstract**

In this paper we explore some use cases for the Monte-Carlo (MC) method in option pricing. Monte-Carlo can be used to price a lot of different styled options. To proof its efficiency and short coming we compare it to some analytical results. We plot the results of the MC method for pricing European, Asian and digital options. In addition we approximate the greek delta with Monte-Carlo. We specifically examine the bump-and-revalue and Likelihood Ratio method for estimating delta. Since variance is a well known short coming of these methods we show a very effective variance reduction technique for closely correlated options. The results show the effect of the law of large numbers and converges to a true analytical solution and delta can be approximated. The bump-and-revalue method performs better if a fixed seed is used, but fails to estimate the delta for a digital option. The LR method, however, can estimate the delta more accurate and converges to the (theoretical) true value. Finally we see the variance reduction technique control variate effectively reduces variance.

## 1. Introduction

In 1973, the Chicago Board Option Exchange firstly introduced a structured market for trading options J. C. Hull (2003). This latest type of contract consists of an agreement between the option's writer and the holder on the price of the underlying assets at a certain time called the maturity date. The price of the underlying asset which can be bought/sold by the holder at maturity time is called the strike price. Two types of options exist: (i) a call option in which the holder has the right to purchase the underlying asset at the agreed strike price; (ii) a put option in which the holder has the right to sell the underlying asset at an agreed strike price. Options are traded in different markets such as the European, American and Asian market with each following their own market rules. For example, the holder of a European contract can only honour the contract at maturity while with an American option, the holder also can exercise the contract before maturity. In Asian markets holder can enter either a European or an American option. However, the payoff of the Asian option is defined by the average stock price and not the difference between the stock price and the strike price as it is the case for simple European/American option.

The decision of the holder to exercise the contract depends on the stock price of the underlying asset at maturity time and the type of contract. A call option is only exercised if the stock price is above the strike price and visa versa for a put. However, to avoid going away empty-handed every time, the writer imposes a price on the derivative so that regardless of the holder's decision, the writer always earns the option's price. A common technique for option pricing is the Black-Scholes formula (Black & Scholes, 1973). This formula yields an analytical solution for the stock price at maturity by assuming a log-normal distribution of the stock price and no dividend payments. Unfortunately, for options other than European options, the formula produces difficulties, and consequently, new (numerical) methods are required.

Boyle (1977) was one of the first to use Monte Carlo (MC) methods to determine the price of (European) options and nowadays, it is widely adopted in the field of financial computation. The MC pricing method is a random sampling technique and consists of a sequence of random numbers drawn from a known distribution. Similar to any other MC method, it converges to a (theoretical) true value, as stated by the law of large numbers. Convergence can be considered slow, but variance reducing techniques are available to reduce this inconvenience.

In this report, we use the MC method to price a European put option and test its convergence, sensitivity to specific parameters and compare it to the Black-Scholes solution. We also examine the bump-and-revalue method and the Likelihood Ratio (LR) method for determining the greeks, in particular, the hedging parameter $\Delta$. Furthermore, we apply the control variates as a variance reduction technique on the Monte Carlo method to determine the option price of an Asian option.

## 2. Background & Theory

### 2.1. Black-Scholes equation

For simple option styles such as the European option, an analytical solution can be used to determine the option price of a risk-free portfolio (Black & Scholes, 1973). To price the option, it is assumed that no dividends are paid, the market is arbitrage-free and the interest rate $r$ is constant. The option price C can be computed using the following equation

$$C_t = N(d1)S_t - N(d_2)Ke^{-r(T-t)} \tag{1}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and N being the cumulative normal density function.

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_x^{-\infty} e^{-\frac{1}{z^2}} dz$$

$S_T$ is the stock price at maturity, $K$ is the strike price, $T$ represents the maturity time, $t$ is the time step and $\sigma$ is the volatility.

### 2.2. Monte Carlo

For options other than the European one, the analytical solution is not applicable due to the complex trading mechanics of the options. In such cases, the option price is determined numerically using the Monte Carlo method (Boyle, 1977).

#### 2.2.1 Wiener process

First, we assume that the stock price (S) evolves in the risk-neutral world. Therefore it can be model by the Wiener process, which results in a geometric brownian motion.

The Wiener process is defined by equation 2 with $r$ being the drift term and $\sigma$ being the volatility term. $dz$ refers to the Brownian motion.

$$dS = rSdt + \sigma Sdz \tag{2}$$

To simulate the stock price movement, the Wiener process is discretized using Euler scheme (Rouah, 2020). The euler method is resumed by equation 3 for $t \in 1, ..., T$, $r$ being the interest rate, $S_t$ being the current stock price, $\sigma$ being the volatility. $\epsilon$ is a random number from a normal distribution $\phi(0, 1)$.

$$S_t = S_{t-1} + rS_{t-1} + \sigma S_{t-1}\epsilon\sqrt{\Delta t} \tag{3}$$

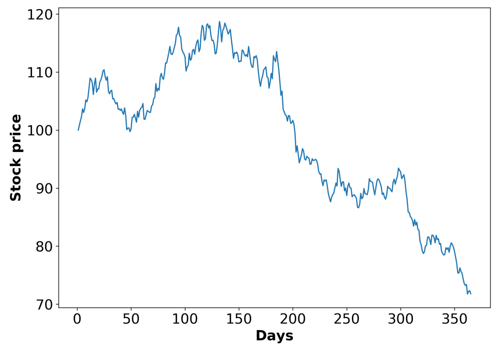



**Figure 1:** Example of a Wiener process price path using a volatility of 20% and and a interest rate of 6%.

The discretized euler method introduce time stepping errors which affect the end price at maturity time. However if the interest rate $r$ and the volatility $\sigma$ are constant, equation 3 can be integrated to stochastic differential equations (SDE) using Euler–Maruyama method (equation 4)

$$S_t = S_0 e^{(r-0.5\sigma^2)T + \sigma\sqrt{T}Z} \tag{4}$$

with $S_t$ being the stock price at maturity, $S_0$ the stock price at the moment and $Z$ being standard normal variable. Using equation 4, one doesn't need to compute the stock movements until maturity time but can directly determine $S_T$. The payoff at maturity depends on the type of the option. Table 1 resumes the different payoffs.

**Table 1:** Payoff formulas. For an Asian option, the payoff is equal to the difference between the mean of the stock price over a defined period and the strike price.

| Name | Payoff |
|---|---|
| Call | $\max\{S(T) - K, 0\}$ |
| Put | $\max\{K - S(T), 0\}$ |
| Spread | $\max\{S_1(T) - S_2(T) - K, 0\}$ |
| Asian call | $\max\{0, mean(S) - K\}$ |
| Asian put | $\max\{0, K - mean(S)\}$ |

Thus, the option price is determined using the discounted payoff equation as shown below, taking the risk-free rate $r$ into account and $N$ being the number of samples.

$$V(S^n, t = 0) = e^{-rT}\frac{\sum_{n=1}^{N} payoff^m(S^n)}{N} \tag{5}$$

The payoff matrix can take many forms and therefore makes this applicable for many processes. The standard error of this formula 6 is given as followed.

$$\frac{\sigma(payoff)}{\sqrt{N}} \tag{6}$$

### 2.2.2 Bump-and-revalue method

To measure the validity of the generated Monte Carlo model, a sensitivity analysis needs to be applied. By measuring the uncertainty of the output to the uncertainty of the input parameters, the robustness of the model can be computed. One method is the Bump-and-revalue method which introduces finite perturbation to the model. The idea is to compare the outcome of the model with and without a perturbation term $\epsilon$. Equation 7 resumes the bump-and-revalue method with $f$ returning the discounted payoff of the option using equation 5 (Cathcart et al., 2011).

$$\Delta = \frac{E\left[f(S_0+\epsilon)\right] + E\left[f(S_0)\right]}{\epsilon} \tag{7}$$

$$\begin{aligned} Var(\Delta) = \frac{1}{\epsilon^2} * (&Var\left(E\left[f(S_0+\epsilon)\right]\right) + \\ &Var\left(E\left[f(S_0)\right]\right) - \\ &2Cov\left(E\left[f(S_0+\epsilon)\right], E\left[f(S_0)\right]\right) \end{aligned} \tag{8}$$

$\Delta$ is a greek which capture the change of the option price to the change of the underlying stock price (J. C. Hull, 2003). According to literature, the bias which is due to the finite difference approximation $\epsilon$ can be reduced using small $\epsilon$. Furthermore, as can be seen from equation 8, one can use the same stream of random number for the bumped and unbumped value to increase the covariance between the two samples and therefore decreasing the variance of $\Delta$ (Cathcart et al., 2011).

For a digital option, the bump-and-revalue method as state above can't be used due to the fact that the payoff in such an option is discretized. At maturity time, if the option is beneficial, the payoff is 1 and if the option is not worth being exercised, the payoff is 0. An alternative would be the likelihood ratio method which doesn't take the payoff into account but the probability density (Glasserman, 2013).

$$C(S_0) = E[f(S_T)] = \int f(x)g(x,S_0)dx$$

with $C(S_0)$ being the price of the option, $f(x)$ being the payoff function using the Euler integration (equation 4) and $g(x,S_0)$ being the probability density of the stock price at maturity. Using the following relation $\Delta = \frac{dC}{dS_0}$, the integral can be rewriten as following by multiplying it by $\frac{g(x,S_0)}{g(x,S_0)}$.

$$\int_0^\infty f(x)\frac{\partial g(x,S_0)}{\partial S_0}\frac{g(x,S_0)}{g(x,S_0)}dx$$

Thus the expectation of the option price can be rewritten as follow.

$$C(S_0) = E\left[f(S_T)\frac{\partial log(g(S_T))}{\partial S_0}\right] \tag{9}$$

### 2.2.3 Variance Reduction

Due to the random nature of a Monte Carlo method, the variance is quite important. Thus, the variance decreases with an increasing number of samples. However, generating a high number of sampling is a compute-intensive task. Therefore, it is important to reduce the variance for a low number of samples. A variety of techniques exist to overcome this problem, a commonly used method in Monte-Carlo is antithetic variables. A more advanced method is control variates, this requires a close and known related process to compare it with. Related in this sense means with a high co-variance. Since we have such a process we use this technique to reduce our variance.

To reduced the variance of a random variable $X$ with control variates, another random variable $Y$ is used from which we know the expectation value $E[Y] = \mu$ is used. This expectation value is in our case calculated analytically. Thus the expected value for X can be expressed the following equation (J. Hull & White, 1988):

$$\tilde{X} = E[X + c(Y-\mu)]$$

The variance will result in the following equation.

$$Var[X+c(Y-\mu)] = Var[X]+c^2Var[Y]+2cCov[X,Y]$$

If $c$ is chosen to be $c = -\frac{Cov[X,Y]}{Var[Y]}$ The Variance reduce to the following where an high correlation between $X$ and $Y$ reduced the variance of $Y$.

$$Var[X+c(Y-\mu)] = Var[X] - \frac{Cov[X,Y]^2}{Var[Y]}$$

## 3. Methodology

The numerical analysis for this report was done with Python 3 and the scientific computing package NumPy. The generated results are plotted with the matplotlib package.

Firstly, the Monte Carlo method was analysed on a European put option with a spot price of 100 €, a strike price $K$ of 99€, an interest rate $r$ of 6%

and a volatility $\sigma$ of 20%. As mentioned in the section 2, the MC method uses equation 5 to evaluate the option price. To investigate the accuracy of the Monte Carlo method, a different number of samples were taken and compared to the well-known Black-Scholes solution. Each sample generated the stock price at maturity based on the Euler integration method (equation 4). The respective payoff is computed. Finally, the mean of the payoff is computed for a series of samples and used in the discount equation. Once the optimal number of samples needed to compute an accurate estimation of the option price was determined, the method was tested for a varying volatility and strike price.

The $2nd$ part of the report consists of determining the greek $\Delta$ using the Bump-and-revalue method. Thus the method is compared to the theoretical value. Using equation 7, the epsilon were chosen to vary from 0.01 to 0.5 by steps of 0.01. The stock price of 100 €, a strike price $K$ of 99€, an interest rate $r$ of 6% and a volatility $\sigma$ of 20%. For each *epsilon*, the effect of a different number of samples was investigated into. Furthermore, the same analysis was done using once a random seed and once fixed seed for $f(S_0+\epsilon)$ and $f(S_0)$. This means that the function $f$ used the same random numbers for $Z$ used in equation equation 4. Finally, the bump-and-revalue method was used to evaluate the $\Delta$ of a digital option. Due to the discontinuous payoff, the likelihood ration method (equation 9) was used. The distribution of the stock price at maturity is assumed to be log-normal (Black & Scholes, 1973). Therefore $g(S_T)$ can be expressed the following

$$g(S_T) = \frac{1}{S_T\sigma\sqrt{2\pi T}} e^{\left[-\frac{1}{2}\left(\frac{log(S_T/S_0)-(r-0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\right]}$$

Thus

$$\frac{\partial(g(S_T))}{\partial S_T} = \frac{Z}{S_0\sigma\sqrt{T}}$$

with $Z$ being the standard normal variable. Using the derived equation with the payoff being either one or zero, the $\Delta$ for the call option was determined by taking the mean of the different outcomes.

$$C = E\left[e^{-rT}max(S_T-K,0)\frac{Z}{S_0\sigma\sqrt{T}}\right]$$

The $3rd$ part of the report is about variance reduction of the Monte Carlo method applied to Asian options. Also, an analytical solution was derived from the geometric mean Asian option. We tried to reduce the variance on the expected payoff by using the control variates method. As stated in table 1, the payoff of an Asian option is dependent on the mean of the stock price over a certain period. The mean can be determined by the arithmetic or geometric mean. Due to the high correlation, it is expected that the covariance between the two means reduces the variance of the payoff with the arithmetic mean. The control variate is compared to the Monte-Carlo with a normal mean, but first we proved that the analytical close form solution of the Asian option price derived in the Appendix 6 is equal to the geometric mean Monte-Carlo in order to conclude that the Monte-Carlo is correct. For this experiment, the stock price $S$ is 100 €, the strike price $K$ is 99 €, the interest rate $r$ is 0.06 and the volatility $\sigma$ is 0.20%, the same as the European.

## 4. Results & Discussion

In this section, we discuss the results of the experiments. First, we look at the Monte-Carlo for European. We compare it with the analytical solution and see how it reacts on changes some key variables. Secondly, we calculate delta with the Monte-Carlo and take a look at the digital option with different techniques.

### 4.1. European Option

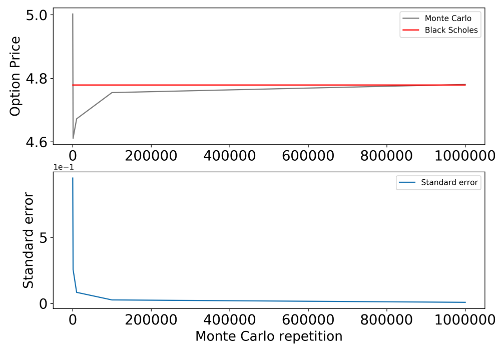



**Figure 2:** Monte Carlo method using different number of samples. $S_0$ = 100 €, the strike price $K$=99 €, the volatility $\sigma$ = 0.20% and the interest rate is 0.06%

In figure 2 we see the Monte-Carlo option price convergence to the analytical price. In addition the variance is reducing with the increasing amount of steps, thus following the predictions given by the law of large numbers.

**Table 2:** Pricing the option using the Monte Carlo method with different number of samples. The stock price $S_0 = 100$ €, the strike price $K$=99 €, the volatility $\sigma = 0.20\%$ and the interest rate is 0.06%.

| Number of samples | Mean | ±Var |
|---|---|---|
| 100 | 5 | ± 0.94 |
| 1000 | 4.61 | ± 0.25 |
| 10000 | 4.67 | ± 0.08 |
| 10000 | 4.75 | ± 0.027 |
| 1000000 | 4.78 | ± 0.008 |
| Black-Scholes | 4.77 | / |

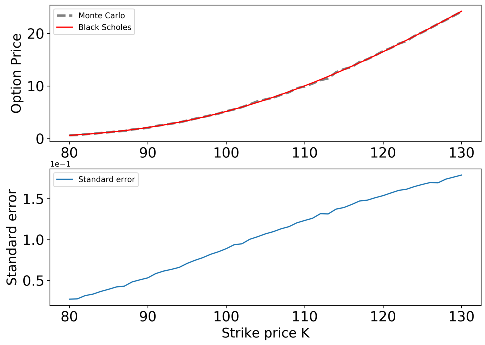



**Figure 3:** Monte Carlo method with 10 000 samples on varying strike price in order to determine an European call option $K$. $S_0 = 100$ €, the volatility $\sigma = 0.20\%$ and the interest rate is 0.06%.

In figure 3 we see the effect of the strike price for the European put, it behaves as we expect by increasing option price when strike prices rise, since profit changes.The standard error also seems to behave in the same way, this because higher prices change more due to the volatility being defined as a part and not a fixed value and lower prices have more zero profits which also decrease variance.

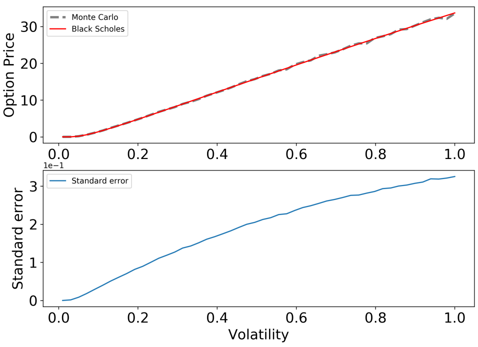
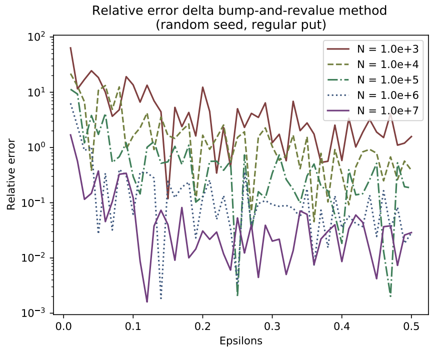



**Figure 4:** Monte Carlo method with 10 000 samples on varying Volatility in order to determine an European call option. $S_0 = 100$ €, the strike price $K$=99 €and the interest rate is 0.06%.

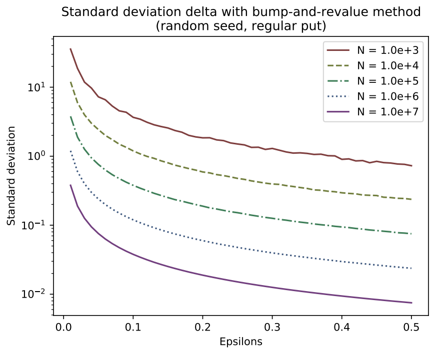
In figure 4 we see that volatility is linear related to the option price. The mean is deviating more from the analytical solution with higher volatility. This and the fact that the standard error is increasing with volatility are both related to an increase in variance due to the fact that volatility is correlated/equal to the variance.

## 4.2. Bump-and-revalue

As mentioned earlier, we examine the bump-and-revalue method for estimating the greeks and in particular the $\Delta$. To explore the consistency of the method we apply it for various values of $\epsilon$ and the amount of iterations and also experiment with the random seeds used.



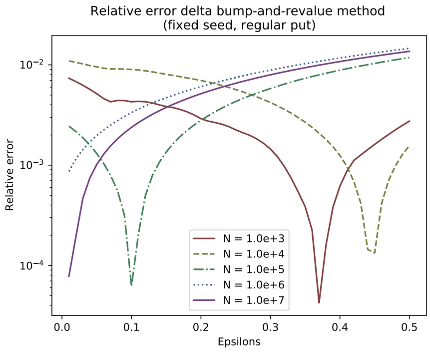

**(a)** Relative error w.r.t. to analytical solution



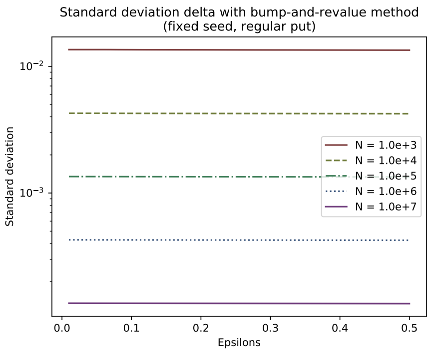

**(b)** Standard deviation $\Delta$

**Figure 5:** Relative error and standard deviation of the estimated $\Delta$ with the bump and revalue method initialised with random seeds for a European put option. The amount of iterations differs from 1000 till 10,000,000 and the epsilons range from 0.01 to 0.5 by steps of 0.01. Other parameters can be found in the theoretical background.

Figure 5a captures the results for an European put option. We see that increasing the number of iterations seems to improve the accuracy on average. It, however, contains still a lot of fluctuation, and the estimation can be considered unstable. Furthermore, we note a downwards trend for an increasing $\epsilon$ suggesting that the estimation becomes more accurate for bigger perturbations of the initial stock price. This is also confirmed by figure 5b where we examine a decreasing standard deviation for increasing $\epsilon$. Note also that an increase in the number of iterations also causes the standard deviation to drop. We already know that we can achieve a better estimation by using the same seed. The results for the bump-and-revalue method with fixed seed are represented in figure 6. We examine a drop in the relative error as the highest error found is approximately 0.01% while the highest error with the random seed is approximately 100%. Another notable appearance is what appears to be a tipping point; the relative error decreases for increasing $\epsilon$ until a particular value and then it stars to decrease. Furthermore, we see a stable standard deviation for each $\epsilon$, but we clearly examine a decrease in the standard deviation for an increasing number of iterations.



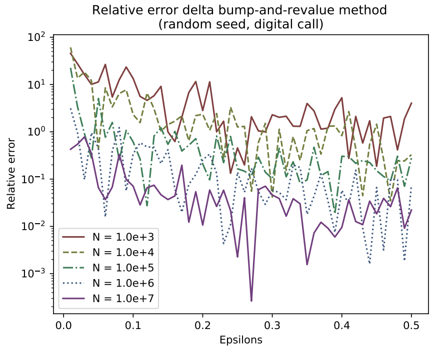

**(a)** Relative error w.r.t. to analytical solution



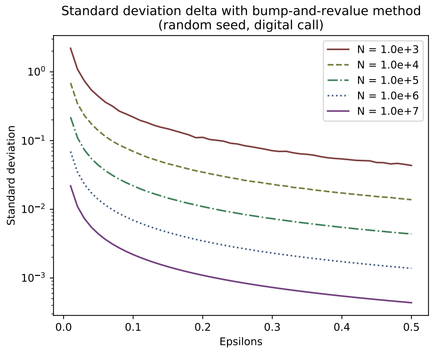

**(b)** Standard deviation $\Delta$

**Figure 6:** Relative error and standard deviation of the estimated $\Delta$ with the bump and revalue method initialised with fixed seeds for a European put option. The amount of iterations differs from 1000 till 10,000,000 and the epsilons ranges from 0.01 to 0.5 by steps of 0.01. Other parameters can be found in the theoretical background.

The bump-and-revalue method is also applied to an European digital call option in order to estimate the hedging parameter $\Delta$. The results with random seeds can be found in figure 7. The results are rather similar to those of the European put option with random seeds and the same observations can be made. If we now apply the bump-and-revalue method with fixed seeds, the relative error and standard deviation, which we can find in figure 8, tend to drop to a lower maximum value. However, the results do not appear to become significantly more stable as we see from the fluctuations in the relative error. A reason is the payoff structure of the digital option. As the option either pays nothing-at-all or one euro a change in the initial stock price can cause the option to shift from an out-of-the-money situation towards an in-of-the money situation strongly influencing the estimation of $\Delta$.

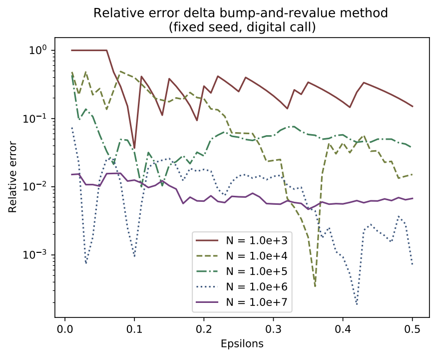



**(a)** Relative error w.r.t. the analytical solution

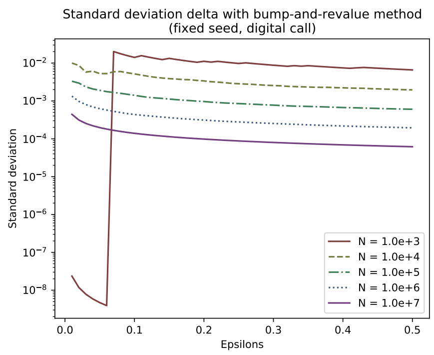



**(b)** Standard deviation

**Figure 7:** Relative error and standard deviation of the estimated $\Delta$ with the bump and revalue method initialised with random seeds for an European digital call option. The amount of iterations differs from 1000 till 10,000,000 and the epsilons ranges from 0.01 to 0.5 by steps of 0.01. Other parameters can be found in the theoretical background.




**(a)** Relative error w.r.t. the analytical solution




**(b)** Standard deviation

**Figure 8:** Relative error and standard deviation of the estimated $\Delta$ with the bump and revalue method initialised with fixed seeds for an European digital call option. The amount of iterations differs from 1000 till 10,000,000 and the epsilons ranges from 0.01 to 0.5 by steps of 0.01. Other parameters can be found in the theoretical background.

A possible solution to the unstable behaviour of the bump-and-revalue method for the digital option is the Likelihood Ratio (LR) method. The results are presented in figure 9. The results seem to be more stable compared to the bump-and-revalue method. We examine a decrease in both the relative error and standard deviation for an increasing number of iterations, suggesting that the LR method converges to the (theoretical) true value. Note that the standard deviation decreases linearly which is logical as the standard deviation for MC methods combined with LR method is $\frac{\sigma_{payoff}}{\sqrt{N}}$ and is a linear function in N.

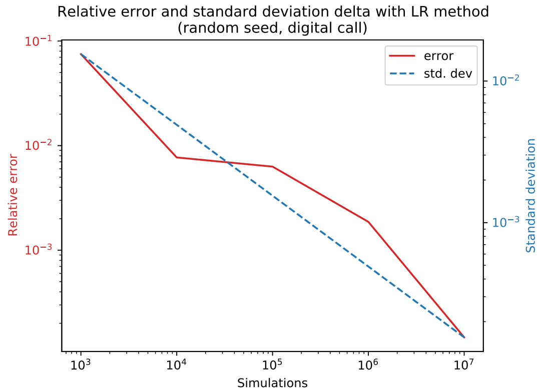



**Figure 9:** LR method applied to an European digital call option. The parameters can be found in the corresponding subsection in the theoretical background. The relative error and standard deviation of $\Delta$ are presented for different number of iterations.

## 4.3. Asian Option

In the background, we spoke about the payoff of the Asian option. As shown in table 1 we can see that the payoff is calculated by the mean of the price. Because of the different payoff, we get a different price. The price is calculated by 4 different methods that are explained above. Figure 10 shows the price for the standard variable values varying the number of iterations,

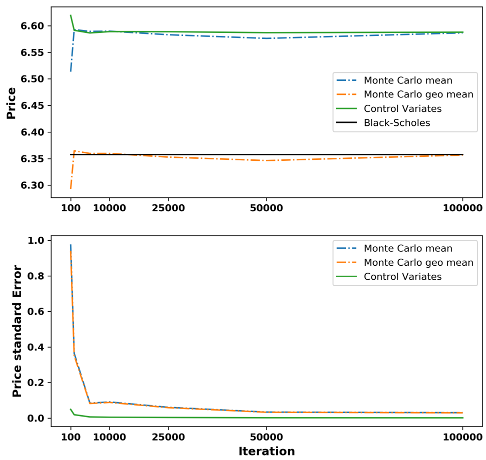



**Figure 10:** The mean option price and below the standard error. For the following methods, Monte Carlo with the geometric mean, Monte Carlo with normal mean and the control variates method. For a total comparison, we also added the black line, which means the analytically derived option price.

Here we see clearly that both Monte-Carlo mean and Control variates converge to the same value, the same can be said about Monte-Carlo geometric mean and the analytical derivation. This behaves as expected, as they both model the same Asian option. If we look at the standard error, we clearly see the control variates having a smaller error in comparison with the Monte-Carlo methods. This behaviour was also as expected. The reason for the geometric mean for having a lower option price is given by the fact that the geometric mean is lower than the normal mean, thus undervaluing the stock average.

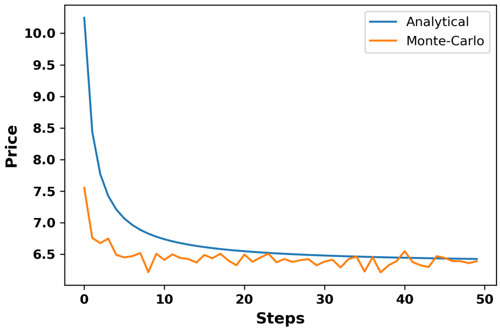



**Figure 11:** The option price for different amount of steps, the steps are the number of averaging moments in the asian option. We look at both geometric Monte-Carlo and the analytical version.

In figure 11 we see that for steps going to infinity the option price converges to a minimum value, both for Monte-Carlo and the analytical solution. This tells us, that less evaluations, or higher evaluation periods rise the option price.

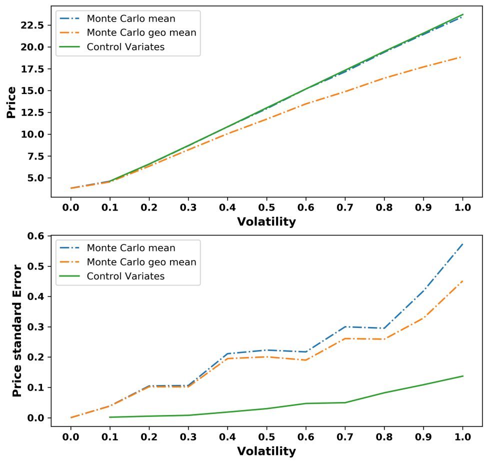



**Figure 12:** The three

In figure 12 we confirm the same behaviour as the European option, where the price of the option rises with the volatility. Again the geometric mean in undervalued thus at higher volatility will rise less quick and at volatility zero will be exactly the same.

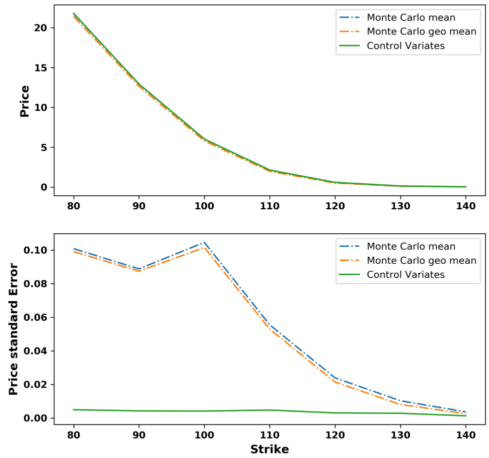



**Figure 13:** The three

In figure 13 we that all methods act similarly on the change of strike price for the Asian call option. The error of the control variates is again lower than the Monte-Carlo. The reason that the error decreases is because of the certainty of no profit increase, so if fewer samples are profitable, many are just zero and a low error is observed.

## 5. Conclusion

Monte-Carlo can model every option and, by definition of the law of large numbers, give a good estimation on a option price. Asian options can be priced analytically, when using geometric mean. However, this undervalues the option price. Variance of a Monte-Carlo method can be dramatically reduced by control variates. Asian option behave similar to European options only operates at a lower price since the average reduces large deviation in the price and thus large profits.

We also discussed two methods to estimate the hedging parameter $\Delta$: (i) bump-and-revalue; (ii) LR. We found that using a random seed with the bump-and-revalue method the estimations are rather unstable, but that the variance decreases for larger $\epsilon$. The estimations become more accurate if the same seed is used for the simulations. Furthermore, it appears that the method is not suitable for an European digital call option and other sophisticated methods, such as the LR method, are necessary. Not only is the LR method applicable to more types of options, it also appears to be more accurate, stable and decreases for the number of iterations.

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## 6. Appendix

### 6.1. Derivation of an analytical expression for the price of an Asian option

*Proof.* The geometric mean is defined as follow

$$G(\bar{S}) = \left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{n}} = \sqrt[n]{x_1 x_2 \cdots x_n}$$

$$ln\left(\frac{G(\bar{S})}{S_0}\right) = ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{n}}}{S_0}\right) = ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)}{S_0^n}\right)^{\frac{1}{n}} = \frac{1}{n} ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)}{S_0^n}\right)$$

Thus we can rewrite $\prod_{i=1}^{n} S_i$ as

$$\frac{\prod_{i=1}^{n} S_i}{S_0^n} = \frac{S_n}{S_{n-1}}\left(\frac{S_{n-1}}{S_{n-2}}\right)^2 \ldots \left(\frac{S_2}{S_1}\right)^{n-1}\left(\frac{S_1}{S_0}\right)^n \frac{\cancel{S_0^n}}{\cancel{S_0^n}}$$

where the last term $S_0^n$ is cancelled by the denominator. Furthermore $ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)}{S_0^n}\right)$ can be rewritten as a sum.

$$\begin{aligned}
\frac{1}{n} ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)}{S_0^n}\right) &= \frac{1}{n}\left[ln\left(\frac{S_n}{S_{n-1}}\right) + ln\left(\frac{S_{n-1}}{S_{n-2}}\right)^2 + \ldots + ln\left(\frac{S_1}{S_0}\right)^n\right] \\
&= \frac{1}{n}\left[ln\left(\frac{S_n}{S_{n-1}}\right) + 2ln\left(\frac{S_{n-1}}{S_{n-2}}\right) + \ldots + nln\left(\frac{S_1}{S_0}\right)\right]
\end{aligned}$$

Since the price path $S_i$ is based on the Wiener process, the price can be expressed by the euler scheme (equation 4, section 2), with $T = \frac{T}{n}$. Hence we have

$$\frac{S_i}{S_{i-1}} = \frac{S_0 e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}}{S_0 e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}} = \frac{e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}}{e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}}$$

Since $Z$ is a standard normal variable and independent from the other generated $Z$, the division results in a Cauchy distribution which is also normal distributed. Thus

$$\frac{S_i}{S_{i-1}} = e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}$$

$$\begin{aligned}
\frac{1}{n} ln\left(\frac{\left(\prod_{i=1}^{n} S_i\right)}{S_0^n}\right) =& \frac{1}{n}\left[ln\left(e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}\right) + 2ln\left(e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}\right) \ldots + nln\left(e^{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{T}Z}\right)\right] \\
=& \frac{1}{n}\left[\left((r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z\right) + 2\left((r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{T}Z\right) \ldots + n\left((r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z\right)\right] \\
=& \frac{1}{n}\left[(1 + 2 + \ldots + n)\left((r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z\right)\right] \\
=& \frac{\sum_{i=1}^{n} x_i}{n}\left(\underbrace{(r-0.5\sigma^2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}Z}_{\text{Wiener process}}\right)
\end{aligned}$$

According to J. C. Hull (2003), the mean is equal :

$$\begin{aligned}\mu =&\frac{\sum_{i=1}^{n} x_i}{n}(r-0.5\sigma^2)\frac{T}{n}\\ =&\frac{(1+n)\cancel{n}}{2\cancel{n}}(r-0.5\sigma^2)\frac{T}{n}\\ =&\frac{(1+n)}{2n}(r-0.5\sigma^2)T\end{aligned}$$

According to J. C. Hull (2003), the variance is equal :

$$\begin{aligned}var =&\frac{\sum_{i=1}^{n} x_i^2}{n^2}(\sigma^2\frac{T}{n})\\ =&\frac{(2n+1)(n+1)}{6n}(\sigma^2\frac{T}{n})\\ =&\frac{(2n+1)(n+1)}{6n^2}\sigma^2 T\end{aligned}$$

$\square$

Thus the change between the stock price $S_0$ at $t=0$ and the stock price at $S_T$ at maturity time $T$ is governed by a normally distribution $\phi$ with the determined mean and variance

$$ln(S_T)-ln(S_0)\sim\phi\left[\frac{(1+n)}{2n}(r-0.5\sigma^2)T,\frac{(2n+1)(n+1)}{6n^2}\sigma^2 T\right]$$
$$ln(S_T)\sim\phi\left[\frac{(1+n)}{2n}(r-0.5\sigma^2)T+ln(S_0),\frac{(2n+1)(n+1)}{6n^2}\sigma^2 T\right]$$

Thus if $n\to\infty$, $\frac{1+n}{2n}\to\frac{1}{2}$ and $\frac{(1+n)(2n+1)}{6n^2}\to\frac{1}{3}$.

$$ln(S_T)\sim\phi\left[\frac{1}{2}(r-0.5\sigma^2)T+ln(S_0),\frac{1}{3}\sigma^2 T\right]$$

For a stock price at maturity time $T$ with a lognormal distribution, the generalised Black-Scholes equation can be used to derived the price of an Asian option

$$c=e^{-rt}\left[e^{\mu+0.5var^2}n\left(\frac{\mu+var^2-logK}{var}\right)-Kn\left(\frac{\mu-K}{var}\right)\right]$$

Using as $\mu=\frac{1}{2}(r-0.5\sigma^2)T+ln(S_0)$ and $var=\frac{1}{3}\sigma^2 T$:

$$e^{\mu+0.5\sigma^2}=S_0 e^{\frac{1}{2}(r-\frac{2}{6})T}$$

Using the previously derived equation, we can also rewrite

$$\begin{aligned}\left(\frac{\mu+var^2-log(K)}{var}\right)=&\left(\frac{loge^{\mu+\frac{var^2}{2}}+\frac{var^2}{2}-log}{var}\right)\\ =&\left(\frac{log(\frac{S_0}{40})^{\frac{1}{2}(r-\frac{2}{6})T})}{K}+\frac{\sigma^2}{6}T\sqrt{\frac{T}{3}}\sigma\right)= \qquad d_1\end{aligned}$$

$$\begin{aligned}\left(\frac{\mu-K}{var}\right)=&d_1-\sigma\sqrt{\frac{T}{3}}\\ =&d_2\end{aligned}$$

Plugging all this back in the general solution we get for an call option

$$c = e^{-rT}\left(S_0 e^{\frac{1}{2}(r-\frac{2}{6})T} n(d_1) - S_0 e^{\frac{1}{2}(r-\frac{2}{6})T} K n(d_2)\right)$$

and for a put option

$$p = e^{-rT}\left(-S_0 e^{\frac{1}{2}(r-\frac{2}{6})T} n(-d_1) + S_0 e^{\frac{1}{2}(r-\frac{2}{6})T} K n(-d_2)\right)$$