Numerical verification of the Black-Scholes Model and Cox-Ross-Rubinstein market model

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Abstract

Options are a type of derivatives which give the holder the right but not the obligation to exercise the contract at maturity time. The price of the derivative depends on the underlying asset and can be determined using either the numerical Cox-Ross-Rubinstein market model (Cox et al., 1979) or the analytical Black-Scholes Model (Black & Scholes, 1973). In this report, the numerical model is compared to the analytical solution. The effect of the volatility and the size on the binomial tree of the Cox-Ross-Rubinstein is analysed. Furthermore, the greek Δ is studied in case of a dynamical hedging. For the binomial tree, it has been shown that a higher volatility increases the option price. Additionally, the size of the tree revealed to be crucial for the precision of the pricing. In the case of the dynamical hedging, it has been shown that the frequency of applying the strategy and the implied volatility is a crucial element for achieving a riskfree portfolio.

1. Introduction

In 1973, the Chicago Board Option Exchange firstly introduced a structured market for trading options (Hull, 2003). This latest type of contract consists of an agreement between the option's writer and the holder on the price of the underlying assets at a certain time called the maturity date. The price of the underlying asset which can be bought/sold by holder at maturity time is is called the strike price. Two types of options defined as call option and put option exists. In a call option, the holder is given the right to purchase the underlying asset at the agreed strike price. In a put option, the holder has the right to sell the underlying asset at an agreed strike price. The option contract differs from the forward contract in that the option doesn't need to be honoured. Multiple call/put option markets exists such as the European, the American and the Asian. In a European option the holder can only honour the contract at maturity time whereas in an American option the contract can be honoured before maturity time. The decision of the holder to exercise the contract depends on the stock price of the underlying asset at maturity time (or before for an American option) and the type of contract. A call option is only exercised if the stock price is above the strike price. This allows the holder to buy the underlying stocks for cheaper than its current value and sell it on the market to reap the profits. A put option is only exercised if the stock price is below the strike price. The holder can buy the underlying asset and sell them at a higher price which corresponds to the strike price. If the holder can't make any profit from the option, the contract is not honoured. However, in order to avoid going away empty-handed every time, the writer imposes a certain price on the derivative. In case of the contract being honoured or not, the writer always earns the price put on the option. In this report, the mechanics which determines the option price are analysed. More specific the relation between the Cox-Ross-Rubinstein market model (CRR model) (Cox et al., 1979) and the analytical solution of the Black-Scholes equation (Black & Scholes, 1973) are analysed. Furthermore, the dynamical hedging from an option writer's point of view analysed in order to guarantee a riskless portfolio.

2. Background & Theory

In order to study the Black-Scholes Model and the Cox-Ross-Rubinstein market model, a few assumptions need to be formulated. We state that the financial system is composed of a simple two-state economy with no transaction cost and a constant interest rate r. Furthermore, no dividend is paid to the holders of the stock. Finally, we assume that the market is arbitrage-free; no miss-pricing of the option relatives to each other.

2.1. Cox-Ross-Rubinstein market model

In a risk-free world, the expected stock price at the first time step Δt is computed using continuous compounding with

$$S_0 e^{r\Delta t} \tag{1}$$

where S_0 is the initial stock price, r is the risk-free interest rate (Kandhai et al., 2020).

In a Cox-Ross-Rubinstein market model, the value of a stock can either increase with a probability p or decrease with a probability $1-p$. Consequently, the price of the stock either increase with by pS_0u or decreases by $(1-p)S_0d$ (figure 1) with u and d being

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$

σ is the volatility term which defines the uncertainty of the return of an underlying asset due to fluctuations.

The expected return of the stock needs to match the expected return of the stock computed the binomial tree. Consequently from equation 1,

$$pS_0u + (1-p)S_0d = S_0 e^{r\Delta t}$$

Thus p is the risk neutral probability of the stock price moving up ward (Cox et al., 1979)

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2}$$

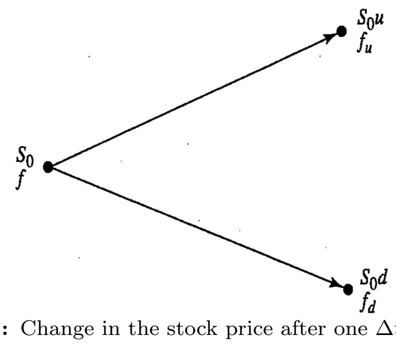

Figure 1: Change in the stock price after one Δt step (Hull, 2003).

In order to calculate the option price f at node 0, the following statement needs to be true. In a riskless portfolio, the value of the two resulting branches needs to be identical. In other words,

$$S_0u\Delta - f_u = S_0d\Delta - f_d \tag{3}$$

where f_u and f_d represent the payoff of the option in case of an upwards and downwards movement respectively (Hull, 2003). Δ represents the amount of shares needed to build a riskless portfolio. The fundamental property of such a portfolio is that it should earn the risk-free rate of interest r. Using the continuous compounding (equation 1), the cost of a portfolio at node 0 should be

$$S_0\Delta - f = (S_0u\Delta - f_u)e^{-rT} \tag{4}$$

Using equation 3 for Δ and substituting into equation 4, the following equation obtained

$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{5}$$

where p is the risk-neutral probability. f_u and f_d is the difference between the stock price at maturity and the strike price. In case of a call option, if the stock price S_0d is below the strike price, the option is not honoured and f_d=0. In case of a put option, if the stock price S_0u is above the strike price, the option is not honoured and f_u=0.

Figure 1 shows a binomial tree consisting of a node representing the spot time (left) and the two nodes representing the maturity time (right). However, the binomial tree normally consists of multiple layers between the moment the contract is set up and the maturity time. For each node, two more branches emerge which create two more alternatives. This would yield a total of 2^{n+1} nodes and, in terms of time, 2^n forward and backward walks must be computed. However, as there are fixed downward and upward movements the binomial tree is considered to be recombinant i.e. if the underlying asset moves up and then down (u,d), the price will be the same as if it had moved down and then up (d,u). Hence, for n periods a recombinant tree requires only $\frac{1}{2}(n^2+n)$ nodes and has a total of (n^2+n) forward and backward walks . This means that the computational complexity is brought down to $O(n^2)$.

2.2. Black-Scholes equation

The Black-Scholes equation is an analytical solution to determine the option price for a European style option (Black & Scholes, 1973). For an American option, the solution is not applicable due to the fact that in an American option, the holder has the right to honour the option before maturity time. Based on the assumptions mentioned at the beginning of this section, the value of the option will only depend on the underlying stock S_T. Furthermore, the distribution of all the outcomes the stock price can have at maturity time is said to be lognormal (Black & Scholes, 1973).

While the full derivation of the Black-Scholes equation is beyond the scope of this paper, the final solutions are given below. The option price C is defined by the differential equation 6 with K being the strike price, S_t being the stock price at maturity time T and t being the step size.

$$C_t = N(d1)S_t - N(d_2)Ke^{-r(T-t)} \tag{6}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and N being the cumulative normal density function.

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_x^{-\infty} e^{-\frac{1}{z^2}}dz$$

2.3. Dynamical Delta hedging

In order to guarantee a risk-less portfolio, the option's writer can apply a dynamical hedging strategy. The approach is to evaluate Δ at every update of the price of the underlying stock and make sure to hold Δ units of the underlying stock. Any remaining cash or debt is put in a risk-free account. At the exercising time, the writer needs to be able to deliver the underlying asset to the holder of the option if required and sell the remaining Δ shares. By doing so, the chance to be even at maturity time increases. Thus the hedging can be achieved by differentiating equation 6 and using the chain rule which results in the following equation.

$$\Delta_t = \frac{\partial C_t}{\partial S_t} = N(d_1) \tag{7}$$

As we see from equation 7 Δ represents the rate of change in the option price with respect to the price of the underlying stock.

3. Methodology

For this report, the numerical analysis was done using Python 3 with the scientific computing package NumPy.

To compute the Cox-Ross-Rubinstein market model, a back-ward induction scheme was used. At each step $x \in 1, ..., n$ with n being the total number of steps, the stock price at the respective nodes can be computed using the according formula S_0u^x and S_0d^x. Consequently, all the possible stock prices at maturity are known (figure 2).

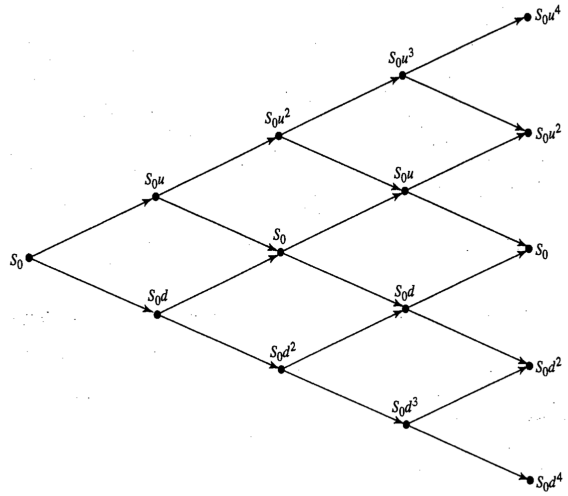

Figure 2: Binomial Tree with 4 steps. At each node the stock price is computed using the respective formula S_0u^x and S_0d^x (Hull, 2003)

Thus, the option price for the nodes at maturity are simple the difference between the strike price and the stock price at maturity S_T. For a call option the price f is

$$f_u = max(S_T - K, 0)$$

and for a put option

$$f_d = max(K - S_T, 0)$$

From the nodes at maturity, the option price f of the the previous node can be computed using equation 5. This process is repeated until the option price is determined for the node representing the spot time.

In this report the results of the binomial tree are compared to the Black-Scholes equation varying different parameters such as the volatility, the number of steps in the tree and the option style. The strike price K is set to €99 and the stock price S_0 at spot time is set to €100. The interest rate r is fixed to 6% and the volatility σ to 20%.

For the dynamical hedging, a simulation of the market was modelled. The movement of the stock was determined by the Wiener process, which resulted in an geometric brownian motion (figure 3).

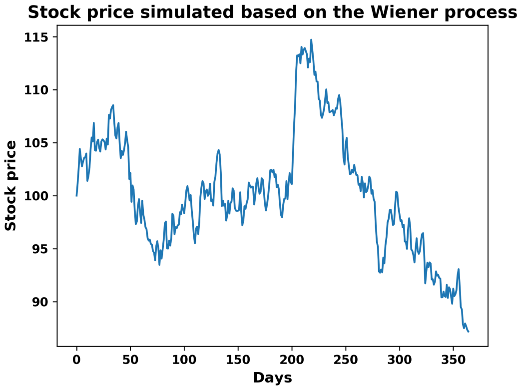

Figure 3: Example of a Wiener process price path.

The Wiener process is resumed by equation 8 for $t \in 1, ..., T$, r being the interest rate, S_t being the current stock price, σ being the volatility. ϵ is a random number from a normal distribution $\phi(0, 1)$.

$$S_t = rS_{t-1} + \sigma S_{t-1}\epsilon\sqrt{\Delta t} \tag{8}$$

The interest rate r is defined as the drift rate and $\sigma\sqrt{\Delta t}$ as the standard deviation.

In a later stage, the dynamical hedging was applied to real data such as the AAPL Stock Price (Apple Inc.) and Royal Dutch Shell Plc Stock Price.

4. Results & Discussion

Figure 4 shows the impact of the volatility on the option price for a European call option determined by the binomial tree (blue) and the Black-Scholes solution (orange).

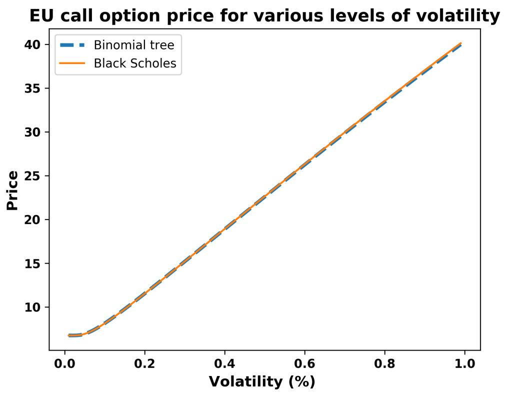

Figure 4: European call option: Effect of the volatility on the option price determined with a binomial tree of 50 steps with an interest rate of 6%, a spot price of €100 and a strike price of €99 and the Black-Scholes equation.

Both methods yield approximately similar result. As we see, the option price increases with the level of volatility. The high volatility induces important fluctuation in the stock price path. Consequently, high fluctuations increase the possibility of making higher profits as an option holder. Therefore, the option price increases.

Figure 5 shows the results of the option price based on the Black-Scholes solution (orange) and the binomial tree (blue) for different steps.

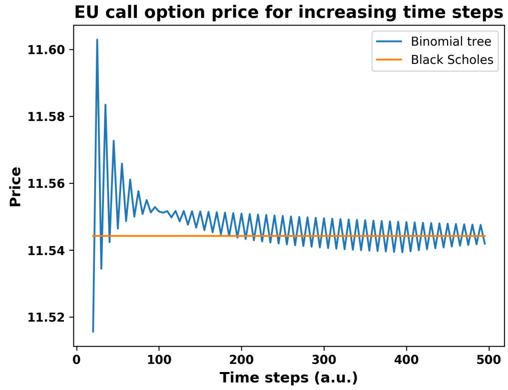

Figure 5: European call option: Effect of the number of step in a binomial tree on defining the price of an option. The generated option prices are compared to the analytical solution of the Black-Scholes equation.

The Black-Scholes equation yields an option price of €11.54. The binomial trees with a step size smaller than 100 showed high fluctuation in determining the option price. From 100 time steps on, the fluctuations are less important with a convergence to the Black-Scholes solution. The convergence shows the beginning of scalloped lines with a cusp point around 100 number of steps. The demonstration of the scalloped lines and the convergence by $\frac{1}{n}$ to the analytically computed option price is limited by the computational power available (Diener & Diener, 2004). Figure 6 shows the average computational time needed for various amount of time steps. One can see that the time increases exponential with the time steps. The minor fluctuations are due to interference of the background tasks on the running machine which slightly influences the speed of the computer. However, this main message is clear that the running time rises exponentially.

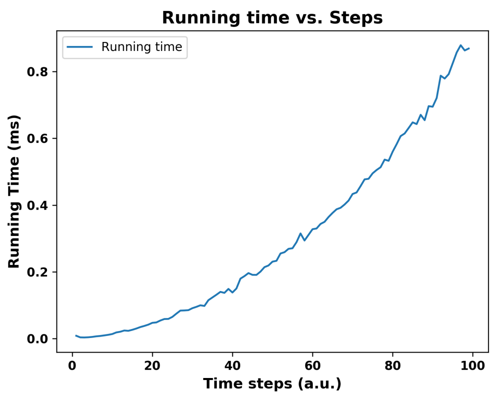

Figure 6: Exponential increasing computing time of a binomial tree with increasing step size.

Figure 19 shows the option price of an American call option with different steps. The results are identical to figure 5. This can be explained by the fact that for an American call option is not wise to exercise the option before maturity time and thus can be treated as a European call option (Hull, 2003). This is due to the fact that the option's holder can earn on the interest rate of the strike price over the whole period of the contract. Furthermore, since no dividends are paid, the holder can't expect any income from the stock. Finally, an early exercise runs the risk that the stock price could eventually drop further which prevents higher profits. Consequently, figure 17 yields similar results for an varying volatility as for figure 4.

On the other hand, the results for an American put option differs from the Black-Scholes and thus from the European put option (figure 16). Figure 7 shows that the American put option price converges to 5.34. As a comparison, the Black-Scholes returns for the European put option a price of 4.78. The reason for the higher pricing for the American put option is because for the latest option it is wiser to exercise before maturity time if the contract is profitable. This has to do with the fact that a stock price can never be negative. Thus the maximal profit of a put option will always be the strike price whereas the profit in a call option is not limited. Thus in some cases where the stock price is at some point, it might be wiser to exercise earlier to maximise the profit.

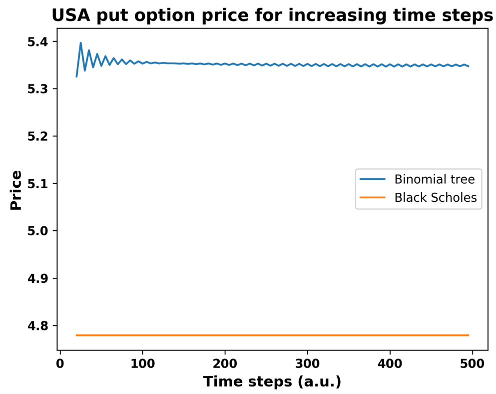

Figure 7: American put option: Effect of the number of step in a binomial tree on defining the price of an option. The generated option prices are compared to analytical solution of the Black-Scholes equation.

Figure 8 visualises the hedging parameter Δ in terms of the volatility σ. We clearly see a peculiar development of the hedge parameter. For $\sigma \in [0, 0.4]$ an increase in volatility results in a decrease of Δ, but for higher volatilities we note an increase. The two regions represents different situations for the option holder. At first the option holder is *in-the-money*, most of its payoffs are already positive. Hence, increasing volatility causes an increase in the likelihood for zero payoffs from a shift in the underlying stock price i.e. Δ increases. However, if the option holder is *out-of-the-money*, for example with $\sigma \in (0.4, 1)$, most of its payoffs are zero and a rise in volatility increases the likelihood for positive payoffs from a change in the underlying price of the stock i.e. Δ increases.

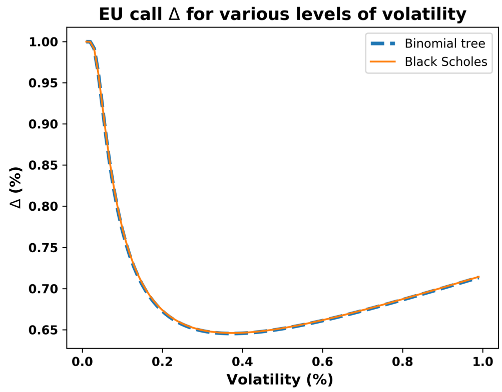

Figure 8: European call option: Effect of the volatility on the hedging parameter Δ. The Δ has been computed based on a binomial tree of 50 steps, an interest rate of 6%, a strike price K=€99 and a stock price S_0=€100.

In the second part of the report, the dynamical hedging was analysed by applying a daily hedging to real data from AAPL Stock Price (Apple Inc.) and RDS-A Stock Price (Royal Dutch Shell). Figure 10 and 9 show the results of the hedging (blue) on the stock movement (red) for a European call option. As one can see, when the stock moves upward, the hedging parameter Δ increases to 1 whereas if the stock decreases, Δ decreases. The meaning of Δ tells the option's writer how much of the stock to hold. In case of the stock price losing in value, the writer should sell first sell 1 stock and then buy Δ shares in order to assure a riskless portfolio at maturity time. However, achieving a riskless portfolio is not always possible and depends highly on the last movement before maturity. In case of hedging the real data, both portfolios were not riskless and enhanced €3.21 (Apple) and €2.45 (Shell) losses.

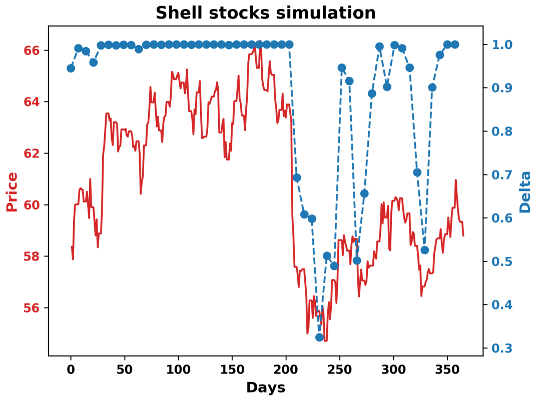

Figure 9: Dynamical hedging in case of the Shell stock from $2019-01-01$ to '$2020-02-01$.

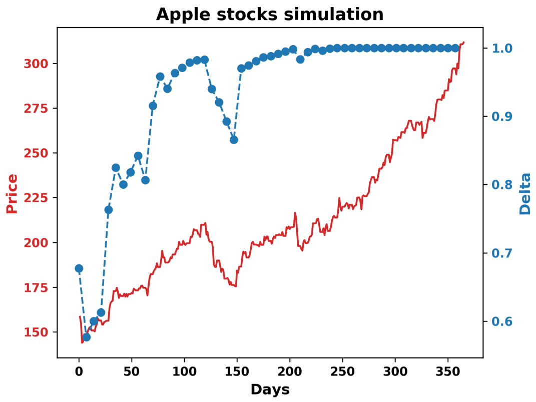

Figure 10: Dynamical hedging in case of the Shell stock from $2019-01-01$ to $2020-02-01$

Figure 12 and 11 show the results of the profit at maturity time for the writer of the option using daily and weekly hedging respectively. The hedging was applied to a simulated stock movement based on the Wiener process. For each strategy, the process was repeated 1000 times in order to create a distribution. Both strategies result in a distribution of around 0 profit, creating occasionally losses or profits. However, the daily hedging yield a mean value of 0.017 and a standard deviation of 0.359 whereas the weekly hedging yields a mean value of 0.122 with a standard deviation of 0.910.

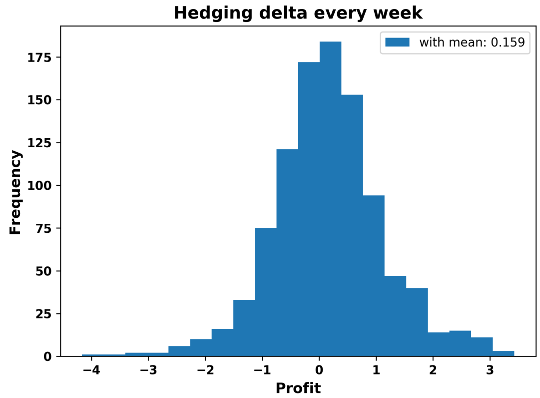

Figure 11: 1000 repetition of daily hedging with a profit mean of 0.017 and an standard deviation of 0.359.

Thus a daily hedging strategy increases the chances of being even at maturity time.

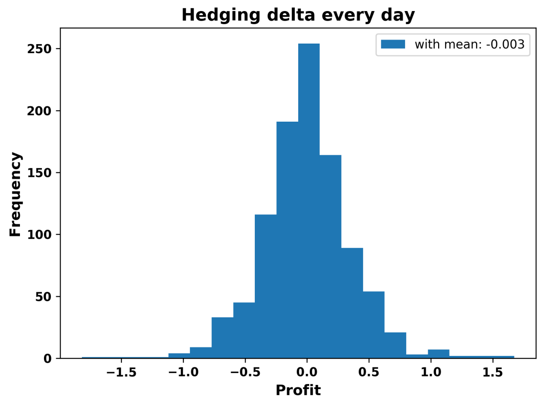

Figure 12: 1000 repetition of weekly hedging with a profit mean of 0.122 and an standard deviation of 0.359.

Figure 13 reinforce the statement that a higher number of hedging n along the time to maturity decrease the uncertainty of running a risk at maturity. Figure 14 shows how the mean approaches zero with a higher number of hedging and how the respective variance decreases.

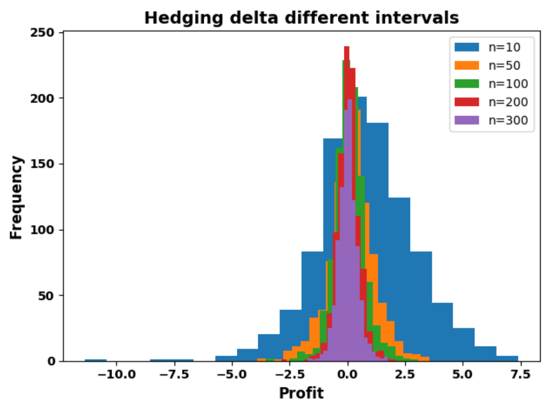

Figure 13: Dynamical hedging on a call option with different number of hedging n. For each strategy, the end profit was computed 1000 times.

In other words, in order to the likelihood of having a riskless portfolio, one should practice the hedging with a high frequency.

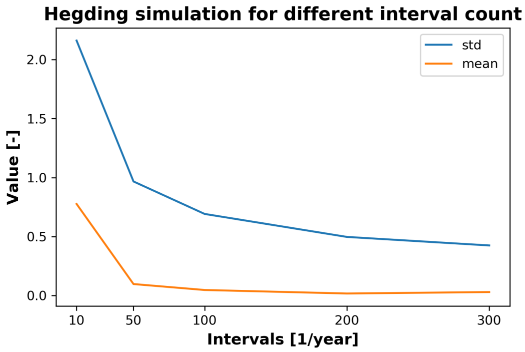

Figure 14: Computed mean and standard deviation of the different profit distribution shown in figure 13.

Figure 15 shows the relation between the implied volatility and the volatility used by the stock price. With a stock price of 100 and a volatility of 0.20 or 20%, we see that using a lower implied volatility will reduce the profits and visa versa. Therefore we need to mach the volatility as close as possible to ensure a risk free portfolio. Since the price of the stock is set at 100, the profit can also be interpreted as a percentage. We can approximate the change as followed, a 10% change in volatility will have a 4% change in the profit.

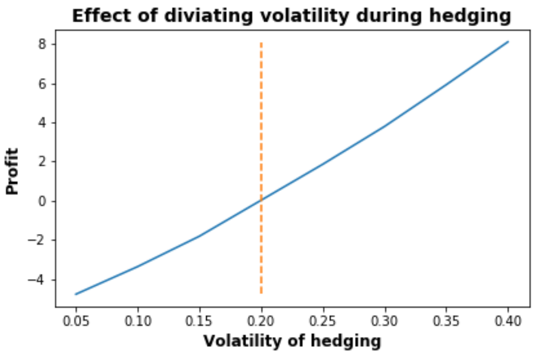

Figure 15: Mean profit in comparison with different hedging volatility's. The orange stripped line represents the volatility of the price path.

5. Conclusion

Through out the report it has been demonstrated that the volatility and the complexity of the binomial tree affects the pricing of the options. An high volatility rises the option price because the possibility for the holder to gain higher profits is more likely. Furthermore, the importance of the length of the binomial tree has been demonstrated. The more levels a binomial tree has, the more accurate is the option pricing in respect to the analytical solution. In case of applying a dynamical hedging strategy, it has been shown that a higher frequency improves the outcome of running risk-free at maturity time. In addition we see the importance of the implied volatility during the hedging on running risk free.

6. Appendix

6.1. European style option

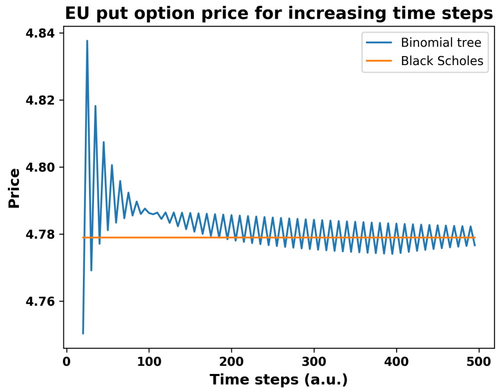

Figure 16: European put option: Effect of the number of step in a binomial tree on defining the price of an option. The generated option prices are compared to analytical solution of the Black-Scholes equation.

6.2. USA style option

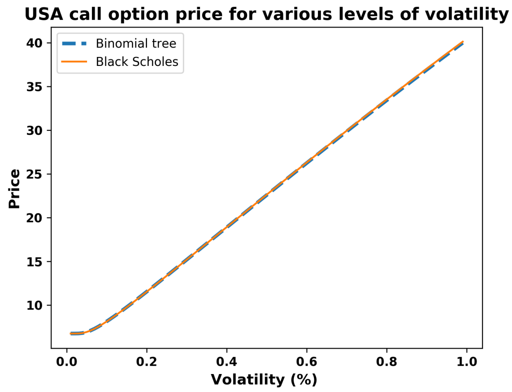

Figure 17: American call option: Effect of the volatility on the option price determined with a binomial tree of 50 steps with an interest rate of 6%, spot price of 100e and strike price of 99e and the Black-Scholes equation.

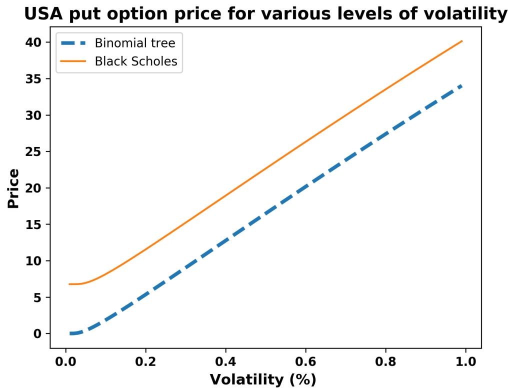

Figure 18: American put option: Effect of the volatility on the option price determined with a binomial tree of 50 steps with an interest rate of 6%, spot price of $100e$ and strike price of $99e$ and the Black-Scholes equation.

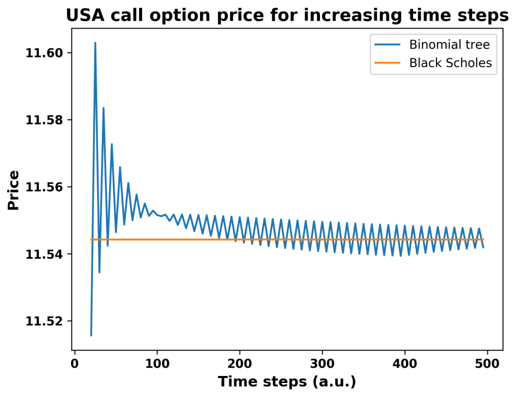

Figure 19: American call option: Effect of the number of step in a binomial tree on defining the price of an option. The generated option prices are compared to analytical solution of the Black-Scholes equation

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