Kürzeste Verbindungs-Linie in 3 Dimensionen

Problem

Kürzeste Verbindung zwischen zwei Linien finden.

$$L_1(s) = \vec{P}_1 + s * \vec{V}_1$$

$$L_2(s) = \vec{P}_2 + s * \vec{V}_2$$

➔ Kürzeste Verbindung ist eine Linie die senkrecht zu beiden anderen Linien steht ($\vec{A} \cdot \vec{B} = 0 \rightarrow senkrecht$)

$$\vec{V}_I \cdot \vec{V}_1 = \vec{V}_I \cdot \vec{V}_2 = 0$$

$$\vec{V}_I = \vec{P}_{I2} - \vec{P}_{I1}$$

$$L_I(s) = \vec{P}_{I1} + s * \vec{V}_I$$

➔ Verbindungs-Linie muss beide Linien schneiden, sonst keine Verbindung (Punkte müssen auf L_1 bzw. L_2 liegen)

$$\vec{P}_{I1} = L_1(a)$$

$$\vec{P}_{I2} = L_2(b)$$

➔ Gesucht sind die Punkt „Indexe“ a und b, beschrieben durch die Gleichung

$$\left(\left(\vec{P}_2 + b * \vec{V}_2\right) - \left(\vec{P}_1 + a * \vec{V}_1\right)\right) \cdot \vec{V}_1 = \left(\left(\vec{P}_2 + b * \vec{V}_2\right) - \left(\vec{P}_1 + a * \vec{V}_1\right)\right) \cdot \vec{V}_2 = 0$$

Lösung

1. Gleichungssystem aufstellen

$$\begin{cases} \left(\left(\vec{P}_2 + b * \vec{V}_2\right) - \left(\vec{P}_1 + a * \vec{V}_1\right)\right) \cdot \vec{V}_1 = 0 \\ \left(\left(\vec{P}_2 + b * \vec{V}_2\right) - \left(\vec{P}_1 + a * \vec{V}_1\right)\right) \cdot \vec{V}_2 = 0 \end{cases}$$

2. Vektoren als Spalten-Vektoren schreiben

$$\begin{cases} \left(\begin{pmatrix} x_{p2} \\ y_{p2} \\ z_{p2} \end{pmatrix} + b * \begin{pmatrix} x_{v2} \\ y_{v2} \\ z_{v2} \end{pmatrix} - \begin{pmatrix} x_{p1} \\ y_{p1} \\ z_{p1} \end{pmatrix} - a * \begin{pmatrix} x_{v1} \\ y_{v1} \\ z_{v1} \end{pmatrix}\right) \cdot \begin{pmatrix} x_{v1} \\ y_{v1} \\ z_{v1} \end{pmatrix} = 0 \\ \left(\begin{pmatrix} x_{p2} \\ y_{p2} \\ z_{p2} \end{pmatrix} + b * \begin{pmatrix} x_{v2} \\ y_{v2} \\ z_{v2} \end{pmatrix} - \begin{pmatrix} x_{p1} \\ y_{p1} \\ z_{p1} \end{pmatrix} - a * \begin{pmatrix} x_{v1} \\ y_{v1} \\ z_{v1} \end{pmatrix}\right) \cdot \begin{pmatrix} x_{v2} \\ y_{v2} \\ z_{v2} \end{pmatrix} = 0 \end{cases}$$

3. In lineare Form bringen/Vektoren auflösen

$$\begin{cases} \left(x_{p2} + b * x_{v2} - x_{p1} - a * x_{v1}\right) * x_{v1} + \left(y_{p2} + b * y_{v2} - y_{p1} - a * y_{v1}\right) * y_{v1} + \left(z_{p2} + b * z_{v2} - z_{p1} - a * z_{v1}\right) * z_{v1} = 0 \\ \left(x_{p2} + b * x_{v2} - x_{p1} - a * x_{v1}\right) * x_{v2} + \left(y_{p2} + b * y_{v2} - y_{p1} - a * y_{v1}\right) * y_{v2} + \left(z_{p2} + b * z_{v2} - z_{p1} - a * z_{v1}\right) * z_{v2} = 0 \end{cases}$$

4. Klammern lösen

$$\begin{cases} x_{p2} * x_{v1} + b * x_{v2} * x_{v1} - x_{p1} * x_{v1} - a * x_{v1} * x_{v1} + y_{p2} * y_{v1} + b * y_{v2} * y_{v1} - y_{p1} * y_{v1} - a * y_{v1} * y_{v1} + z_{p2} * z_{v1} + b * z_{v2} * z_{v1} - z_{p1} * z_{v1} - a * z_{v1} * z_{v1} = 0 \\ x_{p2} * x_{v2} + b * x_{v2} * x_{v2} - x_{p1} * x_{v2} - a * x_{v1} * x_{v2} + y_{p2} * y_{v2} + b * y_{v2} * y_{v2} - y_{p1} * y_{v2} - a * y_{v1} * y_{v2} + z_{p2} * z_{v2} + b * z_{v2} * z_{v2} - z_{p1} * z_{v2} - a * z_{v1} * z_{v2} = 0 \end{cases}$$

5. Terme mit Unbekannten sortieren

$$\begin{cases} b*x_{v2}*x_{v1}+b*y_{v2}*y_{v1}+b*z_{v2}*z_{v1}-a*x_{v1}*x_{v1}-a*y_{v1}*y_{v1}-a*z_{v1}*z_{v1}+x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}=0 \\ b*x_{v2}*x_{v2}+b*y_{v2}*y_{v2}+b*z_{v2}*z_{v2}-a*x_{v1}*x_{v2}-a*y_{v1}*y_{v2}-a*z_{v1}*z_{v2}+x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}=0 \end{cases}$$

6. Terme mit unbekannten auf rechte Seite bringen

$$\begin{cases} x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}=-b*x_{v2}*x_{v1}-b*y_{v2}*y_{v1}-b*z_{v2}*z_{v1}+a*x_{v1}*x_{v1}+a*y_{v1}*y_{v1}+a*z_{v1}*z_{v1} \\ x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}=-b*x_{v2}*x_{v2}-b*y_{v2}*y_{v2}-b*z_{v2}*z_{v2}+a*x_{v1}*x_{v2}+a*y_{v1}*y_{v2}+a*z_{v1}*z_{v2} \end{cases}$$

7. Gemeinsame Faktoren auf rechter Seite mit Klammern beseitigen

$$\begin{cases} x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}=a*(x_{v1}*x_{v1}+y_{v1}*y_{v1}+z_{v1}*z_{v1})-b*(x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1}) \\ x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}=a*(x_{v1}*x_{v2}+y_{v1}*y_{v2}+z_{v1}*z_{v2})-b*(x_{v2}*x_{v2}+y_{v2}*y_{v2}+z_{v2}*z_{v2}) \end{cases}$$

8. Je einen Term mit einer der Unbekannten wider auf linke Seite bringen

$$\begin{cases} x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}+b*(x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1})=a*(x_{v1}*x_{v1}+y_{v1}*y_{v1}+z_{v1}*z_{v1}) \\ x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}-a*(x_{v1}*x_{v2}+y_{v1}*y_{v2}+z_{v1}*z_{v2})=-b*(x_{v2}*x_{v2}+y_{v2}*y_{v2}+z_{v2}*z_{v2}) \end{cases}$$

9. Durch Faktor (klammer) von Unbekannten auf rechter Seite teilen

$$\begin{cases} \dfrac{x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}+b*(x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1})}{x_{v1}*x_{v1}+y_{v1}*y_{v1}+z_{v1}*z_{v1}}=a \\ -\dfrac{x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}-a*(x_{v1}*x_{v2}+y_{v1}*y_{v2}+z_{v1}*z_{v2})}{x_{v2}*x_{v2}+y_{v2}*y_{v2}+z_{v2}*z_{v2}}=b \end{cases}$$

10. Gleichungen ineinander einsetzen

$$\begin{cases} \dfrac{x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}-\frac{x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}-a*(x_{v1}*x_{v2}+y_{v1}*y_{v2}+z_{v1}*z_{v2})}{x_{v2}*x_{v2}+y_{v2}*y_{v2}+z_{v2}*z_{v2}}*(x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1})}{x_{v1}*x_{v1}+y_{v1}*y_{v1}+z_{v1}*z_{v1}}=a \\ -\dfrac{x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}-\frac{x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}+b*(x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1})}{x_{v1}*x_{v1}+y_{v1}*y_{v1}+z_{v1}*z_{v1}}*(x_{v1}*x_{v2}+y_{v1}*y_{v2}+z_{v1}*z_{v2})}{x_{v2}*x_{v2}+y_{v2}*y_{v2}+z_{v2}*z_{v2}}=b \end{cases}$$

11. Zur Vereinfachung Hilfsvariablen für sich widerholende Terme definieren

$$S_1 = x_{p2}*x_{v1}-x_{p1}*x_{v1}+y_{p2}*y_{v1}-y_{p1}*y_{v1}+z_{p2}*z_{v1}-z_{p1}*z_{v1}$$

$$S_2 = x_{p2}*x_{v2}-x_{p1}*x_{v2}+y_{p2}*y_{v2}-y_{p1}*y_{v2}+z_{p2}*z_{v2}-z_{p1}*z_{v2}$$

$$S_3 = x_{v2}^2+y_{v2}^2+z_{v2}^2$$

$$S_4 = x_{v1}^2+y_{v1}^2+z_{v1}^2$$

$$S_5 = x_{v2}*x_{v1}+y_{v2}*y_{v1}+z_{v2}*z_{v1}$$

$$\begin{cases} \dfrac{S_1-\frac{S_2-a*S_5}{S_3}*S_5}{S_4}=a \\ -\dfrac{S_2-\frac{S_1+b*S_5}{S_4}*S_5}{S_3}=b \end{cases}$$

12. Bruch mit Faktor zusammen fassen

$$\begin{cases} \dfrac{S_1-\frac{S_2*S_5-a*S_5^2}{S_3}}{S_4}=a \\ -\dfrac{S_2-\frac{S_1*S_5+b*S_5^2}{S_4}}{S_3}=b \end{cases}$$

13. Unbekannte auf linker Seite aus Bruch raus holen (x2)

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3} + \dfrac{S_5^2}{S_3} * a}{S_4} = a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4} - \dfrac{S_5^2}{S_4} * b}{S_3} = b \end{cases}$$

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} + \dfrac{\dfrac{S_5^2}{S_3}}{S_4} * a = a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} + \dfrac{\dfrac{S_5^2}{S_4}}{S_3} * b = b \end{cases}$$

14. Vereinfachen

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} + \dfrac{S_5^2}{S_3 * S_4} * a = a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} + \dfrac{S_5^2}{S_4 * S_3} * b = b \end{cases}$$

15. Term mit Unbekannter auf rechte Seite bringen

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} = a - \dfrac{S_5^2}{S_3 * S_4} * a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} = b - \dfrac{S_5^2}{S_4 * S_3} * b \end{cases}$$

16. Umschreiben und gemeinsamen Faktor mit Klammer beseitigen

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} = 1 * a - \dfrac{S_5^2}{S_3 * S_4} * a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} = 1 * b - \dfrac{S_5^2}{S_4 * S_3} * b \end{cases}$$

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} = \left(1 - \dfrac{S_5^2}{S_3 * S_4}\right) * a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} = \left(1 - \dfrac{S_5^2}{S_4 * S_3}\right) * b \end{cases}$$

17. Vereinfachen

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4} = \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4} * a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3} = \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4} * b \end{cases}$$

18. Bruch auf linke Seite bringen

$$\begin{cases} \dfrac{S_1 - \dfrac{S_2 * S_5}{S_3}}{S_4 * \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4}} = a \\ -\dfrac{S_2 - \dfrac{S_1 * S_5}{S_4}}{S_3 * \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4}} = b \end{cases}$$

19. Vereinfachen (oft …)

$$\begin{cases} \dfrac{\dfrac{S_1 * S_3 - S_2 * S_5}{S_3}}{S_4 * \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4}} = a \\ -\dfrac{\dfrac{S_2 * S_4 - S_1 * S_5}{S_4}}{S_3 * \dfrac{S_3 * S_4 - S_5^2}{S_3 * S_4}} = b \end{cases}$$

$$\begin{cases} \dfrac{\dfrac{S_1 * S_3 - S_2 * S_5}{S_3}}{\dfrac{S_3 * S_4 - S_5^2}{S_3}} = a \\ -\dfrac{\dfrac{S_2 * S_4 - S_1 * S_5}{S_4}}{\dfrac{S_3 * S_4 - S_5^2}{S_4}} = b \end{cases}$$

$$\begin{cases} \dfrac{S_1 * S_3 - S_2 * S_5}{S_3 * S_4 - S_5^2} = a \\ -\dfrac{S_2 * S_4 - S_1 * S_5}{S_3 * S_4 - S_5^2} = b \end{cases}$$

20. Hilfs-Gleichungen vereinfachen

$$\begin{aligned}S_1 &= x_{p2} * x_{v1} - x_{p1} * x_{v1} + y_{p2} * y_{v1} - y_{p1} * y_{v1} + z_{p2} * z_{v1} - z_{p1} * z_{v1}\\ &= x_{p2} * x_{v1} + y_{p2} * y_{v1} + z_{p2} * z_{v1} - x_{p1} * x_{v1} - y_{p1} * y_{v1} - z_{p1} * z_{v1}\\ &= \left(x_{p2} * x_{v1} + y_{p2} * y_{v1} + z_{p2} * z_{v1}\right) - \left(x_{p1} * x_{v1} + y_{p1} * y_{v1} + z_{p1} * z_{v1}\right)\\ &= \begin{pmatrix}x_{p2}\\ y_{p2}\\ z_{p2}\end{pmatrix} \cdot \begin{pmatrix}x_{v1}\\ y_{v1}\\ z_{v1}\end{pmatrix} - \begin{pmatrix}x_{p1}\\ y_{p1}\\ z_{p1}\end{pmatrix} \cdot \begin{pmatrix}x_{v1}\\ y_{v1}\\ Z_{v1}\end{pmatrix} = \vec{P}_2 \cdot \vec{V}_1 - \vec{P}_1 \cdot \vec{V}_1\end{aligned}$$

$$\begin{aligned}S_2 &= x_{p2} * x_{v2} - x_{p1} * x_{v2} + y_{p2} * y_{v2} - y_{p1} * y_{v2} + z_{p2} * z_{v2} - z_{p1} * z_{v2}\\ &= x_{p2} * x_{v2} + y_{p2} * y_{v2} + z_{p2} * z_{v2} - x_{p1} * x_{v2} - y_{p1} * y_{v2} - z_{p1} * z_{v2}\\ &= \left(x_{p2} * x_{v2} + y_{p2} * y_{v2} + z_{p2} * z_{v2}\right) - \left(x_{p1} * x_{v2} + y_{p1} * y_{v2} + z_{p1} * z_{v2}\right)\\ &= \begin{pmatrix}x_{p2}\\ y_{p2}\\ z_{p2}\end{pmatrix} \cdot \begin{pmatrix}x_{v2}\\ y_{v2}\\ z_{v2}\end{pmatrix} - \begin{pmatrix}x_{p1}\\ y_{p1}\\ z_{p1}\end{pmatrix} \cdot \begin{pmatrix}x_{v2}\\ y_{v2}\\ Z_{v2}\end{pmatrix} = \vec{P}_2 \cdot \vec{V}_2 - \vec{P}_1 \cdot \vec{V}_2\end{aligned}$$

$$S_3 = x_{v2}^2 + y_{v2}^2 + z_{v2}^2 = length\left(\vec{V}_2\right)^2 = \left|\vec{V}_2\right|^2$$

$$S_4 = x_{v1}^2 + y_{v1}^2 + z_{v1}^2 = length\left(\vec{V}_1\right)^2 = \left|\vec{V}_1\right|^2$$

$$S_5 = x_{v2} * x_{v1} + y_{v2} * y_{v1} + z_{v2} * z_{v1} = \begin{pmatrix}x_{v2}\\ y_{v2}\\ z_{v2}\end{pmatrix} \cdot \begin{pmatrix}x_{v1}\\ y_{v1}\\ z_{v1}\end{pmatrix} = \vec{V}_2 \cdot \vec{V}_1$$

21. Hilfsgleichungen einsetzen

$$\begin{cases}\dfrac{\left(\vec{P}_2 \cdot \vec{V}_1 - \vec{P}_1 \cdot \vec{V}_1\right) * \left|\vec{V}_2\right|^2 - \left(\vec{P}_2 \cdot \vec{V}_2 - \vec{P}_1 \cdot \vec{V}_2\right) * \left(\vec{V}_2 \cdot \vec{V}_1\right)}{\left|\vec{V}_2\right|^2 * \left|\vec{V}_1\right|^2 - \left(\vec{V}_2 \cdot \vec{V}_1\right)^2} = a\\ -\dfrac{\left(\vec{P}_2 \cdot \vec{V}_2 - \vec{P}_1 \cdot \vec{V}_2\right) * \left|\vec{V}_1\right|^2 - \left(\vec{P}_2 \cdot \vec{V}_1 - \vec{P}_1 \cdot \vec{V}_1\right) * \left(\vec{V}_2 \cdot \vec{V}_1\right)}{\left|\vec{V}_2\right|^2 * \left|\vec{V}_1\right|^2 - \left(\vec{V}_2 \cdot \vec{V}_1\right)^2} = b\end{cases}$$