Solving Models with Numerical Methods

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Outline

- **Part I:** Solving Deterministic Models
    1. Solow Model
    2. Life-Cycle Model
- **Part II:** Solving Stochastic Models
    1. Discretizing Shocks and Expectations
    2. Optimal Portfolios

What are Numerical Methods?

- *Numerical Methods* – using programming to solve economic problems
- Potential applications:
  1. Write 'Do Files' in Stata
  2. Bootstrapping & Monte Carlo simulations
  3. Solve models with *no analytical solution*
  4. Check analytical insights

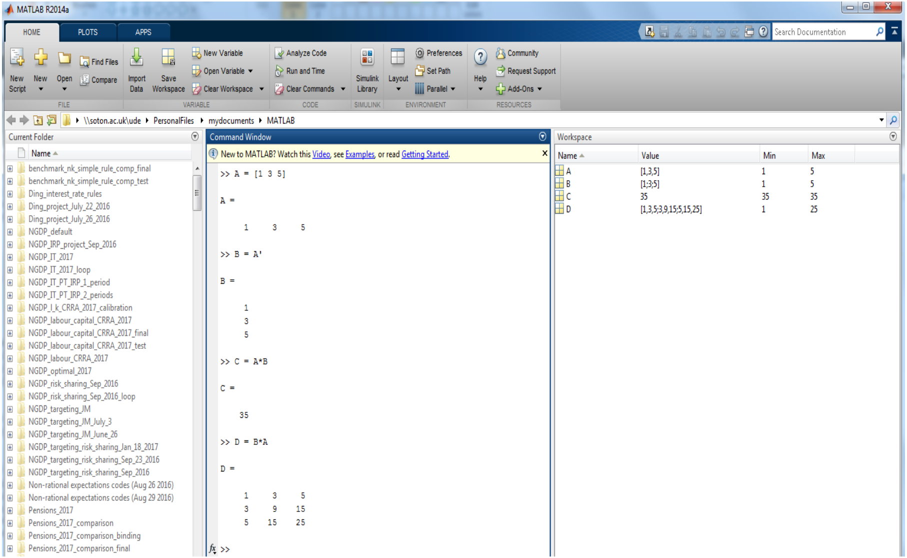
Programming Languages

- Matlab/Python/Julia useful for solving economic models
- R/Julia good for econometrics

Programming Languages cont'd

- I will use Matlab in this talk. Easy to learn additional languages.
- Open-Source alternatives:
  1. R for time-series analysis
  2. Octave is a Matlab clone
  3. Python – an easy all-purpose language

Welcome to Matlab!



Part I: Solving Deterministic Models

- By *deterministic* we mean models without risk or uncertainty

- **Example:** Solow Model

  1. $y_t = k_t^{\alpha}$

  2. $k_{t+1} - k_t = \frac{1}{1+n}\left[sk_t^{\alpha} - (n+\delta)k_t\right]$

  where lowercase means *per-person*

- **Note:** $k_0 = \overline{k}$ given

Solow Model

- Want to solve for $t = 0, 1, 2, \ldots T$ the system:

$$y_t = k_t^\alpha \tag{1}$$

$$k_{t+1} - k_t = \frac{1}{1+n}\left[sk_t^\alpha - (n+\delta)k_t\right] \tag{2}$$

- Start in period 0:

$$k_0 = \overline{k} \quad \Rightarrow \quad y_0 = k_0^\alpha$$

$$k_1 = \frac{1}{1+n}\left[sk_0^\alpha - (n+\delta)k_0\right] + k_0 \quad \Rightarrow \quad y_1 = k_1^\alpha$$

$$k_2 = \frac{1}{1+n}\left[sk_1^\alpha - (n+\delta)k_1\right] + k_1 \quad \Rightarrow \quad y_2 = k_2^\alpha$$

$$\ldots$$

Computer code

- Translating the equations into computer code

```
%Initial capital and output
k(1) = 1;
y(1) = k(1)^alpha;

%Number of simulated periods
T = 150;

%Simulate economy using a loop
for t=2:T

    k(t) = (1/(1+n))*(s*k(t-1)^alpha - (n+delta)*k(t-1) ) + k(t-1);
    y(t) = k(t)^alpha;
    Growth(t) = 100*(y(t) - y(t-1))/y(t-1);

end
```

Numerical solution

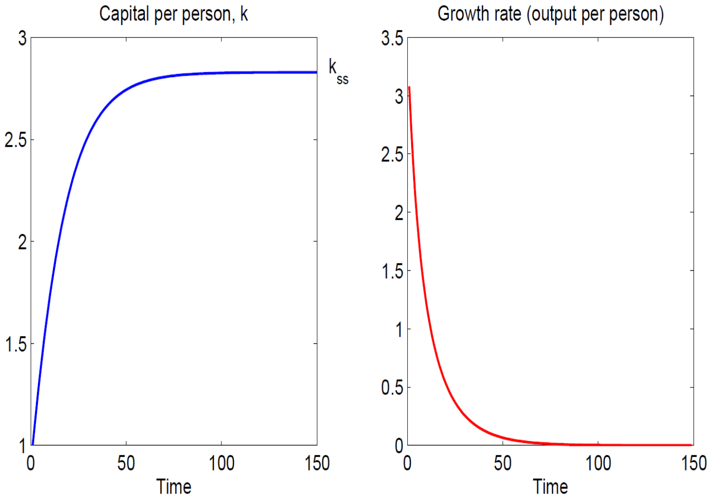


Figure: $\alpha = 1/3$, $\delta = 0.05$, $n = 0.05$, $s = 0.20$

Life Cycle Model

- Households live T periods and solve:

$$\max_{A_1, A_2, \dots, A_{T-1}} U = \log(C_1) + \beta \log(C_2) + \dots + \beta^{T-1} \log(C_T)$$

s.t.

$$C_t = Y_t + (1+r)A_{t-1} - A_t \qquad \text{for} \qquad t = 1, 2, \dots T$$

$$Y_t, r \text{ given} \qquad \text{and} \qquad A_0 = A_T = 0$$

- The first-order conditions are:

$$\frac{1}{C_t} = \beta(1+r)\frac{1}{C_{t+1}} \qquad \text{for} \qquad t = 1, 2, \dots T-1$$

Solution steps

- The FOCs imply

$$C_{t+1} = \beta(1+r)C_t \qquad \text{for} \qquad t = 1, 2, \ldots T-1$$

- Budget constraints tell us that

$$A_t = Y_t - C_t + (1+r)A_{t-1} \qquad \text{for} \qquad t = 1, 2, \ldots T-1$$

- In the final period:

$$A_T = 0 \qquad \Rightarrow \qquad S_T \equiv A_T - A_{T-1} = -rA_{T-1}$$

Life Cycle Model cont'd

- We want to solve for the paths of C and A
- **Algorithm**
  1. Guess $A_1 = A^*$, compute $C_1 = Y_1 - A_1$
  2. Compute $C_2, C_3, ...C_T$ using FOCs
  3. Find the A^* such that $A_T = 0$

Numerical solution

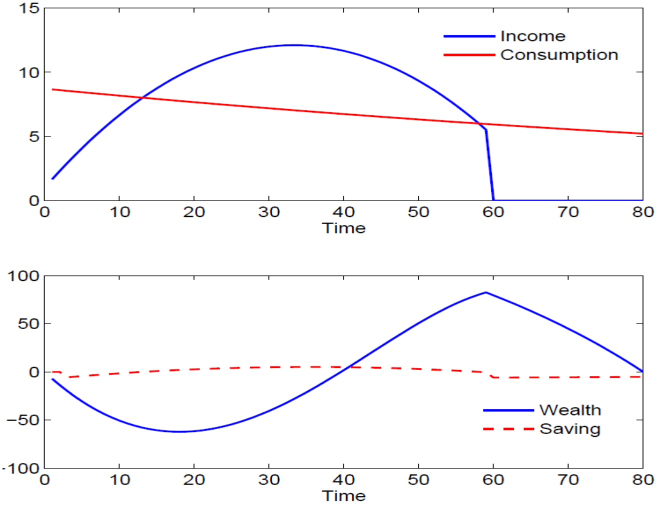


Figure: $T = 80$, $\beta = 0.96$, $r = 0.035$; Retire at $t = 60$

Part II: Solving Stochastic Models

- By *stochastic* we mean models with risk
- Economy subject to random shocks which cannot be forecast
- In such models, agents take *expectations* over future outcomes
- **How to deal with this?**

Discretizing shocks and expectations

- The easiest way is to *discretize*
- That is, assume shocks take on finite number of values
- **Example:**

$$\epsilon = \begin{cases} \epsilon_1 & \text{with probability } p \\ \epsilon_2 & \text{with probability } 1 - p \end{cases}$$

- We can then build expectations *by hand*

A toy model

- Suppose utility is:
$$U = log(C)$$
where $C(\epsilon) = \epsilon$

- Expected utility is given by
$$E[U] = E[log(C)] = p\log\left[C(\epsilon_1)\right] + (1-p)\log\left[C(\epsilon_2)\right]$$
where $C(\epsilon_1) = \epsilon_1$ and $C(\epsilon_2) = \epsilon_2$

- Since we know ϵ_1 and ϵ_2 (given), we are done!

Two-period life-cycle model

- Households live 2 periods and solve:

$$\max_{B} U = \log(C_1) + \beta E\left[\log(C_2)\right]$$

s.t.

$$C_1 = \overline{Y}_1 - B, \qquad C_2 = Y_2 + (1+r)B$$

$$\overline{Y}_1, r \text{ given}$$

$$Y_2 = \begin{cases} \epsilon_1 & \text{with probability } p \\ \epsilon_2 & \text{with probability } 1-p \end{cases}$$

- Note that:

$$E\left[\log(C_2)\right] = p\log(Y_2(\epsilon_1) + (1+r)B) + (1-p)\log(Y_2(\epsilon_2) + (1+r)B)$$

Two-period life-cycle model cont'd

- The first-order condition is:

$$\frac{1}{C_1} = \beta(1+r)\left(\frac{p}{C_2(\epsilon_1)} + \frac{(1-p)}{C_2(\epsilon_2)}\right) \tag{3}$$

- **Algorithm 1**
  1. Guess $B = \overline{B}$
  2. Compute $C_1 = \overline{Y}_1 - B$ and $E\left[\log(C_2)\right]$
  3. Find the $\overline{B}$ that maximises U
  4. Check FOC (3) is satisfied

Computer code: Algorithm 1

- Steps 1-3 can be coded as follows:

```
beta = 0.96; r = 0.05; Y1 = 1; p = 0.5;
eps1 = 0.5; eps2 = 1.05; Nguess = 5000;

for i=1:Nguess

    Bguess(i) = (i-1)/Nguess;
    B = Bguess(i);
    C1 = Y1 - B;
    C2_1 = eps1 + (1+r)*B;
    C2_2 = eps2 +(1+r)*B;

    U(i) = log(C1) + beta*( p*log(C2_1) + (1-p)*log(C2_2) );

end

%Find maximum of utility w.r.t. B guesses
[Umax, IndexU] = max(U);      %IndexU is the location of the optimal value
```

Numerical solution: Algorithm 1

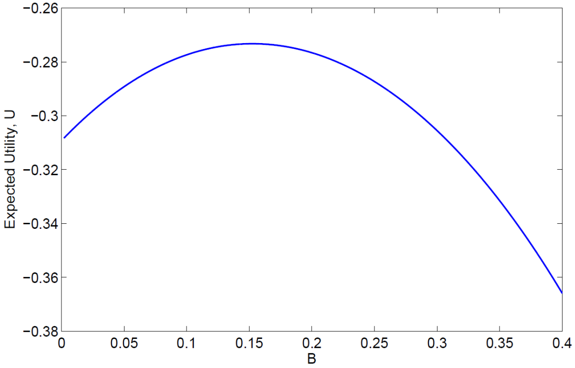


Figure: $\beta = 0.96,\ r = 0.05,\ Y_1 = 1,\ p = 0.5,\ \epsilon_1(\epsilon_2) = 0.5(1.05)$

- We find $B^* = 0.153$ and $Resid^{FOC} = 6.8 \times 10^{-8} \approx 0$

An alternative algorithm

- **Algorithm 2**

  1. Guess $B = \overline{B}$
  2. Iterate on B until FOC (3) is satisfied

- FOC implies:

$$\frac{1}{C_1} - \beta(1+r)\left(\frac{p}{C_2(\epsilon_1)} + \frac{(1-p)}{C_2(\epsilon_2)}\right) = 0$$

- Define

$$Resid^{FOC} = \left|\frac{1}{\overline{Y}_1 - B} - \beta(1+r)\left(\frac{p}{\epsilon_1 + (1+r)B} + \frac{(1-p)}{\epsilon_2 + (1+r)B}\right)\right|$$

and loop over B until $Resid^{FOC} \approx 0$

Optimal portfolio choice I

- Households choose bonds B and risky stocks S:

$$\max_{S,B} U = \log(C_1) + \beta E\left[\log(C_2)\right]$$

s.t.

$$C_1 = \overline{Y}_1 - B - S, \qquad C_2 = (1+r)B + (1+r+d)S$$

$$\overline{Y}_1, r \ \text{ given}$$

$$d = \begin{cases} d_1 = 0.01 & \text{with probability } p \\ d_2 = -0.70 & \text{with probability } 1-p \end{cases}$$

where $p > 1 - p$

Optimal portfolio choice II

- The first-order conditions are:

$$\frac{1}{C_1} = \beta(1+r)\left(\frac{p}{C_2(d_1)} + \frac{(1-p)}{C_2(d_2)}\right) \tag{4}$$

for bonds, B

$$\frac{1}{C_1} = \beta\left(\frac{p(1+r+d_1)}{C_2(d_1)} + \frac{(1-p)(1+r+d_2)}{C_2(d_2)}\right) \tag{5}$$

for stocks, S

Optimal portfolio choice III

- A similar procedure to the above can be used
- **Algorithm**
  1. Guess $S = \overline{S}$ and $B = \overline{B}$
  2. Compute $C_1 = \overline{Y}_1 - S - B$ and $E\left[\log(C_2)\right]$
  3. Find the $\overline{S}, \overline{B}$ combo that maximises U
  4. Check FOCs (4) & (5) are satisfied

Numerical solution

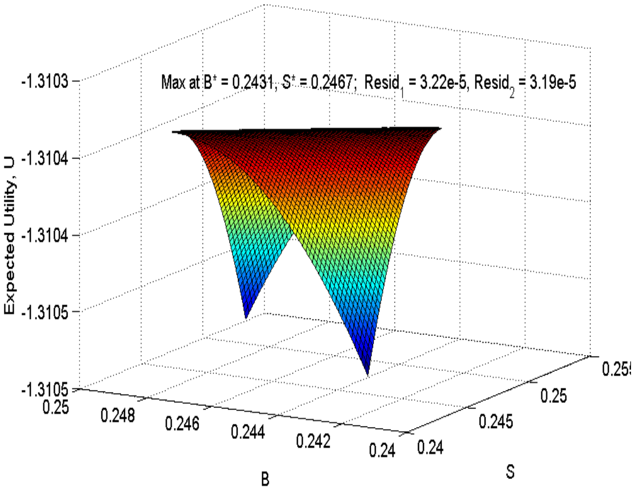


Figure: $\beta = 0.96,\ r = 0.05,\ Y_1 = 1,\ p = 0.99,\ d_1(d_2) = 0.0107(-0.700)$

ANY QUESTIONS?