Supplementary Appendix for
Indexed versus nominal government debt under inflation and price-level targeting

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(This Appendix is for publication online)

Section 1 – Derivations and proofs

1.A – Derivation of first-order conditions (baseline model)

Consumers solve a maximisation problem of the form

$$\max_{\{c_{t,Y},\, z_t,\, c_{t+1,O}\}} U_t = \frac{1}{1-\gamma}\left[c_{t,Y}^{\varepsilon} + \beta \left[E_t c_{t+1,O}^{1-\gamma} \right]^{\frac{\varepsilon}{1-\gamma}} \right]^{\frac{1-\gamma}{\varepsilon}} \tag{A1}$$

subject to

$$c_{t,Y} = (1-\tau) w_t - k_{t+1} - b_{t+1}^{i} - b_{t+1}^{n} - m_t \qquad \text{(Budget constraint of young)}$$

$$c_{t+1,O} = (1-\tau^k) r_{t+1}^{k} k_{t+1} + r_{t+1}^{i} b_{t+1}^{i} + r_{t+1}^{n} b_{t+1}^{n} + r_{t+1}^{m} m_t \qquad \text{(Budget constraint of old)}$$

$$m_t = \delta \qquad \text{(Cash constraint)}$$

where $z_t \equiv (k_{t+1}, b_{t+1}^{i}, b_{t+1}^{n}, m_t)$ is the vector of assets chosen by households.

The Lagrangian for this problem is as follows:

$$L_t = E_t \begin{Bmatrix} U_t + \lambda_{t,Y} [\, (1-\tau) w_t - k_{t+1} - b_{t+1}^{i} - b_{t+1}^{n} - m_t - c_{t,Y}] \\ + \lambda_{t+1,O} [(1-\tau^k) r_{t+1}^{k} k_{t+1} + r_{t+1}^{i} b_{t+1}^{i} + r_{t+1}^{n} b_{t+1}^{n} + r_{t+1}^{m} m_t - c_{t+1,O}] + \mu_t (m_t - \delta) \end{Bmatrix} \tag{A2}$$

First-order conditions are as follows:

$$c_{t,Y}:\ \frac{\partial U_t}{\partial c_{t,Y}} = \lambda_{t,Y}, \qquad c_{t+1,O}:\ \frac{\partial U_t}{\partial c_{t+1,O}} = \lambda_{t+1,O}, \qquad k_{t+1}:\ \lambda_{t,Y} = E_t [\lambda_{t+1,O} (1-\tau^k) r_{t+1}^{k}]$$

$$b_{t+1}^{i}:\ \lambda_{t,Y} = E_t [\lambda_{t+1,O} r_{t+1}^{i}], \qquad b_{t+1}^{n}:\ \lambda_{t,Y} = E_t [\lambda_{t+1,O} r_{t+1}^{n}], \qquad m_t:\ \lambda_{t,Y} = E_t [\lambda_{t+1,O} r_{t+1}^{m}] + \mu_t$$

By substitution, this system can be reduced to four Euler equations:

$$\frac{\partial U_t}{\partial c_{t,Y}} = E_t \left[\frac{\partial U_t}{\partial c_{t+1,O}} (1-\tau^k) r_{t+1}^{k} \right], \qquad \frac{\partial U_t}{\partial c_{t,Y}} = E_t \left[\frac{\partial U_t}{\partial c_{t+1,O}} r_{t+1}^{n} \right], \qquad \frac{\partial U_t}{c_{t,Y}} = E_t \left[\frac{\partial U_t}{\partial c_{t+1,O}} r_{t+1}^{i} \right],$$

$$\frac{\partial U_t}{\partial c_{t,Y}} = E_t \left[\frac{\partial U_t}{\partial c_{t+1,O}} r_{t+1}^{m} \right] + \mu_t$$

The partial derivatives of the utility function are as follows:

$$\frac{\partial U_t}{\partial c_{t,Y}} = \left[c_{t,Y}^{\varepsilon} + \beta \left[E_t c_{t+1,O}^{1-\gamma} \right]^{\frac{\varepsilon}{1-\gamma}} \right]^{\frac{1-\gamma-\varepsilon}{\varepsilon}} c_{t,Y}^{-(1-\varepsilon)} \tag{A3}$$

$$\frac{\partial U_t}{\partial c_{t+1,O}} = \left[c_{t,Y}^{\varepsilon} + \beta \left[E_t c_{t+1,O}^{1-\gamma} \right]^{\frac{\varepsilon}{1-\gamma}} \right]^{\frac{1-\gamma-\varepsilon}{\varepsilon}} \beta \left[E_t (c_{t+1,O}^{1-\gamma}) \right]^{\frac{\varepsilon-(1-\gamma)}{1-\gamma}} c_{t+1,O}^{-\gamma} \qquad \text{(A4)}$$

Dividing (A4) by (A3) gives

$$\frac{\partial U_t / \partial c_{t+1,O}}{\partial U_t / \partial c_{t,Y}} = \frac{\beta \left[E_t (c_{t+1,O}^{1-\gamma}) \right]^{\frac{\varepsilon-(1-\gamma)}{1-\gamma}} c_{t+1,O}^{-\gamma}}{c_{t,Y}^{-(1-\varepsilon)}} = \beta \left(\frac{c_{t,Y}}{c_{t+1,O}} \right)^{1-\varepsilon} \left(\frac{c_{t+1,O}}{(E_t c_{t+1,O}^{1-\gamma})^{1/(1-\gamma)}} \right)^{1-\gamma-\varepsilon} \qquad \text{(A5)}$$

Defining $sdf_{t+1} \equiv \dfrac{\partial U_t / \partial c_{t+1,O}}{\partial U_t / \partial c_{t,Y}}$, the four Euler equations above can be written as in the main text:

$$1 = E_t[sdf_{t+1}(1-\tau^k) r_{t+1}^k] \qquad \text{(A6)}$$

$$1 = E_t[sdf_{t+1} r_{t+1}^n] \qquad \text{(A7)}$$

$$1 = E_t[sdf_{t+1} r_{t+1}^i] \qquad \text{(A8)}$$

$$1 = E_t[sdf_{t+1} r_{t+1}^m] + \tilde{\mu}_t \qquad \text{(A9)}$$

where $\tilde{\mu}_t \equiv \mu_t / \lambda_{t,Y}$.

1.B – The binding legal constraint on money holdings

Proposition: The constraint binds with strict equality when $R_t > 1$

Proof.

By equations (A7) and (A9), the Lagrange multiplier on the cash constraint is given by

$$\mu_t = \lambda_{t,Y} E_t[sdf_{t+1}(r_{t+1}^n - r_{t+1}^m)] \qquad \text{(B1)}$$

Since the real return on nominal bonds is $r_{t+1}^n = R_t /(1+\pi_{t+1}) = R_t r_{t+1}^m$, we can say that

$$\mu_t = \lambda_{t,Y} E_t[sdf_{t+1}(R_t - 1) r_{t+1}^m)] = \lambda_{t,Y} (R_t - 1) E_t[sdf_{t+1} r_{t+1}^m] \qquad \text{(B2)}$$

since R_t is known at the end of period t.

The Kuhn-Tucker conditions associated with μ_t are as follows:

$$\{ \mu_t \geq 0 \quad \text{and} \quad \mu_t (m_t - \delta) = 0 \} \qquad \text{(B3)}$$

The second condition in (B3) is the complementary slackness condition. It implies that the cash constraint will be strictly binding iff $\mu_t > 0$ for all t.

Dividing (B2) by $1 = E_t[sdf_{t+1} r_{t+1}^n] = R_t E_t[sdf_{t+1} r_{t+1}^m]$, it follows that $\mu_t = \lambda_{t,Y}(R_t - 1)/R_t$.

Since $\lambda_{t,Y} > 0$ (as the budget constraint of the young will always hold with equality), it follows that $\mu_t > 0$ iff $R_t > 1$ for all t. **Q.E.D.**

1.C – Approximate analytical expressions for long run inflation risk under IT and PT

This appendix derives approximate expressions for the inflation variance under IT and PT.

Inflation Targeting (IT)

Under IT, inflation is period t is given by

$$1+\pi_t = (1+\pi^*)^{20}\prod_{j=1}^{20}(1+\varepsilon_{j,t}) \tag{C1}$$

where $\varepsilon_{j,t}$ are IID-normal innovations with mean zero and variance σ^2.

A general non-linear function $g(\varepsilon)$, where ε is a vector of variables, can be approximated by $\mathrm{var}(g(\varepsilon)) \approx \sum[g_j'(\mu)]^2\mathrm{var}(\varepsilon_j)$ using the 'Delta method'. Here, μ is the unconditional mean of the vector ε, and g_j' is the first derivative of $g(\varepsilon)$ with respect to variable ε_j.

The inflation variance under IT can therefore be approximated as follows:

$$\mathrm{var}(\pi_t) \approx \sum_{j=1}^{20}(1+\pi^*)^{40}\sigma^2 = (1+\pi^*)^{40}20\sigma^2 \tag{C2}$$

Price-level targeting (PT)

Under PT, inflation in period t is given by

$$1+\pi_t = (1+\pi^*)^{20}\frac{(1+\varepsilon_{20,t})}{(1+\varepsilon_{20,t-1})} \tag{C3}$$

where $\varepsilon_{20,t}$ and $\varepsilon_{20,t-1}$ are IID-normal innovations with mean zero and variance σ^2.

Using the same approximation method as above, the inflation variance under PT is given by

$$\mathrm{var}(\pi_t) \approx [(1+\pi^*)^{20}]^2\sigma^2 + [-(1+\pi^*)^{20}]^2\sigma^2 = (1+\pi^*)^{40}2\sigma^2 \tag{C4}$$

Hence the unconditional variance of inflation under IT is (approx.) 10 times that under PT.

1.D – First-order conditions under imperfect credibility

In this case, consumers solve the following problem where $s \in \{IT, PT\}$:

$$\max_{\{c_{t,Y},\, z_t,\, c_{t+1,O}\}} U_t^{IC} = \frac{1}{1-\gamma}\left[c_{t,Y}^{\varepsilon} + \beta\Big[p_{IT}E[c_{t+1,O(IT)}^{1-\gamma}\,|\,\Omega_t] + (1-p_{IT})E[c_{t+1,O(PT)}^{1-\gamma}\,|\,\Omega_t]\Big]^{\frac{\varepsilon}{1-\gamma}}\right]^{\frac{1-\gamma}{\varepsilon}} \tag{D1}$$

subject to

$$c_{t,Y} = (1-\tau)w_t - k_{t+1} - b_{t+1}^i - b_{t+1}^n - m_t \quad \text{(Budget constraint of young)}$$

$$c_{t+1,O(IT)} = (1-\tau_{IT}^k)r_{t+1}^k k_{t+1} + r_{t+1(IT)}^i b_{t+1}^i + r_{t+1(IT)}^n b_{t+1}^n + r_{t+1(IT)}^m m_t \quad \text{(Budget constraint of old with IT)}$$

$$c_{t+1,O(PT)} = (1-\tau_{PT}^k)r_{t+1}^k k_{t+1} + r_{t+1(PT)}^i b_{t+1}^i + r_{t+1(PT)}^n b_{t+1}^n + r_{t+1(PT)}^m m_t \quad \text{(Budget constraint of old with PT)}$$

$$m_t = \delta \quad \text{(Cash constraint)}$$

where $E[X_{t+1(s)}\,|\,\Omega_t]$ is the expectation of X_{t+1} in regime s, conditional upon period-t information, Ω_t.

The Lagrangian for this problem is as follows:

$$L_t = E\left\{\left.\begin{pmatrix} U_t^{IC} + \lambda_{t,Y}((1-\tau)w_t - k_{t+1} - b_{t+1}^i - b_{t+1}^n - m_t - c_{t,Y}) + \mu_t(m_t - \delta) \\ + \lambda_{t+1,O(IT)}((1-\tau_{IT}^k)r_{t+1}^k k_{t+1} + r_{t+1(IT)}^i b_{t+1}^i + r_{t+1(IT)}^n b_{t+1}^n + r_{t+1(IT)}^m m_t - c_{t+1,O(IT)}) \\ + \lambda_{t+1,O(PT)}((1-\tau_{PT}^k)r_{t+1}^k k_{t+1} + r_{t+1(PT)}^i b_{t+1}^i + r_{t+1(PT)}^n b_{t+1}^n + r_{t+1(PT)}^m m_t - c_{t+1,O(PT)}) \end{pmatrix}\right| \Omega_t \right\} \quad \text{(D2)}$$

First-order conditions are as follows:

$$c_{t,Y} : \frac{\partial U_t^{IC}}{\partial c_{t,Y}} = \lambda_{t,Y}, \quad c_{t+1,O(IT)} : \frac{\partial U_t^{IC}}{\partial c_{t+1,O(IT)}} = \lambda_{t+1,O(IT)}, \qquad c_{t+1,O(PT)} : \frac{\partial U_t^{IC}}{\partial c_{t+1,O(PT)}} = \lambda_{t+1,O(PT)}$$

$$k_{t+1} : \lambda_{t,Y} = E\left([\lambda_{t+1,O(IT)}(1-\tau_{IT}^k)r_{t+1}^k + \lambda_{t+1,O(PT)}(1-\tau_{PT}^k)r_{t+1}^k] \,\middle|\, \Omega_t \right)$$

$$b_{t+1}^i : \lambda_{t,Y} = E\left([\lambda_{t+1,O(IT)} r_{t+1(IT)}^i + \lambda_{t+1,O(PT)} r_{t+1(PT)}^i] \,\middle|\, \Omega_t \right)$$

$$b_{t+1}^n : \lambda_{t,Y} = E\left([\lambda_{t+1,O(IT)} r_{t+1(IT)}^n + \lambda_{t+1,O(PT)} r_{t+1(PT)}^n] \,\middle|\, \Omega_t \right)$$

$$m_t : \lambda_{t,Y} = E\left([\lambda_{t+1,O(IT)} r_{t+1(IT)}^m + \lambda_{t+1,O(PT)} r_{t+1(PT)}^m] \,\middle|\, \Omega_t \right) + \mu_t$$

By substitution, this system can be reduced to four Euler equations:

$$\frac{\partial U_t^{IC}}{\partial c_{t,Y}} = E\left[\left(\frac{\partial U_t^{IC}}{\partial c_{t+1,O(IT)}}(1-\tau_{IT}^k)r_{t+1}^k + \frac{\partial U_t^{IC}}{\partial c_{t+1,O(PT)}}(1-\tau_{PT}^k)r_{t+1}^k \right) \middle| \Omega_t \right] \quad \text{(D3)}$$

$$\frac{\partial U_t^{IC}}{\partial c_{t,Y}} = E\left[\left(\frac{\partial U_t^{IC}}{\partial c_{t+1,O(IT)}} r_{t+1(IT)}^n + \frac{\partial U_t^{IC}}{\partial c_{t+1,O(PT)}} r_{t+1(PT)}^n \right) \middle| \Omega_t \right] \quad \text{(D4)}$$

$$\frac{\partial U_t^{IC}}{c_{t,Y}} = E\left[\left(\frac{\partial U_t^{IC}}{\partial c_{t+1,O(IT)}} r_{t+1(IT)}^i + \frac{\partial U_t^{IC}}{\partial c_{t+1,O(PT)}} r_{t+1(PT)}^i \right) \middle| \Omega_t \right] \quad \text{(D5)}$$

$$\frac{\partial U_t^{IC}}{\partial c_{t,Y}} = E\left[\left(\frac{\partial U_t^{IC}}{\partial c_{t+1,O(IT)}} r_{t+1(IT)}^m + \frac{\partial U_t^{IC}}{\partial c_{t+1,O(PT)}} r_{t+1(PT)}^m \right) \middle| \Omega \right] + \mu_t \quad \text{(D6)}$$

The partial derivatives of the utility function are as follows:

$$\frac{\partial U_t^{IC}}{\partial c_{t,Y}} = \Phi c_{t,Y}^{-(1-\varepsilon)} \quad \text{(D7)}$$

$$\frac{\partial U_t^{IC}}{\partial c_{t+1,O(s)}} = \beta \Phi p_s (p_{IT} E[c_{t+1,O(IT)}^{1-\gamma} \mid \Omega_t] + (1-p_{IT})E[c_{t+1,O(PT)}^{1-\gamma} \mid \Omega_t])^{\frac{\varepsilon-(1-\gamma)}{1-\gamma}} c_{t+1,O(s)}^{-\gamma} \quad \text{(D8)}$$

where $\Phi \equiv \left[c_{t,Y}^{\varepsilon} + \beta \left[p_{IT} E[c_{t+1,O(IT)}^{1-\gamma} \mid \Omega_t] + (1-p_{IT}) E[c_{t+1,O(PT)}^{1-\gamma} \mid \Omega_t] \right]^{\frac{\varepsilon}{1-\gamma}} \right]^{\frac{1-\gamma-\varepsilon}{\varepsilon}}$ and $p_{PT} \equiv 1 - p_{IT}$.

Dividing (D8) by (D7) gives

$$\frac{\partial U_t^{IC} \big/ \partial c_{t+1,O(s)}}{\partial U_t^{IC} \big/ \partial c_{t,Y}} = \beta p_s \left(\frac{c_{t,Y}}{c_{t+1,O(s)}} \right)^{1-\varepsilon} \left[\frac{c_{t+1,O(s)}}{\left(p_{IT} E[c_{t+1,O(IT)}^{1-\gamma} \mid \Omega_t] + (1-p_{IT}) E[c_{t+1,O(PT)}^{1-\gamma} \mid \Omega_t] \right)^{1/(1-\gamma)}} \right]^{1-\gamma-\varepsilon} \quad \text{(D9)}$$

Defining $sdf_{t+1(s)} \equiv \dfrac{1}{p_s} \dfrac{\partial U_t^{IC} \big/ \partial c_{t+1,O(s)}}{\partial U_t^{IC} \big/ \partial c_{t,Y}}$, the four Euler equations above can be written as

$$1 = R_t \left(p_{IT} E[SDF_{t+1(IT)} \mid \Omega_t] + (1-p_{IT}) E[SDF_{t+1(PT)} \mid \Omega_t] \right) \quad \text{(D10)}$$

$$1 = r_t \left(p_{IT} E[SDF_{t+1(IT)} (1+\pi_{t+1(IT)}^{ind}) \big| \Omega_t] + (1-p_{IT}) E[SDF_{t+1(PT)} (1+\pi_{t+1(PT)}^{ind}) \big| \Omega_t] \right) \quad \text{(D11)}$$

$$1 = \alpha\, k_{t+1}^{\alpha-1} \left([\, p_{IT} (1-\tau_{IT}^{k}) E[sdf_{t+1(IT)} A_{t+1} \mid \Omega_t] + (1-p_{IT})(1-\tau_{PT}^{k}) E[sdf_{t+1(PT)} A_{t+1} \mid \Omega_t] \right) \quad \text{(D12)}$$

$$1 = \left(p_{IT} E[SDF_{t+1(IT)} \mid \Omega_t] + (1-p_{IT}) E[SDF_{t+1(PT)} \mid \Omega_t] \right) + \tilde{\mu}_t \quad \text{(D13)}$$

where $SDF_{t+1(s)} \equiv sdf_{t+1(s)}/(1+\pi_{t+1(s)})$ and $\tilde{\mu}_t \equiv \mu_t \big/ \lambda_{t,Y}$.

End of analytical results; please turn over for the numerical results.

Section 2 – Numerical results

2.A – Results for the baseline model without money

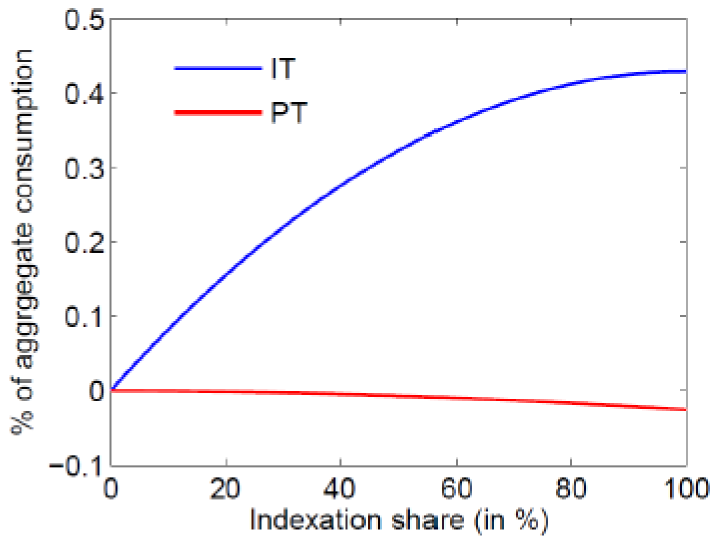

Fig A1 – Social welfare gain from indexation (when money is absent from the model). Figure reports the welfare gain or loss of indexation at x%, as compared to the case of zero indexation. Units: % of aggregate consumption under each regime. All other parameters have the same values as in the baseline model. Optimal indexation shares are equal to 4% under PT and 100% under IT.

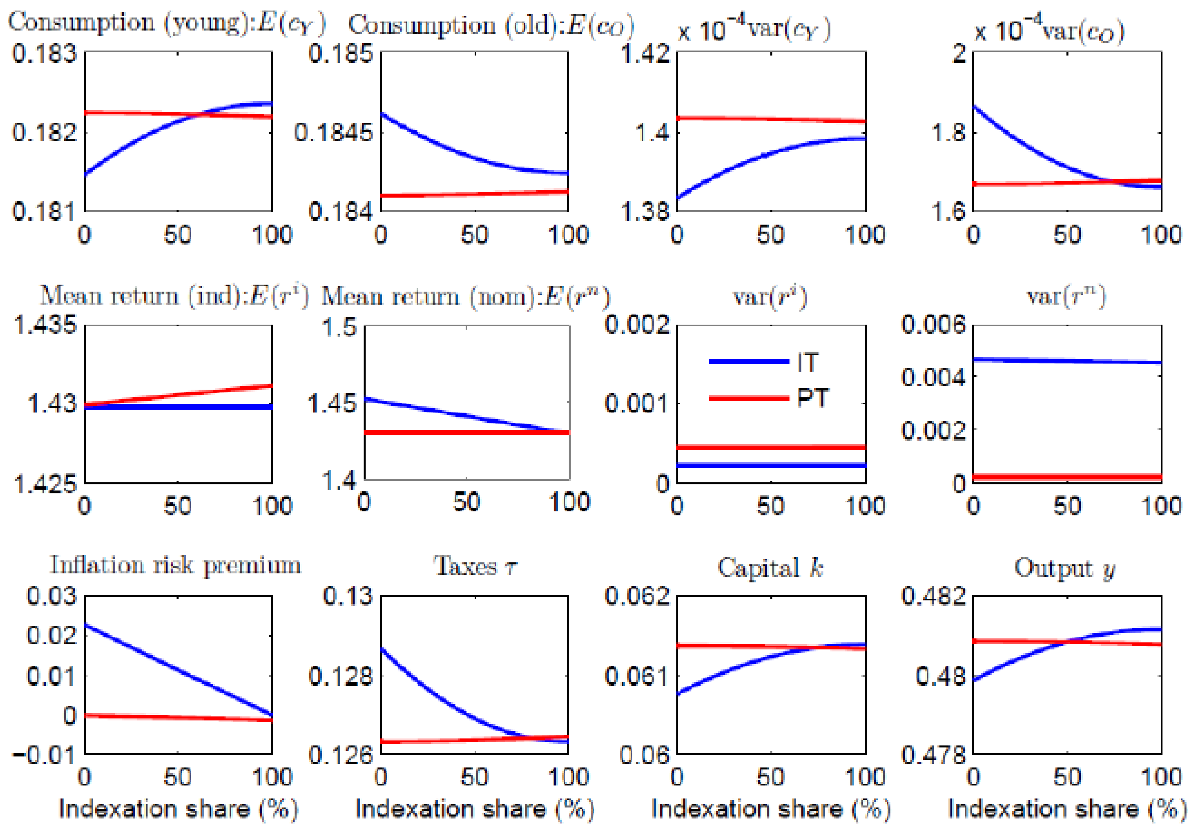

Fig A2 – Indexation and real variables under IT and PT (when money is absent from the model). The inflation risk premium is the difference between the expected real returns on nominal and indexed bonds (i.e. $E(r^n) - E(r^i)$). Returns are non-annualized gross returns and have *not* been converted into percent. All series are unconditional moments. All other parameters have the same values as in the baseline model.

2.B – Results for baseline model without productivity risk

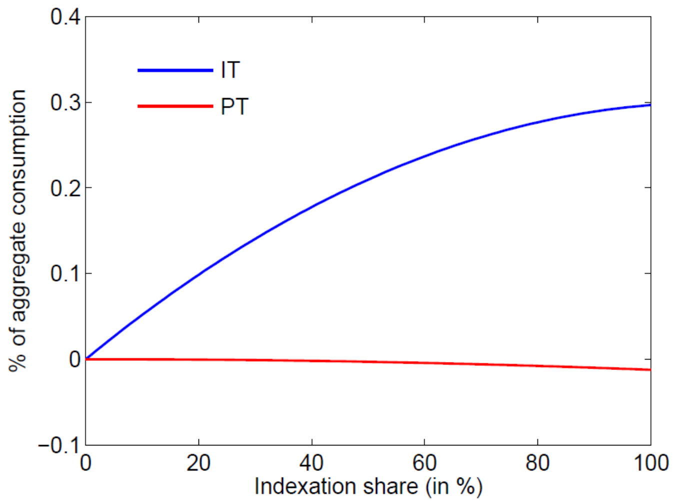

Fig B1 – Social welfare gain from indexation (excluding productivity risk from the model). Figure reports the welfare gain or loss of indexation at x%, as compared to the case of zero indexation. Units: % of aggregate consumption under each regime. All other parameters have the same values as in the baseline model. Optimal indexation shares are equal to 4% under PT and 100% under IT.

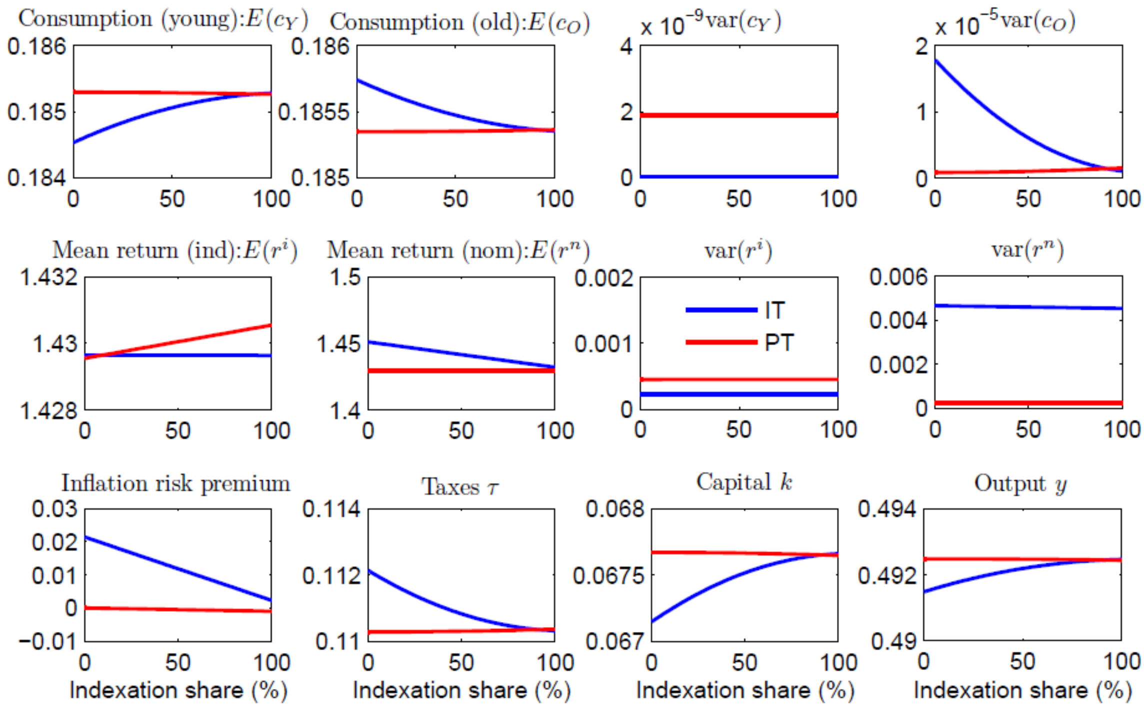

Fig B2 – Indexation and real variables under IT and PT (excluding productivity risk from the model). The inflation risk premium is the difference between the expected real returns on nominal and indexed bonds (i.e. $E(r^n) - E(r^i)$). Returns are non-annualized gross returns and have *not* been converted into percent. All series are unconditional moments. All other parameters have the same values as in the baseline model.