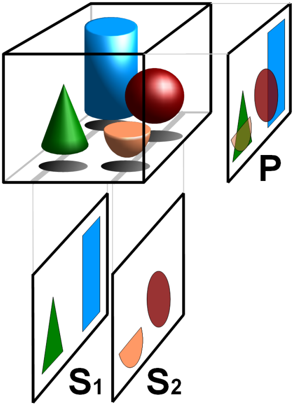
Basics of Seismic Tomography

Introduction

Tomography

Tomography

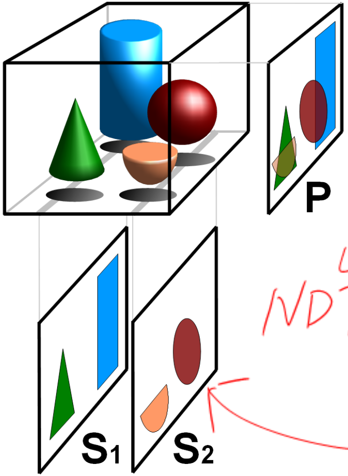
Tomography is imaging by sections or sectioning through the use of any kind of penetrating wave.



The word tomography is derived from Ancient Greek, meaning 'slice, section' and 'to write' or 'to describe.'

https://en.wikipedia.org/wiki/Tomography

Tomography



NDT

Tomography is used in radiology, archaeology, biology, atmospheric science, geophysics, oceanography, plasma physics, materials science, astrophysics, quantum information, and other areas of science. A device used in tomography is called a tomograph, while the image produced is a tomogram.

https://en.wikipedia.org/wiki/Tomography

Tomography

Types of Tomography

Aerial tomography, Atom probe tomography, Computed tomography imaging spectrometer, Computed tomography of chemiluminescence, Confocal microscopy (Laser scanning confocal microscopy), Cryogenic electron tomography, Cryogenic transmission electron microscopy, Electrical capacitance tomography, Electrical capacitance, Electrical capacitance volume tomography, Electrical resistivity tomography, Electrical impedance tomography, Electron tomography, Transmission electron microscopy, Focal plane tomography, Functional magnetic resonance imaging, Hydraulic tomography, Infrared microtomographic imaging, Laser Ablation Tomography, Magnetic induction tomography, Magnetic particle imaging , Magnetic resonance imaging or nuclear magnetic resonance tomography, Muon tomography, Microwave tomography, Neutron tomography, Ocean acoustic tomography, Optical coherence tomography, Optical diffusion tomography, Optical projection tomography, Photoacoustic imaging in biomedicine, Positron emission tomography, Positron emission tomography - computed tomography, Quantum tomography, Single photon emission computed tomography, Seismic tomography, Terahertz tomography, Thermoacoustic imaging, Ultrasound-modulated optical tomography, Ultrasound computer tomography, Ultrasound transmission tomography , X-ray computed tomography, X-ray microtomography, Zeeman-Doppler imaging, etc.

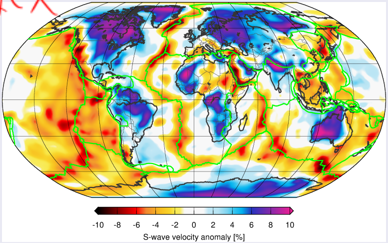
Computed Tomography & Seismic Tomography

- How Does a CT Scan Work? https://www.youtube.com/watch?v=l9swbAtRRbg

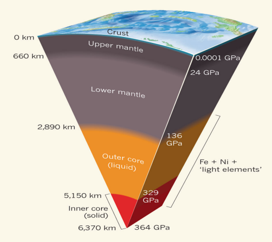
$\int_{\Omega} \nabla \cdot F \, d\vec{x}$

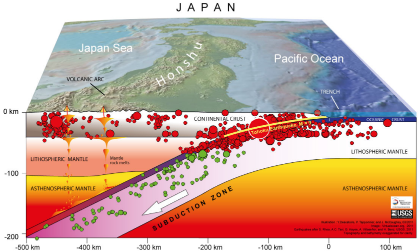
- Seismic Tomography https://www.youtube.com/watch?v=Hrto0nIP8nk

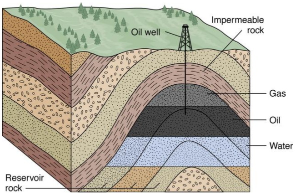
$= \int_{\partial\Omega} F \cdot d\vec{x}$



Seismic Tomography: Image (CT Scan) the Earth's interior

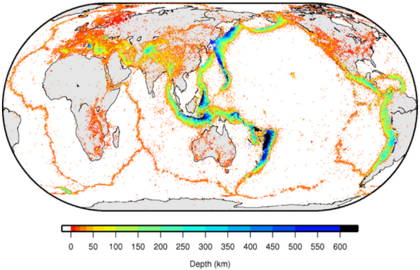


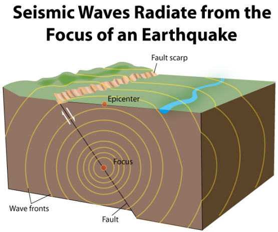


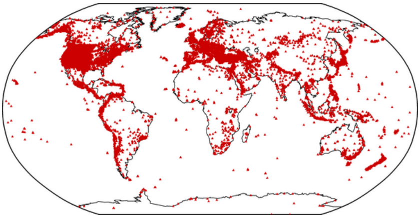


Earthquakes: Millions occur each year

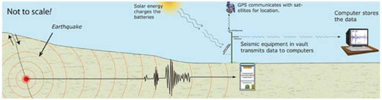
ISC locations: 1960 to present





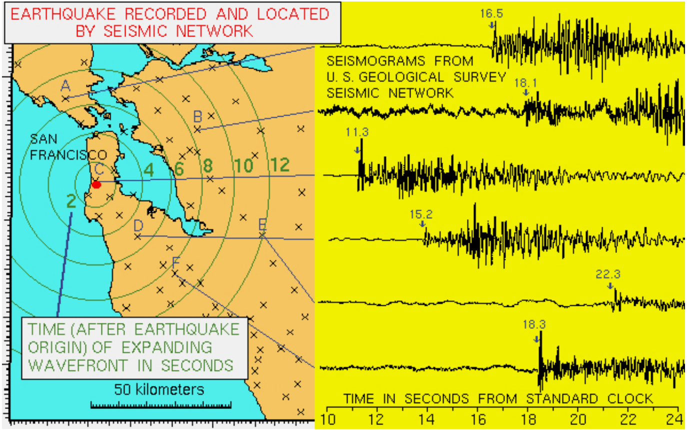


∼15,000 seismic stations worldwide



Seismic data recording

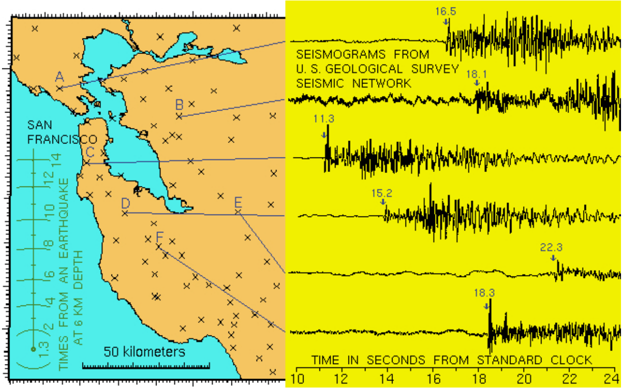
Earthquake and seismic data



The earthquake occurred near San Francisco. Seismograms $\mathbf{s}^{\mathrm{obs}}(t, \mathbf{x}_r^i)$ are recorded at stations A-F. The arrival times T^{obs} of seismic energy are identifiable.

Earthquake and seismic data

Traveltime data are the most robust information that can be extracted from seismograms.



We usually do not know the real locations of earthquakes. But for simplicity, we assume earthquake locations are known in our discussion.

$$T = \frac{d}{v}$$

Traveltime Tomography

$\Rightarrow v(\vec{x})$

$\downarrow$

Robust

Traveltime Tomography

Question 0

Let $f(x) = \cos^2 x$. If $f(x) = 0.29192658171$, find the value of x.

Question 0 non-uniques

Let $f(x) = \cos^2 x$. If $f(x) = 0.29192658171$, find the value of x.

$t = \frac{d}{v} \Leftarrow f$

We know that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cancel{o((x - x_0)^2)}.$$

Ignoring the second and beyond terms gives

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Solve the above approximate equation for x,

$$x = x_0 + \left[f'(x_0)\right]^{-1} \left[f(x) - f(x_0)\right].$$

$f(x_0) = \frac{1}{4}$

$f'(x_0) = -1$ $\quad x_0 = \pi/4$

$x = \pi/4 - 1 \cdot (0.2919 - \frac{1}{4})$

Traveltime Tomography

Question 0

Let $f(x) = \cos^2 x$. If $f(x) = 0.29192658171$, find the value of x.

We know that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + o((x - x_0)^2).$$

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Solve the above approximate equation for x,

$$x = x_0 + \left[f'(x_0)\right]^{-1} \left[f(x) - f(x_0)\right].$$

Question 0 - Extension

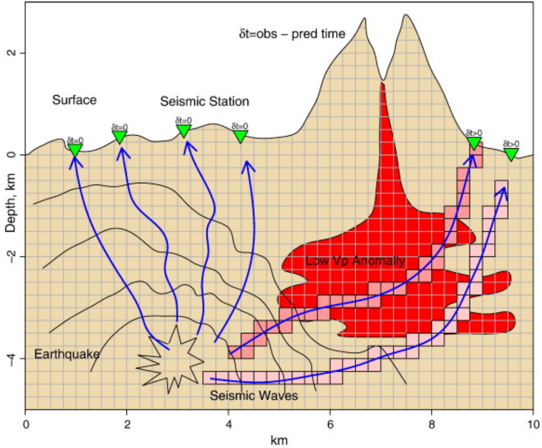
Let $f(x) = \cos^2 x$. If $f(x) = 0.29192658171$ and x is around 2.0, find the value of x.

x_0

Traveltime Tomography

Seismic Ray Theory

Under high-frequency approximation, seismic waves are considered to travel along geometric paths in smoothly varying media. These paths are called **rays**.

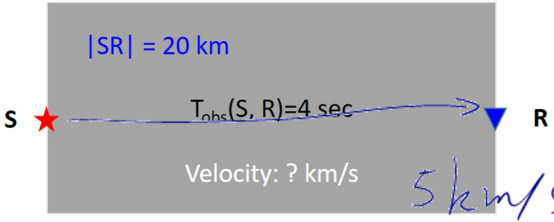


$$t = \frac{L}{v}$$

Traveltime Tomography

Question 1

The traveltime from the earthquake location S (red star) to the seismic station R (blue inverse triangle) is 4 second. The distance $|SR|$ is 20 km. What is the velocity along the line SR?



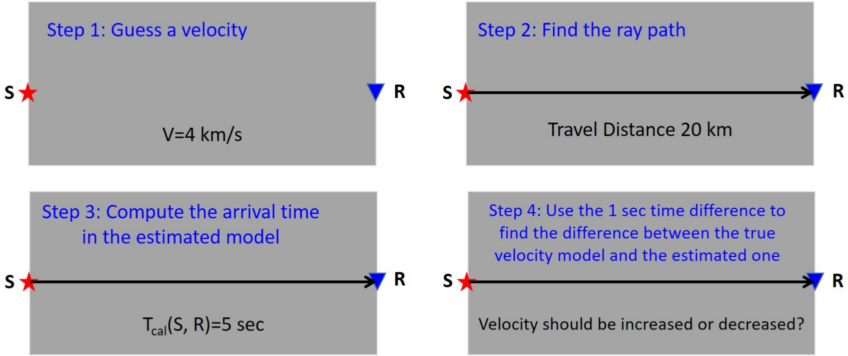
5 km/s

10/3.0

10 km/s

Traveltime Tomography

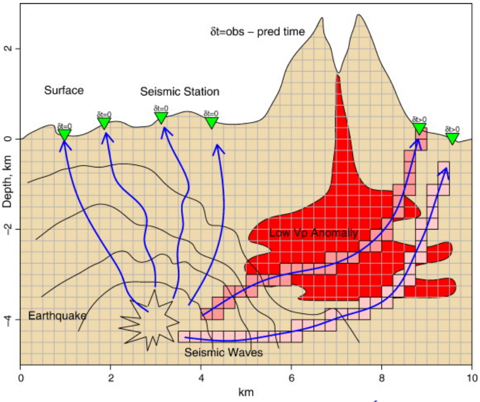
The observed arrival time is 4 second. Find the velocity.



$T_{obs}(S,R) = 4$ sec

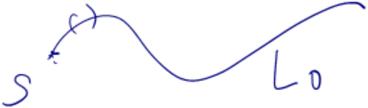
Traveltime tomography

Searching for the true velocity model around an estimated model (Reference model).



Reference model $\mathbf{m}_0$: velocity $c_0(\mathbf{x})$, ray path L_0, traveltime T^{syn};
True model $\mathbf{m}$: velocity $c(\mathbf{x})$, ray path L, traveltime T^{obs}.

Reference model $\mathbf{m}_0$: velocity $c_0(\mathbf{x})$, ray path L_0, traveltime T^{syn}



$$T^{syn} = \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \tag{1}$$

True model $\mathbf{m}$: velocity $c(\mathbf{x})$, ray path L, traveltime T^{obs}

$$T^{obs} = \int_{L} \frac{ds}{c(\mathbf{x})} \tag{2}$$

Assume: $c(\mathbf{x}) = c_0(\mathbf{x}) + \delta c(\mathbf{x})$, $|\delta c(\mathbf{x})| \ll c_0(\mathbf{x})$; $L_0 \approx L$

$$T^{obs} - T^{syn} = \int_{L} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \approx \int_{L_0} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \tag{3}$$

$L \approx L_0$

Traveltime Tomography

Reference model $\mathbf{m}_0$: velocity $c_0(\mathbf{x})$, ray path L_0, traveltime T^{syn}

$$T^{syn} = \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \tag{1}$$

True model $\mathbf{m}$: velocity $c(\mathbf{x})$, ray path L, traveltime T^{obs}

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$$T^{obs} - T^{syn} = \int_{L} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \approx \int_{L_0} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} \tag{3}$$

Fermat's principle

The path taken by a ray between two given points is the path that can be traversed in the least time.

Traveltime Tomography

After applying the Fermat's principle, we have

$$
\begin{aligned}
T^{obs} - T^{syn} &\approx \int_{L_0} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} = \int_{L_0} \left[\frac{1}{c(\mathbf{x})} - \frac{1}{c_0(\mathbf{x})} \right] ds \\
&= \int_{L_0} \frac{c_0(\mathbf{x}) - c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds = \int_{L_0} \frac{c_0(\mathbf{x}) - c_0(\mathbf{x}) - \delta c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds \qquad (4) \\
&= \int_{L_0} \frac{-\delta c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] \frac{\delta c(\mathbf{x})}{c(\mathbf{x})} ds
\end{aligned}
$$

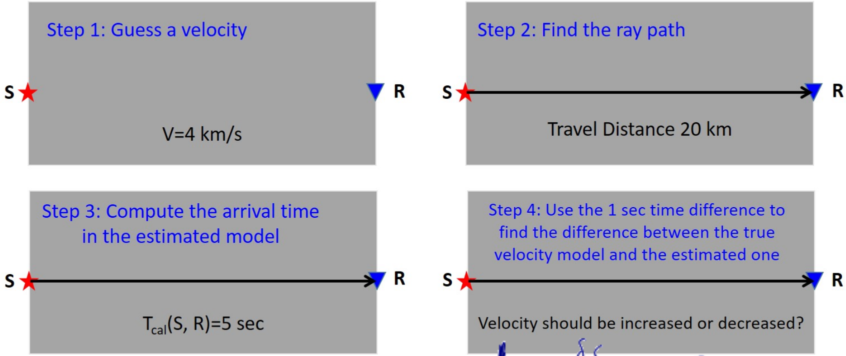
After applying the Fermat's principle, we have

$$
\begin{aligned}
T^{obs} - T^{syn} &\approx \int_{L_0} \frac{ds}{c(\mathbf{x})} - \int_{L_0} \frac{ds}{c_0(\mathbf{x})} = \int_{L_0} \left[\frac{1}{c(\mathbf{x})} - \frac{1}{c_0(\mathbf{x})}\right] ds \\
&= \int_{L_0} \frac{c_0(\mathbf{x}) - c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds = \int_{L_0} \frac{c_0(\mathbf{x}) - c_0(\mathbf{x}) - \delta c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds \\
&= \int_{L_0} \frac{-\delta c(\mathbf{x})}{c_0(\mathbf{x})c(\mathbf{x})} ds = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})}\right] \frac{\delta c(\mathbf{x})}{c(\mathbf{x})} ds
\end{aligned}
\tag{4}
$$

The traveltime difference $T^{obs} - T^{syn}$ is related to the relative velocity difference $\delta c(\mathbf{x})/c(\mathbf{x})$ via the line integral of $-\frac{1}{c_0(\mathbf{x})}$ along the ray path L_0.

$$
T^{obs} - T^{syn} = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})}\right] \frac{\delta c(\mathbf{x})}{c(\mathbf{x})} ds. \tag{5}
$$

The observed arrival time is 4 second. Find the velocity.



$$4 - 5 = -\frac{1}{4} \cdot \frac{\delta c}{c} \cdot 20 \Rightarrow$$

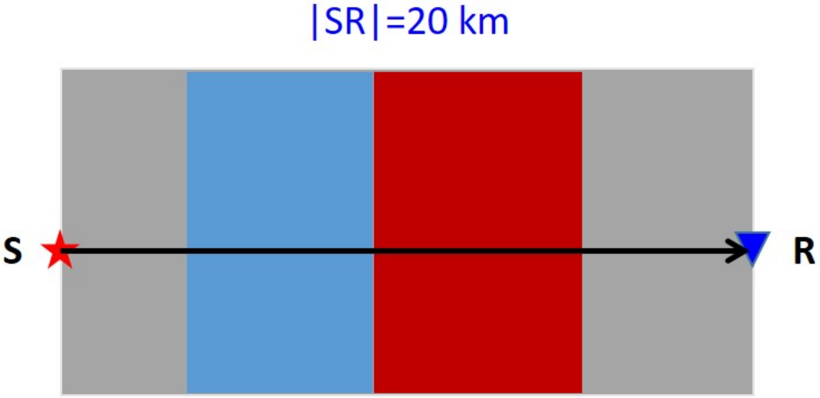
$$T^{obs} - T^{syn} = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] \frac{\delta c(\mathbf{x})}{c(\mathbf{x})} ds. \qquad -1 = -5 \cdot \frac{\delta c}{c}$$

$$\delta c = 1 \Leftarrow C_0 + \delta c = 5 \delta c \leftarrow c = 5 \cdot \delta c$$

Traveltime Tomography

Seismic heterogeneity

In most cases, the velocity model is not homogenous. Seismic heterogeneities commonly exist inside the Earth.



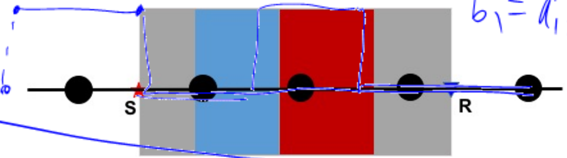
Model Parameterisation: The way of representing $\frac{\delta c(\mathbf{x})}{c(\mathbf{x})}$

Model Parameterisation

Grid Approach

Discretise the continuous space into a finite number of grid points. Each grid point is associated with a basis function $B_k(\mathbf{x})$ ($k = 1, 2, \cdots, n$). Then

$$\frac{\delta c(\mathbf{x})}{c(\mathbf{x})} = \sum_{k=1}^{n} X_k B_k(\mathbf{x})$$



$b_1 = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n$

$$T^{obs} - T^{syn} = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})}\right] \frac{\delta c(\mathbf{x})}{c(\mathbf{x})} ds = \int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})}\right] \left[\sum_{k=1}^{n} X_k B_k(\mathbf{x})\right] ds$$
$$= \sum_{k=1}^{n} \left[\int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})}\right] B_k(\mathbf{x}) ds\right] X_k. \quad (6)$$

Model Parameterisation

One ray gives one equation with n unknowns.

$$T^{obs} - T^{syn} = \sum_{k=1}^{n} \left[\int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] B_k(\mathbf{x}) ds \right] X_k. \tag{7}$$

The coefficient of the k-th unknown is

$$\int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] B_k(\mathbf{x}) ds.$$

Model Parameterisation

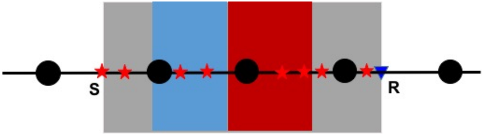
One ray gives one equation with n unknowns.

$$T^{obs} - T^{syn} = \sum_{k=1}^{n} \left[\int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] B_k(\mathbf{x}) ds \right] X_k. \tag{7}$$

The coefficient of the k-th unknown is

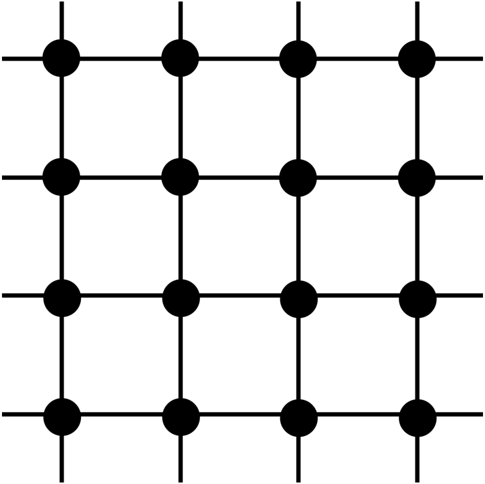
$$\int_{L_0} \left[-\frac{1}{c_0(\mathbf{x})} \right] B_k(\mathbf{x}) ds.$$

To determine N unknowns, we must have at least n independent equations. Thus, we need at least n different rays.



Model Parameterisation

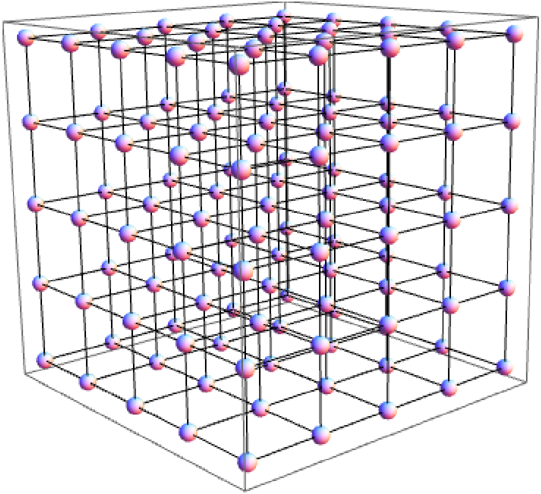
Two-dimensional Case



$B_x \cdot B_y \cdot B_z$

Model Parameterisation

Three-dimensional Case



Model Parameterisation

The i-th ray gives an equation

$$b_i = T_i^{obs} - T_i^{syn} = \sum_{k=1}^{n} a_{ik} X_k.$$

Model Parameterisation

The *i*-th ray gives an equation

D

R

$$b_i = T_i^{obs} - T_i^{syn} = \sum_{k=1}^{n} a_{ik} X_k.$$

With a total of m rays, we have a linear system

$$AX = b,$$

∂

where $\quad m \times n \quad n \times 1 \quad m \times 1$

S

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

Ray Tracing

$$T^{obs} - T^{syn} = \int_{L_0} \left[-\frac{1}{C_0(\vec{x})} \right] \cdot \frac{\delta C(\vec{x})}{C(\vec{x})} \, ds$$

Ray Tracing

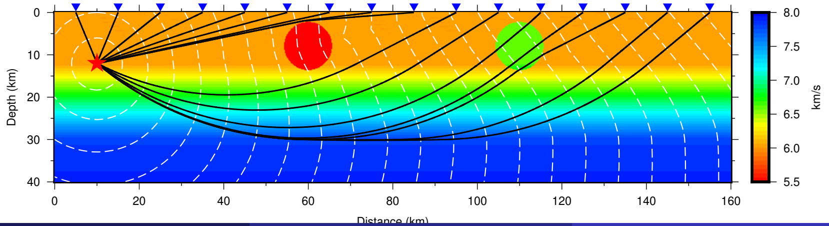
For the i-th ray, the coefficient of the k-th unknown is

$$a_{ik} = \int_{L_{0,i}} \left[-\frac{1}{c_0(\mathbf{x})} \right] B_k(\mathbf{x}) ds.$$

For the reliability of seismic traveltime tomography, it is essential to accurately determine ray paths.

- Pseudo-bending method (Um & Thurber, BSSA, 1987)
- Eikonal ray-tracing (Rawlinson et al., GJI, 2004; Tong et al., GRL, 2017)

$\nabla T \cdot \nabla T = s^2(\vec{x})$

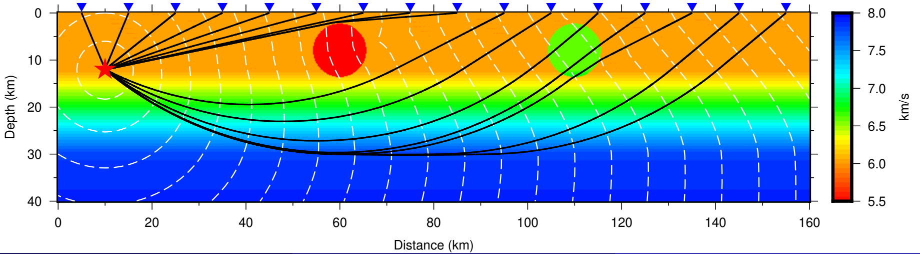


Ray Tracing

Seismic ray tracing is usually complicated, and sometimes computationally expensive.
$L_{0,i}$ can be accurately determined by solving the eikonal equation

$$c_0(\mathbf{x})|\nabla T(\mathbf{x})| = 1$$

with the fast marching method (Rawlinson et al., GJI, 2004; Liu et al., GRL, 2018).



$$T^{obs} = T_{true}^{obs} + e$$

Solving the linear system $AX = b$

$$Ax = b$$

$$n = 0.5\ m \qquad m > n$$

$$\Downarrow$$

$$n = 2000 \qquad m >> n$$

Question 2

Solve the following system of equations:

$$\begin{cases} x + y & = 2 \\ x + 2y & = 3 \end{cases}$$

Question 2

Solve the following system of equations:

$m = 2$

$n = 2$

$$\begin{cases} x + y &= 2 \\ x + 2y &= 3 \end{cases}$$

$x = 1$

$y = 1$

Question 2 - Extension

Solve the following system of equations:

$$\begin{cases} x + y &= 2 + e_1 \\ x + 2y &= 3 + e_2 \\ x + 3y &= 6 + e_3 \end{cases}$$

Solving $AX = b$

A is an $m \times n$ matrix. If $m \neq n$, the inverse of A doesn't exist. To find a solution,

Multiply by A^T

$m \times n$

$$AX = b$$

$$\Rightarrow \underbrace{A^TA}_{n\times n}\underbrace{X}_{n\times 1} = \underbrace{A^Tb}_{n\times 1}$$

A^TA is a positive-definite matrix or positive semi-definite matrix. One can prove that the solution of $A^TAX = A^Tb$ is the minimiser of $||AX - b||_2^2$ if exists.

$$X = (A^TA)^{-1}A^Tb$$

Solving $AX = b$

A is an $m \times n$ matrix. If $m \neq n$, the inverse of A doesn't exist. To find a solution,

Multiply by A^T

$$
\begin{aligned}
&AX = b \\
\Rightarrow &A^T AX = A^T b
\end{aligned}
$$

$A^T A$ is a positive-definite matrix or positive semi-definite matrix. One can prove that the solution of $A^T AX = A^T b$ is the minimiser of $||AX - b||_2^2$ if exists.

Question 2 - Extension

Solve the following system of equations:

$$
\begin{cases}
x + y &= 2 \\
x + 2y &= 3 \\
x + 3y &= 6
\end{cases}
$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

Solving $AX = b$

Question 3

Solve the following system of equations:

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} x + y &= 9 \\ x + y &= 10 \\ x + y &= 11 \end{cases}$$

$$A^T A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

Add λI to $A^T A$ ($\lambda > 0$)

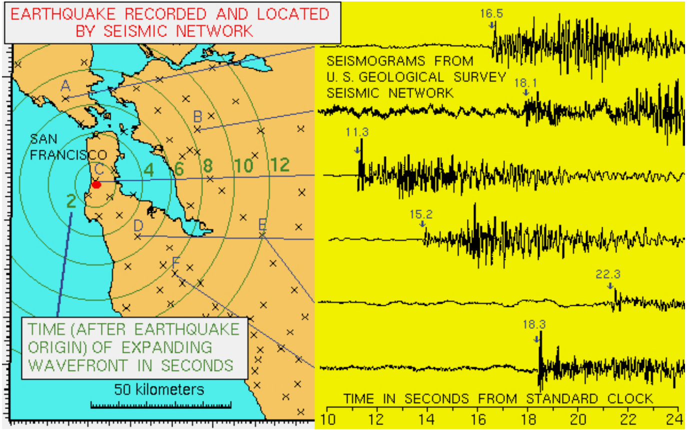
$$x = \frac{a}{\lambda_1} \qquad y = \frac{b}{\lambda_2}$$

$$\begin{aligned} & AX = b \\ \Rightarrow & A^T AX = A^T b \\ \Rightarrow & \left[A^T A + \lambda I\right] X = A^T b \end{aligned}$$

$$\lambda > 0$$

$A^T A + \lambda I$ is a positive-definite matrix. Its condition number is smaller than that of $A^T A$. Good for a stable solution.

Solving $AX = b$



$T^{obs} \neq T^{obs}_{true}$

The picked arrival times always contain errors. T^{obs} is inaccurate.

Solving $AX = b$

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{2000\lambda}{\lambda^2+2000\lambda}\begin{pmatrix}1\\1\end{pmatrix}$$

Question 4

Solve the following system of equations:

$$\frac{1}{\lambda^2+2000\lambda}\begin{pmatrix}1000+\lambda & -1000\\ -1000 & 1000+\lambda\end{pmatrix}$$

$$\begin{cases} x+y &= 2+e_1 \\ x+y &= 2+e_2 \\ \cdots & \cdots \\ x+y &= 2+e_{1000} \end{cases}$$

$$A^T b \approx \begin{pmatrix}2000\\2000\end{pmatrix}$$

$$= (A^T A + \lambda I)^{-1}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \frac{2000}{\lambda+2000}\begin{pmatrix}1\\1\end{pmatrix}$$

e_i $(i = 1, 2, \cdots, 1000)$ are random errors following a normal distribution. Find the value of x and y.

$$A^T A = \begin{pmatrix}1000 & 1000\\ 1000 & 1000\end{pmatrix} \qquad A^T b = \begin{pmatrix}2000+\sum_{i=1}^{1000} e_i\\ 2000+\sum_{i=1}^{1000} e_i\end{pmatrix}$$

$$\begin{pmatrix}1000+\lambda & 1000\\ 1000 & 1000+\lambda\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2000\\2000\end{pmatrix}$$

Exercise

$$C(\vec{x})$$

$$T^{syn} \approx T^{obs}$$

$$C_0(\vec{x}) \xrightarrow{\delta c(\vec{x})} C_0+\delta c(\vec{x})$$

$$T^{syn} \neq T^{obs}$$

Exercise

Find the velocity model $c(x)$ $(0 \leq x \leq 20$ km$)$

There are 10 earthquakes occurring at 10 different locations in the interval $(0, 20)$. The only seismic station is located at $x = 20$ km. The location and corresponding observed traveltime of every earthquake are listed below. Determine the velocity model $c(x)$ on the interval [0, 20]

	Eq. 1	Eq. 2	Eq. 3	Eq. 4	Eq. 5
Location	x=0	x=2	x=4	x=6	x=8
T^{obs} (sec)	3.61	3.32	2.86	2.50	2.04

	Eq. 6	Eq. 7	Eq. 8	Eq. 9	Eq. 10
Location	x=10	x=12	x=14	x=16	x=18
T^{obs} (sec)	1.66	1.30	1.02	0.64	0.33

Exercise

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