A I I E Transactions
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/uiie19

On Solving Multifacility Location Problems using a Hyperboloid Approximation Procedure

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Published online: 09 Jul 2007.

On Solving Multifacility Location Problems using a Hyperboloid Approximation Procedure

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Abstract: An iterative solution method is presented for solving multifacility location problems involving rectilinear and/or Euclidean distances. The iterative procedure is based on the use of an approximating function involving hyperboloids, which in the limit approach the cones in the original objective function. Given that the hyperboloid approximation procedure converges, it is shown to converge to the optimum solution. Computational experience with the procedure is described.

■ Considerable interest in facility location problems has developed since Kuhn and Kuenne (8) addressed the generalized Weber problem, also referred to as the Fermat problem and the Steiner problem. In the past decade over two hundred papers have either been published or presented at national and international meetings on the subject of facilities location.

At present, a strong interdisciplinary interest in facility location exists within the fields of applied mathematics, architecture, civil engineering, economics, industrial engineering, management science, operations research, regional science, systems engineering, transportation systems, and urban design, among others. As expected, the term "facility" can be defined very broadly. For example, a facility can be a manned space vehicle, a school, a student to be bused to a school in an urban environment, a machine tool, a hospital, a remote computer terminal, an ambulance, an airport, a police cruiser, a home appliance, a pump in a pipeline, a sewage treatment plant, a planned community, a warehouse, an office within an office building, and an office building. For a recent survey of the facility location literature, see (1), (5) and (9).

Received October 1972; revised February 1973. This paper was presented in the Location Theory Session, Joint ORSA/TIMS/AIIE National Meeting, Atlantic City, New Jersey, November 1972. The paper is contributed by the Facilities Planning and Design Division of AIIE.

In this paper, we treat a multifacility location problem which has been studied previously by others. Specifically, an efficient solution procedure is presented for solving the location problem which can be formulated mathematically as follows:

$$\text{P1.} \quad \textit{Minimize } f(X) = \sum_{1 \leqslant j < k \leqslant n} v_{jk} d(X_j, X_k) + \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} d(X_j, P_i)$$

where

X_j = vector location of new facility j, $1 \leqslant j \leqslant n$

P_i = vector location of existing facility i, $1 \leqslant i \leqslant m$

$d(X_j, X_k)$ = distance between new facilities j and k, $1 \leqslant j < k \leqslant n$

$d(X_j, P_i)$ = distance between new facility j and existing facility i, $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$

v_{jk} = nonnegative weight between new facilities j and k, $1 \leqslant j < k \leqslant n$

w_{ji} = nonnegative weight between new facility j and existing facility i, $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$.

In the case of rectilinear distances, Cabot, Francis, and Stary (1) and Wesolowsky and Love (15) show that P1 can be expressed as a linear programming problem. In the case of Euclidean distances, Love (9) solves P1 as a convex programming problem. Independent of this effort, Wesolowsky and Love (16) define an approximating function and solve the rectilinear problem using a gradient search procedure.

The purpose of this paper is to present a solution procedure which can be used to solve P1 for the case of rectilinear and/or Euclidean distances. The procedure is an extension of that employed independently by Kuhn and Kuenne (8), Cooper (2), and Weiszfeld (14) in solving the single facility location problem with Euclidean distances. To facilitate the presentation we treat first the Euclidean problem, followed by a generalized distance formulation of the multifacility location problem.

Euclidean Problem

In the case of Euclidean distances, letting $X_j = (x_j, y_j)$ and $P_i = (a_i, b_i)$, P1 becomes

$$\text{P2.}\quad \textit{Minimize}\, f(X) = \sum_{1 \leq j < k \leq n} v_{jk}[(x_j - x_k)^2 + (y_j - y_k)^2]^{1/2}$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}[(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2}$$

Notice, P2 can be easily expressed as a three dimensional location problem if desired.

It is assumed that the location problem is well-formulated in the sense that all new facilities are chained (5). New facility j is chained if it is connected to some existing facility either directly via a positive valued w_{ij} or indirectly via a positive valued v_{jk} where new facility k is chained. For a formal definition of a chained facility, see Francis and Cabot (5).

Taking the partial derivatives of $f(X)$ in P2 with respect to x_j and y_j, respectively, gives

$$\partial f/\partial x_j = \sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}(x_j - x_k)/D_{jk} + \sum_{i=1}^{m} w_{ji}(x_j - a_i)/E_{ji}, 1 \leq j \leq n \qquad [1]$$

and

$$\partial f/\partial y_j = \sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}(y_j - y_k)/D_{jk} + \sum_{i=1}^{m} w_{ji}(y_j - b_i)/E_{ji}, 1 \leq j \leq n \qquad [2]$$

where

$$D_{jk} = [(x_j - x_k)^2 + (y_j - y_k)^2]^{1/2} \qquad [3]$$

and

$$E_{ji} = [(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2}. \qquad [4]$$

Normally, Eqs. [1] and [2] are set equal to zero and some appropriate numerical technique is used to solve the resultant set of nonlinear equations. As an illustration, Love (9) employed a modified gradient serach procedure to solve Eqs. [1] and [2]. For the case of a single new facility, an iterative procedure has been used extensively to solve Eqs. [1] and [2]; e.g., see Cooper (2), (3), (4), Kuhn and Kuenne (8), Miehle (10), and Weiszfeld (4).

The use of numerical methods in solving [1] and [2] is hindered by the property that the partial derivatives are not defined at all points in the solution space. In particular, notice that if either new facilities j and k have the same location or new facility j and existing facility i have the same location then both $\partial f/\partial x_j$ and $\partial f/\partial y_j$ are undefined. In order to resolve the difficulty, it is convenient to consider a geometrical interpretation of P2.

Geometric Interpretation, P2

Geometrically, each cost term included in P2 represents the equation of a right circular cone. To see why this is true suppose existing facility i is located at the origin and assume w_{ji} is positive valued. By letting new facility j move along the x-axis a distance r_{ji}, the weighted distance function, f_{ji}, between new facility j and existing facility i can be described by the following linear relationship,

$$f_{ji} = w_{ji} r_{ji}. \qquad [5]$$

Since new facility j is not restricted to move in any specified direction, r_{ji} can be interpreted as describing the locus of points equi-distant from existing facility i. As shown in Fig. 1, the locus of points a distance r_{ji} from new facility i is a circle of radius r_{ji}. Consequently, f_{ji} can be given as

$$f_{ji} = w_{ji}(x_j^2 + y_j^2)^{1/2}. \qquad [6]$$

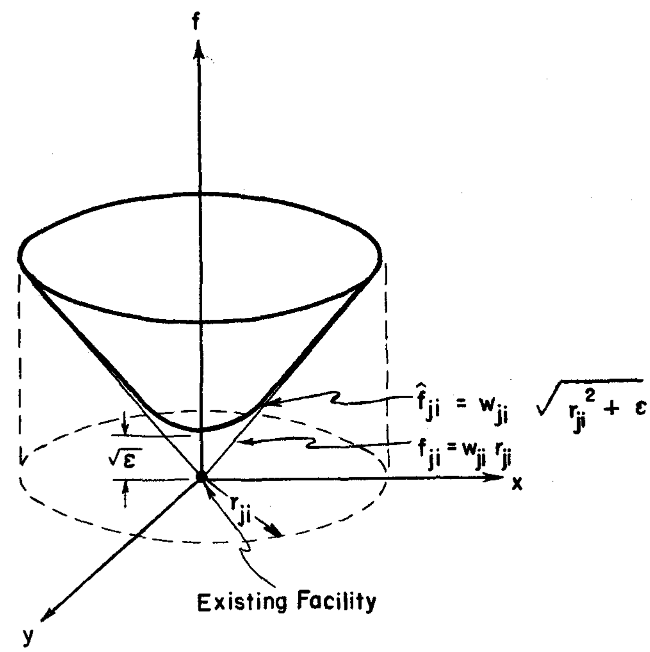

Fig. 1. Geometrical interpretation of Euclidean location problem

Thus Eq. [6] is a right circular cone generated by revolving the straight line given by Eq. [5] about the f-axis. If new facility i has the location (a_i,b_i) then Eq. [6] becomes

$$f_{ji} = w_{ji} [(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2}. \quad [7]$$

Similarly, if new facility j is located at (x_j,y_j), new facility k is located at (x_k,y_k), and v_{jk} is positive valued, then relative to new facility j, the weighted distance function between new facilities j and k generates a right circular cone centered at the point (x_j,y_j). Consequently, $f(X)$ in P2 represents the sum of cones.

The sharp points of the cones involved in the summation result in the undefined derivatives and produce the "knife-edged surface" referred to by Vergin and Rogers (13). Since a cone is a limiting form of a hyperboloid, if the cones are replaced by hyperboloids, a smooth approximating function, $\hat{f}(X)$ is obtained. Furthermore, since hyperboloids are strict convex functions and $\hat{f}$ is the sum of hyperboloids, $\hat{f}(X)$ is a strict convex function.

As noted in Fig. 1, the equation for a hyperbola lying in the first quadrant of the f-x plane is given by

$$\hat{f}_{ji} = w_{ji}(r_{ji}^2 + \epsilon)^{1/2} \quad [8]$$

where ϵ is a positive valued constant. The hyperboloid centered on the point (a_i, b_i) in the x-y plane can be generated by rotating Eq. [8] about the translated axes and can be expressed as

$$\hat{f}_{ji} = w_{ji}[(x_j - a_i)^2 + (y_j - b_i)^2 + \epsilon]^{1/2} \quad [9]$$

Hyperboloid Approximation Procedure

From Fig. 1, it is seen that the addition of the constant, ϵ, essentially results in the replacement of the point of a cone by a smooth hyperbolic surface. Consequently, by introducing ϵ the partial derivatives exist everywhere. Furthermore, the smaller the value of ϵ the closer the hyperboloid approximates the cone.

Letting

$$\hat{D}_{jk} = [(x_j - x_k)^2 + (y_j - y_k)^2 + \epsilon]^{1/2} \quad [10]$$

and

$$\hat{E}_{ji} = [(x_j - a_i)^2 + (y_j - b_i)^2 + \epsilon]^{1/2} \quad [11]$$

then the new optimization problem can be given as

$$\text{P3.} \quad \textit{Minimize } \hat{f}(X) = \sum_{1 \leq j < k \leq n} v_{jk}\hat{D}_{jk} + \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\hat{E}_{ji}$$

where $\lim_{\epsilon\to 0} \hat{f}(X) = f(X)$. Therefore, solving P3 using a very small value of ϵ yields a solution which is approximately the same as that obtained by solving P2.

Taking the partial derivatives of $\hat{f}(X)$ with respect to x_j and y_j gives

$$\partial\hat{f}/\partial x_j = x_j[\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}] - [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}x_k/\hat{D}_{jk} + \sum_{i=1}^{m} w_{ji}a_i/\hat{E}_{ji}] \quad [12]$$

$$\partial\hat{f}/\partial y_j = y_j[\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}] - [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}y_k/\hat{D}_{jk} + \sum_{i=1}^{m} w_{ji}b_i/\hat{E}_{ji}]. \quad [13]$$

Since Eqs. [12] and [13] are always defined, the partial derivatives can be set equal to zero and the resultant set of nonlinear equations solved numerically. Using the same approach employed by Kuhn and Kuenne (8), Cooper (2), and Weiszfeld (14) for the case of a single facility, the following iterative expressions are obtained by setting Eqs. [12] and [13] equal to zero and solving for x_j and y_j:

$$x_j^{(h+1)} = [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}x_k^{(h)}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}a_i/\hat{E}_{ji}^{(h)}] \Big/ [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}^{(h)}] \quad [14]$$

and

$$y_j^{(h+1)} = [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}y_k^{(h)}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}b_i/\hat{E}_{ji}^{(h)}] \Big/ [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}^{(h)}] \quad [15]$$

where the superscript indicates the iteration number. Alternatively, from Eq. [12]

$$\sum_{k=1}^{n} v_{jk}x_k^{(h)}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}a_i/\hat{E}_{ji}^{(h)} = x_j^{(h)}[\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}^{(h)}] - \partial\hat{f}/\partial x_j^{(h)}. \quad [16]$$

Substituting Eq. [16] in Eq. [14] and reducing gives

$$x_j^{(h+1)} = x_j^{(h)} - \rho^{(h)}\, \partial\hat{f}/\partial x_j^{(h)} \qquad [17]$$

where

$$\rho^{(h)} = [\sum_{\substack{k=1\\ \neq j}}^{n} v_{jk}/\hat{D}_{jk}^{(h)} + \sum_{i=1}^{m} w_{ji}/\hat{E}_{ji}^{(h)}]^{-1}. \qquad [18]$$

Similarly,

$$y_j^{(h+1)} = y_j^{(h)} - \rho^{(h)}\, \partial\hat{f}/\partial y_j^{(h)}. \qquad [19]$$

The iteration given by either Eqs. [14] and [15] or [17] and [19] is subsequently referred to as the hyperboloid approximation procedure or HAP.

It is interesting to compare [14] and [15] with the iterative procedure used in gradient search (18). The size of the gradient step is given by $\rho^{(h)}$. HAP differs from steepest descent gradient techniques since the value of $\rho^{(h)}$ given by [18] need not be an "optimal" step (6). However, the ease with which the value of $\rho^{(h)}$ is calculated using [18] results in a solution procedure which has been observed to be quite efficient in comparison with steepest descent gradient procedures. More iterations are usually required using HAP than with steepest descent gradient techniques; however, HAP requires less computation time to perform an iteration than steepest descent gradient procedures.

In addition to its computational efficiency, HAP is appealing due to its similarity to the single facility iteration procedure so widely used. Also, HAP takes advantage of the specialized structure of the location problem. Furthermore, no background in nonlinear programming is required to apply HAP in solving location problems.

Based on Eqs. [17], [18], and [19], $(x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$ is the optimum solution to P3 if and only if the following convergent solution is obtained, $(x_1^{(h+1)}, y_1^{(h+1)}, \cdots, x_n^{(h+1)}, y_n^{(h+1)}) = (x_1^{(h)}, y_1^{(h)}, \cdots, x_n^{(h)}, y_n^{(h)}) = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$. To establish the claim first let $(x_1^*, y_1^*, \cdots, x_n^*, \cdots, x_n^*, y_n)$ be the optimum solution to P3. Since $\hat{f}$ is a convex and differentiable function, all first partials evaluated at the optimum solution will equal zero. Hence, from Eqs. [17] and [19] a convergent solution is obtained. Next, suppose the following convergent solution is obtained, $(x_1^{(h+1)} y_1^{(h+1)}, \cdots, x_n^{(h+1)} y_n^{(h+1)}) = (x_1^{(h)}, y_1^{(h)}, \cdots, x_n^{(h)}, y_n^{(h)}) = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$. Therefore, from Eqs. [17] and [18], $\rho^{(h)}\, \partial\hat{f}/\partial x_j^{(h)} = 0$. But, from Eq. [18], $\rho^{(h)} > 0$. Consequently, it must be true that the convergent solution is a stationary point. Since $\hat{f}$ is a convex function the convergent solution is the optimum solution.

It remains to show that HAP converges. Katz (7) and Weiszfeld (14) have established the convergence of the iterative procedure for the case of a single new facility. Based on considerable computational experience, it is believed that HAP always converges. However, no proof of convergence exists for the multifacility problem.

Example Problem

As an illustration of the use of the hyperboloid approximation procedure, consider a location problem involving two new facilities and five existing facilities. Since distances between facilities are Euclidean, the facilities might be machines connected by straight-line conveyors; military bases connected by air travel; or industrial plants and warehouses where straight line travel can be reasonably approximated. For the example, assume the existing facilities are located at the coordinate points:
$P_1 = (0,0)$; $P_2 = (2,4)$; $P_3 = (6,2)$; $P_4 = (6,10)$; and $P_5 = (8,8)$.
Also, let $v_{12} = 2$ and

$$W = (w_{ji}) = \begin{pmatrix} 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 1 & 3 & 2 \end{pmatrix}$$

With a starting location of $X_1^0 = (0,0)$ and $X_2^0 = (0,0)$ and a hyperboloid constant of $\epsilon = 10^{-4}$, the successive values of X_1 and X_2 shown in Table 1 are obtained using Eqs [14] and [15].

Table 1: Iterative solution to the example problem

h	$X_1^{(h)}$	$X_2^{(h)}$	$f(X)$
0	(0.0, 0.0)	(0.0, 0.0)	100.800
1	(0.006, 0.005)	(0.036, 0.043)	100.458
2	(0.015, 0.012)	(0.106, 0.128)	99.795
3	(0.026, 0.022)	(0.328, 0.407)	97.713
4	(0.044, 0.038)	(0.980, 1.235)	91.692
5	(0.072, 0.062)	(2.223, 2.863)	80.799
6	(0.117, 0.102)	(2.936, 4.158)	75.853
7	(0.185, 0.166)	(3.132, 4.526)	75.113
8	(0.292, 0.264)	(3.408, 4.738)	74.406
9	(0.455, 0.407)	(3.677, 4.934)	73.586
10	(0.688, 0.612)	(3.914, 5.119)	72.615
15	(2.411, 2.226)	(4.647, 5.762)	67.690
20	(2.816, 2.674)	(4.921, 6.092)	67.264
30	(2.839, 2.687)	(5.087, 6.327)	67.240
40	(2.840, 2.687)	(5.121, 6.376)	67.239
45	(2.840, 2.687)	(5.126, 6.383)	67.239*

*The SC-I stopping criterion, discussed subsequently, was used to terminate the solution.

Distance Formulation

A generalized distance formulation of P1 can be obtained by incorporating the hyperboloid constant, ϵ. The resulting formulation becomes

P4. *Minimize* $f(X) = \sum_{1 \le j < k \le n} v_{jk} \hat{d}(x_j, x_k) + \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} \hat{d}(x_j, P_i)$

where

$$\hat{d}(x_j, x_k) = \begin{cases} [(x_j - x_k)^2 + (y_j - y_k)^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(x_j - x_k)^2 + \epsilon]^{1/2} + [(y_j - y_k)^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

and

$$\hat{d}(x_j, P_i) = \begin{cases} [(x_j - a_i)^2 + (y_j - b_i)^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(x_j - a_i)^2 + \epsilon]^{1/2} + [(y_j - b_i)^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

The HAP iteration equations reduce to

$$x_j^{(h+1)} = \left[\sum_{\substack{k=1 \\ \neq j}}^{n} v_{jk} x_k^{(h)} / \hat{D}_{jk}^{(h)}(x) + \sum_{i=1}^{m} w_{ji} a_i / \hat{E}_{ji}^{(h)}(x) \right] \Bigg/ \left[\sum_{\substack{k=1 \\ \neq j}}^{n} v_{jk} / \hat{D}_{jk}^{(h)}(x) + \sum_{i=1}^{m} w_{ji} / \hat{E}_{ji}^{(h)}(x) \right] \quad [20]$$

and

$$y_j^{(h+1)} = \left[\sum_{\substack{k=1 \\ \neq j}}^{n} v_{jk} y_k^{(h)} / \hat{D}_{jk}^{(h)}(y) + \sum_{i=1}^{m} w_{ji} b_i / \hat{E}_{ji}^{(h)}(y) \right] \Bigg/ \left[\sum_{\substack{k=1 \\ \neq j}}^{n} v_{jk} / \hat{D}_{jk}^{(h)}(y) + \sum_{i=1}^{m} w_{ji} / \hat{E}_{ji}^{(h)}(y) \right] \quad [21]$$

where

$$\hat{D}_{jk}^{(h)}(x) = \begin{cases} [(x_j^{(h)} - x_k^{(h)})^2 + (y_j^{(h)} - y_k^{(h)})^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(x_j^{(h)} - x_k^{(h)})^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

$$\hat{E}_{ji}^{(h)}(x) = \begin{cases} [(x_j^{(h)} - a_i)^2 + (y_j^{(h)} - b_i)^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(x_j^{(h)} - a_i)^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

$$\hat{D}_{jk}^{(h)}(y) = \begin{cases} [(x_j^{(h)} - x_k^{(h)})^2 + (y_j^{(h)} - y_k^{(h)})^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(y_j^{(h)} - y_k^{(h)})^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

and

$$\hat{E}_{ji}^{(h)}(y) = \begin{cases} [(x_j^{(h)} - a_i)^2 + (y_j^{(h)} - b_i)^2 + \epsilon]^{1/2} & \text{if Euclidean} \\ [(y_j^{(h)} - b_i)^2 + \epsilon]^{1/2} & \text{if rectilinear} \end{cases}$$

The generalized distance formulation of the multifacility location problem allows a consideration of situations in which some distances are Euclidean and others are rectilinear, as well as the more common case where all distances are either rectilinear or Euclidean. When all distances are rectilinear, P4 reduces to the formulation treated by Wesolowsky and Love (16).

Computational Experience

Although the convergence of HAP has not been established mathematically, our computational experience has failed to produce a problem in which HAP has not converged. Since we have shown that if a convergent solution is obtained it is an optimum solution, an optimum solution has been obtained for all problems solved.

A stopping criterion, which is more efficient than the one based on successive changes in the locations of the new facilities (convergent solution), is one based on successive changes in the value of the objective function. The latter stopping criterion, even though commonly used in conjunction with search procedures, can produce solutions in which the new facilities are far removed from their optimum locations. However, the resulting objective function value is approximately equal to the minimum value. For all problems solved, the stopping criterion based on successive changes in the value of the objective function yielded solutions having objective function values approximately the same as were obtained using a stopping criterion based on successive changes in the locations of the new facilities. However, it has not been proved that a convergent solution is optimum when a stopping criterion based on successive changes in the value of the objective function is used. A comparison of computational efficiencies based on the two stopping criteria is given subsequently.

It has been observed that the larger the value of ϵ the faster the convergence to the optimum value of the approximating function; the accuracy of the approximation decreases with an increasing value of ϵ. Consequently, in solving location problems using HAP a large value of ϵ is used initially; the solution obtained serves as the starting solution for a problem involving a smaller value of ϵ; the process is continued by successively reducing the value of ϵ until a convergent solution is obtained. Specifically, HAP is programmed using an initial value of ϵ equal to 10^{-2}. After each complete iteration the value of ϵ is decreased by 10^{-2} until a final value of 10^{-12} is assigned to ϵ.

A large number of sample problems have been run to test the convergence and solution efficiency of HAP. In addition to solving example problems provided in the literature by Love (9), Wesolowsky and Love (14), Pritsker and Ghare (11), and Rao (12), over 100 problems were generated randomly using the following probability distributions:

$$p(n) = \frac{1}{49} \quad ; \quad n = 2, 3, \ldots, 50$$

$$p(m|n) = \frac{1}{51-n} \quad ; \quad m = n, n+1, \ldots, 50$$

$$p(v_{jk}) = \begin{cases} 0.40 & ; \ v_{jk} = 0 \\ 0.05 & ; \ v_{jk} = 1, 2, \ldots, 12 \end{cases}$$

$$p(w_{ji}) = \begin{cases} 0.300 & ; \ w_{ji} = 0 \\ 0.028 & ; \ w_{ji} = 1, 2, \ldots, 25 \end{cases}$$

$$f(a_i) = 0.04 \quad ; \ 0 < a_i < 25$$

$$f(b_i) = 0.04 \quad ; \ 0 < b_i < 25$$

Table 2: HAP solution times for Euclidean distance problems based on samples of 5

n	m	Mean Time (seconds)	Maximum (seconds)	Minimum (seconds)
3	20	2.86 (2.12)*	5.17 (3.87)	1.84 (1.41)
3	31	2.97 (2.52)	4.56 (3.96)	1.83 (1.57)
5	12	4.45 (3.77)	6.09 (5.92)	2.43 (1.98)
5	42	7.74 (5.04)	12.83 (6.32)	5.90 (4.45)
7	27	11.39 (7.07)	14.79 (7.61)	7.71 (5.45)
9	16	16.47 (12.23)	19.69 (15.90)	15.70 (9.97)
15	48	51.74 (34.79)	80.49 (57.73)	37.13 (23.53)
18	38	73.44 (54.76)	104.77 (78.34)	56.49 (37.58)
25	44	167.05 (126.43)	227.42 (135.69)	88.26 (105.79)
49	50	426.01 (415.52)	533.08 (510.00)	299.58 (285.68)

*Numbers in parentheses refer to SC-II times.

In Table 2 representative solution times are given for the Euclidean distance location problem. The entries are based on samples of size 5 for each combination of n and m shown and were determined using an IBM 370/155 computer. Two stopping criteria were examined. Stopping Criterion (SC) I was based on successive changes in the locations of new facilities; Stopping Criterion (SC) II was based on successive changes in the value of the objective function. Specifically, the search terminated under SC I when

$$|x_j^{(h)} - x_j^{(h-1)}| \leq .001 \qquad j = 1, \ldots, n,$$

and

$$|y_j^{(h)} - y_j^{(h-1)}| \leq .001 \qquad j = 1, \ldots, n.$$

Under SC II the search terminated when

$$|\hat{f}^{(h)} - \hat{f}^{(h-1)}| \leq 10^{-6} \hat{f}^{(h)}$$

Also, sample problems have been solved for both the case of rectilinear distances and the case of a mixture of rectilinear and Euclidean distances. Solution times do not appear to be significantly affected by the distance measure employed.[1]

Acknowledgement

The authors wish to express their appreciation to Dr. Richard L. Francis of the University of Florida for encouraging an investigation of a multifacility extension to the single facility iterative procedure.

[1]A listing of the HAP computer program can be obtained by contacting the authors.

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