Trajectory Generation with Tension constraints
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r	derivative to minimize in cost function
	implies r initial conditions at each keyframe (position and $r-1$ derivatives, indexed from 0 (constant term)
n	order of desired trajectory, minimum order is $2r-1$
	implies $n+1$ coefficients, indexed from 0 (constant term)
d	number of dimensions to optimize, indexed from 1
	for example, $d=3$ when optimizing $[xyz]$ position of a trajectory,
m	number of pieces in trajectory
	implies $m+1$ keyframes in trajectory, indexed from 1
t_{des}	vertical vector of desired arrival times at keyframes, indexed from 0
p_{des}	matrix of desired positions, each row represents a derivative, each column represents a keyframe
	Inf represents unconstrained

1 Spline Interpolation

To analytically solve for the piecewise cubic spline $X(t)$ of m pieces going through positions $p_{des} = [X(t_0) \;\; X(t_1) \;\; X(t_2) \;\; ... \;\; X(t_m)]^T$:

$$X(t) = \begin{cases} X_1(t) = c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \le t < t_1 \\ X_2(t) = c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + ... + c_{2,0}, & t_1 \le t < t_2 \\ ... \\ X_m(t) = c_{m,3}t^3 + c_{m,2}t^2 + c_{m,1}t + ... + c_{m,0}, & t_{m-1} \le t < t_m \end{cases}$$

We can set up a system of $4m$ equations to solve for each of the $4m$ coefficients.

$2m$ Position Constraints:

$$\begin{aligned} X_1(t_0) &= X(t_0) \\ X_1(t_1) &= X(t_1) \\ X_2(t_1) &= X(t_1) \\ X_2(t_2) &= X(t_2) \\ &... \\ X_m(t_{m-1}) &= X(t_{m-1}) \\ X_m(t_m) &= X(t_m) \end{aligned}$$

$m-1$ Velocity Constraints:

$$\begin{aligned} \dot{X}_1(t_1) &= \dot{X}_2(t_1) \\ \dot{X}_2(t_2) &= \dot{X}_3(t_2) \\ &... \\ \dot{X}_{m-1}(t_{m-1}) &= \dot{X}_m(t_{m-1}) \end{aligned}$$

$m-1$ Acceleration Constraints:

$$\begin{aligned} \ddot{X}_1(t_1) &= \ddot{X}_2(t_1) \\ \ddot{X}_2(t_2) &= \ddot{X}_3(t_2) \\ &... \\ \ddot{X}_{m-1}(t_{m-1}) &= \ddot{X}_m(t_{m-1}) \end{aligned}$$

2 Endpoint Constraints (for example, velocity):

$$\begin{aligned} \dot{X}_1(t_0) &= \dot{X}(t_0) \\ \dot{X}_m(t_m) &= \dot{X}(t_m) \end{aligned}$$

The resulting $X(t)$ corresponds to the solution to the optimization problem of finding $X = [c_{1,3} \;\; c_{1,2} \;\; c_{1,1} \;\; c_{1,0} \;\; ... \;\; c_{m,0} \;\;]^T$ that minimizes the cost functional $J = \int_{t_0}^{t_1} \| \frac{d^2X(t)}{dt} \|^2 dt$ subject to $3m+1$ equality constraints $Ax = b$, where the equality constraints come from position constraints, endpoint constraints, and velocity continuity constraints.

In the general case, the minimum-order of the piece-wise polynomial to minimize the cost functional $J = \int_{t_0}^{t_1} \| \frac{d^{(r)}X(t)}{dt} \|^2 dt$ is $n = 2r - 1$. To analytically solve for the coefficients $X = [c_{1,n} \;\; c_{1,n-1} \;\; c_{1,n-2} \;\; ... \;\; c_{1,0} \;\;]^T$, we need $(n+1)m$ constraints. These constraints come from:

$2m$ Position Constraints
$(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $r-1$
$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$

This gives a total of $2m+(m-1)(r-1)+2(r-1) = 2m+(m-1)(\frac{n+1}{2}-1)+2(\frac{n+1}{2}-1) = \frac{mn}{2}+\frac{m}{2}+m+\frac{n}{2}-\frac{1}{2}$ constraints. We thus need $((n+1)m) - \left(\frac{mn}{2}+\frac{m}{2}+m+\frac{n}{2}-\frac{1}{2}\right) = (\frac{n-1}{2})(m-1) = (r-1)(m-1)$ more constraints. This corresponds to constraining derivatives at intermediate points to be continuous up until the $2(r-1)$ derivative.

To solve for the minimum-order piecewise polynomial that of minimizes the cost functional of the rth derivative, we solve for coefficients using the constraints:

$2m$ Position Constraints
$2(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $2(r-1)$
$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$

2 Trajectory Generation

We aim to design a 1-dimensional desired trajectory for a load with keyframe positions $p_{L,des} = [X_L(t_0) \;\; X_L(t_1) \;\; X_L(t_2) \;\; ... \;\; X_L(t_m)]^T$ at times $t_{des} = [t_0 \;\; t_1 \;\; t_2 \;\; ... \;\; t_m]^T$ with tensions $T_{des} = [T_0 \;\; T_1 \;\; T_2 \;\; ... \;\; T_m]^T$. For m keyframes, we want a piecewise trajectory $X_L(t)$:

$$X_L(t) = \begin{cases} X_{L,1}(t) = c_{1,n}t^n + c_{1,n-1}t^{n-1} + c_{1,n-2}t^{n-2} + ... + c_{1,0}, & t_0 \leq t < t_1 \\ X_{L,2}(t) = c_{2,n}t^n + c_{2,n-1}t^{n-1} + c_{2,n-2}t^{n-2} + ... + c_{2,0}, & t_1 \leq t < t_2 \\ ... \\ X_{L,m}(t) = c_{m,n}t^n + c_{m,n-1}t^{n-1} + c_{m,n-2}t^{n-2} + ... + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

We can solve for this by first finding a piecewise trajectory for the quadrotor, $X(t)$:

$$X(t) = \begin{cases} X_1(t) = c_{1,n}t^n + c_{1,n-1}t^{n-1} + c_{1,n-2}t^{n-2} + ... + c_{1,0}, & t_0 \leq t < t_1 \\ X_2(t) = c_{2,n}t^n + c_{2,n-1}t^{n-1} + c_{2,n-2}t^{n-2} + ... + c_{2,0}, & t_1 \leq t < t_2 \\ ... \\ X_m(t) = c_{m,n}t^n + c_{m,n-1}t^{n-1} + c_{m,n-2}t^{n-2} + ... + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

The quadrotor and load states are related by:

$$X(t) = X_L(t) + l$$
$$X^{(i)}(t) = X_L^{(i)}(t)$$

For continuity, these relations hold at all keyframes, regardless of T_{des}. This implies that the new position constraints for $X(t)$ are $p_{des} = [X(t_0) \;\; X(t_1) \;\; X(t_2) \;\; ... \;\; X(t_m)]^T =$ $p_{L,des} = [X_L(t_0) + l \;\; X_L(t_1) + l \;\; X_L(t_2) + l \;\; ... \;\; X(t_m) + l]^T$.

The vector T_{des} is defined such that values are either ∞ or 0, indicating a non-zero or zero tension value, respectively. For trajectories $X_{L,i}(t)$ where $T_i > 0$, we want to minimize the 6th derivative of $X_{L,i}(t)$. For trajectory pieces $X_{L,i}(t)$ where $T_i = 0$, the quadrotor trajectory is independent of the load trajectory and we want to minimize the 4th derivative.

Let $x(t)$ be the non-dimensionalized quadrotor trajectory:

$$x(\tau) = \begin{cases} x_1(\tau) = c_{1,n}\tau^n + c_{1,n-1}\tau^{n-1} + ...c_{1,1}\tau + c_{1,0}, & t_0 \leq t < t_1, \tau = \frac{t-t_0}{t_1-t_0} \\ x_2(\tau) = c_{2,n}\tau^n + c_{2,n-1}\tau^{n-1} + ...c_{2,1}\tau + c_{2,0}, & t_1 \leq t < t_2, \tau = \frac{t-t_1}{t_2-t_1} \\ ... \\ x_m(\tau) = c_{m,n}\tau^n + c_{m,n-1}\tau^{n-1} + ...c_{m,1}\tau + c_{m,0}, & t_{m-1} \leq t < t_m, \tau = \frac{t-t_{m-1}}{t_m-t_{m-1}} \end{cases} , 0 \leq \tau < 1$$

Assume there is only one keyframe s where $T_s = 0$. Let the state vector $x = [x_I \;\; x_{II} \;\; x_{III}]^T$, where $x_I = [c_{1,n} \;\; c_{1,n-1} \;\; ... \;\; c_{s,0}]^T$, $x_{II} = [c_{s+1,n} \;\; c_{s+1,n-1} \;\; ... \;\; c_{s+1,0}]^T$, $x_{III} = [c_{s+2,n} \;\; c_{s+2,n-1} \;\; ... \;\; c_{s+2,0}]^T$, and

$x = [c_{1,n} \;\; c_{1,n-1} \;\; ... \;\; c_{m,1} \;\; ... \;\; c_{m,0}]^T$, $x_k = [c_{k,n} \;\; c_{k,n-1} \;\; ... \;\; c_{k,0}]^T$. The cost function to minimize is then:

$$
\begin{aligned}
J &= \int_{t_0}^{t_m} \|\frac{d^6 X_I(t)}{dt}\|^2 + \|\frac{d^4 X_{II}(t)}{dt}\|^2 + \|\frac{d^6 X_{III}(t)}{dt}\|^2 dt \\
&= \sum_{k=1}^{s} \int_{t_{k-1}}^{t_k} \|\frac{d^6 X_k(t)}{dt}\|^2 dt + \sum_{k=s+1}^{s+1} \int_{t_{k-1}}^{t_k} \|\frac{d^4 X_k(t)}{dt}\|^2 dt + \sum_{k=s+2}^{m} \int_{t_{k-1}}^{t_k} \|\frac{d^6 X_k(t)}{dt}\|^2 dt \\
&= \sum_{k=1}^{s} \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^{11}} \|\frac{d^6 x_k(\tau)}{d\tau}\|^2 d\tau + \\
&\sum_{k=s+1}^{s+1} \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^{7}} \|\frac{d^4 x_k(\tau)}{d\tau}\|^2 d\tau + \sum_{k=s+2}^{m} \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^{11}} \|\frac{d^6 x_k(\tau)}{d\tau}\|^2 d\tau \\
&= \sum_{k=1}^{s} x_k^T \frac{1}{(t_k - t_{k-1})^{11}} Q_{(0,1)} x_k + \sum_{k=s+1}^{s+1} x_k^T \frac{1}{(t_k - t_{k-1})^{7}} Q_{(0,1)} x_k + \sum_{k=s+2}^{m} x_k^T \frac{1}{(t_k - t_{k-1})^{11}} Q_{(0,1)} x_k \\
&= x^T Q x
\end{aligned}
$$

subject to: $Ax = b$

TO FIND Q:
When $x' = [c_0 \;\; c_1 \;\; ... \;\; c_{n-1} \;\; c_n]^T$, we can find $Q'_{(0,1)}$ with:

$$
Q'[i,j]_{(t_0,t_1)} = \begin{cases} \prod_{k=0}^{r-1} (i-k)(j-k) \frac{t_1^{i+j-2r+1} - t_0^{i_j-2r+1}}{i+j-2r+1}, & i \geq r \wedge j \geq r \\ 0, & i < r \vee j < r \end{cases}, i = 0...n, j = 0...n \tag{1}
$$

Since our $x_k = [c_{k,n} \;\; c_{k,n-1} \;\; ... \;\; c_{k,1} \;\; c_{k,0}]^T$, reflecting Q' horizontally and vertically will give us the desired Q for the form of x_k. We can then create the block diagonal matrix:

$$
Q = \begin{bmatrix} \frac{1}{(t_1-t_0)^{2r-1}} Q_{(0,1)} & 0 & 0 & ... & 0 \\ 0 & \frac{1}{(t_2-t_1)^{2r-1}} Q_{(0,1)} & 0 & ... & 0 \\ & & ... & & \\ 0 & ... & 0 & \frac{1}{(t_{m-1}-t_{m-2})^{2r-1}} Q_{(0,1)} & 0 \\ 0 & ... & 0 & 0 & \frac{1}{(t_m-t_{m-1})^{2r-1}} Q_{(0,1)} \end{bmatrix}, \tag{2}
$$

where $r = 6$ if $k \neq s+1$ and $r = 4$ if $k = s+1$.

TO FIND A_{EQ}:

At keyframes 0 and m, we have a full set of constraints for position and derivatives 1-5. At t_s and t_{s+1}, where $T_s = 0$, $\ddot{X} = -g$, $X^{(3)} = X^{(4)} = X^{(5)} = 0$. At all other points, the keyframe positions are constrained.While $X(t_i)$ is specified for $i \neq s+1$, $X(t_{s+1})$ is calculated from free fall conditions of the load. The equation for free fall of the load is:

$$
X_L(t_{s+1}) = -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}_L(t_s)(t_{s+1} - t_s) + X_L(t_s)
$$

Translating this to the quadrotor equation:

$$
\begin{aligned}
X(t_{s+1}) &= X_L(t_{s+1}) + l \\
&= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}_L(t_s)(t_{s+1} - t_s) + X_L(t_s) + l \\
\text{From } X = X_L + l \text{ and } \dot{X} = \dot{X}_L & \\
&= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}(t_s)(t_{s+1} - t_s) + X(t_s)
\end{aligned}
$$

Nondimensionalizing with:

$$
\begin{aligned}
X(t_{s+1}) = x_{s+1}(\tau_1) &= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}(t_s)(t_{s+1} - t_s) + X(t_s) \\
&= -\frac{g}{2}(t_{s+1} - t_s)^2 + \frac{\dot{x}_{s+1}(\tau_0)}{t_{s+1} - t_s}(t_{s+1} - t_s) + x_{s+1}(\tau_0) \\
&= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{x}_{s+1}(\tau_0) + x_{s+1}(\tau_0)
\end{aligned}
$$

At keyframe t_{s+1} when the quad and load collide, there is a change in load velocity from the collision. Assuming a completely inelastic collision:

$$
\begin{aligned}
(\dot{X}^+ - \dot{X}_L^+) &= e(\dot{X}^- + \dot{X}_L^-) \\
m_L\dot{X}_L^+ + m\dot{X}^+ &= m_L\dot{X}_L^- + m\dot{X}^- \\
\text{For } e = 0, & \\
\dot{X}^+ &= \dot{X}_L^+ \\
m_L\dot{X}_L^- + m\dot{X}^- &= (m_L + m)\dot{X}^+ \\
\text{If } m >> m_L,\ (m_L + m) \approx m, m_L \approx 0 & \\
\dot{X}^+ &= \dot{X}^-
\end{aligned}
$$

We want the quadrotor velocity at t_{s+1} to be continuous and also match the load velocity at that time for a smooth transition. At the end of free-fall, the load velocity is:

$$
\begin{aligned}
\dot{X}_L(t_{s+1}) &= -g(t_{s+1} - t_s) + \dot{X}_L(t_s) \\
\frac{\dot{x}_{L,s+1}(\tau_1)}{t_{s+1} - t_s} &= -g(t_{s+1} - t_s) + \frac{\dot{x}_{L,s+1}(\tau_0)}{t_{s+1} - t_s} \\
\dot{x}_{L,s+1}(\tau_1) &= -g(t_{s+1} - t_s)^2 + \dot{x}_{L,s+1}(\tau_0) \\
&= -g(t_{s+1} - t_s)^2 + \dot{x}_{s+1}(\tau_0)
\end{aligned}
$$

We thus want to set

$$
\begin{aligned}
&\dot{x}_{s+1}(\tau_1) = \dot{x}_{s+2}(\tau_0) = \dot{x}_{L,s+1}(\tau_1) = -g(t_{s+1} - t_s)^2 + \dot{x}_{L,s+1}(\tau_0) \\
&\dot{x}_{s+1}(\tau_1) - \dot{x}_{s+1}(\tau_0) = -g(t_{s+1} - t_s)^2 \\
&\dot{x}_{s+2}(\tau_0) - \dot{x}_{s+1}(\tau_0) = -g(t_{s+1} - t_s)^2
\end{aligned}
$$

There $26 + 2m$ constraints - 6 from each endpoint, $2(m-1)$ additional position constraints, (4)(4) constraints to account for constraints on derivatives 2-5 at t_s and t_{s+1}. They are:

$$A_{endpoint}x = b_{endpoint}$$

$$\begin{bmatrix} x_1(\tau_0) \\ \dot{x}_1(\tau_0) \\ \ddot{x}_1(\tau_0) \\ x_1^{(3)}(\tau_0) \\ x_1^{(4)}(\tau_0) \\ x_1^{(5)}(\tau_0) \\ x_1(\tau_1) \\ x_2(\tau_0) \\ x_2(\tau_1) \\ x_3(\tau_0) \\ ... \\ x_s(\tau_0) \\ x_s(\tau_1) \\ \ddot{x}_s(\tau_1) \\ x_s^{(3)}(\tau_1) \\ x_s^{(4)}(\tau_1) \\ x_s^{(5)}(\tau_1) \\ x_{s+1}(\tau_0) \\ \ddot{x}_{s+1}(\tau_0) \\ x_{s+1}^{(3)}(\tau_0) \\ x_{s+1}^{(4)}(\tau_0) \\ x_{s+1}^{(5)}(\tau_0) \\ x_{s+1}(\tau_1) - \dot{x}_{s+1}\tau_0(\tau_1-\tau_0) - x_{s+1}(\tau_0) \\ \ddot{x}_{s+1}(\tau_1) \\ x_{s+1}^{(3)}(\tau_1) \\ x_{s+1}^{(4)}(\tau_1) \\ x_{s+1}^{(5)}(\tau_1) \\ x_{s+2}(\tau_0) - \dot{x}_{s+1}\tau_0(\tau_1-\tau_0) - x_{s+1}(\tau_0) \\ \ddot{x}_{s+2}(\tau_0) \\ x_{s+2}^{(3)}(\tau_0) \\ x_{s+2}^{(4)}(\tau_0) \\ x_{s+2}^{(5)}(\tau_0) \\ x_{s+2}(\tau_1) \\ x_{s+3}(\tau_0) \\ ... \\ x_m(\tau_0) \\ x_m(\tau_1) \\ \dot{x}_m(\tau_1) \\ \ddot{x}_m(\tau_1) \\ x_m^{(3)}(\tau_1) \\ x_m^{(4)}(\tau_1) \\ x_m^{(5)}(\tau_1) \end{bmatrix} = \begin{bmatrix} X(t_0) \\ (t_1-t_0)\dot{X}(t_0) \\ (t_1-t_0)^{(2)}X^{(2)}(t_0) \\ (t_1-t_0)^{(3)}X^{(3)}(t_0) \\ (t_1-t_0)^{(4)}X^{(4)}(t_0) \\ (t_1-t_0)^{(5)}X^{(5)}(t_0) \\ X(t_1) \\ X(t_1) \\ X(t_2) \\ X(t_2) \\ ... \\ X(t_{s-1}) \\ X(t_s) \\ -g(t_s-t_{s-1})^{(2)} \\ 0 \\ 0 \\ 0 \\ X(t_s) \\ -g(t_{s+1}-t_s)^{(2)} \\ 0 \\ 0 \\ 0 \\ -\frac{g}{2}(\tau_1-\tau_0)^2(t_{s+1}-t_s)^2 \\ -g(t_{s+1}-t_s)^{(2)} \\ 0 \\ 0 \\ 0 \\ -\frac{g}{2}(\tau_1-\tau_0)^2(t_{s+1}-t_s)^2 \\ -g(t_{s+2}-t_{s-1})^{(2)} \\ 0 \\ 0 \\ 0 \\ X(t_{s+2}) \\ X(t_{s+2}) \\ ... \\ X(t_{m-1}) \\ X(t_m) \\ (t_m-t_{m-1})\dot{X}(t_m) \\ (t_m-t_{m-1})^{(2)}X^{(2)}(t_m) \\ (t_m-t_{m-1})^{(3)}X^{(3)}(t_m) \\ (t_m-t_{m-1})^{(4)}X^{(4)}(t_m) \\ (t_m-t_{m-1})^{(5)}X^{(5)}(t_m) \end{bmatrix} \tag{3}$$

We also need continuity of derivatives lower than 6 at all intermediate keyframes, which gives an additional $5(m-3)+2$ constraints - $5(m-3)$ continuity constraints at each keyframe i such that $i \neq 0, m, ss+1$ and

2 for velocity continuity at t_s and t_{s+1}. Explicitly:

$$A_{cont}x = b_{cont}$$

$$\begin{bmatrix} \frac{1}{(t_1-t_0)}\dot{x}_1(\tau_1) - \frac{1}{(t_2-t_1)}\dot{x}_2(\tau_0) \\ \dots \\ \frac{1}{(t_1-t_0)^{(5)}}x_1^{(5)}(\tau_1) - \frac{1}{(t_2-t_1)^{(5)}}x_2^{(5)}(\tau_0) \\ \dots \\ \frac{1}{(t_{s-1}-t_{s-2})^{(5)}}x_{s-1}^{(5)}(\tau_1) - \frac{1}{(t_s-t_{s-1})^{(5)}}x_s^{(5)}(\tau_0) \\ \frac{1}{(t_s-t_{s-1})}\dot{x}_s(\tau_1) - \frac{1}{(t_{s+1}-t_s)}\dot{x}_{s+1}(\tau_0) \\ \frac{1}{(t_{s+1}-t_s)}\dot{x}_{s+1}(\tau_1) - \frac{1}{(t_{s+2}-t_{s+1})}\dot{x}_{s+2}(\tau_0) \\ \frac{1}{(t_{s+2}-t_{s+1})}\dot{x}_{s+2}(\tau_1) - \frac{1}{(t_{s+3}-t_{s+2})}\dot{x}_{s+3}(\tau_0) \\ \dots \\ \frac{1}{(t_{s+2}-t_{s+1})^{(5)}}x_{s+2]}^{(5)}(\tau_1) - \frac{1}{(t_{s+3}-t_{s+2})^{(5)}}x_{s+3}^{(5)}(\tau_0) \\ \dots \\ \frac{1}{(t_{m-1}-t_{m-2})^{(5)}}x_{m-1}^{(5)}(\tau_1) - \frac{1}{(t_m-t_{m-1})^{(5)}}x_m^{(5)}(\tau_0) \end{bmatrix} = 0$$

The full set of $7m + 13$ constraints, $Ax = b$, are:

$$Ax = b$$
$$\begin{bmatrix} A_{endpoint} \\ A_{cont} \end{bmatrix} x = \begin{bmatrix} b_{endpoint} \\ 0 \end{bmatrix} \tag{4}$$

Note that for m pieces, we have $12m$ coefficients. Thus, we are missing $5m - 13$ constraints for a fully constrained system.

TO FIND A_{INEQ}:
We need 3 sets of inequality constraints. In general, we choose N_c intermediate points between t_f and t_i, located at times $t_c = t_i + \frac{k}{N_c+1}$ for $k = 1, 2, ..., N_c$ and evaluate the inequality constraints at these points. Note that the inequality constraints are not evaluated at trajectory endpoints.

When the rope is taut, tension at t_c is always greater than 0, or $T_i(t_c) > 0$ for all $i \neq s+1$:

$$\begin{aligned} T &> 0 \\ -T &< 0 \\ -(m_L(\ddot{X}_L(t) + g) &\leq -\epsilon \\ -(m_L(\frac{\ddot{x}(t_c)}{(t_f - t_i)^2} + g) &\leq -\epsilon \\ -\ddot{x}(t_c) &\leq (-\frac{\epsilon}{m_L} + g)(t_f - t_i)^2 \end{aligned}$$

During this free-fall phase, the position of the load $X_L(t_c)$ can be explicitly calculated as a function of time and the quadrotor state at $X(t_s)$:

$$\begin{aligned}
X_L(t_c) &= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}_L(t_i)(t_c - t_i) + X_L(t_i) \\
&= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}(t_i)(t_c - t_i) + X(t_i) - l
\end{aligned}$$

$$\tau = \frac{t - t_i}{t_f - t_i}, t = \tau(t_f - t_i) + t_i$$

$$\begin{aligned}
t_c &= t_i + \frac{p}{N_c + 1}(t_f - t_i) \\
&= (\tau_0(t_f - t_i) + t_i) + \frac{p}{N_c + 1}\left((\tau_1(t_f - t_i) + t_i) - (\tau_0(t_f - t_i) + t_i)\right) \\
&= \tau_0(t_f - t_i) + t_i + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0) \\
t_c - t_i &= \tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0) \\
X_L(\tau_c) &= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}(t_i)(t_c - t_i) + X(t_i) - l \\
&= -\frac{g}{2}\left(\tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0)\right)^2 + \frac{\dot{x}(\tau_0)}{t_f - t_i}\left(\tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0)\right) + x(\tau_0) - l \\
&= -\frac{g}{2}\tau_c^2(t_f - t_i)^2 + \dot{x}(\tau_0)\tau_c + x(\tau_0) - l
\end{aligned}$$

When the rope is slack, the quad and load shouldn't collide with each other, giving the constraint:

$$\begin{aligned}
X(t_c) - X_L(t_c) &> 0 \\
X_L(t_c) - X(t_c) &\leq -\epsilon \\
-X(t_c) &\leq -\epsilon - X_L(t_c) \\
&\leq -\epsilon + \frac{g}{2}\tau_c^2(t_f - t_i)^2 - \dot{x}(\tau_0)\tau_c - x(\tau_0) + l \\
-x(\tau_c) + \dot{x}(\tau_0)\tau_c + x(\tau_0) &\leq -\epsilon + \frac{g}{2}\tau_c^2(t_f - t_i)^2 + l
\end{aligned}$$

In addition, the rope should remain slack:

$$\begin{aligned}
X(t_c) - X_L(t_c) &< l \\
X(t_c) - X_L(t_c) &\leq l - \epsilon \\
X(t_c) &\leq l - \epsilon + X_L(t_c) \\
&\leq l - \epsilon - \frac{g}{2}\tau_c^2(t_f - t_i)^2 + \dot{x}(\tau_0)\tau_c + x(\tau_0) - l \\
x(\tau_c) - \dot{x}(\tau_0)\tau_c - x(\tau_0) &\leq -\epsilon - \frac{g}{2}\tau_c^2(t_f - t_i)^2
\end{aligned}$$