

Trajectory Generation with Tension constraints

August 22, 2013

r	derivative to minimize in cost function
	implies r initial conditions at each keyframe (position and $r - 1$ derivatives, indexed from 0 (constant term)
n	order of desired trajectory, minimum order is $2r - 1$
	implies $n + 1$ coefficients, indexed from 0 (constant term)
d	number of dimensions to optimize, indexed from 1
	for example, $d = 3$ when optimizing $[xyz]$ position of a trajectory,
m	number of pieces in trajectory
	implies $m + 1$ keyframes in trajectory, indexed from 1
t_{des}	vertical vector of desired arrival times at keyframes, indexed from 0
p_{des}	matrix of desired positions, each row represents a derivative, each column represents a keyframe
	Inf represents unconstrained

1 Spline Interpolation

To analytically solve for the piecewise cubic spline $X(t)$ of m pieces going through positions $p_{des} = [X(t_0) \ X(t_1) \ X(t_2) \ \dots \ X(t_m)]^T$:

$$X(t) = \begin{cases} X_1(t) = c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \leq t < t_1 \\ X_2(t) = c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + \dots + c_{2,0}, & t_1 \leq t < t_2 \\ \dots \\ X_m(t) = c_{m,3}t^3 + c_{m,2}t^2 + c_{m,1}t + \dots + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

We can set up a system of $4m$ equations to solve for each of the $4m$ coefficients.

$2m$ Position Constraints:

$$\begin{aligned} X_1(t_0) &= X(t_0) \\ X_1(t_1) &= X(t_1) \\ X_2(t_1) &= X(t_1) \\ X_2(t_2) &= X(t_2) \\ &\dots \\ X_m(t_{m-1}) &= X(t_{m-1}) \\ X_m(t_m) &= X(t_m) \end{aligned}$$

$m - 1$ Velocity Constraints:

$$\begin{aligned} \dot{X}_1(t_1) &= \dot{X}_2(t_1) \\ \dot{X}_2(t_2) &= \dot{X}_3(t_2) \\ &\dots \\ \dot{X}_{m-1}(t_{m-1}) &= \dot{X}_m(t_{m-1}) \end{aligned}$$

$m - 1$ Acceleration Constraints:

$$\begin{aligned} \ddot{X}_1(t_1) &= \ddot{X}_2(t_1) \\ \ddot{X}_2(t_2) &= \ddot{X}_3(t_2) \\ &\dots \\ \ddot{X}_{m-1}(t_{m-1}) &= \ddot{X}_m(t_{m-1}) \end{aligned}$$

2 Endpoint Constraints (for example, velocity):

$$\begin{aligned} \dot{X}_1(t_0) &= \dot{X}(t_0) \\ \dot{X}_m(t_m) &= \dot{X}(t_m) \end{aligned}$$

The resulting $X(t)$ corresponds to the solution to the optimization problem of finding

$X = [c_{1,3} \ c_{1,2} \ c_{1,1} \ c_{1,0} \ \dots \ c_{m,0}]^T$ that minimizes the cost functional $J = \int_{t_0}^{t_1} \left\| \frac{d^{(r)}X(t)}{dt} \right\|^2 dt$ subject to $3m + 1$ equality constraints $Ax = b$, where the equality constraints come from position constraints, endpoint constraints, and velocity continuity constraints.

In the general case, the minimum-order of the piece-wise polynomial to minimize the cost functional

$J = \int_{t_0}^{t_1} \left\| \frac{d^{(r)}X(t)}{dt} \right\|^2 dt$ is $n = 2r - 1$. To analytically solve for the coefficients

$X = [c_{1,n} \ c_{1,n-1} \ c_{1,n-2} \ \dots \ c_{1,0}]^T$, we need $(n + 1)m$ constraints. These constraints come from:

$2m$ Position Constraints

$(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $r-1$

$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$

This gives a total of $2m + (m-1)(r-1) + 2(r-1) = 2m + (m-1)(\frac{n+1}{2} - 1) + 2(\frac{n+1}{2} - 1) = \frac{mn}{2} + \frac{m}{2} + m + \frac{n}{2} - \frac{1}{2}$ constraints. We thus need $((n+1)m) - (\frac{mn}{2} + \frac{m}{2} + m + \frac{n}{2} - \frac{1}{2}) = (\frac{n-1}{2})(m-1) = (r-1)(m-1)$ more constraints. This corresponds to constraining derivatives at intermediate points to be continuous up until the $2(r-1)$ derivative.

To solve for the minimum-order piecewise polynomial that of minimizes the cost functional of the r th derivative, we solve for coefficients using the constraints:

$2m$ Position Constraints

$2(m-1)(r-1)$ Constraints for continuity of derivatives 1 to $2(r-1)$

$2(r-1)$ Endpoint Constraints, for derivatives 1 to $r-1$

2 Trajectory Generation

We aim to design a 1-dimensional desired trajectory for a load with keyframe positions $p_{L,des} = [X_L(t_0) \ X_L(t_1) \ X_L(t_2) \ \dots \ X_L(t_m)]^T$ at times $t_{des} = [t_0 \ t_1 \ t_2 \ \dots \ t_m]^T$ with tensions $T_{des} = [T_0 \ T_1 \ T_2 \ \dots \ T_m]^T$. For m keyframes, we want a piecewise trajectory $X_L(t)$:

$$X_L(t) = \begin{cases} X_{L,1}(t) = c_{1,n}t^n + c_{1,n-1}t^{n-1} + c_{1,n-2}t^{n-2} + \dots + c_{1,0}, & t_0 \leq t < t_1 \\ X_{L,2}(t) = c_{2,n}t^n + c_{2,n-1}t^{n-1} + c_{2,n-2}t^{n-2} + \dots + c_{2,0}, & t_1 \leq t < t_2 \\ \dots \\ X_{L,m}(t) = c_{m,n}t^n + c_{m,n-1}t^{n-1} + c_{m,n-2}t^{n-2} + \dots + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

We can solve for this by first finding a piecewise trajectory for the quadrotor, $X(t)$:

$$X(t) = \begin{cases} X_1(t) = c_{1,n}t^n + c_{1,n-1}t^{n-1} + c_{1,n-2}t^{n-2} + \dots + c_{1,0}, & t_0 \leq t < t_1 \\ X_2(t) = c_{2,n}t^n + c_{2,n-1}t^{n-1} + c_{2,n-2}t^{n-2} + \dots + c_{2,0}, & t_1 \leq t < t_2 \\ \dots \\ X_m(t) = c_{m,n}t^n + c_{m,n-1}t^{n-1} + c_{m,n-2}t^{n-2} + \dots + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

The quadrotor and load states are related by:

$$\begin{aligned} X(t) &= X_L(t) + l \\ X^{(i)}(t) &= X_L^{(i)}(t) \end{aligned}$$

For continuity, these relations hold at all keyframes, regardless of T_{des} . This implies that the new position constraints for $X(t)$ are $p_{des} = [X(t_0) \ X(t_1) \ X(t_2) \ \dots \ X(t_m)]^T = p_{L,des} = [X_L(t_0) + l \ X_L(t_1) + l \ X_L(t_2) + l \ \dots \ X_L(t_m) + l]^T$.

The vector T_{des} is defined such that values are either ∞ or 0, indicating a non-zero or zero tension value, respectively. For trajectories $X_{L,i}(t)$ where $T_i > 0$, we want to minimize the 6th derivative of $X_{L,i}(t)$. For trajectory pieces $X_{L,i}(t)$ where $T_i = 0$, the quadrotor trajectory is independent of the load trajectory and we want to minimize the 4th derivative.

Let $x(t)$ be the non-dimensionalized quadrotor trajectory:

$$x(\tau) = \begin{cases} x_1(\tau) = c_{1,n}\tau^n + c_{1,n-1}\tau^{n-1} + \dots c_{1,1}\tau + c_{1,0}, & t_0 \leq t < t_1, \tau = \frac{t-t_0}{t_1-t_0} \\ x_2(\tau) = c_{2,n}\tau^n + c_{2,n-1}\tau^{n-1} + \dots c_{2,1}\tau + c_{2,0}, & t_1 \leq t < t_2, \tau = \frac{t-t_1}{t_2-t_1} \\ \dots \\ x_m(\tau) = c_{m,n}\tau^n + c_{m,n-1}\tau^{n-1} + \dots c_{m,1}\tau + c_{m,0}, & t_{m-1} \leq t < t_m, \tau = \frac{t-t_{m-1}}{t_m-t_{m-1}} \end{cases}, 0 \leq \tau < 1$$

Assume there is only one keyframe s where $T_s = 0$. Let the state vector $x = [x_I \ x_{II} \ x_{III}]^T$, where $x_I = [c_{1,n} \ c_{1,n-1} \ \dots \ c_{s,0}]^T$, $x_{II} = [c_{s+1,n} \ c_{s+1,n-1} \ \dots \ c_{s+1,0}]^T$, $x_{III} = [c_{s+2,n} \ c_{s+2,n-1} \ \dots \ c_{s+2,0}]^T$, and

$x = [c_{1,n} \ c_{1,n-1} \ \dots \ c_{m,1} \ \dots \ c_{m,0}]^T$, $x_k = [c_{k,n} \ c_{k,n-1} \ \dots \ c_{k,0}]^T$. The cost function to minimize is then:

$$\begin{aligned}
J &= \int_{t_0}^{t_m} \left\| \frac{d^6 X_I(t)}{dt} \right\|^2 + \left\| \frac{d^4 X_{II}(t)}{dt} \right\|^2 + \left\| \frac{d^6 X_{III}(t)}{dt} \right\|^2 dt \\
&= \sum_{k=1}^s \int_{t_{k-1}}^{t_k} \left\| \frac{d^6 X_k(t)}{dt} \right\|^2 dt + \sum_{k=s+1}^{s+1} \int_{t_{k-1}}^{t_k} \left\| \frac{d^4 X_k(t)}{dt} \right\|^2 dt + \sum_{k=s+2}^m \int_{t_{k-1}}^{t_k} \left\| \frac{d^6 X_k(t)}{dt} \right\|^2 dt \\
&= \sum_{k=1}^s \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^{11}} \left\| \frac{d^6 x_k(\tau)}{d\tau} \right\|^2 d\tau + \\
&\quad \sum_{k=s+1}^{s+1} \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^7} \left\| \frac{d^4 x_k(\tau)}{d\tau} \right\|^2 d\tau + \sum_{k=s+2}^m \int_0^1 \frac{t_k - t_{k-1}}{(t_k - t_{k-1})^{11}} \left\| \frac{d^6 x_k(\tau)}{d\tau} \right\|^2 d\tau \\
&= \sum_{k=1}^s x_k^T \frac{1}{(t_k - t_{k-1})^{11}} Q_{(0,1)} x_k + \sum_{k=s+1}^{s+1} x_k^T \frac{1}{(t_k - t_{k-1})^7} Q_{(0,1)} x_k + \sum_{k=s+2}^m x_k^T \frac{1}{(t_k - t_{k-1})^{11}} Q_{(0,1)} x_k \\
&= x^T Q x
\end{aligned}$$

subject to: $Ax = b$

TO FIND Q:

When $x' = [c_0 \ c_1 \ \dots \ c_{n-1} \ c_n]^T$, we can find $Q'_{(0,1)}$ with:

$$Q'[i, j]_{(t_0, t_1)} = \begin{cases} \prod_{k=0}^{r-1} (i-k)(j-k) \frac{t_1^{i+j-2r+1} - t_0^{i+j-2r+1}}{i+j-2r+1}, & i \geq r \wedge j \geq r, i = 0 \dots n, j = 0 \dots n \\ 0, & i < r \vee j < r \end{cases} \quad (1)$$

Since our $x_k = [c_{k,n} \ c_{k,n-1} \ \dots \ c_{k,1} \ c_{k,0}]^T$, reflecting Q' horizontally and vertically will give us the desired Q for the form of x_k . We can then create the block diagonal matrix:

$$Q = \begin{bmatrix} \frac{1}{(t_1 - t_0)^{2r-1}} Q_{(0,1)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{(t_2 - t_1)^{2r-1}} Q_{(0,1)} & 0 & \dots & 0 \\ & & \dots & & \\ 0 & \dots & 0 & \frac{1}{(t_{m-1} - t_{m-2})^{2r-1}} Q_{(0,1)} & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{(t_m - t_{m-1})^{2r-1}} Q_{(0,1)} \end{bmatrix}, \quad (2)$$

where $r = 6$ if $k \neq s+1$ and $r = 4$ if $k = s+1$.

TO FIND A_{EQ} :

At keyframes 0 and m , we have a full set of constraints for position and derivatives 1-5. At t_s and t_{s+1} , where $T_s = 0$, $\ddot{X} = -g$, $X^{(3)} = X^{(4)} = X^{(5)} = 0$. At all other points, the keyframe positions are constrained. While $X(t_i)$ is specified for $i \neq s+1$, $X(t_{s+1})$ is calculated from free fall conditions of the load. The equation for free fall of the load is:

$$X_L(t_{s+1}) = -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}_L(t_s)(t_{s+1} - t_s) + X_L(t_s)$$

Translating this to the quadrotor equation:

$$\begin{aligned} X(t_{s+1}) &= X_L(t_{s+1}) + l \\ &= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}_L(t_s)(t_{s+1} - t_s) + X_L(t_s) + l \end{aligned}$$

From $X = X_L + l$ and $\dot{X} = \dot{X}_L$

$$= -\frac{g}{2}(t_{s+1} - t_s)^2 + \dot{X}(t_s)(t_{s+1} - t_s) + X(t_s)$$

Nondimensionalizing with:

$$\begin{aligned} \tau &= \frac{t - t_i}{t_f - t_i}, t = \tau(t_f - t_i) + t_i \\ t_{s+1} - t_s &= \tau_1(t_{s+1} - t_s) + t_i - (\tau_0(t_{s+1} - t_s) + t_i) \\ &= (\tau_1 - \tau_0)(t_{s+1} - t_s) \\ X(t_{s+1}) &= x_{s+1}(\tau_1) = -\frac{g}{2}(\tau_1 - \tau_0)^2(t_{s+1} - t_s)^2 + \dot{X}(t_s)(\tau_1 - \tau_0)(t_{s+1} - t_s) + X(t_s) \\ &= -\frac{g}{2}(\tau_1 - \tau_0)^2(t_{s+1} - t_s)^2 + \frac{\dot{x}_{s+1}(\tau_0)}{t_{s+1} - t_s}(\tau_1 - \tau_0)(t_{s+1} - t_s) + x_{s+1}(\tau_0) \\ &= -\frac{g}{2}(\tau_1 - \tau_0)^2(t_{s+1} - t_s)^2 + \dot{x}_{s+1}(\tau_0)(\tau_1 - \tau_0) + x_{s+1}(\tau_0) \end{aligned}$$

At keyframe t_{s+1} when the quad and load collide, there is a change in load velocity from the collision. Assuming a completely inelastic collision:

$$\begin{aligned} (\dot{X}^- - \dot{X}_L^-) &= e(\dot{X}^+ + \dot{X}_L^+) \\ m_L \dot{X}_L^+ + m \dot{X}^+ &= m_L \dot{X}_L^- + m \dot{X}^- \\ \text{For } e &= 0, \\ \dot{X}^- &= \dot{X}_L^- \\ m_L \dot{X}_L^+ + m \dot{X}^+ &= (m_L + m) \dot{X}^- \\ \text{If } m &\gg m_L, \\ \dot{X}^+ &= \dot{X}^- \end{aligned}$$

We want the quadrotor velocity at t_{s+1} to be continuous and can be chosen independently of the load velocity.

There $26 + 2m$ constraints - 6 from each endpoint, $2(m - 1)$ additional position constraints, (4)(4)

constraints to account for constraints on derivatives 2-5 at t_s and t_{s+1} . They are:

$$A_{\text{endpoint}}x = b_{\text{endpoint}} \quad (3)$$

$$\begin{bmatrix}
x_1(\tau_0) \\
\dot{x}_1(\tau_0) \\
\ddot{x}_1(\tau_0) \\
x_1^{(3)}(\tau_0) \\
x_1^{(4)}(\tau_0) \\
x_1^{(5)}(\tau_0) \\
x_1(\tau_1) \\
x_2(\tau_0) \\
x_2(\tau_1) \\
x_3(\tau_0) \\
\vdots \\
x_s(\tau_0) \\
x_s(\tau_1) \\
\ddot{x}_s(\tau_1) \\
x_s^{(3)}(\tau_1) \\
x_s^{(4)}(\tau_1) \\
x_s^{(5)}(\tau_1) \\
x_{s+1}(\tau_0) \\
\ddot{x}_{s+1}(\tau_0) \\
x_{s+1}^{(3)}(\tau_0) \\
x_{s+1}^{(4)}(\tau_0) \\
x_{s+1}^{(5)}(\tau_0) \\
x_{s+1}(\tau_1) - \dot{x}_{s+1}\tau_0(\tau_1 - \tau_0) - x_{s+1}(\tau_0) \\
\ddot{x}_{s+1}(\tau_1) \\
x_{s+1}^{(3)}(\tau_1) \\
x_{s+1}^{(4)}(\tau_1) \\
x_{s+1}^{(5)}(\tau_1) \\
x_{s+2}(\tau_0) - \dot{x}_{s+1}\tau_0(\tau_1 - \tau_0) - x_{s+1}(\tau_0) \\
\ddot{x}_{s+2}(\tau_0) \\
x_{s+2}^{(3)}(\tau_0) \\
x_{s+2}^{(4)}(\tau_0) \\
x_{s+2}^{(5)}(\tau_0) \\
x_{s+2}(\tau_0) \\
x_{s+2}(\tau_1) \\
x_{s+3}(\tau_0) \\
\vdots \\
x_m(\tau_0) \\
x_m(\tau_1) \\
\dot{x}_m(\tau_1) \\
\ddot{x}_m(\tau_1) \\
x_m^{(3)}(\tau_1) \\
x_m^{(4)}(\tau_1) \\
x_m^{(5)}(\tau_1)
\end{bmatrix}
=
\begin{bmatrix}
X(t_0) \\
(t_1 - t_0)\dot{X}(t_0) \\
(t_1 - t_0)^{(2)}X^{(2)}(t_0) \\
(t_1 - t_0)^{(3)}X^{(3)}(t_0) \\
(t_1 - t_0)^{(4)}X^{(4)}(t_0) \\
(t_1 - t_0)^{(5)}X^{(5)}(t_0) \\
X(t_1) \\
X(t_1) \\
X(t_2) \\
X(t_2) \\
\vdots \\
X(t_{s-1}) \\
X(t_s) \\
-g(t_s - t_{s-1})^{(2)} \\
0 \\
0 \\
0 \\
X(t_s) \\
-g(t_{s+1} - t_s)^{(2)} \\
0 \\
0 \\
0 \\
-g(t_s - \tau_0)^2(t_{s+1} - t_s)^2 \\
-g(t_{s+1} - t_s)^{(2)} \\
0 \\
0 \\
0 \\
-g(t_s - \tau_0)^2(t_{s+1} - t_s)^2 \\
-g(t_{s+2} - t_{s-1})^{(2)} \\
0 \\
0 \\
0 \\
X(t_{s+2}) \\
X(t_{s+2}) \\
\vdots \\
X(t_{m-1}) \\
X(t_m) \\
(t_m - t_{m-1})\dot{X}(t_m) \\
(t_m - t_{m-1})^{(2)}X^{(2)}(t_m) \\
(t_m - t_{m-1})^{(3)}X^{(3)}(t_m) \\
(t_m - t_{m-1})^{(4)}X^{(4)}(t_m) \\
(t_m - t_{m-1})^{(5)}X^{(5)}(t_m)
\end{bmatrix}$$

We also need continuity of derivatives lower than 6 at all intermediate keyframes, which gives an additional $5(m-3) + 2$ constraints - $5(m-3)$ continuity constraints at each keyframe i such that $i \neq 0, m, ss+1$ and

2 for velocity continuity at t_s and t_{s+1} . Explicitly:

$$A_{cont}x = b_{cont}$$

$$\begin{bmatrix} \frac{1}{(t_1-t_0)}\dot{x}_1(\tau_1) - \frac{1}{(t_2-t_1)}\dot{x}_2(\tau_0) \\ \dots \\ \frac{1}{(t_1-t_0)^{(5)}}x_1^{(5)}(\tau_1) - \frac{1}{(t_2-t_1)^{(5)}}x_2^{(5)}(\tau_0) \\ \dots \\ \frac{1}{(t_{s-1}-t_{s-2})^{(5)}}x_{s-1}^{(5)}(\tau_1) - \frac{1}{(t_s-t_{s-1})^{(5)}}x_s^{(5)}(\tau_0) \\ \frac{1}{(t_s-t_{s-1})}\dot{x}_s(\tau_1) - \frac{1}{(t_{s+1}-t_s)}\dot{x}_{s+1}(\tau_0) \\ \frac{1}{(t_{s+1}-t_s)}\dot{x}_{s+1}(\tau_1) - \frac{1}{(t_{s+2}-t_{s+1})}\dot{x}_{s+2}(\tau_0) \\ \frac{1}{(t_{s+2}-t_{s+1})}\dot{x}_{s+2}(\tau_1) - \frac{1}{(t_{s+3}-t_{s+2})}\dot{x}_{s+3}(\tau_0) \\ \dots \\ \frac{1}{(t_{s+2}-t_{s+1})^{(5)}}x_{s+2}^{(5)}(\tau_1) - \frac{1}{(t_{s+3}-t_{s+2})^{(5)}}x_{s+3}^{(5)}(\tau_0) \\ \dots \\ \frac{1}{(t_{m-1}-t_{m-2})^{(5)}}x_{m-1}^{(5)}(\tau_1) - \frac{1}{(t_m-t_{m-1})^{(5)}}x_m^{(5)}(\tau_0) \end{bmatrix} = 0$$

The full set of $7m + 13$ constraints, $Ax = b$, are:

$$Ax = b$$

$$\begin{bmatrix} A_{endpoint} \\ A_{cont} \end{bmatrix} x = \begin{bmatrix} b_{endpoint} \\ 0 \end{bmatrix} \quad (4)$$

Note that for m pieces, we have $12m$ coefficients. Thus, we are missing $5m - 13$ constraints for a fully constrained system.

TO FIND A_{INEQ} :

We need 3 sets of inequality constraints. In general, we choose N_c intermediate points between t_f and t_i , located at times $t_c = t_i + \frac{k}{N_c+1}$ for $k = 1, 2, \dots, N_c$ and evaluate the inequality constraints at these points. Note that the inequality constraints are not evaluated at trajectory endpoints.

When the rope is taut, tension at t_c is always greater than 0, or $T_i(t_c) > 0$ for all $i \neq s + 1$:

$$\begin{aligned} T &> 0 \\ -T &< 0 \\ -(m_L(\ddot{X}_L(t) + g) &\leq -\epsilon \\ -(m_L(\frac{\ddot{x}(t_c)}{(t_f - t_i)^2} + g) &\leq -\epsilon \\ -\ddot{x}(t_c) &\leq -(\frac{\epsilon}{m_L} + g)(t_f - t_i)^2 \end{aligned}$$

During this free-fall phase, the position of the load $X_L(t_c)$ can be explicitly calculated as a function of time

and the quadrotor state at $X(t_s)$:

$$\begin{aligned}
X_L(t_c) &= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}_L(t_i)(t_c - t_i) + X_L(t_i) \\
&= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}(t_i)(t_c - t_i) + X(t_i) - l \\
\tau &= \frac{t - t_i}{t_f - t_i}, t = \tau(t_f - t_i) + t_i \\
t_c &= t_i + \frac{p}{N_c + 1}(t_f - t_i) \\
&= (\tau_0(t_f - t_i) + t_i) + \frac{p}{N_c + 1}((\tau_1(t_f - t_i) + t_i) - (\tau_0(t_f - t_i) + t_i)) \\
&= \tau_0(t_f - t_i) + t_i + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0) \\
t_c - t_i &= \tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0) \\
X_L(\tau_c) &= -\frac{g}{2}(t_c - t_i)^2 + \dot{X}(t_i)(t_c - t_i) + X(t_i) - l \\
&= -\frac{g}{2}(\tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0))^2 + \frac{\dot{x}(\tau_0)}{t_f - t_i}(\tau_0(t_f - t_i) + \frac{p}{N_c + 1}(t_f - t_i)(\tau_1 - \tau_0)) + x(\tau_0) - l \\
&= -\frac{g}{2}\tau_c^2(t_f - t_i)^2 + \dot{x}(\tau_0)\tau_c + x(\tau_0) - l
\end{aligned}$$

When the rope is slack, the quad and load shouldn't collide with each other, giving the constraint:

$$\begin{aligned}
X(t_c) - X_L(t_c) &> 0 \\
X_L(t_c) - X(t_c) &\leq -\epsilon \\
-X(t_c) &\leq -\epsilon - X_L(t_c) \\
&\leq -\epsilon + \frac{g}{2}\tau_c^2(t_f - t_i)^2 - \dot{x}(\tau_0)\tau_c - x(\tau_0) + l \\
-x(\tau_c) + \dot{x}(\tau_0)\tau_c + x(\tau_0) &\leq -\epsilon + \frac{g}{2}\tau_c^2(t_f - t_i)^2 + l
\end{aligned}$$

In addition, the rope should remain slack:

$$\begin{aligned}
X(t_c) - X_L(t_c) &< l \\
X(t_c) - X_L(t_c) &\leq l - \epsilon \\
X(t_c) &\leq l - \epsilon + X_L(t_c) \\
&\leq l - \epsilon - \frac{g}{2}\tau_c^2(t_f - t_i)^2 + \dot{x}(\tau_0)\tau_c + x(\tau_0) - l \\
x(\tau_c) - \dot{x}(\tau_0)\tau_c - x(\tau_0) &\leq -\epsilon - \frac{g}{2}\tau_c^2(t_f - t_i)^2
\end{aligned}$$