Trajectory Generation for Quadrotor with Cable-Suspended Load in 1D
August 14th, 2013

r	derivative to minimize in cost function
	implies r initial conditions at each keyframe (position and $r-1$ derivatives, indexed from 0 (constant term)
n	order of desired trajectory, minimum order is $2r-1$
	implies $n+1$ coefficients, indexed from 0 (constant term)
d	number of dimensions to optimize, indexed from 1
	$d=1$ for this document
m	number of pieces in trajectory
	implies $m+1$ keyframes in trajectory, indexed from 1
t_{des}	vertical vector of desired arrival times at keyframes, indexed from 0
pos_{des}	matrix of desired positions, each row represents a derivative, each column represents a keyframe
	Inf represents unconstrained
T_{des}	desired tensions at keyframes

1 Optimization of a trajectory between $m+1$ keyframes in one dimension

We seek the piece-wise trajectory:

$$X(t) = \begin{cases} X_1(t), & t_0 \leq t < t_1 \\ X_2(t), & t_1 \leq t < t_2 \\ ... \\ X_m(t), & t_{m-1} \leq t < t_m \end{cases}$$

that will minimize the cost function:

$$\begin{aligned} J &= \int_{t_0}^{t_m} \| \frac{d^r X(t)}{dt} \|^2 dt \\ &= X^T Q_{(t_m, t_0)} X \\ \text{subject to: } & A_t X = b_t \end{aligned}$$

We again look for the non-dimensionalized trajectory:

$$x(\tau) = \begin{cases} x_1(\tau) = c_{1,n}\tau^n + c_{1,n-1}\tau^{n-1} + ...c_{1,1}\tau + c_{1,0}, & t_0 \leq t < t_1, \tau = \frac{t-t_0}{t_1-t_0} \\ x_2(\tau) = c_{2,n}\tau^n + c_{2,n-1}\tau^{n-1} + ...c_{2,1}\tau + c_{2,0}, & t_1 \leq t < t_2, \tau = \frac{t-t_1}{t_2-t_1} \\ ... \\ x_m(\tau) = c_{m,n}\tau^n + c_{m,n-1}\tau^{n-1} + ...c_{m,1}\tau + c_{m,0}, & t_{m-1} \leq t < t_m, \tau = \frac{t-t_{m-1}}{t_m-t_{m-1}} \end{cases} , 0 \leq \tau < 1$$

Let $x_k = [c_{k,n} \;\; c_{k,n-1} \;\; ... \;\; c_{k,1} \;\; c_{k,0}]^T$ and $x = [x_1; x_2; ...; x_m] = [c_{1,n} \;\; c_{1,n-1} \;\; c_{1,n-2} \;\; ... \;\; c_{1,1} \;\; c_{1,0} \;\; c_{2,n} \;\; c_{2,n-1} \;\; ... \;\; c_{m,1} \;\; c_{m,0}]^T$. Here, $d = 1$. Each piece of the trajectory is individually optimized between $\tau_0 = 0$ and $\tau_1 = 1$. We evaluate a time t on trajectory $x_k(\tau)$ by finding k such that $t_{k-1} \leq t < t_k$ at time $\tau = \frac{t-t_{k-1}}{t_k-t_{k-1}}$.

We want to minimize:

$$\begin{aligned} J &= \int_{t_0}^{t_m} \| \frac{d^r X(t)}{dt} \|^2 dt \\ &= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \| \frac{d^r X_k(t)}{dt} \|^2 dt \\ &= \sum_{k=1}^{m} \int_0^1 \frac{1}{(t_k - t_{k-1})^{2r}} \| \frac{d^r x_k(\tau)}{d\tau} \|^2 d\tau \\ &= \sum_{k=1}^{m} x_k^T \frac{1}{(t_k - t_{k-1})^{2r}} Q_{(0,1)} x_k \\ &= x^T Q x \\ \text{subject to: } & Ax = b \end{aligned}$$

Note that alternatively, we could have non-dimensionalized the entire trajectory between 0 and 1 and evaluate a time t on trajectory $x_k(\tau)$ by finding k such $\frac{t_{k-1}-t_0}{t_m-t_0} \leq \tau < \frac{t_k-t_0}{t_m-t_0}$, where $\tau = \frac{t-t_0}{t_m-t_0}$.

TO FIND Q:

Recall that for each $x'_k = [c_{k,0} \;\; c_{k,1} \;\; ... \;\; c_{k,n-1} \;\; c_{k,n}]^T$, where $k = 1...m$, $Q'_{(0,1)}$ is given by Eq. **??**. Since our $x_k = [c_{k,n} \;\; c_{k,n-1} \;\; ... \;\; c_{k,1} \;\; c_{k,0}]^T$, reflecting Q' horizontally and vertically will give us the desired Q for the form of x_k. We can then create the block diagonal matrix:

$$Q = \begin{bmatrix} \frac{1}{(t_1-t_0)^{2r}}Q_{(0,1)} & 0 & 0 & ... & 0 \\ 0 & \frac{1}{(t_2-t_1)^{2r}}Q_{(0,1)} & 0 & ... & 0 \\ & & ... & & \\ 0 & ... & 0 & \frac{1}{(t_{m-1}-t_{m-2})^{2r}}Q_{(0,1)} & 0 \\ 0 & ... & 0 & 0 & \frac{1}{(t_m-t_{m-1})^{2r}}Q_{(0,1)} \end{bmatrix} \tag{1}$$

TO FIND A:

First, we need to account for endpoint constraints:

$$A_{endpoint_t} X = b_{endpoint_t}$$

$$\begin{bmatrix} A(t_0) & 0 & 0 & ... & 0 \\ A(t_1) & 0 & 0 & ... & 0 \\ 0 & A(t_1) & 0 & ... & 0 \\ 0 & A(t_2) & 0 & ... & 0 \\ 0 & 0 & A(t_2) & ... & 0 \\ 0 & 0 & A(t_3) & ... & 0 \\ & & ... & & \\ 0 & 0 & ... & 0 & A(t_{m-1}) \\ 0 & 0 & ... & 0 & A(t_m) \end{bmatrix} X = \begin{bmatrix} X_1(t_0) \\ \dot{X}_1(t_0) \\ ... \\ X_1^{(r-1)}(t_0) \\ X_1(t_1) \\ \dot{X}_1(t_1) \\ ... \\ X_1^{(r-1)}(t_1) \\ X_2(t_1) \\ \dot{X}_2(t_1) \\ ... \\ X_2^{(r-1)}(t_1) \\ X_2(t_2) \\ \dot{X}_2(t_2) \\ ... \\ X_2^{(r-1)}(t_2) \\ ... \\ X_m(t_{m-1}) \\ \dot{X}_m(t_{m-1}) \\ ... \\ X_m^{(r-1)}(t_{m-1}) \\ X_m(t_m) \\ \dot{X}_m(t_m) \\ ... \\ X_m^{(r-1)}(t_m) \end{bmatrix}$$

In the non-dimensionalized case, we have, $\tau_0 = 0$, $\tau_1 = 1$, and:

$$A_{endpoint}x = b_{endpoint}$$

$$\begin{bmatrix} A(\tau_0) & 0 & 0 & ... & 0 \\ A(\tau_1) & 0 & 0 & ... & 0 \\ 0 & A(\tau_0) & 0 & ... & 0 \\ 0 & A(\tau_1) & 0 & ... & 0 \\ 0 & 0 & A(\tau_0) & ... & 0 \\ 0 & 0 & A(\tau_1) & ... & 0 \\ & & ... & & \\ 0 & 0 & ... & 0 & A(\tau_0) \\ 0 & 0 & ... & 0 & A(\tau_1) \end{bmatrix} x = \begin{bmatrix} X_1(t_0) \\ (t_1 - t_0)\dot{X}_1(t_0) \\ ... \\ (t_1 - t_0)^{(r-1)} X_1^{(r-1)}(t_0) \\ X_1(t_1) \\ (t_1 - t_0)\dot{X}_1(t_1) \\ ... \\ (t_1 - t_0)^{(r-1)} X_1^{(r-1)}(t_1) \\ X_2(t_1) \\ (t_2 - t_1)\dot{X}_2(t_1) \\ ... \\ (t_2 - t_1)^{(r-1)} X_2^{(r-1)}(t_1) \\ X_2(t_2) \\ (t_2 - t_1)\dot{X}_2(t_2) \\ ... \\ (t_2 - t_1)^{(r-1)} X_2^{(r-1)}(t_2) \\ ... \\ X_m(t_{m-1}) \\ (t_m - t_{m-1})\dot{X}_m(t_{m-1}) \\ ... \\ (t_m - t_{m-1})^{(r-1)} X_m^{(r-1)}(t_{m-1}) \\ X_m(t_m) \\ (t_m - t_{m-1})\dot{X}_m(t_m) \\ ... \\ (t_m - t_{m-1})^{(r-1)} X_m^{(r-1)}(t_m) \end{bmatrix} \tag{2}$$

Note that again, we omit rows where a condition is unconstrained. Also, except for constraints at t_0 and t_m, every other constraint must be included twice - a constraint at t_k must be applied as a final condition to $x_k(\tau_1)$ and an initial condition $x_{k+1}(\tau_0)$. The equation for $A[i, j](t)$ is given in Eq. **??**.

We must also account for continuity constraints, which ensure that when the trajectory switches from one piece to another at the keyframes, position and all derivatives lower than r remain continuous, for a smooth path. In other words, we require:

$$A_{cont_t}X = b_{cont_t}$$

$$\begin{bmatrix} X_1(t_1) - X_2(t_1) \\ \dot{X}_1(t_1) - \dot{X}_2(t_1) \\ ... \\ X_1^{(r-1)}(t_1) - X_2^{(r-1)}(t_1) \\ ... \\ X_{m-1}(t_{m-1}) - X_m(t_{m-1}) \\ \dot{X}_{m-1}(t_{m-1}) - \dot{X}_m(t_{m-1}) \\ ... \\ X_{m-1}^{(r-1)}(t_{m-1}) - X_m^{(r-1)}(t_{m-1}) \end{bmatrix} = 0$$

Translating to the nondimensionalized case, $\tau_0 = 0$, $\tau_1 = 1$, and:

$$A_{cont}x = b_{cont}$$

$$\begin{bmatrix} x_1(\tau_1) - x_2(\tau_0) \\ \frac{1}{(t_1-t_0)}\dot{x}_1(\tau_1) - \frac{1}{(t_2-t_1)}\dot{x}_2(\tau_0) \\ ... \\ \frac{1}{(t_1-t_0)^{(r-1)}}x_1^{(r-1)}(\tau_1) - \frac{1}{(t_2-t_1)^{(r-1)}}x_2^{(r-1)}(\tau_0) \\ ... \\ x_{m-1}(\tau_1) - x_m(\tau_0) \\ \frac{1}{(t_{m-2}-t_{m-1})}\dot{x}_{m-1}(\tau_1) - \frac{1}{(t_m-t_{m-1})}\dot{x}_m(\tau_0) \\ ... \\ \frac{1}{(t_{m-2}-t_{m-1})^{(r-1)}}x_{m-1}^{(r-1)}(\tau_1) - \frac{1}{(t_m-t_{m-1})^{(r-1)}}x_m^{(r-1)}(\tau_0) \end{bmatrix} = 0$$

$$\begin{bmatrix} A_{cont}(t_1) & 0 & 0 & ... & 0 \\ 0 & A_{cont}(t_2) & 0 & ... & 0 \\ 0 & 0 & A_{cont}(t_3) & ... & 0 \\ & & ... & & \\ 0 & 0 & ... & 0 & A_{cont}(t_{m-1}) \end{bmatrix} x = 0$$

where:

$$A_{cont}[i,j](t_k) = \begin{cases} \frac{1}{(t_k-t_{k-1})^i}\prod_{k=0}^{i-1}(n-k-j)\tau_1^{n-j-i}, & n-j \geq i \wedge j \leq n \\ 0, & n-j < i \wedge j \leq n \\ -\frac{1}{(t_{k+1}-t_k)^i}\prod_{k=0}^{i-1}(1-k-j)\tau_0^{1-j-i}, & 1-j \geq i \wedge j > n \\ 0, & 1-j < i \wedge j > n \end{cases}, i = 0...(r-1), j = 0...2(n+1) \tag{3}$$

Our constraints, $Ax = b$, take the form:

$$\begin{gathered} Ax = b \\ \begin{bmatrix} A_{endpoint} \\ A_{cont} \end{bmatrix} x = \begin{bmatrix} b_{endpoint} \\ 0 \end{bmatrix} \end{gathered} \tag{4}$$

TO EVALUATE:

$$X(t) = \begin{cases} x_1(0), & t < t_0 \\ x_1(\tau) = c_{1,n}\tau^n + c_{1,n-1}\tau^{n-1} + ...c_{1,1}\tau + c_{1,0}, & t_0 \leq t < t_1, \tau = \frac{t-t_0}{t_1-t_0} \\ x_2(\tau) = c_{2,n}\tau^n + c_{2,n-1}\tau^{n-1} + ...c_{2,1}\tau + c_{2,0}, & t_1 \leq t < t_2, \tau = \frac{t-t_1}{t_2-t_1} \\ ... \\ x_m(\tau) = c_{m,n}\tau^n + c_{m,n-1}\tau^{n-1} + ...c_{m,1}\tau + c_{m,0}, & t_{m-1} \leq t < t_m, \tau = \frac{t-t_{m-1}}{t_m-t_{m-1}} \\ x_m(1), & t_m \leq t \end{cases}$$

$$X^{(k)}(t) = \begin{cases} \frac{1}{(t_1-t_0)^k}x_1^{(k)}(0), & t < t_0 \\ \frac{1}{(t_1-t_0)^k}x_1^{(k)}(\tau), & t_0 \leq t < t_1, \tau = \frac{t-t_0}{t_1-t_0} \\ \frac{1}{(t_2-t_1)^k}x_2^{(k)}(\tau), & t_1 \leq t < t_2, \tau = \frac{t-t_1}{t_2-t_1} \\ ... \\ \frac{1}{(t_m-t_{m-1})^k}x_m^{(k)}(\tau), & t_{m-1} \leq t < t_m, \tau = \frac{t-t_{m-1}}{t_m-t_{m-1}} \\ \frac{1}{(t_m-t_{m-1})^k}x_m^{(k)}(1), & t_m \leq t \end{cases}$$