

Quadrotor with a Cable-Suspended Load in 1-Dimension  
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$m_Q, m_L \in \mathbb{R}$	Mass of quadrotor, load
$f \in \mathbb{R}$	Magnitude of thrust for quadrotor
$l \in \mathbb{R}$	Length of suspension cable
$T \in \mathbb{R}$	Magnitude of tension in cable
$x_Q, \mathbf{x}_L \in R^2$	Position of center of mass of quadrotor, load
$v_Q, \mathbf{v}_L \in R^2$	Velocity of center of mass of quadrotor, load

# 1 Equations of Motion

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)\mathbf{u}_1, & \mathbf{x}_1 \notin \mathcal{S}_1 \\ \mathbf{x}_2^+ = \Delta_1(\mathbf{x}_1^-), & \mathbf{x}_1^- \in \mathcal{S}_1 \\ \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2) + g_2(\mathbf{x}_2)\mathbf{u}_2, & \mathbf{x}_2 \notin \mathcal{S}_2 \\ \mathbf{x}_1^+ = \Delta_2(\mathbf{x}_2^-), & \mathbf{x}_2^- \in \mathcal{S}_2 \end{cases}$$

$$\Sigma : \begin{cases} \dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{x}_L \\ \dot{v}_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_L \\ v_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_L+m_Q} \end{bmatrix} f + \begin{bmatrix} 0 \\ -g \end{bmatrix}, & \mathbf{x}_1 \notin \{\mathbf{x}_1 \mid T \equiv \|m_L(\dot{v}_L + g)\| = 0\} \\ \mathbf{x}_2^+ = \begin{bmatrix} x_L^+ \\ v_L^+ \\ x_Q^+ \\ v_Q^+ \end{bmatrix} = \begin{bmatrix} x_L^- \\ v_L^- \\ x_L^- + l \\ v_L^- + l \end{bmatrix}, & \mathbf{x}_1^- \in \{\mathbf{x}_1 \mid T = 0\} \\ \dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{x}_L \\ \dot{v}_L \\ \dot{x}_Q \\ \dot{v}_Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_L \\ v_L \\ x_Q \\ v_Q \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_Q} \end{bmatrix} f + \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \end{bmatrix}, & \mathbf{x}_2 \notin \{\mathbf{x}_2 \mid \|x_Q - x_L\| = L\} \\ \mathbf{x}_1^+ = \begin{bmatrix} x_L^+ \\ v_L^+ \end{bmatrix} = \begin{bmatrix} x_L^- \\ v_Q^- \end{bmatrix}, & \mathbf{x}_2^- \in \{\mathbf{x}_2 \mid \|x_Q - x_L\| = L\} \end{cases}$$

## 1.1 When cable is taut

Assume that the quadrotor is moving in the  $y - z$  plane, where its only degree of freedom is in the  $z$  direction.

Constraint:  $x_Q = x_L + l$

$$\begin{aligned}\sum \mathbf{F} &= m\ddot{x} \\ -m_Q g - T + f &= m_Q \dot{v}_Q \\ -m_L g + T &= m_L \dot{v}_L\end{aligned}$$

Solving for  $\dot{v}_L$  in terms of only  $f$ :

$$\begin{aligned}T &= -m_Q \dot{v}_Q - m_Q g + f \\ m_L \dot{v}_L &= -m_L g + (-m_Q \dot{v}_Q - m_Q g + f)\end{aligned}$$

Differentiating the constraint gives:

$$\begin{aligned}\dot{x}_Q &= v_Q = \dot{x}_L = v_L \\ \dot{v}_Q &= \dot{v}_L \\ (m_L + m_Q) \dot{v}_L &= -(m_L + m_Q)g + f \\ \dot{v}_L &= -g + \frac{f}{m_L + m_Q}\end{aligned}$$

This gives our equations of motion:

$$\begin{aligned}\dot{x}_L &= v_L \\ \dot{v}_L &= -g + \frac{f}{m_L + m_Q}\end{aligned}$$

Note that the system transitions to  $\mathbf{x}_2$  when the tension is 0. The tension force in the cable can be explicitly from the equation of motion:

$$\begin{aligned}-m_L g + T &= m_L \dot{v}_L \\ T &= m_L(\dot{v}_L + g)\end{aligned}$$

Note that this force will always go from positive to negative at the transition.

## 1.2 When cable is slack

Assume that the quadrotor is moving in the  $y - z$  plane, where its only degree of freedom is in the  $z$  direction.

No constraints. Load is in free fall while quad is controlled by its input force  $f$ .

$$\begin{aligned}\sum \mathbf{F} &= m\ddot{x} \\ -m_Q g + f &= m_Q \dot{v}_Q \\ -m_L g &= m_L \dot{v}_L\end{aligned}$$

Solving for  $\dot{v}_L$  and  $\dot{v}_Q$  in terms of only  $f$ :

$$\begin{aligned}\dot{v}_L &= -g \\ \dot{v}_Q &= -g + \frac{f}{m_Q}\end{aligned}$$

This makes our equations of motion:

$$\begin{aligned}\dot{x}_L &= v_L \\ \dot{v}_L &= -g \\ \dot{x}_Q &= v_Q \\ \dot{v}_Q &= -g + \frac{f}{m_Q}\end{aligned}$$

Note that the system transitions to  $\mathbf{x}_2$  when the quad and load are at a distance  $l$  apart again, with the load falling away from the quad. In other words, we transition when:

$$(x_Q - x_L) - l = 0$$

Note that this force will always go from negative to positive at the transition.

## 2 Differential Flatness

### 2.1 $x_1$ system:

Recall the equation of motion:

$$\begin{aligned}\dot{x}_L &= v_L \\ \dot{v}_L &= -g + \frac{f}{m_L + m_Q}\end{aligned}$$

Choose flat output  $\mathbf{y} = [x_L]$

Derive  $\dot{x}_L = v_L$ ,  $\ddot{x}_L = \dot{v}_L$  from differentiation of  $x_L$

We can simply find the nominal input force  $f$  from the equation of motion:

$$f = (m_L + m_Q)\dot{v}_L + g$$

### 2.2 $x_2$ system:

Recall the equations of motion:

$$\begin{aligned}\dot{x}_L &= v_L \\ \dot{v}_L &= -g \\ \dot{x}_Q &= v_Q \\ \dot{v}_Q &= -g + \frac{f}{m_Q}\end{aligned}$$

Flat output  $\mathbf{y} = [x_Q]$

$x_L$  and  $v_L$  are known from initial conditions because load is in free fall:

$$\begin{aligned}v_L(t) &= -gt + v_L(t_0) \\ x_L(t) &= -gt^2 + v_L(t_0)t + x_L(t_0)\end{aligned}$$

Derive  $\dot{x}_Q = v_Q$ ,  $\ddot{x}_Q = \dot{v}_Q$ , and all higher derivatives from differentiation of  $x_Q$

We can simply derive the input  $f$  from the equation of motion:

$$f = m_Q(\dot{v}_Q + g)$$