Math 74 Course Notes (Rough)

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1 Induction 1/29

Proposition 1.1 (Principle of Induction). *Let $P(n)$ be a statement indexed by $n \in \mathbb{N}$. To show $P(n)$ is true for all n show that*

1. *$P(1)$ is true.*
2. *Assume $P(k)$ is true, then show $P(k+1)$ is true.*

Let us see an example of this in practice.

Example 1. Let us show that $S_n = 1 + 3 + \ldots + 2n + 1$ is a perfect square $\forall n$. $S_1 = 4$ is a perfect square so the first step is good. Now note

$$S_{k+1} = 1 + 3 + \ldots + 2k + 1 + 2k + 3 = S_k + 2k + 3$$

By assumption S_k is a perfect square so $S_{k+1} = x^2 + 2k + 3$ for some $x \in \mathbb{N}$. Now this doesn't seem to be a perfect square, but how about $x^2 + 2k + 1$. At first glance you might think that this is the perfect square $(x+1)^2 = x^2 + 2x + 1$, but the middle factor doesn't quite match up. It seems there there are two obstacles in proving the statement, (1) we have the expression $x^2 + 2k + 3$ instead of $x^2 + 2k + 1$ and (2) $x \neq k$. We can resolve the first by replacing the statement S_n with $S'_n = 1 + 3 + \ldots + 2n - 1$. Notice now the induction step reads

$$S'_{k+1} = S'_k + 2k + 1$$

Now computing the value of S_k for small values of k, you might notice a pattern and conjecture that $S'_k = k^2$. This will take care of the second obstacle because it will "strengthen the inductive hypothesis." Let us now prove the statement $S'_n = n^2$ by induction. $S'_1 = 1$ so we are good. Now,

$$S'_{k+1} = 1 + \ldots + 2k - 1 + 2k + 1 = S'_k + 2k + 1 = k^2 + 2k + 1 = (k+1)^2$$

Let's return to the original problem of showing S_n is a perfect square. We see now that we should also strengthen the inductive hypothesis, specifically $S_n = (n+1)^2$ and you should check that the algebra works out.
There are two very important principles you should pick up from this example, namely solve an easier problem and do small cases.

Let us rephrase the principle of induction a bit

Lemma 1.2 (POI). *Let S be a subset of $\mathbb{N}$ such that*

(i) $1 \in S$

(ii) $k \in S \implies k + 1 \in S$

Then $S = \mathbb{N}$

Here's another statement you might have heard sometime in your life.

Lemma 1.3. *(WOP) The Well Ordering Principle states that any non-empty subset of $\mathbb{N}$ has a least element.*

Now here's something that is somewhat surprising.

Theorem 1.4. *The Well Ordering Principle and the Principle of Induction are the same, that is*

$$WOP \iff POI$$

Proof. $\implies$ Assume WOP. Given a set S satisfying (i) and (ii) of Lemma 1.2 we want to show $S = \mathbb{N}$. Certainly S is non-empty as $1 \in S$ so applying WOP to it gives us ... nothing, as 1 is the smallest element[1] in $\mathbb{N}$ so we already know S has a smallest element. Now it seems we are stuck. In these scenarios when the problem is hard, there is one method that gives you a new direction, namely proceeding via contradiction or contrapositive. There seemed to be confusion during lecture what exactly is the negation of the POI because the POI is an implication! So how does this go, that is what is the negation of $P \implies Q$? Well, it turns out that $P \implies Q$ is the same as $\neg P \vee Q$ (Check the truth tables). But now, it is easier to see the negation of $P \implies Q$ is $P \wedge \neg Q$. Returning to our problem, suppose POI isn't true which means by our previous discussion that (i) and (ii) are true and $S \neq \mathbb{N}$. Because $S \neq \mathbb{N}$ this means that $\mathbb{N} \setminus S$ is nonempty so applying the well ordering principle we have a minimal element of $\mathbb{N} \setminus S$, say k. What does this mean? Well for any other element smaller than k such as $k-1$ it's not in $\mathbb{N} \setminus S$, meaning $k-1 \in S$. But by (ii) we see that $(k-1)+1 = k \in S$ which is a contradiction and thus this direction is done.
$\impliedby$ Given POI we now want to show WOP, so given a nonempty subset S of $\mathbb{N}$, I want to show it has a least element. I want to apply POI, but I sorta have no information about S, in particular S may not satisfy conditions (i) and (ii) of Lemma 1.2 so we can't apply POI and appear to be stuck. Again this makes us think that maybe we should proceed via contradiction and this is what we do. Suppose S does not have a least element, then we see that $1 \notin S$ as otherwise S would have a least element. By the same reasoning $2 \notin S$ as otherwise S would have a least element[2]. We keep repeating this argument and see that if $k \notin S$ then $k+1 \notin S$. But this means that $\mathbb{N} \setminus S$ satisfies conditions (i) and (ii) of Lemma 1.2 and therefore by POI, $\mathbb{N} \setminus S = \mathbb{N} \implies S = \emptyset$ which is a contradiction. ■

Example 2. Here is a fun example, show

$$\Gamma(n) = \int_0^\infty x^n e^{-x}\, dx = n!$$

Proceed by induction, if you get stuck a hint is to use integration by parts with $u = x^{n+1} \dots$

False Proof Using Induction: I claim $\frac{d}{dx}(x^n) = 0$. Clearly $\frac{d}{dx}(1) = 0$. Our inductive hypothesis will be $(x^k)' = 0\ \forall k \leq n$. For the inductive step,

$$(x^{n+1})' = (x^n \cdot x)' = x^n(x)' + x(x^n)' = x^n 0 + x0 = 0$$

So what went wrong? Well it turns out that while the above mainpulation is valid for all $n \geq 1$ it isn't for $n = 0$ aka x^1 and this incorrect step allowed the rest to follow.

[1]Some might argue that the natural numbers start with 0, but we aren't logicians so who cares.
[2]It is crucial that we knew that $1 \notin S$ before concluding this

Homework 1. Show that $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Homework 2. Show that $1^3 + 2^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

Homework 3. Show a polynomial of degree n over a field has at most n roots.

Homework 4. Show that $1 + \dfrac{1}{4} + \ldots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$

Homework 5. Let F_n be a sequence defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0, F_1 = 1$, show

$$F_0 - F_1 + \ldots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$$