Collaborative Notes - Math 185

Mathematics Undergraduate Student Assocation, UC Berkeley

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Contents

The following notes are collaborative notes meant for use by undergraduates at UC Berkeley, taking Math 185 (complex analysis). However, they may be read or edited by anyone interested in the material.

The original draft of these notes is loosely based on the Spring 2018 section taught by Charles Hadfield, and was written by Aidan Backus. We do not claim that any of the proofs given are original work.

Chapter 1

Topological preliminaries

TODO: Stop using Φ to mean both a homotopy and the natural identification of $\mathbb{C}$ and $\mathbb{R}$.

$\mathbb{C}$ is a vector space over $\mathbb{R}$ of dimension 2. We can think of an element $z \in \mathbb{C}$ as a vector $(x, y) \in \mathbb{R}^2$ by the identification $z = x + iy$, where $i^2 = -1$ is the imaginary unit.

Definition 1.1. If $z = x + iy \in \mathbb{C}$, we write

$$x = \operatorname{Re} z$$

for the *real part* of z and

$$y = \operatorname{Im} z$$

for the *imaginary part*.

$\mathbb{C}$ has a norm, and its norm satisfies the identities $|x+iy| = \sqrt{x^2 + y^2}$, $|zw| = |z||w|$, and $|z+w| \leq |z|+|w|$ for each $z = x + iy$ and w.

The last estimate is the triangle inequality, which gives $\mathbb{C}$ a metric (and therefore topological) structure: its metric $d : \mathbb{C}^2 \to [0, \infty)$ is $d(z, w) = |z - w|$. Thus, open sets in $\mathbb{C}$ are those which can be written as a union of $B_r z$, where $B_r z$ denotes the ball around z of radius $r > 0$. The closed sets in $\mathbb{C}$ are complements of open sets.

Because of its structure as a normed vector space, $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, and therefore is *complete* in the sense that Cauchy sequences converge.

Throughout these notes, U will denote an arbitrary open subset of $\mathbb{C}$. We'll usually think of it as the domain of whichever function we're studying (and usually $\mathbb{C}$ will be the codomain).

1.1 Review of 104

As proven in Math 104 for $\mathbb{R}^n$, the following results hold. To prove them for $\mathbb{C}$, just identify $\mathbb{R}^2$ with $\mathbb{C}$ by the homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{C}$ given by $\Phi(x, y) = x + iy$, and its inverse $\Psi = \Phi^{-1}$.

We'll start by examining the properties of sequences:

Lemma 1.2. *Let $\{a_n : n \in \mathbb{N}\}$ be a sequence in $\mathbb{C}$ and $L \in \mathbb{C}$. Then*

$$\lim_{n\to\infty} a_n = L$$

if and only if both

$$\lim_{n\to\infty} \operatorname{Re} a_n = \operatorname{Re} L$$

and

$$\lim_{n\to\infty} \operatorname{Im} a_n = \operatorname{Im} L.$$

Lemma 1.3. *$F \subseteq \mathbb{C}$ is closed iff each sequence in F which converges in $\mathbb{C}$ converges in F.*

Theorem 1.4. *Cauchy sequences converge in $\mathbb{C}$.*

Definition 1.5. If a_n is a sequence in $\mathbb{C}$ and $|a_n|$ is convergent, we say that a_n is *absolutely convergent*.

Lemma 1.6. *Absolutely convergent sequences converge.*

However, we don't just want to talk about convergence of sequence of points. Sequences of functions are also useful.

Definition 1.7. Let $D \subseteq \mathbb{C}$ and let f_n be a sequence of functions $D \to \mathbb{C}$. Further, let $z \in D$ and k, m, n be natural numbers.

We say $f_n \to f$ *uniformly* if

$$\forall \epsilon > 0 \ \exists N > 0 \ \forall x \in D \ \forall n > N \ |f_n(x) - f(x)| < \epsilon.$$

Furthermore, $\sum f_n$ is *uniformly convergent* if its sequence of partial sums is.

Theorem 1.8 (Weierstrass M-test)**.** *Suppose $\{a_i\}$ is a sequence in $(0, \infty)$. If $\forall k \ \forall z \ |f_k(z)| < a_k$ and $\sum a_k$ converges, then $\sum f_k$ is uniformly convergent.*

Sequences allow us to give a convenient definition of compactness:

Definition 1.9. Let K be a metric space. K is *compact* if each sequence in K has a convergent subsequence in K.

Theorem 1.10 (Heine-Borel)**.** *$K \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

Lemma 1.11. *If $\mathbb{C} \supset K_1 \supseteq K_2 \supseteq \dots$ and the K_is are compact, then their intersection is nonempty.*

One of the most useful properties of compact spaces is their behavior with respect to open covers.

Definition 1.12. Let $\mathcal{U}$ be a family of open sets of some topological space X. If $\bigcup \mathcal{U} = X$, we say that $\mathcal{U}$ is an *open cover* of X. The elements of $\mathcal{U}$ are called *scraps*.

Lemma 1.13. *Let K be a metric space. K is compact if and only if for each open cover $\mathcal{U}$, there is a finite subcover $\mathcal{V} \subseteq \mathcal{U}$.*

Topologists take 1.13 as the *definition* of compactness, because it generalizes better from metric spaces to Hausdorff spaces. However, open covers won't be as important as sequences to us, so we use the other definition. The actual importance of open covers for us is that they allow us to define the notion of a Lebesgue number.

Definition 1.14. Let $\mathcal{U}$ be an open cover of a metric space X and $\mu > 0$. We say that μ is a *Lebesgue number* for $\mathcal{U}$ if, for each $x \in X$, there is a scrap $U \in \mathcal{U}$ such that $B_\mu(x) \subseteq U$.

Lemma 1.15. *If K is compact, then there exists an open cover of K with Lebesgue number > 0.*

One of the most useful notions that sequences give rise to is that of a cluster point. Some topologists call cluster points *accumulation points*, *limit points*, or *close points*.

Definition 1.16. Let $Y \subseteq X$, where X is a metric space. Y *clusters* at $x \in X$ if there is a sequence in $X \setminus \{x\}$ which converges to x.

Theorem 1.17 (Bolzano-Weierstrass)**.** *Let K be a compact metric space and $A \subseteq K$ be infinite. There is a point $x \in A$ such that A clusters at a.*

Sequences also allow us to define what it means for a function to be continuous.

Lemma 1.18. *Suppose X is a metric space, $f : X \to \mathbb{C}$, $L \in \mathbb{C}$, and $z_0 \in \mathbb{C}$.*

If $z_0 \in X$, then the following are equivalent:

1. *If $a_n \to a$ is a convergent sequence, then $f(a_n) \to f(a)$.*
2. $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall d \in D \ |d - z_0| < \delta \implies |f(d) - f(z_0)| < \varepsilon.$

3. $f^{-1}(U)$ is open.

Definition 1.19. If $f : X \to \mathbb{C}$ satisfies the hypotheses of 1.18, then f is *continuous.*

Lemma 1.20. *Suppose $K \subset \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous. Then $f(K)$ is compact.*

Definition 1.21. Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$. $f : D \to \mathbb{C}$ has a *limit* $L \in \mathbb{C}$ at z_0 if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \text{ s.t. } \forall d \in D \;\; 0 < |d - z_0| < \delta \implies |f(d) - L| < \varepsilon.$$

Continuous functions are especially well-behaved when their domains are compact.

Lemma 1.22. *Let K be compact and $f : K \to \mathbb{C}$ be continuous. Then $f(K)$ is compact.*

Now we examine some "special" subsets of $\mathbb{C}$.

Definition 1.23. A topological space X is *discrete* if every subset of X is open.

Definition 1.24. Let $D \subseteq \mathbb{C}$. D is *convex* if for each line segment $[a, b]$ where $a, b \in D$, $[a, b] \subseteq D$.

Lemma 1.25. *Let $a \in U$ and $r > 0$. Then $\overline{B_r a} \subset U$ iff $\exists R > r$ with $B_R a \subseteq U$.*

Lemma 1.26. *The following are equivalent:*

1. For each $V, W \subseteq U$ open and disjoint with $V \cup W = U$ either $V = U$ or $W = U$.

2. For each $p, q \in U$ there is a path between them.

3. Each $f : U \to \{0, 1\}$ is constant.

Definition 1.27. If the hypotheses of 1.26 are met we say that U is *connected.*

This definition of connectedness only makes sense if U is open in a Euclidean space, for there are are pathological spaces, like the topologist's sine curve, where they are not equivalent!

None of this should be new material. If it's unfamiliar, refer to any 104 text, such as Rudin, Pugh, or Ross. It would be fruitful to come up with a few explicit examples of each type of space and set presented above.

1.2 Curves

Often in complex analysis, we want to compute the integral of a function not along an interval, but along a much more general path through $\mathbb{C}$. Curves allow us to do this.

Definition 1.28. Let X be a topological space and $a < b$ be real numbers. A *curve* in X is a continuous map $\gamma : [a, b] \to X$.

$\gamma(a)$ is called the *initial point* and $\gamma(b)$ is called the *final point.* We write $\gamma^* = \gamma([a, b])$.

If $\gamma(a) = \gamma(b)$ we say that γ is *closed.*

γ is *constant* if γ^* consists of a point.

If $\mu : [c, d] \to X$ is also a curve and $\gamma(b) = \mu(c)$ then we define the curve

$$\gamma \oplus \mu : [a, b + d - c] \to \mathbb{C}$$

by $\gamma \oplus \mu(t) = \gamma(t)$ if $t \leq b$ or $\mu(t)$ otherwise.

Notice that this definition only really has teeth when the space X "looks like" $\mathbb{C}$, for example if $X = U$. Consider the case when X is discrete, for example. Because $[a, b]$ is connected, the only curves in X are constant!

On the other hand, when the codomain X has a notion of differentiability, we can define especially nice curves:

Definition 1.29. A curve γ in $\mathbb{C}$ is *smooth* if γ' exists and is continuous.

If each γ_i is a smooth curve and

$$\gamma = \bigoplus_{i \leq N} \gamma_i = \gamma_1 \oplus \cdots \oplus \gamma_N$$

then we say γ is *piecewise smooth*.

Notice that these definitions conflict with the usual usage of the words closed and smooth! Any curve is necessary closed (in fact, compact) by 1.20, and a smooth function is C^∞, not C^1.

Also notice that curves aren't just their image: they come with a parametrization, which equips them with an orientation (a "direction") and a speed. As we will see, the parametrization won't matter much, only the orientation.

Definition 1.30. If $\gamma : [0, 1] \to X$ is a curve, then $-\gamma$ is given by $-\gamma(t) = \gamma(1 - t)$.

The following is just the usual arc length formula from calculus.

Definition 1.31. If γ is a curve in $\mathbb{C}$ then the *length* of γ is

$$\ell(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Curves which wind around a point once will prove themselves to be especially useful.

Definition 1.32. If $r > 0$, define $\Gamma_r(z_0)$ by the curve $t \mapsto z_0 + re^{it}$ for $t \in [0, 2\pi]$.

Notice that if γ is a curve, then $\gamma'(t)$ is the tangent vector at t, multiplied by the speed. Dividing out by the speed will give us the direction the curve is facing. This motivates the following definition.

Definition 1.33. Suppose γ_1 and $\gamma_2 : [-1, 1] \to \mathbb{C}$ are curves, $\gamma_1(0) = \gamma_2(0) = z$, and $\gamma_1'(0) \neq 0$ and $\gamma_2'(0) \neq 0$.

The *angle* between γ_1 and γ_2 at 0 is the unique $\theta \in (-\pi, \pi]$ such that

$$\frac{\gamma_1'(0)}{\gamma_2'(0)} = \left| \frac{\gamma_1'(0)}{\gamma_2'(0)} \right| re^{i\theta}.$$

We often want to talk about ways to deform a curve into another. Homotopy makes this rigorous, furnishing continuous functions which morph a curve into another.

Definition 1.34. Suppose that X is a topological space and

$$\begin{cases} \gamma_0 : [0, 1] \to X \\ \gamma_1 : [0, 1] \to X \end{cases}$$

are closed curves in X. They are *X-homotopic* if there exists $\Phi : [0, 1]^2 \to X$ such that

$$\begin{cases} \Phi(t, 0) = \gamma_0(t) \\ \Phi(t, 1) = \gamma_1(t) \\ \Phi(0, s) = \Phi(1, s) \end{cases}$$

and γ_0 is *X-nullhomotopic* if γ_1 is constant. The function Φ is called a *homotopy*, and if, if γ_1 is constant, a *nullhomotopy*.

The third condition guarantees that each "intermediate" curve $t \mapsto \Phi(t, s)$ is closed. Also notice that X-homotopy is an equivalence relation.

Definition 1.35. If U is connected and each closed curve in U is U-nullhomotopic, then we say that U is *simply connected*.

The Riemann mapping theorem, which we'll prove much later on, will show that up to an angle-preserving transformation, there are only two simply connected sets: $B_1 0$ and $\mathbb{C}$ itself!

Chapter 2

Complex calculus

We're ready to generalize calculus to the complex setting.

Recall that $U \subseteq \mathbb{C}$ is assumed to be an open set.

2.1 Cauchy-Riemann equations

Definition 2.1. Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$.

$f : D \to \mathbb{C}$ is *differentiable* at z_0 if, for some $L \in \mathbb{C}$, the function

$$f'(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0 \\ L, & z = z_0 \end{cases}$$

is continuous.

We say that the function f' is the *derivative* of f.

Multiplying through by the denominator $z - z_0$ shows that the following lemma characterizes differentiability.

Lemma 2.2. *Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$. The following are equivalent:*

1. *f is differentiable at z_0 and $f'(z_0) = L$.*
2. *$\exists \phi : D \to \mathbb{C}$ continuous at z_0 with $\phi(z_0) = L$ and*

$$\forall z \in D \ f(z) = f(z_0) + (z - z_0)\phi(z).$$

3. *$\exists \psi : D \to \mathbb{C}$ continuous at z_0 with $\psi(z_0) = 0$ and*

$$\forall z \in D \ f(z) = f(z_0) + (z - z_0)(L + \psi(z)).$$

We think of ψ as the sublinear Taylor remainder of f at z_0 and $\phi = L + \psi$ where we are viewing L as a linear operator on $\mathbb{C}$.

Recall from 104:

Lemma 2.3. *Differentiability implies continuity.*

Lemma 2.4. *If $f, g : U \to \mathbb{C}$ are differentiable, $\lambda \in \mathbb{C}$, and $z \in U$ then:*

1. *$(f + g)' = f' + g'$,*
2. *$(fg)' = f'g + g'f$,*
3. *$(\lambda f)' = \lambda f'$,*

4. $(f \circ g)'(z) = f'(g(z))g'(z)$*, and*

5. if $|f| > 0$ *then* $1/f = -f'/f^2$*.*

Definition 2.5. A *holomorphic* function is one which is differentiable on an open set.

Recall that a function is analytic if it has a Taylor series. Our goal is to prove that a function is holomorphic iff it is analytic. This is much stronger than differentiability on an open set in $\mathbb{R}^2$ – which doesn't even imply second-differentiability, let alone analyticity!

To see how this could be possible, let's start by identifying functions in $\mathbb{R}^2$ with their counterparts in $\mathbb{C}$ and seeing why differentiability in $\mathbb{C}$ is so special.

Recall the homeomorphisms $\Phi : \mathbb{R}^2 \to \mathbb{C}$ and $\Psi = \Phi^{-1}$. Let $\tilde{U} = \Psi(U)$. Then for each map $f : U \to \mathbb{C}$ we have a natural $\tilde{f} : \tilde{U} \to \mathbb{R}^2$.

In 105, it is defined that $\tilde{f} = (u, v)$ is differentiable at $\Phi(z_0)$ if there is a linear map $M : \mathbb{R}^2 \to \mathbb{R}^2$ (the derivative) and sublinear map $r : \tilde{U} \to \mathbb{R}^2$ satisfying the analogue of 2.2, namely

$$\forall z \in \tilde{U}\ \tilde{f}(\Psi(z)) = \tilde{f}(\Psi(z_0)) + M(\Psi(z - z_0)) + r(\Psi(z)).$$

Now let $z \in \mathbb{C}$, and $\Psi(z) = (x, y)$. If we multiply z by $w = a + ib$, this corresponds to multiplying $\Psi(z)$ by the *matrix*

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

for

$$\Psi(wz) = \Psi((a + ib)(x + iy)) = \Psi(ax - by + i(ay + bx)) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = Mz.$$

In the language of linear algebra, complex numbers are nothing more than antisymmetric linear operators!

Theorem 2.6 (Cauchy-Riemann equations)**.** *Let* Ψ *and* Φ *be as above. Then if* $f : U \to \mathbb{C}$*,* $z_0 \in U$*, and* $\tilde{f} = (u, v)$*, the following are equivalent:*

1. f *is differentiable at* z_0*.*

2. u *and* v *are differentiable at* $\Psi(z_0)$ *and*

$$\begin{cases} \partial_1 u(\Psi(z_0)) = \partial_2 v(\Psi(z_0)) \\ \partial_2 u(\Psi(z_0)) = -\partial_1 v(\Psi(z_0)). \end{cases}$$

Proof. By 2.2, the derivative M of $\tilde{f}$ is given by $r(\Psi(z)) = \Psi(z - z_0)\psi(\Psi(z))$ and $(a, b) = (\operatorname{Re} f'(z_0), \operatorname{Im} f'(z_0))$. Then we use the argument above, taking close note of the fact that the partial derivatives of $\tilde{f}$ are precisely the entries in M, and thus must satisfy certain relations. □

In particular $z \mapsto \operatorname{Re} z$, $z \mapsto \operatorname{Im} z$, and $z \mapsto |z|$ are not holomorphic *anywhere*! Your first line of attack in proving that a function is not holomorphic is to show that the Cauchy-Riemann equations fail.

The Cauchy-Riemann equations are the most important equations in the class. (In fact, when Aidan took 185, the entire first midterm came down to proving consequences of them.) As we will later see, the Cauchy-Riemann equations imply that f is holomorphic iff $\Delta \tilde{f} = 0$ (where Δ is the Laplacian). In the language of PDE, holomorphic functions are precisely those which are harmonic.

2.2 Integration on curves

Just as holomorphic functions in complex analysis correspond to harmonic functions in multivariable calculus, complex integrals will correspond to line integrals. We'll start by defining a "preintegral" that we can use to define the actual integral we care about.

Definition 2.7. If $f : [a, b] \to \mathbb{C}$ and $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable, then

$$\int_a^b f = \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

Recall from 104:

Lemma 2.8. *Integration is linear.*

The following integral estimate is quite useful.

Lemma 2.9. *If $f : [a, b] \to \mathbb{C}$ is continuous, then $|f|$ is integrable and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. $|f|$ is continuous, so there exist $r \geq 0$ and $\theta \in [0, 2\pi)$ such that

$$\int_a^b f = r(\cos\theta + i \sin\theta).$$

Put $\zeta = \cos\theta - i\sin\theta$. Then $|\zeta| = 1$ and

$$\begin{aligned} \left| \int_a^b f \right| &= \zeta \int f = \operatorname{Re} \zeta \int_a^b f = \int_a^b \operatorname{Re} \zeta f \\ &\leq \int_a^b |\operatorname{Re} \zeta f| \leq \int_a^b |\zeta f| = \int_a^b |f|. \end{aligned}$$

□

Now we can define the true integral:

Definition 2.10. Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth, $\gamma^* \subseteq D \subseteq \mathbb{C}$, and $f : D \to \mathbb{C}$ be continuous.
If γ is smooth, we define the *contour integral* to be

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

If not, then $\gamma = \oplus\gamma_i$ and each γ_i is smooth, so we write

$$\int_\gamma f(z)\ dz = \sum_{i=1}^N \int_{\gamma_i} f(z)\ dz.$$

The whole point of the class is to study the properties of holomorphic functions defined on open sets by taking contour integrals around those open sets.

As an example, which is quite important in its own right, let's integrate $1/z$ around the origin, where it blows up:

Lemma 2.11. *If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by $f(z) = z^{-1}$ then*

$$\int_{\Gamma_1 0} f(z)\ dz = 2\pi i.$$

Proof.

$$\int_0^{2\pi} f(\Gamma_1 0(t))\Gamma_1 0'(t)\ dt = \int_0^{2\pi} e^{-it} i e^{it}\ dt = 2\pi i.$$

□

As a result of this lemma, we'll often see $2\pi i$s pop up in theorems whenever we integrate around a function which blows up.

Lemma 2.12. *Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth, $f, \tilde{f} : \gamma^* \to \mathbb{C}$, and $\lambda \in \mathbb{C}$. We have:*

1.
$$\int_\gamma \lambda f + \tilde{f} = \lambda \int_\gamma f + \int_\gamma \tilde{f}.$$

2.
$$\left| \int_\gamma f \right| \leq \ell(\gamma) \max_{z \in \gamma^*} |f(z)|.$$

3. *If $\phi : [c, d] \to [a, b]$ is smooth and strictly increasing then $\gamma \circ \phi$ is piecewise smooth and*
$$\int_\gamma f = \int_{\gamma \circ \phi} f.$$

4. *If $f_n : \gamma^* \to \mathbb{C}$ and $f_n \to f$ uniformly then*
$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. As an example, we prove (3). You should try to prove the others for practice!

We can assume γ is smooth, for if not, we can just break up γ into finitely many smooth curves. Then

$$\begin{aligned} \int_{\gamma \circ \phi} f &= \int_c^d (f \circ \gamma \circ \phi)(\gamma \circ \phi)' \\ &= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\, dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)\, ds \\ &= \int_\gamma f. \end{aligned}$$

□

So far, nothing too crazy has happened. In fact, the fundamental theorem of calculus holds just fine:

Theorem 2.13 (fundamental theorem of calculus, part I)**.** *If f is holomorphic on U and γ is contained in U, then*
$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)).$$

Proof.
$$\int_\gamma f' = \int_a^b f' \circ \gamma \cdot \gamma' = \int_a^b (f \circ \gamma)' = f(\gamma(b)) - f(\gamma(a)).$$

□

Just like in calculus, *if* a function has an antiderivative, then its integral is determined by the value of the antiderivative on the boundary of the integration domain.

Moreover, if f' vanishes on a connected set U, we can draw a curve γ inside U to reach any point in U, and by the fundamental theorem we have:

Corollary 2.14. *If f is holomorphic and U is connected, with $f' \equiv 0$, then f is constant.*

2.3 Power series

Recall from 104 that a power series is a function of the form

$$f(z) = \sum_{n=k}^{\infty} \alpha_k (z-a)^k.$$

We can and do, without loss of generality, assume $a = 0$; this simplifies notation greatly without affecting convergence.

Definition 2.15. If $\sum \alpha_n z^n$ is a power series and if R is the supremum of all values r such that $\sum |\alpha_n| r^n$ converges, then R is its *radius of convergence*.

This seems kind of sketchy; what if the series only converged on a set which was not circular? Fortunately, this never happens!

Lemma 2.16. *Let R be the radius of convergence of $f(z) = \sum \alpha_k z^k$. Then:*

1. *If $|z| < R$, then $f(z)$ converges.*
2. *If $|z| > R$, then $f(z)$ diverges.*
3. *If $0 < r < R$ and $|z| \leq r$ then $\sum \alpha_n z^n$ converges absolutely uniformly.*

Proof. Take $z \neq 0$.

Suppose that $\sum \alpha_n z^n$ converges. Then $\alpha_n z^n$ is bounded, say $|\alpha_n z^n| \leq M$. If $0 < r < |z|$, then $|\alpha_n| r^n| \leq M(r/|z|)^n$, and therefore $\sum |\alpha_n z^n|$ converges. Therefore $r < R$, so $|z| \leq R$. This proves the contrapositive of (2).

Now, if $0 < r < R$, $\sum |\alpha_n| r^n$ converges whenever $|z| \leq r$. Then $|\alpha_n z^n| \leq |\alpha_n| r^n$ and so (3) follows by the Weierstrass M-test, implying (1). $\square$

We can use the root test and the ratio test, developed in 104, to compute radii of convergence. For example, the radius of convergence of $\sum z^n$ is 1. You should practice this on a few power series of your own devision.

Functions which are expressible by power series are known as analytic functions:

Definition 2.17. $f : U \to \mathbb{C}$ is *analytic* at $a \in U$ if $\exists R > 0$ and a power series $\sum \alpha_n (z-a)^n$ with a radius of convergence $r \geq R$ such that $B_R(a) \subseteq U$ and $\sum \alpha_n (z-a)^n = f(z)$ for each $z \in B_R(a)$.

Note, an analytic function has a power series which is locally valid. If the topology of U isn't too nice, a power series may diverge in parts of U but a different power series will hold there. It's okay; the function is still analytic in that case.

Polynomials are clearly analytic (all but finitely $\alpha_n = 0$), but most functions which we care about turn out to be analytic. Our goal, recall, is to prove every holomorphic function is analytic! Let's begin.

Lemma 2.18. *The power series $\sum \alpha_n z^n$ and $\sum n\alpha_n z^{n-1}$ have the same radius of convergence.*

Proof. Let R and $\hat{R}$ be the respective radii of convergence and $0 < r < R$. Then we can find a $\rho \in (r, R)$. For each such ρ, $\sum |\alpha_n| \rho^n$ converges, but $n(r/\rho)^n \to 0$ and is therefore bounded by a constant $M \geq |n(r\rho)^n|$. Therefore

$$n|\alpha_n| r^n = n \left(\frac{r}{\rho}\right)^n |\alpha_n| \rho^n \leq M |\alpha_n| \rho^n$$

and since the constant M is irrelevant, $\sum n|\alpha_n| r^n$ converges. So $\hat{R} > r$, implying that $\hat{R} \geq R$.

But we can repeat the same argument with R and $\hat{R}$ reversed. So $R \geq \hat{R}$. $\square$

This is what we needed to prove Taylor's theorem, which characterizes analytic functions.

Theorem 2.19 (Taylor)**.** *If $R \in (0,\infty)$ and $\sum \alpha_n z^n$ has a radius of convergence $\geq R > 0$, define $f : B_R(0) \to \mathbb{C}$ by*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$$

Then f is smooth in $B_R(0)$ and for each $z \in B_R(0)$,

$$f'(z) = \sum_{n=1}^{\infty} n\alpha_n z^{n-1}.$$

Moreover,

$$\alpha_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Fix $z_0 \in B_R(0)$, $\epsilon > 0$, and $r \in (|z_0|, R)$. By the above lemma, we can find an N such that

$$\sum_{n=N+1}^{\infty} n|\alpha_n|r^{n-1} < \frac{\epsilon}{4}.$$

Therefore, $\forall z \in B_r(0) \setminus \{z_0\}$, the remainder

$$\begin{aligned}
R(z) &= \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n\alpha_n z_0^{n-1} \right| \\
&= \left| \sum_{n=0}^{\infty} \alpha_n \left(\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1} \right) \right| \\
&= \left| \sum_{n=1}^{\infty} \alpha_n \left(\left(\sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right) - nz_0^{n-1} \right) \right| \\
&\leq \left| \sum_{n=1}^{N} \alpha_n \sum_{k=0}^{n-1} z^k z^{n-1-k} - z_0^{n-1} \right| + \left| \sum_{n=N+1}^{\infty} 2n|\alpha_n|r^{n-1} \right| \\
&= A(z) + B(z)
\end{aligned}$$

and A is continuous, so there exists a δ such that if $z \in B_\delta(z_0) \setminus \{z_0\}$ then $A(z) < \epsilon/2$. We already estimated $B(z) < \epsilon/2$. So $R(z) < \epsilon$.

Therefore f is differentiable and α_1 is as desired. Furthermore, this argument applies to the derivatives of f as well, so by induction f is smooth and each α_n is as desired. ☐

Corollary 2.20. *Analytic functions are holomorphic.*

By Taylor's theorem, if a function is analytic in U, U is connected, and we know its behavior on a small open set in U, we already know its behavior everywhere.

Corollary 2.21. *Suppose $\sum \alpha_n z^n$ and $\sum \beta_n z^n$ are power series with radii of convergence R_1, R_2. If $0 < \epsilon \leq \min R_i$ and $\forall z \in B_\epsilon(0)$ we have $\alpha_n z^n = \beta_n z^n$, then $\forall n$ $\alpha_n = \beta_n$.*

Proof. Look at $\sum(\alpha_n - \beta_n)z^n$. The derivatives of this are all 0. ☐

So we are justified in taking the power series definitions of exp, sin, and cos – if we wanted them to be analytic, we have no choice but to accept them! Moreover, accepting Euler's formula,

$$\exp(i\theta) = \cos\theta + i\sin\theta,$$

a computation verifies that

$$e^z = e^w \iff z - w \in 2\pi i\mathbb{Z}.$$

In particular, log is *not* well-defined as the inverse of exp, because exp is no longer a bijection. We'll have to redefine log later, once we have the necessary topological machinery.

As an aside, here's a way to compute products of analytic functions:

Theorem 2.22. *Let $r > 0$ and $f(z) = \sum \alpha_n z^n$, $g(z) = \sum \beta_n z^n$ be power series with radius of convergence $\geq r$. Put*

$$\gamma_n = \sum_{j \leq n} \alpha_j \beta_{n-j}.$$

Then $\sum \gamma_n z^n$ has radius of convergence $\geq r$ and $\sum \gamma_n z^n = f(z)g(z)$.

Proof. fg is holomorphic, and the proof follows by telescoping and using the binomial formula. $\square$

Chapter 3

Integrals of closed curves

We're going to start off proving a rather technical representation formula which will allow us to construct analytic functions. The rest of the chapter, and indeed the class, is going to be fallout from it.

Lemma 3.1. *Let γ be a curve, $g : \gamma^* \to \mathbb{C}$ be continuous, $z_0 \in U = \mathbb{C} \setminus \gamma^*$, and*

$$f(z) = \int_\gamma \frac{g(w)}{w - z}\, dw.$$

Then f is analytic, and for each $n \in \mathbb{N}$, its coefficients are

$$\alpha_n = \int_\gamma \frac{g(w)}{(w - z_0)^{n+1}}\, dw.$$

Moreover, the radius of convergence of f is at least

$$\inf_{w \in \gamma^*} |w - z_0| > 0.$$

Proof. Let $z \in B_R z_0$ and $w \in \gamma^*$. Then

$$\left|\frac{z - z_0}{w - z_0}\right| \le \left|\frac{z - z_0}{R}\right| < 1$$

so, using a geometric series,

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0 - (z - z_0)} \\ &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n \end{aligned}$$

converges uniformly.

Define $h, h_1, h_2, \cdots : \gamma^* \to \mathbb{C}$ by

$$h_n(w) = \frac{g(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

and

$$h(w) = \frac{g(w)}{w - z_0}.$$

These are continuous on γ^* and $\sum h_i = h$ by another geometric series computation; the limit converges uniformly on γ^*. By compactness of γ^*, we can swap limits, so

$$f(z) = \int_\gamma h = \sum_{n=0}^{\infty} \int_\gamma h_n = \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \frac{g(w)}{w - z_0}\, dw = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

on $B_R z_0$. ∎

3.1 Winding numbers

A special case occurs when $g \equiv 1$.

Definition 3.2. If γ is closed, then define $\mathrm{Ind}_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ by

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z},$$

the *index function* or *winding number* of γ.

By Cauchy's integral formula, Ind_γ is analytic. As we shall see, the winding number of γ at z tells us how many times γ winds around z counterclockwise (minus the times it winds around z clockwise).

Lemma 3.3. *If γ is closed, then $\mathrm{Ind}_\gamma(\mathbb{C}\setminus\gamma^*) \subseteq \mathbb{Z}$, and $\mathrm{Ind}_\gamma(z)$ is identically 0 outside of the region encircled by γ.*

Proof. Put $f : [a, b] \to \mathbb{C}$ by

$$f(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds.$$

Then f is differentiable and

$$f'(t) = \frac{\gamma'(t)}{\gamma(t) - z}.$$

Define $g : [a, b] \to \mathbb{C}$ by

$$g(t) = e^{-f(t)}(\gamma(t) - z).$$

Then $g' \equiv 0$, so g is constant. In particular,

$$e^{f(t)-f(a)} = \frac{\gamma(t) - z}{\gamma(a) - z}$$

but $f(a) = 0$ and $\gamma(b) = \gamma(a)$. Therefore,

$$e^{f(b)} = e^{f(b)-f(a)} = \frac{\gamma(a) - z}{\gamma(a) - z} = 1.$$

So $f(b) \in 2\pi i\mathbb{Z}$, but $f(b) = 2\pi i \,\mathrm{Ind}_\gamma(z)$.

A computation verifies that

$$\lim_{|z|\to\infty} \mathrm{Ind}_\gamma(z) = 0$$

but since Ind_γ takes values in the integers it must be identically 0 if $|z|$ is sufficiently large. But then by analyticity, Ind_γ is 0 on any connected components which include such sufficiently large $|z|$. □

Corollary 3.4. *If γ is closed and $\phi : [0, 1] \to \mathbb{C} \setminus \gamma^*$ is continuous then $\mathrm{Ind}_\gamma \circ\phi$ is constant.*

Proof. $\mathrm{Ind}_\gamma \circ\phi$ is a continuous function mapping a connected set into a discrete set. □

For example, for each $z_0 \in \mathbb{C}$ and $r > 0$, $\mathrm{Ind}_{\Gamma_r(z_0)}(z_0) = 1$: by connectivity of the ball, $\mathrm{Ind}_{\Gamma_r(z_0)(z)}$ is 1 if $z \in B_r z_0$ and 0 otherwise.

Playing with Ind makes its properties clearer.

3.2 Cauchy-Goursat in a convex set

In multivariable calculus one often worries about conservative vector fields, those for which line integrals around closed curves vanish. You might intuit that holomorphic functions $U \to \mathbb{C}$ are "conservative", and if the topology of U isn't too bad, you'd be right.

First, a notational convenience, which we won't need after its use to prove the next few theorems.

Definition 3.5. For $z_1, z_2, z_3 \in \mathbb{C}$, write Δ to mean the triangle with those endpoints, including the interior, and $\partial\Delta$ to indicate the path around the triangle, oriented counterclockwise.

If the z_is are collinear then this is a degenerate triangle.

Theorem 3.6 (Cauchy-Goursat)**.** *If $p \in U$, $\Delta_0 \subseteq U$, f is continuous, and f is holomorphic on $U \setminus \{p\}$ then*

$$\int_{\partial\Delta} f = 0.$$

Proof. There are three cases, depending on where p is in relation to $\partial\Delta_0$.

If $p \notin \Delta_0$, then define z_i' to be the point opposite z_i on $\partial\Delta_0$. Then we have four triangles:

1. Δ_0^1, determined by z_1, z_2', z_3',
2. Δ_0^2, determined by z_1', z_2, z_3',
3. Δ_0^3, determined by z_1', z_2', z_3, and
4. Δ_0^4, determined by z_1', z_2', z_3'.

Then

$$\left|\int_{\partial\Delta_0} f\right| \le \sum_{i=1}^{4} \left|\int_{\partial\Delta_0^i} f\right| \le 4\left|\int_{\partial\Delta_0^k} f\right|$$

for some index k. Set $\Delta_1 = \Delta_0^k$.

By induction we get a nested sequence of triangles

$$\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \dots$$

and

$$\left|\int_{\partial\Delta_k} f\right| \le 4\left|\int_{\partial\Delta_{k+1}} f\right|$$

with $\ell(\partial\Delta_k) = \ell(\partial\Delta_{k+1})/2$, so

$$\left|\int_{\partial\Delta_0} f\right| \le 4^n\left|\int_{\partial\Delta_n} f\right|.$$

Since the Δ_n are compact there is a point in their intersection, say $z_0 \neq p$ and so in a sufficiently small ball around that point,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)$$

for a sublinear ψ. Integrating the affine part, we get a map

$$z \mapsto f(z_0)z + \frac{1}{2}f'(z_0)(z - z_0)^2$$

and so for each n,

$$\int_{\partial\Delta_n} f(z_0) + f'(z_0)(z - z_0)\, dz = 0.$$

Fix an $\epsilon > 0$. We can find $\delta > 0$ such that

$$||\psi||_{C^0(B_\delta z_0 \cap U)} < \epsilon$$

and $\Delta_n \subseteq B_\delta z_0$. From there,

$$\begin{aligned}\left|\int_{\partial\Delta_n} f\right| &= \left|\int_{\partial\Delta_n} f(z) - (f(z_0 + f'(z_0)(z-z_0))\ dz\right| \\ &= \left|\int_{\partial\Delta_n} (z-z_0)\psi(z)\ dz\right| \\ &\leq \ell(\partial\Delta_n) \sup_{z\in\partial\Delta_n} |z-z_0||\psi(z)| \\ &\leq \ell(\partial\Delta_n)^2 ||\psi||_{C^0} \\ &< \epsilon\ell(\partial\Delta_n)^2.\end{aligned}$$

In the second case, p is a vertex of Δ_0, say $p = z_1$. For $\epsilon > 0$ choose p_2, p_3 along the segments $[z_1, z_2]$ and $[z_2, z_3]$ with $|z_1, p_i| < \epsilon$. This determines a triangle Δ_1 by (z_1, p_2, p_3), and $\Delta_0 \setminus \Delta_1$ is a polygon which can be triangulated, say by $\Delta_2, \ldots$.

Then if $k \geq 2$, f is holomorphic on Δ_k and so its integral along $\partial\Delta_k$ vanishes. In particular,

$$\left|\int_{\partial\Delta_0} f\right| \leq \int_{\partial\Delta_1} |f| < 3\epsilon||f||_{C^0}$$

so its integral vanishes.

Finally if $p \in \Delta_0$ and it is not a vertex, then we can triangulate Δ_0 such that p is a vertex and the theorem follows. □

We need continuity at p because if not, then f might be horribly behaved. If we define

$$f(z) = \begin{cases} z^{-1}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

then Cauchy-Goursat fails. See 2.11 for proof.

Theorem 3.7 (fundamental theorem of calculus, part II)**.** *Suppose that U is convex, and $f : U \to \mathbb{C}$ is continuous.*

If for each triangle $\Delta \subseteq U$,

$$\int_{\partial\Delta} f = 0$$

then for each $a \in U$, the function

$$F(z) = \int_a^z f(t)\ dt$$

is holomorphic on U and $F' = f$.

Proof. Fix $z_0 \in U$ and allow z to range over U. Then if Δ is the triangle given by (z, z_0, a) we have

$$0 = \int_{\partial\Delta} f = \int_a^z f + \int_z^{z_0} f + \int_{z_0}^a f = F(z) - F(z_0) + \int_z^{z_0} f$$

while

$$\int_{z_0}^z f(z_0)\ dw = f(z_0)(z - z_0)$$

and

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{z_0}^z f(w) - f(z_0)\ dw.$$

Fix $\epsilon > 0$. By continuity we can find a $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &\leq \frac{1}{|z - z_0|}\int_z^{z_0} |f(w) - f(z_0)|\ dw \\ &< \frac{1}{|z - z_0|}\ell(z, z_0)\epsilon C = \epsilon C\end{aligned}$$

for some constant C. Convexity guarantees that all these segments are in fact contained in U. □

3.3 Cauchy's integral formula

It will be useful to adopt some nonstandard terminology for dealing with the corollaries of Cauchy-Goursat. The true power of Cauchy-Goursat is that it holds even if we don't know that f is holomorphic everywhere; we only need f to satisfy these conditions:

Definition 3.8. If $f : U \to \mathbb{C}$ is continuous and holomorphic on all but finitely many points of U, we say that f is *cofinitely holomorphic.*

As it turns out, a cofinitely holomorphic function will end up being holomorphic, which is why nobody actually uses this terminology.

Corollary 3.9. *Suppose γ is closed in U convex, $a \in U$ and $f : U \to \mathbb{C}$ is cofinitely holomorphic.*
Then the function

$$F(z) = \int_a^z f$$

has $F' = f$ and is holomorphic and moreover

$$\int_\gamma f = 0.$$

Corollary 3.10 (Cauchy's integral formula)**.** *Suppose γ is closed in U convex and $f : U \to \mathbb{C}$ is cofinitely holomorphic. Then*

$$f(z)\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw.$$

Proof. Define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(w), & w = z. \end{cases}$$

g is continuous and holomorphic away from z so

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma g \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dw - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dw - f(z)\operatorname{Ind}_\gamma(z). \end{aligned}$$

□

Cauchy's integral formula tells us that if f is cofinitely holomorphic on γ, we already know everything there is to know about f in the region enclosed by γ. This will be a theme we see over and over.

Now we're ready for the sockdolager:

Corollary 3.11. *Suppose U is convex and $f : U \to \mathbb{C}$.*
If f is holomorphic, then it is analytic.
If $z_0 \in U$, $r > 0$ and $B_r(z_0) \subseteq U$, we can find α_n such that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

with radius of convergence $\geq r$.
Moreover, there is a unique g holomorphic with the maximal radius of convergence afforded by this theorem, such that when g is restricted to the appropriate ball, $g = f$.

Proof. Fix $\rho \in (0, r)$. Then $\Gamma_\rho z_0 \subseteq U$ and if $z \in B_\rho z_0$, then $\operatorname{Ind}_{\Gamma_\rho} z = 1$. Moreover, if $\eta \in (\rho, r)$, then the ball $B_\eta z_0$ is convex.

So,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho z_0} \frac{f(w)}{w - z}\, dw.$$

This is analytic by 3.1 with the desired power series. □

Definition 3.12. With hypotheses as in 3.11, g is the *analytic continuation* of f.

Now we see why the notion of a cofinitely holomorphic function is utterly useless: we can prove results about open balls in U, which are convex. (This is why we demand that the domain U is open, so that it will be locally convex.) By induction, we have:

Corollary 3.13 (Morera). *Let $f : U \to \mathbb{C}$. The following are equivalent:*

1. *f is holomorphic.*
2. *For each $\Delta \subseteq U$, $\int_{\partial\Delta} f = 0$.*
3. *$f^{(n)}$ is holomorphic.*
4. *f is analytic.*
5. *$f^{(n)}$ is analytic.*

This is huge; it's definitely not true in $\mathbb{R}^n$!

Corollary 3.14 (Cauchy's integral formula for derivatives). *Suppose $f : U \to \mathbb{C}$ is holomorphic, $a \in U$, and $r > 0$ with $\overline{B_r a} \subset U$. Then*

$$|f^{(n)}(z)| = \frac{n!}{2\pi i} \int_{\Gamma_r a} \frac{f(w)}{(w - z)^{n+1}}\, dw.$$

Proof. Apply Cauchy's formula and induct on the Taylor coefficients, using 1.25 to ensure all the balls make sense. □

3.4 Cauchy's estimate

By using the estimate 2.9, the following, very useful estimate follows.

Corollary 3.15 (Cauchy's estimate). *With hypotheses as in 3.14,*

$$|f^{(n)}(a)| \le \frac{C}{r^n} \max_{w \in \partial B_r a} |f(w)|$$

for some constant C which only depends on n.

This has important implications in algebra, of all places.

Definition 3.16. A holomorphic function is *entire* if its domain is $\mathbb{C}$.

Theorem 3.17 (Liouville). *An entire function is constant or unbounded.*

Proof. Suppose $f : \mathbb{C} \to \mathbb{C}$ and $|f(z)| \le M$. Then for each $r > 0$, the estimate

$$|f'(z)| \le \frac{M}{r}$$

holds. In particular $f' \equiv 0$. □

The following estimate is true clearly for linear polynomials, and then by induction and iterated differentiation one has:

Lemma 3.18. *Let $f \neq 0$ be a polynomial. Then we have $\mu > 0$ and $R \geq 1$ such that for all z with $|z| > R$,*

$$|f(z)| \geq \mu |z|.$$

So polynomials necessarily grow without bound.

The punchline is that polynomials must be zero *somewhere* in $\mathbb{C}$, though this is clearly not true in $\mathbb{R}$.

Theorem 3.19 (Fundamental theorem of algebra)**.** *A nonconstant polynomial has a root.*

Proof. If not, then $1/f$ is bounded on $\mathbb{C} \setminus B_r 0$ for some $r > 0$. But $1/f$ is bounded on the compact set $\overline{B_r 0}$, because the lack of roots guarantees its continuity. So $1/f$ is constant. □

3.5 Locally uniform convergence

Another consequence of Cauchy-Goursat is that we don't actually care that much about uniform convergence in $\mathbb{C}$: it's locally uniform convergence that's interesting.

Definition 3.20. Suppose $f_n \to f$ is a convergent sequence of functions in U. If, for each $z \in U$, there is a neighborhood $V \ni z$ on which $f_n \to f$ uniformly, then $f_n \to f$ *locally uniformly.*

Theorem 3.21 (Weierstrass)**.** *Suppose $f_n \to f$ locally uniformly and each f_n is holomorphic. Then f is holomorphic and convergence of its derivatives is locally uniform.*

Proof. For $a \in U$ there exists $r > 0$ with $B_r a \subseteq U$ and $f_n \to f$ uniformly on $B_r a$. So f is continuous on $B_r a$. By Morera's theorem, if $\Delta \subset B_r a$, then

$$\oint_{\partial \Delta} f = 0$$

whence f is holomorphic.

Suppose now that $f_n \to f$ uniformly on $B_{2r} a$, then $f_n \to f$ uniformly on $\partial B_r a$. By Cauchy's estimate, we can bound differences in f' by differences in $\partial B_r a$. □

Corollary 3.22. *Suppose that $r > 0$ and $f_n \to f$ locally uniformly on $B_r 0$, with each f_n holomorphic. Suppose further that*

$$f_m(z) = \sum_{n=0}^{\infty} \alpha_n^{(m)} z^n$$

and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n;$$

then $\alpha_n^{(m)} \to \alpha_n$.

Proof. By induction on Weierstrass's theorem. □

The strategy to show that a limit function f is holomorphic is as follows: use the M-test to show uniform convergence on each open set, which is enough to apply Weierstrass's theorem.

Chapter 4

Zeroes and poles

Nothing too exciting has happened yet, because we've been hampered by the assumption that U is convex, and that f is continuous even when it fails to be holomorphic. As it turns out, local convexity is all we really need, and complex analysis becomes more interesting when f behaves more spectacularly at points where it's not holomorphic.

Recall that U is assumed open.

4.1 Zeroes of holomorphic functions

Definition 4.1. If $a \in U$, f holomorphic with $f(a) = 0$, we say that a is an *isolated zero* if it is contained in a punctured ball on which f is nonzero. We further say that a is of *order* $N > 0$ if each $f^{(n)}(a) = 0$ for $n < N$ and $f^{(N)} \neq 0$. If, for each $n > 0$, $f^{(n)}(a) = 0$, then a is a *zero of infinite order.*

We'll soon see that the only interesting holomorphic functions are those which have only finite-order, isolated zeroes; if not, they'll just be identically zero!

We can "factor" out the zeroes of a holomorphic function, just like a polynomial.

Theorem 4.2. *If $a \in U$, f holomorphic with $f(a) = 0$, then $\exists r > 0$ with $B_r a \subseteq U$ such that exactly one of the following is true:*

1. *$f \equiv 0$ on $B_r a$ and a is a zero of infinite order.*
2. *a is an isolated zero and there exists a unique $g : B_r a \to \mathbb{C}$ holomorphic with $|g| > 0$ and $f(z) = (z-a)^N g(z)$ where N is the order of a.*

Proof. We can pick $B_R a \subseteq U$ on which $f(z) = \sum \alpha_n (z-a)^n$ uniformly and

$$\alpha_n = \frac{f^{(n)}(a)}{n!}.$$

Then if for each n, $f^{(n)}(a) = 0$, $\alpha_n = 0$, so $f \equiv 0$.

Otherwise it's of order $N < \infty$ and

$$f(z) = \sum_{n=N}^{\infty} \alpha_n (z-a)^n = (z-a)^N \sum_{k=0}^{\infty} \alpha_{N+k} (z-a)^k$$

and this last series is g. Moreover, for R sufficiently small, $g \neq 0$, because a is isolated by continuity of g. $\square$

Because locally, holomorphic functions act like polynomials, we can rattle off several corollaries which resemble properties of polynomials.

Corollary 4.3. *If $a \in U$, f holomorphic then $f(a) = 0$ of order N iff*

$$\lim_{z \to a} \frac{f(z)}{(z-a)^N}$$

exists and is nonzero.

Proof. One direction is obvious. Otherwise we have $f(a) = 0$, and the order is finite. So we can factor out a g, which is the limit, which is continuous and nonzero. $\square$

The following proof is an example of an "open and closed" argument, which will be a powerful tool for proving theorems about holomorphic functions on connected sets.

Corollary 4.4. *If U is connected, $B_r a \subseteq U$, and f is holomorphic with $f|_{B_r a} \equiv 0$ then $f \equiv 0$.*

Proof. The set

$$V = \{p \in U : \exists s > 0 \ f|B_s p \equiv 0\}$$

is a union of open balls, thus open. On the other hand, the set

$$W = \{p \in U : \exists s > 0 \ \forall z \in B_s p \setminus \{0\} \ f(z) \neq 0\}$$

is also a union of open balls, thus also open, and $U = V \cup W$, $\emptyset = V \cap W$. Since U is connected and $a \in V$, $W = \emptyset$. $\square$

Corollary 4.5. *Consider U connected $\supseteq V \neq \emptyset$ open, and f, g holomorphic, if $f|_V = g|_V$ then $f = g$.*

Proof. $(f-g)|_V \equiv 0$ so $f - g \equiv 0$. $\square$

In other words, if U is connected, then the behavior of f is completely determined by its behavior on any open set in U. This isn't too surprising, as an open set is all we need to construct all the Taylor coefficients of f.

In more extreme cases, we can characterize f by its behavior on a *sequence* in U.

Corollary 4.6. *For U connected, if there exists a sequence $z_n \to z$ of distinct points with $z \in U$, and $f(z_n) = 0$ for each z_n then $f \equiv 0$.*

Proof. The zeroes of f cluster at z, so z is not an isolated zero; thus, z must be a zero of infinite order. $\square$

Notice that we need $z \in U$; the counterexample is $z \mapsto \sin \pi/z$, $z_n = 1/n$, whose zeroes cluster at 0, where the map is not holomorphic.

4.2 Isolated singularities

Definition 4.7. If f is holomorphic except at $a \in U$, then a is an *isolated singularity* of f. Moreover, if there is a holomorphic function $g : U \to \mathbb{C}$ such that $f \equiv g$ off of a, then a is *removable.*

For convenience, we'll say that g is a *holomorphic extension*, but this is nonstandard terminology.

The classic example of a removable singularity is -1 for the function

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}$$

which morally is $g(x) = x + 1$. Removable singularities are mostly harmless.

Theorem 4.8 (Riemann)**.** *If f has an isolated singularity at $a \in U$ and there is a neighborhood $V \ni a$ on which f is bounded, then a is removable.*

Proof. Consider $h : U \to \mathbb{C}$ with $h(z) = (z-a)f(z)$ away from a and $h(a) = 0$. h is continuous because f is bounded, and is clearly holomorphic away from a, thus cofinitely holomorphic, so Morera's theorem applies and h is holomorphic.

So, by 2.2, there exists g continuous with $h(z) = (z-a)g(z)$ (since $h(a) = 0$). But then by Cauchy-Goursat, g is holomorphic. But $g \equiv f$ away from a. $\square$

Since, in order for an isolated singularity a to fail to be removable, f must be unbounded near a, the behavior of f near a must be calamitous. In particular, case (3) of the following theorem is shocking. (And I could make a good YouTuber... "10 shocking facts about isolated singularities you won't believe! (3) is INSANE!)"

Theorem 4.9 (Casorati-Weierstrass). *If f has an isolated singularity at $a \in U$ then exactly one of the following is true.*

1. *a is removable.*
2. *There exists $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in \mathbb{C}$ such that $c_m \neq 0$ and the function*
$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$
has a removable singularity at a.
3. *For each neighborhood V of a, $f(V)$ is dense in $\mathbb{C}$.*

Before the proof, we define what's going on in (2) and (3).

Definition 4.10. If (2) holds, then a is a *pole* of order m. A pole of order 1 is *simple*.
If (3) holds, then a is an *essential singularity.*

Proof of Casorati-Weierstrass. Suppose that (3) does not hold. Then $f(V)$ isn't dense somewhere in $\mathbb{C}$, so there exist $w \in \mathbb{C}$, $r > 0$, and $\mu > 0$ with $B_r a \subseteq U$ and for $z \in B_r a \setminus \{a\}$, $|f(z) - w| > \mu$.

Define $g : B_r a \setminus \{a\} \to \mathbb{C}$ by
$$g(z) = \frac{1}{f(z) - w}.$$
g is holomorphic, but also bounded because
$$\frac{1}{|f(z) - w|} < \frac{1}{\mu},$$
so by Riemann, g has a holomorphic extension $h : B_r a \to \mathbb{C}$.

Either $h(a) = 0$ or not. If not, then $f(z) - w$ does not tend to ∞ at a, so f is bounded and therefore has a removable singularity. So (1) holds.

Otherwise, a is a zero of order $m < \infty$ for h. So $\exists b : B_r a \to \mathbb{C}$ holomorphic such that $b(a) \neq 0$ and $h(z) = (z-a)^m b(z)$ and
$$f(z) = w + \frac{1}{b(z)(z-a)^m}.$$
But $1/b \neq 0$ so $1/b$ is holomorphic and has a power series,
$$\frac{1}{b(z)} = \sum_{n=0}^{\infty} \alpha_n (z-a)^n$$
whence
$$f(z) = w + \sum_{n=0}^{\infty} \alpha_n (z-a)^{n-m}.$$

When $n > m$ these terms are benign and can be absorbed into the Taylor series for f. Otherwise, $c_i = \alpha_i$, implying (2). $\square$

As it turns out, if (3) holds, then something much stronger is true:

Theorem 4.11 (Picard's great theorem)**.** *Suppose that $f : U \to \mathbb{C}$ is holomorphic and has an essential singularity at $a \in \mathbb{C}$. Then for each neighborhood $V \ni a$, $\mathbb{C} \setminus f(V)$ is finite.*

Sadly, we won't develop the machinery to prove this theorem. But indeed, as promised, essential singularities are calamitous.

A quick glance at the series in (2) confirms:

Corollary 4.12. *If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then a is a pole of order m iff*

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

and is finite.

The next theorem will show why we care about isolated singularities so much: integrals around closed curves only are interesting if the curve encloses singularities.

Theorem 4.13 (homotopy theorem)**.** *If $f : U \to \mathbb{C}$ is holomorphic and γ_0 and γ_1 are U-homotopic, closed curves then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

In particular, if γ_0 is U-nullhomotopic then

$$\int_{\gamma_0} f = 0.$$

Proof. Consider a homotopy $\Phi : [0,1]^2 \to U$ between γ_0 and γ_1. $[0,1]^2$ is compact, so there are $\varepsilon > 0$ and $N > 0$ such that for each (t_1, s_1) and $(t_2, s_2) \in [0,1]^2$, if $|(t_1, s_1) - (t_2, s_2)| < 2/N$ then $|\Phi(t_1, s_1) - \Phi(t_2, s_2)| < \varepsilon$. (That is, if N is large, then we can divide $[0,1]^2$ into tiles of size $\leq 2/N$ whose images will be within ε of each other.)

Put $z_{nm} = \Phi(n/N, m/N)$ for $n, m \leq N$. In particular if $k, j \leq 1$ then $|z_{nm} - z_{n-k,m-j}| < \varepsilon$ and therefore is contained in U. The ball around z_{nm} is *convex*, so

$$\int_{z_{n-1,m-1}}^{z_{n,m-1}} f + \int_{z_{n,m-1}}^{z_{nm}} f - \int_{z_{n-1,m-1}}^{z_{n-1,m}} f - \int_{z_{n-1,m}}^{z_{nm}} f = 0$$

because the curve generated when the last two integrals are reversed is closed.

The z_{nm} tile $[0,1]^2$ so

$$\sum_{j=1}^{N} \int_{z_{j-1,0}}^{z_{j,0}} f = \sum_{j=1}^{N} \int_{z_{j-1,N}}^{z_{j,N}} f.$$

Each of these is a discretization of γ_0 or γ_1, so the result holds. $\square$

4.3 Cauchy-Goursat's true power

We needed to painstakingly prove Cauchy-Goursat and related theorems for convex sets precisely as lemmata to the homotopy theorem: we used that U was locally convex in its proof.

Now that we know the homotopy theorem, all those old results generalize trivially to simply connected sets.

Corollary 4.14 (Cauchy-Goursat)**.** *Suppose that U is simply connected and $f : U \to \mathbb{C}$ is holomorphic. Then if γ is closed,*

$$\int_\gamma f = 0.$$

Corollary 4.15 (fundamental theorem of calculus)**.** *If U is simply connected, and $f : U \to \mathbb{C}$ is holomorphic, then the function $F : U \to \mathbb{C}$ given by*

$$F(z) = \int_\gamma f$$

is holomorphic with $F'(z) = f(z)$, where $\gamma(0)$ is constant and $\gamma(1) = z$.

4.4 The complex logarithm

TODO: Rewrite all of this mess.

Now we can define the logarithm function, but we have to be a bit careful.

Corollary 4.16. *If U is connected and simply connected and $f : U \to \mathbb{C}$ is holomorphic and nonzero, then there exists $g : U \to \mathbb{C}$ such that $e^g = f$.*

Proof. f'/f is holomorphic, so it has an antiderivative F. Then $(fe^F)' = 0$. So, fe^F is constant. For $a \in U$, there exists $w \in \mathbb{C}$ such that $e^w = f(a)$ since the image of exp is $\mathbb{C} \setminus \{0\}$. So

$$f(z) = f(z)e^{-F}e^F = e^{-w-F(a)-F(z)}.$$

□

Corollary 4.17. *If $f : U \to \mathbb{C}$ is holomorphic, γ_0 and $\gamma_1 : [0,1] \to U$, $\gamma_0(0) = \gamma_1(0)$, and $\gamma_0(1) = \gamma_1(1)$, and there is a continuous $\Phi : [0,1]^2 \to U$ such that $\Phi(t,j) = \gamma_j(t)$, $\Phi(j,s) = \gamma_0(j)$ for each $s \in [0,1]$, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. There exists a homotopy $\tilde{\Phi}$ constructible just by drawing a picture of Φ. □

Recall that $z \mapsto 1/z$ doesn't have an antiderivative, but we want to define log to be its antiderivative. Still, the integral makes sense if the domain of log is simply connected.

Definition 4.18.

$$\log z = \int_1^z \frac{dw}{w}$$

for $z \notin (-\infty, 0]$.

Lemma 4.19.

1. log *is holomorphic.*
2. $\log' = 1/z$.
3. $e^{\log z} = z$.
4. $$\log z = \log|z| + 2i\tan^{-1}\frac{\operatorname{Im} z}{\operatorname{Re} z + |z|}.$$
5. log *is a bijection* $\mathbb{C} \setminus (-\infty, 0] \to \mathbb{R} \times i[-\pi, \pi]$.

Proof. Most of this is high school trig. The hard part is to show that if $\log z = x + iy$ then $|y| < \pi$. Indeed, $y \neq \pi + 2\pi n$ for $n \neq 0$, so it follows by continuity. □

Now we can define exponentiation.

Lemma 4.20. $a^n = e^{n \log a}$.

Proof. Use calculus. □

Definition 4.21. For $a \in \mathbb{C} \setminus (-\infty, 0]$, and $z \in \mathbb{C}$, define

$$a^z = e^{z \log a}.$$

But this function behaves very pathologically away from $\mathbb{R}$.

Chapter 5

The residue calculus

The residue calculus is a powerful technique for dealing with poles. It will allow us to compute integrals around closed curves just by summing up a few easy limits – and to compute integrals around curves which aren't closed, including integrals in $\mathbb{R}$, by extending them to a closed curve without affecting their value. It will also allow us to characterize holomorphic functions further as geometric transformations which "preserve angle".

Recall that U is assumed open. Furthermore we will assume $0 \leq R_1 < R_2 \leq \infty$. That's so we can make the following definition:

Definition 5.1. If $a \in \mathbb{C}$ define the *annulus*

$$A(a, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z - a| < R_2\}.$$

The annulus $A(a, 0, \infty)$ is just the plane punctured at $\mathbb{C}$.

5.1 Laurent series

Analogously to the definition of the improper integral $\int_{-\infty}^{\infty}$, we can define a "two sided" series:

Definition 5.2. For each $k \in \mathbb{Z}$, let $z_k \in \mathbb{C}$. The series $\sum z_k$ *converges* if $\sum_{k \geq 0} z_k$ and $\sum_{k > 0} z_{-k}$ converges. Moreover, its sum is

$$\sum_{k=-\infty}^{\infty} z_k = \sum_{k=0}^{\infty} z_k + \sum_{k=1}^{\infty} z_{-k}.$$

We'll need to be able to sum in both directions to deal with poles, and to be able to sum in both directions infinitely far to deal with essential singularities.

Definition 5.3. If $a \in \mathbb{C}$, a *Laurent series* about a is a series

$$\sum_{k=-\infty}^{\infty} \alpha_k (z - a)^k$$

with coefficients $\alpha_k \in \mathbb{C}$.

Laurent series are used to describe functions which are holomorphic away from a singularity.

Just like when we developed Taylor series, it was convenient to assume that the "center" $a = 0$. Here the series will always be centered on 0. So, for the next theorem, we'll assume f is singular at the origin.

Lemma 5.4. *If f is holomorphic on $A(0, R_1, R_2)$ and $r \in (R_1, R_2)$ then define, for each $k \in \mathbb{Z}$,*

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_r 0} \frac{f(w)}{w^{k+1}} \, dw.$$

Then α_k is independent of r, and the series

$$\sum_{k=-\infty}^{\infty} \alpha_k z^k = f(z).$$

Proof. Independence of r follows by the homotopy theorem: we could choose any curve which winds around $B_{R_1}0$.

For $z \in A(0, R_1, R_2)$, choose r_1, r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$, and define

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

on $A(0, R_1, R_2) \setminus \{z\}$. Then g has a removable singularity at z, since f is holomorphic and thus has

$$|f'(z)| = \lim_{w \to z} \left| \frac{f(w) - f(z)}{w - z} \right| < \infty.$$

We'll identify g with its holomorphic extension.

$\Gamma_{r_1}0$ and $\Gamma_{r_2}0$ are homotopic so

$$\int_{\Gamma_{r_1}0} g = \int_{\Gamma_{r_2}0} g.$$

Splitting up g, we have

$$\int_{\Gamma_{r_2}0} \frac{f(w)}{w - z}\ dw - \int_{\Gamma_{r_1}0} \frac{f(w)}{w - z}\ dw = f(z) \left[\int_{\Gamma_{r_2}0} \frac{dw}{w - z} - \int_{\Gamma_{r_1}0} \frac{dw}{w - z} \right].$$

Notice that $(w - z)^{-1}$ is not holomorphic, but it is holomorphic away from z, and that $\Gamma_{r_1}0$ does not loop around w. So the final integral vanishes, and a now familiar computation shows that

$$\left[\int_{\Gamma_{r_2}0} - \int_{\Gamma_{r_1}0} \right] \frac{f(w)}{w - z}\ dw = 2\pi i f(z).$$

On $\partial B_{r_1}0$, $h_n(w) = (w/z)^n$ has $\sum h_n$ uniformly convergent.

Thus,

$$\begin{aligned}
-\frac{1}{2\pi i} \int_{\Gamma_{r_1}0} \frac{f(w)}{w - z}\ dw &= \frac{1}{2\pi i} \int_{\Gamma_{r_1}0} \frac{f(w)}{1 - w/z}\ dw \\
&= \frac{1}{2\pi i} \frac{1}{z} \int_{\Gamma_{r_1}0} f(w) \sum_{n=0}^{\infty} \frac{w^n}{z}\ dw \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \frac{1}{z^{n+1}} \int_{\Gamma_{r_1}0} f(w) w^n\ dw \\
&= \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}.
\end{aligned}$$

On the other hand,

$$\int_{\Gamma_{r_2}0} \frac{f(w)}{w - z}\ dw = 2\pi i \sum_{n=0}^{\infty} \alpha_n z^n$$

as desired. $\square$

In other words, 3.14 holds even when f has singularities, as long as we allow n to range over $\mathbb{Z}$.

Definition 5.5. With notation as in 5.4, $\sum \alpha_k(z-a)^k$ is the *Laurent series* of f about a. Define $h : A(a, R_1, \infty) \to \mathbb{C}$ by

$$h(z) = \sum_{k=-\infty}^{-1} \alpha_k(z-a)^k.$$

Then h is called the *principal part* of f about a. If $R_1 = 0$, we say that α_{-1} is the *residue* of f at a and write $\alpha_{-1} = \operatorname{Res}_a f$.

We can now rephrase the Casorati-Weierstrass theorem in terms of Laurent series.

Corollary 5.6 (Casorati-Weierstrass)**.** *Let f have an isolated singularity at $a \in U$ and*

$$f(z) = \sum_{k=-\infty}^{\infty} \alpha_k(z-a)^k.$$

Let $m \in \mathbb{N}$ and h be the principal part of f. Then:

1. *a is removable iff $\forall k \leq -1$, $\alpha_k = 0$.*
2. *a is an isolated pole of order m iff $\forall k < -m$, $\alpha_k = 0$ and $\alpha_{-m} \neq 0$.*
3. *a is an essential singularity iff there are infinitely many nonzero α_k $(k < 0)$.*
4. *$f - h$ has a removable singularity at a.*

We can also rephrase some of the results about factoring out zeroes and poles from holomorphic functions. This will be our main tool for computing residues.

Corollary 5.7. *Suppose that f has an isolated pole of order m at a. Then*

$$\operatorname{Res}_a f = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z).$$

In particular, if f has a simple pole at a, then

$$\operatorname{Res}_a f = \lim_{z \to a} (z-a) f(z).$$

5.2 The residue theorem

Notice that each of the terms in the Laurent series other than the residue have an antiderivative. In particular, if γ is closed and U-nullhomotopic, and f is holomorphic off of a, then each term of the Laurent series except the residue vanishes, and

$$\frac{1}{2\pi i} \int_\gamma f = \frac{\alpha_{-1}}{2\pi i} \int_\gamma \frac{dz}{w-z} = \alpha_{-1} \operatorname{Ind}_\gamma a = \operatorname{Res}_a f \operatorname{Ind}_\gamma a$$

where α_k are the terms of the Laurent series of f.

As a result, if f has no essential singularities, then the only interesting term of the principal part is the residue, which encodes valuable information about f. Moreover, the following theorem is motivated:

Theorem 5.8 (residue theorem)**.** *Let $A \subset U$ be discrete and $f : U \setminus A \to \mathbb{C}$ be holomorphic. Suppose that for each $a \in A$, f has an isolated singularity at a.*

If $\gamma^* \subset U \setminus A$ and γ is closed and U-nullhomotopic, then

$$A' = \{a \in A : |\operatorname{Ind}_\gamma a| > 0\}$$

is finite. Moreover,

$$\frac{1}{2\pi i} \int_\gamma f = \sum_{a_k \in A'} \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

Proof. Let Φ be a nullhomotopy of γ with image K. Then $K = \Phi([0,1]^2)$, which is compact.

Moreover, if $a \in A \setminus K$, then $\operatorname{Ind}_\gamma a = 0$. Therefore $A' \subseteq A \cap K$. Moreover, $A \cap K$ is infinite it must cluster somewhere in K by the Bolzano-Weierstrass theorem – but we assumed that A was discrete, so $A \cap K$ is finite. In particular A' is finite.

$V = (U \setminus A) \cup A'$ is therefore open and γ is therefore V-nullhomotopic. Let h_k be the principal part of f centered on a_k. Then $g = f - \sum h_k$ is holomorphic on V, so its integral vanishes. Then

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_\gamma \sum_{k=1}^N h_k = \sum_{k=1}^N \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

□

The following lemma will allow us to construct closed curves so we can actually apply the residue theorem. The idea is that we can start with a curve that is not closed and draw a harmless arc through the half-plane, on which the integral will vanish.

Lemma 5.9 (Jordan)**.** *Let $f : \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \to \mathbb{C}$ be continuous and*

$$\lim_{R\to\infty} \sup_{\theta\in[0,\pi]} |f(Re^{i\theta})| = 0.$$

Define $\gamma_R : [0,\pi] \to \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ by $\gamma_R(\theta) = Re^{i\theta}$. For $m > 0$,

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z)\ dz = 0.$$

Proof. TODO: This. □

Before Mathematica, mathematicians would use the residue theorem to compute integrals from real analysis. Even now, we tend to want to use some concrete examples to illustrate the power of the residue theorem, rather than the way algebraists learn new theorems, which is to rephrase everything in terms of category theory.

Example 5.10. Let's compute

$$\int_0^\infty \frac{\sin x}{x}\ dx.$$

Put $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = z^{-1}e^{iz}$. For $\theta \in [0,\pi]$, consider

$$\begin{cases} \mu_\varepsilon(\theta) & = \varepsilon e^{i(\pi-\theta)} \\ \gamma_R(\theta) & = Re^{i\theta}. \end{cases}$$

Then $\gamma_R \oplus [-R,-\varepsilon] \oplus \mu_\varepsilon \oplus [\varepsilon, R]$ is closed and $\mathbb{C} \setminus \{0\}$-nullhomotopic.

By Jordan's lemma applied to γ_R and the residue theorem,

$$\left[\int_\varepsilon^R + \int_{\gamma_R} + \int_{-R}^\varepsilon + \int_{\mu_\epsilon}\right] f = \left[\int_\varepsilon^R + \int_{-R}^\varepsilon + \int_{\mu_\epsilon}\right] f = 0.$$

But

$$\left[\int_\varepsilon^R + \int_{-R}^{-\varepsilon}\right] f = 2i\int_\varepsilon^R \frac{\sin x}{x}\ dx$$

and

$$\lim_{\varepsilon\to 0}\int_{\mu_\varepsilon} f = \lim_{\varepsilon\to 0}\int_0^\pi \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\ d\theta = \lim_{\varepsilon\to 0} -i\int_0^\pi e^{i\varepsilon e^{i\theta}}\ d\theta = -i\pi.$$

So,

$$\int_0^R \frac{\sin x}{x}\ dx = \frac{i\pi}{2i} = \frac{\pi}{2}.$$

Recall that for $\operatorname{Re} z > 1$, the Riemann ζ-function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Example 5.11. We will compute $\zeta(2)$. This works for any $2n$, $n \geq 1$.

Let $g : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ be $g(z) = \cot \pi z$. Then g is holomorphic, and if $z \in (0,1) \times i\mathbb{R}$, $k \in \mathbb{Z}$, $g(z+k) = g(z)$. Moreover, for each compact $K \subseteq \mathbb{C} \setminus \mathbb{Z}$, g is bounded. In particular, for $\delta > 0$, g is bounded on $K_k = [k+\delta, k+1-\delta]$. g is also bounded outside of $\{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$, as is clear if one rewrites g in terms of the complex exponential. So g has poles precisely on $\mathbb{Z}$.

If $f(z) = z^{-2} g(z)$ then

$$\lim_{N \to \infty} \int_{\Gamma_{N+\delta} 0} f = 0$$

since by Jordan's lemma applied twice. Moreover, close to 0, g satisfies

$$\frac{1 + o(z^2)}{\pi z + o(z^2)} = \frac{1}{z} \frac{1}{\pi}.$$

In particular, $\operatorname{Res}_0 g = \pi^{-1}$, so

$$\operatorname{Res}_n f = \frac{1}{\pi n^2}.$$

At 0, f has a pole of third order, so by the zero-factoring corollary, $\operatorname{Res}_0 f = -\pi/3$.

It follows that

$$0 = \lim_{N \to \infty} \lim_{\delta \to 0} \int_{\Gamma_{N+\delta} 0} f = \lim_{N \to \infty} 2 \sum_{k=1}^{N} \frac{1}{\pi k^2} - \frac{\pi}{3} = \lim_{N \to \infty} 2 \sum_{k=1}^{\infty} \frac{1}{\pi k^2} - \frac{\pi}{3} = 2\zeta(2) - \frac{\pi}{3}$$

whence

$$\zeta(2) = \frac{\pi^2}{6}.$$

5.3 The argument principle

The residue theorem will allow us to prove the argument principle, a powerful combinatorial theorem with lots of consequences.

Definition 5.12. γ is *simple* if $\operatorname{Ind}_\gamma(\mathbb{C} \setminus \gamma^*) = \{0, 1\}$.

Simple curves are those which wind around exactly once, counterclockwise. They're deformations of counterclockwise circles.

Theorem 5.13 (argument principle)**.** *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \operatorname{Ind}_\gamma(a) = 1\},$$

$A \subset U$ is finite, $f : U \setminus A \to \mathbb{C}$ is holomorphic, and f has N zeroes and P poles in U_1 (repeated by order).

If, for each $a \in A$, f has a finite-order pole at a, and has no poles and zeroes on γ^*, then

$$N - P = \operatorname{Ind}_{f \circ \gamma}(0).$$

Proof. Since the poles are isolated, $U \setminus A$ is connected. Similarly

$$V = U \setminus (A \cup \{z \in U : f(z) = 0\})$$

is open, and $|f| > 0$ on V. Define $g : V \to \mathbb{C}$ by $g = f'/f$ which is holomorphic.

Suppose that f has a zero at a of order m. Then there exists a *locally* holomorphic h, nonzero at a, given by $f(z) = (z-a)^m h(z)$. Then, locally,

$$g(z) = \frac{mh(z)}{(z-a)h(z)} + \frac{h'(z)}{h(z)}$$

and the second term is holomorphic. The first term has a residue of m.

Repeating the argument for poles (with signs swapped), we can apply the residue theorem to the integral

$$\frac{1}{2\pi i}\int_\gamma g = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

□

Definition 5.14. Suppose that $A \subset U$ is discrete, $f : U \setminus A \to \mathbb{C}$ is holomorphic, and f has no essential singularities. Then we say that f is *meromorphic*.

So, meromorphic functions have poles, but no essential or nonisolated singularities.

The argument principle is a counting principle: if γ winds around U once, and f is meromorphic on U, then we can count the zeroes and poles, and their difference will be $\mathrm{Ind}_{f\circ\gamma}(0)$. Why 0? Because we're counting zeroes! If we were counting the points where $f = b$, we would have $\mathrm{Ind}_{f\circ\gamma}(b)$.

Lemma 5.15. *Suppose $f : U \to \mathbb{C}$ is holomorphic, γ closed in U, and $b \in U \setminus \gamma^*$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}\,dz = \mathrm{Ind}_{f\circ\gamma}(b).$$

Proof. We can assume γ is smooth, so $f\circ\gamma$ is holomorphic, and that the domain of γ is $[0,1]$. Then

$$\begin{aligned}\mathrm{Ind}_{f\circ\gamma}(b) &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dz}{z-b} = \frac{1}{2\pi i}\int_0^1 \frac{(f\circ\gamma)'(t)\,dt}{(f\circ\gamma)(t)-b}\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t))-b} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-b}.\end{aligned}$$

□

Corollary 5.16. *With hypotheses as in 5.13, $A = \emptyset$, and $b \in \mathbb{C}\setminus f(\gamma^*)$, $f - b$ has $\mathrm{Ind}_{f\circ\gamma}(b)$ zeroes in U_1.*

5.4 Rouche's theorem

We can now see that adding a function g which is smaller than f to f won't affect its number of zeroes.

Theorem 5.17 (Rouche)**.** *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \mathrm{Ind}_\gamma(a) = 1\},$$

$f, g : U \to \mathbb{C}$ are holomorphic, and for each $z \in \gamma^*$, $|g(z)| < |f(z)|$. Let N_h denote the number of zeroes of h repeated by multiplicity. Then $N_f = N_{f+g}$.

Proof. Let $\tau \in [0,1]$ and $f_\tau = f + \tau g$. Now $f_\tau \neq 0$ on γ^* because if not then $|f(z)| = |\tau g(z)| \leq |g(z)|$ somewhere on γ^*. So we can apply the argument principle.

Put $\phi : [0,1] \to \mathbb{C}$ by

$$\phi(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f_\tau'}{f_\tau} = \mathrm{Ind}_{f_\tau\circ\gamma}(0)$$

which is constant. In particular $\phi(0) = \phi(1)$ so $N_f = N_{f+\tau g}$. □

An amusing corollary is that we can one-liner the fundamental theorem of algebra, not that the proof by Liouville's theorem was particularly difficult.

Corollary 5.18 (fundamental theorem of algebra, again)**.** *If f is a polynomial over $\mathbb{C}$, then f has a zero or f is constant.*

Proof. Let $f(x) = \sum a_k x^k$ be degree n. For R sufficiently large, $|x| > R$, $\sum_{k<n} a_k x^k < a_n x^n$, but $a_n x^n$ has a zero at 0, so f has a zero on $B_R 0$ by Rouche. □

More importantly, we can understand the behavior of zeroes under the effect of locally uniform convergence. if f is the locally uniform limit of a sequence of functions f_n, then each zero of f forms by zeroes of the f_ns "converging" together to a single point!

Theorem 5.19. *Suppose U is connected, $f_k : U \to \mathbb{C}$ are holomorphic, $f_k \to f$ locally uniformly, f is not identically zero, $a \in U$, $m \in \mathbb{N}$, and $f(a) = 0$.*

a is a zero of order m iff there is a neighborhood V such that $a \in V \subseteq U$ and for each $s > 0$, if $B_s a \subseteq V$ then cofinitely many of the f_n have m zeroes in $B_s a$, counted by multiplicity.

Proof. Choose $r > 0$ such that $f_k \to f$ uniformly on $B_r a$ and $\forall z \in B_r a \setminus \{a\} \subseteq U$, $f(z) \neq 0$. Let $s < r$ such that

$$\varepsilon = \min_{\partial B_s a} |f| > 0.$$

Then

$$\exists N \in \mathbb{N}\ \forall n > N |f_n - f| < \varepsilon$$

on $B_r a$. In particular, for $z \in \partial B_s a$, $|f_n(z) - f(z)| < |f(z)| \leq \varepsilon$. Let $\gamma = \Gamma_s a$ and apply Rouche. Then

$$N_{f_n} = N_{f_n - f + f} = N_f.$$

□

Corollary 5.20 (Hurwitz)**.** *If U is connected, $f_k : U \to \mathbb{C}$ are holomorphic, and $f_k \to f$ locally uniformly, and $\forall k\ |f_k| > 0$, then either $f \equiv 0$ or $|f| > 0$.*

As with the argument principle, there's nothing special about 0: we could translate f and do this for any $b \in \mathbb{C}$. Doing this for *every* such b, and requiring that each f_n hits b at most once, gives the following:

Corollary 5.21. *If U is connected, $f_k : U \to \mathbb{C}$ are holomorphic and injective, $f_k \to f$ locally uniformly, then f is constant or injective.*

5.5 Open maps and maximum moduli

The higher-dimensional version of the next theorem, proven in functional analysis, is one of the most powerful theorems in mathematics, an analyst's Zorn's lemma.

Theorem 5.22 (open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and not constant. Then $f(U)$ is open.*

To prove it, let's rephrase the open mapping theorem as a statement about zeroes in neighborhoods, so we can apply the argument principle.

Lemma 5.23. *With hypotheses as in 5.22, suppose $a \in U$, $f(a) = b$, and $f - b$ has a zero at a of order $N < \infty$.*

Then there are open neighborhoods U_0 of a and V_0 of b such that $f(U_0) = V_0$. Moreover, if $w \in V_0 \setminus \{b\}$, then $f - w$ has N simple zeros in U_0 and no other zeroes.

Proof. $\exists r > 0$ such that $B_{2r} a \subseteq U$ and for $z \in B_{2r} a$, $f(z) - b \neq 0$ and $f'(z) \neq 0$. Now do the argument principle with $\gamma = \Gamma_r a$. Then $\mathrm{Ind}_{f \circ \gamma}(b) = N$.

Let $V_0 = \mathrm{Ind}_{f \circ \gamma}^{-1}(\{N\})$ which is open because $\{N\}$ is open in the topology of $\mathbb{Z}$. Similarly let $U_0 = B_r a \cap f^{-1}(V_0)$, which is clearly open. Then $f(U_0) \subseteq V_0$.

Let $w \in V_0$. If $w = b$ then $f(a) = w$. Otherwise, $\operatorname{Ind}_{f\circ\gamma}(w) = N$ so $f - w$ has N zeroes by multiplicity in $B_r a$. Let z be such a zero; then $f(z) = w$ so $V_0 \subseteq f(U_0)$. Also, $f'(z) \neq 0$, so

$$0 \neq \frac{d}{dz}(f'(z) - w)$$

so f has a simple zero there. ☐

Proof of 5.22. $U = \bigcup U_0$, so $V = \bigcup V_0$. Open sets remain open after arbitrary unions. ☐

The open mapping theorem has important corollaries for dealing with extrema.

Corollary 5.24 (maximum modulus principle). *If U is connected, $f : U \to \mathbb{C}$ holomorphic, and $|f|$ attains its max on U, then f is constant.*

Proof. If $|f|$ has a maximum at $a \in U$ then there are U_0 and V_0, as in the open mapping theorem. But then $\exists c \in V_0$ with $|c| > |b|$. ☐

Similarly:

Corollary 5.25. *Suppose that U is connected, and $f : U \to \mathbb{C}$ is holomorphic.*

If $\operatorname{Re} f$ or $\operatorname{Im} f$ attains a max or min, or $|f|$ attains its min, then f is constant.

Moreover, if U is bounded and f is also defined and continuous on ∂U, then

$$\sup_U |f| = \max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

5.6 Conformal maps

We're in the position to give a geometric interpretation of holomorphicity.

Definition 5.26. f is *biholomorphic* if $f : U \to V$ is a bijection and f and f^{-1} are both holomorphic.

If such a biholomorphic function exists then U and V are *conformally equivalent* or *conformal*.

By Liouville's theorem, $B_1 0$ and $\mathbb{C}$ are not conformal, but on the other hand, this shocking result is true:

Theorem 5.27 (Riemann mapping theorem). *If U is connected and simply connected, then U is conformal with $B_1 0$ or $\mathbb{C}$.*

The only biholomorphic functions from $B_1 0$ to itself are rotations, by the maximum modulus principle:

Lemma 5.28 (Schwarz). *Let $f : B_1 0 \to B_1 0$ be holomorphic and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $a \neq 0$ and there exists $a \in B_1 0$ with $f|(a)| = |a|$ then $f(z) = \lambda z$ on $B_1 0$ where $\lambda \in S^1$.*

Proof. Let $g : B_1 0 \to \mathbb{C}$ be given by the holomorphic continuation of $g(z) = z^{-1} f(z)$. Then, for $r < 1$, $|z| = r$,

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

and in particular max modulus implies $|g|_{B_r 0} \leq r^{-1}$. It follows that $|g| \leq 1$. So $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

On the other hand, if $|f(z)| = |z|$ then g is constant by max modulus. So $\lambda = g(z)$. ☐

Corollary 5.29. *Let $f : B_1 0 \to B_1 0$ be biholomorphic. Then $\exists \lambda \in S^1$ such that $f(z) = \lambda z$.*

Proof. We have $|f^{-1}(z)| \leq |z| \leq |f(z)|$, but also $|f(z)| \leq |z| \leq |f^{-1}(z)|$. ☐

The open mapping theorem allows us to construct lots of biholomorphic functions.

Theorem 5.30 (inverse function theorem). *Let $f : U \to \mathbb{C}$ be injective and holomorphic, $V = f(U)$. Then f is biholomorphic and $|f'| > 0$.*

Proof. By the open mapping theorem, V is open. For $a \in U$, $b = f(a)$, $r > 0$, $f(U \cap B_r a)$ is open, and the $U \cap B_r a$ are a basis for the topology of V, so g is continuous.

If $f'(a) = 0$ then at a, $f - b$ has an order-2 zero, so we have $r > 0$ such that if $w \in B_1 b \setminus \{b\}$ then $f - w$ has two simple zeroes. So we have $\beta \neq \gamma$ such that $f(\beta) = w = f(\gamma)$, which is a contradiction. So $|f'| > 0$.

If h is the continuous extension of $\frac{z-a}{f(z)-b}$ then h is continuous. So

$$f^{-1}(b) = \lim_{v \to b} \frac{f^{-1}(u) - f^{-1}(b)}{u - b} = \lim_{u \to b} (h \circ f^{-1})(u) = \frac{1}{f'(a)}.$$

So f^{-1} is holomorphic. □

If one plots a holomorphic function in Mathematica, it looks very pretty, with lots of symmetries and loops. This is why.

Lemma 5.31. *Let $\gamma_1(0) = \gamma_2(0) = z \in U$ and $f : U \to \mathbb{C}$ be holomorphic. Let θ be the angle between two curves at 0.*

If $|f'| > 0$ then $\theta(\gamma_1, \gamma_2) = \theta(f \circ \gamma_1, f \circ \gamma_2)$.

Proof. TODO: This. □

The lemma tells us what a holomorphic function is: locally, it's nothing more than a translation, dilation, and rotation, up to some deformation by a small ε.

Definition 5.32. If $f : U \to \mathbb{C}$ is holomorphic and $|f'| > 0$, then f is *conformal* or *angle-preserving.*

If f is holomorphic and injective, then f is conformal by the inverse function theorem.

In special relativity, a particle P's equations of motion are preserved under a group known as the Lorentz group. However, if P's mass vanishes, then the equations of motion are preserved under *any* conformal mapping.

Chapter 6

Harmonic functions

6.1 The mean value property

6.2 Harnack's inequality

6.3 Applications to PDE

6.4 Riemann mappings

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