Collaborative Notes - Math 185

Mathematics Undergraduate Student Assocation, UC Berkeley

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Contents

The following notes are collaborative notes meant for use by undergraduates at UC Berkeley, taking Math 185 (complex analysis). However, they may be read or edited by anyone interested in the material.

The original draft of these notes is loosely based on the Spring 2018 section taught by Charles Hadfield, and was written by Aidan Backus. We do not claim that any of the proofs given are original work.

As these notes are a "perpetual draft", they may contain errors. As such, they can be edited on ShareLaTeX; however, for particularly egregious errors, you are encouraged to contact Aidan at aidan-backus@berkeley.edu.

Chapter 1

Topological preliminaries

$\mathbb{C}$ is a vector space over $\mathbb{R}$ of dimension 2. We can think of an element $z \in \mathbb{C}$ as a vector $(x, y) \in \mathbb{R}^2$ by the identification $z = x + iy$, where $i^2 = -1$ is the imaginary unit.

Definition 1.1. If $z = x + iy \in \mathbb{C}$, we write

$$x = \operatorname{Re} z$$

for the *real part* of z and

$$y = \operatorname{Im} z$$

for the *imaginary part*.

$\mathbb{C}$ has a norm, and its norm satisfies the identities $|x+iy| = \sqrt{x^2+y^2}$, $|zw| = |z||w|$, and $|z+w| \leq |z|+|w|$ for each $z = x + iy$ and w.

The last estimate is the triangle inequality, which gives $\mathbb{C}$ a metric (and therefore topological) structure: its metric $d : \mathbb{C}^2 \to [0, \infty)$ is $d(z, w) = |z - w|$. Thus, open sets in $\mathbb{C}$ are those which can be written as a union of $B_r z$, where $B_r z$ denotes the ball around z of radius $r > 0$. The closed sets in $\mathbb{C}$ are complements of open sets.

Definition 1.2. We will write $\Phi : \mathbb{R}^2 \to \mathbb{C}$ to mean

$$\Phi(x, y) = x + iy$$

and $\Psi = \Phi^{-1}$.

Φ and Ψ are vector space isomorphisms, but stronger than that, they are isometries: they preserve distance and angle as well as vector space structure. In particular, they are homeomorphisms: they preserve topological structure. So, as a normed real vector space, a metric space, or as a topological space, $\mathbb{R}^2$ and $\mathbb{C}$ are identical.

But make no mistake: $\mathbb{C}$ is a field (it has multiplication) while $\mathbb{R}^2$ is not. Moreover, $\mathbb{C}$ has a very different differential structure than $\mathbb{R}^2$, and it is this difference, given by the Cauchy-Riemann equations that we'll consider later, that will make complex analysis worthy of study.

Throughout these notes, U will denote an arbitrary open subset of $\mathbb{C}$. We'll usually think of it as the domain of whichever function we're studying (and usually $\mathbb{C}$ will be the codomain). Occasionally we'll need to identify U with its counterpart in $\mathbb{R}^2$, in which case we'll write $\tilde{U} = \Psi(U)$.

1.1 Review of 104

As proven in Math 104 for $\mathbb{R}^n$, the following topological results hold. To prove them for $\mathbb{C}$, just identify $\mathbb{R}^2$ with $\mathbb{C}$ by the homeomorphism Φ.

We'll start by examining the properties of sequences:

Lemma 1.3. *Let $\{a_n : n \in \mathbb{N}\}$ be a sequence in $\mathbb{C}$ and $L \in \mathbb{C}$. Then*

$$\lim_{n\to\infty} a_n = L$$

if and only if both

$$\lim_{n\to\infty} \operatorname{Re} a_n = \operatorname{Re} L$$

and

$$\lim_{n\to\infty} \operatorname{Im} a_n = \operatorname{Im} L.$$

Lemma 1.4. *$F \subseteq \mathbb{C}$ is closed iff each sequence in F which converges in $\mathbb{C}$ converges in F.*

Theorem 1.5. *Cauchy sequences converge in $\mathbb{C}$.*

Definition 1.6. If a_n is a sequence in $\mathbb{C}$ and $|a_n|$ is convergent, we say that a_n is *absolutely convergent*.

Lemma 1.7. *Absolutely convergent sequences converge.*

However, we don't just want to talk about convergence of sequence of points. Sequences of functions are also useful.

Definition 1.8. Let $D \subseteq \mathbb{C}$ and let f_n be a sequence of functions $D \to \mathbb{C}$. Further, let $z \in D$ and k, m, n be natural numbers.

We say $f_n \to f$ *uniformly* if

$$\forall \epsilon > 0\ \exists N > 0\ \forall x \in D\ \forall n > N\ |f_n(x) - f(x)| < \epsilon.$$

Furthermore, $\sum f_n$ is *uniformly convergent* if its sequence of partial sums is.

Theorem 1.9 (Weierstrass M-test)**.** *Suppose $\{a_i\}$ is a sequence in $(0, \infty)$. If $\forall k\ \forall z\ |f_k(z)| < a_k$ and $\sum a_k$ converges, then $\sum f_k$ is uniformly convergent.*

Sequences allow us to give a convenient definition of compactness:

Definition 1.10. Let K be a metric space. K is *compact* if each sequence in K has a convergent subsequence in K.

Theorem 1.11 (Heine-Borel)**.** *$K \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

Lemma 1.12. *If $\mathbb{C} \supset K_1 \supseteq K_2 \supseteq \dots$ and the K_is are compact, then their intersection is nonempty.*

One of the most useful properties of compact spaces is their behavior with respect to open covers.

Definition 1.13. Let $\mathcal{U}$ be a family of open sets of some topological space X. If $\bigcup \mathcal{U} = X$, we say that $\mathcal{U}$ is an *open cover* of X. The elements of $\mathcal{U}$ are called *scraps*.

Lemma 1.14. *Let K be a metric space. K is compact if and only if for each open cover $\mathcal{U}$, there exists an open subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ only has finitely many scraps.*

Topologists take 1.14 as the *definition* of compactness, because it generalizes better from metric spaces to Hausdorff spaces. However, open covers won't be as important as sequences to us, so we use the other definition. The actual importance of open covers for us is that they allow us to define the notion of a Lebesgue number, which we'll discuss later in the preliminaries.

One of the most useful notions that sequences give rise to is that of a cluster point. Some topologists call cluster points *accumulation points*, *limit points*, or *close points*.

Definition 1.15. Let $Y \subseteq X$, where X is a metric space. Y *clusters* at $x \in X$ if there is a sequence in $X \setminus \{x\}$ which converges to x.

Theorem 1.16 (Bolzano-Weierstrass)**.** *Let K be a compact metric space and $A \subseteq K$ be infinite. There is a point $x \in A$ such that A clusters at a.*

Sequences also allow us to define what it means for a function to be continuous.

Lemma 1.17. *Suppose X is a metric space, $f : X \to \mathbb{C}$, $L \in \mathbb{C}$, and $z_0 \in \mathbb{C}$.*
If $z_0 \in X$, then the following are equivalent:

1. *If $a_n \to a$ is a convergent sequence, then $f(a_n) \to f(a)$.*
2. $\forall \varepsilon > 0\ \exists \delta > 0\ \forall d \in D\ |d - z_0| < \delta \implies |f(d) - f(z_0)| < \varepsilon$.
3. *$f^{-1}(U)$ is open.*

Definition 1.18. If $f : X \to \mathbb{C}$ satisfies the hypotheses of 1.17, then f is *continuous.*

Lemma 1.19. *Suppose $K \subset \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous. Then $f(K)$ is compact.*

Definition 1.20. Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$. $f : D \to \mathbb{C}$ has a *limit* $L \in \mathbb{C}$ at z_0 if

$$\forall \varepsilon > 0\ \ \exists \delta > 0 \text{ s.t. } \forall d \in D\ 0 < |d - z_0| < \delta \implies |f(d) - L| < \varepsilon.$$

Continuous functions are especially well-behaved when their domains are compact.

Lemma 1.21. *Let K be compact and $f : K \to \mathbb{C}$ be continuous. Then $f(K)$ is compact.*

Continuity is also a means for giving a topological definition of connectivity.

Definition 1.22. Suppose that X is a topological space, and there does not exist a surjective, continuous function $X \to \{0, 1\}$. Then we say that X is *connected.*

Lemma 1.23. *X is connected if and only if the only subsets of X which are both open and closed are X itself and $\emptyset$.*

Lemma 1.24. *Intervals $[a, b]$ and their Cartesian products $[a, b] \times [c, d] \times \cdots \times [e, f]$ are connected and compact.*
Moreover, $\mathbb{C}$ is connected but not compact.

Now we examine some "special" subsets of $\mathbb{C}$.

Definition 1.25. A topological space X is *discrete* if every subset of X is open.

Definition 1.26. Let $D \subseteq \mathbb{C}$. D is *convex* if for each line segment $[a, b]$ where $a, b \in D$, $[a, b] \subseteq D$.

Lemma 1.27. *Let $a \in U$ and $r > 0$. Then $\overline{B_r a} \subset U$ iff $\exists R > r$ with $B_R a \subseteq U$.*

None of this should be new material. If it's unfamiliar, refer to any 104 text, such as Rudin, Pugh, or Ross. It would be fruitful to come up with a few explicit examples of each type of space and set presented above.

1.2 Curves

Often in complex analysis, we want to compute the integral of a function not along an interval, but along a much more general path through $\mathbb{C}$. Curves allow us to do this.
Recall that U is assumed to be an open set in $\mathbb{C}$.

Definition 1.28. Let X be a topological space and $a < b$ be real numbers. A *curve* in X is a continuous map $\gamma : [a, b] \to X$.
$\gamma(a)$ is called the *initial point* and $\gamma(b)$ is called the *final point.* We write $\gamma^* = \gamma([a, b])$.
If $\gamma(a) = \gamma(b)$ we say that γ is *closed.*
γ is *constant* if γ^* consists of a point.
If $\mu : [c, d] \to X$ is also a curve and $\gamma(b) = \mu(c)$ then we define the curve

$$\gamma \oplus \mu : [a, b + d - c] \to \mathbb{C}$$

by $\gamma \oplus \mu(t) = \gamma(t)$ if $t \leq b$ or $\mu(t)$ otherwise.

Notice that this definition only really has teeth when the space X "looks like" $\mathbb{R}^n$ ($n \geq 2$), for example if $X = U$. Consider the case when X is discrete, for example. Because $[a, b]$ is connected, the only curves in X are constant!

On the other hand, when the codomain X has a notion of differentiability, we can define especially nice curves:

Definition 1.29. A curve γ in $\mathbb{C}$ is *smooth* if γ' exists and is continuous.

If each γ_i is a smooth curve and

$$\gamma = \bigoplus_{i \leq N} \gamma_i = \gamma_1 \oplus \cdots \oplus \gamma_N$$

then we say γ is *piecewise smooth.*

Notice that these definitions conflict with the usual usage of the words closed and smooth! Any curve is necessarily closed (in fact, compact) by 1.19, and a smooth function is C^∞, not C^1.

Also notice that curves aren't just their image: they come with a parametrization, which equips them with an orientation (a "direction") and a speed. As we will see, the parametrization won't matter much, only the orientation.

Definition 1.30. If $\gamma : [0, 1] \to X$ is a curve, then $-\gamma$ is given by $-\gamma(t) = \gamma(1 - t)$.

The following is just the usual arc length formula from calculus.

Definition 1.31. If γ is a curve in $\mathbb{C}$ then the *length* of γ is

$$\ell(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Curves which wind around a point once will prove themselves to be especially useful.

Definition 1.32. If $r > 0$, define $\Gamma_r(z_0)$ by the curve $t \mapsto z_0 + re^{it}$ for $t \in [0, 2\pi]$.

Notice that if γ is a curve, then $\gamma'(t)$ is the tangent vector at t, multiplied by the speed. Dividing out by the speed will give us the direction the curve is facing. Since multiplying by an element of the form $e^{i\theta}$ corresponds by a rotation by θ, we have the following definition:

Definition 1.33. Suppose γ_1 and $\gamma_2 : [-1, 1] \to \mathbb{C}$ are curves, $\gamma_1(0) = \gamma_2(0) = z$, and $\gamma_1'(0) \neq 0$ and $\gamma_2'(0) \neq 0$.

The *angle* between γ_1 and γ_2 at 0 is the unique $\theta \in (-\pi, \pi]$ such that

$$\frac{\gamma_1'(0)}{|\gamma_1'(0)|} = \frac{\gamma_2'(0)}{|\gamma_2'(0)|} e^{i\theta}.$$

That this θ is in fact unique won't become clear until we've studied properties of the exponential function exp in greater detail. That's fine; we won't need the notion of angle until much later on.

Often we will need to draw a curve γ between two points in $\mathbb{C}$, so that we can integrate a function along γ. Path-connectivity allows us to describe when this is possible.

Definition 1.34. Suppose that X is a topological space, and for each $x, y \in X$, there is a curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then we say that X is *path-connected.*

There is a close relationship between path-connectivity and connectivity, which we'll exploit to its full potential.

Lemma 1.35. *If X is a path-connected space, then X is connected.*

Proof. Suppose that X is not connected. Then there is a surjective continuous function $f : X \to \{0, 1\}$. Suppose that $f(x) = 0$ and $f(y) = 1$. Let $\gamma : [0, 1] \to X$ be a curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Now the function $f \circ \gamma : [0, 1] \to \{0, 1\}$ is continuous and surjective, which is a contradiction. $\square$

Lemma 1.36. *If U is connected then U is path-connected.*

Proof. Let $z \in U$ and let $V \subseteq U$ be the set of points $w \in U$ such that there is a curve from z to w. V is nonempty, since $z \in U$. Let $W = U \setminus V$; we'll prove that V and W are both open, so $W = \emptyset$ by 1.23.

Since U is open, there is an open ball $B_r z \subseteq U$. A straight line connects z to any point in $B_r z$, so $B_r z$ is path-connected. So $B_r z \subseteq V$, and this works for any point in V. So V is open. On the other hand, if $w \in W$, then there is an open ball $B_s w \subseteq W$ (for if not, then by the same argument, there's a path from a point in V to w, and thus a path from z to w). So W is open, so V is closed. □

The above proof was an example of an "open and closed argument", an idea we'll see again whenever we have to apply connectivity.

The hypothesis that U is open in a Euclidean space in the above lemma is essential; the lemma fails spectacularly in worse spaces. However, we won't ever need them. As such, we'll be happy to assume that U is connected, when we actually want U to be path-connected.

We often want to talk about ways to deform a curve into another. Homotopy makes this rigorous, furnishing continuous functions which morph a curve into another.

Definition 1.37. Suppose that X is a topological space and

$$\begin{cases} \gamma_0 : [0,1] \to X \\ \gamma_1 : [0,1] \to X \end{cases}$$

are curves in X. They are *X-homotopic* if there exists a continuous $H : [0,1]^2 \to X$ such that

$$\begin{cases} H(t,0) = \gamma_0(t) \\ H(t,1) = \gamma_1(t) \end{cases}$$

If γ_0 is closed, then γ_0 is *X-nullhomotopic* (or *X-contractible*) if γ_1 is constant. The function H is called a *homotopy*, and if, if γ_1 is constant, a *nullhomotopy*.

Notice that X-homotopy is an equivalence relation. In particular, if two closed curves are X-nullhomotopic then they are X-homotopic.

The hypothesis that H is continuous is critical to the definition: if H was not continuous, we could break up the connected set $[0,1]$ into multiple pieces, allowing us to go around "obstacles" in X. For this reason, homotopy can be used to detect "holes" in X, as the following example demonstrates:

Example 1.38. Consider the curves $\gamma_1, \gamma_2 : [0,\pi]$ given by

$$\gamma_1(\theta) = e^{i\theta}$$

and

$$\gamma_2(\theta) = e^{-i\theta}.$$

They are $\mathbb{C}$-homotopic, as witnessed by the homotopy

$$H(\theta, s) = e^{f(s)i\theta}$$

where $f : [0,1] \to [-1,1]$ is a continuous function such that $f(0) = 1$ and $f(1) = -1$.

On the other hand, they are not $\mathbb{C} \setminus \{0\}$-homotopic, as any homotopy must pass through 0.

Definition 1.39. If X is path-connected and each closed curve in X is X-nullhomotopic, then we say that X is *simply connected*.

If you've taken algebraic topology, this definition is equivalent to the statement that the fundamental group is trivial:

$$\pi_1(X) = \{0\}.$$

If not, that's fine; you won't need the fundamental group in 185.

Example 1.40. $\mathbb{C}$ is simply connected, so any two closed curves in $\mathbb{C}$ are $\mathbb{C}$-homotopic.

On the other hand, $\mathbb{C} \setminus \{0\}$ is not simply connected, as $\Gamma_1 0$ is not nullhomotopic.

The above example is critical: anything interesting that happens in complex analysis happens when we integrate along a circle around a point that causes a space to fail to be simply connected.

The Riemann mapping theorem, which we'll prove much later on, will show that up to an angle-preserving transformation, there are only two open, simply connected sets in $\mathbb{C}$: $B_1 0$ and $\mathbb{C}$ itself!

1.3 Lebesgue numbers

We'll use the notion of a Lebesgue number to prove the homotopy theorem. If you're not familiar with Lebesgue numbers, there's no reason to peruse this section until you're trying to understand the proof.

Definition 1.41. Let $\mathcal{U}$ be an open cover of a metric space X and $\mu > 0$. We say that μ is a *Lebesgue number* for $\mathcal{U}$ if, for each $x \in X$, there is a scrap $U \in \mathcal{U}$ such that $B_\mu(x) \subseteq U$.

In other words, while $\mathcal{U}$ might have scraps which are arbitrarily small, as long as the Lebesgue number is positive, most of them are irrelevant: we can fit any sufficiently small ball into a scrap anyways.

Lemma 1.42. *If K is a compact metric space with open cover $\mathcal{U}$, then $\mathcal{U}$ has a Lebesgue number > 0.*

Proof. If $K = \mathcal{U}$ then we're done. Otherwise, since K is compact, there is a finite subcover $A_1, A_2, \ldots, A_n$. For each $i \leq n$, let $C_i = K \setminus A_i$. Since C_i is nonempty and compact, the function $f : K \to \mathbb{R}$ given by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} \min_{y \in C_i} d(x, y)$$

is well-defined and continuous, and > 0 since no point is in every scrap.

Since K is compact, f attains a minimum. So let

$$\mu = \frac{1}{2} \min_K f > 0.$$

For each $B_\mu x \subset K$, $B_\mu x \subseteq A_i$ for at least one A_i, so μ is a Lebesgue number. $\square$

1.4 The Arzela-Ascoli theorem

The Arzela-Ascoli theorem is a mechanism for us to generalize the notion of "compactness" to function spaces, where being simply closed and bounded is not enough: another necessary hypothesis is equicontinuity. We'll use this theorem to prove Montel's theorem on normal families, and ultimately the fabled Riemann mapping theorem.

Definition 1.43. Let K be a compact space. $C^0(K)$ denotes the space of continuous functions $K \to \mathbb{C}$.

Notice that the 0 in C^0 denotes the number of times $f \in C^0$ is differentiable. The reason that we assume K to be compact is so that $f(K)$ is compact, and thus bounded, so that the following definition makes sense.

Definition 1.44. Let $f \in C^0(K)$. The sup-*norm* of f is

$$||f|| = \sup_K |f|.$$

As with any norm, the sup-norm defines a metric, and thus a topology on $C^0(K)$. Topologies on spaces of functions are one of the main areas of concern in functional analysis, but we won't need much functional analysis.

As you should verify if it was not proven in 104, if K is a complete metric space, then the sup-norm turns $C^0(K)$ into a complete metric space as well.

Definition 1.45. Let $\mathcal{F} \subseteq C^0(K)$. $\mathcal{F}$ is *equicontinuous* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $f \in \mathcal{F}$ and $x, y \in K$, if $d(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The key thing about equicontinuity is that δ is not allowed to depend on the individual function, or on the point $x \in K$. Once $\mathcal{F}$ has been chosen, δ is purely a function of ε. One can think of δ as a measure on how rapidly f oscillates, so equicontinuity is morally the statement that $\mathcal{F}$ does not contain elements which oscillate arbitrarily rapidly.

Theorem 1.46 (Arzela-Ascoli). *Let $K \subset \mathbb{C}$ be compact. A subset $\mathcal{F} \subset C^0(K)$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.*

Proof. Suppose that $\mathcal{F}$ is closed, bounded, and equicontinuous and let f_n be a sequence in $\mathcal{F}$. Then f_n is bounded and if it converges, it does so in $\mathcal{F}$. Now let

$$D = \{x + yi \in K : (x, y) \in \mathbb{Q}^2\}$$

be the set of rational points of K. Then D is dense and countable in K, and there is a sequence d_n whose image is D.

$f_n(d_1)$ is bounded, so a subsequence converges, say $f_{1;n}(d_1) \to y_1$. Moreover, $f_{1;n}(d_2)$ is bounded, so a subsequence converges, say $f_{2;n}(d_2) \to y_2$. On the other, hand $f_{2;n}(d_1) \to y_1$ still.

Iterating this construction countably many times, one has a sequence of sequences $f_{m;k}$ such that if $j \leq m$ then $f_{m;k}$ is a subsequence of $f_{j;k}$ and $f_{m;k}(d_j) \to y_j$. So given m we can choose ℓ large enough such that if $j \leq m$ and $k \geq \ell$ then $|f_{m;k}(d_j) - y_j| < m^{-1}$.

Now let $g_m = f_{m;\ell}$. Then g_m is a subsequence of f_m, so to prove that $\mathcal{F}$ is compact we just need to prove that g_m converges. We'll show that g_m is Cauchy in $\mathcal{F}$. For $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x - y| < \delta$, $|g_m(x) - g_m(y)| < \varepsilon/3$, since $\mathcal{F}$ is equicontinuous.

Fix x. By density of D and compactness of K, we can find a J large enough that for each x, there is a d_j with $j \leq J$ such that $|x - d_j| < \delta$. So $|f(x) - f(d_j)| < \varepsilon/3$. On the other hand, $\{d_1, \dots, d_J\}$ is finite, so there is M such that if $\ell, m > M$ and $j \leq J$ then $|g_m(d_j) - g_\ell(d_j)| < \varepsilon/3$. Now apply the triangle inequality.

On the other hand, if $\mathcal{F}$ is compact, let $\varepsilon > 0$. Then there is a finite open cover $\mathcal{U}$ of $\mathcal{F}$ by balls of radius $\varepsilon/3$, say centered on the functions f_k. Given k, there is some $\delta_k > 0$ such that for $|x - y| < \delta_k$, $|f_k(x) - f_k(y)| < \varepsilon/3$. Let δ be the max of the δs (which is finite, since $\mathcal{U}$ is finite). Given any $f \in \mathcal{F}$ and $|x - y| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \varepsilon.$$

Thus $\mathcal{F}$ is equicontinuous with oscillation measure δ. ☐

Exercise 1.47. Let a_n be a sequence in $[1, 2]$. Prove that the sequence

$$f_n(x) = \frac{1}{a_n} \sin(a_n x) + \cos(x + a_n)$$

has a convergent subsequence.

Chapter 2

Complex calculus

We're ready to generalize calculus to the complex setting.
Recall that $U \subseteq \mathbb{C}$ is assumed to be an open set.

2.1 Cauchy-Riemann equations

Definition 2.1. Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$.
$f : D \to \mathbb{C}$ is *differentiable* at z_0 if, for some $L \in \mathbb{C}$, the function

$$f'(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0 \\ L, & z = z_0 \end{cases}$$

is continuous.
We say that the function f' is the *derivative* of f.

Multiplying through by the denominator $z - z_0$ shows that the following lemma characterizes differentiability.

Lemma 2.2. *Suppose $D \subseteq \mathbb{C}$ clusters at $z_0 \in \mathbb{C}$. The following are equivalent:*

1. *f is differentiable at z_0 and $f'(z_0) = L$.*
2. *$\exists \phi : D \to \mathbb{C}$ continuous at z_0 with $\phi(z_0) = L$ and*
$$\forall z \in D \ f(z) = f(z_0) + (z - z_0)\phi(z).$$
3. *$\exists \psi : D \to \mathbb{C}$ continuous at z_0 with $\psi(z_0) = 0$ and*
$$\forall z \in D \ f(z) = f(z_0) + (z - z_0)(L + \psi(z)).$$

We think of ψ as the sublinear Taylor remainder of f at z_0 and $\phi = L + \psi$ where we are viewing L as a linear operator on $\mathbb{C}$.
Recall from 104:

Lemma 2.3. *Differentiability implies continuity.*

Lemma 2.4. *If $f, g : U \to \mathbb{C}$ are differentiable, $\lambda \in \mathbb{C}$, and $z \in U$ then:*

1. $(f + g)' = f' + g'$,
2. $(fg)' = f'g + g'f$,
3. $(\lambda f)' = \lambda f'$,

4. $(f \circ g)'(z) = f'(g(z))g'(z)$, *and*

5. *if* $|f| > 0$ *then* $1/f = -f'/f^2$.

Definition 2.5. A *holomorphic* function is one which is differentiable on an open set U. The space of holomorphic functions on U is written $\mathcal{O}(U)$.

Recall that a function is analytic if it has a Taylor series. Our goal is to prove that a function is holomorphic iff it is analytic. This is much stronger than differentiability on an open set in $\mathbb{R}^2$ – which doesn't even imply second-differentiability, let alone analyticity!

To see how this could be possible, let's start by identifying functions in $\mathbb{R}^2$ with their counterparts in $\mathbb{C}$ and seeing why differentiability in $\mathbb{C}$ is so special.

Recall the homeomorphisms $\Phi : \mathbb{R}^2 \to \mathbb{C}$ and $\Psi = \Phi^{-1}$. Let $\tilde{U} = \Psi(U)$. Then for each map $f : U \to \mathbb{C}$ we have a natural $\tilde{f} : \tilde{U} \to \mathbb{R}^2$ given by

$$\tilde{f} = \Psi \circ f \circ \Phi.$$

In 105, it is defined that $\tilde{f} = (u, v)$ is differentiable at $\Phi(z_0)$ if there is a linear map $M : \mathbb{R}^2 \to \mathbb{R}^2$ (the derivative) and sublinear map $r : \tilde{U} \to \mathbb{R}^2$ satisfying the analogue of 2.2, namely

$$\forall z \in \tilde{U}\ \tilde{f}(\Psi(z)) = \tilde{f}(\Psi(z_0)) + M(\Psi(z - z_0)) + r(\Psi(z)).$$

Now let $z \in \mathbb{C}$, and $\Psi(z) = (x, y)$. If we multiply z by $w = a + ib$, this corresponds to multiplying $\Psi(z)$ by the *matrix*

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

for

$$\Psi(wz) = \Psi((a + ib)(x + iy)) = \Psi(ax - by + i(ay + bx)) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = Mz.$$

In the language of linear algebra, complex numbers are nothing more than antisymmetric linear operators!

Theorem 2.6 (Cauchy-Riemann equations)**.** *Let* Ψ *and* Φ *be as above. Then if* $f : U \to \mathbb{C}$, $z_0 \in U$, *and* $\tilde{f} = (u, v)$, *the following are equivalent:*

1. f *is differentiable at* z_0.

2. u *and* v *are differentiable at* $\Psi(z_0)$ *and*

$$\begin{cases} \partial_1 u(\Psi(z_0)) = \partial_2 v(\Psi(z_0)) \\ \partial_2 u(\Psi(z_0)) = -\partial_1 v(\Psi(z_0)). \end{cases}$$

Proof. By 2.2, the derivative M of $\tilde{f}$ is given by $r(\Psi(z)) = \Psi(z - z_0)\psi(\Psi(z))$ and $(a, b) = (\operatorname{Re} f'(z_0), \operatorname{Im} f'(z_0))$. Then we use the argument above, taking close note of the fact that the partial derivatives of $\tilde{f}$ are precisely the entries in M, and thus must satisfy certain relations. $\square$

Your first line of attack in proving that a function is not holomorphic is to show that the Cauchy-Riemann equations fail.

Example 2.7. $z \mapsto \operatorname{Re} z$, $z \mapsto \operatorname{Im} z$, and $z \mapsto |z|$ are not holomorphic *anywhere*! This may be shocking, because two of those are differentiable (in fact, linear) in and the third is differentiable off of 0 in $\mathbb{R}$.

We'll prove that $f(z) = \operatorname{Re} z$ is not holomorphic; the other two are similar and good practice.

To see this, observe that $u(x, y) = x$ and $v(x, y) = 0$. Then $\partial_1 u(x, y) = 1$ while $\partial_2 v(x, y) = 0$.

The Cauchy-Riemann equations are the most important equations in the class. (In fact, when Aidan took 185, the entire first midterm came down to proving consequences of them.) As we will later see, the Cauchy-Riemann equations imply that f is holomorphic iff $\Delta \tilde{f} = 0$ (where $\Delta = \nabla \cdot \nabla$ is the Laplacian). In the language of PDE, holomorphic functions are precisely those which are harmonic.

2.2 Integration on curves

Just as holomorphic functions in complex analysis correspond to harmonic functions in multivariable calculus, complex integrals will correspond to line integrals. We'll start by defining a "preintegral" that we can use to define the actual integral we care about.

Definition 2.8. If $f : [a, b] \to \mathbb{C}$ and $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable, then

$$\int_a^b f = \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

Recall from 104:

Lemma 2.9. *Integration is linear.*

The following integral estimate is quite useful.

Lemma 2.10. *If $f : [a, b] \to \mathbb{C}$ is continuous, then $|f|$ is integrable and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. $|f|$ is continuous, so there exist $r \geq 0$ and $\theta \in [0, 2\pi)$ such that

$$\int_a^b f = r(\cos\theta + i\sin\theta).$$

Put $\zeta = \cos\theta - i\sin\theta$. Then $|\zeta| = 1$ and

$$\begin{aligned} \left| \int_a^b f \right| &= \zeta \int f = \operatorname{Re} \zeta \int_a^b f = \int_a^b \operatorname{Re} \zeta f \\ &\leq \int_a^b |\operatorname{Re} \zeta f| \leq \int_a^b |\zeta f| = \int_a^b |f|. \end{aligned}$$

□

Now we can define the true integral:

Definition 2.11. Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth, $\gamma^* \subseteq D \subseteq \mathbb{C}$, and $f : D \to \mathbb{C}$ be continuous.

If γ is smooth, we define the *contour integral* to be

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

If not, then $\gamma = \oplus\gamma_i$ and each γ_i is smooth, so we write

$$\int_\gamma f(z)\ dz = \sum_{i=1}^N \int_{\gamma_i} f(z)\ dz.$$

The whole point of the class is to study the properties of holomorphic functions defined on open sets by taking contour integrals around those open sets.

As an example, which is quite important in its own right, let's integrate $1/z$ around the origin, where it blows up:

Lemma 2.12. *If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by $f(z) = z^{-1}$ then*

$$\int_{\Gamma_1 0} f(z)\ dz = 2\pi i.$$

Proof.

$$\int_0^{2\pi} f(\Gamma_1 0(t))\Gamma_1 0'(t)\ dt = \int_0^{2\pi} e^{-it} i e^{it}\ dt = 2\pi i.$$

□

Perhaps unsurprisingly, this is possible precisely because $\mathbb{C} \setminus \{0\}$ is not simply connected. As a result of this lemma, we'll often see $2\pi i$s pop up in theorems whenever we integrate around a function which blows up.

Lemma 2.13. *Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth, $f, \tilde{f} : \gamma^* \to \mathbb{C}$, and $\lambda \in \mathbb{C}$. We have:*

1.
$$\int_\gamma \lambda f + \tilde{f} = \lambda \int_\gamma f + \int_\gamma \tilde{f}.$$

2.
$$\left| \int_\gamma f \right| \leq \ell(\gamma) \max_{z \in \gamma^*} |f(z)|.$$

3. *If $\phi : [c, d] \to [a, b]$ is smooth and strictly increasing then $\gamma \circ \phi$ is piecewise smooth and*
$$\int_\gamma f = \int_{\gamma \circ \phi} f.$$

4. *If $f_n : \gamma^* \to \mathbb{C}$ and $f_n \to f$ uniformly then*
$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. As an example, we prove (3). You should try to prove the others for practice!

We can assume γ is smooth, for if not, we can just break up γ into finitely many smooth curves. Then

$$\begin{aligned}
\int_{\gamma \circ \phi} f &= \int_c^d (f \circ \gamma \circ \phi)(\gamma \circ \phi)' \\
&= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\ dt \\
&= \int_a^b f(\gamma(s))\gamma'(s)\ ds \\
&= \int_\gamma f.
\end{aligned}$$

□

So far, nothing too crazy has happened. In fact, the fundamental theorem of calculus holds just fine:

Theorem 2.14 (fundamental theorem of calculus, part I)**.** *If f is holomorphic on U and γ is contained in U, then*

$$\int_\gamma f' = f(\gamma(b)) - f(\gamma(a)).$$

Proof.

$$\int_\gamma f' = \int_a^b f' \circ \gamma \cdot \gamma' = \int_a^b (f \circ \gamma)' = f(\gamma(b)) - f(\gamma(a)).$$

□

Just like in calculus, *if* a function has an antiderivative, then its integral is determined by the value of the antiderivative on the boundary of the integration domain.

Moreover, if f' vanishes on a connected set U, we can draw a curve γ inside U to reach any point in U, and by the fundamental theorem we have:

Corollary 2.15. *If $f \in \mathcal{O}(U)$ and U is connected, with $f' \equiv 0$, then f is constant.*

2.3 Power series

Recall from 104 that a power series is a function of the form

$$f(z) = \sum_{n=k}^{\infty} \alpha_k (z-a)^k.$$

We can and do, without loss of generality, assume $a = 0$; this simplifies notation greatly without affecting convergence.

Definition 2.16. If $\sum \alpha_n z^n$ is a power series and if R is the supremum of all values r such that $\sum |\alpha_n| r^n$ converges, then R is its *radius of convergence.*

This seems kind of sketchy; what if the series only converged on a set which was not circular? Fortunately, this never happens!

Lemma 2.17. *Let R be the radius of convergence of $f(z) = \sum \alpha_k z^k$. Then:*

1. *If $|z| < R$, then $f(z)$ converges.*
2. *If $|z| > R$, then $f(z)$ diverges.*
3. *If $0 < r < R$ and $|z| \le r$ then $\sum \alpha_n z^n$ converges absolutely uniformly.*

Proof. Take $z \neq 0$.

Suppose that $\sum \alpha_n z^n$ converges. Then $\alpha_n z^n$ is bounded, say $|\alpha_n z^n| \le M$. If $0 < r < |z|$, then $|\alpha_n| r^n| \le M(r/|z|)^n$, and therefore $\sum |\alpha_n z^n|$ converges. Therefore $r < R$, so $|z| \le R$. This proves the contrapositive of (2).

Now, if $0 < r < R$, $\sum |\alpha_n| r^n$ converges whenever $|z| \le r$. Then $|\alpha_n z^n| \le |\alpha_n| r^n$ and so (3) follows by the Weierstrass M-test, implying (1). $\square$

We can use the root test and the ratio test, developed in 104, to compute radii of convergence. For example, the radius of convergence of $\sum z^n$ is 1. You should practice this on a few power series of your own devision.

Functions which are expressible by power series are known as analytic functions:

Definition 2.18. $f : U \to \mathbb{C}$ is *analytic* at $a \in U$ if $\exists R > 0$ and a power series $\sum \alpha_n (z-a)^n$ with a radius of convergence $r \ge R$ such that $B_R(a) \subseteq U$ and $\sum \alpha_n (z-a)^n = f(z)$ for each $z \in B_R(a)$.

Note, an analytic function has a power series which is locally valid. If the topology of U isn't too nice, a power series may diverge in parts of U but a different power series will hold there. It's okay; the function is still analytic in that case.

Polynomials are clearly analytic (all but finitely $\alpha_n = 0$), but most functions which we care about turn out to be analytic. Our goal, recall, is to prove every holomorphic function is analytic! Let's begin.

Lemma 2.19. *The power series $\sum \alpha_n z^n$ and $\sum n\alpha_n z^{n-1}$ have the same radius of convergence.*

Proof. Let R and $\hat{R}$ be the respective radii of convergence and $0 < r < R$. Then we can find a $\rho \in (r, R)$. For each such ρ, $\sum |\alpha_n| \rho^n$ converges, but $n(r/\rho)^n \to 0$ and is therefore bounded by a constant $M \ge |n(r\rho)^n|$. Therefore

$$n|\alpha_n| r^n = n\left(\frac{r}{\rho}\right)^n |\alpha_n| \rho^n \le M |\alpha_n| \rho^n$$

and since the constant M is irrelevant, $\sum n|\alpha_n| r^n$ converges. So $\hat{R} > r$, implying that $\hat{R} \ge R$.

But we can repeat the same argument with R and $\hat{R}$ reversed. So $R \ge \hat{R}$. $\square$

This is what we needed to prove Taylor's theorem, which characterizes analytic functions.

Theorem 2.20 (Taylor). *If $R \in (0, \infty)$ and $\sum \alpha_n z^n$ has a radius of convergence $\geq R > 0$, define $f : B_R(0) \to \mathbb{C}$ by*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$$

Then f is smooth in $B_R(0)$ and for each $z \in B_R(0)$,

$$f'(z) = \sum_{n=1}^{\infty} n\alpha_n z^{n-1}.$$

Moreover,

$$\alpha_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Fix $z_0 \in B_R(0)$, $\epsilon > 0$, and $r \in (|z_0|, R)$. By the above lemma, we can find an N such that

$$\sum_{n=N+1}^{\infty} n|\alpha_n|r^{n-1} < \frac{\epsilon}{4}.$$

Therefore, $\forall z \in B_r(0) \setminus \{z_0\}$, the remainder

$$\begin{aligned}
R(z) &= \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n\alpha_n z_0^{n-1} \right| \\
&= \left| \sum_{n=0}^{\infty} \alpha_n \left(\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right) \right| \\
&= \left| \sum_{n=1}^{\infty} \alpha_n \left(\left(\sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right) - n z_0^{n-1} \right) \right| \\
&\leq \left| \sum_{n=1}^{N} \alpha_n \sum_{k=0}^{n-1} z^k z^{n-1-k} - z_0^{n-1} \right| + \left| \sum_{n=N+1}^{\infty} 2n|\alpha_n|r^{n-1} \right| \\
&= A(z) + B(z)
\end{aligned}$$

and A is continuous, so there exists a δ such that if $z \in B_\delta(z_0) \setminus \{z_0\}$ then $A(z) < \epsilon/2$. We already estimated $B(z) < \epsilon/2$. So $R(z) < \epsilon$.

Therefore f is differentiable and α_1 is as desired. Furthermore, this argument applies to the derivatives of f as well, so by induction f is smooth and each α_n is as desired. ☐

Corollary 2.21. *Analytic functions are holomorphic.*

By Taylor's theorem, if a function is analytic in U, U is connected, and we know its behavior on a small open set in U, we already know its behavior everywhere.

Corollary 2.22. *Suppose $\sum \alpha_n z^n$ and $\sum \beta_n z^n$ are power series with radii of convergence R_1, R_2. If $0 < \epsilon \leq \min R_i$ and $\forall z \in B_\epsilon(0)$ we have $\alpha_n z^n = \beta_n z^n$, then $\forall n$ $\alpha_n = \beta_n$.*

Proof. Look at $\sum(\alpha_n - \beta_n)z^n$. The derivatives of this are all 0. ☐

As an aside, here's a way to compute products of analytic functions:

Theorem 2.23. *Let $r > 0$ and $f(z) = \sum \alpha_n z^n$, $g(z) = \sum \beta_n z^n$ be power series with radius of convergence $\geq r$. Put*

$$\gamma_n = \sum_{j \leq n} \alpha_j \beta_{n-j}.$$

Then $\sum \gamma_n z^n$ has radius of convergence $\geq r$ and $\sum \gamma_n z^n = f(z)g(z)$.

Proof. fg is holomorphic, and the proof follows by telescoping and using the binomial formula. ☐

2.4 Trigonometry, logarithms, and Euler's formula

By 2.22, we are justified in taking the power series definitions of exp, sin, and cos – if we wanted them to be analytic, we have no choice but to accept them!

Definition 2.24.

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

and

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

Recall the following formula from calculus:

Theorem 2.25 (Euler's formula)**.** *If* $\theta \in \mathbb{R}$ *then*

$$\exp i\theta = \cos\theta + i\sin\theta.$$

If you haven't seen the proof of this, you should derive it for practice from the power series definitions.

Corollary 2.26.

$$e^z = e^w \iff z - w \in 2\pi i\mathbb{Z}.$$

Proof. Let $z = x + iy$. Then $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. So exp is periodic in its imaginary part with period $2\pi i$, but injective in its real part. □

In particular, log is *not* well-defined as the inverse of exp, because exp is no longer a bijection.
But there was another definition of log that was developed in 104.

Definition 2.27. Let $z \in \mathbb{C} \setminus (-\infty, 0]$. The *natural logarithm* of z is given by

$$\log z = \int_1^z \frac{dt}{t}.$$

This integral only makes sense if there's a straight line segment from 1 to z, which is why we exclude $(-\infty, 0]$ (since $1/t$ isn't a well-defined function at 0). In spite of this pathology, log is fairly well-behaved.

Lemma 2.28. log *is holomorphic on its domain and its derivative is* $z \mapsto z^{-1}$.

Proof. Apply the fundamental theorem of calculus to the definition of the logarithm. □

Since we can apply calculus, we can mimic the usual proofs from 104:

Corollary 2.29. $\exp \circ \log = \mathrm{id}$. *Moreover,* $\log ab = \log a + \log b$.

Sadly, the opposite relation ($\log \circ \exp = \mathrm{id}$) does not hold, because log's image is not all of $\mathbb{C}$:

Corollary 2.30. $\operatorname{Im} \log z \in (-\pi, \pi]$.

Proof. log is continuous because it is holomorphic, and its domain is connected, so its image

$$V = \log(\mathbb{C} \setminus \{0\})$$

must be connected. Moreover, $\log((0, \infty]) = \mathbb{R}$ so $\mathbb{R} \subseteq V$.

On the other hand, we have already seen that $y \mapsto \exp iy$ is periodic if $y \in \mathbb{R}$ and its period is 2π. Since the trig functions are symmetric about 0 it must be that $V = (-\pi, \pi]$. □

Finally, recall from calculus the definition for exponentiation with arbitrary base:

Definition 2.31. Let $a, z \in \mathbb{C}$. Then
$$a^z = \exp(z \log a).$$

Example 2.32. Let's compute 2^{1+i}. We have

$$2^{1+i} = \exp(1 \log 2 + i \log 2) = \exp(\log 2)(\cos \log 2 + i \sin \log 2) = 2(\cos \log 2 + i \sin \log 2).$$

Chapter 3

Integration in a convex set

Before we can get to the real meat of the course – the residue calculus – we'll prove preliminary versions of the theorems we'll see later under a very strong assumption: that the domain U is *convex*. This will allow us to draw line segments and integrate along them using the fundamental theorem of calculus. Later we'll see that convex sets are rather boring, and the theorems generalize and become much more powerful and natural when U is merely simply connected (or sometimes just connected). However, to generalize these theorems, we'll need the weak versions; we'll exploit the fact that U is *locally* convex because it is open.

Recall that U is assumed open.

We're going to start off proving a rather technical representation formula which will allow us to construct analytic functions. The rest of the chapter, and indeed the class, is going to be fallout from it.

Lemma 3.1. *Let γ be a curve, $g : \gamma^* \to \mathbb{C}$ be continuous, $z_0 \in U = \mathbb{C} \setminus \gamma^*$, and*

$$f(z) = \int_\gamma \frac{g(w)}{w - z}\, dw.$$

Then f is analytic, and for each $n \in \mathbb{N}$, its coefficients are

$$\alpha_n = \int_\gamma \frac{g(w)}{(w - z_0)^{n+1}}\, dw.$$

Moreover, the radius of convergence of f is at least

$$\inf_{w \in \gamma^*} |w - z_0| > 0.$$

Proof. Let $z \in B_R z_0$ and $w \in \gamma^*$. Then

$$\left|\frac{z - z_0}{w - z_0}\right| \leq \left|\frac{z - z_0}{R}\right| < 1$$

so, using a geometric series,

$$\begin{aligned}\frac{1}{w - z} &= \frac{1}{w - z_0 - (z - z_0)} \\ &= \frac{1}{w - z_0}\frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0}\sum_{n=0}^\infty \left(\frac{z - z_0}{w - z_0}\right)^n\end{aligned}$$

converges uniformly.

Define $h, h_1, h_2, \cdots : \gamma^* \to \mathbb{C}$ by

$$h_n(w) = \frac{g(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

and

$$h(w) = \frac{g(w)}{w - z_0}.$$

These are continuous on γ^* and $\sum h_i = h$ by another geometric series computation; the limit converges uniformly on γ^*. By compactness of γ^*, we can swap limits, so

$$f(z) = \int_\gamma h = \sum_{n=0}^{\infty} \int_\gamma h_n = \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \frac{g(w)}{w - z_0} \, dw = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

on $B_R z_0$. □

3.1 Winding numbers

A special case occurs when $g \equiv 1$.

Definition 3.2. If γ is closed, then define $\mathrm{Ind}_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ by

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z},$$

the *index function* or *winding number* of γ.

By 3.1, Ind_γ is analytic. As we shall see, the winding number of γ at z tells us how many times γ winds around z counterclockwise (minus the times it winds around z clockwise).

Lemma 3.3. *If γ is closed, then $\mathrm{Ind}_\gamma(\mathbb{C} \setminus \gamma^*) \subseteq \mathbb{Z}$, and $\mathrm{Ind}_\gamma(z)$ is identically 0 outside of the region encircled by γ.*

Proof. Put $f : [a, b] \to \mathbb{C}$ by

$$f(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, ds.$$

Then f is differentiable and

$$f'(t) = \frac{\gamma'(t)}{\gamma(t) - z}.$$

Define $g : [a, b] \to \mathbb{C}$ by

$$g(t) = e^{-f(t)}(\gamma(t) - z).$$

Then $g' \equiv 0$, so g is constant. In particular,

$$e^{f(t) - f(a)} = \frac{\gamma(t) - z}{\gamma(a) - z}$$

but $f(a) = 0$ and $\gamma(b) = \gamma(a)$. Therefore,

$$e^{f(b)} = e^{f(b) - f(a)} = \frac{\gamma(a) - z}{\gamma(a) - z} = 1.$$

So $f(b) \in 2\pi i \mathbb{Z}$, but $f(b) = 2\pi i \, \mathrm{Ind}_\gamma(z)$.

A computation verifies that

$$\lim_{|z| \to \infty} \mathrm{Ind}_\gamma(z) = 0$$

but since Ind_γ takes values in the integers it must be identically 0 if $|z|$ is sufficiently large. But then by analyticity, Ind_γ is 0 on any connected components which include such sufficiently large $|z|$. □

So why does the winding number behave the way it does? Let's start by looking at an example:

Example 3.4. Let $U = B_1 0$ and $\gamma = \Gamma_1 0$. By 2.12,

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w} = 1$$

and, by connectivity, Ind_γ is identically 1 inside of U. But on the other hand, 3.3 implies that Ind_γ is identically 0 outside of U.

Now suppose that instead of wrapping around U 1 time,

$$\gamma = \bigoplus_{j=1}^{n} \Gamma_1 0;$$

that is, γ winds around U n times. Then, $\mathrm{Ind}_\gamma = n\,\mathrm{Ind}_{\Gamma_1 0} = n$ inside of U.

On the other hand, $-\Gamma_1 0$ has winding number -1 inside of U. So, for $U = B_1 0$, Ind_γ does count the number of times γ winds around U counterclockwise, minus the times it winds around clockwise.

It shouldn't be hard to believe that this result generalizes when γ is a less simple curve, or when U has a less obvious geometry. The homotopy theorem (4.13) makes this rigorous: as long as the deformation of γ isn't "too bad", the integral, and thus the winding number, is unchanged.

Of course, we haven't proven the homotopy theorem, so we can't use this trick in any proofs yet. But it does provide a nice way to visualize the winding number, as well as foreshadowing some of the big ideas from later on.

3.2 Cauchy-Goursat in a convex set

In multivariable calculus one often worries about conservative vector fields, those for which line integrals around closed curves vanish. You might intuit that holomorphic functions $U \to \mathbb{C}$ are "conservative", and if the topology of U isn't too bad, you'd be right.

First, a notational convenience, which we won't need after its use to prove the next few theorems.

Definition 3.5. For $z_1, z_2, z_3 \in \mathbb{C}$, write Δ to mean the triangle with those endpoints, including the interior, and $\partial\Delta$ to indicate the path around the triangle, oriented counterclockwise.

If the z_is are collinear then this is a degenerate triangle.

Theorem 3.6 (Cauchy-Goursat)**.** *If $p \in U$, $\Delta_0 \subseteq U$, f is continuous, and f is holomorphic on $U \setminus \{p\}$ then*

$$\int_{\partial\Delta} f = 0.$$

Proof. There are three cases, depending on where p is in relation to $\partial\Delta_0$.

If $p \notin \Delta_0$, then define z_i' to be the point opposite z_i on $\partial\Delta_0$. Then we have four triangles:

1. Δ_0^1, determined by z_1, z_2', z_3',
2. Δ_0^2, determined by z_1', z_2, z_3',
3. Δ_0^3, determined by z_1', z_2', z_3, and
4. Δ_0^4, determined by z_1', z_2', z_3'.

Then

$$\left|\int_{\partial\Delta_0} f\right| \le \sum_{i=1}^{4}\left|\int_{\partial\Delta_0^i} f\right| \le 4\left|\int_{\partial\Delta_0^k} f\right|$$

for some index k. Set $\Delta_1 = \Delta_0^k$.

By induction we get a nested sequence of triangles

$$\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \dots$$

and

$$\left|\int_{\partial\Delta_k} f\right| \leq 4\left|\int_{\partial\Delta_{k+1}} f\right|$$

with $\ell(\partial\Delta_k) = \ell(\partial\Delta_{k+1})/2$, so

$$\left|\int_{\partial\Delta_0} f\right| \leq 4^n\left|\int_{\partial\Delta_n} f\right|.$$

Since the Δ_n are compact there is a point in their intersection, say $z_0 \neq p$ and so in a sufficiently small ball around that point,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)$$

for a sublinear ψ. Integrating the affine part, we get a map

$$z \mapsto f(z_0)z + \frac{1}{2}f'(z_0)(z - z_0)^2$$

and so for each n,

$$\int_{\partial\Delta_n} f(z_0) + f'(z_0)(z - z_0)\ dz = 0.$$

Fix an $\epsilon > 0$. We can find $\delta > 0$ such that

$$||\psi||_{C^0(B_\delta z_0 \cap U)} < \epsilon$$

and $\Delta_n \subseteq B_\delta z_0$. From there,

$$\begin{aligned}
\left|\int_{\partial\Delta_n} f\right| &= \left|\int_{\partial\Delta_n} f(z) - (f(z_0 + f'(z_0)(z - z_0))\ dz\right| \\
&= \left|\int_{\partial\Delta_n} (z - z_0)\psi(z)\ dz\right| \\
&\leq \ell(\partial\Delta_n) \sup_{z \in \partial\Delta_n} |z - z_0||\psi(z)| \\
&\leq \ell(\partial\Delta_n)^2 ||\psi||_{C^0} \\
&< \epsilon\ell(\partial\Delta_n)^2.
\end{aligned}$$

In the second case, p is a vertex of Δ_0, say $p = z_1$. For $\epsilon > 0$ choose p_2, p_3 along the segments $[z_1, z_2]$ and $[z_2, z_3]$ with $|z_1, p_i| < \epsilon$. This determines a triangle Δ_1 by (z_1, p_2, p_3), and $\Delta_0 \setminus \Delta_1$ is a polygon which can be triangulated, say by $\Delta_2, \ldots$.

Then if $k \geq 2$, f is holomorphic on Δ_k and so its integral along $\partial\Delta_k$ vanishes. In particular,

$$\left|\int_{\partial\Delta_0} f\right| \leq \int_{\partial\Delta_1} |f| < 3\epsilon||f||_{C^0}$$

so its integral vanishes.

Finally if $p \in \Delta_0$ and it is not a vertex, then we can triangulate Δ_0 such that p is a vertex and the theorem follows. $\square$

We need continuity at p because if not, then f might be horribly behaved. If we define

$$f(z) = \begin{cases} z^{-1}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

then Cauchy-Goursat fails. See 2.12 for proof.

Theorem 3.7 (fundamental theorem of calculus, part II)**.** *Suppose that U is convex, and $f : U \to \mathbb{C}$ is continuous.*

If for each triangle $\Delta \subseteq U$,

$$\int_{\partial\Delta} f = 0$$

then for each $a \in U$, the function

$$F(z) = \int_a^z f(t)\ dt$$

is holomorphic on U and $F' = f$.

Proof. Fix $z_0 \in U$ and allow z to range over U. Then if Δ is the triangle given by (z, z_0, a) we have

$$0 = \int_{\partial\Delta} f = \int_a^z f + \int_z^{z_0} f + \int_{z_0}^a f = F(z) - F(z_0) + \int_z^{z_0} f$$

while

$$\int_{z_0}^z f(z_0)\ dw = f(z_0)(z - z_0)$$

and

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{z_0}^z f(w) - f(z_0)\ dw.$$

Fix $\epsilon > 0$. By continuity we can find a $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &\le \frac{1}{|z - z_0|} \int_z^{z_0} |f(w) - f(z_0)|\ dw \\ &< \frac{1}{|z - z_0|} \ell(z, z_0)\epsilon C = \epsilon C \end{aligned}$$

for some constant C. Convexity guarantees that all these segments are in fact contained in U. $\square$

3.3 Cauchy's integral formula

It will be useful to adopt some nonstandard terminology for dealing with the corollaries of Cauchy-Goursat. The true power of Cauchy-Goursat is that it holds even if we don't know that f is holomorphic everywhere; we only need f to satisfy these conditions:

Definition 3.8. If $f : U \to \mathbb{C}$ is continuous and holomorphic on all but finitely many points of U, we say that f is *cofinitely holomorphic*.

As it turns out, a cofinitely holomorphic function will end up being holomorphic, which is why nobody actually uses this terminology.

Corollary 3.9. *Suppose γ is closed in U convex, $a \in U$ and $f : U \to \mathbb{C}$ is cofinitely holomorphic.*

Then the function

$$F(z) = \int_a^z f$$

has $F' = f$ and is holomorphic and moreover

$$\int_\gamma f = 0.$$

Corollary 3.10 (Cauchy's integral formula)**.** *Suppose γ is closed in U convex and $f : U \to \mathbb{C}$ is cofinitely holomorphic. Then*

$$f(z)\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\ dw.$$

Proof. Define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(w), & w = z. \end{cases}$$

g is continuous and holomorphic away from z so

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma g \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dw - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dw - f(z)\operatorname{Ind}_\gamma(z). \end{aligned}$$

□

Cauchy's integral formula tells us that if f is cofinitely holomorphic on γ, we already know everything there is to know about f in the region enclosed by γ. This will be a theme we see over and over.

Now we're ready for the sockdolager:

Corollary 3.11. *Suppose U is convex and $f : U \to \mathbb{C}$.*

If f is holomorphic, then it is analytic.

If $z_0 \in U$, $r > 0$ and $B_r(z_0) \subseteq U$, we can find α_n such that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

with radius of convergence $\geq r$.

Moreover, there is a unique g holomorphic with the maximal radius of convergence afforded by this theorem, such that when g is restricted to the appropriate ball, $g = f$.

Proof. Fix $\rho \in (0, r)$. Then $\Gamma_\rho z_0 \subseteq U$ and if $z \in B_\rho z_0$, then $\operatorname{Ind}_{\Gamma_\rho} z = 1$. Moreover, if $\eta \in (\rho, r)$, then the ball $B_\eta z_0$ is convex.

So,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho z_0} \frac{f(w)}{w-z}\,dw.$$

This is analytic by 3.1 with the desired power series. □

Definition 3.12. With hypotheses as in 3.11, g is the *analytic continuation* of f.

Now we see why the notion of a cofinitely holomorphic function is utterly useless: we can prove results about open balls in U, which are convex. (This is why we demand that the domain U is open, so that it will be locally convex.) By induction, we have:

Corollary 3.13 (Morera)**.** *Let $f : U \to \mathbb{C}$. The following are equivalent:*

1. *f is holomorphic.*
2. *For each $\Delta \subseteq U$, $\int_{\partial\Delta} f = 0$.*
3. *$f^{(n)}$ is holomorphic.*
4. *f is analytic.*
5. *$f^{(n)}$ is analytic.*

This is huge; it's definitely not true in $\mathbb{R}^n$!

Corollary 3.14 (Cauchy's integral formula for derivatives)**.** *Suppose $f : U \to \mathbb{C}$ is holomorphic, $a \in U$, and $r > 0$ with $\overline{B_r a} \subset U$. Then*

$$|f^{(n)}(z)| = \frac{n!}{2\pi i} \int_{\Gamma_r a} \frac{f(w)}{(w-z)^{n+1}}\ dw.$$

Proof. Apply Cauchy's formula and induct on the Taylor coefficients, using 1.27 to ensure all the balls make sense. ☐

3.4 Cauchy's estimate

By using the estimate 2.10, the following, very useful estimate follows.

Corollary 3.15 (Cauchy's estimate)**.** *With hypotheses as in 3.14,*

$$|f^{(n)}(a)| \leq \frac{C}{r^n} \max_{w \in \partial B_r a} |f(w)|$$

for some constant C which only depends on n.

This has important implications in algebra, of all places.

Definition 3.16. A holomorphic function is *entire* if its domain is $\mathbb{C}$.

Theorem 3.17 (Liouville)**.** *An entire function is constant or unbounded.*

Proof. Suppose $f : \mathbb{C} \to \mathbb{C}$ and $|f(z)| \leq M$. Then for each $r > 0$, the estimate

$$|f'(z)| \leq \frac{M}{r}$$

holds. In particular $f' \equiv 0$. ☐

The following estimate is true clearly for linear polynomials, and then by induction and iterated differentiation one has:

Lemma 3.18. *Let $f \neq 0$ be a polynomial. Then we have $\mu > 0$ and $R \geq 1$ such that for all z with $|z| > R$,*

$$|f(z)| \geq \mu|z|.$$

So polynomials necessarily grow without bound.

The punchline is that polynomials must be zero *somewhere* in $\mathbb{C}$, though this is clearly not true in $\mathbb{R}$.

Theorem 3.19 (Fundamental theorem of algebra)**.** *A nonconstant polynomial has a root.*

Proof. If not, then $1/f$ is bounded on $\mathbb{C} \setminus B_r 0$ for some $r > 0$. But $1/f$ is bounded on the compact set $\overline{B_r 0}$, because the lack of roots guarantees its continuity. So $1/f$ is constant. ☐

3.5 Locally uniform convergence

Another consequence of Cauchy-Goursat is that we don't actually care that much about uniform convergence in $\mathbb{C}$: it's locally uniform convergence that's interesting.

Definition 3.20. Suppose $f_n \to f$ is a convergent sequence of functions in U. If, for each $z \in U$, there is a neighborhood $V \ni z$ on which $f_n \to f$ uniformly, then $f_n \to f$ *locally uniformly.*

Theorem 3.21 (Weierstrass)**.** *Suppose $f_n \to f$ locally uniformly and each f_n is holomorphic. Then f is holomorphic and convergence of its derivatives is locally uniform.*

Proof. For $a \in U$ there exists $r > 0$ with $B_r a \subseteq U$ and $f_n \to f$ uniformly on $B_r a$. So f is continuous on $B_r a$. By Morera's theorem, if $\Delta \subset B_r a$, then

$$\oint_{\partial\Delta} f = 0$$

whence f is holomorphic.

Suppose now that $f_n \to f$ uniformly on $B_{2r}a$, then $f_n \to f$ uniformly on $\partial B_r a$. By Cauchy's estimate, we can bound differences in f' by differences in $\partial B_r a$. □

Corollary 3.22. *Suppose that $r > 0$ and $f_n \to f$ locally uniformly on $B_r 0$, with each f_n holomorphic. Suppose further that*

$$f_m(z) = \sum_{n=0}^{\infty} \alpha_n^{(m)} z^n$$

and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n;$$

then $\alpha_n^{(m)} \to \alpha_n$.

Proof. By induction on Weierstrass's theorem. □

The strategy to show that a limit function f is holomorphic is as follows: use the M-test to show uniform convergence on each open set, which is enough to apply Weierstrass's theorem.

Chapter 4

Integration in a simply connected set

Nothing too exciting has happened yet, because we've been hampered by the assumption that U is convex, and that f is continuous even when it fails to be holomorphic. As it turns out, local convexity is all we really need, and complex analysis becomes more interesting when f behaves more spectacularly at points where it's not holomorphic.

The residue calculus is a powerful technique for dealing with poles, points where f fails to be holomorphic or continuous, but not "too badly". It will allow us to compute integrals around closed curves just by summing up a few easy limits – and to compute integrals around curves which aren't closed, including integrals in $\mathbb{R}$, by extending them to a closed curve without affecting their value. It will also allow us to characterize holomorphic functions further as geometric transformations which "preserve angle".

Recall that U is assumed open. Furthermore we will assume $0 \leq R_1 < R_2 \leq \infty$.

4.1 Zeroes of holomorphic functions

Definition 4.1. If $a \in U$, $f \in \mathcal{O}$ with $f(a) = 0$, we say that a is an *isolated zero* if it is contained in a punctured ball on which f is nonzero. We further say that a is of *order* $N > 0$ if each $f^{(n)}(a) = 0$ for $n < N$ and $f^{(N)} \neq 0$. If, for each $n > 0$, $f^{(n)}(a) = 0$, then a is a *zero of infinite order.*

We'll soon see that the only interesting holomorphic functions are those which have only finite-order, isolated zeroes; if not, they'll just be identically zero!

We can "factor" out the zeroes of a holomorphic function, just like a polynomial.

Theorem 4.2. *If $a \in U$, $f \in \mathcal{O}$ with $f(a) = 0$, then $\exists r > 0$ with $B_r a \subseteq U$ such that exactly one of the following is true:*

1. *$f \equiv 0$ on $B_r a$ and a is a zero of infinite order.*
2. *a is an isolated zero and there exists a unique $g : B_r a \to \mathbb{C}$ holomorphic with $|g| > 0$ and $f(z) = (z-a)^N g(z)$ where N is the order of a.*

Proof. We can pick $B_R a \subseteq U$ on which $f(z) = \sum \alpha_n (z-a)^n$ uniformly and

$$\alpha_n = \frac{f^{(n)}(a)}{n!}.$$

Then if for each n, $f^{(n)}(a) = 0$, $\alpha_n = 0$, so $f \equiv 0$.

Otherwise it's of order $N < \infty$ and

$$f(z) = \sum_{n=N}^{\infty} \alpha_n (z-a)^n = (z-a)^N \sum_{k=0}^{\infty} \alpha_{N+k} (z-a)^k$$

and this last series is g. Moreover, for R sufficiently small, $g \neq 0$, because a is isolated by continuity of g. $\square$

Because locally, holomorphic functions act like polynomials, we can rattle off several corollaries which resemble properties of polynomials.

Corollary 4.3. *If $a \in U$, $f \in \mathcal{O}$ then $f(a) = 0$ of order N iff*

$$\lim_{z \to a} \frac{f(z)}{(z-a)^N}$$

exists and is nonzero.

Proof. One direction is obvious. Otherwise we have $f(a) = 0$, and the order is finite. So we can factor out a g, which is the limit, which is continuous and nonzero. $\square$

The following proof is an example of an "open and closed" argument, which will be a powerful tool for proving theorems about holomorphic functions on connected sets.

Corollary 4.4. *If U is connected, $B_r a \subseteq U$, and f is holomorphic with $f|_{B_r a} \equiv 0$ then $f \equiv 0$.*

Proof. The set

$$V = \{p \in U : \exists s > 0 \ f|B_s p \equiv 0\}$$

is a union of open balls, thus open. On the other hand, the set

$$W = \{p \in U : \exists s > 0 \ \forall z \in B_s p \setminus \{0\} \ f(z) \neq 0\}$$

is also a union of open balls, thus also open, and $U = V \cup W$, $\emptyset = V \cap W$. Since U is connected and $a \in V$, $W = \emptyset$. $\square$

Corollary 4.5. *Consider U connected $\supseteq V \neq \emptyset$ open, and f, g holomorphic, if $f|_V = g|_V$ then $f = g$.*

Proof. $(f - g)|_V \equiv 0$ so $f - g \equiv 0$. $\square$

In other words, if U is connected, then the behavior of f is completely determined by its behavior on any open set in U. This isn't too surprising, as an open set is all we need to construct all the Taylor coefficients of f.

In more extreme cases, we can characterize f by its behavior on a *sequence* in U.

Corollary 4.6. *For U connected, if there exists a sequence $z_n \to z$ of distinct points with $z \in U$, and $f(z_n) = 0$ for each z_n then $f \equiv 0$.*

Proof. The zeroes of f cluster at z, so z is not an isolated zero; thus, z must be a zero of infinite order. $\square$

Notice that we need $z \in U$; the counterexample is $z \mapsto \sin \pi/z$, $z_n = 1/n$, whose zeroes cluster at 0, where the map is not holomorphic.

4.2 Isolated singularities

Definition 4.7. If f is holomorphic except at $a \in U$, then a is an *isolated singularity* of f. Moreover, if there is a holomorphic function $g : U \to \mathbb{C}$ such that $f \equiv g$ off of a, then a is *removable*.

For convenience, we'll say that g is a *holomorphic extension*, but this is nonstandard terminology.

The classic example of a removable singularity is -1 for the function

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}$$

which morally is $g(x) = x + 1$. Removable singularities are mostly harmless.

Theorem 4.8 (Riemann)**.** *If f has an isolated singularity at $a \in U$ and there is a neighborhood $V \ni a$ on which f is bounded, then a is removable.*

Proof. Consider $h : U \to \mathbb{C}$ with $h(z) = (z-a)f(z)$ away from a and $h(a) = 0$. h is continuous because f is bounded, and is clearly holomorphic away from a, thus cofinitely holomorphic, so Morera's theorem applies and h is holomorphic.

So, by 2.2, there exists g continuous with $h(z) = (z-a)g(z)$ (since $h(a) = 0$). But then by Cauchy-Goursat, g is holomorphic. But $g \equiv f$ away from a. ☐

Since, in order for an isolated singularity a to fail to be removable, f must be unbounded near a, the behavior of f near a must be calamitous. In particular, case (3) of the following theorem is shocking. (And I could make a good YouTuber... "10 shocking facts about isolated singularities you won't believe! (3) is INSANE!)"

Theorem 4.9 (Casorati-Weierstrass)**.** *If f has an isolated singularity at $a \in U$ then exactly one of the following is true.*

1. *a is removable.*
2. *There exists $m \in \mathbb{N}$ and $c_1, \dots, c_m \in \mathbb{C}$ such that $c_m \neq 0$ and the function*

$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

   has a removable singularity at a.

3. *For each neighborhood V of a, $f(V)$ is dense in $\mathbb{C}$.*

Before the proof, we define what's going on in (2) and (3).

Definition 4.10. If (2) holds, then a is a *pole* of order m. A pole of order 1 is *simple.*

If (3) holds, then a is an *essential singularity.*

Proof of Casorati-Weierstrass. Suppose that (3) does not hold. Then $f(V)$ isn't dense somewhere in $\mathbb{C}$, so there exist $w \in \mathbb{C}$, $r > 0$, and $\mu > 0$ with $B_r a \subseteq U$ and for $z \in B_r a \setminus \{a\}$, $|f(z) - w| > \mu$.

Define $g : B_r a \setminus \{a\} \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - w}.$$

g is holomorphic, but also bounded because

$$\frac{1}{|f(z) - w|} < \frac{1}{\mu},$$

so by Riemann, g has a holomorphic extension $h : B_r a \to \mathbb{C}$.

Either $h(a) = 0$ or not. If not, then $f(z) - w$ does not tend to ∞ at a, so f is bounded and therefore has a removable singularity. So (1) holds.

Otherwise, a is a zero of order $m < \infty$ for h. So $\exists b : B_r a \to \mathbb{C}$ holomorphic such that $b(a) \neq 0$ and $h(z) = (z-a)^m b(z)$ and

$$f(z) = w + \frac{1}{b(z)(z-a)^m}.$$

But $1/b \neq 0$ so $1/b$ is holomorphic and has a power series,

$$\frac{1}{b(z)} = \sum_{n=0}^{\infty} \alpha_n (z-a)^n$$

whence

$$f(z) = w + \sum_{n=0}^{\infty} \alpha_n (z-a)^{n-m}.$$

When $n > m$ these terms are benign and can be absorbed into the Taylor series for f. Otherwise, $c_i = \alpha_i$, implying (2). ☐

As it turns out, if (3) holds, then something much stronger is true:

Theorem 4.11 (Picard's great theorem)**.** *Suppose that $f : U \to \mathbb{C}$ is holomorphic and has an essential singularity at $a \in \mathbb{C}$. Then for each neighborhood $V \ni a$, $\mathbb{C} \setminus f(V)$ is finite.*

Sadly, we won't develop the machinery to prove this theorem. But indeed, as promised, essential singularities are calamitous.

A quick glance at the series in (2) confirms:

Corollary 4.12. *If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then a is a pole of order m iff*

$$\lim_{z \to a} (z-a)^m f(z) \neq 0$$

and is finite.

The next theorem will show why we care about isolated singularities so much: integrals around closed curves only are interesting if the curve encloses singularities.

Theorem 4.13 (homotopy theorem)**.** *If $f : U \to \mathbb{C}$ is holomorphic and γ_0 and γ_1 are U-homotopic, closed curves then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

In particular, if γ_0 is U-nullhomotopic then

$$\int_{\gamma_0} f = 0.$$

Proof. Consider a homotopy $\Phi : [0,1]^2 \to U$ between γ_0 and γ_1. $[0,1]^2$ is compact, so there are $\varepsilon > 0$ and $N > 0$ such that for each (t_1, s_1) and $(t_2, s_2) \in [0,1]^2$, if $|(t_1, s_1) - (t_2, s_2)| < 2/N$ then $|\Phi(t_1, s_1) - \Phi(t_2, s_2)| < \varepsilon$. (That is, if N is large, then we can divide $[0,1]^2$ into tiles of size $\leq 2/N$ whose images will be within ε of each other.)

Put $z_{nm} = \Phi(n/N, m/N)$ for $n, m \leq N$. In particular if $k, j \leq 1$ then $|z_{nm} - z_{n-k,m-j}| < \varepsilon$ and therefore is contained in U. The ball around z_{nm} is *convex*, so

$$\int_{z_{n-1,m-1}}^{z_{n,m-1}} f + \int_{z_{n,m-1}}^{z_{nm}} f - \int_{z_{n-1,m-1}}^{z_{n-1,m}} f - \int_{z_{n-1,m}}^{z_{nm}} f = 0$$

because the curve generated when the last two integrals are reversed is closed.

The z_{nm} tile $[0,1]^2$ so

$$\sum_{j=1}^{N} \int_{z_{j-1,0}}^{z_{j,0}} f = \sum_{j=1}^{N} \int_{z_{j-1,N}}^{z_{j,N}} f.$$

Each of these is a discretization of γ_0 or γ_1, so the result holds. ☐

4.3 Cauchy-Goursat's true power

We needed to painstakingly prove Cauchy-Goursat and related theorems for convex sets precisely as lemmata to the homotopy theorem: we used that U was locally convex in its proof.

Now that we know the homotopy theorem, all those old results generalize trivially to simply connected sets.

Corollary 4.14 (Cauchy-Goursat)**.** *Suppose that U is simply connected and $f : U \to \mathbb{C}$ is holomorphic. Then if γ is closed,*

$$\int_\gamma f = 0.$$

Corollary 4.15 (fundamental theorem of calculus)**.** *If U is simply connected, and $f : U \to \mathbb{C}$ is holomorphic, then the function $F : U \to \mathbb{C}$ given by*

$$F(z) = \int_\gamma f$$

is holomorphic with $F'(z) = f(z)$, where $\gamma(0)$ is constant and $\gamma(1) = z$.

4.4 Laurent series

A Laurent series is a "two-sided" Taylor series, analogous to the improper integral $\int_{-\infty}^{\infty}$. We'll need finitely many negative terms to deal with poles – and infinitely many to deal with essential singularities. They describe functions which are holomorphic away from a singularity.

Recall that $0 \leq R_1 < R_2 \leq \infty$. We made that assumption so that we could make the following definition:

Definition 4.16. If $a \in \mathbb{C}$ define the *annulus*

$$A(a, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z - a| < R_2\}.$$

The annulus $A(a, 0, \infty)$ is just the plane punctured at 0.

Definition 4.17. For each $k \in \mathbb{Z}$, let $z_k \in \mathbb{C}$. The series $\sum z_k$ *converges* if $\sum_{k \geq 0} z_k$ and $\sum_{k>0} z_{-k}$ converges.

Moreover, its sum is

$$\sum_{k=-\infty}^{\infty} z_k = \sum_{k=0}^{\infty} z_k + \sum_{k=1}^{\infty} z_{-k}.$$

Definition 4.18. If $a \in \mathbb{C}$, a *Laurent series* about a is a series

$$\sum_{k=-\infty}^{\infty} \alpha_k (z - a)^k$$

with coefficients $\alpha_k \in \mathbb{C}$.

Just like when we developed Taylor series, it was convenient to assume that the "center" $a = 0$. Here the series will always be centered on 0. So, for the next theorem, we'll assume f is singular at the origin.

Lemma 4.19. *If f is holomorphic on $A(0, R_1, R_2)$ and $r \in (R_1, R_2)$ then define, for each $k \in \mathbb{Z}$,*

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_r 0} \frac{f(w)}{w^{k+1}}\, dw.$$

Then α_k is independent of r, and the series

$$\sum_{k=-\infty}^{\infty} \alpha_k z^k = f(z).$$

Proof. Independence of r follows by the homotopy theorem: we could choose any curve which winds around $B_{R_1} 0$.

For $z \in A(0, R_1, R_2)$, choose r_1, r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$, and define

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

on $A(0, R_1, R_2) \setminus \{z\}$. Then g has a removable singularity at z, since f is holomorphic and thus has

$$|f'(z)| = \lim_{w \to z} \left| \frac{f(w) - f(z)}{w - z} \right| < \infty.$$

We'll identify g with its holomorphic extension.

$\Gamma_{r_1} 0$ and $\Gamma_{r_2} 0$ are homotopic so

$$\int_{\Gamma_{r_1} 0} g = \int_{\Gamma_{r_2} 0} g.$$

Splitting up g, we have

$$\int_{\Gamma_{r_2} 0} \frac{f(w)}{w - z}\, dw - \int_{\Gamma_{r_1} 0} \frac{f(w)}{w - z}\, dw = f(z) \left[\int_{\Gamma_{r_2} 0} \frac{dw}{w - z} - \int_{\Gamma_{r_1} 0} \frac{dw}{w - z} \right].$$

Notice that $(w-z)^{-1}$ is not holomorphic, but it is holomorphic away from z, and that $\Gamma_{r_1}0$ does not loop around w. So the final integral vanishes, and a now familiar computation shows that

$$\left[\int_{\Gamma_{r_2}0} - \int_{\Gamma_{r_1}0}\right] \frac{f(w)}{w-z}\, dw = 2\pi i f(z).$$

On $\partial B_{r_1}0$, $h_n(w) = (w/z)^n$ has $\sum h_n$ uniformly convergent.

Thus,

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{\Gamma_{r_1}0} \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i}\int_{\Gamma_{r_1}0} \frac{f(w)}{1-w/z}\, dw \\
&= \frac{1}{2\pi i}\frac{1}{z}\int_{\Gamma_{r_1}0} f(w)\sum_{n=0}^{\infty}\frac{w}{z}^n\, dw \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{1}{z^{n+1}}\int_{\Gamma_{r_1}0} f(w)w^n\, dw \\
&= \sum_{n=1}^{\infty}\alpha_{-n}z^{-n}.
\end{aligned}$$

On the other hand,

$$\int_{\Gamma_{r_2}0} \frac{f(w)}{w-z}\, dw = 2\pi i\sum_{n=0}^{\infty}\alpha_n z^n$$

as desired. □

In other words, 3.14 holds even when f has singularities, as long as we allow n to range over $\mathbb{Z}$.

Definition 4.20. With notation as in 4.19, $\sum \alpha_k(z-a)^k$ is the *Laurent series* of f about a. Define $h : A(a, R_1, \infty) \to \mathbb{C}$ by

$$h(z) = \sum_{k=-\infty}^{-1}\alpha_k(z-a)^k.$$

Then h is called the *principal part* of f about a. If $R_1 = 0$, we say that α_{-1} is the *residue* of f at a and write $\alpha_{-1} = \operatorname{Res}_a f$.

We can now rephrase the Casorati-Weierstrass theorem in terms of Laurent series.

Corollary 4.21 (Casorati-Weierstrass). *Let f have an isolated singularity at $a \in U$ and*

$$f(z) = \sum_{k=-\infty}^{\infty}\alpha_k(z-a)^k.$$

Let $m \in \mathbb{N}$ and h be the principal part of f. Then:

1. *a is removable iff $\forall k \le -1$, $\alpha_k = 0$.*
2. *a is an isolated pole of order m iff $\forall k < -m$, $\alpha_k = 0$ and $\alpha_{-m} \neq 0$.*
3. *a is an essential singularity iff there are infinitely many nonzero α_k $(k < 0)$.*
4. *$f - h$ has a removable singularity at a.*

We can also rephrase some of the results about factoring out zeroes and poles from holomorphic functions. This will be our main tool for computing residues.

Corollary 4.22 (zero-factoring). *Suppose that f has an isolated pole of order m at a. Then*

$$\operatorname{Res}_a f = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).$$

In particular, if f has a simple pole at a, then

$$\operatorname{Res}_a f = \lim_{z\to a}(z-a)f(z).$$

4.5 The residue theorem

Notice that each of the terms in the Laurent series other than the residue have an antiderivative. In particular, if γ is closed and U-nullhomotopic, and f is holomorphic off of a, then each term of the Laurent series except the residue vanishes, and

$$\frac{1}{2\pi i}\int_\gamma f = \frac{\alpha_{-1}}{2\pi i}\int_\gamma \frac{dz}{w-z} = \alpha_{-1}\operatorname{Ind}_\gamma a = \operatorname{Res}_a f \operatorname{Ind}_\gamma a$$

where α_k are the terms of the Laurent series of f.

As a result, if f has no essential singularities, then the only interesting term of the principal part is the residue, which encodes valuable information about f. Moreover, the following theorem is motivated:

Theorem 4.23 (residue theorem)**.** *Let $A \subset U$ be discrete and $f : U \setminus A \to \mathbb{C}$ be holomorphic. Suppose that for each $a \in A$, f has an isolated singularity at a.*

If $\gamma^* \subset U \setminus A$ and γ is closed and U-nullhomotopic, then

$$A' = \{a \in A : |\operatorname{Ind}_\gamma a| > 0\}$$

is finite. Moreover,

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{a_k \in A'} \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

Proof. Let Φ be a nullhomotopy of γ with image K. Then $K = \Phi([0,1]^2)$, which is compact.

Moreover, if $a \in A \setminus K$, then $\operatorname{Ind}_\gamma a = 0$. Therefore $A' \subseteq A \cap K$. Moreover, $A \cap K$ is infinite it must cluster somewhere in K by the Bolzano-Weierstrass theorem – but we assumed that A was discrete, so $A \cap K$ is finite. In particular A' is finite.

$V = (U \setminus A) \cup A'$ is therefore open and γ is therefore V-nullhomotopic. Let h_k be the principal part of f centered on a_k. Then $g = f - \sum h_k$ is holomorphic on V, so its integral vanishes. Then

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_\gamma \sum_{k=1}^N h_k = \sum_{k=1}^N \operatorname{Res}_{a_k} f \operatorname{Ind}_\gamma a_k.$$

□

Chapter 5

Applications of the residue calculus

The residue calculus is the use of the residue theorem to reduce integrals – which are inheritely analytic objects – to finite sums, which are algebraic in nature. The residue theorem will now serve as a sledgehammer with which we can smash open several problems: integrals over $\mathbb{R}$, combinatorial problems involving counting zeroes and poles, and basic results about the geometry of $\mathbb{C}$, among others.

Recall that U is assumed open.

5.1 Improper integration

Before Mathematica, mathematicians would use the residue theorem to compute integrals from real analysis. Even now, we tend to want to use some concrete examples to illustrate the power of the residue theorem, rather than the way some algebraists learn new theorems, which is to rephrase everything in terms of category theory.

The following lemma will allow us to construct closed curves so we can actually apply the residue theorem. The idea is that we can start with a curve that is not closed and draw a harmless arc through the half-plane, on which the integral will vanish.

Lemma 5.1 (Jordan). *Let $f : \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \to \mathbb{C}$ be continuous and*

$$\lim_{R\to\infty} \sup_{\theta\in[0,\pi]} |f(Re^{i\theta})| = 0.$$

Define $\gamma_R : [0, \pi] \to \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ by $\gamma_R(\theta) = Re^{i\theta}$. For $m > 0$,

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z)\ dz = 0.$$

Proof. TODO: This. □

Now let's make like we're high schoolers and compute some integrals.

Example 5.2. Let's compute

$$\int_0^\infty \frac{\sin x}{x}\ dx.$$

Put $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = z^{-1}e^{iz}$. For $\theta \in [0, \pi]$, consider

$$\begin{cases} \mu_\varepsilon(\theta) & = \varepsilon e^{i(\pi-\theta)} \\ \gamma_R(\theta) & = Re^{i\theta}. \end{cases}$$

Then $\gamma_R \oplus [-R, -\varepsilon] \oplus \mu_\varepsilon \oplus [\varepsilon, R]$ is closed and $\mathbb{C} \setminus \{0\}$-nullhomotopic.

By Jordan's lemma applied to γ_R and the residue theorem,

$$\left[\int_\varepsilon^R + \int_{\gamma_R} + \int_{-R}^\varepsilon + \int_{\mu_\epsilon}\right] f = \left[\int_\varepsilon^R + \int_{-R}^\varepsilon + \int_{\mu_\epsilon}\right] f = 0.$$

But

$$\left[\int_\varepsilon^R + \int_{-R}^{-\varepsilon}\right] f = 2i \int_\varepsilon^R \frac{\sin x}{x}\ dx$$

and

$$\lim_{\varepsilon \to 0} \int_{\mu_\varepsilon} f = \lim_{\varepsilon \to 0} \int_0^\pi \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\ d\theta = \lim_{\varepsilon \to 0} -i \int_0^\pi e^{i\varepsilon e^{i\theta}}\ d\theta = -i\pi.$$

So,

$$\int_0^R \frac{\sin x}{x}\ dx = \frac{i\pi}{2i} = \frac{\pi}{2}.$$

For $\operatorname{Re} z > 1$, the *Riemann ζ-function* is given by

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}.$$

Example 5.3. We will compute $\zeta(2)$. This works for any $2n$, $n \geq 1$.

Let $g : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ be $g(z) = \cot \pi z$. Then g is holomorphic, and if $z \in (0,1) \times i\mathbb{R}$, $k \in \mathbb{Z}$, $g(z+k) = g(z)$. Moreover, for each compact $K \subseteq \mathbb{C} \setminus \mathbb{Z}$, g is bounded. In particular, for $\delta > 0$, g is bounded on $K_k = [k+\delta, k+1-\delta]$. g is also bounded outside of $\{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$, as is clear if one rewrites g in terms of the complex exponential. So g has poles precisely on $\mathbb{Z}$.

If $f(z) = z^{-2} g(z)$ then

$$\lim_{N \to \infty} \int_{\Gamma_{N+\delta} 0} f = 0$$

since by Jordan's lemma applied twice. Moreover, close to 0, g satisfies

$$\frac{1 + o(z^2)}{\pi z + o(z^2)} = \frac{1}{z}\frac{1}{\pi}.$$

In particular, $\operatorname{Res}_0 g = \pi^{-1}$, so

$$\operatorname{Res}_n f = \frac{1}{\pi n^2}.$$

At 0, f has a pole of third order, so by the zero-factoring corollary, $\operatorname{Res}_0 f = -\pi/3$.

It follows that

$$0 = \lim_{N \to \infty} \lim_{\delta \to 0} \int_{\Gamma_{N+\delta} 0} f = \lim_{N \to \infty} 2 \sum_{k=1}^N \frac{1}{\pi k^2} - \frac{\pi}{3} = \lim_{N \to \infty} 2 \sum_{k=1}^\infty \frac{1}{\pi k^2} - \frac{\pi}{3} = 2\zeta(2) - \frac{\pi}{3}$$

whence

$$\zeta(2) = \frac{\pi^2}{6}.$$

The following is a general formula for solving improper integrals of rational functions on $\mathbb{R}$.

Lemma 5.4. *Let $p, q : \mathbb{C} \to \mathbb{C}$ be polynomials. If the degree of q exceeds the degree of p by at least two, $A = \{\operatorname{Im} z > 0\}$ is the upper half plane, and if $x \in \mathbb{R}$ then $q(x) \neq 0$, then*

$$\int_{-\infty}^\infty \frac{p(x)}{q(x)} dx = 2\pi i \sum_{a \in A} \operatorname{Res}_a f.$$

Proof. TODO: This. □

Exercise 5.5. Let $U \subseteq \mathbb{C}$ be open, $\overline{B_1 0} \subseteq U$, and $f : U \to \mathbb{C}$ holomorphic. Compute

$$\int_0^{2\pi} f(e^{i\theta}) \cos^2(\frac{\theta}{2})\ d\theta.$$

5.2 The argument principle

The residue theorem will allow us to prove the argument principle, a powerful combinatorial theorem with lots of consequences.

Definition 5.6. γ is *simple* if $\mathrm{Ind}_\gamma(\mathbb{C} \setminus \gamma^*) = \{0, 1\}$.

Simple curves are those which wind around exactly once, counterclockwise. They're deformations of counterclockwise circles.

Theorem 5.7 (argument principle). *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \mathrm{Ind}_\gamma(a) = 1\},$$

$A \subset U$ is finite, $f : U \setminus A \to \mathbb{C}$ is holomorphic, and f has N zeroes and P poles in U_1 (repeated by order).

If, for each $a \in A$, f has a finite-order pole at a, and has no poles and zeroes on γ^*, then

$$N - P = \mathrm{Ind}_{f\circ\gamma}(0).$$

Proof. Since the poles are isolated, $U \setminus A$ is connected. Similarly

$$V = U \setminus (A \cup \{z \in U : f(z) = 0\})$$

is open, and $|f| > 0$ on V. Define $g : V \to \mathbb{C}$ by $g = f'/f$ which is holomorphic.

Suppose that f has a zero at a of order m. Then there exists a *locally* holomorphic h, nonzero at a, given by $f(z) = (z - a)^m h(z)$. Then, locally,

$$g(z) = \frac{mh(z)}{(z-a)h(z)} + \frac{h'(z)}{h(z)}$$

and the second term is holomorphic. The first term has a residue of m.

Repeating the argument for poles (with signs swapped), we can apply the residue theorem to the integral

$$\frac{1}{2\pi i}\int_\gamma g = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

□

Definition 5.8. Suppose that $A \subset U$ is discrete, $f : U \setminus A \to \mathbb{C}$ is holomorphic, and f has no essential singularities. Then we say that f is *meromorphic.*

So, meromorphic functions have poles, but no essential or nonisolated singularities.

The argument principle is a counting principle: if γ winds around U once, and f is meromorphic on U, then we can count the zeroes and poles, and their difference will be $\mathrm{Ind}_{f\circ\gamma}(0)$. Why 0? Because we're counting zeroes! If we were counting the points where $f = b$, we would have $\mathrm{Ind}_{f\circ\gamma}(b)$.

Lemma 5.9. *Suppose $f : U \to \mathbb{C}$ is holomorphic, γ closed in U, and $b \in U \setminus \gamma^*$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - b}\,dz = \mathrm{Ind}_{f\circ\gamma}(b).$$

Proof. We can assume γ is smooth, so $f \circ \gamma$ is holomorphic, and that the domain of γ is $[0, 1]$. Then

$$\begin{aligned}\mathrm{Ind}_{f\circ\gamma}(b) &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dz}{z-b} = \frac{1}{2\pi i}\int_0^1 \frac{(f\circ\gamma)'(t)\,dt}{(f\circ\gamma)(t) - b} \\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t)) - b} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - b}.\end{aligned}$$

□

Corollary 5.10. *With hypotheses as in 5.7, $A = \emptyset$, and $b \in \mathbb{C} \setminus f(\gamma^*)$, $f - b$ has $\mathrm{Ind}_{f\circ\gamma}(b)$ zeroes in U_1.*

5.3 Rouche's theorem

We can now see that adding a function g which is smaller than f to f won't affect its number of zeroes.

Theorem 5.11 (Rouche)**.** *Suppose U is connected, γ simple, closed, and nullhomotopic in U,*

$$U_1 = \{a \in U \setminus \gamma^* : \mathrm{Ind}_\gamma(a) = 1\},$$

$f, g : U \to \mathbb{C}$ are holomorphic, and for each $z \in \gamma^*$, $|g(z)| < |f(z)|$. Let N_h denote the number of zeroes of h repeated by multiplicity. Then $N_f = N_{f+g}$.

Proof. Let $\tau \in [0,1]$ and $f_\tau = f + \tau g$. Now $f_\tau \neq 0$ on γ^* because if not then $|f(z)| = |\tau g(z)| \leq |g(z)|$ somewhere on γ^*. So we can apply the argument principle.

Put $\phi : [0,1] \to \mathbb{C}$ by

$$\phi(\tau) = \frac{1}{2\pi i} \int_\gamma \frac{f_\tau'}{f_\tau} = \mathrm{Ind}_{f_\tau \circ \gamma}(0)$$

which is constant. In particular $\phi(0) = \phi(1)$ so $N_f = N_{f+\tau g}$. ☐

An amusing corollary is that we can one-liner the fundamental theorem of algebra, not that the proof by Liouville's theorem was particularly difficult.

Corollary 5.12 (fundamental theorem of algebra, again)**.** *If f is a polynomial over $\mathbb{C}$, then f has a zero or f is constant.*

Proof. Let $f(x) = \sum a_k x^k$ be degree n. For R sufficiently large, $|x| > R$, $\sum_{k<n} a_k x^k < a_n x^n$, but $a_n x^n$ has a zero at 0, so f has a zero on $B_R 0$ by Rouche. ☐

More importantly, we can understand the behavior of zeroes under the effect of locally uniform convergence. if f is the locally uniform limit of a sequence of functions f_n, then each zero of f forms by zeroes of the f_ns "converging" together to a single point!

Theorem 5.13. *Suppose U is connected, $f_k : U \to \mathbb{C}$ are holomorphic, $f_k \to f$ locally uniformly, f is not identically zero, $a \in U$, $m \in \mathbb{N}$, and $f(a) = 0$.*

a is a zero of order m iff there is a neighborhood V such that $a \in V \subseteq U$ and for each $s > 0$, if $B_s a \subseteq V$ then cofinitely many of the f_n have m zeroes in $B_s a$, counted by multiplicity.

Proof. Choose $r > 0$ such that $f_k \to f$ uniformly on $B_r a$ and $\forall z \in B_r a \setminus \{a\} \subseteq U$, $f(z) \neq 0$. Let $s < r$ such that

$$\varepsilon = \min_{\partial B_s a} |f| > 0.$$

Then

$$\exists N \in \mathbb{N}\ \forall n > N |f_n - f| < \varepsilon$$

on $B_r a$. In particular, for $z \in \partial B_s a$, $|f_n(z) - f(z)| < |f(z)| \leq \varepsilon$. Let $\gamma = \Gamma_s a$ and apply Rouche. Then

$$N_{f_n} = N_{f_n - f + f} = N_f.$$

☐

Corollary 5.14 (Hurwitz)**.** *If U is connected, $f_k : U \to \mathbb{C}$ are holomorphic, and $f_k \to f$ locally uniformly, and $\forall k$ $|f_k| > 0$, then either $f \equiv 0$ or $|f| > 0$.*

As with the argument principle, there's nothing special about 0: we could translate f and do this for any $b \in \mathbb{C}$. Doing this for *every* such b, and requiring that each f_n hits b at most once, gives the following:

Corollary 5.15. *If U is connected, $f_k : U \to \mathbb{C}$ are holomorphic and injective, $f_k \to f$ locally uniformly, then f is constant or injective.*

Exercise 5.16. Let $a > e$. How many zeroes does

$$e^z = az^n$$

have on $B_1 0$?

5.4 Open maps and maximum moduli

The higher-dimensional version of the next theorem, proven in functional analysis, is one of the most powerful theorems in mathematics, an analyst's Zorn's lemma.

Theorem 5.17 (open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and not constant. Then $f(U)$ is open.*

To prove it, let's rephrase the open mapping theorem as a statement about zeroes in neighborhoods, so we can apply the argument principle.

Lemma 5.18. *With hypotheses as in 5.17, suppose $a \in U$, $f(a) = b$, and $f - b$ has a zero at a of order $N < \infty$.*

Then there are open neighborhoods U_0 of a and V_0 of b such that $f(U_0) = V_0$. Moreover, if $w \in V_0 \setminus \{b\}$, then $f - w$ has N simple zeros in U_0 and no other zeroes.

Proof. $\exists r > 0$ such that $B_{2r}a \subseteq U$ and for $z \in B_{2r}a$, $f(z) - b \neq 0$ and $f'(z) \neq 0$. Now do the argument principle with $\gamma = \Gamma_r a$. Then $\mathrm{Ind}_{f\circ\gamma}(b) = N$.

Let $V_0 = \mathrm{Ind}_{f\circ\gamma}^{-1}(\{N\})$ which is open because $\{N\}$ is open in the topology of $\mathbb{Z}$. Similarly let $U_0 = B_r a \cap f^{-1}(V_0)$, which is clearly open. Then $f(U_0) \subseteq V_0$.

Let $w \in V_0$. If $w = b$ then $f(a) = w$. Otherwise, $\mathrm{Ind}_{f\circ\gamma}(w) = N$ so $f - w$ has N zeroes by multiplicity in $B_r a$. Let z be such a zero; then $f(z) = w$ so $V_0 \subseteq f(U_0)$. Also, $f'(z) \neq 0$, so

$$0 \neq \frac{d}{dz}(f'(z) - w)$$

so f has a simple zero there. ☐

Proof of 5.17. $U = \bigcup U_0$, so $V = \bigcup V_0$. Open sets remain open after arbitrary unions. ☐

The open mapping theorem has important corollaries for dealing with extrema.

Corollary 5.19 (maximum modulus principle)**.** *If U is connected, $f : U \to \mathbb{C}$ holomorphic, and $|f|$ attains its max on U, then f is constant.*

Proof. If $|f|$ has a maximum at $a \in U$ then there are U_0 and V_0, as in the open mapping theorem. But then $\exists c \in V_0$ with $|c| > |b|$. ☐

Similarly:

Corollary 5.20. *Suppose that U is connected, and $f : U \to \mathbb{C}$ is holomorphic.*

If $\mathrm{Re}\, f$ or $\mathrm{Im}\, f$ attains a max or min, or $|f|$ attains its min, then f is constant.

Moreover, if U is bounded and f is also defined and continuous on ∂U, then

$$\sup_U |f| = \max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

By the way, this is part of why we take our domain U to be open rather than compact. For if $f : K \to \mathbb{C}$ is holomorphic and K is compact, then f attains a maximum on K. For example, the only holomorphic functions $S^1 \to \mathbb{C}$ are constant because S^1 does not have a boundary.

5.5 Normal families

Normal families are certain well-behaved spaces of holomorphic functions. These are of a similar flavor to the equicontinuous spaces of functions that you may have studied in 104, and we'll use them to prove the Riemann mapping theorem, which classifies all simply connected open subsets of $\mathbb{C}$.

The below material critically on the Arzela-Ascoli theorem from functional analysis. The statement and proof of Arzela-Ascoli is given in the preliminaries chapter, since it's not directly related to complex analysis. (In fact, many 185 classes skip over Montel's theorem and its consequences, including the Riemann mapping theorem, entirely, to avoid covering Arzela-Ascoli.)

Definition 5.21. Let $\mathcal{F} \subset \mathcal{O}$. $\mathcal{F}$ is said to be *normal* if for each sequence $f_1, f_2, \dots$ in $\mathcal{F}$, there is a subsequence f_{k_n} which converges locally uniformly to a function $f \in \mathcal{F}$.

Notice that the definition of normality feels very much like compactness. In fact, one can put a topology on $\mathcal{O}$ (the *compact-open topology*) in which a function converges iff it converges locally uniformly; then $\mathcal{F}$ is normal iff it is compact.

We shall characterize normal families. The key hypothesis is local uniform bounding:

Definition 5.22. Let $\mathcal{F} \subset \mathcal{O}$. $\mathcal{F}$ is said to be a *locally uniformly bounded family* if for each point $z \in U$ there is an open set $V \ni z$ and a constant $M > 0$ such that for each $f \in \mathcal{F}$, $|f| < M$ on V.

Theorem 5.23 (Montel)**.** *$\mathcal{F}$ is locally uniformly bounded iff it is normal.*

Proof. Let $\mathcal{F}$ be a locally uniformly bounded family on U, $z \in U$ and $\varepsilon > 0$. Then there are $M > 0$ and $r > 0$ such that for each $f \in \mathcal{F}$, $|f| < M$ on $\overline{B_{2r} z_0} \subset U$. Moreover, if $z, w \in B_{2r} z_0$ then

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\Gamma_{2r} z_0} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} \, d\zeta$$

by Cauchy's integral formula. Moreover for each ζ such that $|\zeta - z_0| = 2r$, and $z, w \in B_r z_0$,

$$|(\zeta - z)(\zeta - w)| > r^2.$$

Since $f(\zeta) < M$ anyways,

$$|f(z) - f(w)| \leq \frac{2|z-w|}{r} \sup_{\partial B_{2r} z_0} |f| < \frac{2M|z-w|}{r}.$$

Therefore if $\delta < r$ and $\delta < r\varepsilon M^{-1}/4$, then for $|z - w| < \delta$ one has $f(z) - f(w)| < \varepsilon$.

Since this δ only depended on ε and M and not f, $\mathcal{F}$ is equicontinuous on each $\overline{B_{2r} z_0}$, which are compact, and clearly $\mathcal{F}$ is bounded on each. Taking closures, $\mathcal{F}$ restricted to a sufficiently small compact set satisfies the hypotheses of the Arzela-Ascoli theorem and is thus compact in C^0, thus normal.

On the other hand, if $\mathcal{F}$ is normal, then $\mathcal{F}$ restricted to a sufficiently small compact set K is compact in C^0. On each such K, $\mathcal{F}$ is uniformly bounded. Therefore $\mathcal{F}$ is locally uniformly bounded. ☐

We'll now prove a collection of lemmata which will be used in the proof of the Riemann mapping theorem. TODO: This.

5.6 Conformal maps

We're in the position to give a geometric interpretation of holomorphicity.

Lemma 5.24. *Let $\gamma_1(0) = \gamma_2(0) = z \in U$ and $f : U \to \mathbb{C}$ be holomorphic. Let θ be the angle between two curves at* 0.

If $|f'| > 0$ then $\theta(\gamma_1, \gamma_2) = \theta(f \circ \gamma_1, f \circ \gamma_2)$.

Proof. TODO: This. ☐

If one plots a holomorphic function in Mathematica, it looks very pretty, with lots of symmetries and loops. This is why. The lemma tells us what a holomorphic function is: locally, it's nothing more than a translation, dilation, and rotation, up to some deformation by a small ε.

Definition 5.25. $f : U \to V$ is *biholomorphic*, *conformal*, *angle-preserving*, or a *complex diffeomorphism* if f is a bijection and f and f^{-1} are both holomorphic.

If such a conformal map exists then U and V are *conformally equivalent* or *conformal*.

Notice that some writers only assume that a conformal map is injective, not surjective!

By Liouville's theorem, $B_1 0$ and $\mathbb{C}$ are not conformal, but on the other hand, this shocking result is true:

Theorem 5.26 (Riemann mapping theorem)**.** *If U is connected and simply connected, then U is conformal with $B_1 0$ or $\mathbb{C}$.*

Proof. TODO: This. $\square$

Conformal maps are abundant.

Theorem 5.27 (inverse function theorem)**.** *Let $f : U \to \mathbb{C}$ be injective and holomorphic, $V = f(U)$. Then f is conformal from U to V and $|f'| > 0$.*

Proof. By the open mapping theorem, V is open. For $a \in U$, $b = f(a)$, $r > 0$, $f(U \cap B_r a)$ is open, and the $U \cap B_r a$ are a basis for the topology of V, so g is continuous.

If $f'(a) = 0$ then at a, $f - b$ has an order-2 zero, so we have $r > 0$ such that if $w \in B_1 b \setminus \{b\}$ then $f - w$ has two simple zeroes. So we have $\beta \neq \gamma$ such that $f(\beta) = w = f(\gamma)$, which is a contradiction. So $|f'| > 0$.

If h is the continuous extension of $\frac{z-a}{f(z)-b}$ then h is continuous. So

$$f^{-1}(b) = \lim_{v \to b} \frac{f^{-1}(u) - f^{-1}(b)}{u - b} = \lim_{u \to b} (h \circ f^{-1})(u) = \frac{1}{f'(a)}.$$

So f^{-1} is holomorphic. $\square$

The only conformal functions from the ball to itself are rotations, by the maximum modulus principle:

Lemma 5.28 (Schwarz)**.** *Let $f : B_1 0 \to B_1 0$ be holomorphic and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $a \neq 0$ and there exists $a \in B_1 0$ with $f|(a)| = |a|$ then $f(z) = \lambda z$ on $B_1 0$ where $\lambda \in S^1$.*

Proof. Let $g : B_1 0 \to \mathbb{C}$ be given by the holomorphic continuation of $g(z) = z^{-1} f(z)$. Then, for $r < 1$, $|z| = r$,

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

and in particular max modulus implies $|g|_{B_r 0} \leq r^{-1}$. It follows that $|g| \leq 1$. So $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

On the other hand, if $|f(z)| = |z|$ then g is constant by max modulus. So $\lambda = g(z)$. $\square$

Corollary 5.29. *Let $f : B_1 0 \to B_1 0$ be conformal. Then $\exists \lambda \in S^1$ such that $f(z) = \lambda z$.*

Proof. We have $|f^{-1}(z)| \leq |z| \leq |f(z)|$, but also $|f(z)| \leq |z| \leq |f^{-1}(z)|$. $\square$

If f is holomorphic and bijective, then f is conformal by the inverse function theorem.

Thanks to the inverse function theorem, once one's constructed a bijection it's not to show that the function is, in fact, conformal.

Example 5.30. The half space

$$U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

is conformal with $B_1 0$ as witnessed by the function

$$F(z) = \frac{z - i}{z + i}.$$

To see this, consider that

$$F'(z) = \frac{z + i + z - i}{(z + i)^2} = \frac{2z}{(z + i)^2} \neq 0,$$

so by the inverse function theorem it is injective. Moreover, if $|w| < 1$, then for

$$z = i \frac{1 + w}{1 - w}$$

it's clear that $F(z) = w$, so F is surjective.

You should generalize the above example to show that, up to some details, it completely characterizes conformal maps between the half space and the ball:

Exercise 5.31. Let U be the half space and $F : U \to B_1 0$ be conformal. Show that

$$F(z) = e^{i\theta} \frac{z - a}{z - \overline{a}}$$

for some $\theta \in [0, 2\pi)$ and $a \in U$.

In special relativity, a particle P's equations of motion are preserved under a group known as the Lorentz group. However, if P's mass vanishes, then the equations of motion are preserved under *any* conformal mapping.

Chapter 6

Harmonic functions

We'd like to apply some of the methods of complex analysis to study PDE in $\mathbb{R}^2$. Historically, this was one of the reasons complex analysis was invented in the first place.

Throughout this chapter, we'll assume $V \subseteq \mathbb{R}^2$ is open, and $\Psi(V) = U$, where $\Psi : \mathbb{C} \to \mathbb{R}^2$ is the identification homeomorphism 1.2 ($\Psi(x + iy) = (x, y)$). Thus we identify $\mathbb{R}^2$ with $\mathbb{C}$. Moreover, when it's convenient, we'll abuse notation and write $e^{i\theta}$ to mean the point $(\cos\theta, \sin\theta) \in \mathbb{R}^2$.

6.1 The Laplacian

Recall from multivariable calculus the notion of a Laplacian.

Definition 6.1. Let $f : V \to \mathbb{R}$ be a twice-differentiable function. The *Laplacian* of f is

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Physically, one can think of the Laplacian as measuring curvature or "gradient". One generally thinks of systems whose Laplacians are zero as being in "steady state" (and indeed a steady-state chemical gradient or heat gradient will have zero Laplacian).

In linear algebra or elsewhere you probably learned some of the following PDE. They justify

Definition 6.2. A twice-differentiable function $f : V \times [0, \infty) \to \mathbb{R}$ *solves the heat equation* if

$$(\Delta - \partial_t) f = 0.$$

Here we're taking Δ in the spatial variables V only.

Definition 6.3. A twice-differentiable function $f : V \times [0, \infty) \to \mathbb{R}$ *solves the wave equation* if

$$(\Delta - \partial_t^2) f = 0.$$

Definition 6.4. A twice-differentiable function $f : V \times [0, \infty) \to \mathbb{C}$ *solves Schroedinger's equation* if

$$(\Delta + i\partial_t) f = H,$$

where $H : V \to \mathbb{R}$ is called the *potential*.

In all of these equations, the time derivative is a function of curvature: as $\Delta f \to 0$, the function stabilizes in time. This justifies our previous claim that $\Delta f = 0$ means that f is in a "steady state". It also suggests that functions whose Laplacians vanish are of especial interest.

Definition 6.5. A twice-differentiable function $f : V \to \mathbb{R}$ is said to be *harmonic* or *solves Laplace's equation* if

$$\Delta f = 0.$$

If we have some way of studying harmonic functions, we understand the behavior of the above PDE. For example, we can solve the *one-dimensional* wave equation using harmonic functions:

Example 6.6. TODO: This.

So what's the connection to complex analysis? Recall the Cauchy-Riemann equations: we identified a function $f : U \to \mathbb{C}$ with

$$\tilde{f} = \Psi \circ f \circ \Phi.$$

Then $\tilde{f} : V \to \mathbb{R}^2$ could be written as $\tilde{f} = (u, v)$, and f was holomorphic if and only if

$$\begin{cases} \partial_1 u = \partial_2 v \\ \partial_2 u = -\partial_1 v. \end{cases}$$

Lemma 6.7. *Let notation be as above. If f is holomorphic, then u and v are harmonic.*

Proof. It follows by Cauchy-Riemann and equality of mixed partials;

$$\partial_1^2 u = \partial_1 \partial_2 v = \partial_2 \partial_1 v = -\partial_2^2 u$$

and

$$\partial_1^2 v = -\partial_1 \partial_2 u = -\partial_2 \partial_1 u = -\partial_2^2 v.$$

□

To turn this into an "if and only if", we need to see that u and v aren't any old harmonic functions: they are married by the Cauchy-Riemann equations.

Definition 6.8. Suppose that $u, v : V \to \mathbb{R}$ are harmonic and solve the Cauchy-Riemann equations. Then we say that u and v are *harmonic conjugates.*

Given u, we'll show that if the topology of V isn't too bad, then u has a harmonic conjugate v. Thus, given *any* harmonic function on V we'll be able to construct a natural holomorphic function $u + iv$ on U. That way, PDE problems on V reduce to complex analysis on U.

Theorem 6.9. *Let V be simply connected. If $u : V \to \mathbb{R}$ is harmonic then it has a harmonic conjugate.*

Proof. Let $g : U \to \mathbb{C}$ be given by

$$g(z) = \partial_1 u(\Psi(z)) - i\partial_2 u(\Psi(z)).$$

One can verify the Cauchy-Riemann equations for g, so g is holomorphic. Since U is simply connected, Cauchy-Goursat furnishes a holomorphic function $h : U \to \mathbb{C}$ where $h' = g$. Put $v = \operatorname{Im} h$. Now the function $u + iv$ satisfies the Cauchy-Riemann equations.

TODO: Actually verify Cauchy-Riemann. □

We rattle off some corollaries, which use the fact that V is locally simply connected. This shows that we have solved two of the three basic PDE problems (existence, regularity, and uniqueness) for Laplace's equation.

Corollary 6.10 (regularity of Laplace's equation)**.** *Each harmonic function is smooth.*

Corollary 6.11 (maximum modulus principle)**.** *If V is connected and $u : V \to \mathbb{R}$ is harmonic and attains its maximum on V, then u is constant. Moreover, if V is bounded, then*

$$\sup_V u = \max_{\overline{V}} u = \max_{\partial V} u.$$

Harmonic functions are completely determined by their behavior on the boundary of an open set.

Corollary 6.12 (uniqueness of Laplace's equation)**.** *Let V be connected and $u, v : \overline{V} \to \mathbb{R}$ be harmonic. If $u = v$ on ∂V then $u = v$ on V.*

Proof. $u - v$ and $v - u$ are harmonic by linearity of Δ. But $u - v = v - u = 0$ on ∂V, so $u - v \leq 0$ and $v - u \leq 0$. Thus $u - v = v - u = 0$ on V. □

6.2 The mean value property

Harmonic functions satisfy the following miraculous equation, which says that harmonic functions are always equal to their averages on circles centered at each point.

Theorem 6.13 (mean value property)**.** *Let $u : V \to \mathbb{R}$ be harmonic and $z \in V$. If γ is a circle centered on z of radius $r > 0$ such that $B_r z \subset V$, then*

$$u(z) = \frac{1}{2\pi r} \int_\gamma u(w)\ dw.$$

Proof. Let $f = u + iv$, where v is the harmonic conjugate of u. Then f is holomorphic, and

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_r z} \frac{f(w)}{w - z} dw = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

and it follows when we take the real part of the integral. □

The above characterization of the mean value property is particularly useful for us, since complex analysis is particularly interested in integrals around curves. However, there is a *different* notion of a mean value property, which uses an area integral, as defined in multivariable calculus: why integrate around the ball? Why not integrate on the ball itself?

Lemma 6.14. *Let $u : V \to \mathbb{R}$. If u satisfies the mean value property, then, for each $z \in V$ and $r > 0$,*

$$u(z) = \frac{1}{\pi r^2} \int_{B_r z} u\ dA.$$

Proof. Without loss of generality, assume $z = 0$. Then define

$$\phi(r) = \frac{1}{2\pi r} \int_{\Gamma_r 0} u(w)\ dw.$$

Then ϕ is constant, so $\phi' = 0$. Switching to polar coordinates, one has

$$\frac{1}{\pi r^2} \int_{\Gamma_r 0} u(w)\ dw = \frac{1}{\pi r^2} \int_0^r \phi(s)\ ds$$

and $\phi(s)$ is independent of s, so we can pull out $\phi(s) = u(z)$, which completes the proof. □

From this, it follows that the mean value property completely characterizes harmonic functions in C^2: if the mean value property holds for u, then Δu necessarily is 0.

Theorem 6.15 (mean value property, converse)**.** *Let $u : V \to \mathbb{R}$ be C^2. If, for each $z \in V$ and $r > 0$,*

$$u(z) = \frac{1}{\pi r^2} \int_{B_r z} u(w)\ dA,$$

then u is harmonic.

Proof. Without loss of generality, we can assume $\Delta u > 0$ on an open $W \subseteq V$ for the sake of contradiction (and if not, just add a negative sign to the relevant equalities). Furthermore, since there were no hypotheses on V except that it is open, we may assume $W = V$. Let ϕ and z be as in the proof of 6.14. Then $\phi' = 0$ and by Green's theorem,

$$0 = \phi'(r) = \frac{1}{2\pi r} \int_{\Gamma_r 0} u'(w)\ dw = \frac{1}{\pi r^2} \int_{B_r 0} \Delta u\ dA > 0$$

which is a contradiction. □

All this should further the same intuition that Cauchy's estimate gave: a holomorphic (equivalently, harmonic) function cannot grow too fast, but it cannot also oscillate too wildly. Its growth is controlled by the mollifying effects of the mean value property, which is really just Cauchy's integral formula unmasked.

In fact, control over the mean value is so useful in analysis one introduces the following, much weaker notion.

Definition 6.16. Let $u : V \to \mathbb{R}$ be continuous and, for each $z \in V$ and $r > 0$,

$$u(z) \leq \frac{1}{2\pi r} \int_{\Gamma_r z} u(w)\ dw.$$

Then we say that u is *subharmonic*.

Of course, there is an analogous notion of superharmonic functions.

By mimicking the proofs of the usual claims about harmonic functions, you can recover quite a lot about subharmonic functions (and, analogously, superharmonic functions).

Exercise 6.17. Show that if u is subharmonic then it satisfies a maximum principle, and that if u is furthermore C^2, then $\Delta u \geq 0$.

6.3 Boundary value problems

A common problem in PDE theory is the boundary-value problem, or Dirichlet problem. Given a PDE, and the value of a function on the boundary of a space, we want to construct a solution on the whole space. If such a solution exists, then it is unique; and we'll be able to show that, for simply connected sets, a solution does in fact exist.

Definition 6.18. Let $R > 0$. The function $P_R : B_R 0 \times \partial B_R 0$ given by

$$P_R(re^{i\varphi}, e^{i\theta}) = \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\varphi}|^2}$$

is called *Poisson's kernel* for the ball of radius R.

Our method for solving the boundary-value problem is to smash the boundary condition against Poisson's kernel. First we'll solve Laplace's equation in a ball.

Theorem 6.19. *Let $R > 0$ and $h : \partial B_R 0 \to \mathbb{R}$ be a continuous function. Let $u : \overline{B_R 0} \to \mathbb{R}$ be*

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

on $B_R 0$ and $u = h$ on $\partial B_R 0$.

Then u is harmonic on $B_R 0$ and continuous on $\overline{B_R 0}$.

Proof. Without loss of generality, assume $R = 1$.

Let $f : B_1 0 \to \mathbb{C}$ be given by

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1 0} \frac{h(w)}{w} \left(2\frac{w}{w - z} - 1\right)\ dw.$$

Then f is holomorphic and, if $z = re^{i\varphi}$,

$$\begin{aligned}
\operatorname{Re} f(z) &= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} \frac{h(w)}{w} \left(2\frac{w}{w - z} - 1\right) \\
&= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} h(w) \frac{e^{i\theta} + re^{i\varphi}}{e^{i\theta} - re^{i\varphi}}\ d\theta \\
&= \operatorname{Re} \frac{1}{2\pi} \int_{\Gamma_1 0} h(w) \frac{1 - r^2}{|e^{i\theta} - re^{i\varphi}|^2}\ d\theta \\
&= u(re^{i\varphi})
\end{aligned}$$

so u is harmonic.

However, there is the possibility that u fails to be continuous on the boundary. Let $\alpha \in [0, 2\pi)$. We'll show continuity at $e^{i\alpha}$. By compactness, h is bounded, say $|h| < M$. Then if $\varepsilon > 0$ we can find $\delta > 0$ such that whenever $\theta \in [\alpha - 2\varepsilon, \alpha + 2\varepsilon]$,

$$|h(e^{i\theta}) - h(e^{i\alpha})| < \varepsilon.$$

We derive from Cauchy's integral formula

$$2\pi = \int_0^{2\pi} \frac{1 - r^2}{|e^{i\theta} - re^{i\varphi}|^2} d\theta$$

TODO: Verify this.

whence

$$|u(re^{i\varphi}) - u(e^{i\alpha})| \leq \frac{1}{2\pi} \int_\alpha^{\alpha+2\pi} |h(e^{i\theta}) - he^{i\alpha}| P_1(re^{i\varphi}, e^{i\theta}) d\theta.$$

TODO: Finish these estimates. ☐

Though we've only solved the problem on the ball, we'll be able to solve this problem on any simply connected V except for $\mathbb{C}$ itself, by the Riemann mapping theorem. As an example, we'll solve Laplace's equation in a half space.

Corollary 6.20 (solving Laplace's equation in a half space)**.** *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and*

$$U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

is the half space. Let $F : \overline{U} \to \overline{B_1 0}$ be

$$F(z) = \frac{z - i}{z + i}$$

as in 5.30, $\tilde{\phi} = \phi \circ F^{-1}$, and

$$\tilde{u}(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \phi(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

as in 6.19.

If $u = F^{-1} \circ \tilde{u} \circ F$, then u is harmonic on U and continuous on $\overline{U}$.

More generally, if V is open and simply connected with nonempty boundary and $F : V \to B_1 0$ is angle-preserving then, given $\phi : \partial V \to \mathbb{R}$, one can construct $u : \overline{V} \to \mathbb{R}$ by $u = F^{-1} \circ \tilde{u} \circ F$ where $\tilde{u}$ solves the Dirichlet problem on $\overline{V}$ with boundary condition $\phi \circ F^{-1}$.

Exercise 6.21. Solve the Dirichlet problem on the quadrant

$$V = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \ \operatorname{Im} z > 0\}.$$

Poisson's kernel guarantees that locally uniform limits of harmonic functions are harmonic.

Theorem 6.22 (Harnack)**.** *Let $u_1, u_2, \cdots : V \to \mathbb{R}$ be harmonic and u be their locally uniform limit. Then u is harmonic.*

Proof. u is continuous because the limit is locally uniform. Let $p \in V$ and assume without loss of generality that $p = 0$. Then there exists $R > 0$ such that $B_R 0 \subseteq V$ and the limit is uniform on $\overline{B_R 0}$. So by Poisson's kernel, if $r < R$ then

$$u_n(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} u_n(Re^{i\theta}) P_R(re^{i\varphi}, e^{i\theta}) d\theta$$

and this equality holds even as $n \to \infty$. ☐

6.4 Harnack's inequality

The mean value property restricts the possible growth of a harmonic function. As a result, the following inequality, which describes the growth of *positive* harmonic functions, shouldn't be too much of a surprise.

Theorem 6.23 (Harnack's inequality)**.** *Suppose that $R > 0$, $u : \overline{B_R 0} \to \mathbb{R}$ is continuous on $\overline{B_R 0}$ and harmonic on $B_R 0$, and $u \geq 0$ on $B_R 0$. Then for each $r \in (0, 1)$ and $\theta \in [0, 2\pi)$,*

$$\frac{R-r}{R+r} u(0) \leq u(re^{i\theta}) \leq \frac{R+r}{R-r} u(0).$$

Proof. For each $\varphi \in [0, 2\pi)$,

$$R - r \leq |Re^{i\theta} - re^{i\varphi}| \leq R + r;$$

this follows by application of the triangle inequality to the ball. One then has

$$\frac{R-r}{R+r} = \frac{R^2 - r^2}{(R+r)^2} \leq \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\varphi}|^2} = P_R(re^{i\theta}, e^{i\varphi}) \leq \frac{R^2 - r^2}{(R-r)^2} = \frac{R+r}{R-r}.$$

By the mean value property,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) \, d\varphi$$

but on the other hand, Poisson's kernel yields

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) P_R(re^{i\theta}, e^{i\varphi}) d\varphi \leq \frac{R+r}{R-r} \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) \, d\varphi = \frac{R+r}{R-r} u(0).$$

Similarly for the other inequality. □

Thus, there are no nontrivial positive entire harmonic functions.

Corollary 6.24. *Suppose that $u : \mathbb{R}^2 \to (0, \infty)$ is harmonic. Then u is constant.*

Proof. u restricts to $\overline{B_R 0}$ for each $R > 0$. But also

$$\lim_{R \to \infty} \frac{R-r}{R+r} = \lim_{R \to \infty} \frac{R+r}{R-r} = 1.$$

So for each $z \in \mathbb{C}$, $u(z) \leq u(0) \leq u(z)$. □

Harnack's inequality also takes a more general form, which is clear by dividing up a space into overlapping open balls and applying Harnack's inequality to each one.

Corollary 6.25 (Harnack's inequality, generalized)**.** *Suppose that V is connected and bounded. Then there is a constant $C > 0$, which only depends on V, such that for each harmonic function $u : V \to (0, \infty)$ and each $z, w \in V$,*

$$C^{-1} u(z) \leq u(w) \leq C u(z).$$

Harnack's inequality gives us another convergence theorem for harmonic functions.

Theorem 6.26 (Harnack's monotone convergence theorem)**.** *Suppose $u_1, u_2, \dots : V \to \mathbb{R}$ are harmonic and $u_1 \leq u_2 \leq \dots$. If $p_0 \in V$ and the sequence $u_n(p_0)$ converges in $\mathbb{R}$, then there is a function $u : V \to \mathbb{R}$ such that $u_n \to u$ locally uniformly. In particular, u is harmonic.*

Proof. Without loss of generality, we may assume that for each $n \geq 1$, $u_n \geq 0$; if not, just translate.

For each such harmonic function $v \geq 0$, Harnack's inequality gives, on a sufficiently small ball of radius R,

$$\frac{R - |p-q|}{R + |p-q|} v(p) \leq v(q) \leq \frac{R + |p-q|}{R - |p-q|} v(p)$$

whence, for p and q sufficiently close,

$$\frac{1}{4}v(p) \leq v(q) \leq 4v(p).$$

Suppose that W_1 is the set of points p in V such that the sequence $u_n(p)$ is bounded. Because of the above estimates, W_1 is open. Similarly, the set W_2 of points p such that $u_n(p)$ is unbounded is open. Since W_1 is nonempty and V is connected, $V = W_1$. Thus we can put

$$u(z) = \sup_{n\in\mathbb{N}} u_n(z).$$

Thus $u_n \leq u$. If $m \geq n$, then $u_m - u_n \geq 0$ is harmonic and if $|p - q|$ is sufficiently small (say an element of a set X), then

$$(u_m - u_n)(q) \leq 4(u_m - u_n)(p)$$

which is uniformly Cauchy on X, thus converges locally uniformly on V.

By Harnack's theorem, then, u is harmonic. □

Harnack's inequality also gives us a metric which we can apply to any bounded connected open subset of $\mathbb{R}^2$, which is, from the point of a hyperbolic geometer, the natural metric on $B_1 0$.

Definition 6.27. Let V be connected and bounded and H be the space of harmonic functions on V. Put

$$\tau(z, w) = \sup_{h\in H} \left|\frac{h(w)}{h(z)}\right|.$$

The function $\log \circ \tau : V \times V \to [0, \infty)$ is called the *Harnack metric* of V.

It's not clear that this function is even well-defined (why is the sup always finite?), let alone that it's a metric – and the most difficult question of all, that it's a metric which is actually worthy of study.

Lemma 6.28. *The Harnack metric $d : V \times V \to [0, \infty)$ is well-defined and a metric.*

Proof. TODO: This. □

The value of the Harnack metric is that from its point of view, the boundary S^1 is infinitely far away from any point in the ball. In this sense, we can identify the ball with the plane; the "boundary" of $\mathbb{C}$ is the point at infinity ∞, which is infinitely far away from any point in $\mathbb{C}$.

Lemma 6.29. *If $d : B_1 0 \times B_1 0 \to [0, \infty)$ is the Harnack metric, then for $\zeta \in S^1$ and $w \in B_1 0$,*

$$\lim_{z\to\zeta} d(z, w) = \infty.$$

Proof. TODO: This. □

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