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Equations of Riemann surfaces with automorphisms

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Abstract. We present an algorithm for computing equations of canonically embedded Riemann surfaces with automorphisms. This is used to produce equations of Riemann surfaces with large automorphism groups for genus 7. The main tools are the Eichler trace formula for the character of the action of the automorphism group on holomorphic differentials, algorithms for producing matrix generators of a representation of a finite group with a specified irreducible character, and Gröbner basis techniques for computing flattening stratifications.

Riemann surfaces (or algebraic curves) with automorphisms have been important objects of study in complex analysis, algebraic geometry, number theory, and mathematical physics for more than a century, as their symmetries often permit us to do calculations that would otherwise be intractable.

Such Riemann surfaces are special in the sense that a general Riemann surface of genus $g \geq 3$ has no nontrivial automorphisms. Moreover, the group of automorphisms of a Riemann surface of genus $g \geq 2$ is finite.

A great deal of progress has been made on classifying and studying these Riemann surfaces. Notably, for the important special cases of cyclic curves or superelliptic curves, the automorphism groups that may occur and affine plane equations of these curves have been published. See [**1**, **20**] for recent work by Sanjeewa, Shaska, Beshaj, and Zhupa on cyclic and superelliptic curves, and [**3**, **4**, **22**] for some prior work on hyperelliptic and cyclic trigonal curves.

In the general case, Breuer and Conder performed computer searches that for each genus g list the Riemann surfaces of genus g with large automorphism groups (that is, $|\operatorname{Aut}(X)| > 4(g_X - 1)$). Specifically, they list sets of surface kernel generators (see Definition 1.2 below), which describe these Riemann surfaces as branched covers of $\mathbb{P}^1$. Breuer's list extends to genus $g = 48$, and Conder's list extends to genus $g = 101$ [**2**, **5**]. Even for small values of g, these lists are extremely large, as a surface X may appear several times for various subgroups of its full automorphism group. In [**15**], Magaard, Shaska, Shpectorov, and Völklein refined Breuer's list by determining which surface kernel generators correspond to the full automorphism group of the Riemann surface. This reveals that many, but not all, of the Riemann surfaces with large automorphism groups are cyclic or superelliptic.

To our knowledge, at this time there is no algorithm published in the literature for producing equations of the noncyclic Riemann surfaces. Here, we present

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an algorithm to compute canonical equations of an arbitrary nonhyperelliptic Riemann surfacewith automorphisms. The main tools are the Eichler trace formula for the character of the action of $\mathrm{Aut}(X)$ on holomorphic differentials, algorithms for producing matrix generators of a representation of a finite group with a specified irreducible character, and Gröbner basis techniques for computing flattening stratifications. This algorithm has been used to produce equations of the nonhyperelliptic Riemann surfaces with genus $4 \leq g_X \leq 7$ satisfying $|\mathrm{Aut}(X)| > 4(g_X - 1)$.

A few more remarks may help distinguish Algorithm 1.1 from previous work. For cyclic or superelliptic curves, the equations in [**1**, **20**] are more concise than those obtained by Algorithm 1.1, so Algorithm 1.1 may be considered complementary to these papers. The input to Algorithm 1.1 need not be the full automorphism group, so it can be run from Breuer or Conder's data (though we recommend using the full automorphism group from [**15**] whenever possible). Furthermore, Algorithm 1.1 can be applied to arbitrary families of Riemann surfaces with automorphisms (not just those with 0- or 1-dimensional Hurwitz loci). Finally, the canonical ideal of a Riemann surface contains some information that the author does not know how to obtain using techniques of Fuchsian group theory alone. For example, the Betti table can be computed from the canonical ideal, and for low genus curves, this yields information about the linear series on the curve (see [**21**]).

Here is an outline of the paper. In Section 1, we describe the main algorithm. In Section 2, we describe one example in detail, a genus 7 Riemann surface with 54 automorphisms. In Section 3, we give equations of selected canonically embedded Riemann surfaces of genus 7 along with matrix surface kernel generators. The calculations were performed in `Magma`, `GAP`, and `Macaulay2` [**11**, **13**, **16**].

Online material. The webpage [**24**] contains links to the latest version of the author's `Magma` code, files detailing the calculations for specific examples, and many equations that are not printed in Section 3.

1. The main algorithm

We begin by stating the main algorithm. Then, in the following subsections, we discuss the steps in more detail, including precise definitions and references for terms and facts that are not commonly known.

ALGORITHM 1.1.
INPUTS:

(1) *A finite group G;*
(2) *an integer $g \geq 3$;*
(3) *a set of surface kernel generators $(a_1, \ldots, a_{g_0}; b_1, \ldots, b_{g_0}; g_1, \ldots, g_r)$ determining a family of nonhyperelliptic Riemann surfaces X of genus g with $G \subset \mathrm{Aut}(X)$*

OUTPUT: *A locally closed set $B \subset \mathbb{A}^n$ and a family of smooth curves $\mathcal{X} \subset \mathbb{P}^{g-1} \times B$ such that for each closed point $b \in B$, the fiber $\mathcal{X}_b$ is a smooth genus g canonically embedded curve with $G \subset \mathrm{Aut}(\mathcal{X}_b)$.*

Step 1. *Compute the conjugacy classes and character table of G.*
Step 2. *Use the Eichler trace formula to compute the character of the action on differentials and on quadrics and cubics in the canonical ideal.*
Step 3. *Obtain matrix generators for the action on holomorphic differentials.*
Step 4. *Use the projection formula to obtain candidate quadrics and cubics.*
Step 5. *Compute a flattening stratification and select the locus yielding smooth algebraic curves with degree $2g-2$ and genus g.*

We comment on Steps 2-5 below.

1.1. Step 2: Counting fixed points and the Eichler trace formula. We begin by defining surface kernel generators.

Definition 1.2 (cf. [**2**] Theorem 3.2, Theorem 3.14). A *signature* is a list of integers $(g_0; e_1, \ldots, e_r)$ with $g_0 \geq 0$, $r \geq 0$, and $e_i \geq 2$.

A set of *surface kernel generators* for a finite group G and signature $(g_0; e_1, \ldots, e_r)$ is a sequence of elements $a_1, \ldots, a_{g_0}, b_1, \ldots, b_{g_0}, g_1, \ldots, g_r \in G$ such that

(1) $\langle a_1, \ldots, a_{g_0}, b_1, \ldots, b_{g_0}, g_1, \ldots, g_r \rangle = G$;
(2) $\mathrm{Order}(g_i) = e_i$; and
(3) $\prod_{j=1}^{g_0} [a_j, b_j] \prod_{i=1}^{r} g_i = \mathrm{Id}_G$.

Surface kernel generators are called *ramification types* in [**15**] and *generating vectors* in [**19**].

As explained in [**2**, Section 3.11], surface kernel generators describe the quotient morphism $X \to X/G$ as a branched cover. Here X is a Riemann surface of genus g, G is a subgroup of $\mathrm{Aut}(X)$, the quotient X/G has genus g_0, the quotient morphism branches over r points, and the integers e_i describe the ramification over the branch points.

Surface kernel generators are used in the following formula for the number of fixed points of an automorphism:

Theorem 1.3 ([**2**, Lemma 11.5]). *Let σ be an automorphism of order $h > 1$ of a Riemann surface X of genus $g \geq 2$. Let $(g_1, \ldots, g_r)$ be part of a set of surface kernel generators for X, and let $(m_1, \ldots, m_r)$ be the orders of these elements. Let $Fix_{X,u}(\sigma)$ be the set of fixed points of σ, where σ acts on a neighborhood of the fixed point by $z \mapsto \exp(2\pi i u/h) z$. Then*

$$|\operatorname{Fix}_{X,u}(\sigma)| = |C_G(\sigma)| \sum_{\substack{g_i \ s.t. \\ h | m_i \\ \sigma \sim g_i^{m_i u/h}}} \frac{1}{m_i}$$

Here $C_G(\sigma)$ is the centralizer of σ in G, and $\sim$ denotes conjugacy.

Next we recall the Eichler Trace Formula. For a Riemann surface X, let Ω_X be the holomorphic cotangent bundle, and let $\omega_X = \bigwedge \Omega_X$ be the sheaf of holomorphic differentials. The Eichler Trace Formula gives the character of the action of $\mathrm{Aut}(X)$ on $\Gamma(\omega_X^{\otimes d})$.

THEOREM 1.4 (Eichler Trace Formula [**9**, Theorem V.2.9]). *Suppose $g_X \geq 2$, and let σ be a nontrivial automorphism of X of order h. Write χ_d for the character of the representation of* $\operatorname{Aut}(X)$ *on* $\Gamma(\omega_X^{\otimes d})$. *Then*

$$\chi_d(\sigma) = \begin{cases} 1 + \displaystyle\sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\operatorname{Fix}_{X,u}(\sigma)| \frac{\zeta_h^u}{1-\zeta_h^u} & \text{if } d = 1 \\ \displaystyle\sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\operatorname{Fix}_{X,u}(\sigma)| \frac{\zeta_h^{u(d\% h)}}{1-\zeta_h^u} & \text{if } d \geq 2 \end{cases}$$

Here $d\%h$ denotes the unique integer x between 0 *and $h-1$ such that $d \equiv x \pmod{h}$.*

Together, the previous two results give a group-theoretic method for computing the character of the $\operatorname{Aut}(X)$ action on $\Gamma(\omega_X^{\otimes d})$ starting from a set of surface kernel generators.

We can use the character of $\operatorname{Aut}(X)$ on $\Gamma(\omega_X^{\otimes d})$ to obtain the character of $\operatorname{Aut}(X)$ on quadrics and cubics in the canonical ideal as follows. Let S be the coordinate ring of $\mathbb{P}^{g-1}$, let $I \subset S$ be the canonical ideal, and let S_d and I_d denote the degree d subspaces of S and I.

By Noether's Theorem, the sequence

$$0 \to I_d \to S_d \to \Gamma(\omega_X^{\otimes d}) \to 0$$

is exact for each $d \geq 2$, and by Petri's Theorem, the canonical ideal is generated either by quadrics or by quadrics and cubics. Thus, beginning with the character of the action on $\Gamma(\omega_X) \cong S_1$, we may compute the characters of the actions on $S_2 = \operatorname{Sym}^2 S_1$ and $S_3 = \operatorname{Sym}^3 S_1$ and $\Gamma(\omega_X^{\otimes 2})$ and $\Gamma(\omega_X^{\otimes 3})$, and then obtain the characters of the actions on I_2 and I_3.

1.2. Step 3: matrix generators for a specified irreducible character. From Step 2 we have the character of the action on $\Gamma(\omega_X)$. We seek matrix generators for this action.

Finding efficient algorithms to produce matrix generators of a representation of a finite group G with a specified character is a subject of ongoing research [**6**, **7**]. For lack of a suitably general reference, in [**24**], we outline an algorithm that was suggested by Valery Alexeev and James McKernan. This algorithm is not expected to perform efficiently; it is given merely to establish that Step 3 in Algorithm 1.1 can be performed algorithmically.

Finally, we note that in [**23**], Streit describes a method for producing matrix generators for the action of $\operatorname{Aut}(X)$ on $\Gamma(\omega_X)$ for some Belyĭ curves.

1.3. Step 4: the projection formula. Recall the projection formula for representations of finite groups (see for instance [**10**] formula (2.31)).

THEOREM 1.5 (Projection formula). *Let V be a finite-dimensional representation of a finite group G over $\mathbb{C}$. Let $V_1, \ldots, V_k$ be the irreducible representations of G, let χ_i be their characters, and let $V \cong \bigoplus_{i=1}^k V_i^{\oplus m_i}$. Let $\pi_i : V \to V_i^{\oplus m_i}$ be the projection onto the i^{th} isotypical component of V. Then*

$$\pi_i = \frac{\dim(V_i)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} g.$$

From Step 3, we have matrix generators for the G action on $\Gamma(\omega_X) = S_1$. Thus, we can compute matrix generators for the actions on S_2 and S_3, and use the projection formula to compute the isotypical subspace $S_{d,p}$ of degree d polynomials on which G acts with character χ_p. In some examples, we have $I_{d,p} = S_{d,p}$, but more commonly, we have strict containment $I_{d,p} \subset S_{d,p}$. In this case we write elements of $I_{d,p}$ as generic linear combinations of the basis elements of $S_{d,p}$ and then seek coefficients that yield a smooth algebraic curve with the correct degree and genus.

The coefficients used to form these generic linear combinations form the base space $\mathbb{A}^n$ of the family $\mathcal{X}$ produced by the main algorithm.

1.4. Step 5: Flattening stratifications.

Theorem 1.6 ([**17**, Lecture 8]). *Let $f : X \to S$ be a projective morphism with S a reduced Noetherian scheme. Then there exist locally closed subsets $S_1, \dots, S_n$ such that $S = \sqcup_{i=1}^n S_i$ and $f|_{f^{-1}(S_i)}$ is flat.*

The stratification $S = \sqcup_{i=1}^n S_i$ is called a flattening stratification for the map f. Since S is reduced, flatness implies that over each stratum, the Hilbert polynomial of the fibers is constant. We find the stratum with Hilbert polynomial $P(t) = (2g-2)t - g + 1$, then intersect this stratum with the locus where the fibers are smooth. This completes Algorithm 1.1.

Flattening stratifications have been an important tool in theoretical algebraic geometry for over 50 years. There exist Gröbner basis techniques for computing flattening stratifications; in the computational literature, these are typically called *comprehensive* or *parametric Gröbner bases*, or *Gröbner systems*. The foundational work on this problem was begun by Weispfenning, and many authors, including Manubens and Montes, Suzuki and Sato, Nabeshima, and Kapur, Sun, and Wang, have made important improvements on the original algorithm [**12**, **18**, **25**].

The size of a Gröbner basis can grow very quickly with the number of variables and generators of an ideal, and unfortunately, even the most recent software cannot compute flattening stratifications for the examples we consider. We discuss a strategy for circumventing this obstacle below.

The algorithms for comprehensive Gröbner bases described in Section 1.4 all begin with the same observation. Let $S = K[x_0, \dots, x_m]$ be a polynomial ring over a field. Let $\preceq$ be a multiplicative term order on S. Then a theorem of Macaulay states that the Hilbert function of I is the same as the Hilbert function of its initial ideal with respect to this term order (see [**8**, Theorem 15.26]).

Therefore, whenever two ideals in S have Gröbner bases with the same leading monomials with respect to some term order, they will have the same initial ideal for that term order, hence they must have the same Hilbert function and Hilbert polynomial, and therefore they will lie in the same stratum of a flattening stratification. To reach a different stratum of the flattening stratification, it is necessary to alter the leading terms of the Gröbner basis — for instance, by restricting to the locus where that coefficient vanishes.

Here is a brief example to illustrate this idea. Let $\mathbb{A}^2$ have coordinates c_1, c_2, and let $\mathbb{P}^3$ have coordinates x_0,x_1,x_2,x_3. The ideal

$$I = \langle c_1x_0x_2 - c_2x_1^2, c_1x_0x_3 - c_2x_1x_2, c_1x_1x_3 - c_2x_2^2 \rangle$$

defines a 2-parameter family of subschemes of $\mathbb{P}^3$. A Gröbner basis for I in $\mathbb{C}[c_1, c_2][x_0, x_1, x_2, x_3]$ with respect to the lexicographic term order is

$$\begin{gathered} c_1x_0x_2 - c_2x_1^2, c_1x_0x_3 - c_2x_1x_2, c_1x_1x_3 - c_2x_2^2, \\ (c_1c_2 - c_2^2)x_1x_2^2, c_2x_1^2x_3 - c_2x_1x_2^2, (c_1c_2 - c_2^2)x_1^2x_2, c_2x_0x_2^2 - c_2x_1^2x_2, \\ (c_1c_2^2 - c_2^3)x_1^4, (c_1c_2^2 - c_2^3)x_2^4, c_2^2x_1x_2^2x_3 - c_2^2x_2^4, c_2^2x_0x_1^2x_2 - c_2^2x_1^4. \end{gathered}$$

Over the locus where c_1, c_2, and $c_1 - c_2$ are invertible, the initial ideal is $\langle x_0x_2, x_0x_3, x_1x_3, x_1x_2^2, x_1^2x_2, x_1^4, x_2^4\rangle$ with Hilbert polynomial $P(t) = 8$. On the other hand, when $c_1 = 0$, or $c_2 = 0$, or $c_1 - c_2 = 0$, we get a different initial ideal and Hilbert polynomial. For example, the locus $c_1 = c_2 \neq 0$ yields the twisted cubic with $P(t) = 3t + 1$.

2. Example: A genus 7 Riemann surface with 54 automorphisms

Magaard, Shaska, Shpectorov, and Völklein's tables show that there exist two smooth, compact genus 7 Riemann surfaces with full automorphism group G given by the group labeled $(54, 6)$ in the `GAP` library of small finite groups [**15**]. For each of these two Riemann surfaces, $X/G \cong \mathbb{P}^1$. The quotient morphisms are branched over 3 points of $\mathbb{P}^1$, and the ramification indices over these points are 2, 6, and 9.

A polycyclic presentation of the group G is

$$\langle g_1, g_2, g_3, g_4 | g_1^2, g_2^3, g_3^3g_4^{-2}, g_4^3, g_1^{-1}g_3g_1(g_3^2g_4)^{-1}, g_2^{-1}g_3g_2(g_3g_4)^{-1}, g_1^{-1}g_4g_1(g_4)^{-2}\rangle.$$

Breuer's surface kernel generators for the first of these two surfaces are $a = g_1$, $b = g_1g_2g_3^2g_4$, and $c = g_2^2g_3g_4^2$. This Riemann surface is not cyclic or superelliptic.

Step 1. We use `Magma` to compute the conjugacy classes and character table of G. There are 10 conjugacy classes, represented by the following elements: Id_G, g_1, g_4, g_2^2, g_2,g_1g_2, $g_1g_2^2$, $g_2g_3g_4$, g_3, $g_2^2g_3g_4^2$.

The irreducible characters are given below by their values on the ten conjugacy classes.

Class	1	2	3	4	5	6	7	8	9	10
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	-1	1	1	1
χ_3	1	-1	1	ζ_3	ζ_3^2	$-\zeta_3^2$	$-\zeta_3$	ζ_3^2	1	ζ_3
χ_4	1	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	ζ_3	1	ζ_3^2
χ_5	1	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	ζ_3^2	1	ζ_3
χ_6	1	-1	1	ζ_3^2	ζ_3	$-\zeta_3$	$-\zeta_3^2$	ζ_3	1	ζ_3^2
χ_7	2	0	2	2	2	0	0	-1	-1	-1
χ_8	2	0	2	$2\zeta_3$	$2\zeta_3^2$	0	0	$-\zeta_3^2$	-1	$-\zeta_3$
χ_9	2	0	2	$2\zeta_3^2$	$2\zeta_3$	0	0	$-\zeta_3$	-1	$-\zeta_3^2$
χ_{10}	6	0	-3	0	0	0	0	0	0	0

Step 2. Let V_i be the irreducible G-module with character χ_i given by the table above. For any G-module V, let $V \cong \bigoplus_{i=1}^r V_i^{\oplus m_i}$ be its decomposition into irreducible G-modules.

We use the Eichler trace formula to compute these multiplicities m_i for several relevant G-modules. Let $S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$, and let S_d denote polynomials of degree d. Let I_d be the kernel of $\to S_d \to \Gamma(\omega_X^{\otimes d})$. Then we have

$$S_1 \cong \Gamma(\omega_X) \cong V_3 \oplus V_{10}$$

$$I_2 \cong V_1 \oplus V_4 \oplus V_8 \oplus V_{10}$$

$$S_2 \cong V_1 \oplus V_4^{\oplus 2} \oplus V_5 \oplus V_7 \oplus V_8 \oplus V_9 \oplus V_{10}^{\oplus 3}$$
$$\Gamma(\omega_X^{\otimes 2}) \cong V_4 \oplus V_5 \oplus V_7 \oplus V_9 \oplus V_{10}^{\oplus 2}$$

STEP 3. We use `GAP` to obtain matrix representatives of a G action with character equal to the character of the G action on S_1. Such a representation is obtained by mapping the surface kernel generators $a = g_1$ and $b = g_1g_2g_3^2g_4$ to the matrices below.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_9^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_9^2 \\ 0 & 0 & 0 & 0 & \zeta_9^5 & 0 & 0 \\ 0 & \zeta_9^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_9^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_9 & 0 & 0 & 0 \end{bmatrix}$$

STEP 4. We use the projection formula to decompose the G-module of quadrics S_2 into its isotypical components. When an isotypical component has multiplicity greater than 1, we (noncanonically) choose ordered bases so that the G action is given by the same matrices on each ordered basis.

$$\begin{array}{llclcl} S_{2,1} & \cong & V_1 & = & \langle x_1x_6 + x_2x_4 + x_3x_5 \rangle \\ S_{2,4} & \cong & V_4^{\oplus 2} & = & \langle x_0^2 \rangle \oplus \langle x_1x_6 - \zeta_6x_2x_4 + \zeta_3x_3x_5 \rangle \\ S_{2,8} & \cong & V_8 & = & \langle x_1x_4 + \zeta_3x_2x_5 - \zeta_6x_3x_6, x_1x_5 + \zeta_3x_2x_6 - \zeta_6x_3x_4 \rangle \\ S_{2,10} & \cong & V_{10}^{\oplus 3} & = & \langle x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_0x_5, x_0x_6 \rangle \\ & & & \oplus & \langle \zeta_6x_5^2, -x_6^2, -\zeta_3x_4^2, x_1^2, \zeta_3x_2^2, -\zeta_6x_3^2 \rangle \\ & & & \oplus & \langle -x_4x_6, -\zeta_3x_4x_5, \zeta_6x_5x_6, \zeta_3x_2x_3, -\zeta_6x_1x_3, x_1x_2 \rangle \end{array}$$

The second isotypical subspace yields a polynomial of the form

$$c_1(x_0^2) + c_2(x_1x_6 - \zeta_6x_2x_4 + \zeta_3x_3x_5).$$

Assuming that c_1 and c_2 are nonzero, we may scale x_0 to get $c_1 = c_2 = 1$.

The fourth isotypical subspace yields polynomials of the form

$$\begin{array}{l} c_3x_0x_1 + c_4\zeta_6x_5^2 - c_5x_4x_6, \\ c_3x_0x_2 - c_4x_6^2 - c_5\zeta_3x_4x_5, \\ c_3x_0x_3 - c_4\zeta_3x_4^2 + c_5\zeta_6x_5x_6, \\ c_3x_0x_4 + c_4x_1^2 + c_5\zeta_3x_2x_3, \\ c_3x_0x_5 + c_4\zeta_3x_2^2 - c_5\zeta_6x_1x_3, \\ c_3x_0x_6 - c_4\zeta_6x_3^2 + c_5x_1x_2. \end{array}$$

Assuming that the leading coefficient is nonzero, we may set it to 1.

Thus, this Riemann surface has a canonical ideal of the form

$$\begin{array}{l} x_1x_6 + x_2x_4 + x_3x_5, \\ x_0^2 + x_1x_6 - \zeta_6x_2x_4 + \zeta_3x_3x_5, \\ x_1x_4 + \zeta_3x_2x_5 - \zeta_6x_3x_6, \\ x_1x_5 + \zeta_3x_2x_6 - \zeta_6x_3x_4, \\ x_0x_1 + c_4\zeta_6x_5^2 - c_5x_4x_6, \\ x_0x_2 - c_4x_6^2 - c_5\zeta_3x_4x_5, \\ x_0x_3 - c_4\zeta_3x_4^2 + c_5\zeta_6x_5x_6, \\ x_0x_4 + c_4x_1^2 + c_5\zeta_3x_2x_3, \\ x_0x_5 + c_4\zeta_3x_2^2 - c_5\zeta_6x_1x_3, \\ x_0x_6 - c_4\zeta_6x_3^2 + c_5x_1x_2 \end{array}$$

For generic values of c_4 and c_5, the intersection of these ten quadrics is empty.

STEP 5. To find values of the coefficients c_4, c_5 that yield a smooth curve, we partially compute a flattening stratification. Begin Buchberger's algorithm with respect to the lexicographic term order. We compute the S-pair reductions between the generators and find that

$$S(f_1, f_5) \to (\zeta_6 - 1)(c_4 + c_5)x_4^2 x_5 + \cdots$$
$$S(f_2, f_5) \to (-c_4 c_5 + 1)x_1^2 x_6 + \cdots$$

Therefore, in Buchberger's algorithm, these polynomials are added to the Gröbner basis. This suggests that the locus given by the equations $c_4+c_5 = 0$ and $-c_4c_5+1 = 0$ may be an interesting stratum in the flattening stratification.

We may check in `Magma` that the values $c_4 = i$ and $c_5 = -i$ yield a smooth genus 7 curve in $\mathbb{P}^6$ with the desired automorphism group. (Notice that further scaling x_0 by i allows us to write these equations over a smaller field.)

From these equations, we can compute the minimal free resolution and Betti table of this ideal:

1					
	10	16			
			16	10	
					1

Schreyer has classified Betti tables of genus 7 canonical curves in [**21**]. This Betti table implies that the curve is not hyperelliptic, trigonal, or tetragonal, and it has no degree 6 morphism $C \to \mathbb{P}^2$.

3. Selected results

In the following table we give equations for the Riemann surfaces of genus 7 with large automorphism groups (that is, $|\operatorname{Aut}(X)| > 4(g_X - 1)$) that are isolated in moduli ($\delta = 0$, in the notation of [**15**]). Equations for the 1-dimensional families as well as many lower genus examples can be found at [**24**]. We order the examples as they appear in Table 4 in [**15**].

For hyperelliptic Riemann surfaces, we give an equation of the form $y^2 = f(x)$. Many of these are classically known, and all of them can be found in [**22**].

For nonhyperelliptic curves, we give equations of the canonical ideals and surface kernel generators as elements of $\mathrm{GL}(g, \mathbb{C})$. When a matrix $M \in \mathrm{GL}(g, \mathbb{C})$ is sufficiently sparse, we frequently write the product $[x_0, \ldots, x_{g-1}]M$ to save space. For the cyclic trigonal curves, we also give a cyclic trigonal equation.

Throughout the tables below, canonical ideals are shown in the polynomial ring $\mathbb{C}[x_0, \ldots, x_6]$. The symbol ζ_n denotes $e^{2\pi i/n}$, and we write i for ζ_4.

After computing the canonical equations of the nonhyperelliptic Riemann surfaces, we can compute the Betti tables of these ideals and use the results of [**21**] to classify the curve as having a g_4^1, g_6^2, g_3^1, or none of these.

Finally, note that some of the examples below are not cyclic or superelliptic.

Genus 7, Locus 1: Group $(504, 156)$, signature (2,3,7)

Ideal: Macbeath,[**14**]:

$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2,$
$x_0^2 + \zeta_7 x_1^2 + \zeta_7^2 x_2^2 + \zeta_7^3 x_3^2 + \zeta_7^4 x_4^2 + \zeta_7^5 x_5^2 + \zeta_7^6 x_6^2,$
$x_0^2 + \zeta_7^{-1} x_1^2 + \zeta_7^{-2} x_2^2 + \zeta_7^{-3} x_3^2 + \zeta_7^{-4} x_4^2 + \zeta_7^{-5} x_5^2 + \zeta_7^{-6} x_6^2,$
$(\zeta_7^{-3} - \zeta_7^3)x_0x_6 - (\zeta_7^{-2} - \zeta_7^2)x_1x_4 + (\zeta_7 - \zeta_7^{-1})x_3x_5,$
$(\zeta_7^{-3} - \zeta_7^3)x_1x_0 - (\zeta_7^{-2} - \zeta_7^2)x_2x_5 + (\zeta_7 - \zeta_7^{-1})x_4x_6,$
$(\zeta_7^{-3} - \zeta_7^3)x_2x_1 - (\zeta_7^{-2} - \zeta_7^2)x_3x_6 + (\zeta_7 - \zeta_7^{-1})x_5x_0,$
$(\zeta_7^{-3} - \zeta_7^3)x_3x_2 - (\zeta_7^{-2} - \zeta_7^2)x_4x_0 + (\zeta_7 - \zeta_7^{-1})x_6x_1,$
$(\zeta_7^{-3} - \zeta_7^3)x_4x_3 - (\zeta_7^{-2} - \zeta_7^2)x_5x_1 + (\zeta_7 - \zeta_7^{-1})x_0x_2,$
$(\zeta_7^{-3} - \zeta_7^3)x_5x_4 - (\zeta_7^{-2} - \zeta_7^2)x_6x_2 + (\zeta_7 - \zeta_7^{-1})x_1x_3,$
$(\zeta_7^{-3} - \zeta_7^3)x_6x_5 - (\zeta_7^{-2} - \zeta_7^2)x_0x_3 + (\zeta_7 - \zeta_7^{-1})x_2x_4$

Maps: $(x_0, \ldots, x_6) \mapsto (x_0, -x_1, -x_2, -x_3, x_4, x_5, -x_6),$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Genus 7, Locus 2: Group $(144, 127)$, signature (2,3,12). Has g_6^2

Ideal: $x_0^2 + x_3x_4 - \zeta_6x_3x_5 - \zeta_6x_5x_6,$
$2ix_1^2 + x_3x_4 + \zeta_6x_3x_5 + 2x_4x_6 - \zeta_6x_5x_6,$
$2ix_1x_2 + (-2\zeta_6 + 1)x_3x_4 + \zeta_6x_3x_5 - 2\zeta_6x_4x_6 + \zeta_6x_5x_6,$
$2ix_2^2 - x_3x_4 + (-\zeta_6 + 2)x_3x_5 + (2\zeta_6 - 2)x_4x_6 + (-\zeta_6 + 2)x_5x_6,$
$x_1x_3 - \zeta_6x_2x_6 + \zeta_{12}x_4^2 + (\zeta_{12}^3 - 2\zeta_{12})x_4x_5 + \zeta_{12}x_5^2,$
$x_1x_4 - \zeta_3x_2x_5 - x_3x_6 - x_6^2,$
$x_1x_5 - x_2x_5 + x_3^2 + (-\zeta_6 + 2)x_3x_6,$
$x_1x_6 + x_2x_6 - \zeta_{12}x_4^2 + \zeta_{12}x_4x_5 - \zeta_{12}x_5^2,$
$x_2x_3 - \zeta_6x_2x_6 + \zeta_{12}x_4^2,$
$x_2x_4 + (-\zeta_6 - 1)x_2x_5 + \zeta_6x_3^2 + 2\zeta_6x_3x_6 + \zeta_6x_6^2$

Maps: $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{12}^{-1} & -\zeta_{12} & 0 & 0 & 0 & 0 \\ 0 & -\zeta_{12} & -\zeta_{12}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6 & \zeta_6 & 0 \\ 0 & 0 & 0 & -\zeta_3 & 0 & 0 & \zeta_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\zeta_3 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6 & 0 \end{bmatrix}$,
$\begin{bmatrix} \zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & \zeta_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_3 & -\zeta_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Genus 7, Locus 3: Group $(64, 41)$, signature (2,3,16), tetragonal

Ideal: $x_0^2 + x_1x_2$,
$x_3x_4 - \zeta_8x_5x_6$,
$x_1^2 + x_3x_6 + ix_4x_5$,
$x_2^2 + ix_3x_6 + x_4x_5$,
$x_0x_1 + \zeta_{16}^7x_3^2 - \zeta_{16}^5x_5^2$,
$x_0x_2 - \zeta_{16}^7x_4^2 + \zeta_{16}^5x_6^2$,
$x_0x_3 - \zeta_{16}x_2x_6$,
$x_0x_4 + \zeta_{16}x_1x_5$,
$x_0x_5 + \zeta_{16}^7x_2x_4$,
$x_0x_6 - \zeta_{16}^7x_1x_3$

Maps: $(x_0, \dots, x_6) \mapsto (-x_0, x_2, x_1, x_4, x_3, x_6, x_5)$,
$(x_0, \dots, x_6) \mapsto (ix_0, -\zeta_8^3x_2, -\zeta_8x_1, x_6, -\zeta_8^3x_5, -\zeta_8x_4, x_3)$

Genus 7, Locus 4: Group $(64, 38)$, signature (2,4,16), hyperelliptic

$y^2 = x^{16} - 1$

Genus 7, Locus 5: Group $(56, 4)$, signature (2,4,28), hyperelliptic

$y^2 = x^{15} - x$

Genus 7, Locus 6: Group $(54, 6)$, signature (2,6,9)

Ideal: $x_1x_6 + x_2x_4 + x_3x_5$,
$x_0^2 - x_1x_6 + \zeta_6x_2x_4 - \zeta_3x_3x_5$,
$x_1x_4 + \zeta_3x_2x_5 - \zeta_6x_3x_6$,
$x_1x_5 + \zeta_3x_2x_6 - \zeta_6x_3x_4$,
$x_0x_1 - \zeta_6x_5^2 - x_4x_6$,
$x_0x_2 + x_6^2 - \zeta_3x_4x_5$,
$x_0x_3 + \zeta_3x_4^2 + \zeta_6x_5x_6$,
$x_0x_4 - x_1^2 + \zeta_3x_2x_3$,
$x_0x_5 - \zeta_3x_2^2 - \zeta_6x_1x_3$,
$x_0x_6 + \zeta_6x_3^2 + x_1x_2$

Maps: $(x_0, \dots, x_6) \mapsto (-x_0, \zeta_9^5x_6, \zeta_9^8x_4, \zeta_9^2x_5, \zeta_9x_2, \zeta_9^7x_3, \zeta_9^4x_1)$,
$(x_0, \dots, x_6) \mapsto (\zeta_6x_0, \zeta_3^2x_4, \zeta_3^2x_5, \zeta_3^2x_6, \zeta_3x_3, \zeta_3x_1, \zeta_3x_2)$

Genus 7, Locus 7: Group $(54, 6)$, signature (2,6,9)
Complex conjugate of the previous curve

Genus 7, Locus 8: Group $(54, 3)$, signature (2,6,9) cyclic trigonal
Trigonal equation: $y^3 = x^9 - 1$

Ideal: 2×2 minors of $\begin{bmatrix} x_0 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}$, and
$x_0^3 - x_6^2 x_3 + x_2^3$,
$x_0^2 x_1 - x_6^2 x_4 + x_2^2 x_3$,
$x_0 x_1^2 - x_6^2 x_5 + x_2^2 x_4$,
$x_1^3 - x_6^3 + x_2^2 x_5$

Maps: $(x_0, \dots, x_6) \mapsto (x_1, x_0, -x_6, -x_5, -x_4, -x_3, -x_2)$,
$(x_0, \dots, x_6) \mapsto (\zeta_9 x_1, \zeta_9^2 x_0, -\zeta_9 x_6, -\zeta_9^2 x_5, -\zeta_9^3 x_4, -\zeta_9^4 x_3, -\zeta_9^5 x_2)$

Genus 7, Locus 9: Group $(48, 32)$, signature (3,4,6). Has g_6^2
Ideal:
$x_0^2 + x_3x_5 + \zeta_6 x_3 x_6 - \zeta_3 x_4 x_6$,
$\sqrt{3}i(x_1x_3 - x_2x_4 + x_1x_5) - x_1x_6 + x_2x_5 - x_2x_6$,
$\sqrt{3}i(2x_1x_4 - x_2x_5 + x_2x_6) + x_1x_5 - x_1x_6$,
$\sqrt{3}i(2x_2x_3 + x_1x_6 - x_2x_5) + 3x_1x_5 + x_2x_6$,
$-3(x_1^2 + x_3x_5 - x_3x_6) + \sqrt{3}i(x_4x_5 - x_4x_6) + 2x_5^2 + 2(\zeta_6 - 1)x_5x_6 - 2\zeta_6 x_6^2$,
$-3(x_1x_2 - x_4x_5) + \sqrt{3}i(x_3x_5 - x_3x_6 + x_4x_6 - x_5^2) + 2\zeta_6 x_5x_6 - x_6^2$,
$-3(x_2^2 - x_3x_5 - x_4x_6) + \sqrt{3}i(x_3x_6 - 3x_4x_5) + 2(\zeta_6 + 1)x_5x_6 + 2(\zeta_6 - 1)x_6^2$,
$-3(2x_4^2 - x_3x_5 + x_3x_6) + \sqrt{3}i(-x_4x_5 + x_4x_6) + 2x_5^2 + 2(\zeta_6 - 1)x_5x_6 - 2\zeta_6 x_6^2$,
$-3(-2x_3x_4 + x_4x_5) + \sqrt{3}i(-x_3x_5 + x_3x_6 - x_4x_6 - x_5^2) + 2\zeta_6 x_5x_6 - x_6^2$,
$-3(2x_3^2 + x_3x_5 + x_4x_6) + \sqrt{3}i(-x_3x_6 + 3x_4x_5) + 2(\zeta_6 + 1)x_5x_6 + 2(\zeta_6 - 1)x_6^2$
Maps:
$(x_0, \dots, x_6) \mapsto (\zeta_3 x_0, -\zeta_6 x_2, -\zeta_3 x_1 - x_2, -x_3 + \zeta_3 x_4, \zeta_6 x_3, \zeta_3 x_5, -\zeta_6 x_5 + x_6)$,
$(x_0, \dots, x_6) \mapsto (-x_0, -x_2, x_1, x_4, -x_3, \zeta_3 x_5 - \zeta_6 x_6, -\zeta_6 x_5 - \zeta_3 x_6)$

Genus 7, Locus 10: Group $(42, 4)$, signature (2,6,21) cyclic trigonal
Trigonal equation: $y^3 = x^8 - x$

Ideal: 2×2 minors of $\begin{bmatrix} x_0 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}$, and
$x_0^3 - x_6^2 x_2 + x_2^2 x_3$
$x_0^2 x_1 - x_6^2 x_3 + x_2^2 x_4$,
$x_0 x_1^2 - x_6^2 x_4 + x_2^2 x_5$,
$x_1^3 - x_6^2 x_5 + x_2^2 x_6$

Maps:
$(x_0, \dots, x_6) \mapsto (\zeta_7 x_1, \zeta_7^6 x_0, -\zeta_7^4 x_6, -\zeta_7^2 x_5, -x_4, -\zeta_7^5 x_3, -\zeta_7^3 x_2)$,
$(x_0, \dots, x_6) \mapsto (\zeta_{21}^{-2} x_1, \zeta_{21}^{-5} x_0, -\zeta_{21}^{-1} x_6, -\zeta_{21}^{-4} x_5, (\zeta_3 + 1)x_4, -\zeta_{21}^{11} x_3, -\zeta_{21}^{8} x_2)$

Genus 7, Locus 11: Group $(32, 11)$, signature (4,4,8). Has g^2_6

Ideal: $x_3x_5 + x_4x_6$,
$x_0^2 + x_1x_5 + ix_2x_6$,
$x_1x_4 + ix_2x_3 + x_5x_6$,
$x_1x_2 + x_3x_4$,
$x_1x_6 + \zeta_8^3 x_4x_5$,
$x_2x_5 + \zeta_8 x_3x_6$,
$x_1^2 - ix_3^2 - \zeta_8^3 x_5^2$,
$x_2^2 + ix_4^2 + \zeta_8^3 x_6^2$,
$-ix_2x_4 + \zeta_8^3 x_3^2$,
$x_1x_3 - \zeta_8^3 x_4^2$

Maps: $(x_0, \ldots, x_6) \mapsto (-x_0, x_2, -x_1, -ix_4, -ix_3, ix_6, ix_5)$,
$(x_0, \ldots, x_6) \mapsto (ix_0, x_1, ix_2, -x_3, -ix_4, -x_5, ix_6)$

Genus 7, Locus 12: Group $(32, 10)$, signature (4,4,8). Has g^2_6

Ideal: $x_1x_6 + \zeta_{16}^6 x_2x_5 + x_3x_4$,
$x_1x_2 + x_5x_6$,
$x_0^2 + x_1x_6 - \zeta_{16}^6 x_2x_5$,
$x_3x_6 - \zeta_{16}^4 x_4x_5$,
$x_1^2 - \zeta_{16}^7 x_4^2 - \zeta_{16}^6 x_5^2$,
$x_2^2 + \zeta_{16}^3 x_3^2 - \zeta_{16}^{10} x_6^2$,
$-\zeta_{16}^2 x_2x_6 - \zeta_{16}^7 x_5^2$,
$x_1x_5 + (-\zeta_{16}^{12} - \zeta_{16}^4)x_3^2 - \zeta_{16}^{11} x_6^2$,
$x_1x_3 + \zeta_{16}^7 x_4x_6$,
$x_2x_4 + \zeta_{16} x_3x_5$

Maps: $(x_0, \ldots, x_6) \mapsto (-x_0, x_2, -x_1, -\zeta_{16}^6 x_4, -\zeta_{16}^2 x_3, -\zeta_{16}^2 x_6, -\zeta_{16}^6 x_5)$,
$(x_0, \ldots, x_6) \mapsto (ix_0, -\zeta_{16}^2 x_2, -\zeta_{16}^6 x_1, ix_4, -ix_3, -ix_6, -ix_5)$

Genus 7, Locus 13: Group $(30, 4)$, signature (2,15,30), hyperelliptic
$y^2 = x^{15} - 1$

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