CASTELNUOVO-MUMFORD REGULARITY IN BIPROJECTIVE SPACES

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Abstract. We define the concept of regularity for bigraded modules over a bigraded polynomial ring. In this setting we prove analogs of some of the classical results on m-regularity for graded modules over polynomial algebras.

1. Introduction

In chapter 14 of [13] Mumford introduced the concept of *regularity* for a coherent sheaf $\mathcal{F}$ on projective space $\mathbf{P}^n$: $\mathcal{F}$ is p-regular if, for all $i \geq 1$ we have vanishing for the twists

$$H^i(\mathbf{P}^n, \mathcal{F}(k)) = 0, \quad \text{for all } k + i = p.$$

This in turn implies the stronger condition of vanishing for $k + i \geq p$. Regularity was investigated later by several people, notably Bayer and Mumford [1], Bayer and Stillman [2], Eisenbud and Goto [4], and Ooishi [14]. Let $R = K[x_0, ..., x_n]$ be the polynomial algebra in $n + 1$ variables over a field K, graded in the usual way. If M is a finitely generated graded R-module, then the local cohomology groups $H^i_{\mathbf{m}}(M)$ with respect to the ideal $\mathbf{m} = (x_0, ..., x_n)$ are graded in a natural way and we say that M is p-regular if

$$H^i_{\mathbf{m}}(M)_k = 0 \quad \text{for all } k + i \geq p + 1.$$

If $\mathcal{F}$ is the coherent sheaf on $\mathbf{P}^n$ associated with M in the usual way, we have

$$H^{i+1}_{\mathbf{m}}(M)_k = H^i(\mathbf{P}^n, \mathcal{F}(k)) \quad \text{for all } i \geq 1,$$

which shows the compatibility of these definitions. An important result in this theory is:

Theorem 1.1. *Suppose K is a field and $I \subset R$ is a graded ideal. Then I is p-regular if and only if the minimal free graded resolution of I has the form*

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \longrightarrow \bigoplus_{\alpha=1}^{r_1} Re_{\alpha,1} \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \longrightarrow I \longrightarrow 0$$

where $\deg(e_{\alpha,i}) \leq p + i$ *for all* $i \geq 0$.

The conditions of p-regularity can be derived quasi-axiomatically from the following considerations. One seeks a condition in the form of

$$(1) \qquad H^i_{\mathbf{m}}(M)_k = 0 \text{ all } i \geq 0, \text{ all } k \in C_i(p) \Longrightarrow R_s M_p = M_{p+s} \text{ all } s \geq 0,$$

for certain regions $C_i(p) \subset \mathbb{Z}$. One postulates:

1. For each i, region $C_i(p)$ is independent of the number $n + 1$ of variables.
2. If M is p-regular in the sense of the left-hand side of (1), then for a generic linear form $x \in R_1$, $\bar{M} = M/xM$ is p-regular.

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First, when $n+1=0$, that is, we are considering graded K-modules, since $\mathbf{m}=(0)$, we have $H^0_{\mathbf{m}}(M)=M$, and since $R_s=0$ for $s\geq 1$, property (1) forces $M_k=H^0_{\mathbf{m}}(M)_k=0$ for $k\geq p+1$ in this case. By principle 1., this must hold for all n. Assuming that $\mathbf{m}\notin \mathrm{Ass}(M)$ where $\mathrm{Ass}(M)$ denotes the associated primes for M, and K is infinite, then x may be chosen so that we have an exact sequence

$$0 \longrightarrow xM=M(-1) \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0$$

which gives rise to the long exact sequence in cohomology. We have

$$H^0_{\mathbf{m}}(M)_k \longrightarrow H^0_{\bar{\mathbf{m}}}(\bar{M})_k \longrightarrow H^1_{\mathbf{m}}(M(-1))_k = H^1_{\mathbf{m}}(M)_{k-1}.$$

In order that we have $H^0_{\bar{\mathbf{m}}}(\bar{M})_k=0$ for $k\geq p+1$, as is demanded by principle 2., we must have $H^1_{\mathbf{m}}(M)_k=0$ for $k\geq p$. In a similar way, we obtain the vanishing region for $H^2_{\mathbf{m}}(M)_k$ from that of $H^1_{\mathbf{m}}(M)_k$, etc., and we find that they are exactly the conditions of p-regularity given. Of course, one deduces property (1) from the definition of p-regularity, by induction on the number of variables $n+1$, by a reversal of the above steps.

The other essential feature of p-regularity is that

3. R is 0-regular.

This follows from Serre's calculations of the cohomology of the invertible sheaves $\mathcal{O}(k)$ on $\mathbf{P}^n$ ([17]), as reinterpreted by Grothendieck in the language of local cohomology (combine [8, Prop. (2.1.5)] with [9, Exp. II, Prop. 5]).

Our definition of regularity for bigraded modules follows this pattern. Let $R=K[x,y]=K[x_0,...,x_m,y_0,...,y_n]$, which is bigraded in the usual way. Let $\mathbf{m}=(xy)=(x_iy_j)$ be the irrelevant ideal. We seek regions $C_i(p,p')\subset\mathbb{Z}^2$ with the property that

(2)
$$H^i_{\mathbf{m}}(M)_{k,k'}=0 \text{ all } i\geq 0, \text{ all } (k,k')\in C_i(p,p') \Longrightarrow R_{s,s'}M_{p,p'}=M_{p+s,p'+s'} \text{ all } s\geq 0, s'\geq 0$$

One postulates the analogs of 1. and 2. above. For 2. we need regularity for both M/xM and M/yM for generic $x\in R_{1,0}$ and $y\in R_{0,1}$. This leads to the regions called $Reg_{i-1}(p,p')$ (the shift $i\to i-1$ is explained later). We are able to prove analogs in this setting of the many of the classical results for graded modules (see theorem (3.4) and proposition (3.5)). Actually, we first do a separate treatment for sheaves, the way Mumford did (propositions (2.7) and (2.8)). However, in attempting to generalize theorem (1.1) to a structure theorem for free resolutions for bigraded modules, the conditions we have proposed are seen to be inadequate. Therefore, we define a new concept of *strong* regularity and prove that it does indeed give the structure theorem that we want (theorem (4.10)). This involves vanishing conditions for all three of $H^*_I(M)$ for the ideals $I=(x),(y),(x,y)$. The previous notion of regularity is now called *weak* regularity. We show that strong regularity implies weak regularity, and that R itself is strongly $(0,0)$-regular. As far as we can determine, there is no simple vanishing condition for $H^*_{(xy)}(M)$ alone that implies the structure theorem that we want.

In the last section we write down a free resolution that permits computation of $H^i_{\mathbf{m}}(M)$. In a sequel to this work applications and examples will be discussed. Also, it is clear that the methods in this paper may be extended to a multigraded module over a multigraded polynomial algebra. This will also be addressed in a future work.

2. Regularity for coherent sheaves

First, we will give definition and some properties of regularity of a coherent sheaf similar to [13, Ch. 14]. Let K be a field, and $R=K[x_0,\cdots,x_m,y_0,\cdots,y_n]$ be the polynomial ring, bigraded with variables x having bidegree $(1,0)$ and variables y having bidegree $(0,1)$. We let

$$\mathbf{m} = R_+ = \bigoplus_{a>0,b>0} R_{a,b},$$

the *irrelevant ideal*. Some of the general theory of graded and multigraded algebras used here can be found in [6], [7].

Let $X = \mathbf{P}^m \times \mathbf{P}^n$, which when regarded as a scheme is $\mathrm{Proj}\,(R)$, where by definition, this is the set of bigraded prime ideals $\mathbf{p}$ that do not contain the irrelevant ideal $\mathbf{m}$. There are projections p_1 and p_2 of X onto its two factors. If $\mathcal{F}_1$ is sheaf of $\mathcal{O}_{\mathbf{P}^m}$-modules, and $\mathcal{F}_2$ is sheaf of $\mathcal{O}_{\mathbf{P}^n}$-modules, we denote

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 = p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2, \text{ an } \mathcal{O}_X\text{-module.}$$

As in the usual case of projective space there is a functor $M \to \tilde{M}$ from bigraded R-modules to quasi-coherent sheaves on X, and every quasi-coherent sheaf $\mathcal{F}$ arises this way, in a nonunique fashion. In fact, if

$$M = \bigoplus_{(a,b)\in\mathbb{Z}^2} H^0(X,\, \mathcal{F}(a,b))$$

then $\mathcal{F} \cong \tilde{M}$. Here, for any sheaf of $\mathcal{O}_X$- modules $\mathcal{F}$, we denote

$$\mathcal{F}(a,b) = \mathcal{F} \otimes \mathcal{O}_X(a,b)$$

where $\mathcal{O}_X(a,b) = \mathcal{O}_{\mathbf{P}^m}(a) \boxtimes \mathcal{O}_{\mathbf{P}^n}(b)$ is the invertible sheaf associated to the graded R-module $R(a,b)$. Recall that if M is any graded R-module, $M(a,b)$ is the graded module with degrees shifted via $M(a,b)_{d,e} = M_{d+a,e+b}$. If Z is a scheme, tensor products involving $\mathcal{O}_Z$-modules will always be relative to $\mathcal{O}_Z$ unless otherwise stated

When $m \geq 1$, and $n \geq 1$, the Picard group $\mathrm{Pic}(X)$ is isomorphic with $\mathbb{Z}^2$ with (a,b) corresponding to $\mathcal{O}_X(a,b)$. Interpreting the Picard group as the group of divisor-classes, $\mathcal{O}_X(a,b)$ corresponds to the divisor $aL_1 + bL_2$, where $L_1 = H_1 \times \mathbf{P}^n$, $H_1 \subset \mathbf{P}^m$ being any hyperplane, and $L_2 = \mathbf{P}^m \times H_2$, $H_2 \subset \mathbf{P}^n$ being any hyperplane.

Note the special case: if m or n is 0, the biprojective space reduces to a projective space. Except in the case where both are 0, the Picard group $\mathrm{Pic}(X)$ is isomorphic with $\mathbb{Z}$. If both are 0, the space reduces to a point, and its Picard group is trivial. Even in these degenerate cases we still use notations such as $\mathcal{F}(a,b)$, where one or other twisting by a or b might be trivial.

Definition 2.1. For each integer $i > 0$, let

$$\begin{aligned} St_i &= \{(r,s) \in \mathbb{Z}^2 : r+s = -i-1,\ r<0,\ s<0\} \\ &= \{(-i,-1),\, (-i+1,-2),\, \ldots,\, (-2,-i+1),\, (-1,-i)\}, \end{aligned}$$

for $i \leq 0$, let

$$\begin{aligned} St_i &= \{(r,s) \in \mathbb{Z}^2 : r+s = -i,\ r \geq 0,\ s \geq 0\} \\ &= \{(-i,0),\, (-i-1,1),\, \ldots,\, (1,-i-1),\, (0,-i)\}. \end{aligned}$$

For each $(p,p') \in \mathbb{Z}^2$ let $St_i(p,p') = (p,p') + St_i$.
For $i \geq 0$, let $Reg_i(p,p') = \mathbb{Z}_+^2 + St_i(p,p')$ where $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$.
For $i = -1$, let $Reg_{-1} = \mathbb{Z}_+^2 + (p+1, p'+1)$.
Let $Reg'_{-1}(p,p') = (p+1,p') + \mathbb{Z}_+^2$.
Let $Reg''_{-1}(p,p') = (p,p'+1) + \mathbb{Z}_+^2$.
For $i \geq 0$, define $DReg_i(p,p') = \mathbb{Z}_-^2 + St_{-i}(p,p')$ where $\mathbb{Z}_- = \{n \in \mathbb{Z} : n \leq 0\}$.
Note that, for all $i \geq -1$,

$$Reg_i(p,p') \ = \ \mathbb{Z}^2 \cap \{(x,y) \in \mathbb{R}^2 \mid x \geq p-i,\ y \geq p'-i,\ x+y \geq p+p'-i-1\}$$

and, for all $i \geq 0$,

$$DReg_i(p, p') = -Reg_{i+1}(-p + 1, -p' + 1)$$

Remark 2.2. For $i \geq 0$, and for all p, p', we have

1. $(k, k') \in St_i(p, p') \Rightarrow (k - 1, k'), (k, k' - 1) \in St_{i+1}(p, p')$.
2. $St_i(p, p') \in Reg_i(p, p')$.
3. $(k, k') \in Reg_i(p, p') \Rightarrow (k - 1, k'), (k, k' - 1) \in Reg_{i+1}(p, p')$.
4. $Reg_i(q, q') \subset Reg_i(p, p')$, if $q \geq p, q' \geq p'$.
5. $(k, k') \in Reg'_{-1}(p, p') \Longrightarrow (k - 1, k') \in Reg_0(p, p')$.
6. $(k, k') \in Reg''_{-1}(p, p') \Longrightarrow (k, k' - 1) \in Reg_0(p, p')$.

Here is a picture:

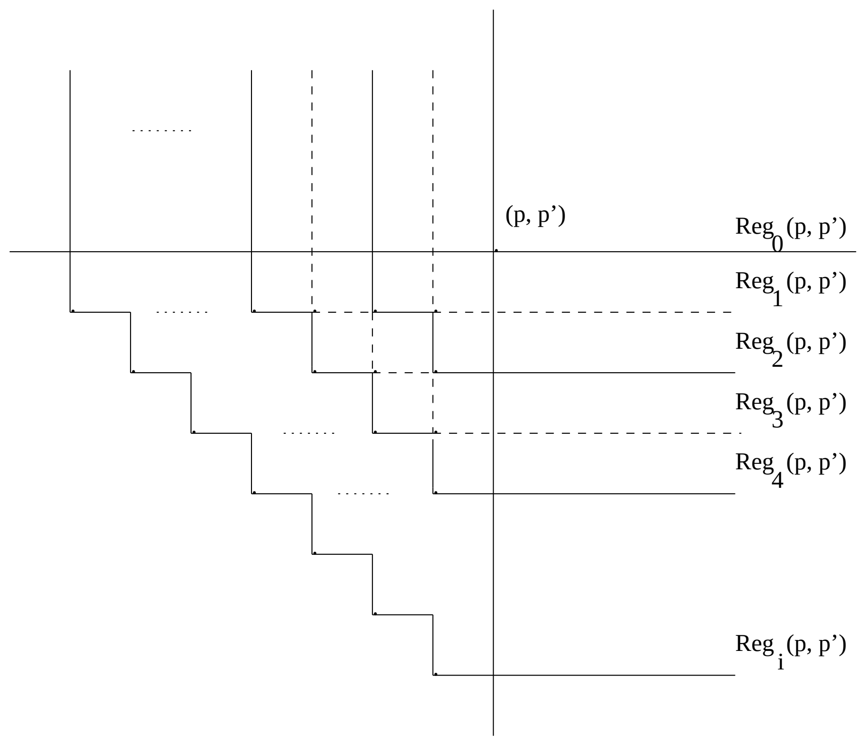

FIGURE 1. $Reg_i(p, p')$

Using these notations, we make the following definition.

Definition 2.3. Let $\mathcal{F}$ be a coherent sheaf on X. We will say that $\mathcal{F}$ is (p, p')-*regular* if, for all $i \geq 1$,

$$H^i(X, \mathcal{F}(k, k')) = 0$$

whenever $(k, k') \in St_i(p, p')$.

Remark 2.4. If $n = 0$, $\mathbf{P}^m \times \mathbf{P}^0 \cong \mathbf{P}^m$, so every coherent sheaf on $\mathbf{P}^m \times \mathbf{P}^0$ is naturally identified with a sheaf on $\mathbf{P}^m$. The sheaf $\mathcal{F}(p, p')$ is independent of p'. Under this identification, $\mathcal{F}$ is (p, p')-regular on $\mathbf{P}^m \times \mathbf{P}^0$ in the sense of Definition 2.3, if and only if $\mathcal{F}$ is p-regular on $\mathbf{P}^m$ in the sense of Mumford.

Proof. First, we will show that (p, p')-regular implies p-regular.

In this case, $\mathcal{F}(k, k') \cong \mathcal{F}(k)$. $\mathcal{F}$ is (p, p')-regular means that for all $i \geq 1$,

$$H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k, k')) = H^i(\mathbf{P}^m, \mathcal{F}(k)) = 0,$$

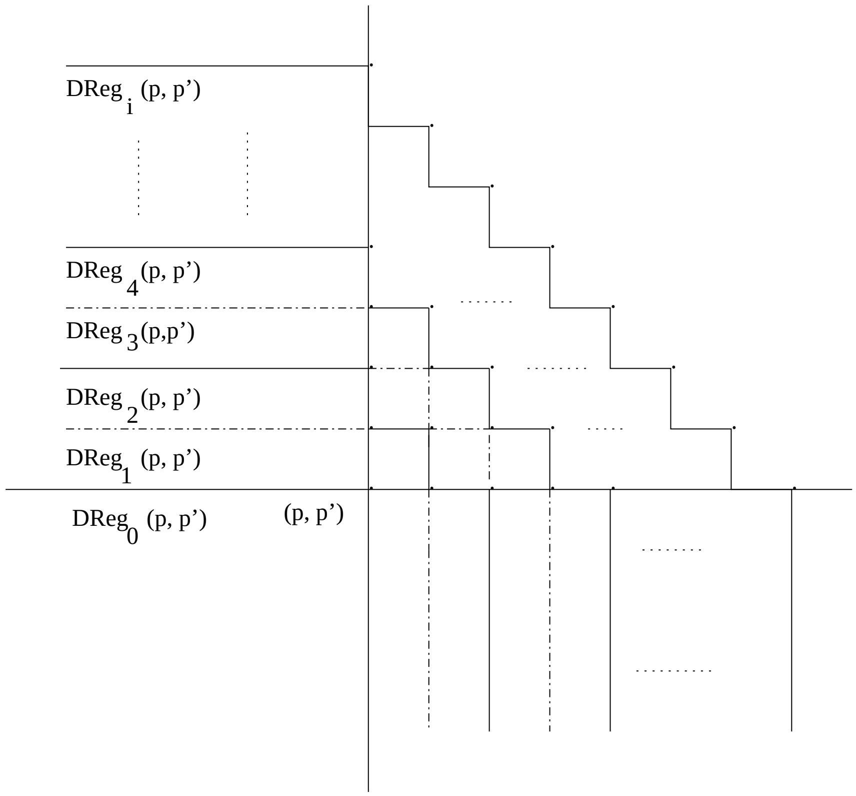

FIGURE 2. $DReg_i(p, p')$

where $p - i \leq k \leq p - 1$. Since $k + i \geq p$, according to [13, p. 100], $\mathcal{F}$ is p-regular.

Second, we will show that $\mathcal{F}$ is p-regular implies (p, p')-regular.

If $\mathcal{F}$ is p-regular, then $H^i(\mathbf{P}^m, \mathcal{F}(k)) = 0$ whenever $k + i \geq p$, this implies

$$H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k, k')) = 0$$

for any $k' \in \mathbb{Z}$. In particular, $H^i(\mathbf{P}^m \times \mathbf{P}^0, \mathcal{F}(k, k')) = 0$ for all $(k, k') \in St_i(p, p')$. Therefore, $\mathcal{F}$ is (p, p')-regular. $\square$

Proposition 2.5. $\mathcal{O}_X$ *is* $(0, 0)$*-regular.*

Proof. If m or $n = 0$, by the previous remark, $\mathcal{O}_X$ is $(0, 0)$-regular $\Leftrightarrow$ $\mathcal{O}_X$ is 0-regular. But $\mathcal{O}_{\mathbf{P}^m}$ is 0-regular since

$$H^a(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(k)) = 0, \text{ if } a \geq 1 \text{ and } a + k \geq 0 \tag{3}$$

$$H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(k)) = 0, \text{ if } k \leq -1. \tag{4}$$

These formulas are a consequence of Serre's results on the cohomology of projective space. [10]

If m and $n \geq 1$, we can apply the Künneth formula [16],

$$H^i(X, \mathcal{O}_{\mathbf{P}^m}(k) \boxtimes \mathcal{O}_{\mathbf{P}^n}(k')) = \bigoplus_{a+b=i} H^a(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(k)) \otimes H^b(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k')).$$

We will show that $H^a(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(k)) = 0$ or $H^b(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k')) = 0$ whenever $a + b = i$ and $(k, k') \in St_i(0, 0)$. If $(k, k') \in St_i(0, 0)$, then $k = -i + l$ and $k' = -1 - l$ where $0 \leq l \leq i - 1$. If $a = 0$ or $b = 0$, we are done by Equation (4), since $k, k' < 0$. If $a > 0$, and $b > 0$, we only need to show $a - i + l \geq 0$ or $b - 1 - l \geq 0$. Suppose both $a - i + l \leq -1$ and $b - 1 - l \leq -1$. Since

$a+b=i$,

$$-1=(a-i+l)+(b-1-l)\leq -2.$$

This contradiction shows that either $a-i+1\geq 0$ or $b-1-l\geq 0$, and the proof is completed by Equation (4). □

Lemma 2.6. *Assume that K is infinite, and that $m\geq 1$. Let $\mathcal{F}$ be a coherent sheaf on X. Let L_1 be a hyperplane defined by $\sum_{i=0}^m a_i x_i = 0$, and let $\mathcal{F}_{L_1} = \mathcal{F}\otimes\mathcal{O}_{L_1}$ denote the sheaf $\mathcal{F}$ restricted to L_1. If $\mathcal{F}$ is (p,p')-regular, then $\mathcal{F}_{L_1}$ is (p,p')-regular for a generic L_1. The similar statement is true for hyperplanes L_2 defined by a form $\sum_{i=0}^n b_i y_i = 0$ assuming $n\geq 1$.*

Proof. Given $\mathcal{F}$, choose a hyperplane L_1, where L_1 is defined by an equation of the form $f=\sum_{i=0}^m a_i x_i = 0$, such that L_1 does not contain any of points of the finite set of associated primes $A(\mathcal{F})$ (for the definition of this, see [13, p.40]). Note that this is possible: $A(\mathcal{F})$ is finite, and because K is infinite, we can find a linear form missing the p_1-projections of the associated primes. Tensor the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1,0) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_{L_1} \longrightarrow 0$$

with $\mathcal{F}(k,k')$. For all $x\in X$, multiplication by f is injective in $\mathcal{F}_x$, since by construction, f is a unit at all associated primes of $\mathcal{F}_x$. Therefore the resulting sequence is exact:

$$0 \longrightarrow \mathcal{F}(k-1,k') \xrightarrow{f} \mathcal{F}(k,k') \longrightarrow \mathcal{F}\otimes\mathcal{O}_{L_1}(k,k') = \mathcal{F}_{L_1}(k,k') \longrightarrow 0 \tag{5}$$

This gives an exact cohomology sequence:

$$\cdots \longrightarrow H^i(\mathcal{F}(k,k')) \longrightarrow H^i(\mathcal{F}_{L_1}(k,k')) \longrightarrow H^{i+1}(\mathcal{F}(k-1,k')) \longrightarrow \cdots$$

If $(k,k')\in St_i(p,p')$, then $(k-1,k')\in St_{i+1}(p,p')$ by 2.2, and the first and the last groups vanish when $i\geq 1$, since we are assuming that $\mathcal{F}$ is (p,p')-regular. This forces the second group to vanish, thus proving that $\mathcal{F}_{L_1}$ is (p,p')-regular. □

Proposition 2.7. *If $\mathcal{F}$ is a (p,p')-regular coherent sheaf on $X=\mathbf{P}^m\times\mathbf{P}^n$, then for all $i\geq 1$,*

$$H^i(X,\mathcal{F}(k,k'))=0 \tag{6}$$

whenever $(k,k')\in Reg_i(p,p')$. That is, $\mathcal{F}$ is (q,q')-regular for $q\geq p,\ q'\geq p'$.

Proof. We will prove (6) by double induction on (m,n). If $m=0$ or $n=0$, by Remark 2.4 (p,p')-regularity reduces to ordinary p-regularity or p'-regularity for projective space, and (6) is true by Mumford's result [13]. So assume $m\geq 1$ and $n\geq 1$. Every element of $Reg_i(p,p')$ is of the form $(k+r,k'+s)$ for some $(k,k')\in St_i(p,p')$, and $(r,s)\geq(0,0)$. Now we will do double induction on the pair (r,s). The case $(r,s)=(0,0)$ is true by assumption of (p,p')-regularity for $\mathcal{F}$. Choose a hyperplane L_1 as in Lemma 2.6 such that $\mathcal{F}_{L_1}$ is (p,p')-regular. Consider the cohomology exact sequence attached to (5) with (k,k') replaced by $(k+r+1,k'+s)$:

$$H^i(\mathcal{F}(k+r,k'+s)) \longrightarrow H^i(\mathcal{F}(k+r+1,k'+s)) \longrightarrow H^i(\mathcal{F}_{L_1}(k+r+1,k'+s))$$

Since $\mathcal{F}_{L_1}$ is (p,p')-regular, and since L_1 is a biprojective space of lower dimension, the induction hypothesis says that the right-hand term is 0. The left-hand side also vanishes, by induction hypothesis on (r,s). Hence the middle term vanishes, as required. A symmetric argument shows vanishing for $(k+r,k'+s+1)$. □

Proposition 2.8. *If $\mathcal{F}$ is a (p,p')-regular coherent sheaf on X, then $H^0(X,\mathcal{F}(k,k'))$ is spanned by*

$$H^0(X,\mathcal{F}(k-1,k'))\otimes H^0(X,\mathcal{O}(1,0)),$$

if $k>p, k'\geq p'$; and it is spanned by

$$H^0(X,\mathcal{F}(k,k'-1))\otimes H^0(X,\mathcal{O}(0,1)),$$

if $k \geq p, k' > p'$.

Proof. We use induction on $\dim(X)$: for $\dim(X) = 0$, the result is true. By Lemma 2.6, we know that $\mathcal{F}_{L_1}$ is (p, p')-regular. Consider the following diagram:

$$\begin{array}{ccccccc} & & H^0(\mathcal{F}(k-1,k')) \otimes H^0(\mathcal{O}_X(1,0)) & \xrightarrow{\sigma} & H^0(\mathcal{F}_{L_1}(k-1,k')) \otimes H^0(\mathcal{O}_{L_1}(1,0)) \\ & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} \\ H^0(\mathcal{F}(k-1,k')) & \xrightarrow{\alpha} & H^0(\mathcal{F}(k,k')) & \xrightarrow{\nu} & H^0(\mathcal{F}_{L_1}(k,k')) \end{array}$$

If $k > p$ and $k' \geq p'$, σ is surjective because $\mathcal{F}$ is (p, p')-regular, and thus $H^1(\mathcal{F}(k-2, k')) = 0$. τ is surjective by induction hypothesis. ν is also surjective, since $H^1(\mathcal{F}(k-1, k)) = 0$.

Let $t \in H^0(\mathcal{F}(k, k'))$, we have $\nu(t) = \tau(s) = \tau\sigma(s')$ for some

$$s \in H^0(\mathcal{F}_{L_1}(k-1, k')) \otimes H^0(\mathcal{O}_{L_1}(1, 0)), \text{ and } s' \in H^0(\mathcal{F}(k-1, k')) \otimes H^0(\mathcal{O}_Q(1, 0)).$$

We have $\nu(\mu(s')) = \tau(\sigma(s')) = \nu(t)$, and $t - \mu(s') \in ker(\nu)$. Since the last row of the diagram is exact in the middle, so we have $t' \in H^0(\mathcal{F}(k-1, k'))$ such that $\alpha(t') = t - \mu(s')$. This says that $H^0(\mathcal{F}(k, k'))$ is spanned by the image of μ and the image of α. But the image of α is in $H^0(\mathcal{F}(k-1, k')) \otimes H^0(\mathcal{O}(1, 0))$, because map α is the multiplication by f, and $f \in H^0(\mathcal{O}(1, 0))$. This means that $H^0(\mathcal{F}(k, k'))$ is spanned by

$$H^0(\mathcal{F}(k-1, k')) \otimes H^0(\mathcal{O}(1, 0))$$

By symmetry, we can show that if $k \geq p, k' > p'$, $H^0(\mathcal{F}(k, k'))$ is spanned by

$$H^0(\mathcal{F}(k, k'-1)) \otimes H^0(\mathcal{O}(0, 1)).$$

□

3. Weak regularity for bigraded modules

We will give the definition and some properties of regularity for a bigraded module similar to [14] and [12]. Let A be a noetherian ring, and let now $R = \oplus_{a,b \geq 0} R_{a,b}$ be any bigraded ring over A, with $R_{0,0} = A$. We assume that it is finitely generated by homogeneous elements of bidegrees $(1, 0)$ and $(0, 1)$. Such a ring will be called a bihomogeneous A-algebra. Previously we considered only the case of a polynomial ring in two sets of variables over a field. Let $\mathbf{m} = R_+ = \oplus_{a>0,b>0} R_{a,b}$ be the irrelevant ideal; it is a bigraded R-module. There is a scheme $X = \mathrm{Proj}\,(R)$, whose points are the bihomogeneous prime ideals $\mathbf{p}$ of R that do not contain the irrelevant ideal. We also have a functor $M \to \tilde{M}$ from bigraded modules to quasicoherent $\mathcal{O}_X$-modules with similar properties to those discussed in section 2. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. If we set $M = \oplus_{a,b \in \mathbb{Z}} H^0(X, \mathcal{F}(a, b))$, then we have $\mathcal{F} = \tilde{M}$.

If R is a bigraded A-algebra, then it defines a graded A-algebra

$$R^\sharp_n = \bigoplus_{i+j=n} R_{i,j}$$

and similarly we have a graded $R^\sharp$-module $M^\sharp$ associated to a bigraded R-module M.

Let $M = \oplus_{a,b \in \mathbb{Z}} M_{a,b}$ be a bigraded R-module. The local cohomology groups $H^i_{\mathbf{m}}(M)$ are bigraded R-modules, and let $H^i_{\mathbf{m}}(M)_{a,b}$ denote the (a, b) part. The general theory of local cohomology is found in [9]. Note that, if $J \subset A$ is an ideal in a ring, and $V(J) \subset C = \mathrm{Spec}(A)$ is the corresponding closed subset, then

$$H^*_J(M) = H^*_{V(J)}(C, \tilde{M})$$

where $\tilde{M}$ is the quasi-coherent sheaf on C associated with the A-module M.

We have

$$H^i_{\mathbf{m}}(M)^\sharp = H^i_{\mathbf{m}^\sharp}(M^\sharp), \text{ ie., } H^i_{\mathbf{m}^\sharp}(M^\sharp)_n = \bigoplus_{k+k'=n} H^i_{\mathbf{m}}(M)_{k,k'}$$

Generally we omit the $\sharp$ from $\mathbf{m}$, as it is clear in context that we are referring to the graded, as opposed to the bigraded, structure.

We recall the following fact: Let R be any ring, $I \subset R$ an ideal and M an R-module. If $\mathrm{Supp}(M) \subset V(I)$ then

$$H^0_I(M) = M, \text{ and } H^i_I(M) = 0 \text{ for } i \geq 1.$$

Also, if R is Noetherian and M is finitely generated,

$$\mathrm{Ass}(M) \subset \mathrm{Supp}(M),$$

and both have the same minimal elements. $\mathrm{Ass}(M)$ is finite.

We allow the case where $R_{a,b} = 0$ for all $a > 0$, or $R_{a,b} = 0$ all $b > 0$. For then $\mathbf{m} = 0$, and thus for all R-modules M,

$$H^0_{\mathbf{m}}(M) = M, \text{ and } H^i_{\mathbf{m}}(M) = 0 \text{ for } i \geq 1.$$

since $V(\mathbf{m}) = \mathrm{Spec}(R)$, so $\mathrm{Supp}(M) \subset V(\mathbf{m})$ always holds. This extreme case plays an important role in the proofs of the main theorems about regularity, which are by induction on the number of variables.

Definition 3.1. We say that a bigraded R-module M is *weakly (p,p')-regular*, if for all $i \geq 0$,

$$H^i_{\mathbf{m}}(M)_{k,k'} = 0 \text{ for all } (k,k') \in Reg_{i-1}(p,p')$$

The connection with the previous concept of regularity for coherent sheaves is established by the following:

Proposition 3.2. *(see [11]) Let $X = \mathrm{Proj}(R)$. For any finitely generated bigraded R-module M we have an exact sequence of bigraded R-modules*

$$0 \longrightarrow H^0_{\mathbf{m}}(M) \longrightarrow M \longrightarrow \bigoplus_{(a,b)\in\mathbb{Z}^2} H^0(X,\, \mathcal{M}(a,b)) \longrightarrow H^1_{\mathbf{m}}(M) \longrightarrow 0$$

and an isomorphism of bigraded R-modules

$$H^{i+1}_{\mathbf{m}}(M) = \bigoplus_{(a,b)\in\mathbb{Z}^2} H^i(X,\, \mathcal{M}(a,b)), \quad \forall i \geq 1$$

Corollary 3.3. *Let $\tilde{M}$ be the sheaf on X associated to the bigraded R-module M, if M is weakly (p,p')-regular, then $\tilde{M}$ is (p,p')-regular in the sense of definition 2.3. This explains the shift in index from i to $i-1$ in the definition of weak regularity for modules.*

The main result for weak regularity is the following:

Theorem 3.4. *Let R be bihomogeneous A-algebra, M a finitely generated bigraded R-module. Fix (p,p').*

1. *Suppose that $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ for all $i \geq 1$ and all $(k,k') \in St_{i-1}(p,p')$, then*

$$H^i_{\mathbf{m}}(M)_{k,k'} = 0 \text{ for all } i \geq 1 \text{ and all } (k,k') \in Reg_{i-1}(p,p')$$

2. *Moreover,*
   a. *if $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ for $(k,k') \in Reg'_{-1}(p,p')$, then we have $R_{d,0}M_{k,k'} = M_{d+k,k'}$ for every $d \geq 0, k \geq p, k' \geq p'$;*
   b. *if $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ for $(k,k') \in Reg''_{-1}(p,p')$, then we have $R_{0,d'}M_{k,k'} = M_{k,k'+d'}$ for every $d' \geq 0, k \geq p, k' \geq p'$.*

3. *if M is weakly (p,p')-regular, and if it satisfies $H^0_{\mathbf{m}}(M)_{k,k'} = 0$ for $(k,k') \in Reg'_{-1}(p,p') \cup Reg''_{-1}(p,p')$, then $R_{d,d'}M_{k,k'} = M_{k+d,k'+d'}$ for all $d,d' \geq 0$, $k \geq p, k' \geq p'$.*

Proof. First, by the same argument as in [14, Theorem 2], we may reduce to the case where A is a local ring with infinite residue field, and assume that $R = A[x_0,\cdots,x_m,y_0,\cdots,y_n]$, with irrelevant ideal $\mathbf{m}$ generated by the x_iy_j. We will prove the claim by induction on (m,n). If either $m=-1$ or $n=-1$ (ie., either x or y variables are missing), or if

$$\mathrm{Ass}_+(M) = \{\mathbf{p} \in \mathrm{Ass}(M) : \mathbf{p} \not\supseteq \mathbf{m}\} = \emptyset$$

the claim is true: in the first case the irrelevant ideal $\mathbf{m} = 0$, so that the remark before the statement of proposition 3.2 applies; in the second case, we have $\mathrm{Supp}(M) \subset V(\mathbf{m})$. In either case, $H^0_{\mathbf{m}}(M) = M$ and $H^i_{\mathbf{m}}(M) = 0$ for every $i \geq 1$.

Suppose that both $m \geq 0$ and $n \geq 0$, and $\mathrm{Ass}_+(M) = \{\mathbf{p}_1,\cdots,\mathbf{p}_r\}$. By our assumptions, A is a um-ring in the terminology of [15], and by theorem 2.3 of that paper we conclude that if we had an equality of A-modules

$$R_{1,0} = \max(A)R_{1,0} \cup (\mathbf{p}_1 \cap R_{1,0}) \cup \cdots \cup (\mathbf{p}_r \cap R_{1,0})$$

then $R_{1,0}$ would have to be equal to one of the terms in the union. It clearly is not the first term. If, say $R_{1,0} = \mathbf{p}_1 \cap R_{1,0}$ we would have

$$\mathbf{m} \subset (x_0,...,x_m) = R \cdot R_{1,0} \subset \mathbf{p}_1$$

which is contrary to the fact that $\mathbf{p}_1$ does not contain $\mathbf{m}$. Thus we can find an element

$$x \in R_{1,0} - \max(A)R_{1,0} \cup (\mathbf{p_1} \cap R_{1,0}) \cup \cdots \cup (\mathbf{p_r} \cap R_{1,0})$$

which we can take as part of a free basis of $R_{1,0}$, and by change of coordinate, we may assume that $x = x_m$.

(1.) Consider the following exact sequence:

$$0 \longrightarrow M_1 \longrightarrow M \xrightarrow{x} xM(1,0) \longrightarrow 0.$$

This implies:

$$H^i_{\mathbf{m}}(M_1) \longrightarrow H^i_{\mathbf{m}}(M) \longrightarrow H^i_{\mathbf{m}}(xM(1,0)) \longrightarrow H^{i+1}_{\mathbf{m}}(M_1). \tag{7}$$

Since x was chosen not to belong to any of the $\mathbf{p}_i$, $\mathrm{Supp}(M_1) \subset V(\mathbf{m})$, and so by the remarks above, the first and last terms above vanish when $i \geq 1$, and so $H^i_{\mathbf{m}}(M) \cong H^i_{\mathbf{m}}(xM(1,0))$ for every $i \geq 1$. Set $\bar{R} = R/xR = A[x_0,\cdots,x_{m-1},y_0,\cdots,y_n]$, $\bar{\mathbf{m}} = \bar{R}_+$ and $\bar{M} = M/xM$. From

$$0 \longrightarrow xM \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0,$$

we have the exact sequence:

$$H^i_{\mathbf{m}}(M)_{k,k'} \longrightarrow H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} \longrightarrow H^{i+1}_{\mathbf{m}}(xM)_{k,k'} = H^{i+1}_{\mathbf{m}}(M)_{k-1,k'}. \tag{8}$$

If $(k,k') \in St_{i-1}(p,p')$, then the first term is 0, by our assumption on M.

Now assume that $i \geq 2$. Then, $(k-1,k') \in St_i(p,p')$ by remark (2.2), and so the last term above is 0, also by our assumption on M, so that $H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$. By induction hypothesis $H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$ for every $i \geq 1$ and $(k,k') \in Reg_{i-1}(p,p')$. If $i \geq 1$ and $(k,k') \in St_{i-1}(p+1,p')$, then in the exact sequence

$$H^i_{\mathbf{m}}(M)_{k-1,k'} = H^i_{\mathbf{m}}(xM)_{k,k'} \longrightarrow H^i_{\mathbf{m}}(M)_{k,k'} \longrightarrow H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} \tag{9}$$

the first and last terms are 0 because $(k-1,k') \in St_{i-1}(p,p')$ and $(k,k') \in Reg_{i-1}(p,p')$, so $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+1,p')$. Repeating the argument we get $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+d,p')$ for every $d \geq 0$, and by symmetry, arguing with a $y \in R_{0,1}$, we get $H^i_{\mathbf{m}}(M)_{k,k'} = 0$ when $i \geq 1$ and $(k,k') \in St_{i-1}(p+d,p'+d')$ for every $d,d' \geq 0$, which is the first claim for $i \geq 2$.

When $i = 1$, the only changes to make in the argument are the following. If $(k, k') \in St_0(p, p')$, then $(k-1, k') \in Reg_1(p, p')$, by formula (2.2). But then $H^2_{\mathbf{m}}(M)_{k-1,k'} = 0$ has been established by the argument in the previous paragraph. Also, when $(k, k') \in St_0(p+1, p')$, we have $(k-1, k') \in St_0(p, p')$ and $(k, k') \in Reg_0(p, p')$, so that the first and last terms in the sequence (9) vanish when $i = 1$, too.

(2a.) Let $\mathrm{Ass}_+(M) = \{\mathbf{p} \in \mathrm{Ass}(M) : \mathbf{p} \not\supseteq \mathbf{m}\}$. Suppose $m, n \geq 0$ and $\mathrm{Ass}_+(M) = \{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$. As before, we change coordinates so that $x = x_m \notin \mathbf{p}_i$, for any i. Set $\bar{R} = R/xR = A[x_0, \cdots, x_{m-1}, y_0, \cdots, y_n]$, $\bar{\mathbf{m}} = \bar{R}_+$ and $\bar{M} = M/xM$. We claim that the induction hypothesis can be applied to $\bar{M}$. First, by the argument proving (1.), we saw that

$$H^i_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0 \text{ for } i \geq 1 \text{ and } (k, k') \in Reg_{i-1}(p, p').$$

From the sequence (8) above with $i = 0$, we see that $H^0_{\bar{\mathbf{m}}}(\bar{M})_{k,k'} = 0$ for every $(k, k') \in Reg'_{-1}(p, p')$, because the extreme terms vanish: the left-hand one because of our assumption on M, the right-hand one because $(k-1, k') \in Reg_0(p, p')$ by remark (2.2) and vanishing of this term has been established above. Thus by induction hypothesis applies to $\bar{M}$, and we have $\bar{R}_{d,0}\bar{M}_{k,k'} = \bar{M}_{d+k,k'}$, which implies $R_{d,0}M_{k,k'} + xM_{d+k-1,k'} = M_{d+k,k'}$. Reasoning by induction on $d \geq 1$, we assume that $M_{d+k-1,k'} = R_{d-1,0}M_{k,k'}$ has been established, the case $d = 1$ being trivial. Then

$$M_{d+k,k'} = R_{d,0}M_{k,k'} + xM_{d+k-1,k'} = R_{d,0}M_{k,k'} + xR_{d-1,0}M_{k,k'} = R_{d,0}M_{k,k'}.$$

This proves our claim. By symmetry, arguing with a y_n, we get the assertion $M_{k,d'+k'} = R_{0,d'}M_{k,k'}$.

(3.) This follows by repeated application of (2a) and (2b). □

For bigraded ideals in the polynomial ring $R = K[x_0, \dots, x_m, y_0, \dots, y_n]$, we have:

Proposition 3.5. *Let K be a field, let $I \subset R$ be any ideal generated by bihomogeneous polynomials, let $\mathcal{I}$ be the corresponding sheaf of ideals in $\mathcal{O}_X$. The following properties are equivalent.*

- (I) *The ideal I is weakly (p, p')-regular in the sense of Definition 3.1.*
- (II) *The natural map $I_{p,p'} \to H^0(\mathcal{I}(p, p'))$ is an isomorphism and $\mathcal{I}$ is (p, p')-regular in the sense of Definition 2.3.*
- (III) *The natural map $I_{d,d'} \to H^0(\mathcal{I}(d, d'))$ is an isomorphism and $\mathcal{I}$ is (d, d')-regular, for all $d \geq p, d' \geq p'$.*

Proof. No loss in generality in assuming that K is infinite, because we may tensor the whole situation by the algebraic closure of K.

(I $\Rightarrow$ II) If I is weakly (p, p')-regular in the sense of Definition 3.1, then we have $H^i_{\mathbf{m}}(I)_{k,k'} = 0$ for $(k, k') \in Reg_{i-1}(p, p')$ for $i \geq 1$. But for an ideal in a polynomial ring, we also have $H^0_{\mathbf{m}}(I) = 0$, since there are no 0-divisors in ring R. By Proposition 3.2, $H^i(\mathcal{I}(k, k')) = H^{i+1}_{\mathbf{m}}(I)_{k,k'} = 0$ for $i \geq 1$, $(k, k') \in Reg_i(p, p')$ for $i \geq 1$; and $I_{p,p'} \cong H^0(\mathcal{I}(p, p'))$.

(II $\Rightarrow$ I) If $\mathcal{I}$ is (p, p')-regular in the sense of Definition 2.3, then $H^i(\mathcal{I}(k, k')) = 0$ for $(k, k') \in Reg_i(p, p')$, $i \geq 1$ by Proposition 2.7. I is an ideal, $H^0_{\mathbf{m}}(I) = 0$, in particular, $H^0_{\mathbf{m}}(I)_{k,k'} = 0$ for $(k, k') \in Reg'_{-1}(p, p') \cup Reg''_{-1}(p, p')$. Since $I_{p,p'} \cong H^0(\mathcal{I}(p, p'))$, by Proposition 3.2, we have $H^1(\mathcal{I})_{p,p'} = 0$, and $H^{i+1}_{\mathbf{m}}(I)_{k,k'} = H^i(\mathcal{I}(k, k')) = 0$ for $(k, k') \in Reg_i(p, p')$, $i \geq 1$. Therefore $H^i_{\mathbf{m}}(I)_{k,k'} = 0$ for all $(k, k') \in Reg_{i-1}(p, p')$, $i \geq 0$, i.e. I is weak (p, p')-regular in the sense of Definition 3.1.

(II $\Rightarrow$ III) follows from Proposition 2.7, and Proposition 2.8.

(III $\Rightarrow$ II) is obvious, we just take $d = p, d' = p'$. □

4. Strong regularity for bigraded modules

From now on, K is a field and $R = K[x_0, \cdots, x_m, y_0, \cdots, y_n] = K[x, y]$ is a polynomial algebra, bigraded in the usual way. We will be using the ideals $(x) = (x_0, \cdots, x_m)$, $(y) = (y_0, \cdots, y_n)$, $(x, y) = (x_0, \cdots, x_m, y_0, \cdots, y_n)$, and $(xy) = \mathbf{m} = (x_i y_j)$.

In addition to the graded $K[x, y]$-module $M^\sharp$ introduced above, we need to consider graded modules as follows. Fix j', and let $M^{[1]}_{j'} = \oplus_j M_{j,j'}$, which is a $K[x] = K[x_0, \cdots, x_m]$-module; fix j, and let $M^{[2]}_j = \oplus_{j'} M_{j,j'}$, which is a $K[y] = K[y_0, \cdots, y_m]$-module. Observe that

$$M = \bigoplus_{j'} M^{[1]}_{j'} = \bigoplus_{j} M^{[2]}_{j}$$

as $K[x]$-module (resp. as $K[y]$-module). Also, each $H^i_{(x)}(M^{[1]}_{j'})$ is a graded $K[x]$-module (resp. each $H^i_{(y)}(M^{[2]}_j)$ is a graded $K[y]$-module), but both $H^i_{(x)}(M)$ and $H^i_{(y)}(M)$ are bigraded $K[x, y]$-modules.

$$H^i_{(x)}(M) = \bigoplus_{j'} H^i_{(x)}(M^{[1]}_{j'})$$
$$H^i_{(y)}(M) = \bigoplus_{j} H^i_{(y)}(M^{[2]}_{j})$$
$$H^i_{(x)}(M)_{j,j'} = H^i_{(x)}(M^{[1]}_{j'})_j$$
$$H^i_{(y)}(M)_{j,j'} = H^i_{(y)}(M^{[2]}_{j})_{j'}$$

Definition 4.1. Let M be a bigraded R-module and let $d \geq 0$.

(I) M satisfies the *vanishing condition* $VC_d(p, p')$ if for all $i \geq 0$

$$\begin{aligned} H^i_{(x)}(M)_{k,k'} &= H^i_{(x)}(M^{[1]}_{k'})_k = 0, \ \forall k \geq p + d + 1 - i, \forall k'; \\ H^i_{(y)}(M)_{k,k'} &= H^i_{(y)}(M^{[2]}_{k})_{k'} = 0, \ \forall k' \geq p' + d + 1 - i, \forall k; \\ & \quad H^i_{(x,y)}(M^\sharp)_{k+k'} = 0, \ \forall k + k' \geq p + p' + d + 1 - i. \end{aligned}$$

(II) M is *(p, p')-regular* if M satisfies $VC_0(p, p')$.

Remark 4.2. For all p, p', we have

1. If M satisfies $VC_0(p, p')$, then M satisfies $VC_d(p, p')$ for all $d \geq 0$.
2. If M satisfies $VC_d(p, p')$, then $M(a, b)$ satisfies $VC_d(p - a, p' - b)$.
3. For all $(\alpha, \alpha') \in DReg_d(p, p')$, if M satisfies $VC_0(\alpha, \alpha')$, then M satisfies $VC_d(p, p')$.

Proposition 4.3. *Let $R = K[x_0, ..., x_m, y_0, ..., y_n]$ be a bigraded polynomial algebra over a field K. Assume that $m, n \geq 0$. Then R is strongly $(0, 0)$-regular.*

Proof. If $R = K[z_1, ..., z_s]$ is any polynomial algebra over a field K, it is a classical fact, due essentially to Serre, that $H^i_{(z)}(R)_k = 0$ whenever $i + k \geq 1$. This verifies the vanishing statement for $H^i_{(x,y)}(R^\sharp)_{k+k'}$ for $R = K[x, y]$. For the case $H^i_{(x)}(R^{[1]}_{k'})_k$, note that

$$R^{[1]}_{k'} = \bigoplus_{|\beta| = k'} K[x] y^\beta.$$

As each term is a free module over $K[x]$ and local cohomology commutes with direct sum, the requisite vanishing follows from Serre's result. $\square$

Proposition 4.4. *If a bigraded R-module M satisfies $VC_d(p,p')$, then $H^i_{(xy)}(M)_{k,k'}=0$ for all $(k,k')\in Reg_0(p+d+1-i,p'+d+1-i), 0\le i\le d+2$; and for all $(k,k')\in Reg_{i-d-1}(p,p'), i\ge d+2$.*

Proof. By the Mayer-Vietoris sequence, we have

$$H^i_{(x,y)}(M) \longrightarrow H^i_{(x)}(M)\oplus H^i_{(y)}(M) \longrightarrow H^i_{(xy)}(M) \longrightarrow H^{i+1}_{(x,y)}(M)$$

(see [10, Exercise 2.4, Ch. III, p. 212]; Note that if $Y_1=V(x)$, $Y_2=V(y)$, then $Y_1\cup Y_2=V(xy)$ and $Y_1\cap Y_2=V(x,y)$ as subsets of $C=\mathrm{Spec}(K[x,y])$.) Assuming that M satisfies $VC_d(p,p')$ we see that $H^i_{(xy)}(M)_{k,k'}=0$ for all (k,k') that satisfy the inequalities:

$$k\ge p+d+1-i,\ \ k'\ge p'+d+1-i,\ \ k+k'\ge p+p'+d-i.$$

If $0\le i\le d+2$ the last condition above is redundant, and so we obtain vanishing in the region described by the first two inequalities, which is just $Reg_0(p+d+1-i,p'+d+1-i)$. If $i\ge d+2$, these three inequalities describe $Reg_{i-d-1}(p,p')$. □

Corollary 4.5. *If M is strongly (p,p')-regular, then it is weakly (p,p')-regular.*

Proof. We have $H^i_{\mathbf{m}}(M)_{k.k'}=0$ for all $(k,k')\in Reg_{i-1}(p,p')$, according to the proposition, whenever $i\ge 2$. For $i=0,1$, this is zero for $(k,k')\in Reg_0(p+1-i,p'+1-i)$, but these are exactly the regions $Reg_{-1}(p,p')$ and $Reg_0(p,p')$. Thus we have the conditions for weak (p,p')-regularity. □

Remark 4.6. If M is strongly (resp. weakly) (p,p')-regular, then $M(a,b)$ is strongly (resp. weakly) $(p-a,p'-b)$-regular.

Proposition 4.7. *If a finitely generated bigraded R-module M satisfies $VC_d(p,p')$, then M is generated by elements of bidegree $(k,k')\in DReg_d(p,p')$.*

Proof. Let A be a homogeneous algebra in the sense of Ooishi's paper (see introduction to [14]), with maximal ideal P. If N is a finitely generated graded module over A, then [14, Thm. 2] asserts that if $H^i_P(N)_k=0$ for all $i+k\ge m+1$, then N is generated in degrees $\le m$.

We first apply this to the graded module $N=M^\sharp$ over the graded ring $A=R^\sharp$. Since M satisfies $VC_d(p,p')$, we have

$$H^i_{(x,y)}(M^\sharp)_{k+k'}=0,\ \forall k+k'\ge p+p'+d+1-i$$

so that by the previous remark, $M^\sharp$ can be generated by elements of degree $\le p+p'+d$. This means that the bigraded M can be generated by bihomogeneous elements of bidegree (k,k') with $k+k'\le p+p'+d$. Now let $A=K[x]$, and for a fixed k', regard $N=M^{[1]}_{k'}$ as an A-module. That M satisfies $VC_d(p,p')$ means here that

$$H^i_{(x)}(M^{[1]}_{k'})_k=0,\ \forall k\ge p+d+1-i$$

and thus by Ooishi's result, that $M^{[1]}_{k'}$ can be generated as $K[x]$-module by elements of degree $\le p+d$. This being true for every k', we see that

$$R_{s,0}M_{p+d,k'}=M_{p+d+s,k'}\quad \text{for all } s\ge 0,k'.$$

Similar reasoning applied to $M^{[2]}_k$ as an $K[y]$-module leads to

$$R_{0,s}M_{k,p'+d}=M_{k,p'+d+s}\quad \text{for all } s\ge 0,k.$$

Combining this information gives that M can be generated by bihomogeneous elements of degree (k,k') where

$$k\le p+d,\quad k'\le p'+d\quad k+k'\le p+p'+d$$

This is the description of the region $DReg_d(p,p')$. □

If M_d is a bigraded R module which satisfies $VC_d(p,p')$, by Proposition 4.7, M_d is generated by elements of bidegree $e_{\alpha,d} = (\alpha_d, \alpha'_d) \in DReg_d(p,p')$. We can find an exact sequence:

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0,$$

where $M_{d+1} = \ker \phi_d$.

Proposition 4.8. *Let M_d be as above. If M_d satisfies $VC_d(p,p')$, then M_{d+1} satisfies $VC_{d+1}(p,p')$, and therefore are generated by elements of bidegree in $DReg_{d+1}(p,p')$.*

Proof. For the case $i = 0$, we have an injection

$$H^0_{(x)}(M_{d+1}) \subset \bigoplus_{\alpha=1}^{r_d} H^0_{(x)}(R)_{k-\alpha_d,k'-\alpha'_d} = 0,$$

so we can assume that $i \geq 1$. Consider the local cohomology sequence with $I = (x)$ of the above exact sequence:

$$H^{i-1}_I(M_d)_{k,k'} \longrightarrow H^i_I(M_{d+1})_{k,k'} \longrightarrow \bigoplus_{\alpha=1}^{r_d} H^i_I(R)_{k-\alpha_d,k'-\alpha'_d}$$

Suppose that $k + i \geq p + (d+1) + 1$. Then the left-hand side above vanishes by assumption on M_d, because $k + (i-1) \geq p + d + 1$. That (α_d, α'_d) belongs to $DReg_d(p,p')$ means that $\alpha_d \leq p + d$. Thus, $k - \alpha_d + i \geq 2$, and since R is $(0,0)$-regular by Proposition (4.3), the last term vanishes.

By similar reasoning, we get the vanishing of $H^i_{(y)}(M_{d+1})_{k,k'}$ for $k' + i \geq p' + (d+1) + 1$, for all k.

Now look at the local cohomology sequence with $I = (x,y)$. Again we may assume that $i \geq 1$. If (k,k') satisfies $k + k' + i \geq p + p' + (d+1) + 1$, our assumption on M_d shows the vanishing of the left-hand side because $k + k' + (i-1) \geq p + p' + d + 1$. That (α_d, α'_d) belongs to $DReg_d(p,p')$ means that $\alpha_d + \alpha'_d \leq p + p' + d$, so that $k + k' - \alpha_d - \alpha'_d + i \geq 2$. Thus the right-hand side vanishes because R is $(0,0)$-regular.

In all three cases we have verified vanishing in the appropriate region to satisfy $VC_{d+1}(p,p')$. □

Conversely:

Proposition 4.9. *Let M_{d+1} be a finitely generated bigraded R-module. If M_{d+1} satisfies $VC_{d+1}(p,p')$ and if there is an exact sequence:*

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0,$$

where $M_{d+1} = \ker \phi_d$, and $e_{\alpha,d} = (\alpha_d, \alpha'_d) \in DReg_d(p,p')$, then M_d satisfies $VC_d(p,p')$. Therefore M_d is generated by elements of bidegree in $DReg_d(p,p')$.

Proof. Let I be any one of the ideals (x), (y), (x,y). Look at the segment of the local cohomology sequence associated with the above exact sequence:

$$\bigoplus_{\alpha=1}^{r_d} H^i_I(R)_{k-\alpha_d,k'-\alpha'_d} \longrightarrow H^i_I(M_d)_{k,k'} \longrightarrow H^{i+1}_I(M_{d+1})_{k,k'}$$

Let $I = (x)$. and suppose $k + i \geq p + d + 1$, we have $k + (i+1) \geq p + (d+1) + 1$, and the last group vanishes by assumption on M_{d+1}. Also, in this region, $k - \alpha_d + i \geq 1$, and the first term vanishes by Proposition (4.3).

By similar reasoning, we obtain the vanishing of $H^i_{(y)}(M_d)_{k,k'}$ if $k'+i \geq p'+d+1$.

For $I=(x,y)$, suppose $k+k'+i \geq p+p'+d+1$. We have $k+k'+(i+1) \geq p+p'+(d+1)+1$, so that the last group vanishes by assumption on M_{d+1}. Also, because $\alpha_d+\alpha'_d \leq p+p'+d$, we get $k+k'-\alpha_d-\alpha'_d-i \geq 1$, last term vanishes because R is $(0,0)$-regular.

In all three cases we have verified vanishing in the appropriate region to satisfy $VC_d(p,p')$. □

We prove some equivalent conditions for regularity of a module, as in [2] and [1]. In the formulation below, R is a polynomial algebra over K in two sets of variables x and y bigraded in the usual way. We assume both variable sets are nonempty.

Theorem 4.10. *Let M be a finitely generated bigraded module over R. The following properties are equivalent.*

(I) *M is (p,p')-regular in the sense of definition 4.1.*

(II) *The minimal resolution of M by free bigraded $R=K[x,y]$-modules:*

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \longrightarrow M \longrightarrow 0,$$

satisfies $e_{\alpha,d}=(\alpha_d,\alpha'_d) \in DReg_d(p,p')$.

(III) *There exists a free resolution with the properties above.*

Proof. (I ⇒ II) Let $M_0=M$. We will inductively construct a sequence of bigraded modules M_d that satisfy $VC_d(p,p')$ and that fit into an exact sequence

$$0 \longrightarrow M_{d+1} \longrightarrow \bigoplus_{\alpha=1}^{r_d} Re_{\alpha,d} \xrightarrow{\phi_d} M_d \longrightarrow 0 \tag{10}$$

where $e_{\alpha,i}=(\alpha_i,\alpha'_i) \in DReg_d(p,p')$. By Proposition (4.8), we know that M_{d+1} will satisfy $VC_{d+1}(p,p')$ and therefore we can find generators for it whose bidegrees are in $DReg_{d+1}(p,p')$. In other words, we may construct the above exact sequence with but d replaced by $d+1$. By Hilbert's syzygy theorem, M_d will become a free bigraded module, with generators in $DReg_d(p,p')$, and by splicing these short sequences together, we get our resolution. We can start this induction at $d=0$, because by hypothesis, $M=M_0$ is (p,p')-regular, and by Proposition (4.7), we know M_0 is generated by elements whose bidegrees are in $DReg_0(p,p')$.

(II ⇒ III) is trivial.

(III ⇒ I) Break the given resolution into short sequences as in equation (10) above. We will show by descending induction on d that M_d satisfies $VC_d(p,p')$. Since the last stage of this, namely M_0, is the module M itself, we will be done, since the condition $VC_0(p,p')$ is exactly (p,p')-regularity. The starting point of the induction is the extreme left-hand term of the resolution $M_s=\oplus_{\alpha=1}^{r_s} Re_{\alpha_s,\alpha'_s}$. Because R is $(0,0)$-regular by Proposition (4.3), and because of Remark (4.2), we see that M_s satisfies $VC_s(p,p')$. If $d<s$ and we assume by induction that M_{d+1} satisfies $VC_{d+1}(p,p')$, from the exact sequence (10) and Proposition (4.9), we find that M_d satisfies $VC_d(p,p')$, verifying the induction step. □

Corollary 4.11. *Any finitely generated bigraded module over $K[x,y]$ is (p,p')-regular for some p,p'.*

Proof. Look at the minimal free bigraded resolution of M, which we know exists and is unique up to isomorphism. Whatever are the bidegrees $e_{\alpha,d}$ of the generators of the various terms in this, it is rather clear that by taking p and p' sufficiently large, for all d these will belong to the region $DReg_d(p,p')$. □

Remark 4.12. Let $I \subset K[x_0, y_0, \cdots, y_n]$ be an ideal such that $I = x_0^m J$ where $J \subset K[y_0, \cdots, y_n]$ is a homogeneous ideal, where K is an infinite filed. Then I is (p, p')-regular if and only if $p \geq m$ and J is p'-regular.

Proof. Suppose J is p'-regular and $p \geq m$, we would like to show that I is (p, p')-regular. Let $R = K[y_0, \cdots, y_n]$, and take a minimal free resolution of J as follows:

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Re_{\alpha,s} \longrightarrow \cdots \xrightarrow{d_1} \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0. \tag{11}$$

Since J is p'-regular, then we have $e_{\alpha,i} \leq p' + i$. Also, note the map d_i is represented by a matrix. Since the free resolution is minimal, then the matrix has no entry in K^* where $K^* = K \setminus \{0\}$. [18, Proposition 11.5]

We can break this exact sequence into two exact sequences:

$$0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow \ker(d_0) \longrightarrow 0, \tag{12}$$

and

$$0 \longrightarrow \ker(d_0) \longrightarrow \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0. \tag{13}$$

If we let $S = K[x_0, y_0, \cdots, y_n]$, we have

$$0 \longrightarrow \ker(x_0^m d_0) \longrightarrow \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d_0} S \longrightarrow S/I \longrightarrow 0. \tag{14}$$

If we tensor the exact sequence (12) with S over R, since S is flat, then we will have an exact sequence:

$$0 \longrightarrow C_n \otimes S \longrightarrow \cdots \longrightarrow C_1 \otimes S \longrightarrow \ker(d_0) \otimes S \longrightarrow 0. \tag{15}$$

Note, at each stage, the matrix which represents the map has no entry in K^*. Since $\ker(x_0^m d_0) = S \otimes_R \ker(d_0)$, we can piece exact sequence (14) and (15) together, we will form a free resolution of I as follow:

(16)

$$0 \longrightarrow C_n \otimes S \longrightarrow \cdots \longrightarrow C_1 \otimes S \longrightarrow \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d_0} S \longrightarrow S/I \longrightarrow 0.$$

This free resolution is minimal, since the matrix represents the map has no entry in K^*. And we can rewrite the minimal free resolution (16) as follow:

$$0 \longrightarrow \bigoplus_{\alpha=1}^{r_s} Se'_{\alpha,s} \longrightarrow \cdots \xrightarrow{d'_1} \bigoplus_{\alpha=1}^{r_0} Se'_{\alpha,0} \xrightarrow{d'_0} S \longrightarrow S/I \longrightarrow 0 \tag{17}$$

where $d'_0 = x_0^m d_0$, and $d'_1 = d_1$, and $e'_{\alpha,i} = (-m, e_{\alpha,i})$. If $m \leq p$ and $e_{\alpha,i} \leq p' + i$, by the equivalent relation of minimal free resolution and (p, p')-regular, we know that I is (p, p')-regular.

On the other hand, suppose I is (p, p')-regular, there is a minimal free resolution of I as (17), where $e'_{\alpha,i} = (a_{\alpha,i}, e_{\alpha,i})$ and $a_{\alpha,i} \leq p$ and $e_{\alpha,i} \leq p' + i$. Note, at each stage, the matrix represents the map has no entry in K^*. And we can split the free resolution into two exact sequences: the free resolution of I (14) and

$$0 \longrightarrow D_n \xrightarrow{d'_n} \cdots \longrightarrow D_1 \xrightarrow{d'_1} \ker(x_0^m d_0) \longrightarrow 0, \tag{18}$$

We always have a resolution of J as (13). Since $\ker(x_0^m d_0) = S \otimes_R \ker(k_0)$, we will have an exact sequence as follow:

$$0 \longrightarrow C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} \ker(d_0) \longrightarrow 0, \tag{19}$$

where $d_i = d'_i$. We can piece the two exact sequences (19) and (13) together to get:

$$0 \longrightarrow C_n \xrightarrow{d_n} \cdots \longrightarrow C_1 \xrightarrow{d_1} \bigoplus_{\alpha=1}^{r_0} Re_{\alpha,0} \xrightarrow{d_0} R \longrightarrow R/J \longrightarrow 0,$$

which can be written as (11). Since the matrix represents d_i has no entry in K^*, the free resolution (11) is minimal, and $e_{\alpha,i} \leq p' + i$. As is well-known, the existence of a free resolution of this type implies that J is p'-regular. □

5. Resolutions

Let $R = K[x_0, ..., x_m, y_0, ..., y_n]$, a polynomial algebra over a field, bigraded in the usual way. Let $\mathbf{m} = (x_i y_j)$, the irrelevant ideal. We will define a complex that allows the computation of the local cohomology modules $H^i_{\mathbf{m}}(M)$. Recall that for any ideal I in a ring R we have

$$H^i_I(M) = \operatorname{inj}\lim_{\nu} \operatorname{Ext}^i_R(R/I^{(\nu)}, M)$$

where $I^{(\nu)}$ is any sequence of ideals cofinal with the collection of powers I^ν. If $R = K[z_1, ..., z_n]$ is a polynomial ring and $I = (z_1, ..., z_n)$, then we can take $I^{(\nu)} = (z_1^\nu, ..., z_n^\nu)$. Since this is generated by a regular sequence, we can compute the Ext groups by using the Koszul complex on $z^\nu = \{z_1^\nu, ..., z_n^\nu\}$ as a free resolution of $R/I^{(\nu)}$:

$$K_*(z^\nu) \longrightarrow R/I^{(\nu)}.$$

This means that we have a free resolution by the truncated complex:

$$K_{\geq 1}(z^\nu) \longrightarrow I^{(\nu)}$$

Let, as before, $\mathbf{m}_1 = (x_0, ..., x_m) \subset R_1 = K[x_0, ..., x_m]$, $\mathbf{m}_2 = (y_0, ..., y_n) \subset R_2 = K[x_0, ..., y_n]$, with $\mathbf{m}_1^{(\nu)} = (x_0^\nu, ..., x_m^\nu)$, with similar notation for $\mathbf{m}_2^{(\nu)}$, $\mathbf{m}^{(\nu)}$ If M_1 is an R_1-module and M_2 is an R_2-module, we let $M_1 \boxtimes M_2$ be the R-module $(M_1 \otimes_{R_1} R) \otimes_R (M_2 \otimes_{R_1} R)$.

Lemma 5.1. *There is a canonical isomorphism $\mathbf{m}_1^{(\nu)} \boxtimes \mathbf{m}_2^{(\nu)} \cong \mathbf{m}^{(\nu)}$ for every $\nu \geq 0$.*

Proof. Clearly there is an epimorphism $\mathbf{m}_1^{(\nu)} \boxtimes \mathbf{m}_2^{(\nu)} \to \mathbf{m}^{(\nu)}$, so the issue is the injectivity of this map. Because R is a flat R_1-module, we have an exact sequence

$$0 \longrightarrow \mathbf{m}_1^{(\nu)} \otimes_{R_1} R = \mathbf{m}_1^{(\nu)} R \longrightarrow R \longrightarrow R/\mathbf{m}_1^{(\nu)} R \longrightarrow 0.$$

Applying $- \otimes_R \mathbf{m}_2^{(\nu)} R$, we see that the lemma holds if we can show that

$$\operatorname{Tor}_1^R(\mathbf{m}_2^{(\nu)} R,\ R/\mathbf{m}_1^{(\nu)} R) = 0$$

We can use the truncated Koszul complex on $\mathbf{m}_2^{(\nu)} R$ to compute the Tor. Since the elements $y_0^\nu, ..., y_n^\nu$ are a regular sequence on $R/\mathbf{m}_1^{(\nu)} R$, this Tor vanishes as claimed. □

As we have mentioned, there are Koszul resolutions

$$K_{\geq 1}(x^\nu) \longrightarrow \mathbf{m}_1^{(\nu)}, \quad K_{\geq 1}(y^\nu) \longrightarrow \mathbf{m}_2^{(\nu)}$$

Proposition 5.2.

$$K_{\geq 1}(x^\nu) \boxtimes K_{\geq 1}(y^\nu) \longrightarrow \mathbf{m}_1^{(\nu)} \boxtimes \mathbf{m}_2^{(\nu)} = \mathbf{m}^{(\nu)}$$

is a finite resolution by free R-modules of finite type.

Proof. The only thing to be proved is that it actually is a resolution. Since R is a flat R_j-module, $j = 1, 2$, we get free resolutions of R-modules:

$$A_* := K_{\geq 1}(x^\nu) \otimes_{R_1} R \longrightarrow \mathbf{m}_1^{(\nu)} \otimes_{R_1} R = \mathbf{m}_1^{(\nu)} R, \quad B_* := K_{\geq 1}(y^{(\nu)}) \otimes_{R_2} R \longrightarrow \mathbf{m}_2^{(\nu)} \otimes_{R_2} R = \mathbf{m}_2^{(\nu)} R$$

Clearly we have an epimorphism $K_{\geq 1}(x^\nu) \boxtimes K_{\geq 1}(y^\nu) = A_* \otimes_R B_* \to \mathbf{m}^{(\nu)}$, so we must prove that

$$\mathrm{H}_i(A_* \otimes_R B_*) = 0, \text{ for all } i \geq 1.$$

Since the A_r are free R-modules, we have the Künneth spectral sequence (see [3, Ch. XVII, 5.2a], [5, I, 5.5.2], [8, Lemme (1.1.4.1)]):

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(A_* \otimes_R \mathrm{H}_q(B_*))) \Longrightarrow \mathrm{H}_{p+q}(A_* \otimes_R B_*)$$

These are zero when $q \geq 1$ since B_* is a resolution. We must see that they are 0 when $p \geq 1$, or that

$$\mathrm{H}_p(A_* \otimes_R \mathbf{m}_2^{(\nu)} R) = \mathrm{H}_{p+1}(K_*(x^\nu) \otimes_R \mathbf{m}_2^{(\nu)} R) = \mathrm{Tor}^R_{p+1}(R/\mathbf{m}_1^{(\nu)} R, \mathbf{m}_2^{(\nu)} R) = 0, \ p \geq 1.$$

This follows because as noted in the previous lemma, the elements $y_0^\nu, ..., y_n^\nu$ form a regular sequence in $R/\mathbf{m}_1^{(\nu)} R$. □

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