The paper under review is an introduction to the Macaulay2 package `VirtualResolutions`. The first half of the paper introduces the following functionality.

(1) The function `multigradedRegularity` computes the minimal elements of the multigraded regularity of a graded module over S, the cox ring of products of projective spaces.
(2) The function `virtualOfPair` computes the virtual resolution of a graded module over S with respect to an element d of the multigraded regularity.
(3) The function `isVirtual` checks whether a complex is a virtual resolution.
(4) The function `resolveViaFatPoint` gives an alternative way of producing virtual resolutions in the zero dimensional case.

The second half of the paper is concerned with demonstrating the above functionality on the examples of curves in $\mathbf{P}^2 \times \mathbf{P}^1$. Essentially the authors produced "random" ideals of curves, and then computed their multigraded regularity.

The overall conclusion of the reviewer is that this paper did not do a satisfactory job in introducing this package. Therefore the reviewer cannot recommend this paper for publication before a significant rewrite of both the package and the paper is made. The following is a summary of the major issues.

(1) While the paper demonstrates each functionality via an example, almost nothing is said about the inner workings. The reader is presented with a few blackboxes and has very little idea on what each function actually does to produce the outputs.
(2) Some current implementations of the functions are not satisfactory (for example `virtualOfPair`), and in some cases wrong (for example `isVirtual`). See the detailed comments below.
(3) The second half of the paper produces some outputs, but no effort is made to address the following: Why do we produce these multigraded regularities? What is a plausible conjecture here? What experimental evidence can we provide for this conjecture? Since there is no investigation of the origins of the curves in $\mathbf{P}^3$, no reasonable conjectures of their images in $\mathbf{P}^2 \times \mathbf{P}^1$ can be and should be made. See more detailed comments below.

Here are the more detailed comments.

Many references to specific theorems in [BES17] are no longer accurate, presumably due to an update in the reference.

While the paper illustrates what `multigradedRegularity` does in an example, nothing is said about how it does it. The current explanations are contained within remark 2.3, which basically says that it uses the `cohomologyHashTable` function in another package. More mathematical details need to be included here to give the reader an idea what the computations involve (without including too much technical details).

As the reviewer understands from the submitted codes, the current implementation of `virtualOfPair` simply selects the subcomplex of multi-degrees in a given list (or less than a given multi-degree). Therefore, the full computation of a virtual resolution of a module M actually requires one to first compute its full resolution, then compute its multigraded regularity, and finally select the correct subcomplex from the full resolution to give the virtual resolution. This to an extent defeats the purpose of virtual resolutions. Instead, algorithm 4.4 in [BES17] suggests a different way to approach this computation. Implemented naively, algorithm 4.4 may not defeat the current method, simply because how optimized M2 is at computing free resolutions using Schreyer's method (and its improved versions). Therefore, the natural thing to do is to write a modification of Schreyer's method of finding syzygies, taking into account that only generators of less than certain multidegrees are of interest. It could possibly turn out that this may not end up faster (a comparison is nice nevertheless), but at least an attempt needs to be made here, which is missing from this package.

The function `isVirtual` suffers from a major problem. In the case of ideals, it is relatively easy to test whether I and J has the same B-saturation. However, in the case of modules, it is not enough to check whether $\operatorname{ann} M$ and $\operatorname{ann} N$ has the same B-saturation in order to conclude whether or not $\tilde{M} \cong \tilde{N}$. In fact, as far as the reviewer knows, there is no method to check whether or not two modules are isomorphic. A way out of this problem is to simply test whether $F_\bullet$ is a virtual resolution (naturally of $H_0(F_\bullet)$ or any module with the same B-saturation), instead of testing whether $F_\bullet$ is a virtual resolution of a given module M.

There are two methods of checking whether a resolution is virtual implemented within `isVirtual`. One is based on [Lop19], and the other checks whether $H_i(F_\bullet)$ is supported only on B for all $i > 0$. A brief description and comparison of these two methods are missing in the paper. More importantly, anywhere the paper states "supported on B" should be replaced with "supported *only* on B". Given that a module M is $\mathbb{Z}^n$ graded, so will be $\operatorname{ann} M$. Thus M is supported only on B iff $\operatorname{ann} M$ is B-primary, in other words, it is enough to check whether m^N annihilates M for each generator m in B for some N big. This should be better than computing the saturation using elimination. If it does turn out that the current method were faster, then a paragraph pointing this out would be nice.

For the second half of the paper, the authors used "random curves in $\mathbf{P}^3$" to produce "random curves" in $\mathbf{P}^2 \times \mathbf{P}^1$, and computed their multigraded regularity. However, since there is no explanation of where the "random curves" of degree 7 and genus 3 in $\mathbf{P}^3$ come from, no meaningful information can be possibly drawn from the tally of the multigraded regularity. No attempted explanation is given regarding whether the different multigraded regularity has to do with the different types of curves on $\mathbf{P}^3$, i.e. those on a quadric surface, on a smooth cubic surface and on a quartic surface etc?

All curves in $\mathbf{P}^3$ in the `SpaceCurves` package are produced using planar models with prescribed multiplicities at given points. So essentially all "random curves" in $\mathbf{P}^2 \times \mathbf{P}^1$ produced in this package are the images of plane curves under a rational map $\mathbf{P}^2 \to \mathbf{P}^2 \times \mathbf{P}^1$. This can be generalized to arbitrary products of projective spaces. One could write a function that maps (rationally) plane curves with suitable multiplicities at given points to $\mathbf{P}^{n_i}$ by certain linear systems (of the blowup of $\mathbf{P}^2$ at these points). This way, meaningful conclusions may be drawn from the multigraded regularity of their images.

For example, if C is a plane curve of degree d with multiplicity p_i at the point P_i. Let $f_j : \mathbf{P}^2 \to \mathbf{P}^{n_j}$ be given by the linear system of plane curves of degree m_j passing through P_i with multiplicity q_i^j for each $j = 1, \dots, r$. Do all such C have images with the same multigraded regularity in $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_r}$?