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Algebraic and Analytic Geometry of Fans

Carlos Andradas
Jesús M. Ruiz

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Abstract

A set which can be defined by systems of polynomial inequalities is called *semialgebraic*. When such a description is possible locally around every point, by means of analytic inequalities varying with the point, the set is called *semianalytic*. If one single system of strict inequalitites is enough, either globally or locally at every point, the set is called *basic*. The topic of this work is the relationship between these two notions. Namely, we describe and characterize, both algebraically and geometrically, the obstructions for a basic semianalytic set to be basic semialgebraic. Then, we describe a special family of obstructions that suffices to recognize whether or not a basic semianalytic set is basic semialgebraic. Finally, we use the preceding results to discuss the effect on basicness of birational transformations.

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Key words and phrases: Semialgebraic and semianalytic sets, basic and generically basic sets; algebraic and analytic fans, trivialization and centers of an algebraic fan; henselization, amalgamation, going-down for fans; fan approximation lemma, birational blowing-down.

Introduction

Let X be a real algebraic variety. A subset $S \subset X$ is called semialgebraic if it can be written as

$$S = \bigcup_{i=1}^{r} \{x \in X \mid f_{i1}(x) > 0, \ldots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

for some polynomial functions f_{ij}, g_i on X. The "pieces"

$$S_i = \{x \in X \mid f_{i1}(x) > 0, \ldots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

are called *basic* semialgebraic sets, and are seen as the most elementary type of them. In fact, many properties of semialgebraic sets are first shown for basic sets, and then extended to the general case. Now, notice that the basic piece S_i can also be seen as a basic open semialgebraic subset of the algebraic set $X \cap \{g_i = 0\}$. This makes basic open semialgebraic sets the natural object of study. Quite a lot of literature has been devoted to them, specially since Bröcker published his celebrated paper [Br2] containing a characterization of basic semialgebraic sets as well as some estimates on the number of polynomials needed to describe them. He brought into evidence a beautiful interplay between semialgebraic sets, real spectra, spaces of orderings and reduced quadratic forms. These results have also been extended to the analytic category, both in the global setting, [AnBrRz1], for the so-called global semianalytic sets, as well as in the local one, that is, for semianalytic germs, [Rz2], [AnBrRz2]. In both cases, the definitions of basic open sets are obvious and similar to the given above. Also, the spirit of the proofs is identical to the one of Bröcker in the algebraic case, showing the depth of his ideas and the relationship just mentioned.

In these notes we apply once more the same general philosophy, but in a different context: instead of working either in the algebraic or analytic side, we want to understand the behaviour of basicness of semialgebraic sets when we go back and forth into the analytic category, considering them as semianalytic germs at all different points. The results obtained here have

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been announced in [AnRz5] under a purely geometric form which, hidding all technical algebraic aspects, tries to convey intuitive support for the main ideas involved.

To be more precise, let $S \subset X$ be a basic open semialgebraic set,

$$S = \{x \in X \mid f_1(x) > 0, \ldots, f_s(x) > 0\}$$

where $f_1, \ldots, f_s$ are polynomial functions on X. Obviously, for all $a \in X$, the germ S_a is basic open in the semianalytic category. This work originated from the question of when the converse is true. That is, suppose that $S \subset X$ is a semialgebraic set such that for all $a \in X$ the germ S_a is basic open seminanalytic,

Is then S basic semialgebraic?

The answer, in that generality, is in the negative. In fact, being basic semialgebraic is a "global property", and the obstructions for an open semialgebraic set S to be basic can very well be of a "non-local" nature, in the sense that they are not concentrated at any point, see Example 1.1 *b)*. There is no hope of recognizing these obstructions from our hypothesis that the S_a's are basic open semianalytic. Hence, the problem we are interested in can be stated in a much more accurate way as:

Assume that S_a is basic open semianalytic for all $a \in X$. Do there exist "local" obstructions for S to be basic open semialgebraic?

We have been consciously ambiguous about the meaning of "local" and "non-local" obstructions. The precise definition requires the use of *fans* and *valuations*. In fact, first of all, the question of whether S is basic open semialgebraic can be translated, by means of the *tilde* operator (Sections 3,4), into the question of whether the corresponding constructible set $\tilde{S}$ of the *real spectrum* $\operatorname{Spec}_r(\mathcal{P}(X))$, is basic open, where $\mathcal{P}(X)$ denotes the ring of polynomial functions on X. Now, Bröcker's result, [Br3], characterizes basic open constructible subsets C of the real spectrum $\operatorname{Spec}_r(A)$ of any commutative ring A, in terms of distinguished subsets of $\operatorname{Spec}_r(A)$ with a certain combinatorial structure, called *fans.* In particular, the fans of $\operatorname{Spec}_r(\mathcal{P}(X))$ can be seen as the obstructions mentioned above for a constructible set to be basic. A well known deep result asserts that the elements of a fan can specialize to at most two points of X. Thus, a *local fan* should mean that all its elements specialize to the same point $a \in X$, while a *non-local fan* is one whose elements specialize to two different points (may be at infinite). Since the expressions "local" and "non-local" may induce to confussions with other notions, we will not use them anymore, talking instead of *fans with 1 specialization point* and *fans with 2 specialization points*, or for short *1pt-fans* and *2pt-fans.* This gives a precise meaning to the above sentences.

Similarly, the question of whether the germ S_a is basic semianalytic, is translated into the question of whether the *constructible* set $\tilde{S}_a$ of the *real spectrum* $\operatorname{Spec}_r(\mathcal{O}_{X,a})$ is basic, where $\mathcal{O}_{X,a}$ denotes the ring of germs of analytic functions on X_a. Here we can apply, again, Bröcker's fan criterium, this time dealing with fans of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$. Thus we have the following diagram of related topics:

$$\begin{array}{ccccc}
\boxed{\begin{array}{c}\text{Semialgebraic}\\ \text{subsets}\\ \text{of } X\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{Constructible}\\ \text{subsets}\\ \text{of } \operatorname{Spec}_r(\mathcal{P}(X))\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{Fans}\\ \text{of}\\ \operatorname{Spec}_r(\mathcal{P}(X))\end{array}}
\end{array}$$

$$\begin{array}{ccccc}
\boxed{\begin{array}{c}\text{Semianalytic}\\ \text{germs}\\ \text{at } a\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{Constructible}\\ \text{subsets}\\ \text{of } \operatorname{Spec}_r(\mathcal{O}_{X,a})\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{Fans}\\ \text{of}\\ \operatorname{Spec}_r(\mathcal{O}_{X,a})\end{array}}
\end{array}$$

Hence, comparing basicness between the analytic and the algebraic categories becomes comparing the fans of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$ and $\operatorname{Spec}_r(\mathcal{P}(X))$. Thus, the question above can be more properly stated as:

Which are the fans of $\operatorname{Spec}_r(\mathcal{P}(X))$ *that are restriction of fans of* $\operatorname{Spec}_r(\mathcal{O}_{X,a})$*?*

These fans are called *analytic.* As pointed out above, an easy necessary condition for a fan $F \subset \mathrm{Spec}_r(\mathcal{P}(X))$ to be analytic is that F must be a 1pt-fan, so that we can state the final form of our problem as:

Which are the 1pt-fans of $\mathrm{Spec}_r(\mathcal{P}(X))$ *that are analytic?*

In this work we give a complete answer to this question, both in algebraic and in geometric terms. However, to reach that point, quite a bit of background and technicalities are needed. This is the reason why these notes have become so thick that we have changed our original idea and have developed them as a monograph. Indeed, first of all, to understand the mentioned "fan criterion", we need to introduce some background on real spectra, the tilde operator, and the relationship between fans and basic semialgebraic sets. All this is done as shortly as possible in Sections 1-5.

Also, being faced to work with fans, we need the most useful tool to handle them: valuation theory. In fact, it is the connection between valuations and fans what allows us to construct and manipulate the latter. The main points about this interplay are collected in Sections 6 and 7. Specially important is another precious result by Bröcker: the so-called *trivialization theorem for fans*, [Br1], which guarantees not only the existence of valuation rings compatible with a given fan, F, but also that some of them, W, trivializes F, that is, the fan F_W induced in the residue field k_W of W has at most two elements. In Section 7 we pay special attention to the notion of centers of a fan: they are the centers at X of the valuation rings compatible with it. Among them, the *trivialization centers*, that is, the centers of the valuation rings trivializing F, will be of utmost importance for the geometric characterization of analytic fans in Section 14.

Going further, remember that our problem is to study whether a given 1pt-fan F of $\mathcal{P}(X)$ extends to the analytic ring $\mathcal{O}_{X,a}$. Thus, as it is becoming customary, we introduce an intermediate step in between, namely the henselization $\mathcal{P}(X)_a^h$ of the localization of $\mathcal{P}(X)$ at a, giving rise to the notion of the *henselization of a local fan*. This is explained in Section 8. So, we study first the extension of F to $\mathcal{P}(X)_a^h$, and second from this ring to $\mathcal{O}_{X,a}$. Quite surprisingly the difficulties arise at the first level, while the extension from the henselian to the analytic ring goes smoothly, as we prove in Section 9. This is a consequence of the powerful M. Artin's approximation theorem.

Thus, continuing with our process, we have to determine which 1pt-fans of $\mathcal{P}(X)$ extend to the henselization. This is done in Section 12, where we obtain the algebraic characterization of analytic fans: a fan F is analytic if and only if the *amalgamation property* holds for some valuation ring compatible with F, or, to put it in short, some valuation ring compatible with F *amalgamates*. This roughly means that there is a suitable extension of the residue field k_V of V to which the induced fan F_V extends. Unfortunately, it turns out that this property, whose definition and basic facts are introduced in Section 11, is quite hard to check and to deal with, except when the fan F_V has at most two elements. However, a detailed analysis of the extension of valuation rings to the henselization, shows that if F is analytic, then amalgamation always holds for the biggest valuation ring W_F trivializing F. The proof of this result appears in Section 12.

This opens the door to the geometric interpretation of amalgamation. Indeed, curiously enough, under these circumstances, amalgamation can be checked by looking at directed families of real algebraic sets Y, such that the centers of W_F at them are finite coverings of the center of W_F in X. This leads to the definition of finite coverings associated to a fan F (Section 13), and later, to the announced geometric characterization of analytic fans (Section 14), which roughly says that a fan is non-analytic if it becomes a 2pt-fan in some finite covering.

Let us explain the last sentence a bit more. Although basicness of semialgebraic sets S is of birational nature, analyticity of fans is not. For instance it may very well happen that a 1pt-fan F becomes a 2pt-fan after blowing-up X, and conversely, 2pt-fans can be made 1pt-fans by blowing-down. This can be used to translate, at convenience, obstructions to basicness from non-local settings to local ones and conversely. In Section 17, we develope an instance of this, showing that all the obstructions for S to be basic can be concentrated at one point a in a suitable model S', so that S is basic semialgebraic if and only if S'_a is basic semianalytic.

Finally, in Sections 15 & 16 we approach the problem from a different point of view: we introduce approximation of fans. Roughly speaking, if a given fan F produces an obstruction for a given semialgebraic set S to be basic semialgebraic, then any F' close enough to F will produce an obstruction too. In particular, if any fan could be approximated by a 2pt-fan, this would imply that analytic fans are redundant to decide whether S is basic semialgebraic or not. Thus, a natural question is to decide which 1pt-fans can be approximated by 2pt-fans. First, we find a dense distinguished family of fans: the so-called *parametric fans*. This is a consequence of the approximation lemma proven in Section 15: any fan can be arbitrarily approximated by a parametric fan. Therefore, parametric fans suffice to check basicness of semialgebraic sets. Without entering in the formal definition, we just say that they are fans obtained by pulling back trivial fans by some special discrete valuation rings.

Then, in Section 16, we prove that approximation by parametric fans behaves also well with respect to analyticity, in the sense that analytic (resp. non-analytic) fans, can be arbitrarily approximated by analytic (resp non-analytic) parametric fans. Also, we see that any fan F can be approximated by a 2pt-fan, except in case all the trivialization centers of F are reduced to one point. Moreover, if F is parametric fan verifying this condition, then any close enough approximation F' also verifies it, and, in particular, it is a 1pt-fan. This indicates, somehow, that these *strictly 1pt-fans* contain the essential analytic information.

In order to put this idea into the geometric terms of the beginning of this introduction, consider the Alexandroff compactification $X^* = X \cup \infty$ of X. Let $S \subset X$ be an open semialgebraic set which is basic semianalytic at all points $a \in X$. Then S is basic semialgebraic if and only if it is basic semianalytic at ∞ and verifies Bröcker's fan criterion for 2pt-fans.

This work was started in 1990, at the Berkeley campus of the University of California, during the Special Year on Real Algebraic Geometry and Quadratic Forms, organized by Professors T.Y. Lam and R. Robson. We enjoyed their hospitality and are glad to thank them the oportunity to participate in the seminars that were held there. We also thank RAGSQUADers who attended to the very first presentation of these ideas which only more than three years later have been brought down into their final form. We also acknowledge to the institutions that made this work possible: the Universidad Complutense of Madrid that supported our stay in Berkeley by a grant del Amo, the N.S.F. and the D.G.I.C.Y.T., inside whose research project PB 89-0379-C02-02 this paper has been developed. Last, we are grateful to our colleagues from Madrid, who, first, took over their shoulders our teaching duties, allowing thereby our leave, and, second, listened patiently our explanations and enriched them with their comments.

1 Basic and generically basic sets

First of all we fix our terminology. Let $X \subset \mathbb{R}^p$ be a (*real*) *algebraic set*, that is, the zero set of finitely many polynomials. A *polynomial function* on X is the restriction of a polynomial $f \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_p]$, and a *regular function* on X is the restriction of a rational function whose denominator has no zero in X. The ring $\mathcal{P}(X)$ of polynomial functions of X is the quotient of the polynomial ring $\mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_p]$ module the ideal of all polynomials vanishing on X, and the ring $\mathcal{R}(X)$ of regular functions of X is the subring of the total ring of fractions of $\mathcal{P}(X)$ consisting of the fractions whose denominators have not zeros in X. Every point $a \in X$ defines a maximal ideal $\mathfrak{m}_a$ in both rings, namely, the ideal of all functions vanishing at a, and the two localizations $\mathcal{P}(X)_{\mathfrak{m}_a}$ and $\mathcal{R}(X)_{\mathfrak{m}_a}$ coincide; this local ring will be denoted by $\mathcal{R}_{X,a}$.

Next, we define what we mean by an *analytic function* on X. Since we are going to discuss analyticity in a purely local sense, we do not enter into coherence matters, and simply define an *analytic function* on an open set U of X as the restriction of an analytic function on some open neighborhood of U in $\mathbb{R}^p$. This defines a sheaf $\mathcal{O} = \mathcal{O}_X$ of analytic function germs on X, whose stalk $\mathcal{O}_{X,a}$ at a point $a \in X$ is the ring of analytic function germs at a. All this will be thoroughly described in Section 8.

A set $S \subset X$ is called *semialgebraic* if there exist regular functions $f_{ij}, g_i \in \mathcal{R}(X)$ such that

$$S = \bigcup_{i=1}^{r} \{x \in X : f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}.$$

Analogously, S is called *semianalytic* if for every $a \in X$ there are a neighborhood U of a and analytic functions (depending on a) $f_{ij}, g_i \in \mathcal{O}(U)$ such that

$$S \cap U = \bigcup_{i=1}^{r} \{x \in U : f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}.$$

We stress the two-fold difference between semialgebraic and semianalytic sets: first we pass from regular to analytic functions, and second, from a global to a local formula.

If S is open, the so-called *finiteness theorem* asserts that the g_i's can be suppressed in the descriptions above. For these and similar results our general references are [BCR, Chapter 2] for the semialgebraic case, and [Ł,§§15-18, pp.65-100], [Rz2] for the semianalytic case. This leads naturally to take into consideration the number of unions needed for the description of S and in particular to study the open sets that can be defined without unions. An open set S is called *s-basic semialgebraic* if there are s regular functions $f_i \in \mathcal{R}(X)$ such that

$$S = \{x \in X : f_1(x) > 0, \dots, f_s(x) > 0\}.$$

An open set S is called *s-basic semianalytic* if for every $a \in X$ there are a neighbourhood U of a and s analytic functions $f_i \in \mathcal{O}(U)$ such that

$$S \cap U = \{x \in U : f_1(x) > 0, \dots, f_s(x) > 0\}.$$

(Note that here the number s of functions does not depend on the point a.) Very often we will write *basic* instead of s-basic for some s.

Basic sets have interested many authors, since the pioneering work of Bröcker, ([Br2], or see [BCR, 7.7, pp.136-141]), and it is nowadays well known that the number of inequalities needed in the descriptions above can always be taken $\leq \dim(X)$ (see [Sch1] and [Br3] for the algebraic case, [AnBrRz1,2] for the semianalytic one). In all these investigations the concept of generically basic set plays an essential role. A semialgebraic set S is called *generically s-basic semialgebraic* if there are an algebraic subset $Z \subset X$ with $\dim(Z) < \dim(X)$ and regular functions $f_i \in \mathcal{R}(X)$ such that

$$S \setminus Z = \{x \in X : f_1(x) > 0, \dots, f_s(x) > 0\} \setminus Z.$$

The condition $\dim(Z) < \dim(X)$ means that Z does not contain any irreducible component of maximal dimension of X. A semianalytic set S is called *generically s-basic semianalytic* if for every $a \in X$ there are a neighborhood U of a, an analytic subset $Z \subset U$ with $\dim(Z) < \dim(X)$ and s analytic functions $f_i \in \mathcal{O}(U)$ such that

$$S \cap U \setminus Z = \{x \in U : f_1(x) > 0, \dots, f_s(x) > 0\} \setminus Z.$$

As above we will use the term *generically basic* to mean generically s-basic for some s. The main reason for the importance of generically basic sets is that this concept is of birational nature and very well behaved with respect to the theory of quadratic forms, as we will see later.

Obviously, if S is semialgebraic then it is also semianalytic, and if S is s-basic (resp. generically s-basic) semialgebraic, it is s-basic (resp. generically s-basic) semianalytic too. Conversely, we have:

Problem I *Let $S \subset X$ be a semialgebraic set which is s-basic (resp. generically s-basic) semianalytic. When is it s-basic (resp. generically s-basic) semialgebraic?*

It is easy to produce examples which show that, with the notations of Problem I, S need not be s-basic semialgebraic. We want to understand where obstructions do appear. So, we start by listing here a few examples which run from the trivial to the more surprising.

Examples 1.1 *a)* The set $S = \{x < 0\} \cup \{y < 0\}$ (*Figure 1*) can be considered the first example of a non basic set. It was shown in [Ł, p. 67] that S is not basic semianalytic, the obstruction being at the origin.

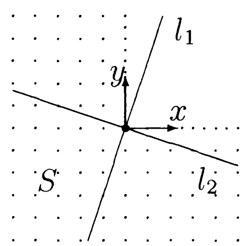


Figure 1

The argument is as follows. Suppose that f is an analytic function defined in some neighborhood U of the origin, and > 0 on $S \cap U$. Then the sign of f cannot change along any line l_2 through the origin with negative slope, and so the expansion of f as power series has an initial form f_0 of even degree. But then the sign of f cannot change along any line outside the zero set of f_0, despite its slope. In particular this includes the lines l_1 with positive slope, which will be either completely contained in or completely disjoint from $f > 0$. Clearly this means that S cannot be described near the origin in the form $\{f_1 > 0, \dots, f_s > 0\}$. This argument works whenever we have two pencils, one completely contained in S and the other exactly half-contained. For instance the sets of *Figure 2* are not basic semianalytic by this argument.

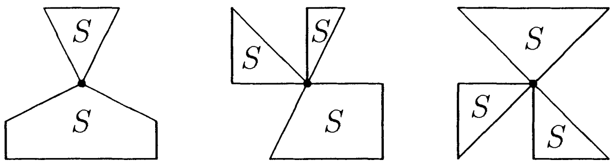


Figure 2

b) Now consider the set $S = \{x < 0\} \cup \{x > 1, y < 0\}$ (*Figure 3*). It is immediate that S is basic semianalytic, and we claim that it is not basic semialgebraic.

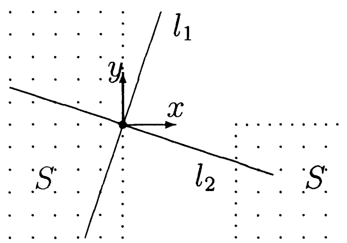


Figure 3

Indeed, the argument of Example *a)* works here if we replace the order of the power expansion of an analytic function f which is > 0 on S around the origin by the degree of a polynomial f which is > 0 on S. Note that from a geometric point of view this means that we are dealing with points at the infinite. In fact, consider the projective plane with coordinates $(x_0 : x_1 : x_2)$, with $x = x_1/x_0, y = x_2/x_0$. If now we look at our sets either in the affine plane $z = x_0/x_1, t = x_2/x_1$ or in $u = x_0/x_2, v = x_1/x_2$, we find the semialgebraic sets of *Figure 4* which are not basic semianalytic according to *a)*. The obstructions are at the infinite points (0:1:0) and (0:0:1) respectively.

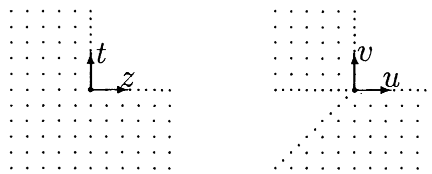


Figure 4

c) Our first example showed an analytic obstruction; the second one an analytic obstruction hidden at infinity, and here we exhibit an example with no analytic obstruction at all, neither finite nor infinite. Consider the sets

$$S_1 = \{3x^2 + y^2 - 4xy + 1 < 0\},\ S_1^+ = S_1 \cap \{y > 0\},\ S_1^- = S_1 \cap \{y < 0\},$$
$$S_2 = \{3x^2 + y^2 + 4xy + 1 < 0\},\ S_2^+ = S_2 \cap \{y > 0\},\ S_2^- = S_2 \cap \{y < 0\},$$

and put $S = S_1 \cup S_2^+$ (*Figure 5*).

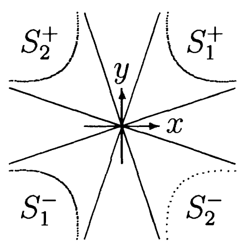


Figure 5

First, S is not basic semialgebraic. For, using pencils and degrees as explained in *a)*, if a polynomial is > 0 on S it must be > 0 also on S_2^-. On the other hand, S is clearly basic semianalytic, and if we look at the infinite points we find the sets of *Figure 6*, which are basic semianalytic too.

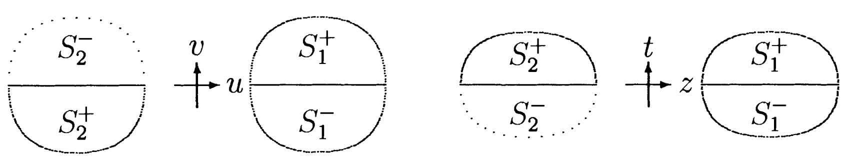


Figure 6

d) Consider the set $S = \{x^2 + y - y^3 < 0, y < 0\}$ (*Figure 7*). It is 2-basic semialgebraic, and 1-basic semianalytic, but it is not 1-basic semialgebraic.

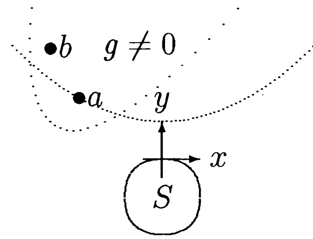


Figure 7

Indeed, suppose that there is a polynomial $f \in \mathbb{R}[\mathrm{x}, \mathrm{y}]$ such that $S = \{f > 0\}$. Then f changes sign along the boundary of S, so that after division we get $f = (\mathrm{x}^2 + \mathrm{y} - \mathrm{y}^3)^p g$, where p is odd and $g \in \mathbb{R}[\mathrm{x}, \mathrm{y}]$ does not vanish on the boundary of S. Since $\mathrm{x}^2 + \mathrm{y} - \mathrm{y}^3$ is irreducible, the latter condition implies that $\{x^2 + y - y^3 = 0, g(x, y) = 0\}$ is a finite set, and consequently we can find a point $a \in \{x^2 + y - y^3 = 0, y > 0\}$ with $g(a) \neq 0$. But then, f must change sign around a, so that there are points $b \in \{y > 0, f(x, y) > 0\}$. This is impossible as $\{f(x, y) > 0\} = S \subset \{y < 0\}$.

e) Consider the subset $S = \{x^2 - y^2 - y^3 < 0, y > 0\} \cup \{x > 0, y > 0\}$ (*Figure 8*). Then S is not basic semialgebraic.

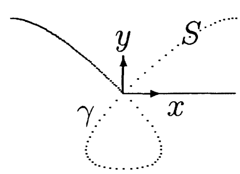


Figure 8

Indeed suppose that $S = \{g_1 > 0, \dots, g_s > 0\}$ for some polynomials. Then some g_i, say g_1, has to change sign in the points of the curve γ situated in the second quadrant, and therefore it must vanish on it. Thus $g_1 = (\mathrm{x}^2 - \mathrm{y}^2 - \mathrm{y}^3)^p h$ where p is an odd positive integer and h does not vanish on γ. But then g_1 changes sign also at the points of γ placed in the first quadrant, which is a contradiction. However, S is basic semianalytic. In fact, this is immediate for all points but the origin, and at it, S can be locally described as $\{x + y\sqrt{1+y} > 0, y > 0\}$. The key point, as the reader will have noticed, is that the curve γ splits at the origin into two different analytic branches, breaking the algebraic obstruction described above. This will be made more precise later with the notion of *fan.*

f) Let $X \subset \mathbb{R}^3$ be the set defined by $x^2 + z^4 - z^2(y^2 + y^3) = 0$, and consider the semialgebraic set $S = X \cap (\{y > 0, x > 0\} \cup \{y > 0, z > 0\})$ (*Figure 9*).

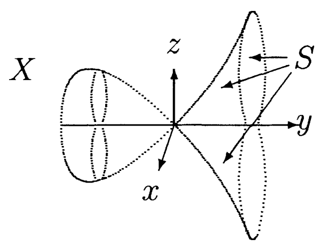


Figure 9

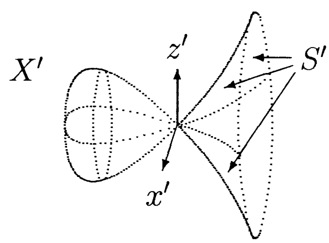


Figure 10

Again S is not basic semialgebraic, but it is basic semianalytic. Unfortunately we cannot see the reason of this right now. To give a clue, notice that if we consider the normalization X' of X, then X' is defined by the equation $x'^2 + z'^2 - (y'^2 + y'^3) = 0$, where $x = x'z', y = y', z = z'$, and the preimage S' of S in X' is described by $S' = X' \cap (\{y' > 0, x' > 0\} \cup \{y' > 0, z' > 0\})$ (*Figure 10*). Now, since $X' \cap \{z' = 0\}$ splits in the origin into two analytic branches, we can play there a similar trick to the one of *e).*

2 The real spectrum

The theory of the real spectrum initiated by Coste and Roy, has shown to be the adequate frame to formalize and make precise the ideas sketched in the preceding sections. Here we will only review the most basic facts, so that our exposition can be followed on a self-contained basis. We rely on [BCR, Chapters 4 and 7] and [AnBrRz2, Chapter III] as general references for proofs and further information.

Let A be a commutative ring with unit. The real spectrum, $\mathrm{Spec}_r(A)$, of A is defined as the collection of all prime cones of A. There are several alternative ways of defining prime cones of A. To start with, a *prime cone of* A is a pair $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $\leq_\alpha$ is an ordering in the residue field $\kappa(\mathfrak{p}_\alpha)$ of $\mathfrak{p}_\alpha$. The ideal $\mathfrak{p}_\alpha$ is called the *support of* α and denoted by $\mathrm{supp}(\alpha)$. Thus, denoting by $\mathbb{F}_3 = \{0, +1, -1\}$ the field of three elements, we can consider a natural *sign* map

$$\mathrm{sign} : \mathrm{Spec}_r(A) \times A \longrightarrow \mathbb{F}_3$$

defined as

$$(\alpha, f) \mapsto \mathrm{sign}(\alpha, f) = \begin{cases} 0 & \text{if } f \in \mathfrak{p}_\alpha \\ +1 & \text{if } f + \mathfrak{p}_\alpha >_\alpha 0 \\ -1 & \text{if } f + \mathfrak{p}_\alpha <_\alpha 0 \end{cases}$$

In this way, each $\alpha \in \mathrm{Spec}_r(A)$ defines a map $\alpha : A \to \mathbb{F}_3$ which verifies the following properties:

(i) α is a multiplicative homomorphism;

(ii) $\alpha^{-1}(0)$ is a prime ideal of A, namely, the support of α;

(iii) $\alpha(-1) = -1$;

(iv) the *positive cone* $\alpha^{-1}(\{0, +1\})$ is an additively closed set.

Conversely, it is immediate to check that any map $\alpha : A \to \mathbb{F}_3$ verifying the above properties defines in a canonical way a unique prime cone with support $\alpha^{-1}(0)$. This provides a description of the points of the real spectrum as *generalized signatures* (that is, maps $\alpha : A \to \mathbb{F}_3$ verifying the four conditions *(i)-(iv)* listed above). This approach is very well suited for our purposes and will be used freely. Also we will adopt indistinctly the notation $\alpha(f) > 0$ or $f(\alpha) > 0$ for $\operatorname{sign}(\alpha, f) = +1$; similarly $\alpha(f) \geq 0$ or $f(\alpha) \geq 0$ for $\operatorname{sign}(\alpha, f) \in \{0, +1\}$; etc.

From a somehow dual viewpoint, the elements of A define functions

$$f : \operatorname{Spec}_r(A) \to \mathbb{F}_3 : \alpha \mapsto \operatorname{sign}(\alpha, f).$$

Notice that in this sense the elements 1 and -1 of A define the constant functions $+1$ and -1 respectively. This is at the basis of the representation of the real spectrum as a *space of signs* ([AnBrRz2, Chapter III]), a topic in which we will not enter here.

Coming back to the notion of real spectrum, a particular case which deserves special attention occurs when A is a field K. Then $\operatorname{Spec}_r(K)$ is the space of orderings of K, a notion that appeared in the literature much earlier than the one of real spectrum and which has been object of study in connection with the theory of quadratic forms. It is, in fact, from this study that we will take profit in what follows.

We will use the notation $\{f_1 > 0, \dots, f_s > 0, g = 0\} \subset \operatorname{Spec}_r(A)$ for the set

$$\{\alpha \in \operatorname{Spec}_r(A) : f_1(\alpha) > 0, \dots, f_s(\alpha) > 0, g(\alpha) = 0\}.$$

Then we define the *constructible sets* of $\operatorname{Spec}_r(A)$ as those $C \subset \operatorname{Spec}_r(A)$ which have a representation

$$C = \bigcup_{i=1}^{p} \{f_{i1} > 0, \dots, f_{is_i} > 0, g_i = 0\},$$

for some $f_{ij}, g_i \in A$. These turn out to be a basis of a topology in $\operatorname{Spec}_r(A)$ —the so-called *constructible topology*— which is nothing but the restriction to $\operatorname{Spec}_r(A)$ of the product topology of $(\mathbb{F}_3)^A$, where in $\mathbb{F}_3$ we consider the discrete topology. Thus $\operatorname{Spec}_r(A)$ is compact with respect to the constructible topology.

Similarly to what was done in the previous section, we say that a constructible set C is *s-basic open* if it can be written as

$$C = \{f_1 > 0, \ldots, f_s > 0\},$$

Sometimes we will call it simply *s-basic*; the plain expression *basic open* or just *basic* means s-basic for some s. Basic open sets generate also a topology which is called the *Harrison topology* of $\operatorname{Spec}_r(A)$, and which is considered as the standard or default topology of $\operatorname{Spec}_r(A)$. Therefore, except when specially mentioned in other sense, all the topological statements refer to this Harrison topology, which is obviously coarser than the constructible one described above. Therefore $\operatorname{Spec}_r(A)$ is also quasi-compact with the Harrison topology. We also define the *Zariski topology* by analogy with the Zariski prime spectrum: a subbasis consists of all sets of the form $\{f \neq 0\}$. We distinguish the closure in this topology with an index Z and using the terms *Zariski closure* and *Zariski closed.*

Let $\alpha, \beta \in \operatorname{Spec}_r(A)$. We say that α *specializes to* β or that β is *a specialization of* α and write $\alpha \to \beta$ if β is in the closure of α, that is, $f(\beta) > 0$ implies $f(\alpha) > 0$ for $f \in A$. More algebraically, seeing α and β as generalized signatures, $\alpha \to \beta$ if and only the positive cone of α is contained in that of β, or, in other words, $\mathfrak{p}_\alpha \subset \mathfrak{p}_\beta$ and the canonical map $A/\mathfrak{p}_\alpha \to A/\mathfrak{p}_\beta$ sends elements $\geq_\alpha 0$ to elements $\geq_\beta 0$. Now suppose that $\alpha \in \operatorname{Spec}_r(A)$ is given, and that $\mathfrak{p}$ is a prime ideal with $\mathfrak{p} \supset \mathfrak{p}_\alpha$. Then we say that α *makes* $\mathfrak{p}$ *convex* if from $0 < f(\alpha) < g(\alpha), f \in A, g \in \mathfrak{p}$ it follows that $f \in \mathfrak{p}$; if this is the case, then $\alpha \to \beta$ for a unique $\beta \in \operatorname{Spec}_r(A)$ with $\mathfrak{p}_\beta = \mathfrak{p}$. The notion of specialization is very important: on the one hand, it has a purely algebraic description, and on the other hand, it reflects topological facts of $\operatorname{Spec}_r(A)$. For instance, if C is open, $\beta \in C$ and $\alpha \to \beta$, then $\alpha \in C$, and this property characterizes openness if C is constructible.

We finish this quick review of the basic definitions of the real spectrum by recalling its functorial character. Namely, if $\phi : A \to B$ is a ring homomorphism, then the map $\phi^* : \operatorname{Spec}_r(B) \to \operatorname{Spec}_r(A)$ defined by $\phi^*(\alpha) = \alpha \circ \phi$, where $\alpha \in \operatorname{Spec}_r(B)$ is seen as a signature $\alpha : B \to \mathbb{F}_3$, is continuous. Clearly, the support of the prime cone $\phi^*(\alpha) \in \operatorname{Spec}_r(A)$ is the ideal $\phi^{-1}(\mathfrak{p}_\alpha)$, and the ordering $\leq_{\phi^*(\alpha)}$ is the restriction of $\leq_\alpha$ via the embedding $\kappa(\phi^{-1}(\mathfrak{p}_\alpha)) \to \kappa(\mathfrak{p}_\alpha)$ induced by ϕ. In particular, if $\mathfrak{p}$ is a prime ideal of A and $\pi : A \longrightarrow \kappa(\mathfrak{p})$ denotes the canonical map of A into the residue field of $\mathfrak{p}$, then π^* defines an immersion of $\operatorname{Spec}_r(\kappa(\mathfrak{p}))$ as the subspace of $\operatorname{Spec}_r(A)$ consisting of the points with support $\mathfrak{p}$. With this identification we have

$$\operatorname{Spec}_r(A) = \bigcup_{\mathfrak{p}} \operatorname{Spec}_r(\kappa(\mathfrak{p})),$$

where the $\mathfrak{p}$'s run among the prime ideals of A.

What is the relationship between the real spectrum and the semialgebraic and semianalytic sets? The answer is given by the "tilde" operator which will be explained in the next section and which will allow us to translate problems and properties from the geometric context to real spectra and viceversa.

Thus, the general philosophy is to seek for a characterization of basic (respectively s-basic) sets among the family of constructible subsets of $\operatorname{Spec}_r(A)$, and try to transport it to the geometric context of semialgebraic and emianalytic sets by means of the tilde operator. To describe such a characterization in $\operatorname{Spec}_r(A)$, we define the *Zariski boundary* of an open set $C \subset \operatorname{Spec}_r(A)$ as the Zariski closure of the boundary of C, that is, $\partial_Z(C) = \overline{\overline{C} \setminus C}^Z$. This Zariski boundary provides a geometric necessary condition for a constructible set to be basic open:

Proposition 2.1 *Let $C \subset \operatorname{Spec}_r(A)$ be a basic open constructible set. Then $C \cap \partial_Z(C) = \emptyset$.*

Proof: Suppose $C = \{f_1 > 0, \dots, f_s > 0\}$ and let $\alpha \in \overline{C} \setminus C$. Then we have $f_i(\alpha) \leq 0$ for some i. Since $\alpha \in \overline{C}$ we cannot have $f_i(\alpha) < 0$, whence $f_i(\alpha) = 0$. This implies that $f_1 \cdots f_s$ vanishes on $\overline{C} \setminus C$ and so $\partial_Z(C) \subset \{f_1 \cdots f_s = 0\}$. Since the latter set does not meet C we are done. $\square$

We will say that C *does not meet its Zariski boundary* instead of writing $C \cap \partial_Z(C) = \emptyset$. With this terminology we can state the main result mentioned above.

Theorem 2.2 *Let C be an open constructible subset of* $\operatorname{Spec}_r(A)$ *which does not meet its Zariski boundary and let s be an integer. Then S is s-basic if and only if for every prime ideal $\mathfrak{p}$ of A, the intersection $C \cap \operatorname{Spec}_r(\kappa(\mathfrak{p}))$ is s-basic.*

This theorem has a long history. It was first obtained by Bröcker, [Br2], in case A was an algebra finitely generated over a real closed field R, but he could not control completely the number of equations involved. This was solved by Scheiderer in [Sch1], who already remarked that the argument worked for any excellent ring A, according to the approach explained in [AnBrRz1]. At the same time Bröcker found a proof that only required A to be noetherian, [Br3]. Finally, Marshall discovered how to modify all these proofs to obtain the result for arbitrary A, [Mr7]. The reader can find a very detailed account of these papers in [Sch2].

Coming back to the statement of Theorem 2.2, one could think that not much has been gained, since we are faced with the same question, but now for a whole family of fields. However, as we have already pointed out, the real spectrum of a field is its space of orderings, and very powerful results are known in this context, which provide an answer to the problem of deciding whether $C \cap \operatorname{Spec}_r(\kappa(\mathfrak{p}))$ is s-basic. This answer uses the notion of *fan*, which we will encounter in the following sections.

The fact that a constructible subset C of the real spectrum $\operatorname{Spec}_r(B)$ of a domain B is basic in the quotient field L of B has a nice geometric translation: the abstract notion of generically basic set. Indeed, consider the map $\operatorname{Spec}_r(L) \to \operatorname{Spec}_r(B)$ associated to the inclusion $B \subset L$, which identifies the elements of $\operatorname{Spec}_r(L)$ with those of $\operatorname{Spec}_r(B)$ with support the zero ideal. Then:

Proposition 2.3 *Let $C \subset \operatorname{Spec}_r(B)$ be a constructible set. Then the restriction $C \cap \operatorname{Spec}_r(L)$ is s-basic (as a constructible subset of $\operatorname{Spec}_r(L)$) if and only if there exist a Zariski closed set $Z \neq \operatorname{Spec}_r(B)$, and elements $f_1, \dots, f_s \in B$ such that $C \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z$.*

Proof: Let $Z = \{h_1 = \cdots = h_r = 0\} \subset \operatorname{Spec}_r(A)$ be a Zariski closed set and suppose that there is $\alpha \in Z \cap \operatorname{Spec}_r(L)$. Then the support of α is the zero ideal, and since $\alpha \in Z$, we have $h_i(\alpha) = 0$, that is, $h_i = 0$ for all i. Whence $Z = \mathrm{Spec}_r(B)$, contradiction. The "if" part follows from this remark.

For the converse, suppose $C \cap \mathrm{Spec}_r(L) = \{f_1 > 0, \dots, f_s > 0\} \cap \mathrm{Spec}_r(L)$, and consider the constructible set

$$D = (C \setminus \{f_1 > 0, \dots, f_s > 0\}) \cup (\{f_1 > 0, \dots, f_s > 0\} \setminus C).$$

We claim that there is an element $h \neq 0$ which vanishes on D. Indeed, otherwise, by the compactness of the constructible topology, the family of non-empty constructible sets $\{h \neq 0\} \cap D$, $h \in B \setminus \{0\}$, would have nonempty intersection and therefore we would find a prime cone $\alpha \in D$ with support zero. Since $D \cap \mathrm{Spec}_r(L) = \emptyset$, this is a contradiction, and we have proved our claim. Now, if $h \neq 0$, we have the Zariski closed set $Z = \{h = 0\} \neq \mathrm{Spec}_r(B)$ and if h vanishes on D, we have $C \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z$. □

To summarize the situation we introduce the notion of *generically s-basic sets* in the real spectrum of a *domain B*. These are the constructible sets $C \subset \mathrm{Spec}_r(B)$ such that for some Zariski closed set $Z \neq \mathrm{Spec}_r(B)$ there are s elements $f_1, \dots, f_s \in A$ such that

$$C \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z.$$

Using this terminology and Proposition 2.3, Theorem 2.2 can be reformulated as follows:

Proposition 2.4 *Let C be an open constructible subset of* $\mathrm{Spec}_r(A)$ *which does not meet its Zariski boundary, and let s be an integer. Then C is s-basic if and only if for every prime ideal $\mathfrak{p}$ of A, the intersection $C \cap \mathrm{Spec}_r(A/\mathfrak{p})$ is generically s-basic.*

In other words, we have reduced the study of basicness to that of generic basicness. Moreover we know that this is a matter of quotient fields, rather than a matter of rings. In some situations however it is convenient to discuss generic basicness without the assumption that the ring involved is a domain.

In this case the definition we adopt is the most immediate generalization: a constructible set $C \subset \mathrm{Spec}_r(A)$ is *s-generically basic* if for every minimal prime ideal $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p})$ maximal, the intersection $C \cap \mathrm{Spec}_r(A/\mathfrak{p})$ is generically s-basic (note that $B = A/\mathfrak{p}$ is a domain).

3 Algebraic and analytic tilde operators

In this section we explain the relation between semialgebraic and semianalytic sets and constructible sets of the real spectrum of suitable rings. This is done by means of the tilde operator.

Let $\mathcal{R}(X)$ be the ring of regular functions of a real algebraic set $X \subset \mathbb{R}^p$. Then we define the *algebraic tilde operator* between the family of semialgebraic subsets of X and the family of constructible subsets of $\mathrm{Spec}_r(\mathcal{R}(X))$ by mapping a semialgebraic set $S \subset X$ to the constructible set $\tilde{S} \subset \mathrm{Spec}_r(\mathcal{R}(X))$ given by any formula that also gives S. Being more precise, if

$$S = \bigcup_{i=1}^{r} \{x \in X \mid f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

then,

$$\tilde{S} = \bigcup_{i=1}^{r} \{\alpha \in \mathrm{Spec}_r(\mathcal{R}(X)) \mid f_{i1}(\alpha) > 0, \dots, f_{is_i}(\alpha) > 0, g_i(\alpha) = 0\}$$

Of course, it is not at all clear that this defines a map, since there is no apparent reason for different formulas of S to produce the same constructible set $\tilde{S}$. However, this statement is exactly the content of the nowadays called Artin-Lang homomorphism theorem ([BCR, 4.1.2, p.76]), which can be formulated by saying that $S = \emptyset$ if and only if $\tilde{S} = \emptyset$.

It follows from this that the definition above is consistent, and gives a bijection which preserves boolean operations: finite unions, intersections and complementations. Also, together with the finiteness theorem quoted in Section 1, it implies that the tilde operator preserves closures and interiors. More subtle topological matters, like connectedness or dimensions, behave equally well, because of the good stratification properties of semialgebraic sets. All these properties can be seen in [BCR, 7.2-7.6, pp.118-135]. Also, concerning our problems, it is obvious from the definition that the notions of s-basic and generically s-basic sets are preserved by this bijection.

Despite the simplicity of the given definition of the tilde operator, to understand its geometrical meaning we have to give a small detour. Each point $a \in X$ defines a unique prime cone of $\mathrm{Spec}_r(\mathcal{R}(X))$, namely the pair consisting of the maximal ideal $\mathfrak{m}_a$ of regular functions vanishing at a, and the unique ordering of the reals, which is the residue field of $\mathfrak{m}_a$. Thus, we have a canonical inclusion $X \hookrightarrow \mathrm{Spec}_r(\mathcal{R}(X))$ which turns out to be a topological embedding, since the open balls defining the usual topology of $\mathbb{R}^p$ are sets of the type $\{f > 0\}$. Hence, $\mathrm{Spec}_r(\mathcal{R}(X))$ can be seen as a compactification of X, which in turn can be thought as the set of "geometric points" of $\mathrm{Spec}_r(\mathcal{R}(X))$. Thus, for any constructible set $C \subset \mathrm{Spec}_r(\mathcal{R}(X))$ we can consider its *restriction* $C \cap X$, which obviously is the semialgebraic subset of X defined by the same formula as C. This gives a *restriction map* which is well defined, and the Artin-Lang homomorphism theorem assures that it is bijective, since it is the inverse of the tilde map. In other words, any constructible set $C \subset \mathrm{Spec}_r(\mathcal{R}(X))$ is completely determined by its set of geometric points, and topological properties of this set determine also the topology of C.

In particular we have:

Proposition 3.1 *Let $S \subset X$ be an open semialgebraic set and s an integer. Then S is s-basic if and only if the constructible set $\tilde{S}$ is s-basic.*

There is also another important consequence of the Artin-Lang homomorphism theorem: the so-called *ultrafilter theorem.* Namely, there is a one to one correspondence between the total orderings of the quotient field of $\mathcal{R}(X)$ and the ultrafilters of open semialgebraic subsets of X of maximum dimension. Given an ordering α, we attach to it the unique ultrafilter $\mathcal{U}_\alpha$ containing all the sets $\{f > 0\}$ for $f >_\alpha 0$. Conversely, given such an ultrafilter $\mathcal{U}$, a function $f \in \mathcal{R}(X)$ is positive in the ordering attached to $\mathcal{U}$ if and only if it is positive in some semialgebraic set of $\mathcal{U}$. This is the justification of the already commonly accepted graphic representation of an ordering which consists of shadowing the attached ultrafilter of semialgebraic sets. For instance, the ordering $(0_+, 0_+)$ of $\mathbb{R}^2$ defined by the ultrafilter containing the interior of the "triangles" bounded by the x-axis, the parabolas of the type $y = x^n$ and the lines $x = \varepsilon$, is usually represented as depicted below:

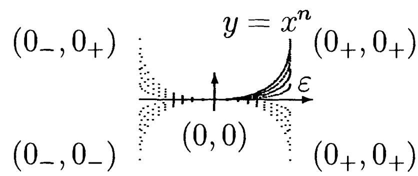


Figure 11

$f(\mathbf{x},\mathbf{y}) = \sum a_{ij}\mathbf{x}^i\mathbf{y}^j$ belongs to this ordering if and only if $a_{i_0j_0} > 0$, where $a_{i_0j_0}$ represents the coefficient of the initial form of f when the monomials are ordered with the lexicographic order, that is, $\mathbf{x}^a\mathbf{y}^b < \mathbf{x}^c\mathbf{y}^d$ if and only if $b < d$, or $b = d$ and $a < c$. In particular notice that $\mathbf{y}$ is infinitely small with respect to $\mathbb{R}(\mathbf{x})$ and $\mathbf{x}$ is infinitely small with respect to $\mathbb{R}$. In a similar way one can define the orderings $(0_+, 0_-)$, $(0_-, 0_+)$, $(0_-, 0_-)$. By translation one defines the corresponding four orderings, $(a_\pm, b_\pm)$, around any point (a, b). We will use this geometric representation of orderings all along this paper.

Correspondingly, there is a tilde operator for germs of semianalytic sets at a given point $a \in X$. Indeed, consider the ring $\mathcal{O}_{X,a}$ of germs at a of analytic functions of X. We define the *analytic tilde operator* as the map which sends a semianalytic germ $S_a \subset X_a$ to the constructible set $\tilde{S}_a \subset \mathrm{Spec}_r(\mathcal{O}_{X,a})$ defined by any formula that also defines S_a. As above, there is a version of the Artin-Lang homomorphism theorem which holds for convergent power series, and that makes consistent the definition just given. Thus, the tilde map gives a bijection between the family of semianalytic germs at a and the family of constructible subsets of $\mathrm{Spec}_r(\mathcal{O}_{X,a})$. Moreover, it preserves inclusions, topological operations and s-basic and generically s-basic sets. All of this follows at once from the good properties of semianalytic germs ([Ł,§§15-18, pp. 65-100], [Rs], [Rz1,2], [FrRcRz], [AnBrRz2, Chapter VIII]).

The geometric interpretation is a bit subtler in this case, since in fact it is not even clear what the geometric points of a germ are. This is usually done by assigning this role to the curve germs, and, in fact, the Artin-Lang homomorphism theorem takes now the form of a curve selection result. However we do no enter in this here, which can be consulted in the quoted references.

As above we summarize these facts in a statement:

Proposition 3.2 *Let $S \subset X$ be a semianalytic set and let s be an integer. Then S is s-basic if and only if for any $a \in X$ the constructible set $\tilde{S}_a \subset$* $\operatorname{Spec}_r(\mathcal{O}_{X,a})$ *is s-basic.*

4 Fans and basic sets

Let A be a commutative ring with unit, and consider its real spectrum $\mathrm{Spec}_r(A)$, which we will see from now on as a space of generalized signatures. Following Knebush ([Kn, Section 7]), a *finite fan of* A is a finite set $F \subset \mathrm{Spec}_r(A)$ whose elements have all the same support $\mathfrak{p}$ and such that for any three $\alpha_1, \alpha_2, \alpha_3 \in F$, their product $\alpha_4 = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ is a well-defined prime cone which belongs to F. The ideal $\mathfrak{p}$ is called the *support* of F and is denoted by $\mathrm{supp}(F)$. We define the *height* of F, denoted by $\mathrm{ht}(F)$, as the height of $\mathfrak{p}$, that is, the Krull dimension of the ring $A_{\mathfrak{p}}$. Also we define the *dimension* of F, denoted by $\dim(F)$, as the Krull dimension of the ring $A/\mathfrak{p}$. Of course, considering the α_i's as orderings in the residue field $\kappa(\mathfrak{p})$, we are just repeating the definition of a fan of $\kappa(\mathfrak{p})$ in the sense of spaces of orderings, see [Mr1-5] or [AnBrRz2, Chapter IV]. Conversely, the fans of $\mathrm{Spec}_r(A)$ are exactly the fans of the real spectra $\mathrm{Spec}_r(\kappa(\mathfrak{p}))$ where $\mathfrak{p}$ runs among the prime ideals of A. Thus, by means of the corresponding identifications, we will think of fans as sets of orderings of the residue fields, and use freely the results about spaces of orderings. In particular, this allows us to see its elements as mappings into $\mathbb{F}_2 = \{-1, +1\}$.

Notice that any subset consisting of one or two orderings of a field K is a fan. These are the so-called *trivial* fans. To give a first example of a non trivial fan, we propose the reader to check that the four orderings, (a_+, b_+), (a_+, b_-), (a_-, b_+), (a_-, b_-) defined in the previous section form a fan of $\mathbb{R}[\mathbf{x}, \mathbf{y}]$ of height 0 and dimension 2.

The notion of fan is probably the most important one in the theory of spaces of orderings, since many properties hold true for $\mathrm{Spec}_r(K)$ if and only if they hold for all fans of K. This is the case, as we will see, for generic s-basicness. Moreover, fans have a natural and important combinatorial structure, which make them easy to handle.

In fact, a fan F has the structure of an affine space over the field of two elements $\mathbb{F}_2 = \{-1, +1\}$, that is, for any $\alpha_0 \in F$, the set $\alpha_0 F$ is a vector space over $\mathbb{F}_2 = \{-1, +1\}$ with the product of signatures as inner operation and the natural scalar multiplication. In particular, it follows that

$\#(F) = 2^k$, where k is the *affine dimension* of F, that is, $k+1$ is the minimal number of elements $\alpha_0, \ldots, \alpha_k \in F$ such that any $\alpha \in F$ is a product of α_i's (necessarily an odd number since $\alpha(-1) = -1$). An important property is that if F' is an affine subspace of F, then F' is again a fan. In particular, given $\alpha_1, \ldots, \alpha_t \in F$, the affine subspace generated by them is a fan, called the fan *generated* by $\alpha_1, \ldots, \alpha_t$.

In connection with basic sets, let us remark the immediate fact that for every basic set $C \subset \mathrm{Spec}_r(A)$, the intersection $F' = F \cap C$ is again a fan, and so $\#(F') = 2^l$ for some $l \leq k$. In fact, if $C = \{f_1 > 0\}$ then f_1 defines an affine map $f_1 : F \to \mathbb{F}_2$: $\alpha \mapsto \alpha(f_1)$. Then $F \cap C = f_1^{-1}(+1)$ which is an affine subspace of F. Moreover, if $F \cap C \neq \emptyset$ then $\#(F \cap C) \geq 2^{k-1}$, and the equality holds if and only if $f_1 > 0$ defines a non vacuous condition over F. By recurrence we have that either $F \cap \{f_1 > 0, \ldots, f_s > 0\} = \emptyset$ or $\#(F \cap \{f_1 > 0, \ldots, f_s > 0\}) \geq 2^{k-s}$ and that the equality holds if and only if $f_1, \ldots, f_s$ are "linearly independent" over F, that is $\{f_1 > 0, \ldots, f_s > 0\} \cap F$ cannot be described with less than s functions.

The fundamental result states that if this holds for all fans, then it holds globally in $\mathrm{Spec}_r(A)$. In fact, we have:

Theorem 4.1 (Marshall, Bröcker) *Let C be a constructible subset of $\mathrm{Spec}_r(A)$ which does not meet its Zariski boundary, and let s be an integer. The following assertions are equivalent:*

a) C is s-basic.

b) For every fan $F \subset \mathrm{Spec}_r(A)$ with $\#(F) = 2^k$ and $F \cap C \neq \emptyset$ we have $\#(F \cap C) = 2^l$ with $0 \leq k - l \leq s$.

Proof: [Br4, Th.5.4]. □

Concerning generically basic sets we deduce:

Proposition 4.2 *Let $C \subset \mathrm{Spec}_r(A)$ be a constructible set and s an integer.*

Then the following assertions are equivalent:

a) *C is generically s-basic.*

b) *For every fan $F \subset \operatorname{Spec}_r(A)$ of height 0 and maximal dimension with $\#(F) = 2^m$ and $F \cap C \neq \emptyset$ we have $\#(F \cap C) = 2^n$ with $0 \leq m - n \leq s$.*

Proof: From Proposition 2.3 we know that C is generically basic if and only if for every minimal prime $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p})$ maximal, the intersection $S \cap \operatorname{Spec}_r(\kappa(\mathfrak{p}))$ is basic. Then the result follows at once from the theorem, since the fans of the fields $\kappa(\mathfrak{p})$ are the fans of A of height 0 and maximal dimension. □

Now we can put the above results together with the ones concerning the tilde operator (cf. Section 3), in order to find characterizations of basic and generically basic semialgebraic sets. To do that we define the *Zariski boundary*, $\partial_Z(S)$, of an open semialgebraic set S, as we did for open constructible sets, that is, $\partial_Z(S) = \overline{\overline{S} \setminus S}^Z$. We say that S *does not meet its Zariski boundary* if $S \cap \partial_Z(S) = \emptyset$. With this terminology:

Proposition 4.3 *Let $S \subset X$ be an open semialgebraic set that does not meet its Zariski boundary, and let s be an integer. Then S is s-basic semialgebraic if and only if for every fan $F \subset \operatorname{Spec}_r(\mathcal{R}(X))$ with $\#(F) = 2^m$ and $F \cap \tilde{S} \neq \emptyset$ we have $\#(F \cap \tilde{S}) = 2^n$ with $0 \leq m - n \leq s$.*

Proposition 4.4 *Let $S \subset X$ be a semialgebraic set and s an integer. Then S is generically s-basic semialgebraic if and only if for every fan $F \subset \operatorname{Spec}_r(\mathcal{R}(X))$ of height 0 and maximal dimension with $\#(F) = 2^m$ and $F \cap \tilde{S} \neq \emptyset$ we have $\#(F \cap \tilde{S}) = 2^n$ with $0 \leq m - n \leq s$.*

We also have the corresponding statements for semianalytic sets, namely:

Proposition 4.5 *Let $S \subset X$ be an open semialgebraic set that does not meet its Zariski boundary, and let s be an integer. Then S is s-basic semi-*

analytic at $a \in X$, if and only if for every fan $F \subset \operatorname{Spec}_r(\mathcal{O}_{X,a})$ with $\#(F) = 2^m$ and $F \cap \tilde{S}_a \neq \emptyset$ we have $\#(F \cap \tilde{S}_a) = 2^n$ with $0 \leq m - n \leq s$.

Proposition 4.6 *Let $S \subset X$ be a semialgebraic set and s an integer. Then S is generically s-basic semianalytic at $a \in X$ if and only if for every fan $F \subset \operatorname{Spec}_r(\mathcal{O}_{X,a})$ of height 0 and maximal dimension with $\#(F) = 2^m$ and $F \cap \tilde{S}_a \neq \emptyset$ we have $\#(F \cap \tilde{S}_a) = 2^n$ with $0 \leq m - n \leq s$.*

Finally, we have the following fundamental result:

Proposition 4.7 *Let X have dimension $\leq d$. Then $\#(F) \leq 2^d$ for every fan $F \subset \operatorname{Spec}_r(\mathcal{R}(X))$ (resp. for every point $a \in X$ and every fan $F \subset \operatorname{Spec}_r(\mathcal{O}_{X,a})$).*

Proof: [Br2, Theorem 6.3] for the algebraic case and [AnBrRz1, Proposition 9.4] for the analytic. □

Using this bound in the previous propositions, we deduce that any (generically) s-basic set is (generically) d-basic, a fact that was already mentioned in Section 1.

Fans are preserved by ring homomorphisms. Namely, consider $\phi : A \to B$ and the corresponding $\phi^* : \operatorname{Spec}_r(B) \to \operatorname{Spec}_r(A)$. Then for any fan $F \subset \operatorname{Spec}_r(B)$, the image $\phi^*(F) \subset \operatorname{Spec}_r(A)$ is again a fan. If $\mathfrak{p}$ is the support of F, then $\phi^{-1}(\mathfrak{p})$ is the support of $\phi^*(F)$. We always have $\#(\phi^*(F)) \leq \#(F)$, with possibly a strict inequality.

At this point, it is clear from all the above statements, that in dealing with basicness questions, we have to know how to produce and handle fans. This will be done by means of valuations. In fact, there is a close relationship between prime cones and valuations, which fans behave specially well. We will devote Section 6 to explain this relationship.

5 Algebraic fans and analytic fans

Let $X \subset \mathbb{R}^p$ be a real algebraic set and $\mathcal{R}(X)$ its ring of regular functions. For every point $a \in X$, consider the ring $\mathcal{O}_{X,a}$ of analytic function germs at a, the canonical homomorphism $\varphi_a : \mathcal{R}(X) \to \mathcal{O}_{X,a}$, and the associated map of real spectra $\varphi_a^* : \operatorname{Spec}_r(\mathcal{O}_{X,a}) \to \operatorname{Spec}_r(\mathcal{R}(X))$. Coming back to our initial problem of comparing basicness when passing from the algebraic category to the analytic one, it follows from the results of the previous section that we are faced to study the relationship between fans in $\mathcal{R}(X)$ and fans in $\mathcal{O}_{X,a}$ for the different points $a \in X$. The aim of this section is to make this idea precise. We begin with the following:

Definition 5.1 *a) An* algebraic fan *of X is a fan of the ring $\mathcal{R}(X)$.*

b) An analytic fan *at $a \in X$ is a fan of the ring $\mathcal{O}_{X,a}$.*

c) An algebraic fan F of X is called analytic *at $a \in X$ if there is an analytic fan F_a at a such that $F = \varphi_a^*(F_a)$.*

d) Suppose that F is analytic at a. A fan F_a of $\mathcal{O}_{X,a}$ is called a lifting *of F if $F = \varphi_a^*(F_a)$ and $\#(F) = \#(F_a)$.*

The next observation is essential:

Proposition 5.2 *Let F be an algebraic fan, which is analytic at $a \in X$. Then there exists a lifting of F.*

Proof: Let F_a be any fan of $\mathcal{O}_{X,a}$ with $F = \varphi_a^*(F_a)$. Obviously we have $\#(F_a) \geq \#(F)$. Suppose $\#(F) = 2^m$, and let $\alpha_1, \dots, \alpha_{m+1}$ be generators of F. Then $\alpha_i = \varphi_a^*(\alpha_i')$ for some $\alpha_i' \in F_a$, $i = 1, \dots, m+1$. Let $F_a' \subset F_a$ be the fan generated by $\alpha_1', \dots, \alpha_{m+1}'$. Clearly $F = \varphi_a^*(F_a')$, whence $\#(F_a') \geq \#(F) = 2^m$, but since F_a' is generated by $m+1$ elements, we must have the equality. $\square$

Now, let $S \subset X$ be a semialgebraic set. Suppose that we are given an algebraic fan F and want to study the intersection $F \cap \tilde{S}$, in order to find out, using Proposition 4.3, whether the set S is s-basic. Suppose also that F is analytic at $a \in X$ and choose a lifting F_a of F. We have the germ S_a of S at a and the constructible set $\tilde{S}_a \subset \operatorname{Spec}_r(\mathcal{O}_{X,a})$. It follows immediately from the definitions and the condition on the number of elements of our fans, that $\#(F \cap \tilde{S}) = \#(F_a \cap \tilde{S}_a)$. Consequently, if F is an obstruction for S to be s-basic semialgebraic, then F_a is an obstruction for S to be s-basic semianalytic at a. This shows that the differences between the algebraic and the analytic categories dwell in the algebraic fans which are not analytic. Thus we come to the final formulation of Problem I:

Problem II *Characterize the algebraic fans which are analytic.*

There is an interesting thing to point out here. The above formulation comes from the discussion above, and concerns basicness. Hence, if we are interested in generic basicness rather than in basicness itself, some modification should be in order. Indeed, we can try to argue as above, with the additional assumption that F has height 0 and maximal dimension, but to conclude the same, we would need to know that the lifting of F can we chosen of height 0 and maximal dimension too. In fact, this is the case, as we will prove that an algebraic fan F which is analytic has always a lifting F_a with the same height and dimension: $\operatorname{ht}(F_a) = \operatorname{ht}(F)$, $\dim(F_a) = \dim(F)$. However the proof is far from obvious and must be postponed till the end of Section 9 (Proposition 9.2).

We start our study of Problem II, by dropping an easy necessary condition for an algebraic fan to be analytic. In order to state it we remember that the points of X are identified with the points of $\operatorname{Spec}_r(\mathcal{R}(X))$ that they define. Thus, it makes sense to say that a prime cone $\alpha \in \operatorname{Spec}_r(\mathcal{R}(X))$ *specializes to* $a \in X$, and this means that $f(\alpha) > 0$ whenever $f(a) > 0$. Note that, obviously, a prime cone cannot specialize to more than one point. Furthermore,

Proposition 5.3 *Let F be an algebraic fan of X. Then:*

a) *The orderings of F cannot specialize to more than two different points of X.*

b) *If F is analytic at the point $a \in X$, then all the prime cones of F specialize to a.*

Proof: Suppose that there are $\alpha_1, \alpha_2, \alpha_3 \in F$ and three different points $a_1, a_2, a_3 \in X$ such that $\alpha_i \to a_i$. Since F is a fan, we have $\alpha_4 = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \in F$, and we claim that α_4 specializes to a_1. Indeed, first note that for every $f \in \mathcal{R}(X)$ positive at a_1, there is a real number $\eta > 0$, such that f is positive on the open ball around a_1 of radius η. Hence

$$\{x \in X : f(x) > 0\} \supset \{x \in X : \eta - \mathrm{d}^2(x) > 0\},$$

where d denotes the euclidean distance to a_1. Using the tilde operator (Section 3) we deduce that

$$\{\alpha \in \mathrm{Spec}_r(\mathcal{R}(X)) : f(\alpha) > 0\} \supset \{\alpha \in \mathrm{Spec}_r(\mathcal{R}(X)) : \eta - \mathrm{d}^2(\alpha) > 0\}.$$

This implies

$$\bigcap_{f(a_1)>0} \{\alpha \in \mathrm{Spec}_r(\mathcal{R}(X)) : f(\alpha) > 0\} \supset \bigcap_{\eta>0} \{\alpha \in \mathrm{Spec}_r(\mathcal{R}(X)) : \eta - \mathrm{d}^2(\alpha) > 0\}.$$

Now suppose that α_4 does not specialize to a_1. We find $\eta > 0$ such that the regular function $h = \eta - \mathrm{d}^2$ is negative in α_4. Clearly we can choose η arbitrarily small, so that also $h(a_2) < 0$, and $h(a_3) < 0$. We conclude that $\alpha_1(h) = +1, \alpha_2(h) = \alpha_3(h) = \alpha_4(h) = -1$ which is impossible since $\alpha_4 = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$. This proves our claim that $\alpha_4 \to a_1$. Since the same reasoning is valid for the other a_i's, we get that α_4 specializes to them too, which is absurd, as a prime cone cannot specialize to more than one point. This proves *a)*.

b) Suppose that there exists a lifting F_a of F. In particular, for every $\alpha \in F$ there is $\beta \in F_a$ such that $\varphi_a^*(\beta) = \alpha$. Hence, to show that α specializes to a, it is enough to prove that $f(\beta) > 0$ if $f(a) > 0$. But if $f(a) > 0$, then f is > 0 in some neighborhood of a, and $\sqrt{f}$ is a well defined analytic function germ at a. It follows that f has a square root in $\mathcal{O}_{X,a}$, and therefore it is positive in β. □

The proposition above reinforces the intuitive idea that the analyticity of a fan is a local question, in the sense that all the elements of F must specialize to one point. According to Proposition 5.3 *a)*, we give the following definition:

Definition 5.4 *Let F be an algebraic fan of X. We say that F is a* 1pt-fan *if all its elements specialize to the same point $a \in X$. We say that F is a* 2pt-fan *if the elements of F specialize to two different points of X.*

It is easy to give examples of 2pt-fans which, therefore, are not analytic.

Example 5.5 Take any two points $(a, 0), (b, 0) \in \mathbb{R}^2$. The four orderings $\alpha_1 = (a_+, 0_+)$, $\alpha_3 = (a_+, 0_-)$, $\alpha_2 = (b_+, 0_+)$, $\alpha_4 = (b_+, 0_-)$ (*Figure 12*) form a fan of $\mathbb{R}[\mathrm{x}, \mathrm{y}]$ of height 0 with two specialization points, namely $(a, 0)$ and $(b, 0)$.

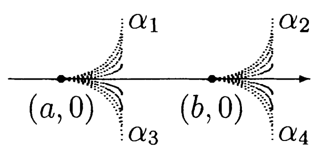


Figure 12

Proposition 5.3 raises the question of when prime cones specialize to geometric points, or in other words whether any fan is either 1pt- or 2pt-. This is always the case if we assume X to be compact. Indeed, we have:

Proposition 5.6 *If X is compact, every prime cone specializes to some point $a \in X$.*

Proof: It is enough to prove the result for maximal prime cones, that is, we assume that X is irreducible, and fix an ordering α of its field $\mathcal{K}(X)$ of rational functions. Using the tilde operator, we see that

$$\{x \in X : f_1(x) \geq 0, \dots, f_s(x) \geq 0\} \neq \emptyset$$

for all $f_1, \dots, f_s$ with $f_1(\alpha) > 0, \dots, f_s(\alpha) > 0$. Since X is compact, it follows that

$$\bigcap_{f(\alpha)>0} \{x \in X : f(x) \geq 0\} \neq \emptyset.$$

Thus, there is a point a at which all functions positive in α take positive value. This exactly means that α specializes to a. $\square$

We deduce from this proposition that, over the reals, every prime cone specializes to some point, maybe at infinity. For, let α be a prime cone of $\mathcal{R}(X)$ and $\mathfrak{p}$ its support. Consider any algebraic compactification Y' of the zero set $Y \subset X$ of $\mathfrak{p}$, that is, Y' is a real algebraic set and there is a birational regular embedding $Y \hookrightarrow Y'$. Then Y' has the same field of rational functions as Y and α can be seen as a prime cone of $\mathcal{R}(Y')$. Now, since Y' is compact, α specializes to some point $a \in Y'$ (Proposition 5.6). If $a \in Y$, then a is a *finite point*, and if $a \in Y' \setminus Y$, then *a is at infinity*. It should be stressed here that we are working over the reals. In fact, over non-archimedean real closed fields, the concept of compactness is useless and must be replaced by an appropiate one, cf. [BCR, Section 2.5]. There are several possible algebraic compactifications that can be used, but the two more natural (and common) are the *projective compactification* and the *Alexandroff compactification*. The former is the Zariski closure of Y in the projective space $\mathbb{P}^p(\mathbb{R}) \subset \mathbb{R}^N$, and the latter is the one point compactification obtained as the euclidean closure of Y in the sphere $\mathbb{S}^p \subset \mathbb{R}^{p+1}$, which is seen as the Alexandroff compactification of $\mathbb{R}^p$ [BCR, Proposition and Definition 2.5.9]).

6 Prime cones and valuations

In this section we develop the relationship between valuations and orderings, paying special attention to fans. Let A be a commutative ring with unit. We follow the terminology of Marshall [Mr6]. Namely, by a valuation ring of A we will understand a pair $(\mathfrak{p}, V)$ where $\mathfrak{p}$ is a prime ideal of A and V is a valuation ring of the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$. The ideal $\mathfrak{p}$ is called the *support of* V and denoted by $\operatorname{supp}(V)$. Whenever no confusion is possible we will denote the pair $(\mathfrak{p}, V)$ just by V. Let $\alpha \in \operatorname{Spec}_r(A)$. We say that α *is compatible with* V if they have the same support, say $\mathfrak{p}$, and V is a valuation ring of $K = \kappa(\mathfrak{p})$ convex with respect to $\leq_\alpha$, that is, if $0 \leq_\alpha f \leq_\alpha g$, with $f \in K$, $g \in V$, then $f \in V$. This is equivalent to say that the maximal ideal $\mathfrak{m}_V$ of V is convex with respect to α.

The valuation rings compatible with α form a chain of subrings of K totally ordered by inclusion, whose biggest element is the trivial valuation $V = K$, and whose smallest element is the so-called *convex hull of* $\mathbb{Q}$ *with respect to* α ([BCR, 10.1.11, p.218]), denoted by V_α:

$$V_\alpha = \{f \in K : -r < f(\alpha) < r \quad \text{for some } \ r \in \mathbb{Q}\}.$$

Note that by construction the residue field k_α of V_α is an archimedean field, and that the maximal ideal $\mathfrak{m}_\alpha$ of V_α consists of the elements of K which are *infinitesimal with respect to* $\mathbb{Q}$, that is, $f \in \mathfrak{m}_\alpha$ if and only if $-\varepsilon < f(\alpha) < \varepsilon$ for every positive rational number ε.

Let V be a valuation compatible with α, and let $\lambda_V : V \to k_V = V/\mathfrak{m}_V$ be its associated place. Set $\mathfrak{p}$ for the support of α and V. Thus, α can be seen, at convenience, either as a point of $\operatorname{Spec}_r(A)$, or as an ordering of $\kappa(\mathfrak{p})$, or, further, as an element of $\operatorname{Spec}_r(V)$ of height 0. The condition that V is compatible with α is equivalent to the fact that $\tau : k_V \setminus \{0\} \to \mathbb{F}_3$ defined by $\tau(f + \mathfrak{m}_V) = \alpha(f)$ is a prime cone of k_V. Identifying τ with the prime cone $\tau \circ \lambda_V \in \operatorname{Spec}_r(V)$, we have that $\operatorname{supp}(\tau) = \mathfrak{m}_V$ and $\alpha \to \tau$ (always in $\operatorname{Spec}_r(V)$). Conversely, given a prime cone τ of k_V one can ask whether there is some $\alpha \in \operatorname{Spec}_r(V)$ of height 0 such that $\alpha \to \tau$. This is completely answered by the Baer-Krull theorem:

Theorem 6.1 (Baer-Krull) *Let V be a valuation ring of the field K with value group Γ and let τ be an ordering of k_V. Then there is a bijection between the set F_τ of orderings α of K compatible with V and specializing to τ and the set of group homomorphisms $\phi : \Gamma \to \{+1, -1\}$. This bijection $\alpha \mapsto \phi$ is defined through any fixed $\alpha_0 \to \tau$ by*

$$\alpha(f) = \alpha_0(f) \cdot \phi(v(f)),$$

where given $f \in V$, $v(f)$ stands for its value in Γ.

Proof: [BCR, 10.1.8, 10.1.10, p.217]. □

More can be said about the structure of the set F_τ of generizations of τ. Indeed, notice that any group homomorphism $\Gamma \to \mathbb{F}_2$ defines a linear map $\Gamma/2\Gamma \to \mathbb{F}_2$ of $\mathbb{F}_2$-vector spaces. Thus, the Baer-Krull theorem states that $\alpha_0 F_\tau$ is bijective with the set of linear maps from $\Gamma/2\Gamma$ into $\mathbb{F}_2$. This bijection endows F_τ with a structure of affine space, encountering, this way, the definition of fan. To be more precise, assume that $\Gamma/2\Gamma$ is of finite dimension and suppose that $\gamma_1, \dots, \gamma_m \in \Gamma$ form a basis of $\Gamma/2\Gamma$ so that the group homomorphisms $\Gamma \to \{+1, -1\}$ correspond bijectively to the mappings $\{\gamma_1, \dots, \gamma_m\} \to \{+1, -1\}$. Let $x_1, \dots, x_m \in V$ have values $\gamma_1, \dots, \gamma_m$, respectively. Then, by Theorem 6.1, all the generizations of τ are completely determined by the signs of $x_1, \dots, x_m$. It can be checked that they form a fan F_τ, whose associated vector space can be identified with $\{-1, +1\}^m$. In fact, if we denote by α_0 the ordering defined by $\alpha_0(x_1) = \dots = \alpha_0(x_m) = +1$, then

$$\varphi : \alpha_0 F \to \{-1, +1\}^m : \quad \alpha_0\alpha \mapsto (\alpha(x_1), \dots, \alpha(x_m))$$

is an isomorphism of $\mathbb{F}_2$-vector spaces. With this identification, the elements $\alpha_0, \alpha_1, \dots, \alpha_m$ defined by the following table form a minimal system of generators of F. Here $*$ can be either $+1$ or -1.

	x_1	x_2	$\cdots$	x_{m-2}	x_{m-1}	x_m
α_0	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$
α_1	$+1$	$+1$	$\cdots$	$+1$	$+1$	-1
α_2	$+1$	$+1$	$\cdots$	$+1$	-1	$*$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
α_{m-1}	$+1$	-1	$\cdots$	$*$	$*$	$*$
α_m	-1	$*$	$\cdots$	$*$	$*$	$*$

The following example is a very important instance of this situation:

Example 6.2 *a)* Let B be a local regular ring with residue field k and quotient field K. Suppose $\dim(B) = m$ and consider a regular system of parameters $x_1, \ldots, x_m$. We construct, by induction, a chain of valuation rings $B \subset V_m \subset V_{m-1} \subset \cdots \subset V_1$ in K, attached to $x_1, \ldots, x_m$, such that for each i, V_i has rank i and residue field $k_i = \mathrm{qf}(B/(x_{m-i+1}, \ldots, x_m))$. If $m = 1$, then $V_1 = B$ is a discrete valuation ring. For $m > 1$, we consider first the discrete valuation ring $V_1 = B_{(x_m)}$. Its residue field k_1 is the quotient field of the local regular ring $B_1 = B/(x_m)$. By induction, we have in k_1 a chain of valuation rings $W_m \subset W_{m-1} \subset \cdots \subset W_2$ with the required conditions. Then, we define V_i as the composite of V_1 and W_i. Clearly, using this construction we can achieve that V_i has value group $\mathbb{Z}^i$. In particular V_m has value group $\mathbb{Z}^m$ and residue field k. Moreover, the values of the elements $x_1, \ldots, x_m$ form a basis of $\mathbb{Z}^m/2\mathbb{Z}^m$.

b) Thus, according to the Baer-Krull theorem, for every ordering τ in k, the set F_τ of orderings α in K compatible with V_m which specialize to τ is a fan with 2^m elements. Moreover, such orderings are completely determined by the signs of $x_1, \ldots, x_m$. This can be seen directly by induction (we use the notations introduced above). If γ is an ordering of k_1 compatible with W_m, we can lift it to two orderings γ_+, γ_- of K compatible with $B_{(x_m)}$ as follows: every $f \in B_{(x_m)}$ can be written as $f = ux_m^n$, where u is a unit of $B_{(x_m)}$ and we define

$$\gamma_+(f) = \gamma(\overline{u}),$$
$$\gamma_-(f) = \gamma(\overline{u})(-1)^n,$$

(here $\overline{u}$ stands for the residue class of u in K'). Since V_m is the composite of W_m and $B_{(x_m)}$, both γ_+ and γ_- are compatible with V_m and specialize to γ.

c) The following is an explicit description of F_τ. First consider the adic completion $\hat{B}$, and the formal power series ring $k[[\mathbf{x}]]$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ are indeterminates. We have a canonical isomorphism $\hat{B} \to k[[\mathbf{x}]]$, that sends the i-th parameter x_i to the i-th indeterminate $\mathbf{x}_i$. This way, each element f of B is a formal power series in the parameters $x_1, \dots, x_m$, say,

$$f = \sum_\nu a_\nu x_1^{\nu_1} \cdots x_m^{\nu_m},$$

with $a_\nu \in k$. Now, we order the exponents lexicographically, so that there is a minimum exponent $\nu^0 = (\nu_1^0, \dots, \nu_m^0)$, with $f_0 = a_{\nu^0} \neq 0$; the corresponding monomial $f_0 x_1^{\nu_1^0} \cdots x_m^{\nu_m^0}$ is called the *initial form* of f. Thus, for any $\alpha \in F_\tau$, we have:

$$\alpha(f) = \tau(f_0)\alpha(x_1)^{\nu_1^0} \cdots \alpha(x_m)^{\nu_m^0};$$

and the map

$$\alpha \mapsto \varepsilon = (\alpha(x_1), \dots, \alpha(x_m)),$$

is a bijection between F_τ and $\{+1, -1\}^m$.

In particular, if two elements $f, g \in B$ have the same initial form, they have the same signs; this happens when $f = g \bmod (x_1, \dots, x_m)^n$ for n large enough.

Notice that all the orderings of the fan F_τ are compatible with V. This leads to the following definition:

Definition 6.3 *A valuation ring V is* compatible with *a fan F if it is compatible with every prime cone $\alpha \in F$.*

It is easily checked that if F is a fan compatible with V, then the specializations of the prime cones of F form a fan $\overline{F}$ in k_V, possibly trivial. The converse of this property is also true and will be of utmost importance in the sequel:

Proposition and Definition 6.4 *Let K be a field and V a valuation ring of K with residue field k_V. Let G be a fan in k_V. Then the collection F of all height zero generizations of the elements of G is a fan compatible with V, which is called the* pull-back *of G by V.*

Proof: We have to show that for any three elements $\alpha_1, \alpha_2, \alpha_3 \in F$, the product signature $\alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ is an ordering which lies in F. Assume for a moment that α is a prime cone. Let $\tau_1, \tau_2, \tau_3 \in G$ with $\alpha_i \rightarrow \tau_i$. It is immediate to check that α is compatible with V and that $\alpha \rightarrow \tau$, where $\tau = \tau_1 \cdot \tau_2 \cdot \tau_3$. Since G is a fan, we have that $\tau \in G$ and therefore α lies in F.

Therefore, all we have to show is that α is a prime cone. Clearly, the non-trivial part of this claim is that $\alpha(x+y) = +1$ if $\alpha(x) = \alpha(y) = +1$. To see this, since V is a valuation ring, either $z = x/y \in V$ or $z = y/x \in V$, and we are reduced to see that $\alpha(1+z) = +1$ if $\alpha(z) = +1$. Now, if z is not a unit of V, then $\alpha_k(1+z) = \tau_k(1) = +1$, and $\alpha(1+z) = +1$. Thus let z be a unit of V. The way α is defined, if $\alpha(z) = +1$, there are only two possibilities for the signs $\alpha_k(z)$: either all of them are $+1$ or only one is $+1$. In the first case the conclusion is immediate, so we suppose, to fix notations, $\alpha_1(z) = +1, \alpha_2(z) = \alpha_3(z) = -1$. Then, since z is a unit of V, the residue class $\overline{z}$ of z is positive in τ_1, and the sum $1 + \overline{z}$ is also > 0. In particular $1 + \overline{z} \neq 0$ and $1 + z$ is a unit of V. But from

$$+1 = \alpha(z) = \tau_1(\overline{z}) \cdot \tau_2(\overline{z}) \cdot \tau_3(\overline{z}) = \tau(\overline{z}),$$

we get that $\overline{z}$ is positive in τ, whence so is $1 + \overline{z}$ too, and

$$\alpha(1+z) = \tau_1(1+\overline{z}) \cdot \tau_2(1+\overline{z}) \cdot \tau_3(1+\overline{z}) = \tau(1+\overline{z}) = +1,$$

as claimed. □

This relationship between fans and valuations will be our main tool to construct fans. For instance, the fan F_τ of all generizations of τ described in Theorem 6.1 above is nothing but the pull-back of the trivial fan consisting of the single ordering τ.

We recall, next, two other important properties of fans and valuations. Let F be a fan of the ring A, $\mathfrak{p}$ its support, and set $K = \kappa(\mathfrak{p})$. Consider the family $\mathfrak{V}$ of valuation rings of A compatible with some element of F. Note that for every $\alpha \in F$, the valuations compatible with α form a chain:

$$V_\alpha \subset \cdots \subset K,$$

whose smallest element V_α is the convex hull of $\mathbb{Q}$ with respect to α and has an archimedian residue field. The first result states that in $\mathfrak{V}$, the number of distinct V_α is quite restricted:

Proposition 6.5 *Among the collection of valuation rings compatible with some element of F there are at most two distinct valuations with archimedean residue field.*

Proof: [Lm, 10.12]. □

In particular, there are at most two distinct convex hulls V_α, say V^1, V^2. Thus the collection of valuation rings in $\mathfrak{V}$ splits into two chains, whose smallest elements are V^1 and V^2. The second result we have in mind assures that these two chains coincide at a certain non-trivial point, and guarantees thereby the existence of a non-trivial valuation ring compatible with the fan. We encounter then the typical tree structure of the family $\mathfrak{V}$ of all valuation rings attached to a fan F:

$$\begin{tikzcd}
V_1^1 \arrow[r] & \cdots \arrow[r] & V_{r1}^1 \arrow[dr] & & & \\
& & & W_1 \arrow[r] & \cdots \arrow[r] & W_s \subset \mathcal{K}(Y) \\
V_1^2 \arrow[r] & \cdots \arrow[r] & V_{r2}^2 \arrow[ur] & & &
\end{tikzcd}$$

where W_i is the first valuation ring compatible with all orderings of F.

Next, we say that a valuation ring V *trivializes* a fan F or that F *trivializes along* V if they are compatible and the induced fan in the residue field k_V is trivial, that is, consists of at most two orderings. Then we have:

Theorem 6.6 (Trivialization theorem) *Let F be a fan of A. Then F trivializes along some valuation ring W, that is, W is compatible with F and the orderings of F have at most two different specializations in k_V.*

Proof: [Br1, 2.7], [AnBrRz2, Chapter IV]. □

We finish this section with an example which will be essential in Sections 16 and 17.

Example 6.7 Continuing with Example 6.2, given two distinct orderings τ_1, τ_2 of k, we consider the pull-back $F_{\tau_1,\tau_2}(x_1, \ldots, x_m)$ by V_m of the trivial fan formed by τ_1 and τ_2. Since the value group of V_m is $\mathbb{Z}^m$, the fan $F_{\tau_1,\tau_2}(x_1, \ldots, x_m)$ has exactly 2^{m+1} elements, and V_m is the biggest valuation ring along which F_{τ_1,τ_2} trivializes. Indeed, notice that if V_j is the valuation ring of K with $V_m \subset V_j$ and rank j, then $F_{\tau_1,\tau_2}(x_1, \ldots, x_m)$ is also compatible with V_j and induces a fan in its residue field with 2^{m+1-j} elements. In Section 17 we will also need the following technical remark: suppose that γ_1, γ_2 are the two orderings of the residue field $k_j = \text{qf}(B/(x_{m-j+1}, \ldots, x_m))$ of V_j, obtained by pulling back τ_1, τ_2 giving a constant sign to $x_1, \ldots, x_{m-j}$. Then, the pull back $F_{\gamma_1,\gamma_2}(x_{m-j+1}, \ldots, x_m)$ of γ_1, γ_2 by V_j, is a fan of 2^{j+1} elements which is a subfan of $F_{\tau_1,\tau_2}(x_1, \ldots, x_m)$, and therefore also compatible with V_m.

7 Centers of an algebraic fan

Here we will try to understand the geometric meaning of some of the results mentioned in the previous section about algebraic fans. Let $X \subset \mathbb{R}^p$ be a real algebraic set and $\mathcal{R}(X)$ its ring of regular functions. Let V be a valuation ring of $\mathcal{R}(X)$, $\mathfrak{p}$ its support and $Y \subset X$ the zero set of $\mathfrak{p}$. Hence V is a valuation ring of the field $\mathcal{K}(Y) = \kappa(\mathfrak{p})$ of rational functions of Y. We say that V is *finite on* X if it contains the ring $\mathcal{R}(Y) = \mathcal{R}(X)/\mathfrak{p}$. In this case the ideal $\mathfrak{q}/\mathfrak{p} = \mathfrak{m}_V \cap \mathcal{R}(Y)$, is called the *center of* V *in* $\mathcal{R}(Y)$, or more generally, the ideal $\mathfrak{q} \subset \mathcal{R}(X)$ is called the *center of* V *in* $\mathcal{R}(X)$. The zero set $Z \subset Y$ of $\mathfrak{q}$ is called the *center of* V *in* Y or *in* X.

Now let $F \subset \mathrm{Spec}_r(\mathcal{R}(X))$ be an algebraic fan of X with support also $\mathfrak{p}$. Thus F can be seen as a fan of the field $\mathcal{K}(Y)$ of rational functions of Y. Since the rank of a valuation of $\mathcal{K}(Y)$ is always $\leq \dim(Y)$, the chains of valuations of $\mathcal{K}(Y)$ have length $\leq \dim(Y)$. Hence as was remarked in the preceding section, the collection $\mathfrak{V}$ of all valuations compatible with some prime cone of F can be numbered in the form

$$\begin{array}{ccc}
V_1^1 \longrightarrow \cdots \longrightarrow V_{r_1}^1 & & \\
& \searrow & \\
& & W_1 \longrightarrow \cdots \longrightarrow W_s \subsetneqq \mathcal{K}(Y) \\
& \nearrow & \\
V_1^2 \longrightarrow \cdots \longrightarrow V_{r_2}^2 & &
\end{array}$$

where V_1^1, V_1^2 are the smallest valuations compatible with some prime cone of F and W_1 stands for the smallest valuation compatible with all prime cones of F. Moreover, by the trivialization theorem (Proposition 6.6), some W_i trivializes F.

For any j, let $\mathfrak{m}_j$ be the maximal ideal of W_j, k_j its residue field and $\lambda_j : \mathcal{K}(Y) \to k_j \cup \infty$ its associated place. If $W_j \subset W_i$, then $W_j/\mathfrak{m}_i$ is a valuation ring of k_i compatible with the fan F_i induced by F in k_i and we have $\lambda_j = \widetilde{\lambda_j} \circ \lambda_i$, where $\widetilde{\lambda_j} : k_i \to k_j \cup \infty$ is the place corresponding to $W_j/\mathfrak{m}_i$. Clearly, the residue field of $W_j/\mathfrak{m}_i$ is k_j, and the fans induced in it by F and F_i coincide. Therefore, if F trivializes along W_i, it trivializes also along W_j. In particular F always trivializes along W_1. The diagram above will be essential to understand when F is analytic.

First, suppose that $W_{s_0} \subset \cdots \subset W_s \subset \mathcal{K}(Y)$ are the valuations compatible with F which are finite on X, that is, which contain the ring $\mathcal{R}(Y) = \mathcal{R}(X)/\mathfrak{q}$ of regular functions of Y. Then for every $i = s_0, \ldots, s$ the valuation ring W_i has the center $\mathfrak{m}_i \cap \mathcal{R}(Y)$ in $\mathcal{R}(Y)$; these centers define prime ideals of $\mathcal{R}(X)$: $\mathfrak{q}_{s_0} \supset \cdots \supset \mathfrak{q}_s \supset \mathfrak{q}$, which are the *centers of F in $\mathcal{R}(X)$*. They are *real* prime ideals and their zero sets $Z_{s_0} \subset \cdots \subset Z_s \subset Y$ are the centers of F in X. In case F trivializes along W_i we call $\mathfrak{q}_i$ and Z_i *trivialization centers.* Furthermore, all prime cones of F make convex every center $\mathfrak{q}_i$, and F specializes to a fan in every residue field $\kappa(\mathfrak{q}_i)$, which consists of at most two elements if $\mathfrak{q}_i$ is a trivialization center. It is important to distinguish the centers of F from the centers of its elements. In fact, it may happen that all the orderings of F are centered at the same point a (that is, F is a 1pt-fan), but F itself is not centered at a, see Examples 7.2 *b), c)*, below (*Figures 16* and *18).*

Note that in general we do not know which W_i's are finite, and it may even happen that none of them is. However, the situation is quite good over the reals, where in addition to Proposition 5.6 we have:

Proposition 7.1 *Let α be a prime cone of $\mathcal{R}(X)$ that specializes to $a \in X$. Then, the convex hull V_α is finite on X and its center is the point a. Consequently, any valuation ring compatible with α is finite on X.*

Proof: Let $\mathfrak{p} = \operatorname{supp}(\alpha)$. Since V_α is compatible with α, it is convex with respect to $\leq_\alpha$, which implies $\mathbb{R} \subset V_\alpha$. Also, since α specializes to a, we have $f(a) - \varepsilon <_\alpha f(\alpha) <_\alpha f(a) + \varepsilon$ for every $f \in \mathcal{R}(X)$ and $\varepsilon > 0$. This equation just means that $f \pmod{\mathfrak{p}} \in \mathcal{R}(X)/\mathfrak{p}$, is bounded by $f(a) \pm \varepsilon \in \mathbb{R} \subset V_\alpha$. By the convexity of V_α we deduce $\mathcal{R}(X)/\mathfrak{p} \subset V_\alpha$. Moreover, if $f(a) = 0$ we have $-\varepsilon <_\alpha f(\alpha) <_\alpha \varepsilon$ for all $\varepsilon > 0$, that is, $f \pmod{\mathfrak{p}}$ is infinitesimal with respect to $\mathbb{R}$. This shows that $\mathfrak{m}_a/\mathfrak{p} \subset \mathfrak{m}_\alpha$. The result follows immediately from these inclusions. □

Now let us come back to our algebraic fan F with support Y and consider any algebraic compactification Y' of Y. Then all valuation rings of $\mathfrak{V}$ are valuation rings of $\mathcal{K}(Y')$ and by Propositions 5.6 and 7.1 they are finite on Y'. Furthermore, if one of them has the center Z' in Y', it is finite on Y if and only if $Z = Z' \cap Y \neq \emptyset$ and in that case Z is its center in Y. Hence we can say again that the centers of F always exist, maybe at infinity, which conforms our remarks after Proposition 5.6. Note also that the last proposition, together with Proposition 6.5, gives a different proof of the fact that the orderings of a fan cannot specialize to more than two points (Proposition 5.3 *a)*).

Next we present some examples for the construction of fans and its centers. Some of them are the fans that we will use later to decide which of the semialgebraic sets of Examples 1.1 are not basic semianalytic. For simplicity we work in $\mathbb{R}^p$ which can be seen as an affine chart of $X = S^p$ after removing one point.

Examples 7.2 *a)* Consider $X = Y = \mathbb{R}^2$, the x-axis Z, and the point $a = (0,0)$. Let F be the fan consisting of the orderings $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ represented in *Figure 13*. This F is the pull-back of the trivial fan τ_1, τ_2 of $\mathcal{R}(Z) = \mathbb{R}(\mathrm{x})$ by the valuation ring $W_2 = \mathbb{R}[\mathrm{x},\mathrm{y}]_{(\mathrm{y})}$, where τ_1, τ_2 are the points of $\mathrm{Spec}_r(\mathcal{R}(Z))$ defined by the half-branches of the x-axis at a. We have $\alpha_1, \alpha_3 \to \tau_1$ and $\alpha_2, \alpha_4 \to \tau_2$. The chain of valuation rings compatible with F is $W_1 \subset W_2 \subset \mathbb{R}(\mathrm{x},\mathrm{y})$, where $W_2 = \mathbb{R}[\mathrm{x},\mathrm{y}]_{(\mathrm{y})}$ and W_1 is the discrete, rank 2 valuation ring contained in W_2, obtained as the comp site of W_1 with $\mathbb{R}[\mathrm{x}]_{(\mathrm{x})}$. In particular, W_1 is centered at a and F can also be seen as the pull-back by W_1 of the unique order of $k_{W_1} = \mathbb{R}$. Then F is centered at both the point a and the x-axis and trivializes along both valuation rings.

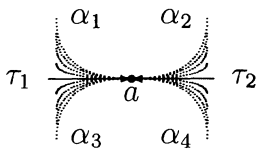


Figure 13

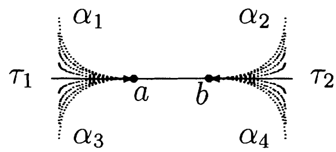


Figure 14

Let now τ_1, τ_2 be half-branches of the x-axis in two different points a, b, and let $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be, as above, the pull back of $\{\tau_1, \tau_2\}$ by W_2 (*Figure 14*). Then F is a fan which has W_2 as the unique non-trivial compatible valuation ring. Therefore in this case the unique center of F is the x-axis.

b) Let, again $X = Y = \mathbb{R}^2$, $a = (0,0)$, and let Z be the nodal cubic $x^2 - y^2 - y^3 = 0$ with the singular point at a. Set $\mathfrak{q} = (\mathrm{x}^2 - \mathrm{y}^2 - \mathrm{y}^3)\mathbb{R}[\mathrm{x},\mathrm{y}]$. Let τ_1, τ_2 be the half-branches of Z at a represented in *Figure 15.* Let α_1, α_3 (resp. α_2, α_4) be the two generizations of τ_1 (resp. τ_2) in $\mathrm{Spec}_r(\mathbb{R}[\mathrm{x},\mathrm{y}])$. Then $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a fan, which is the pull-back of $\{\tau_1, \tau_2\}$ by $W_2 = \mathbb{R}[\mathrm{x},\mathrm{y}]_{\mathfrak{q}}$. It has two nontrivial compatible valuation rings, $W_1 \subset W_2$, where W_1 is the rank two valuation ring of $\mathbb{R}(\mathrm{x},\mathrm{y})$ obtained as composite of W_2 and the valuation ring $\overline{W}_1$ of $\mathcal{K}(Y)$ corresponding to the analytic branch of Z on which both τ_i's are supported. Clearly F trivializes along both valuation rings and has two centers: the nodal curve Y and its singular point a.

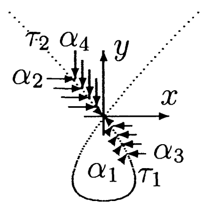


Figure 15

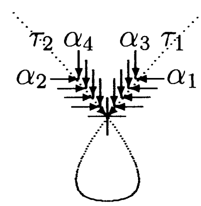


Figure 16

With the same X and Z, let now τ_1, τ_2 be the half-branches of Z represented in *Figure 16* and $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ its pull-back by W_2. Then F has $W_2 = \mathbb{R}[\mathrm{x},\mathrm{y}]_{\mathfrak{q}}$ as the unique compatible valuation ring, since there is no valuation ring of $\mathcal{K}(Y)$ compatible simultaneously with τ_1 and τ_2. The family $\mathfrak{V}$ of valuations compatible with some element of F is:

$$\begin{array}{ccc} V_1^1 & & \\ & \searrow & \\ & & W_2 \subset \mathcal{K}(X) \\ & \nearrow & \\ V_1^2 & & \end{array}$$

where V_1^1 and V_1^2 are the composite of W_2 with the valuation rings of $\mathcal{K}(Z)$ corresponding to the two analytic branches of Z at a. Therefore the curve Z is the only center of F even though the four orderings of F are centered at a, since both rings V_1^1 and V_1^2 are centered at the maximal ideal of the point a.

c) Let $X \subset \mathbb{R}^3$ be the algebraic subset of equation $z^2 - y^2 + y^3 = 0$, and

$a = (0,0,0) \in X$. Let Z be the nodal cubic $X \cap \{x = 0\}$, and $\mathfrak{q}$ its ideal in $\mathcal{R}(X)$. Let τ_1 and τ_2 be the two half-branches of Z at a in the same analytic branch, and $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ the pull-back of $\{\tau_1, \tau_2\}$ to $\mathcal{K}(X)$ by the valuation ring $W_2 = \mathcal{R}(X)_{\mathfrak{q}}$ (*Figure 17*). F is a fan compatible with the valuation rings $W_1 \subset W_2$ of $\mathcal{K}(X)$, where W_1 is the composite of V_2 with the valuation of $\mathcal{K}(Z)$ defined by the analytic branch of Z on which the τ_i's are supported. The centers of F in X are Z and a.

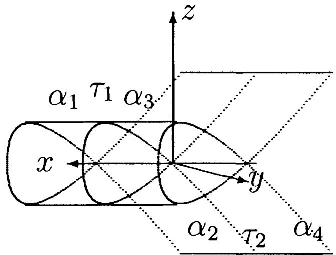


Figure 17

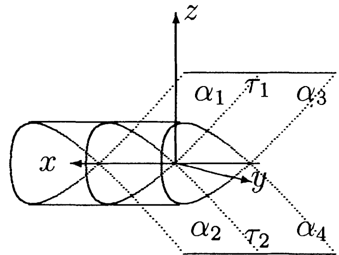


Figure 18

With the same X and Z, consider now the fan F depicted in *Figure 18*, that is, τ_1 and τ_2 are placed now over different analytic branches of Z at a. The situation varies, since F is only compatible with the valuation ring $W_2 = \mathcal{R}(X)_{\mathfrak{q}}$. Indeed, the family $\mathfrak{V}$ of valuations compatible with some element of F is:

$$\begin{tikzcd} V_1^1 \arrow[dr] & \\ & W_2 \subset \mathcal{K}(X) \\ V_1^2 \arrow[ur] & \end{tikzcd}$$

where V_1^1 and V_1^2 are the composite of W_2 with the valuation rings of $\mathcal{K}(Z)$ corresponding to the two analytic branches of Z at a. Thus F has only one center in X: the curve Z.

d) Now, let $X \subset \mathbb{R}^3$ be Whitney's umbrella, that is, X is defined by the equation $x^2 z - y^2 = 0$. Set $a = (0,0,0)$, and let Z be the z-axis, whose ideal in $\mathcal{R}(X)$ will be denoted by $\mathfrak{q}$. Let τ be the upward half-branch of Z at a, and let $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \operatorname{Spec}_r(\mathcal{K}(X))$ be the set of generizations of τ (*Figure 19*).

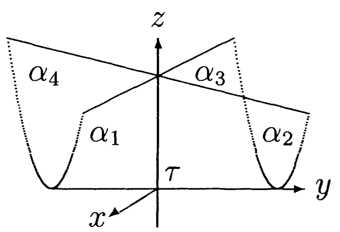


Figure 19

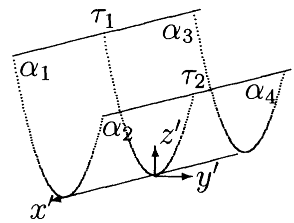


Figure 20

F is a fan with two compatible valuation rings $W_1 \subset W_2$. To see it, consider the localization $\mathcal{R}(X)_{\mathfrak{q}}$; its integral closure W_2 in $\mathcal{K}(X)$ is a discrete valuation ring compatible with F and with residue field $k_2 = \mathcal{K}(Z)(\sqrt{\mathsf{z}})$. Then F induces a trivial fan in k_2 consisting of the two extensions of τ to k_2: one making $\sqrt{\mathsf{z}}$ positive and the other making it negative. Finally, W_1 is the composite of W_2 with the valuation ring $\overline{W}_1 = \mathbb{R}[\sqrt{\mathsf{z}}]_{(\sqrt{\mathsf{z}})}$ of k_2. Thus, F has Z and a as centers, and trivializes along both of them.

All this can be seen more geometrically in the following way, which is in fact the way we produced F. Consider the normalization $X' : z' - y'^2 = 0$ of $\mathcal{R}(X)$, where $x = x', y = x'y', z = z'$ (*Figure 20*). We will construct F on X'. Let Z' be the parabola $X' \cap \{x' = 0\}$ and $\mathfrak{q}$ the ideal of Z' in $\mathcal{R}(X')$; then, $W_2 = \mathcal{R}(X')_{\mathfrak{q}}$ and the residue field k_2 of W_2 is $\mathcal{K}(Z')$. Let τ_1 and τ_2 be the orderings of $\mathcal{K}(Z')$ represented by the half branches of Z' at a. Notice that they both contract to τ over Z. Now, F is the pull-back of $\{\tau_1, \tau_2\}$ by W_2. Finally, notice that W_1 is the composite of W_2 with the localization of $\mathcal{R}(Z')$ at the origin.

e) Let $X \subset \mathbb{R}^3$ be the set $x^2 + z^4 - z^2(y^2 + y^3) = 0$, Z the y-axis, $\mathfrak{q}$ the ideal of Z in $\mathcal{R}(X)$ and τ the right-ward half-branch of Z at $a = (0,0,0)$. Let $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of generizations of τ in $\mathrm{Spec}_r(\mathcal{K}(X))$ (*Figure 21*).

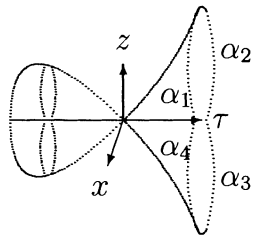


Figure 21

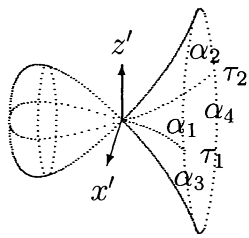


Figure 22

Then F is a fan with a unique compatible valuation ring W. This can be seen as in Example *d)* above: consider the normalization $X' : x'^2 + z'^2 - (y'^2 + y'^3) = 0$ of $\mathcal{R}(X)$, where $x = x'z', y = y', z = z'$. Set $Z' = X' \cap \{z' = 0\}$ and let $\mathfrak{q}'$ be the ideal of Z' in $\mathcal{R}(X')$. Let τ_1 and τ_2 be the two right-ward half-branches of Z' and put $W = \mathcal{R}(X')'_{\mathfrak{q}}$ (*Figure 22*). Then F is the pull-back of $\{\tau_1, \tau_2\}$ by W, and therefore W is compatible with F. Moreover, the residue field of W is the field of rational functions of Z', which is the nodal cubic of Example *b)* above. Since τ_1 and τ_2 are supported on different analytic branches, it follows that there are no other valuation ring compatible with F. Thus, although at a first glance this example looked quite similar to the previous one, the structure of the centers is different, and to appreciate it, one has to climb to some finite extension of the given variety. This idea of considering finite extensions will be crucial for the characterization of analytic fans.

f) Finally, consider in $\mathbb{R}^4$ the hyperplane $Z_2 : t = 0$; the plane $Z_1 : z = t = 0$, and the point $a = (0,0,0,0)$. Let τ_1, τ_2 be any two orderings in Z_1 which specialize to a, but do not specialize to the same 1-dimensional point in $\mathrm{Spec}_r(\mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{t}])$. Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the set of generizations of τ_1, τ_2 in $\mathrm{Spec}_r(\mathcal{R}(Z_2))$ and denote by $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ the set of generizations of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ making $\mathtt{t} > 0$ (*Figure 23*).

Then F is a fan with two compatible valuation rings $W_1 \subset W_2$ where $W_2 = \mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{t}]_{(\mathtt{t})}$ and W_1 is the composite of W_2 with the valuation ring $\mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z}]_{(\mathtt{z})}$ in the residue field $\mathbb{R}(\mathtt{x},\mathtt{y},\mathtt{z})$ of V_2. Then F has both Z_1 and Z_2 as centers, but only trivializes along W_1, since in the residue field $\mathbb{R}(\mathtt{x},\mathtt{y},\mathtt{z})$, the induced fan consists of the four orderings $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and therefore it is not trivial.

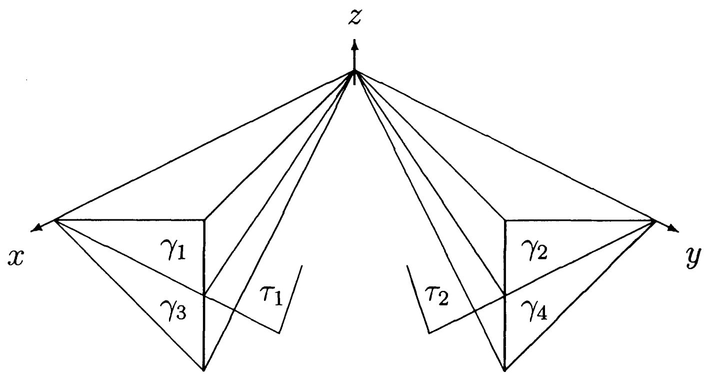


Figure 23

Although we have not presented all results needed to do it, we invite the reader to prove that the fans described in *Figures 14, 16, 18, 21* and *22* are *not* analytic.

8 Henselization of algebraic fans

In the examples of the previous section we have talked of analytic branches. It is time to make this more precise and to analyze the importance of them in the characterization of analytic fans. To that end we will decompose the extension from the local ring of regular functions at a point a to the ring of germs of convergent power series into two steps, introducing an intermediate ring: the henselization.

Let $X \subset \mathbb{R}^p$ be a real algebraic set and $\mathcal{R}(X)$ its ring of regular functions. Consider a point $a \in X$ and its maximal ideal $\mathfrak{m}_a \subset \mathcal{R}(X)$; we recall the notation $\mathcal{R}_{X,a} = \mathcal{R}(X)_{\mathfrak{m}_a}$. Let $\mathcal{O}_{X,a}$ be the ring of analytic function germs at a. We have the canonical homomorphism $\varphi = \varphi_a : \mathcal{R}(X) \to \mathcal{O}_{X,a}$ and the associated map $\varphi^* = \varphi_a^* : \mathrm{Spec}_r(\mathcal{O}_{X,a}) \to \mathrm{Spec}_r(\mathcal{R}(X))$ as in Section 4. To deal with φ and φ^* we can suppose that a is the origin of $\mathbb{R}^p$. Furthermore, we will need an explicit description of the rings and homomorphisms involved.

To start with let $\mathbb{R}\{\mathrm{x}\}$ denote the ring of convergent power series in the variables $\mathrm{x} = (\mathrm{x}_1, \dots, \mathrm{x}_p)$. Consider the ideal $I \subset \mathbb{R}[\mathrm{x}]$ of $X \subset \mathbb{R}^p$, and the extension $I^{\mathrm{con}} = I\mathbb{R}\{\mathrm{x}\}$. Since $a \in X$ is the origin, I^{con} is a proper ideal. Then there is a canonical homomorphism $\mathbb{R}\{\mathrm{x}\}/I^{\mathrm{con}} \to \mathcal{O}_{X,a}$, which induces an isomorphism $\mathbb{R}\{\mathrm{x}\}/J^{\mathrm{con}} \to \mathcal{O}_{X,a}$, where J^{con} is the real-radical I^{con} (Risler's real Nullstellensatz, [Rs, 4.1]). Therefore, the epimorphism $\mathbb{R}\{\mathrm{x}\}/I^{\mathrm{con}} \to \mathbb{R}\{\mathrm{x}\}/J^{\mathrm{con}}$ induces an identification between their real spectra.

Now let $\mathbb{R}\{\mathrm{x}\}_{\mathrm{alg}}$ be the ring of *algebraic power series*, that is, the series of $\mathbb{R}\{\mathrm{x}\}$ which are algebraic over the ring of polynomials $\mathbb{R}[\mathrm{x}]$ ([BCR, 8.1-8.3, pp.143-155]). It is well known that $\mathbb{R}\{\mathrm{x}\}_{\mathrm{alg}}$ is the henselization of the polynomial ring $\mathbb{R}[\mathrm{x}]$ localized at the maximal ideal (x) ([BCR, 8.7.8, p.170]). Moreover the extension $\mathbb{R}\{\mathrm{x}\}_{\mathrm{alg}} \to \mathbb{R}\{\mathrm{x}\}$ is regular and faithfully flat. As above we have the extension $I^{\mathrm{alg}} = I\mathbb{R}\{\mathrm{x}\}_{\mathrm{alg}}$ and the ring $\mathbb{R}\{\mathrm{x}\}_{\mathrm{alg}}/I^{\mathrm{alg}}$ is the henselization of the localization $\mathcal{R}_{X,a}$ and will be denoted by $\mathcal{R}^h_{X,a}$. For the definition and the general properties of henselizations, extremely important in this work, we refer to [EGA, §18, pp.109-185], [Ng, Chapter VII, §§43-45, pp.179-202] and [Rd, Chapters VIII-X, pp.80-111]. On the other hand we consider the real-radical J^{alg} of I^{alg}, and the ring $\mathcal{N}_{X,a} = \mathbb{R}\{\mathrm{x}\}_{\text{alg}}/J^{\text{alg}}$. Note that the elements of the ring $\mathcal{N}_{X,a}$ are analytic function germs at a; they are called *algebraic function germs at a*. From the good properties of the extension $\mathbb{R}\{\mathrm{x}\}_{\text{alg}} \to \mathbb{R}\{\mathrm{x}\}$ it follows $J^{\text{con}} = J^{\text{alg}}\mathbb{R}\{\mathrm{x}\}$. Summing up, we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{R}(X) \subset \mathcal{R}_{X,a} & \xrightarrow{\theta} & \mathcal{R}^h_{X,a} & \xrightarrow{\phi'} & \mathbb{R}\{\mathrm{x}\}/I^{\text{con}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{N}_{X,a} & \xrightarrow{\phi} & \mathcal{O}_{X,a} \end{array}$$

where the horizontal homomorphisms θ, ϕ', ϕ are regular and faithfully flat, and the vertical homomorphisms induce identifications between the real spectra of the corresponding rings. From all these details we keep the following decomposition of φ:

$$\mathcal{R}(X) \xrightarrow{\theta} \mathcal{R}^h_{X,a} \to \mathcal{N}_{X,a} \xrightarrow{\phi} \mathcal{O}_{X,a}$$

and the induced one of φ^*:

$$\operatorname{Spec}_r(\mathcal{O}_{X,a}) \xrightarrow{\phi^*} \operatorname{Spec}_r(\mathcal{N}_{X,a}) \cong \operatorname{Spec}_r(\mathcal{R}^h_{X,a}) \xrightarrow{\theta^*} \operatorname{Spec}_r(\mathcal{R}(X))$$

We have the following important fact:

Proposition 8.1 *a) ϕ^* is surjective.*

b) Let $\mathcal{G}_a$ be the set of all prime cones of $\operatorname{Spec}_r(\mathcal{R}(X))$ specializing to a. Then θ^* is a homeomorphism from $\operatorname{Spec}_r(\mathcal{R}^h_{X,a})$ onto $\mathcal{G}_a$.

c) Property b) is preserved by base change, that is, let B be any $\mathcal{R}(X)$-algebra and denote by $\mathcal{G}'_a$ the set of prime cones of B lying over some prime cone of $\mathcal{G}_a$. Then the homomorphism obtained by base change $\theta' : B \to \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}(X)} B$ induces a homeomorphism from $\operatorname{Spec}_r(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}(X)} B)$ onto $\mathcal{G}'_a$.

Proof: *a)* See [Rz4], [AnBrRz2, Chapter VII].

b) It follows from the general properties of the real spectrum, see [AlRy, 3.3.4], [AnBrRz2, Chapter III], [Ry, 3.7].

c) First of all, replacing B by $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}(X)} B$ we may assume B is a $\mathcal{R}_{X,a}$-algebra and we have $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}(X)} B = \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} B$. Let us see that θ' is bijective. Let $\beta \in \mathcal{G}'_a$ be a prime cone of $\operatorname{Spec}_r(B)$, lying over $\alpha \in \mathcal{G}_a$. Set $\mathfrak{p} \subset B$ for the support of β. It is known ([CsRy, Proposition 4.3]) that the fiber of $\operatorname{Spec}_r(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} B)$ over β coincides with the real spectrum of $(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} B) \otimes_B \kappa(\mathfrak{p}) = \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} \kappa(\mathfrak{p})$. Therefore, we have to check that β extends to a unique point of $\operatorname{Spec}_r(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} \kappa(\mathfrak{p}))$.

Let R be the real closure of $\kappa(\mathfrak{p})$ with respect to $\leq_\beta$. In particular β defines an order preserving homomorphism from $\mathcal{R}_{X,a}$ into R obtained by the composition of the canonical homomorphism into $\kappa(\mathfrak{p})$ and the inclusion of this field into R. Let W_0 and W denote, respectively the convex hull of $\mathbb{Q}$ in $\kappa(\mathfrak{p})$ and R. Since R is real closed, W is a henselian valuation ring dominating W_0 (the local monomorphism is represented by the dotted line in the diagram). Moreover, since β lies over $\alpha \in \mathcal{G}_a$, W_0 is finite on X and the homomorphism $\mathcal{R}_{X,a} \to W_0$ is local. In conclusion we have a local homomorphism of $\mathcal{R}_{X,a}$ into the henselian ring W, which extends uniquely to a local homomorphism $\beta' : \mathcal{R}^h_{X,a} \to W$ making the following diagram commutative:

$$\begin{tikzcd}
 & W \arrow[r, phantom, "\subset"] & R \\
\mathcal{R}^h_{X,a} \arrow[r] \arrow[ur, "\beta'"] & \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} \kappa(\mathfrak{p}) \arrow[ur, "\gamma"] & \\
\mathcal{R}_{X,a} \arrow[u] \arrow[r] & W_0 \arrow[uu, dotted, bend left] \arrow[r, phantom, "\subset"] & \kappa(\mathfrak{p}) \arrow[ul] \arrow[uu, dotted, bend right, "\beta"']
\end{tikzcd}$$

Thus from the universal property of the tensor product we get a homomorphism $\gamma : \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} \kappa(\mathfrak{p}) \to R$ keeping everything commutative. This γ defines the required extension of β. Moreover, since by the properties of the henselization there is a unique β' as above extending β, it follows that γ is unique. This shows that θ' is bijective. We skip the fact that θ' is a homeomorphism, since we will not use it the future. $\square$

Thus, from Proposition 8.1 *b)*, we can complete the above row of mappings to

$$\mathrm{Spec}_r(\mathcal{O}_{X,a}) \xrightarrow{\phi^*} \mathrm{Spec}_r(\mathcal{N}_{X,a}) \cong \mathrm{Spec}_r(\mathcal{R}^h_{X,a}) \xrightarrow{\theta^*} \mathcal{G}_a \subset \mathrm{Spec}_r(\mathcal{R}(X)).$$

Let us explain how this preparation applies to our problem of characterizing the algebraic fans which are analytic. Consider an algebraic fan $F \subset \mathrm{Spec}_r(\mathcal{R}(X))$ and assume that it is analytic at a. Then, by Proposition 5.3 *b)* $F \subset \mathcal{G}_a$. Furthermore, let $F_a \subset \mathrm{Spec}_r(\mathcal{O}_{X,a})$ be a lifting of F (see Definition 5.1 and Proposition 5.2). Then $\phi^*(F_a) = \theta^{*-1}(F)$ is a fan of $\mathcal{R}^h_{X,a}$ and since θ^* is a homeomorphism, this is independent of F_a. It will be denoted by F^h_a and called the *henselization of F at a*. With this terminology, we have just seen that *if an algebraic fan is analytic at a, then its henselization at a is a fan.* It is important to stress that although the henselization of a fan always exists as a set, *it need not be a fan.* Going further we will see in the next section that to decide whether F^h_a is a fan will be the hard task to do.

Let $\mathfrak{p}$ be the support of the fan F, and $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ the associated prime ideals of $\mathfrak{p}\mathcal{R}^h_{X,a}$. By the properties of the henselization, this latter ideal is radical and $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ have all height $\mathrm{ht}(\mathfrak{p})$; furthermore, they are the unique prime ideals of $\mathcal{R}^h_{X,a}$ lying over $\mathfrak{p}$ ([Ng, 43.20, p.187]), and every extension $\mathcal{R}_{X,a}/\mathfrak{p} \subset \mathcal{R}^h_{X,a}/\mathfrak{p}_i$ is algebraic. We call these $\mathfrak{p}_i$'s the *henselian branches of $\mathfrak{p}$ at a.* From a geometric point of view, if Y denotes the subvariety of X defined by $\mathfrak{p}$ and $Y_1, \dots, Y_s$ denote the zero set germs defined by the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ we will say that $Y_1, \dots, Y_s$ are the henselian branches of Y. Obviously, the supports of the orderings of the henselization F^h_a are among the $\mathfrak{p}_i$'s, but they may be different. However, if F^h_a is a fan it follows from the very definition that the support of all orderings of F^h_a is the same prime ideal. Putting all this together we have proved a first important necessary condition for a fan to be analytic:

Proposition 8.2 *Let F be an algebraic fan of X with support $\mathfrak{p} \subset \mathcal{R}(X)$. If F is analytic at $a \in X$, then its henselization F^h_a at a is a fan. In particular, all orderings of F are centered at a, and extend to the same henselian branch of $\mathfrak{p}$ at a.*

Since, as we have pointed out, the homomorphism $\mathcal{R}^h_{X,a} \to \mathcal{N}_{X,a}$ induces an identification between real spectra we can think of the henselization F_a^h of our fan F as living in $\mathrm{Spec}_r(\mathcal{N}_{X,a})$. Concerning the more algebraic matter of computing the height and the dimension of F_a^h seen as fan of $\mathrm{Spec}_r(\mathcal{N}_{X,a})$, we have initially to be careful since the homomorphism $\mathcal{R}_{X,a} \to \mathcal{N}_{X,a}$ only behaves well with respect to real ideals. However, we have:

Proposition 8.3 *Let $\alpha \in \mathcal{G}_a$. Let $\alpha^h = (\theta^*)^{-1}(\alpha)$ and let α' be the point defined by α^h in $\mathrm{Spec}_r(\mathcal{N}_{X,a})$. Then we have $\mathrm{ht}(\alpha) = \mathrm{ht}(\alpha^h) = \mathrm{ht}(\alpha')$ and $\dim(\alpha) = \dim(\alpha^h) = \dim(\alpha')$.*

Proof: Let $\mathfrak{p}$ be the support of α. Consider first the support $\mathfrak{p}^h$ of α^h in $\mathcal{R}^h_{X,a}$. Since this ring is the henselization of $\mathcal{R}_{X,a}$ and $\mathfrak{p}^h$ lies over $\mathfrak{p}$, we deduce that $\mathfrak{p}$ is one of the associated prime ideals of $\mathfrak{p}\mathcal{R}^h_{X,a}$ and by flatness $\mathrm{ht}(\mathfrak{p}^h) = \mathrm{ht}(\mathfrak{p}^h \cap \mathcal{R}_{X,a}) = \mathrm{ht}(\mathfrak{p})$, so that $\mathrm{ht}(\alpha^h) = \mathrm{ht}(\alpha)$. Next, by Chevalley's theorem the two rings $\mathcal{R}_{X,a}/\mathfrak{p}$ and $\mathcal{R}^h_{X,a}/\mathfrak{p}^h$ have the same dimension, so that $\dim(\alpha) = \dim(\alpha^h)$.

Now let $\mathfrak{p}'$ be the support of α' in $\mathcal{N}_{X,a}$. Since $\mathfrak{p}^h$ is the support of α^h, it is a real prime ideal. Hence $\mathfrak{p}^h \supset J^h$ and therefore $\mathfrak{p}' = \mathfrak{p}^h/J^h$. In other words, it follows that the epimorphism $\mathcal{R}^h_{X,a} \to \mathcal{N}_{X,a}$ induces an isomorphism $\mathcal{R}^h_{X,a}/\mathfrak{p}^h \to \mathcal{N}_{X,a}/\mathfrak{p}'$. Consequently α^h and α' have the same dimension. Finally, it is clear that $\mathrm{ht}(\mathfrak{p}^h) \geq \mathrm{ht}(\mathfrak{p}')$, but the equality is not obvious because in passing from $\mathcal{R}^h_{X,a}$ to $\mathcal{N}_{X,a}$ we might loose some non-real prime ideals contained in $\mathfrak{p}^h$. However, since $\alpha \to a$, by the dimension theorem for real spectra ([Rz3, Th.I]), there is a specialization chain $\beta \to \cdots \to a$ of length exactly $\mathrm{ht}(\mathfrak{p}^h)$. The supports of the prime cones of this chain are all real prime ideals, and so they give a chain of prime ideals of $\mathcal{N}_{X,a}$ of length $\mathrm{ht}(\mathfrak{p}^h)$, whose biggest element is $\mathfrak{p}'$. Thus, $\mathrm{ht}(\mathfrak{p}^h) \leq \mathrm{ht}(\mathfrak{p}')$, and we are done. $\square$

An immediate consequence of the proposition is:

Corollary 8.4 *Let F be an algebraic fan of X which is analytic at $a \in X$. Then the henselization F_a^h of F at a has the same height and dimension as F, either in $\mathcal{R}^h_{X,a}$ or in $\mathcal{N}_{X,a}$.*

From the above results it is now apparent why the algebraic fans F in Examples 7.2 *a) Figure 14* and *c) Figure 18* are not analytic. In the first the orderings of F specialize to two different points. In the latter the orderings specialize to the same point a, but they extend to two different henselian branches at a: α_1, α_3 extend to $X_{1a} : z^2 - y^2 + y^3 = 0, yz \geq 0$, and α_2, α_4 to $X_{2a} : z^2 - y^2 + y^3 = 0, yz \leq 0$. However we do not see yet why Examples 7.2 *b) Figure 16* and *e)* are not analytic. As a matter of fact, this involves the most subtle condition in the characterization of analyticity, which will be described in Section 9. But before doing that, we have to clarify completely the behaviour of henselizations.

9 A going-down theorem for fans

Let X be a real algebraic set and $a \in X$. In the previous section we have seen that a necessary condition for an algebraic fan $F \subset \mathcal{G}_a$ to be analytic at a is that the henselization F_a^h is a fan. Here we will show that this is indeed also sufficient, since any fan from $\mathcal{N}_{X,a}$ extends to a fan in $\mathcal{O}_{X,a}$. Therefore our goal from next section on will be to characterize in which conditions the set F_a^h –which is completely determined by F and has its same cardinality– is a fan. The main tool to prove the result of this section is M. Artin's approximation theorem, and we will work in the most abstract context to single out the essential parts of the argument.

Let $\phi : A \to B$ be a local homomorphism of (local) rings, that is, $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where $\mathfrak{m}_A, \mathfrak{m}_B$ denote respectively the maximal ideals of A, B. We will say that ϕ *has the approximation property* if given a system of polynomial equations $f_i(\mathbf{x}_1, \dots, \mathbf{x}_q) = 0,\ 1 \leq i \leq p$, with coefficients in A, any solution $\mathbf{x}_j = b_j \in B$ can be arbitrarily approximated in the adic topology by solutions $\mathbf{x}_j = a_j \in A$. The relevant example for us will be the homomorphism $\phi : \mathcal{N}_{X,a} \to \mathcal{O}_{X,a}$ of the preceding section, since it has the approximation property. The main result is:

Proposition 9.1 *Let $\phi : A \to B$ be a local homomorphism which has the approximation property. Suppose that A and B are domains and let K and L be their quotient fields. Then any fan F of K extends to a fan F' of L with $\#(F') = \#(F)$.*

Proof: Let $F = \{\alpha_i, 1 \leq i \leq 2^m\}$ and set P_i for the positive cone of α_i in A, that is, P_i is the set of elements of A which are positive in α_i. We seek 2^m orderings β_i of L such that $\beta_i|_K = \alpha_i$ (that is with $g(\beta_i) > 0$ for $g \in P_i$) and $\beta_i \cdot \beta_j \cdot \beta_k = \beta_l$ whenever $\alpha_i \cdot \alpha_j \cdot \alpha_k = \alpha_l$. To that end, we consider the product space

$$\Sigma = \mathrm{Spec}_r(L) \times \cdots \times \mathrm{Spec}_r(L)$$

of 2^m copies of $\mathrm{Spec}_r(L)$, and set $E = E_1 \cap E_2 \subset \Sigma$ where E_1 is the set

$$\{(\beta_i)_{1 \leq i \leq 2^m} \in \Sigma : g(\beta_i) > 0 \text{ for } g \in P_i\ 1 \leq i \leq 2^m\}$$

and E_2 is the set

$$\{(\beta_i)_{1\leq i\leq 2^m} \in \Sigma : f(\beta_i)f(\beta_j)f(\beta_k)f(\beta_l) > 0 \text{ for } f \in L,\ \alpha_i\cdot\alpha_j\cdot\alpha_k = \alpha_l\}.$$

We have to check that $E \neq \emptyset$. Rewriting, we have $E_1 = \bigcap_{\mathbf{g}} E_{1\mathbf{g}}$ where the intersection runs over all tuples $\mathbf{g} = (g_i)_{1\leq i\leq 2^m} \in P_1 \times \cdots \times P_{2^m}$, and $E_{1\mathbf{g}}$ is defined as

$$\{\beta_1 \in \operatorname{Spec}_r(L) : g_1(\beta_1) > 0\} \times \cdots \times \{\beta_{2^m} \in \operatorname{Spec}_r(L) : g_{2^m}(\beta_{2^m}) > 0\}.$$

Similarly $E_2 = \bigcap_f E_{2f}$, where the intersection runs over all $f \in L$ and E_{2f} is given by

$$\bigcup_{\varepsilon\in\mathcal{E}} \{\beta_1 \in \operatorname{Spec}_r(L) : \varepsilon_1 f(\beta_1) > 0\} \times\cdots\times \{\beta_{2^m} \in \operatorname{Spec}_r(L) : \varepsilon_{2^m} f(\beta_{2^m}) > 0\},$$

where $\mathcal{E}$ represents the set of all 2^m-tuples $\varepsilon = (\varepsilon_i)_{1\leq i\leq 2^m}$ with $\varepsilon_i = \pm 1$ for all i and $\varepsilon_i\varepsilon_j\varepsilon_k = \varepsilon_l$ whenever $\alpha_i\cdot\alpha_j\cdot\alpha_k = \alpha_l$.

Now suppose $E = \emptyset$. Then, since Σ is compact and all the sets $E_{1\mathbf{g}}, E_{2f}$ are closed, there exist $\mathbf{g}^1, \ldots, \mathbf{g}^p, f_1, \ldots, f_q$ such that

$$E_{1\mathbf{g}^1} \cap \cdots \cap E_{1\mathbf{g}^p} \cap E_{2f_1} \cap \cdots \cap E_{2f_q} = \emptyset.$$

Rewriting this we get

$$\bigcup_{\varepsilon\in\mathcal{E}^q} E_\varepsilon(1) \times \cdots \times E_\varepsilon(2^m) = \emptyset,$$

where $\varepsilon = (\varepsilon_i^j)_{\substack{1\leq i\leq 2^m\\ 1\leq j\leq q}}$ and

$$E_\varepsilon(i) = \{g_i^1 > 0, \ldots, g_i^p > 0, \varepsilon_i^1 f_1 > 0, \ldots, \varepsilon_i^q f_q > 0\}.$$

Hence, for every $\varepsilon \in \mathcal{E}^q$ there is some i such that $E_\varepsilon(i) = \emptyset$. By the abstract Positivstellensatz, ([BCR, 4.4.1, p.82]), this equality is equivalent to the fact that the equation

$$\sum_{\nu,\mu} \mathrm{y}_{i\nu\mu}^2 (g_i^1)^{\nu_1}\cdots(g_i^p)^{\nu_p}(\varepsilon_i^1 f_1)^{\mu_1}\cdots(\varepsilon_q^{q_i})^{\mu_q} =$$
$$= -(g_i^1)^{2r_1}\cdots(g_i^p)^{2r_p}(\varepsilon_i^1 f_1)^{2s_1}\cdots(\varepsilon_i^q f_1)^{2s_q},$$

where $\nu_k, \mu_l = 0, 1$, has a solution, say

$$\mathbf{y}_{i\nu\mu} = y_{i\nu\mu} \in B.$$

Collecting these equations for all $\varepsilon \in \mathcal{E}^q$ and replacing the f_j's by indeterminates $\mathbf{x}_j$ we get a system

$$\begin{aligned}\sum_{\nu\mu} \mathbf{y}_{i\nu\mu}^2 (g_i^1)^{\nu_1} \cdots (g_i^p)^{\nu_p} (\varepsilon_i^1 \mathbf{x}_1)^{\mu_1} \cdots (\varepsilon_i^q \mathbf{x}_q)^{\mu_q} = \\ = -(g_i^1)^{2r_1} \cdots (g_i^p)^{2r_p} (\varepsilon_i^1 \mathbf{x}_1)^{2s_1} \cdots (\varepsilon_i^q \mathbf{x}_1)^{2s_q}, \\ \varepsilon \in \mathcal{E}^q,\ 1 \le i \le 2^m,\end{aligned}$$

which has in B the solution

$$\mathbf{y}_{i\nu\mu} = y_{i\nu\mu}, \quad \mathbf{x}_j = f_j.$$

Since the homomorphism ϕ has the approximation property, this solution can be approximated by other in A, say

$$\mathbf{y}_{i\nu\mu} = z_{i\nu\mu}, \quad \mathbf{x}_j = h_j.$$

Going backwards in the Positivstellensatz that means that there are $h_1, \dots, h_q \in A$ such that for every $\varepsilon \in \mathcal{E}^q$ the set

$$D_\varepsilon(i) = \{g_i^1 > 0, \dots, g_i^p > 0, \varepsilon_i^1 h_1 > 0, \dots, \varepsilon_i^q h_q > 0\}$$

is empty for some i. Consequently, the product $D_\varepsilon(1) \times \cdots \times D_\varepsilon(2^m)$ is empty too. Hence

$$\bigcup_{\varepsilon \in \mathcal{E}^q} D_\varepsilon(1) \times \cdots \times D_\varepsilon(2^m) = \emptyset,$$

which is a contradiction, since by construction $(\alpha_i, 1 \le i \le 2^m)$ belongs to that set. □

Now we come back to the geometric context and find:

Proposition 9.2 *Let $X \subset \mathbb{R}^p$ be a real algebraic set and F an algebraic fan of X. Then F is analytic at a point $a \in X$ if and only if the henselization of F at a is a fan. Moreover, if this is the case, then F has an analytic lifting F_a with the same height and dimension as F.*

Proof: We already know (Proposition 8.2) that if F is analytic at a, then F_a^h is a fan. Conversely, suppose F_a^h is a fan, and let $\mathfrak{p}^h$ be its support in $\mathcal{N}_{X,a}$. Then the extension $\mathfrak{p}^h\mathcal{O}_{X,a}$ is a prime ideal and the canonical homomorphism $\mathcal{N}_{X,a}/\mathfrak{p}^h \to \mathcal{O}_{X,a}/\mathfrak{p}^h\mathcal{O}_{X,a}$ has the approximation property ([Ar1, 1.2 and comments following; Ar2, 1.10, p.26], [BCR, 8.3.1, p.154], [Tg, Chapter III, 4.2, p.59]). Then by Theorem 8.1 there is a fan F_a of $\mathcal{O}_{X,a}$ with support $\mathfrak{p}^h\mathcal{O}_{X,a}$ that extends F_a^h and $\#(F_a) = \#(F_a^h)$. This shows that F is analytic and F_a is a lifting. The assertion concerning heights and dimensions follows from Proposition 8.4 and the equalities $\dim(\mathcal{N}_{X,a}) = \dim(\mathcal{O}_{X,a})$, $\mathrm{ht}(\mathfrak{p}^h) = \mathrm{ht}(\mathfrak{p}^h\mathcal{O}_{X,a})$. □

10 Extension of real valuation rings to the henselization

In the previous section we have learnt that in order to decide if an algebraic fan $F \subset \mathcal{G}_a$ is analytic we have to check if the set of orderings $F_a^h = \theta^*(F) \subset \mathrm{Spec}_r(\mathcal{R}_{X,a}^h)$ is a fan. The key step to be able to do this is a detailed analysis of the extension of real valuation rings from $\mathcal{R}_{X,a}$ to $\mathcal{R}_{X,a}^h$, which is the goal of this section.

Let X be real algebraic set and fix a point $a \in X$. Let $\alpha \in \mathcal{G}_a$ and let V be a valuation ring of $\mathcal{R}(X)$ compatible with α. We denote by $\Gamma(V)$ its value group, by k_V its residue field, and by λ_V the place defined by V. Let $\alpha_V \in \mathrm{Spec}_r(k_V)$ be the ordering induced by α in the residue field of V. Thus we have a homomorphism

$$\lambda_V : \mathcal{R}_{X,a} \to k_V$$

and by the base change property, Proposition 8.1 *c)*, α_V has a unique extension to $\mathrm{Spec}_r(\mathcal{R}_{X,a}^h \otimes_{\mathcal{R}_{X,a}} k_V)$, which we denote by α_V^h. Moreover, since $\mathcal{R}_{X,a}^h \otimes_{\mathcal{R}_{X,a}} k_V$ is a noetherian algebraic extension of k_V (by the properties of henselizations, [Rd, Chapter VIII, §4, Th.3, p.94]), it is a product of fields, say $k_1 \times \cdots \times k_r$. In particular, α_V^h is an ordering of one of the k_i's, which will be denoted by k^*.

We begin with the following easy remark:

Lemma 10.1 *Let X be real algebraic set and fix $a \in X$. Let $\alpha \in \mathcal{G}_a$ and let V be a valuation ring of $\mathcal{R}(X)$ compatible with α. Let α^h be the unique extension of α to* $\mathrm{Spec}_r(\mathcal{R}_{X,a}^h)$. *Then there is at most one extension of V to $\mathcal{R}_{X,a}^h$ compatible with α^h.*

Proof: Suppose that there are two such extensions V_1, V_2. Since $\mathcal{R}_{X,a}^h$ is algebraic over $\mathcal{R}_{X,a}$, we have $\mathrm{rank}(V_1) = \mathrm{rank}(V_2)$. Also, since they are compatible with α^h, either $V_1 \subset V_2$ or $V_2 \subset V_1$, which implies the equality. □

The following proposition is of fundamental importance:

Proposition 10.2 *Let X be real algebraic set and fix $a \in X$. Let $\alpha \in \mathcal{G}_a$ and let V be a valuation ring of $\mathcal{R}(X)$ compatible with α. Then there is a unique valuation ring V_a^h of $\mathcal{R}_{X,a}^h$ such that:*

i) *V_a^h is an extension of V;*

ii) *the value group of V_a^h coincides with the one of V;*

iii) *The residue field of V_a^h is k^*, the factor of $\mathcal{R}_{X,a}^h \otimes_{\mathcal{R}_{X,a}} k_V$ to which α_V extends.*

Moreover, V_a^h is compatible with the unique extension α^h of α to $\mathrm{Spec}_r(\mathcal{R}_{X,a}^h)$ *and α^h specializes to the extension α_V^h of α_V to k^*.*

Proof: First of all we have a commutative square

$$\begin{array}{ccc} \mathcal{R}_{X,a}^h & \xrightarrow{\lambda^*} & k^* \\ \uparrow & & \uparrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda_V} & k_V \end{array}$$

Let $\mathfrak{p} \subset \mathcal{R}_{X,a}$ be the common support of V and α. We set $Y \subset X$ for the zero set of $\mathfrak{p}$, so that $\mathcal{R}(Y) = \mathcal{R}(X)/\mathfrak{p}$. Then V is a valuation ring of $\mathcal{K}(Y) = \kappa(\mathfrak{p})$ and α is a total orderings of $\mathcal{K}(Y)$. Obviously, our problem is equivalent to see that there is a unique valuation ring V^* of $\mathcal{R}_{Y,a}^h$ in the conditions of the statement. Therefore we work from now on with $\mathcal{R}_{Y,a}$.

We start by showing the existence of V^*. Let us see that we may assume also that Y is normal at a, that is, that the ring $\mathcal{R}_{Y,a}$ is integrally closed. Indeed, consider the normalization Y' of Y, that is, the ring $\mathcal{R}(Y')$ is the integral closure of $\mathcal{R}(Y)$ in its total ring of fractions. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of $\mathcal{R}(Y')$ lying over the maximal ideal of a in $\mathcal{R}(Y)$. Then the normalization $\mathcal{R}'_{Y,a}$ of $\mathcal{R}_{Y,a}$ is the semilocalization $\mathcal{R}(Y')_{\mathfrak{m}_1,\dots,\mathfrak{m}_r}$. Since V is integrally closed, it contains $\mathcal{R}'_{Y,a}$ and consequently λ_V extends to $\mathcal{R}'_{Y,a} \to k_V$. By the properties relating normalizations and henselizations, we know ([Ng, 43.20, p.187]) that the normalization of $\mathcal{R}_{Y,a}^h$ is the ring

$$C = \mathcal{R}_{Y,a}^h \otimes_{\mathcal{R}_{Y,a}} \mathcal{R}'_{Y,a} = \mathcal{R}(Y')_{\mathfrak{m}_1}^h \times \cdots \times \mathcal{R}(Y')_{\mathfrak{m}_r}^h.$$

Thus, from the two homomorphisms $\mathcal{R}'_{Y,a}\xrightarrow{\lambda_V}k_V\longrightarrow k^*$ and $\mathcal{R}^h_{Y,a}\xrightarrow{\lambda^*}k^*$, we obtain a canonical homomorphism $\mu : C \to k^*$. Now, the prime ideal $\ker(\mu)$ of C can be seen as a prime ideal of some $\mathcal{R}(Y')^h_{\mathfrak{m}_i}$, so that μ factorizes factorizes as a composition

$$C\longrightarrow\mathcal{R}(Y')^h_{\mathfrak{m}_i}\longrightarrow C/\ker(\mu)\longrightarrow k^*$$

where the first two maps are the canonical epimorphisms. We claim that $\mathcal{R}(Y')_{\mathfrak{m}_i} \subset V$, and $\lambda_i^*|_{\mathcal{R}(Y')_{\mathfrak{m}_i}} = \lambda_V|_{\mathcal{R}(Y')_{\mathfrak{m}_i}}$, where $\lambda_i^* : \mathcal{R}(Y')^h_{\mathfrak{m}_i} \to k^*$ denotes the composition of the second and third maps of the line above. Indeed, recall that $\mathcal{R}(Y') \subset V$; now, if $g \in \mathcal{R}(Y') \subset \mathcal{R}'_{Y,a}$, $g \notin \mathfrak{m}_i$, we have:

$$\lambda_V(g) = \mu(1 \otimes g) = \mu(g_1, \dots, g_i, \dots, g_r) = \lambda_i^*(g_i),$$

where $g_j \in \mathcal{R}(Y')^h_{\mathfrak{m}_j}$ denotes the j-th component of $1 \otimes g$ in C. Now, since $g \notin \mathfrak{m}_i$, g_i is a unit in $\mathcal{R}(Y')^h_{\mathfrak{m}_i}$ and $\lambda_i^*(g_i) \neq 0$. The claim follows readily from this. Hence we obtain a commutative diagram

$$\begin{array}{ccc}
\mathcal{R}(Y')^h_{\mathfrak{m}_i} & \xrightarrow{\lambda_i^*} & k^* \\
\uparrow & & \uparrow \\
\mathcal{R}(Y')_{\mathfrak{m}_i} & \xrightarrow{\lambda_V} & k_V
\end{array}$$

Note also that since V dominates $\mathcal{R}(Y')_{\mathfrak{m}_i}$, $\mathfrak{m}_i$ is a real ideal, and corresponds to a point $a' \in Y'$. Thus, we get a diagram

$$\begin{array}{ccccccccc}
\mathcal{R}^h_{X,a} & \to & \mathcal{R}^h_{Y,a} & \to & \mathcal{R}^h_{Y',a'} & \xrightarrow{\lambda_i^*} & k^* & & F' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{X,a} & \to & \mathcal{R}_{Y,a} & \to & \mathcal{R}_{Y',a'} & \xrightarrow{\lambda_V} & k_V & & F_V
\end{array}$$

In view of this construction we may assume that $\mathcal{R}_{Y,a}$ is normal as announced, and we will do so henceforth. It follows that $\mathcal{R}^h_{Y,a}$ is normal too.

Let $\mathfrak{q} = \ker(\lambda_V)$ be the center of V in $\mathcal{R}(Y)$ and let $\mathfrak{q}^h = \ker(\lambda^*)$. Thus, $\mathfrak{q}^h$ is one of the henselian branches of $\mathfrak{q}$ at a. We set $A = (\mathcal{R}_{Y,a})_{\mathfrak{q}}$, $B = (\mathcal{R}^h_{Y,a})_{\mathfrak{q}^h}$. Then $A \subset B$ is a local algebraic extension of normal rings, and therefore so is the extension $A^h \subset B^h$ of their respective henselizations.

Let V^h be the henselization of V. Since the valuation ring V dominates A, we have also an inclusion $A^h \subset V^h$, where V^h denotes the henselization of V. Since k^* is an algebraic extension of $k_{V^h} = k_V$, by [En, p.217] there is an extension V' of V^h with the same value group $\Gamma_{V^h} = \Gamma_V$ as V^h and with residue field $k_{V'} = k^*$. After replacing V' by its henselization we may assume that V' is henselian, so that we have a diagram:

$$\begin{tikzcd}
 & & B^h & V' \arrow[r] & k^* & F' \\
\mathcal{R}^h_{Y,a} \arrow[r, phantom, "\subset"] & B \arrow[ur] & A^h \arrow[u] \arrow[r] & V^h \arrow[u] \arrow[r] & k_{V^h} \arrow[u] & F_V \\
\mathcal{R}_{Y,a} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & A \arrow[u] \arrow[ur] \arrow[r] & V \arrow[ur] \arrow[rr] & & k_V \arrow[u, equal] & F_V
\end{tikzcd}$$

Hence, all we have to show is that there is a homomorphism $B^h \to V'$ closing the upper side of the square in the diagram above, that is, which extends $A^h \to V'$. Indeed, then $V^* = V' \cap \mathcal{R}^h_{Y,a}$ is the valuation we sought, since it obviously verifies the conditions *i)* ,*ii)* and *iii)* of the statement. So, let us go for the construction of such a mapping $B^h \to V'$. To do it we need a rather technical description of B^h as a suitable inductive limit of local A^h-algebras.

First of all, note that the residue field $\kappa(\mathfrak{q}^h)$ of B^h is algebraic over the one $\kappa(\mathfrak{q})$ of A^h. Let $k \supset \kappa(\mathfrak{q})$ be a finite subextension of $\kappa(\mathfrak{q}^h) \supset \kappa(\mathfrak{q})$. Then by the primitive element theorem we have $k = \kappa(\mathfrak{q})[\theta]$ for some $\theta \in \kappa(\mathfrak{q}^h)$. Let $P \in A^h[t]$ be a monic polynomial whose class in $\kappa(\mathfrak{q})[t]$ is the irreducible polynomial of θ over $\kappa(\mathfrak{q})$. Then we consider the unique homomorphism $A^h[t] \to k$ that extends $A^h \to \kappa(\mathfrak{q}) \subset k$ and maps t to θ. Its kernel $\mathfrak{n}$ contains P, and we consider the localization $C_k = A^h[t]_{\mathfrak{n}}/(P)$. Note that $\mathfrak{n} \cap A^h$ is the maximal ideal of A^h, and therefore the homomorphism $A^h \to C_k$ is local. Therefore, following [Rd, Chapter VIII, Def.2, p.80], C_k is a *local-etale A^h-algebra*, with residue field k. Ve consider the family $\{C_k\}$ of all these local-etale A^h-algebras when k runs among all finite subextensions of $\kappa(\mathfrak{q}^h) \supset \kappa(\mathfrak{q})$.

Let us see that this family is directed. We say that $C_{k_i} \prec C_{k_j}$ if $k_i \subset k_j$ and there is a local homomorphism of A^h-algebras $\xi_{ij} : C_{k_i} \to C_{k_j}$ making commutative the diagram

$$\begin{tikzcd}
C_{k_i} \arrow[r] \arrow[d] & C_{k_j} \arrow[d] \\
k_i \arrow[r, phantom, "\subset"] & k_j
\end{tikzcd}$$

Then, given any two elements $C_{k_1} = A^h[t_1]_{\mathfrak{n}_1}/(P_1)$, $C_{k_2} = A^h[t_2]_{\mathfrak{n}_2}/(P_2)$ of the family, with residue fields $k_1 = \kappa(\mathfrak{q})[\theta_1]$ and $k_2 = \kappa(\mathfrak{q})[\theta_2]$ respectively, we consider the ring $C = A^h[t_1, t_2]_{\mathfrak{a}}/(P_1, P_2)$, where $\mathfrak{a}$ is the kernel of the homomorphism $A^h[t_1, t_2] \to \kappa(\mathfrak{q})[\theta_1, \theta_2]$ defined by sending $t_1 \mapsto \theta_1$ and $t_2 \mapsto \theta_2$. It follows from the jacobian criterion ([Rd, Chapter V, Theorem 1, p. 51, and Theorem 5, p. 60]) that C belongs to our family and obviously we have $C_{k_1} \prec C$ and $C_{k_2} \prec C$.

Thus, let D be the direct limit of the family $\{C_k\}$. Then D is a *local ind-etale A^h-algebra* ([Rd, Chapter VIII, §1, Definition 3, p. 80]) with residue field $\kappa(\mathfrak{q}^h)$. In particular since A^h is an excellent normal henselian domain, it follows that so is D ([Rd, Chapter VIII, §4, Theorem 3, p.94] and [Gr, 5.2, 5.3]). We claim that $D = B^h$. Indeed, we have a trivial commutative triangle of residue fields

$$\begin{tikzcd} \kappa(\mathfrak{q}^h) \arrow[r, equal] & \kappa(\mathfrak{q}^h) \\ \kappa(\mathfrak{q}) \arrow[u] \arrow[ur] & \end{tikzcd}$$

Since B^h is henselian, by the universal property of local ind-etale A^h-algebras, this triangle lifts to another

$$\begin{tikzcd} D \arrow[r] & B^h \\ A^h \arrow[u] \arrow[ur] & \end{tikzcd}$$

([Rd, Chapter VIII, Proposition 1, p. 81]), where the kernel of the horizontal arrow is (0), since it lies over (0) of A^h and D is algebraic over A^h. Furthermore, since the maximal ideal of A^h generates the ones of D and B^h, the maximal ideal of D generates the one of B^h and $\dim(D) = \dim(B^h)$. In conclusion, the injection $D \to B^h$ induces an isomorphism between the respective completions. Finally, since B^h is algebraic over A^h, it is also algebraic over D, which is algebraically closed in its completion ([Ng, 44.1, p. 188]). Thus, we conclude that this injection is in fact an isomorphism, as claimed.

After this preparation, the universal property of B^h as an ind-etale A^h-algebra will give the required homomorphism $B^h \to V'$. Indeed, the commutative triangle of residue fields

$$\begin{array}{ccc} \kappa(\mathfrak{q}^h) & \longrightarrow & k^* \\ \uparrow & \nearrow & \\ \kappa(\mathfrak{q}) & & \end{array}$$

lifts to another

$$\begin{array}{ccc} B^h & \longrightarrow & V' \\ \uparrow & \nearrow & \\ A^h & & \end{array}$$

and the horizontal arrow is the homomorphism we sought.

Let us see now that any valuation ring V^* verifying conditions *i)*, *ii)* and *iii)* is compatible with the extension α^h of α to $\mathrm{Spec}_r \mathcal{R}^h_{X,a}$, from which the uniqueness follows also at once by Lemma 10.2. Indeed, we know by Baer-Krull (Theorem 6.1), that the pull-back $F_{\alpha^h_V}$ by the valuation ring V^* of the extension of α_V to k^*,is determined by the value group Γ_{V^*} of V^*. On the other hand the pull-back F_{α_V} of α_V by the valuation ring V is determined by the value group Γ_V. Since $\Gamma_{V^*} = \Gamma_V$, we get that $F_{\alpha_V} = F_{\alpha^*_V} \cap \mathcal{R}_{X,a}$. Since $\alpha \in F_{\alpha_V}$ and α^h is its unique extension, we havet $\alpha^h \in F_{\alpha^*_V}$, which shows that α^h and V^* are compatible. Finally, α^h specializes to α^h_V in k^* by the uniqueness of the extension of α_V (Proposition 8.1 *c)*). □

Remarks 10.3 *a)* We stress the fact that we are not claiming that the valuation ring V has a unique extension. We are just saying that there is only one compatible with α^h.

b) The existence of an extension V^* of V compatible with α^h (and therefore unique by Lemma 9.2) can be seen by the following easy argument, suggested by E. Becker. Let $\mathfrak{p} \subset \mathcal{R}(X)$ be the support of α, and $\mathfrak{q} \subset \mathcal{R}^h_{X,a}$ the support of α^h. Thus V is a valuation ring of $\kappa(\mathfrak{p})$ and we look for an extension V^* in $\kappa(\mathfrak{q})$. Let W and W^* be, respectively, the convex hull of $\mathbb{Q}$ in $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$. Then an easy computation shows that

$$V \cdot W^* = \{vw \mid v \in V,\ w \in W^*\}$$

is a subring of $\kappa(\mathfrak{q})$ containing W^*, and therefore it is a valuation ring of $\kappa(\mathfrak{q})$ compatible with α^*. Moreover, one can easily see that W^* is an immediate extension of W (and in fact we will do it later), from which it follows that $V \cdot W^*$ has the same value group as W. However, we have not been able to check directly that the residue field of $V \cdot W^*$is our field k^*, This is the reason behind the long detour of our proof of the proposition.

11 The amalgamation property

Let X be a real algebraic set, $a \in X$ and $F \subset \mathcal{G}_a$ an algebraic fan. We are directly faced with the problem of deciding whether F_a^h is a fan. A general strategy will be following: suppose that there is a fan G in $\operatorname{Spec}_r(\mathcal{R}_{X,a}^h)$ which contains F_a^h. Then it follows that F_a^h is a fan. Indeed, let $\gamma_1, \gamma_2, \gamma_3 \in F_a^h$, and set $\alpha_i = \gamma_i \cap \mathcal{R}_{X,a} \in F$. Then the product $\gamma_4 = \gamma_1\gamma_2\gamma_3$ lies in G because G is a fan. But since θ^* is a homeomorphism and the product $\gamma_1\gamma_2\gamma_3$ lies over $\alpha_4 = \alpha_1\alpha_2\alpha_3$ it follows that γ_4 is the unique point of $\operatorname{Spec}_r(\mathcal{R}_{X,a}^h)$ lying over α_4. But $\alpha_4 \in F$ since F is a fan, and therefore $\gamma_4 \in F_a^h$ as wanted. Now to produce such a G we will try to use the only standard method that we have seen for the construction of fans: pulling back some already given fan by a valuation ring. In fact, we have:

Proposition 11.1 *Let $F \subset \mathcal{G}_a$ be an algebraic fan of X. Then, F_a^h is a fan if and only if there is a valuation ring W^h of $\mathcal{R}_{X,a}^h$ such that its restriction W to $\mathcal{R}_{X,a}$ verifies*

a) W is compatible with F, and

b) The fan F_W induced by F in the residue field k_W of W extends to a fan in the residue field of W^h.

Proof: Suppose that F_a^h is a fan and let $\mathfrak{q}$ be its support. Let $\mathfrak{p} = \mathfrak{q} \cap \mathcal{R}_{X,a}$ be the support of F. By the trivialization theorem there is a valuation ring W^h of $\mathcal{R}_{X,a}^h$ which is compatible with F_a^h. Then $W = W^h \cap \kappa(\mathfrak{p})$ is a valuation ring of $\mathcal{R}_{X,a}$ which is compatible with F. If we denote by $(F_a^h)_{W^h}$ and F_W respectively the induced fans in the residue fields of W^h and W, then $(F_a^h)_{W^h}$ is an extension of F_W, and the following square is commutative:

$$\begin{array}{ccccc} \mathcal{R}_{X,a}^h & \xrightarrow{\lambda_{W^h}} & k_{W^h} & & (F_a^h)_{W^h} \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda_W} & k_W & & F_W \end{array}$$

This shows that the condition of the statement is necessary.

Conversely, let W be a valuation ring of $\mathcal{R}_{X,a}$ compatible with F and assume that there is a valuation ring W^h of $\mathcal{R}^h_{X,a}$ extending W and such that in its residue field there is a fan G_{W^h} extending the fan F_W induced by F in the residue field k_W of W. Thus we have a commutative square

$$\begin{array}{ccccc}
\mathcal{R}^h_{X,a} & \xrightarrow{\lambda_{W^h}} & k_{W^h} & & G_{W^h} \\
\uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{X,a} & \xrightarrow{\lambda_W} & k_W & & F_W
\end{array}$$

In particular W^h is a real valuation ring. Let $\beta' \in \operatorname{Spec}_r \mathcal{R}^h_{X,a}$ be any ordering compatible with W^h, and let β be its restriction to $\mathcal{R}_{X,a}$. Obviously β is compatible with W and it follows from Proposition 10.2 that W^h is the unique extension of W compatible with β'. In particular the value groups of W^h and W coincide. From this, it follows that the pull-back G of G_{W^h} by λ_{W^h} is a fan which contains the set of orderings F^h_a, and according to what was explained above, we get that F^h_a is a fan.

□

Although the above result gives a characterization of analytic fans, it is unsatisfactory since we want to decide it in terms of the algebraic data that is, of the lower line of the above diagrams. This leads to the following definition:

Definition 11.2 *Given a field k, a fan F of k, a point $a \in X$, and a homomorphism $\lambda : \mathcal{R}_{X,a} \to k$, we say that $\lambda : \mathcal{R}_{X,a} \to (k, F)$ is an* amalgamation problem. *Then, we say that $\lambda : \mathcal{R}_{X,a} \to (k, F)$* amalgamates *if there exist a field extension $k' \supset k$, a fan F' of k' that extends F and a homomorphism $\lambda' : \mathcal{R}^h_{X,a} \to k'$ making commutative the diagram*

$$\begin{array}{ccccc}
\mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k' & & F' \\
\uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{X,a} & \xrightarrow{\lambda} & k & & F
\end{array}$$

We will say that such a diagram is an amalgamation square *or that the pair (k', F') is an* amalgamation corner.

With this terminology, the proof of the necessary condition in the above proposition gives immediately:

Corollary 11.3 *If the henselization F_a^h of F at $a \in X$ is a fan, then F trivializes along a valuation ring W such that $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ amalgamates.*

In the situation of the last corollary we will briefly say that W *amalgamates.* The study of amalgamation deserves a detailed analysis, and we will prove that in fact it is not only necessary, but also a sufficient condition for F_a^h to be a fan. Moreover, such an analysis will lead to a nice geometric interpretation of the property, and, correspondingly, to a good geometric characterization of the analyticity of fans.

Remarks 11.4 *a)* Although we have used the ring $\mathcal{R}_{X,a}^h$ in the definition of the amalgamation property, we could have equally used the ring $\mathcal{N}_{X,a}$ instead. Indeed, since the kernel of the homomorphism $\mathcal{R}_{X,a}^h \to \mathcal{N}_{X,a}$ is the real-radical of (0), any homomorphism λ' from $\mathcal{R}_{X,a}^h$ to an ordered field factorizes through $\mathcal{N}_{X,a}$.

b) In case $k = \kappa(\mathfrak{p})$, where $\mathfrak{p}$ is the support of an algebraic fan F, we recover the problem of deciding whether the henselization F_a^h is a fan. Indeed, if F_a^h is a fan, $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates by taking $F' = F_a^h$ and $k' = \kappa(\mathfrak{p}^h)$, where $\mathfrak{p}^h$ is the support of F_a^h. Conversely, suppose that there is an amalgamation square

$$\begin{array}{ccccc}
\mathcal{R}_{X,a}^h & \xrightarrow{\lambda'} & k' & & F' \\
\uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{X,a} & \xrightarrow{\lambda} & k & & F
\end{array}$$

By construction F_a^h is the restriction of F' and, consequently, it is a fan.

c) In general, suppose that we have an amalgamation problem

$$\lambda : \mathcal{R}_{X,a} \to (k, F).$$

Consider the ideal $\mathfrak{q} = \ker(\lambda)$ and its zero set $Z \subset X$. Then λ factorizes through $\mathcal{R}_{Z,a}$ and the initial problem is equivalent to $\lambda : \mathcal{R}_{Z,a} \to (k, F)$.

In particular, if $\mathfrak{q}$ is the maximal ideal of $\mathcal{R}_{Z,a}$, then $\lambda : \mathcal{R}_{Y,a} \to (k, F)$ amalgamates.

Indeed, if we are given an amalgamation square

$$\begin{array}{ccccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k' & & F' \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda} & k & & F \end{array}$$

the ideal $\mathfrak{q}^h = \ker(\lambda')$ lies over $\mathfrak{q}$, and is a henselian branch of $\mathfrak{q}$ at a, that is, $\mathfrak{q}^h$ is an associated prime ideal of $\mathfrak{q}\mathcal{R}^h_{X,a}$. Consequently, λ' factorizes through $\mathcal{R}^h_{Z,a}$ and the diagram above gives an amalgamation square for $\lambda : \mathcal{R}_{Z,a} \to (k, F)$. The converse direction is immediate.

Finally note that if $\mathfrak{q}$ is maximal, then $\mathcal{R}_{Z,a} = \mathcal{R}^h_{Z,a} = \mathbb{R}$, and $\lambda : \mathcal{R}_{Z,a} \to (k, F)$ amalgamates trivially.

d) Suppose that

$$\begin{array}{ccccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k' & & F' \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda} & k & & F \end{array}$$

is an amalgamation square. Then there is an amalgamation corner (k'', F'') with $k'' \subset k'$ algebraic over k.

Indeed, let $\mathfrak{q}$ and $\mathfrak{q}^h$ be respectively the kernels of λ and λ'. Then we have a commutative square

$$\begin{array}{ccc} (\kappa(\mathfrak{q}^h), F'|_{\kappa(\mathfrak{q}^h)}) & \xrightarrow{\lambda'} & (k', F') \\ \uparrow & & \uparrow \\ (\kappa(\mathfrak{q}), F|_{\kappa(\mathfrak{q})}) & \xrightarrow{\lambda} & (k, F) \end{array}$$

and $\kappa(\mathfrak{q}^h)$ is an algebraic extension of $\kappa(\mathfrak{q})$. Therefore we can replace (k', F') by $(k'', F'|_{k''})$, where k'' is the algebraic closure of k in k'. □

The next result shows which are the natural candidates to close amalgamation squares. Let

$$\lambda : \mathcal{R}_{X,a} \to (k, F)$$

be an amalgamation problem, and consider the base change $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$. This ring is product of fields, and we know from Proposition 8.1 *c)*, that there is a bijection between $\mathrm{Spec}_r k$ and $\mathrm{Spec}_r(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k)$. Then, notice that F extends to a fan F' of $\mathrm{Spec}_r \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$ if and only if all the orderings of F extend to the same factor of $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$ and they build a fan in it. Then we have:

Proposition and Definition 11.5 *The homomorphism $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates if and only if F extends to a fan F^* of the base change $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$; in this case the support k^* of F^* is a uniquely determined factor of $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$. The couple (k^*, F^*) is then called the* minimal amalgamation corner *for λ and determines a unique henselian branch of $\mathfrak{q} = \ker(\lambda)$ at a. Sometimes we will say that k^* is the* minimal amalgamation field *and that F^* is the* minimal amalgamation fan.

Proof: Suppose that $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates and let (k', F') be an amalgamation corner. Then we obtain a homomorphism $\mu : \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k \to k'$. Thus, the restriction F^* of F' to $\mathrm{Spec}_r(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k)$ is a fan extending F. Let k^* be the support of F^*. As was remarked at the beginning of Section 10, $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k = k_1 \times \cdots \times k_r$, and so $k^* = (\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k)/\ker(\mu) = k_i$ for some i, and we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{R}^h_{X,a} & \xrightarrow{\lambda^*} & k^* & & F^* \\
\uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{X,a} & \xrightarrow{\lambda} & k & & F
\end{array}$$

Furthermore, $\mathfrak{q}^h = \ker(\lambda^*)$ is a prime ideal of $\mathcal{R}^h_{X,a}$ lying over $\mathfrak{q} = \ker(\lambda)$, that is, $\mathfrak{q}^h$ is a henselian branch of $\mathfrak{q}$ at a. The converse is immediate.

□

12 Algebraic characterization of analytic fans

We can now apply the results on extension of valuations of Section 10 and the basic properties af amalgamation to obtain the algebraic characterization of analytic fans. Again, let $X \subset \mathbb{R}^n$ be an algebraic set and $a \in X$. Then,

Theorem 12.1 (Algebraic characterization of analytic fans) *Let F be an algebraic fan of X, and $a \in X$. The following assertions are equivalent:*

a) The fan F is analytic at a.

b) The henselization of F at a is a fan.

c) The fan F trivializes along a valuation ring W such that $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ amalgamates.

d) The fan F is compatible with a valuation ring W such that $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ amalgamates.

Proof: We already know that *a)* and *b)* are equivalent (Proposition 9.2) and that *b)* implies *c)* (Proposition 11.3). On the other hand, *c)* implies *d)* trivially, and therefore it suffices to show that *d)* implies *b)*.

Consider an amalgamation square

$$\begin{array}{ccccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k' & & F' \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda_W} & k_W & & F_W \end{array}$$

By Proposition 11.5 we may assume that (k', F') is minimal. Then, by Proposition 10.2 there is a valuation ring W^h of $\mathcal{R}^h_{X,a}$ lying over W, with residue field k' and $\Gamma_{W^h} = \Gamma_W$. Therefore the result follows at once from Proposition 11.1. $\square$

Thus, in order to check whether a given algebraic fan $F \subset \mathcal{G}_a$ is analytic we have to study if some valuation ring compatible with F amalgamates. Therefore, it is interesting to analize the structure of the family of amalgamating valuation rings. For this, we need to take a closer look at the families $\mathfrak{V}$ and $\mathfrak{V}^h$ of valuations compatible with some of the orderings of F and F_a^h respectively (despite this is a fan or not).

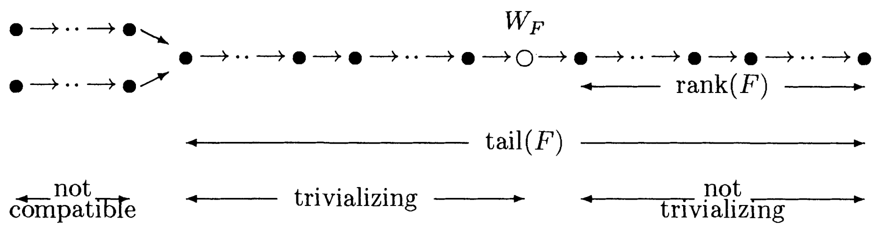
Let $\mathfrak{p}$ be the support of F and set $Y \subset X$ for the zero set of $\mathfrak{p}$. Remember that the set $\mathfrak{V}$ of valuations compatible with some element of F is of the type

$$\begin{array}{ccc} V_1^1 \longrightarrow \cdots \longrightarrow V_{r_1}^1 & & \\ & \searrow & \\ & & W_1 \longrightarrow \cdots \longrightarrow W_s \subset \mathcal{K}(Y) \\ & \nearrow & \\ V_1^2 \longrightarrow \cdots \longrightarrow V_{r_2}^2 & & \end{array}$$

where V_1^1 and V_1^2 are the only two (possibly) different convex hulls of $\mathbb{Q}$ in $\mathcal{K}(Y)$ with respect to the elements of F, and $W_1, \dots, W_s$ are the valuation rings of $\mathcal{K}(Y)$ compatible with F. Moreover, we know from the trivialization theorem that there is an integer $p \leq s$ such that $W_1, \dots, W_p$ trivialize F, while $W_{p+1}, \dots, W_s$ do not. Thus, it makes sense to talk about the largest trivilizing valuation ring of F. In fact this ring will play a key role in the sequel, so that we give the following:

Notation and Definition 12.2. We denote by W_F the *largest trivializing valuation ring* of F. We define the *rank* of F as the rank of the valuation ring W_F, that is, the number of valuation rings compatible with F containing W_F, or equivalently which do not trivialize F. We will denote by λ_F, $\mathfrak{m}_F$, k_F and Γ_F, the place, maximal ideal, residue field and value group of W_F. Also, we denote by $\mathfrak{q}_F = \mathfrak{m}_F \cap \mathcal{R}(X)$ the center of W_F in X, and by Z_F the zero set of $\mathfrak{q}_F$. Finally, we will say that $\mathfrak{V}$ has a *tail of length* s, or simply that $\operatorname{tail}(F) = s$, if there are exactly s valuation rings compatible with F.

Therefore we can represent the set $\mathfrak{V}$ as the following tree:



Let α^h be any ordering of F_a^h, and let V_1^1 be the convex hull of $\mathbb{Q}$ with respect to $\alpha = \alpha^h \cap \mathcal{R}_{Y,a}$, that is, V_1^1 is the left end point of the chain compatible with $\alpha \in F$. Let $(V_1^1)^h$ be its henselization. Since V_1^1 dominates $\mathcal{R}_{Y,a}$, we have a unique extension homomorphism $\mathcal{R}_{Y,a}^h \to (V_1^1)^h$, whose kernel is denoted by $\mathfrak{p}^h$. Thus, the restriction of $(V_1^1)^h$ gives a valuation ring H_1^1 of $\mathcal{R}_{Y,a}^h$, which is an immediate extension of V_1^1. Thus, it follows that H_1^1 is the unique extension of V_1^1 guaranteed by Proposition 10.2, that $\mathfrak{p}^h$ is the support of α^h, and that H_1^1 is compatible with α^h. In particular, since the residue field of H_1^1 is $\mathbb{R}$, we get that it is the convex hull of $\mathbb{Q}$ in $\kappa(\mathfrak{p}^h)$ with respect to α^h.

Altogether, we have that the family of convex hulls of $\mathbb{Q}$ with respect to the orderings of F_a^h consists only of the elements H_1^1 and H_1^2. In particular the whole family $\mathfrak{V}^h$ consists of the valuation rings belonging to the chains

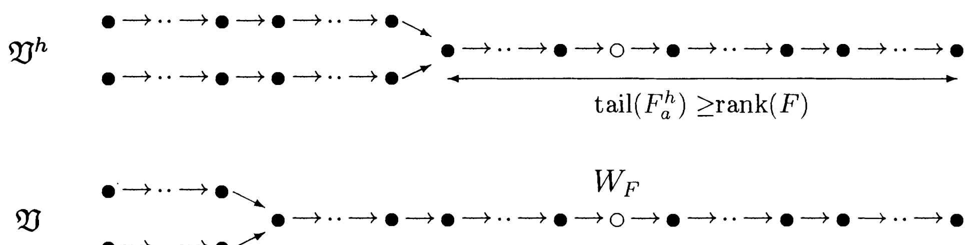
$$H_1^1 \longrightarrow \cdots \longrightarrow H_{r_1}^1 \longrightarrow H_{r_1+1}^1 \longrightarrow \cdots \longrightarrow H_{r_1+s}^1$$

$$H_1^2 \longrightarrow \cdots \longrightarrow H_{r_2}^2 \longrightarrow H_{r_2+1}^2 \longrightarrow \cdots \longrightarrow H_{r_2+s}^2$$

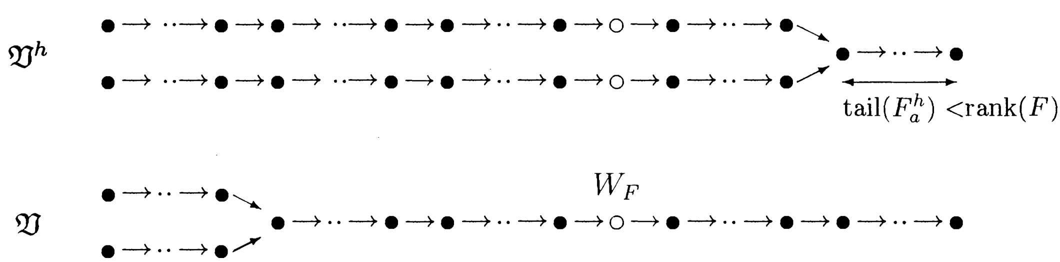
of generizations of H_1^1 and H_1^2. Moreover, since the rank of H_1^i coincides with the rank of V_1^i, it follows from Proposition 10.2, that there is a one to one correspondence between these chains and the respective ones of V_1^1 and V_1^2.

What we do not know is whether these chains will coincide up to some step. Of course, by the trivialization theorem, this will be the case if F_a^h is a fan, recovering then for $\mathfrak{V}^h$ the tree shape of above, that is, with $\operatorname{tail}(\mathfrak{V}^h) > 0$. However, even in this case, it is not clear whether both diagrams are identical, that is, if $\operatorname{tail}(\mathfrak{V}^h) = \operatorname{tail}(\mathfrak{V})$. Notice that we always have $\operatorname{tail}(\mathfrak{V}^h) \leq \operatorname{tail}(\mathfrak{V})$.

Summarizing, we may have the following two cases:



Case 1



Case 2

Notice that in both cases we have a surjective map $\mathfrak{V}^h \to \mathfrak{V}$ whose fibers have at most cardinal 2. The difference between the two cases lies in the number of extensions of W_F. We will see that the first type corresponds to analytic fans, while the second to non-analytic fans. The following lemma is obvious:

Lemma 12.3 *The valuation ring $W \in \mathfrak{V}$ amalgamates (that is, W is compatible with F and $\lambda_W : \mathcal{R}_{X,a} \longrightarrow (k_W, F_W)$ amalgamates) if and only if its fiber in $\mathfrak{V}^h$ has a single point, or, in other words, if it lies under the tail of $\mathfrak{V}^h$.*

□

Moreover, we have

Proposition 12.4 *Let $F \in \mathcal{G}_a$ be an algebraic fan of X, $a \in X$. Suppose that W is a valuation ring in the tail of F which amalgamates. Then W has a unique extension in $\mathfrak{V}^h$ and any $W' \supset W$ amalgamates too.*

Proof: Using a minimal amalgamation corner and Proposition 10.2 as in the proof of Theorem 12.1, we find a valuation ring W^h of $\mathcal{R}^h_{X,a}$ lying over W which is compatible with each single ordering of F^h_a. This implies that W^h is in the tail of $\mathfrak{V}^h$. Now it follows from the tree structure of $\mathfrak{V}$ and $\mathfrak{V}^h$, and Proposition 10.2, that for all $W' \in \mathfrak{V}$, $W' \supset W$, there is a unique extension $(W')^h$ in $\mathfrak{V}^h$. In particular this ring $(W')^h$ is compatible with all the elements of F^h_a (that is, it is in its tail). Denoting by $(F^h_a)_{(W')^h}$ the fan induced in the residue field $k_{(W')^h}$ it is obvious that

$$\begin{array}{ccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda_{(W')^h}} & (k_{(W')^h}, (F^h_a)_{(W')^h}) \\ \uparrow & & \uparrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda_{W'}} & (k_{W'}, F_{W'}) \end{array}$$

is an amalgamation square. □

Corollary 12.5 *The fan F is analytic if and only if W_F amalgamates, or, in other words, if and only if* $\mathrm{rank}(F) \leq \mathrm{tail}(F^h_a)$.

Proof: Suppose first that F is analytic. Then, by Theorem 12.1 there is a valuation ring W of $\mathcal{R}_{X,a}$ which trivializes F and such that λ_W amalgamates. By the very definition we have $W \subset W_F$ and therefore it follows from Proposition 12.4 that W_F amalgamates. Conversely, if W_F amalgamates then F is analytic by Theorem 12.1. Finally notice that the condition $\mathrm{rank}(F) \leq \mathrm{tail}(F^h_a)$ is equivalent to say that W_F lies under the tail of $\mathfrak{V}^h$ which by Lemma 14.3 amounts to say that W_F amalgamates. □

The next example shows the special role of the valuation W_F:

Example 12.6 Let $X = Y = \mathbb{R}^3$, $Z_1 = \{\mathsf{z} = \mathsf{y}^2 - \mathsf{x}^2 - \mathsf{x}^3 = 0\}$ and $Z_2 = \{\mathsf{z} = 0\}$. Set $a = (0,0,0)$. Then $W_2 = \mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}]_{(\mathsf{z})}$ is a discrete valuation ring of $\mathbb{R}(\mathrm{x},\mathrm{y},\mathrm{z})$ whose residue field is $\mathbb{R}(\mathrm{x},\mathrm{y}) = \mathcal{K}(Z_2)$. Now, $\overline{W}_1 = \mathbb{R}[\mathrm{x},\mathrm{y}]_{(\mathrm{y}^2-\mathrm{x}^2-\mathrm{x}^3)}$ is a discrete valuation ring of $\mathcal{K}(Z_2)$ with residue field $\mathcal{K}(Z_1)$. Therefore, the composite of W_2 and $\overline{W}_1$ gives a discrete rank two valuation ring W_1 of $\mathcal{K}(X)$ with $W_1 \subset W_2$. Let $\bar{\tau}_1, \bar{\tau}_2$ be the orderings of $\mathcal{K}(Z_1)$ represented by the half-branches of the curve at a in different analytic branches (*Figure 24*).

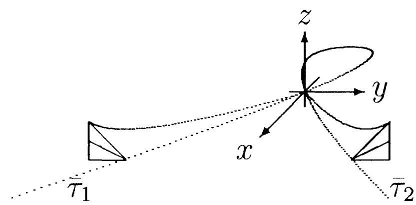


Figure 24

Let $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ be the pull-back of $\bar{\tau}_1, \bar{\tau}_2$ to $\mathcal{K}(Z_2)$ by $\overline{W}_1$ Finally, let F be the fan obtained by pulling-back $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ via by W_2, keeping $z > 0$. Then the family $\mathfrak{V}$ of valuations compatible with some element of F is

$$\begin{array}{ccl} V_1^1 & & \\ & \searrow & \\ & & W_1 \subset W_2 \subset \mathcal{K}(Y) \\ & \nearrow & \\ V_1^2 & & \end{array}$$

where V_1^1 and V_1^2 are the composite of W_1 with the valuation rings of $\mathcal{K}(Z)$ determined by its analytic branches at a. Notice that $W_F = W_1$, since W_2 does not trivializes F.

We claim that W_2 amalgamates, while W_1 does not. Indeed, the family of valuations of the henselization $\mathbb{R}[[\mathrm{x},\mathrm{y},\mathrm{z}]]_{\mathrm{alg}}$ compatibles with F_a^h is

$$\begin{array}{ccl} H_1^1 \subset H_2^1 & & \\ & \searrow & \\ & & H_2 \\ & \nearrow & \\ H_1^2 \subset H_2^2 & & \end{array}$$

where H_1^i is the restriction to $\mathbb{R}[[\mathrm{x},\mathrm{y},\mathrm{z}]]_{\mathrm{alg}}$ of the henselization of V_1^i; $H_2 = (\mathbb{R}[[\mathrm{x},\mathrm{y},\mathrm{z}]]_{\mathrm{alg}})_{(\mathrm{z})}$, and H_2^i are the composite of H_2 with the valuations $(\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathrm{alg}})_{\mathfrak{q}_i}$ where $\mathfrak{q}_1, \mathfrak{q}_2$ are the two prime ideals of $\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathrm{alg}}$ lying over $\mathfrak{q} = (\mathsf{y}^2 - \mathsf{x}^2 - \mathsf{x}^3)$.

In particular we have $\mathrm{tail}(F) = 2$ and $\mathrm{tail}(F^h_a) = 1$. However, W_F does not amalgamate, what shows also that the condition $\mathrm{tail}(F^h_a) > 0$ is not sufficient for F to be analytic. $\square$

As an application, let us look back at Example 7.2 *b) Figure 16*. Keeping its notations consider the unique valuation ring $W = W_F$ trivializing F, so that $k_W = \kappa(\mathfrak{q})$, $F_W = \{\tau_1, \tau_2\}$. Then $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ does not amalgamate. In fact, in this situation, $\mathfrak{q}\mathcal{R}^h_{X,a} = \mathfrak{q}_1 \cap \mathfrak{q}_2$, the $\mathfrak{q}_i$'s being the two branches of the cubic Y at a, and $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k_W = \kappa(\mathfrak{q}_1) \times \kappa(\mathfrak{q}_2)$. Since τ_1 extends to $\kappa(\mathfrak{q}_1)$ and τ_2 extends to $\kappa(\mathfrak{q}_2)$, the fan F_W does not extend to either of these fields. Hence by Proposition 11.5, $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ does not amalgamate, as claimed, and by Corollary 12.3, F^h_a is not a fan.

A similar argument shows that Example 7.2 *c) Figure 18* is not analytic either. Thus, only Example 7.2 *e)* remains open. This example illustrates the most general obstruction to analyticity. In order to understand it we need a geometric interpretation of amalgamation which will be discussed in the next section.

13 Finite coverings associated to a fan

Consider once again an algebraic set $X \subset \mathbb{R}^n$, a point $a \in X$ and a fan $F \subset \mathcal{G}_a$. So far we have seen that the amalgamation of a homomorphism $\lambda_W : \mathcal{R}_{X,a} \longrightarrow (k_W, F_W)$ is equivalent to the fact that F is analytic. We have also seen some geometric necessary conditions for F to be analytic (Proposition 5.3). Thus we want to investigate the geometric information hidden behind the rather algebraic amalgamation condition, in order to find a geometric formulation for it. Although the analyticity of a fan is highly non birational in nature, our geometric characterization comes by studing the behaviour of F in some special birational models of X which we call *finite coverings*. This is mainly due to the good relationship between henselization and finiteness. We start with the following:

Remarks 13.1 *a)* Consider an amalgamation problem $\lambda : \mathcal{R}_{X,a} \to (k, F)$ such that F consists of a single ordering τ. Then $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates.

Indeed, set $\mathfrak{q} = \ker(\lambda)$ and $\zeta = \tau|_{\kappa(\mathfrak{q})}$. Then ζ extends to $\kappa(\mathfrak{q}^h)$ for a unique henselian branch $\mathfrak{q}^h$ of $\mathfrak{q}$ at a (this follows by consideration of the bijection $\theta^* : \operatorname{Spec}_r(\mathcal{R}^h_{X,a}) \to \mathcal{G}_a \subset \operatorname{Spec}_r(\mathcal{R}(X))$ described at the beginning of Section 8). Now if k' is the real closure of k with respect to τ, there is a homomorphism $\kappa(\mathfrak{q}^h) \to k'$ which closes the amalgamation square. $\square$

This remark shows the importance of the final field k_W in the amalgamation problem of Corollary 11.3. For instance, in Examples 7.2 *d)* and *e)* the valuation ring W_2 is centered at the subvariety Z (the z and y-axis respectively) and therefore the homomorphism $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ factorizes through

$$\mathcal{R}_{X,a} \xrightarrow{\eta} (\mathcal{K}(Z), F_W|_{\mathcal{K}(Z)}) \xrightarrow{\iota} (k_W, F_W)$$

and in both cases $F_W|_{\mathcal{K}(Z)}$ consists of one single ordering, while F_W has two. Therefore, according to this remark, η will amalgamate in both cases, although it is not clear whether λ_W will, and in fact it does in case *d)* but not in case *e)*.

b) Consider an amalgamation problem $\lambda : \mathcal{R}_{X,a} \to (k, F)$ and put $\mathfrak{q} = \ker(\lambda)$. We have already seen in *a)* that if F consists of a single ordering, then λ amalgamates. Suppose $F = \{\tau_1, \dots, \tau_s\}$ and consider, for any τ_i, the real closure $\kappa(\tau_i)$ of k with respect to τ_i. By Proposition 8.1 and the comments following, we have a homomorphism $\lambda_i : \mathcal{R}^h_{X,a} \to \kappa(\tau_i)$ making commutative the square:

$$\begin{array}{ccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda_i} & \kappa(\tau_i) \\ \uparrow & & \uparrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda} & k \end{array}$$

Thus, we get a canonical homomorphism $\mu_i : \mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k \longrightarrow \kappa(\tau_i)$ and an obvious necessary condition for λ to amalgamate is that all the kernels $\ker(\mu_i)$ coincide. Suppose that this holds true and denote this common kernel by I. For any $\tau_i \in F$ we have an extension τ_i' in the quotient field K of $\left(\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k\right)/I$, and these τ_i's fail to be a fan if three of them can be separated from any other fourth by a suitable element of K (depending on the latter). Since we are dealing with a finite fan F, it follows that $\lambda : \mathcal{R}_{X,a} \longrightarrow (k, F)$ amalgamates if and only if so does $\lambda : \mathcal{R}_{X,a} \longrightarrow (\kappa, F|\kappa)$ for any finitely generated subextension $\kappa \supset \kappa(\mathfrak{q})$ of $k \supset \kappa(\mathfrak{q})$. □

c) All this is specially easy to check in case F consists of two orderings. Indeed, suppose that $F = \{\tau_1, \tau_2\}$. Since two orderings always build a fan, $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates if and only if $\ker(\mu_1) = \ker(\mu_2)$. □

d) Let again $F = \{\tau_1, \tau_2\}$. We denote by k_0 be the algebraic closure of $\kappa(\mathfrak{q})$ in k and by F_0 the restriction of F to k_0. Then $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates if and only if so does $\lambda : \mathcal{R}_{X,a} \to (k_0, F_0)$.

In fact, given an amalgamation square

$$\begin{array}{cccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k_0' & F_0' \\ \uparrow & & \uparrow & \downarrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda} & k_0 & F_0 \end{array}$$

we can always suppose that k_0' is algebraic over k_0 (Remark 11.4 *d)*). Then $k' = k_0' \otimes_{k_0} k$ is a field: it is a domain (this follows from [Ng, 39.9, p.150] since k_0 is algebraically closed in k) and it is algebraic over k. Hence, keeping the above notation, $\ker(\mu_1) = \ker(\mu_2) = (0)$ and we can extend τ_1, τ_2 to two orderings ζ_1, ζ_2 of k', which form a trivial fan. Thus we get an amalgamation square

$$\begin{array}{ccc} \mathcal{R}^h_{X,a} & \xrightarrow{\lambda'} & k' \\ \uparrow & & \uparrow \\ \mathcal{R}_{X,a} & \xrightarrow{\lambda} & k \end{array} \qquad \begin{array}{c} F' \\ \downarrow \\ F \end{array}$$

where $F' = \{\zeta_1, \zeta_2\}$. □

In particular, this shows that if F is trivial, then $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates if and only if so does $\lambda : \mathcal{R}_{X,a} \to (k^*, F^*)$ for any pure transcendental extension k^* of k such that F^* extends F.

e) Combining *b)* and *d)* we get that in case $\#(F) = 2$, to check whether $\lambda : \mathcal{R}_{X,a} \to (k, F)$ amalgamates it is enough to do it for all $\lambda : \mathcal{R}_{X,a} \to (\kappa, F|_\kappa)$, where $\kappa \supset \kappa(\mathfrak{q})$ is a finite subextension of $k \supset \kappa(\mathfrak{q})$, or, equivalently, when κ runs along an inductive family of finite subextensions with direct limit k_0. □

If $\#(F) > 2$, the situation becomes much more involved. In fact, even if we assume that all the kernels $\ker(\mu_i)$ coincide, or, in other words, that we can extend all the orderings of F to the same factor of $\mathcal{R}^h_{X,a} \otimes_{\mathcal{R}_{X,a}} k$, it is not clear whether these extensions build a fan. In fact, the following example, suggested to us by A. Prestel, shows that this is not the case in general.

Example 13.2 Let $X = \mathbb{R}^4$, $a = (0,0,0,0)$ and set $A = \mathcal{R}(X)$. Consider the ring $W = \mathbb{R}[\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{u}]_{(\mathrm{u})}$, where $\mathrm{z} = \mathrm{v}/\mathrm{u}$; it is a rank one discrete valuation ring of $\mathcal{K}(X) = \mathbb{R}(\mathrm{x}, \mathrm{y}, \mathrm{u}, \mathrm{v})$ with residue field $k_W = \mathbb{R}(\mathrm{x}, \mathrm{y}, \mathrm{z})$. Moreover, W is finite on X and $\mathfrak{m}_W \cap A = (\mathrm{u}, \mathrm{v})A$, so that its center $Z \subset X$ is defined by the equations $\mathrm{u} = \mathrm{v} = 0$. Now, in k_W we consider the 4-element fan $F_W = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ obtained as follows. First, $W_2 = \mathbb{R}[\mathrm{x}, \mathrm{y}, \mathrm{z}]_{(\mathrm{z}^2 - \mathrm{x}^2 - 1)}$ is a discrete rank one valuation ring of $\mathbb{R}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ with residue field $k_{W_2} = \mathcal{K}(Z_2) = \mathrm{qf}\,(\mathbb{R}[\mathrm{x}, \mathrm{y}, \mathrm{z}]/(\mathrm{z}^2 - \mathrm{x}^2 - 1))$, that is, k_{W_2} is the rational function field of the cylinder $Z_2 \subset \mathbb{R}^3$ of equation $\mathrm{z}^2 - \mathrm{x}^2 - 1 = 0$. In this field we consider the rank one discrete valuation ring $W_1 = \mathbb{R}[\mathrm{x}, \mathrm{y}, \mathrm{z}]_{(\mathrm{y}, \mathrm{z}^2 - \mathrm{x}^2 - 1)}/(\mathrm{z}^2 - \mathrm{x}^2 - 1)$. We have $k_{W_1} = \mathrm{qf}\,(\mathbb{R}[\mathrm{x}, \mathrm{z}]/(\mathrm{z}^2 - \mathrm{x}^2 - 1))$, which is the rational function field $\mathcal{K}(Z_1)$ of the hyperbola $Z_1 \subset \mathbb{R}^2$ of equation $z^2 - x^2 - 1 = 0$. Next, in k_{W_1} we consider the trivial fan $\{\tau_1, \tau_2\}$, where τ_1, τ_2 are the orderings defined by the half-branches of the hyperbola ending at the x-axis and defined by the conditions $x < 0, z > 0$ and $x > 0, z < 0$ respectively.

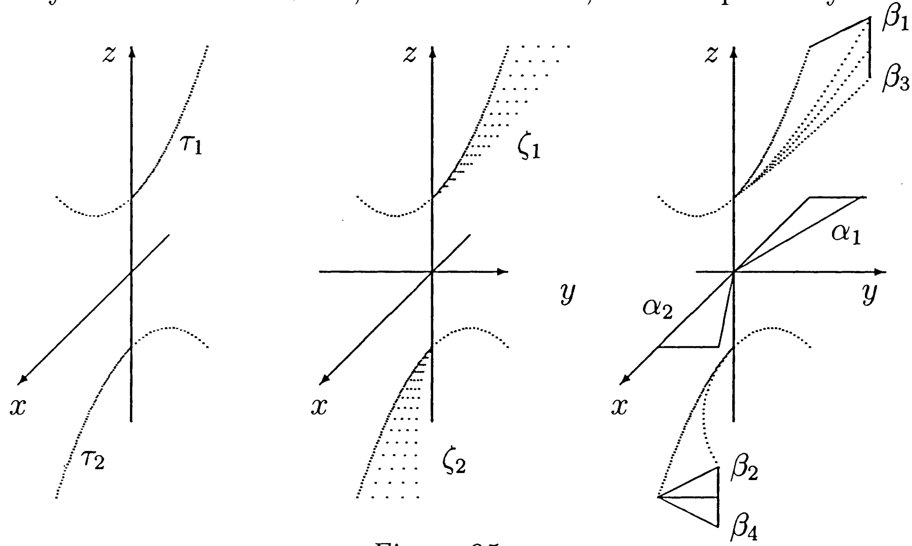


Figure 25

Next, consider the fan $\{\zeta_1, \zeta_2\}$ in k_{W_2} obtained by pulling-back τ_1 and τ_2 by W_1 keeping constant the sign of the uniformization parameter, say, $y > 0$. This way, we obtain a trivial fan on Z_2. Finally, we get F_W by pulling back $\{\zeta_1, \zeta_2\}$ to $\mathbb{R}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ by the valuation ring W_2. Hence F_W consists of four orderings centered on the cilinder Z_2 (*Figure 25*).

To finish, let $F \subset \mathcal{G}_a$ be any fan compatible with W and which induces F_W in k_W (for instance the pull back of F_W by W). We want to study whether $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ amalgamates. Passing to the residue fields, we know that the problem is equivalent to see whether the monomorphism $\mathcal{R}_{Z,a} \longrightarrow (k_W, F_W)$ amalgamates, where Z stands for (x, y)-plane. Notice that the restriction G of F_W to $\mathbb{R}(\mathrm{x}, \mathrm{y})$ is $G = \{\alpha_1, \alpha_2\} = \{(0_+, 0_+), (0_-, 0_+)\}$, where we use the notation introduced in Section 3.

Suppose that there is an amalgamation square

$$\begin{array}{ccccc}
(\mathbb{R}[[\mathrm{x}, \mathrm{y}]]_{\mathrm{alg}}, G^h) & \longrightarrow & k & & \widetilde{F} \\
\uparrow & & \uparrow & & \downarrow \\
(\mathbb{R}[\mathrm{x}, \mathrm{y}], G) & \longrightarrow & k_W = \mathbb{R}(\mathrm{x}, \mathrm{y}, \mathrm{z}) & & F_W
\end{array}$$

Let us see that we may assume that k is the quotient field of $\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}[\mathsf{z}]$. Indeed, first notice that $A = \mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}} \otimes_{\mathbb{R}[\mathsf{x},\mathsf{y}]} k_W$ is a domain (because $\mathbb{R}(\mathsf{x},\mathsf{y})$ is algebraically closed in k_W), and we have a homomorphism $A \longrightarrow k$. On the other hand A is a ring of fractions of

$$\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}} \otimes_{\mathbb{R}[\mathsf{x},\mathsf{y}]} \mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}] = \mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}[\mathsf{z}],$$

so that we get a homomorphism $\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}[\mathsf{z}] \longrightarrow k$ whose kernel I lies over (0) in $\mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}]$. We claim that this implies $I = (0)$. For let $0 \neq f \in I$. Then f verifies an equation of algebraic dependence over $\mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}]$, say $a_0 f^s + a_1 f^{s-1} + \cdots + a_s = 0$, with $a_s \neq 0$, since $\mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}]$ is a domain. But we have $a_s \in I \cap \mathbb{R}[\mathsf{x},\mathsf{y},\mathsf{z}] = (0)$, contradiction. Therefore, the homomorphism $\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}[\mathsf{z}] \longrightarrow k$ is injective, and since $A \subset \mathrm{qf}(\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}[\mathsf{z}])$, we may replace k by this latter quotient field, as claimed.

In particular, $\widetilde{F} = \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ where γ_i is the (unique) extension of β_i to k. However, let us see that $\widetilde{F}$ is not a fan. Indeed, notice that $\sqrt{\mathsf{x}^2+1}$ is an element of $\mathbb{R}[[\mathsf{x},\mathsf{y}]]_{\mathsf{alg}}$, so that the function $f = \mathsf{z} - \sqrt{\mathsf{x}^2+1}$ lies in k and it is immediate to check that it separates three of the orderings of $\widetilde{F}$ from the fourth. $\square$

Notice that this example also shows that Remark 13.1 *d)* is false for non-trivial fans: the reduction of the amalgamation problem $\lambda : \mathcal{R}_{X,a} \longrightarrow (k,F)$ to $\lambda : \mathcal{R}_{X,a} \longrightarrow (k_0,F_0)$, where k_0 is algebraic over $\kappa(\mathfrak{q})$, $\mathfrak{q}$ being the kernel of λ, does not work if $\#(F) > 2$.

For the sequel, let Y denote the support of F, so that F is a fan of the field of rational functions of Y. Also, we recall that we set $Z_F \subset Y$ for the center of the biggest valuation ring, W_F, of $\mathcal{R}(X)$ trivializing F. We set $\mathfrak{q}_F$ for the ideal of functions vanishing on Z_F, so that the function field $\mathcal{K}(Z_F)$ of Z_F, is the quotient field of $\mathcal{R}(X)/\mathfrak{q}_F$. Finally we denote by k_0 the algebraic closure of $\mathcal{K}(Z_F)$ in k_F.

Definition 13.3 *Let $\pi' : Y' \longrightarrow X$ be a regular mapping from an algebraic set $Y' \subset \mathbb{R}^m$ into X, and $Z' \subset Y'$ another algebraic set.*

a) We say that the pair (π', Z'), or simply that (Y', Z'), is a finite covering *(*associated to *F) if* *i)* $\pi'(Y') \subset Y$ *and* $\pi' : Y' \to Y$ *is a birational mapping;*

ii) W_F *is finite on* Y' *and* Z' *is the center of* W_F *in* X'*, and*

iii) $\pi'|_{Z'} : Z' \to Z_F$ *is finite.*

b) Given two finite coverings (Y', Z') *and* (Y'', Z'') *we say that* (Y', Z') precedes strongly *(respectively* weakly*)* (Y'', Z'') *if there is a regular (respectively rational) mapping* $\phi : Y'' \to Y'$ *making commutative the triangles*

$$\begin{tikzcd} Y'' \arrow[rr, "\phi"] \arrow[dr, "\pi''"'] & & Y' \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} Z'' \arrow[rr, "\phi|"] \arrow[dr, "\pi''"'] & & Z' \arrow[dl, "\pi'"] \\ & Z_F & \end{tikzcd}$$

c) We will say that a family of finite coverings is strongly *(respectively* weakly*)* cofinal *if it is cofinal in the strong (respectively weak) preceding relation defined in b).*

The following lemma collects some basic properties of finite coverings:

Lemma 13.4 *a) If* (Y', Z') *precedes strongly to* (Y'', Z'') *then* (Y', Z') *precedes weakly to* (Y'', Z'').

b) The family of all finite coverings is strongly directed.

Proof: *a)* is trivial. For the proof of *b)*, let (Y_1, Z_1) and (Y_2, Z_2) be two finite coverings. We set $A = \mathcal{R}(Y)$ and $\mathfrak{q} = \mathfrak{m}_F \cap A$. Then, we have $A_1 = \mathcal{R}(Y_1) = A[s_1, \ldots, s_r]$ and $A_2 = \mathcal{R}(Y_2) = A[t_1, \ldots, t_p]$ where the s_i and the t_j are elements of the quotient field $\mathcal{K}(Y)$ of A. We also set $\mathfrak{q}_i = \mathfrak{m}_F \cap A_i$, that is, $\mathfrak{q}_i$ is the ideal of functions vanishing on Z_i. Consider the ring $A' = A[s_1, \ldots, s_r, t_1, \ldots, t_p]$. Since A_1 and A_2 are contained in W_F, we have $A' \subset W_F$. We set $\mathfrak{q}' = \mathfrak{m}_F \cap A'$. Obviously A' is the ring of regular functions of an algebraic set $Y' \subset \mathbb{R}^m$ and we denote by Z' the center of W_F in Y'. The ring inclusions induce a commutative square of birational regular mappings:

$$\begin{array}{ccc} Y' & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Y \end{array}$$

Thus, all we have to show is that when we restrict to the centers, the commutative diagram:

$$\begin{array}{ccc} Z' & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ Z_2 & \longrightarrow & Z_F \end{array}$$

consists of finite mappings. But notice that $A'/\mathfrak{q}'$ is generated over $A/\mathfrak{q}$ by the classes of the elements s_i and t_j, which are integral over $A/\mathfrak{q}$, since Z_1 and Z_2 are finite over Z_F. Therefore the result follows at once. □

Next we have:

Proposition 13.5 *a) A family* $\{(Y_i, Z_i)\}$ *of finite coverings is weakly cofinal if and only if* $\varinjlim \mathcal{K}(Z_i) = k_0$.

b) The family of all finite coverings (Y', Z') *with* Z' *normal is strongly cofinal.*

Proof: a) Let $\{(Y_i, Z_i)\}$ be a weakly cofinal family of finite coverings. From the very definition we have that $\mathcal{K}(Z_i)$ is a finite extension of k_0, so that $\varinjlim \mathcal{K}(Z_i) \subset k_0$. For the converse inclusion, take any $\theta \in k_0$ and consider the extension $\kappa = \mathcal{K}(Z_F)[\theta]$. Replacing θ by its product times a suitable denominator we may assume that it is integral over $\mathcal{R}(Z_F)$. Now, $\theta \in k_F$ and therefore there is $t \in W_F \subset \mathcal{K}(Y)$ such that $\theta = t \bmod \mathfrak{m}_F$. We consider $A' = \mathcal{R}(Y)[t] \subset \mathcal{K}(Y)$ and set $\mathfrak{q}' = \mathfrak{m}_F \cap A'$. Then A' is the ring of regular functions of an algebraic set Y' and we have a regular mapping $\pi' : Y' \to Y$. Moreover, if Z' denotes the center of W_F in X', we have

$$\mathcal{R}(Z') = A'/\mathfrak{q}' = \mathcal{R}(Z_F)[t \bmod \mathfrak{q}'] = \mathcal{R}(Z)[\theta]$$

and therefore $\mathcal{K}(Z') = \kappa$. Now, since the given family is weakly cofinal, there is (Y_i, Z_i) weakly "posterior" to (Y', Z'). But this, in particular, implies that $\kappa = \mathcal{K}(Z') \subset \mathcal{K}(Z_i)$, so that $\theta \in \varinjlim \mathcal{K}(Z_i)$.

Conversely, suppose now that $\varinjlim \mathcal{K}(Z_i) = k_0$. Since by the preceding lemma the family of all finite coverings is weakly directed, it is enough to see that given any finite covering (Y', Z') there is an element (Y_i, Z_i) of the family which is weakly posterior to it. We set A and $\mathfrak{q}$ as in the proof of Lemma 13.4 *b)*. Then $A' = \mathcal{R}(Y') = A[s_1, \dots, s_r]$ for some $s_1, \dots, s_r \in \mathcal{K}(Y)$, and we set $\mathfrak{q}' = \mathfrak{m}_F \cap A'$. Thus $\mathcal{R}(Z') = A'/\mathfrak{q}' = \mathcal{R}(Z)[\theta_1, \dots, \theta_r]$ with $\theta_i = s_i (\text{mod } \mathfrak{q}') \in k_0$. Let (Y_i, Z_i) be a member of the given family such that $\theta_1, \dots, \theta_r \in \mathcal{K}(Z_i)$. Then the canonical rational mapping $\phi : Y' \to Y_i$ making the triangle

$$\begin{tikzcd} Y'' \arrow[rr, "\phi"] \arrow[dr, "\pi''"'] & & Y' \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

commutative, induces a mapping $\phi| : Z' \to Z_i$ between the centers which makes commutative the triangle:

$$\begin{tikzcd} Z'' \arrow[rr, "\phi|"] \arrow[dr, "\pi''"'] & & Z' \arrow[dl, "\pi'"] \\ & Z_F & \end{tikzcd}$$

This shows that (Y', Z') precedes weakly (Y_i, Z_i).

b) Again, by the preceding lemma we have to show that given any finite covering (Y', Z') there is another one (Y'', Z'') with Z'' normal. Let B be the integral closure of $\mathcal{R}(Z')$ in $\mathcal{K}(Z')$. Then B is a finite module over $\mathcal{R}(Z')$, say $B = \mathcal{R}(Z')[\theta_1, \dots, \theta_r]$, with $\theta_i \in \mathcal{K}(Z') \subset k_F$. Let $t_i \in W_F \subset \mathcal{K}(Y)$ such that $t_i = \theta_i$ (mod $\mathfrak{m}_F$). Thus, $A'' = \mathcal{R}(Y')[t_1, \dots, t_r]$ is the coordinate ring of an algebraic set Y'', and we denote by Z'' the center of W_F in Y''. Then (Y', Z') precedes (Y, Z), and Z'' is normal, since $\mathcal{R}(Z'') = B$.

$\square$

14 Geometric characterization of analytic fans

Once the notion of finite covering is available, we can understand the geometric meaning of amalgamation:

Theorem 14.1 (Geometric characterization of analytic fans) *Let $X \subset \mathbb{R}^n$ be an algebraic set and $a \in X$. Let $F \in \mathcal{G}_a$ be an algebraic fan. Then the following are equivalent:*

a) F is analytic at a;

b) For any finite covering (Y', Z') associated to F, the two orderings induced by F in $\mathcal{R}(Z')$ specialize to a unique point $a' \in Z'$ and extend to the same henselian branch of Z' at a';

c) For any finite covering (Y', Z') associated to F, the two orderings induced by F in $\mathcal{R}(Z')$ specialize to a unique point $a' \in Z'$;

d) There is a strongly cofinal family $\{(Y_i, Z_i)\}$ of finite coverings associated to F, such that for any i, the two orderings induced by F in $\mathcal{R}(Z_i)$ specialize to a unique point $a' \in Z_i$;

e) There is a strongly cofinal family $\{(Y_i, Z_i)\}$ of finite coverings associated to F, such that for any i, the two orderings induced by F in $\mathcal{R}(Z_i)$ specialize to a unique point $a' \in Z_i$ and extend to the same henselian branch of Z_i at a'.

f) There is a weakly cofinal family $\{(Y_i, Z_i)\}$ of finite coverings associated to F, such that for any i, the two orderings induced by F in $\mathcal{R}(Z_i)$ specialize to a unique point $a' \in Z_i$ and extend to the same henselian branch of Z_i at a'.

Proof: The implications $b) \Rightarrow c)$ and $e) \Rightarrow f)$ are obvious, as well as the equivalence between $d)$ and $c)$. For the implication $c) \Rightarrow e)$ notice that a normal algebraic set is analytically irreducible at every point. Thus, the family of all finite coverings (Y', Z') with Z' normal, which is cofinal (Proposition 13.5), verifies *e)*. Hence, it remains to show that $a) \Rightarrow b)$ and $f) \Rightarrow a)$.

$a) \Rightarrow b)$ Let (Y', Z') be a finite covering associated to F. Thus, $\mathcal{R}(Z') \subset k_F$ and it is a finite extension of $\mathcal{R}(Z)$. Let $a_1, \dots, a_r \in Z'$ be the points of Z' lying over a, and denote by $\mathfrak{m}_1, \dots \mathfrak{m}_r$ their corresponding maximal ideals in $\mathcal{R}(Z')$. We will see that the fan F (considered as a fan in Y') is analytic at some of the a_i's. Thus *b)* will follow from Proposition 8.2. By Corollary 12.5, the homomorphism λ_F amalgamates, so that we have an amalgamation square:

$$\begin{array}{ccccc} \mathcal{R}^h_{Z,a} & \xrightarrow{\lambda'} & k' & & F' \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{Z,a} & \xrightarrow{\lambda_F} & k_F & & F_{W_F} \end{array}$$

Consider the ring $\mathcal{R}(Z')_a = \mathcal{R}(Z') \otimes_{\mathcal{R}(Z)} \mathcal{R}_{Z,a}$. It is a semilocal finite extension of $\mathcal{R}_{Z,a}$, with $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ as maximal real ideals, and possibly some others non real maximal ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_s$ (that is, with $\mathbb{C}$ as residue field). Consequently, by [EGA, 18.6.7, p.138], its henselization is

$$C = \mathcal{R}(Z')^h_a = \mathcal{R}(Z')^h_{\mathfrak{m}_1} \times \cdots \times \mathcal{R}(Z')^h_{\mathfrak{m}_r} \times \mathcal{R}(Z')^h_{\mathfrak{n}_1} \times \cdots \times \mathcal{R}(Z')^h_{\mathfrak{n}_s}.$$

On the other hand, since $\mathcal{R}(Z')_a$ is a finite module over $\mathcal{R}_{Z,a}$, by [EGA, 18.6.8, p. 139], we have that $C = \mathcal{R}^h_{Z,a} \otimes_{\mathcal{R}_{Z,a}} \mathcal{R}(Z')_a$. Thus the two homomorphisms $\lambda' : \mathcal{R}^h_{Z,a} \to k'$ and $\mathcal{R}(Z')_a \subset k_F \to k'$ give a third one $\widetilde{\lambda} : C \to k'$, whose kernel will be denoted by $\widetilde{\mathfrak{p}}$. It follows that $\widetilde{\mathfrak{p}}$ is a prime ideal of some of the factors of C, and we claim that this factor corresponds to a real ideal $\mathfrak{m}_i$. Indeed, we proceed as in the proof of Proposition 8.1 *c)*. Let τ' be any ordering in F' and let W' be the convex hull of $\mathbb{Q}$ in k' with respect to τ'. Then W' dominates $\mathcal{R}^h_{Z,a}$ and since $\mathcal{R}(Z')^h_a$ is finite over it, W' has some center $\widetilde{\mathfrak{m}}$ also in $\mathcal{R}(Z')^h_a$, which selects the sought factor of $\mathcal{R}(Z')^h_a$. As the residue field of W' is $\mathbb{R}$, so is also the residue field $\kappa(\widetilde{\mathfrak{m}})$. This implies that $\widetilde{\mathfrak{m}} = \mathfrak{m}_i$, say for $i = 1$, as claimed. Thus $\widetilde{\lambda}$ factorizes through $C_1 = \mathcal{R}^h_{Z_i,a_1}$ and we get an amalgamation square

$$\begin{array}{ccccc} \mathcal{R}^h_{Z_i,a_1} & \xrightarrow{\widetilde{\lambda}} & k' & & F' \\ \uparrow & & \uparrow & & \downarrow \\ \mathcal{R}_{Z_i,a_1} & \xrightarrow{\lambda_F} & k_F & & F_{W_F} \end{array}$$

which shows that F is analytic at a_1. (Notice that, indeed, $\tilde{\mathfrak{p}}$ lies over (0) in $\mathcal{R}(Z')_a$, and therefore it is a minimal prime ideal of C_1 which determines the henselian branch of Z' at a_1 at which the two orderings induced by F in Z' extend.)

f) $\Rightarrow$ *a)* By Corollary 12.3, we have to show that $\lambda = \lambda_F : \mathcal{R}_{Y,a} \to (k_F, F_{W_F})$ amalgamates, or equivalently, that $\lambda : \mathcal{R}_{Z,a} \to (k_F, F_{W_F})$ does. Since the family is weakly cofinal, by Proposition 13.5 *a)* and Remark 13.1 *b)* and *d)*, it is enough to see that for any element (Y_i, Z_i) in the family, the inclusion

$$\bar{\lambda} : \mathcal{R}_{Z,a} \longrightarrow (\mathcal{K}(Z_i), F' = F_{W_F}|_{\mathcal{K}(Z_i)})$$

amalgamates. By hypothesis, there is a unique point $a' \in Z_i$ so that the two orderings, say τ_1', τ_2' of F' specialize to a'. Thus the local monomorphism

$$\bar{\lambda} : \mathcal{R}_{Z,a} \longrightarrow \mathcal{R}_{Z_i,a'}$$

extends to a (unique) homomorphism

$$\bar{\lambda}^h : \mathcal{R}^h_{Z,a} \longrightarrow \mathcal{R}^h_{Z_i,a'}.$$

Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the associated prime ideals of (0) in $\mathcal{R}^h_{Z_i,a'}$, that is, $C/\mathfrak{q}_1, \dots, C/\mathfrak{q}_r$ are the henselian branches of $\mathcal{R}_{Z_i,a'}$. Next, we also know that the two orderings τ_1', τ_2' extend to the same henselian branch, say $C_1 = C/\mathfrak{q}_1$. We denote by τ_1^*, τ_2^* these extensions. Let k_1 be the quotient field of C_1. Then τ_1^*, τ_2^* define a fan in k_1 and we have an amalgamation square

$$\begin{array}{ccccc}
\mathcal{R}^h_{Z,a} & \xrightarrow{\bar{\lambda}^h} & \mathcal{K}(Z_i)^h & & F^* \\
\uparrow & & \uparrow & & \downarrow \\
\mathcal{R}_{Z,a} & \xrightarrow{\bar{\lambda}} & \mathcal{K}(Z) & & F'
\end{array}$$

Therefore the proof is complete. □

The above characterization is specially easy to check in the two dimensional case. Indeed, let $X \subset \mathbb{R}^n$ be an algebraic surface, $a \in X$, and $F \subset \mathcal{G}_a$ a non-trivial algebraic fan, which, by Proposition 4.7, in dimension 2 implies $\#(F) = 4$ and $\mathrm{ht}(F) = 0$, so that with the notation above we have $Y = X$. Again, denote by Z_F the center of W_F in Y. Let Y' be the normalization of Y and Z' the center of W_F in Y'. Then we have:

Corollary 14.2 *F is analytic at a if and only the two orderings induced by F in Z' specialize to the same point a' and extend to the same henselian branch of Z' at a'.*

Proof: Since $\dim(Y) = 2$ the center Z must have dimension 0 or 1. In the first case we have $Z = \{a\}$ and by Remark 13.1 *a)* F is analytic. Moreover, Z' is also a point and therefore the statement is trivial. So, suppose that Z is an algebraic curve. Then W_F is a rank one discrete valuation ring and, Y' being normal, we have $W_F = \mathcal{R}(Y')_{\mathfrak{q}'}$ where $\mathfrak{q}'$ is the ideal of Z'. In particular $k_F = \mathcal{K}(Z')$ and therefore the family consisting of the single finite covering (Y', Z') is weakly cofinal. Thus the statement follows from the theorem.

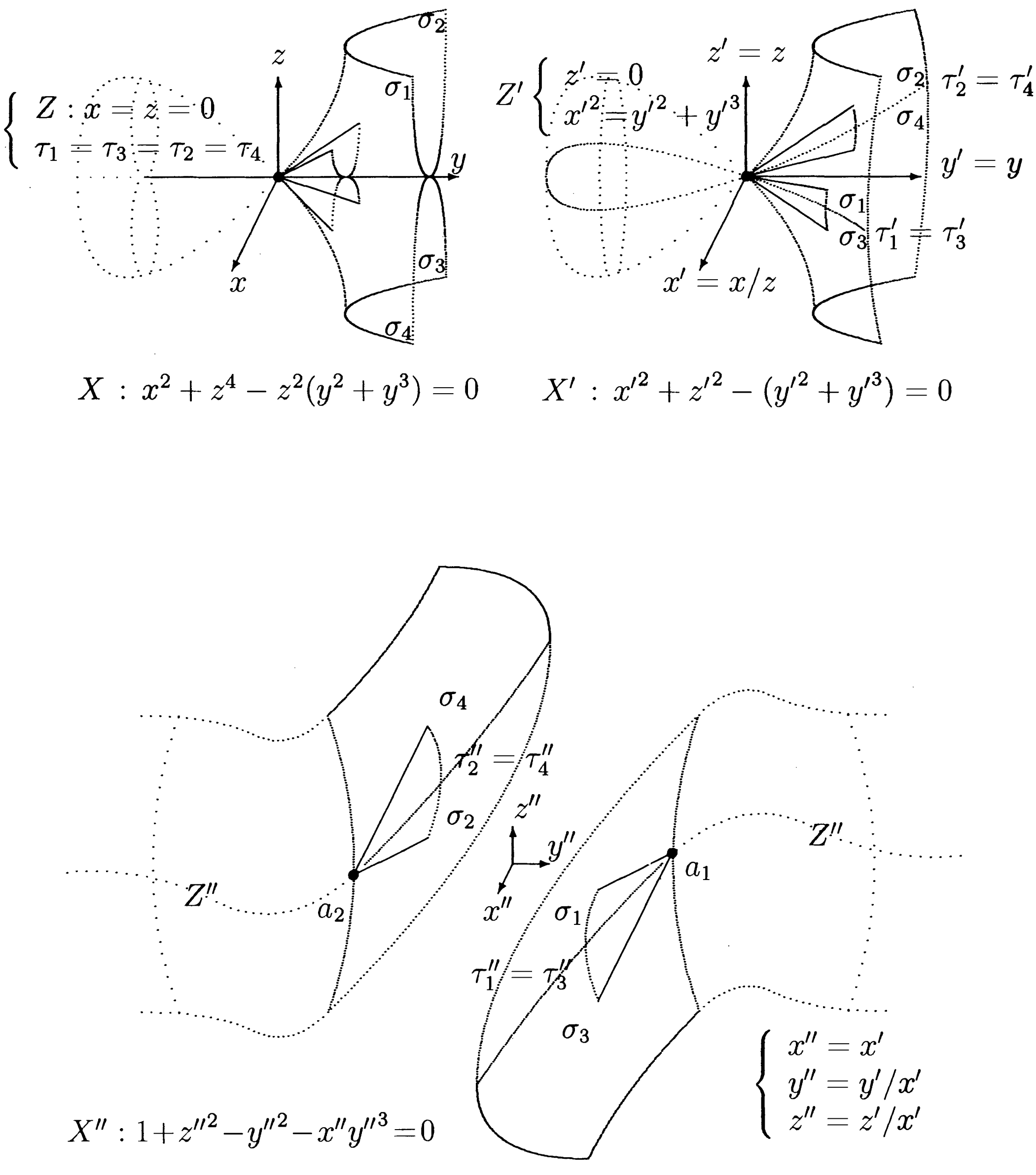
□

As illustrations of the geometric characterization of analytic fans, let us take again a look at Examples 7.2 *e)* and 13.2.

Examples 14.3 *a)* Let us see that the fan F of Example 7.2 *e)* is not analytic, for which we will use all notations introduced there. Remember that $W_F = \mathcal{R}(X')_{\mathfrak{q}}$, where $X' : x'^2 + z'^2 - (y'^2 + y'^3) = 0$ is the normalization of X and $\mathfrak{q}$ is the ideal in $\mathcal{R}(X')$ of the nodal curve $Z' = X' \cap \{z' = 0\}$: $z' = x'^2 - (y'^2 + y'^3) = 0$ (*Figures 22* and *26*). Thus the residue field of W_F is $\mathcal{K}(Z')$, and the fan induced by F in this residue field consists of the orderings τ_1 and τ_2, defined by the two right-ward half branches of Z'. Since they lie in different henselian branches of Z' at the origin, Theorem 14.1 *b)* assures that F is not analytic. In fact, the reader willing to do it can check that the function $f = \mathsf{z}^2 + \mathsf{y} - \mathsf{z}\mathsf{x}\sqrt{1+\mathsf{x}}$ separates three of the orderings of F from the fourth. Going further consider a full desingularization of X,

$$X'' : \; 1 + z''^2 - y''^2 - x''y''^3 = 0$$

(*Figure 26*). Here, the center Z'' of W_F is the curve $z'' = 1 - y''^2 - x''y''^3 = 0$, and the two orderings τ_1', τ_2' induced in the residue field specialize to two different points a_1 and a_2; this is Theorem 3.1 *c)*. □



$Z : x = z = 0$
$\tau_1 = \tau_3 = \tau_2 = \tau_4$
z
y
x
σ_1
σ_2
σ_3
σ_4
$X : x^2 + z^4 - z^2(y^2 + y^3) = 0$
Z'
$z' = 0$
$x'^2 = y'^2 + y'^3$
$z' = z$
$y' = y$
$x' = x/z$
$\tau_2' = \tau_4'$
$\tau_1' = \tau_3'$
$X' : x'^2 + z'^2 - (y'^2 + y'^3) = 0$
$\tau_2'' = \tau_4''$
$\tau_1'' = \tau_3''$
z''
y''
x''
Z''
a_1
a_2
$x'' = x'$
$y'' = y'/x'$
$z'' = z'/x'$
$X'' : 1 + z''^2 - y''^2 - x''y''^3 = 0$

Figure 26

b) Keeping the same notations as in Example 13.2, let us see that F is not analytic. We have that the valuation ring W_F is the composite of W and W_2, so that $k_F = \mathcal{K}(Z_2) = \mathrm{qf}\,(\mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z}]/(\mathtt{z}^2-\mathtt{x}^2-1))$, and the induced fan F_{W_F} is $\{\zeta_1,\zeta_2\}$. Now, the center Z of W_F in $X=\mathbb{R}^4$ is the plane of equations $u=v=0$ (that is, the (x,y)-plane), and the two orderings induced by F in Z (i.e. the restrictions of ζ_1 and ζ_2) are $\alpha_1=(0_+,0_+)$ and $\alpha_2=(0_-,0_+)$. Now it follows that the pair (X',Z_2), where Y' stands for the model of $\mathbb{R}^4$ with $\mathcal{P}(Y')=\mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z},\mathtt{u}]$, is a cofinal family of finite coverings. Since in Z_2 the orderings ζ_1,ζ_2 are centered at different points (namely $(0,0,1)$ and $(0,0,-1)$ respectively), we see that F is not analytic. □

Remark 14.4 The restriction to finite coverings in Theorem 14.1 is essential as the following example shows. Consider $X = Y = \mathbb{R}^2$, so that $\mathcal{P}(Y) = \mathbb{R}[\mathtt{x},\mathtt{y}]$. Let $\mathtt{z} = \mathtt{y}/\mathtt{x}$ and consider the valuation ring $W=\mathbb{R}[\mathtt{x},\mathtt{z}]_{(\mathtt{x})}\subset\mathbb{R}(\mathtt{x},\mathtt{y})$. Notice that if Y' is the affine chart of the blowing-up of Y at the origin with $\mathcal{P}(Y')=\mathbb{R}[\mathtt{x},\mathtt{z}]$, then the center Z' of W in Y' is the z-axis (the exceptional divisor), while the center Z of W in Y is the origin. In particular (Y',Z') is not a finite covering. Next, in $k_W=\mathbb{R}(\mathtt{z})$ we consider two orderings τ_1,τ_2 centered at two different points, say a_1,a_2, respectively. Now let F be the pull-back to $\mathbb{R}(\mathtt{x},\mathtt{y})=\mathbb{R}(\mathtt{x},\mathtt{z})$ of $\{\tau_1,\tau_2\}$ by W (*Figure 26*).

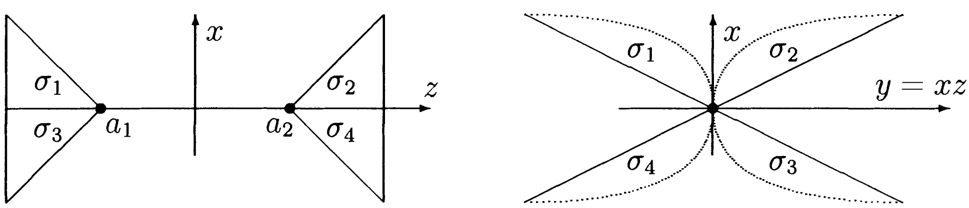


Figure 27

Thus, considered as a fan over Y, F is centered at the origin, and therefore it is analytic. However, in the blowing-up Y' of Y at the origin, F is centered at the excepcional divisor $\{x=0\}$, and the two orderings τ_1,τ_2 induced by F in this center specialize to two different points a_1,a_2. This shows that conditions *c)* and *d)* in Theorem 14.1 are not even necessary for a fan to be analytic, if we remove the finiteness condition.

15 The fan approximation lemma

Let $X \subset \mathbb{R}^n$ be an algebraic set. So far, we have been dealing with all fans of X in order to understand basicness and generic basicness. Now, we want to describe a special class of them which are enough for our purposes, and, in addition, have a very clear geometric meaning (see [AnRz1,2]). This class was already described in Section 6 as an example, but we introduce now its formal definition:

Definition 15.1 *An algebraic fan F of X with support Y is called* para-metric *if there are a blowing-up $Y' \to Y$, a local regular ring $B = \mathcal{R}(Y')_{\mathfrak{q}'}$, a regular system of parameters $x = (x_1, \dots, x_m)$ of B, and two distinct orderings τ_1, τ_2 of the residue field of B, such that F is the pull-back fan $F_{\tau_1,\tau_2}(x)$ described in 6.4, that is, it is the pull-back of τ_1, τ_2 by the valuation ring V_m.*

Notice that although a blowing-up Y' of Y is embedded in $\mathbb{R}^p \times \mathbb{P}^q(\mathbb{R})$, since any real projective space is embedded in some real affine space, we have that Y' is an algebraic subset of some $\mathbb{R}^k$. Also notice that it follows from the very definition that V_m is the biggest valuation ring trivializing F.

The goal of this section is a rather technical lemma that shows how we can replace arbitrary by parametric fans. This replacement is intended in terms of a suitable notion of approximation. To define such a notion, fans are, henceforth, seen as ordered tuples (α_j).

Definition 15.2 *Let $F = (\alpha_j)$ be an algebraic fan of X. We say that another algebraic fan $F' = (\alpha'_j)$ is close to F if*

a) $\#(F) = \#(F')$,

b) F and F' have the same support $Y \subset X$, and

c) for any j, α'_j is close to α_j in $\mathrm{Spec}_r(\mathcal{K}(Y))$.

We can now state:

Lemma 15.3 (Approximation lemma) *Let F be an algebraic fan of X which trivializes along a valuation ring W finite on X. Then, arbitrarily close to F, there is a parametric fan F' and a valuation ring W' such that*

a) W' is finite on X and has the same center as W.

b) F' trivializes along W'.

c) The residue field $k_{W'}$ of W' is a subfield of the residue field k_W of W and F' induces in $k_{W'}$ the same pair of orderings as F.

d) If a prime cone of F specializes to a point $a \in X$, so does the corresponding prime cone of F'.

e) The valuation ring $W_{F'} \supset W'$ has the same center at X as W.

Furthermore, the residue fields $k_{W'}$ form a direct system whose limit is k_W.

Proof: We start by fixing the notation. Let $F = (\alpha_j : 1 \leq j \leq 2^{m+1})$ be the given fan, $\mathfrak{p} \subset \mathcal{R}(X)$ its support, $Y \subset X$ the zero set of $\mathfrak{p}$ and $K = \kappa(\mathfrak{p})$ the field of rational functions of Y. Since W is finite on X, we have $\mathcal{R}(Y) \subset W$; we denote by $\mathfrak{q} = \mathfrak{m}_W \cap \mathcal{R}(Y)$, the center of W, where $\mathfrak{m}_W$ is the maximal ideal of W, and set Z for the zero set of $\mathfrak{q}$. Now, for any j we consider open neighborhoods $U_j = \{f_{j1} > 0, \dots, f_{jr_j} > 0\}$, $f_{jq} \in \mathcal{R}(Y)$. of the α_j's. Thus we want to find $F' = (\alpha'_j)$ with the required properties and $\alpha'_j \in U_j$. After shrinking these neigborhoods, we may assume that they are pairwise disjoint, and we say that the f_{jq}'s *separate* the α_i's. We proceed in several steps.

STEP I: Enlarging the residue field $\kappa(\mathfrak{q})$ by blowings-up.

We will construct F' with the additional property that $k_{W'}$ contains any fixed $\theta_1, \dots, \theta_r \in k_W$, which will imply the last assertion of the statement. To that end, we show how to add an element $\theta = \theta_i$ to $\kappa(\mathfrak{q})$. We choose elements $f, g \in \mathcal{R}(Y)$ such that $f/g \in W$ and its class in k_W is θ. Then, let $Y' \subset Y \times \mathbb{P}^1(\mathbb{R})$ be the blowing-up of Y with center the ideal generated by f

and g. Now, since blowings-up are proper, the valuation ring W dominates a local ring $\mathcal{R}(Y')_{\mathfrak{q}'}$, and f/g is in that localization, which gives $\theta \in \kappa(\mathfrak{q}')$. For, W contains the typical affine chart of Y' with ring of polynomial functions $\mathcal{P}(Y)[fg]$, so that, in fact, $\mathcal{R}(Y')_{\mathfrak{q}'}$ is a localization of $\mathcal{P}(Y)[f/g]$. By this argument, we can suppose that $\theta_1, \ldots, \theta_r$ are already in $\kappa(\mathfrak{q})$, which we do from now on, and do not care about them any more.

STEP II: Splitting of F into halves.

By hypothesis, the α_j's are compatible with the valuation ring $W \subset K$ and specialize to two orderings τ_1, τ_2 in the residue field k_W of W (possibly $\tau_1 = \tau_2$). We are to decompose F in two parts F_1, F_2 using these two orderings. First, we apply resolution of singularities I and II ([Hk]), so that after finitely many blowings-up we may assume that Y is non-singular and all the f_{jq}'s are normal crossings. Then $A = \mathcal{R}(Y)_{\mathfrak{q}}$ is a regular local ring of dimension, say, d, and has a regular system of parameters $x_1, \ldots, x_d$ such that for all j, q

$$f_{jq} = u_{jq} x_1^{\rho_{jq1}} \cdots x_d^{\rho_{jqd}}$$

where the u_{jq}'s are units of A and the ρ_{jql}'s are non-negative integers. Notice that this implies in particular that $m \leq d$, since for any valuation ring of K we have $\dim(\Gamma/2\Gamma) \leq d$.

In this situation the residue field $\kappa(\mathfrak{q})$ of A is a subfield of the residue field k_W of W, and we denote also by τ_1, τ_2 the restriction to $\kappa(\mathfrak{q})$ of τ_1, τ_2. Notice that for each $p = 1, 2$ the signs of the elements f_{jq} in an ordering α with $\alpha \to \tau_p$, are completely determined by the signs in α of the parameters x_l, and the signs, in τ_p, of the units u_{jq} (or, more properly, of their residue classes). Then we distinguish two cases:

Case 1. If $\tau_1 = \tau_2$, let $\alpha_0, \ldots, \alpha_{m+1}$ generate F, and take F_1 to be the fan generated by $\alpha_0, \ldots, \alpha_m$, and $F_2 = F \setminus F_1$. Note that $\#(F_1) = 2^m = \frac{1}{2}\#(F)$.

Case 2. If $\tau_1 \neq \tau_2$, take F_1 to be the fan F_{τ_1} consisting of all orderings of F specializing to τ_1 and, similarly, $F_2 = F_{\tau_2}$. Then $\#(F_1) = 2^k$ and $\#(F_2) = 2^h$. Since, obviously, $F_1 \cup F_2 = F$, we have $\#(F_1) = \#(F_2) = \frac{1}{2}\#(F) = 2^m$. As above we may assume that F_1 is generated by $\alpha_0, \ldots, \alpha_m$, and that $\alpha_{m+1} \in F_2$, so that $\alpha_0 \ldots, \alpha_m, \alpha_{m+1}$ generate the whole F.

STEP III: After some additional blowings-up, we find a regular local ring B dominating A, with the same residue field and a regular system of parameters $y_1, \dots, y_d$ of B such that all the f_{jp}'s are normal crossings with respect to them and for all $j = 0, \dots, m+1$ it holds

$$\alpha_j(y_l) = \begin{cases} +1 & \text{for } 1 \leq l \leq d-j \\ -1 & \text{if } l = d-j+1 \end{cases}$$

Furthermore, any localization of B at a prime ideal containing y_d dominates A.

In fact, after changing x_l by $-x_l$ if necessary, we may assume that $\alpha_0(x_l) = +1$ for all l. Since the functions f_{jq} separate the orderings of F_1 and all these orderings specialize to τ_1, two different $\alpha, \alpha' \in F_1$ cannot have the same sign at all the parameters $x_1, \dots, x_d$. Thus, there is some l such that $\alpha_1(x_l) \neq \alpha_0(x_l) = +1$, and reordering the parameters, we can suppose $\alpha_1(x_d) = -1$. Now, we consider the quadratic transform

$$A' = A[x_1/x_d, \dots, x_{d-1}/x_d]_{(x_1/x_d, \dots, x_{d-1}/x_d, x_d)},$$

and put $z_l = x_l/x_d$ for $1 \leq l < d$, $z_d = x_d$. This quadratic transform is a local regular ring with the same residue field as A, and regular system of parameters $z_1, \dots, z_d$. Furthermore,

$$f_{jq} = u_{jq} x_1^{\rho_{jq1}} \cdots x_d^{\rho_{jqd}} = u_{jq} z_1^{\rho_{jq1}} \cdots z_{d-1}^{\rho_{jq\,d-1}} z_d^{\rho_{jq1} + \cdots + \rho_{jqd}},$$

so that the f_{jq}'s are still normal crossings in A'. Hence, all conditions verified by A are similarly verified by A', and, in addition, any localization of A' at a prime ideal containing z_d dominates A. We reorder the z_l's so that $\alpha_1(z_l) = +1$ for $1 \leq l < r$ and $\alpha_1(z_l) = -1$ for $r \leq l \leq d$.

Next, we consider the extension

$$A^{(1)} = A'[z_r/z_d, \dots, z_{d-1}/z_d]_{(z_1, \dots, z_{r-1}, z_r/z_d, \dots, z_{d-1}/z_d, z_d)}$$

and put $x_l^{(1)} = z_l$ for $1 \leq l < r$, $x_l^{(1)} = z_l/z_d$ for $r \leq l < d$, $x_d^{(1)} = z_d$. Then $A^{(1)}$ is a regular ring dominating A, the residue fields of both rings coincide and $x_1^{(1)}, \dots, x_{d-1}^{(1)}, x_d^{(1)}$ are a regular system of parameters of $A^{(1)}$. As above, the f_{jq}'s are normal crossings in $A^{(1)}$, and all conditions verified by A' are also verified by $A^{(1)}$. Furthermore, $\alpha_1(x_l^{(1)}) = +1$ for $1 \leq l < d$ and $\alpha_1(x_d^{(1)}) = -1$, so that we have completed the first step of the induction process.

Assume now that we have already found a regular local ring $A^{(\ell)}$ dominating A with its same residue field, and a system of parameters $x_1^{(\ell)}, \dots, x_d^{(\ell)} = z_d$ such that the f_{jq}'s are normal crossings for them and $\alpha_j(x_l^{(\ell)}) = +1$ for $1 \leq l \leq d-j$, $\alpha_j(x_{d-j+1}^{(\ell)}) = -1$ $(1 \leq j \leq \ell)$. Then, we look at the prime cone $\alpha_{\ell+1}$ and see that $\alpha_{\ell+1}(x_l) = -1$ for some $l \leq d-\ell$. For, otherwise, $\alpha_{\ell+1}$ would be generated by $\alpha_0, \alpha_1, \dots, \alpha_\ell$, which is impossible by hypothesis. Hence, after reordering $x_1^{(\ell)}, \dots, x_{d-\ell}^{(\ell)}$, we may assume that $\alpha_{\ell+1}(x_l^{(\ell)}) = +1$ for $1 \leq l < r$ and $\alpha_{\ell+1}(x_l^{(\ell)}) = -1$ for $r \leq l \leq d-\ell$. Consider the extension

$$A^{(\ell+1)} = \\ A^{(\ell)}[x_r^{(\ell)}/x_{d-\ell}^{(\ell)}, \dots, x_{d-\ell-1}^{(\ell)}/x_{d-\ell}^{(\ell)}]_{(x_1^{(\ell)},\dots,x_{r-1}^{(\ell)},x_r^{(\ell)}/x_{d-\ell}^{(\ell)},\dots,x_{d-\ell-1}^{(\ell)}/x_{d-\ell}^{(\ell)},x_{d-\ell}^{(\ell)},\dots,x_d^{(\ell)})}$$

and set $x_l^{(\ell+1)} = x_l^{(\ell)}$ for $1 \leq l < r$, $x_l^{(\ell+1)} = x_l^{(\ell)}/x_{d-\ell}^{(\ell)}$ for $r \leq l < d-\ell$, and $x_l^{(\ell+1)} = x_l^{(\ell)}$ for $d-\ell \leq l \leq d$. An immediate computation shows that, for $0 \leq j \leq \ell+1$, $\alpha_j(x_l^{(\ell+1)}) = +1$ if $1 \leq l \leq d-j$, while $\alpha_j(x_{d-j+1}^{(\ell+1)}) = -1$, thus completing the step $\ell+1$. Since the last parameter is z_d, the claim on localizations is fulfilled, and Step III is finished. Notice that we are changing the regular parameters so that the matrix of signs (with entries $+1$ and -1) becomes triangular:

	x_1	$\cdots$	x_{d-m}	x_{d-m+1}	$\cdots$	x_{d-1}	x_d
α_0	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$
α_1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
α_2	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	-1	$*$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
α_m	$+1$	$\cdots$	$+1$	-1	$\cdots$	$*$	$*$

Step IV: Construction of the parametric approximation F'.

Consider any $\alpha \in F$. There are two possibilities:

Case 1. If $\alpha \in F_1$, then $\alpha = \alpha_{j_1} \cdots \alpha_{j_s}$ with $0 \le j_1 < \cdots < j_s \le m$ and s odd. Let $1 \le l \le d-m$; since $\alpha_{j_1}(y_l) = \cdots = \alpha_{j_s}(y_l) = +1$, we get $\alpha(y_l) = +1$.

Case 1. If $\alpha \in F_2$, then $\alpha = \alpha_{j_1} \cdots \alpha_{j_s} \cdot \alpha_{m+1}$ with $0 \le j_1 < \cdots < j_s \le m$ and s even. Let $1 \le l \le d-m$; we get $\alpha(y_l) = \alpha_{j_1}(y_l) \cdots \alpha_{j_s}(y_l) \cdot \alpha_{m+1}(y_l) = \alpha_{m+1}(y_l)$.

This gives two bijections

$$\varphi_p : F_p \to \{-1,+1\}^m : \alpha \mapsto (\alpha(y_{d-m+1}), \ldots, \alpha(y_d)), \quad p = 1,2.$$

In fact, since the f_{jq}'s separate the orderings of each F_p, the two preceding remarks show that the φ_p's are injective, and since all sets involved have 2^m elements, they are bijective.

We have the following diagram

$$\begin{array}{ccc} B & \longrightarrow & C = B_{(y_{d-m+1},\ldots,y_d)} \subset K \\ \downarrow & & \downarrow \\ B/(y_{d-m+1},\ldots,y_d) & \longrightarrow & k_C \\ \downarrow & & \\ k_A = k_B & & \end{array}$$

where k_B, k_C stand for the residue fields of B, C, respectively. Now, we chase orderings through the diagram, starting in $k_B = \kappa(\mathfrak{q})$ with our τ_1, τ_2:

First, since $B/(y_{d-m+1}, \ldots, y_d)$ is local regular with parameters $y_1, \ldots, y_{d-m}$, we can lift τ_1 to an ordering γ_1 of k_C such that $\gamma_1(y_l) = +1$ for $1 \le l \le d-m$ (Example 5.2). Also we can lift τ_2 to an ordering γ_2 of k_C such that $\gamma_2(x_l) = \alpha_{m+1}(y_l)$ for $1 \le l \le d-m$.

Second, since C is a local regular ring of dimension m and $y_{d-m+1}, \ldots, y_d$ are a regular system of parameters, we can built up a parametric fan $F' = F_{\gamma,\gamma_2}(y_{d-m+1}, \ldots, y_d)$ of K starting with the γ_1, γ_2 in k_C. Let F'_p be the set of orderings of F' specializing to γ_p, for $p = 1,2$. We have two bijections $\varphi'_p : F'_p \to \{-1,+1\}^m : \alpha' \mapsto (\alpha'(y_{d-m+1}), \ldots, \alpha'(y_d))$, $p = 1,2$, and consequently we obtain bijections $F_p \to F'_p : \alpha \mapsto \alpha'$ such that:

(i) $\alpha(y_l) = \alpha'(y_l)$ for $d - m < l \leq d$,

(ii) $\alpha(y_l) = +1 = \gamma_1(y_l) = \alpha'(y_l)$ for $\alpha \in F_1$ and $1 \leq l \leq d - m$.
$\alpha(y_l) = \alpha_{m+1}(y_l) = \gamma_2(y_l) = \alpha'(y_l)$ for $\alpha \in F_2$ and $1 \leq l \leq d - m$.

(iii) $\alpha, \alpha' \to \gamma_p$.

This gives another bijection $\psi : F \to F' : \alpha \mapsto \alpha'$, such that $\alpha(y_l) = \alpha'(y_l)$ for $1 \leq l \leq d$ and $\alpha(u) = \alpha'(u)$ for any unit $u \in B$. Consequently, $\alpha(f_{jq}) = \alpha'(f_{jq})$ for all j, q, and since the f_{jq}'s defined the neighborhood U_j of α_j fixed at the beginning, we conclude $\alpha'_j \in U_j$.

STEP V: Verification of conditions a)-d).

a) and *b)* By construction, F' is a subfan of the pull-back $F_{\tau_1,\tau_2}(y)$ of τ_1, τ_2 by the valuation ring V_d of the regular ring B (Example 7.2). We set $W' = V_d$. Thus it is obvious that F' is compatible with W'. Moreover, since V_d dominates B and B dominates $A = \mathcal{R}(Y)_{\mathfrak{q}}$, the center of W' is $\mathfrak{q}$.

c) The residue field $k_{W'}$ of W' is the residue field of B, which coincides with that of A. As W dominates A, $k_{W'} = \kappa(\mathfrak{q}) \subset k_W$. Moreover, the fan induced by F' in $k_{W'}$ is $\{\tau_1, \tau_2\}$.

d) Suppose $\alpha_j \to a \in X$. Since α_j specializes τ_p for $p = 1$ or 2, we have $\tau_p \to a$. But, by *(iii)*, α'_j specializes to the same τ_p and, consequently, $\alpha'_j \to a$.

e) The biggest valuation ring $W_{F'}$ of K along which F' trivializes is the valuation V_m corresponding to the regular local ring C. By construction, V_m dominates C, and, by the last assertion in Step III, C dominates A. Hence, the center of $W_{F'}$ is also $\mathfrak{q}$. □

16 Analyticity and approximation

Let $X \subset \mathbb{R}^p$ be an algebraic set. First of all, we see why the chosen notion of approximation is well adapted to our problem:

Proposition 16.1 *Let $S \subset X$ be a semialgebraic set, and F an algebraic fan of X. Then, for any fan F' close enough to F we have $\#(F \cap \widetilde{S}) = \#(F' \cap \widetilde{S})$.*

Proof: If F' is close enough to F, both fans have the same support $Y \subset X$, and we can replace S by the interior of $S \cap Y$ in Y. In other words, we can suppose $S = \bigcup_{i=1}^{r}\{x \in Y \;:\; f_{i1}(x) > 0, \dots, f_{is}(x) > 0\}$, with $f_{ij} \in \mathcal{R}(X)$. Then:

If $\alpha \in F \cap \widetilde{S}$, there is some i with $f_{i1}(\alpha) > 0, \dots, f_{is}(\alpha) > 0$. This conditions will be verified by any β close enough to α, which, consequently, will also belong to $\widetilde{S}$.

If $\alpha \notin F \cap \widetilde{S}$, for every i there is some j_i with $f_{ij_i}(\alpha) < 0$, and this will hold for any β close to α, which consequently will not belong to $\widetilde{S}$. $\square$

Using this fact and the fan approximation lemma (Lemma 15.3), we can replace arbitrary fans by parametric fans in Proposition 3.3, to obtain a stronger characterization of basicness. The following result shows that approximation by parametric fans behaves equally well with respect to analyticity.

Theorem 16.2 *Arbitrarily close to each analytic (resp. non-analytic) fan of X there are analytic (resp. non-analytic) parametric fans with the same biggest trivialization center.*

Proof: We will prove the two results at a time. Let F be an algebraic fan of X, and W_F the biggest valuation ring along which F trivializes. We want to apply Lemma 16.3 with $W = W_F$, but we can not guarantee that W is finite on X. To overcome this difficulty, we consider the Alexandroff compactification X^* of X. We have the canonical inclusion $\mathcal{R}(X^*) \subset \mathcal{R}(X)$, and since X^* is compact, every prime cone of F specializes to some point $a \in X^*$ (Proposition 4.5). Consequently, W is finite on X^*. We distinguish several possibilities that cover the two statements we want to prove.

Case 1: The prime cones of F specialize to two different points $a_1, a_2 \in X^*$. We choose an approximation F' of F as described in Lemma 16.3, and by condition *d)* the prime cones of F' also specialize to the points a_1, a_2. In this case none of the two fans F and F' is analytic, neither in X^* nor in X.

Case 2: Some prime cone of F specializes to the point at infinity. Pick a close approximation F' of F as provided by Lemma 16.3. Again by condition *d)*, some prime cone of F' specializes to the point at infinity. Hence, F' is not analytic in X, as was not F.

Case 3: All the prime cones of F specialize to a point $a \in X$. Then W is finite on X (Proposition 6.1) and we can choose a close approximation F' of F as in Lemma 16.3. In particular, by condition *c)* once more, all prime cones of F' specialize to a. Now, we recall Corollary 11.4: the fan F is analytic in X if and only if $\lambda_W : \mathcal{R}_{X,a} \to (k_W, F_W)$ amalgamates. Since by condition *c)*, $k_{W'} \subset k_W$, it follows that if λ_W amalgamates, so does $\lambda_{W'} : \mathcal{R}_{X,a} \to (k_{W'}, F_{W'})$. Conversely, if λ_W does not amalgamate, there is a finite subextension $\kappa \supset \kappa(\mathfrak{q})$ of $k_W \supset \kappa(\mathfrak{q})$ such that $\lambda_W : \mathcal{R}_{X,a} \to (\kappa, F_W|_\kappa)$ does not amalgamate (Remark 12.1 *d)*). But, by the last assertion of Lemma 16.3, we can choose F' such that $k_{W'} \supset \kappa$, and then $\lambda_{W'} : \mathcal{R}_{X,a} \to (\kappa, F_W|_\kappa) \to (k_{W'}, F_W|_{k_{W'}})$ cannot amalgamate.

□

The previous result shows the behaviour of approximation with respect to analyticity. Let us see now how approximation behaves with respect to the property of a fan to be 1pt or 2pt.

Lemma 16.3 *a) Let F be a 2pt-algebraic fan of X. Then any fan F' close enough to F is also 2-pt.*

b) Let F be a 1pt-algebraic fan of X. Then any fan F' close enough to F is either 1pt or 2-pt.

Proof: *a)* Let Y be the support of F. The hypothesis means that any element of F specializes to one of two fixed points $a_1, a_2 \in Y$. Now, let Y^* denote the Alexandroff compactification of Y, and let U_1, U_2, be disjoint closed semialgebraic neigborhoods of a_1 and a_2 in X^* completely contained in X. Then, for any $\alpha_j \in F$, $\alpha_j \to a_i$ we have $\alpha_j \in \widetilde{U}_i$. Thus for any approximation such that $\alpha'_j \in \widetilde{U}_i$, since X^* is compact, we have $\alpha'_j \to b_j \in X^*$, and since $\widetilde{U}_i$ is closed, it must be $b_j \in U_i \subset X$. Thus, any ordering α'_j of F' specializes to some point of X and by construction we have at least two different specialization points since $U_1 \cap U_2 = \emptyset$. Thus F' is 2-pt.

b) Is proved similarly to *a)*. □

Notice that part *b)* of the preceding lemma says nothing new in case X is compact, since then any fan is either 1pt or 2pt. Our next result studies the relation between approximation and parametric fans.

Lemma 16.4 *Let A be a regular local ring with residue field k, and let $x_1, \ldots, x_m$ be a regular system of parameters. Let τ_1, τ_2 be orderings of k. If γ_1, γ_2 are close to τ_1, τ_2 in $\mathrm{Spec}_r(k)$ then $F_{\gamma_1,\gamma_2}(x)$ is close to $F_{\tau_1,\tau_2}(x)$ in $\mathrm{Spec}_r(A)$.*

Proof: Indeed, suppose that $F_{\tau_1,\tau_2}(x) = (\alpha_j)$ and $F_{\gamma_1,\gamma_2}(x) = (\alpha'_j)$ are so ordered that the matrices of signs $(\alpha_j(x_l))$ and $(\alpha'_j(x_l))$ coincide, and $\alpha_j \to \tau_i$ if and only if $\alpha'_j \to \gamma^*_i$. Then, by Example 5.2 *c)*, we know that the sign of any element $f \in A$ coincides with the sign the initial form $f_0 x_1^{\nu_1^0} \cdots x_m^{\nu_m^0}$, of f considered as a power series in the parameters x_i, where the exponent are ordered lexicographically and f_0 is a unit in A. Thus, we get

$$\alpha_j(f)\alpha'_j(f) = \tau_i(f_0)\gamma_i(f_0),$$

and from this the claim follows easily. □

The next result shows the announced approximation property:

Proposition 16.5 *Let F be a fan of X with* $\dim(Z_F) \geq 1$. *Then F can be arbitrarily approximated by 2pt-parametric fans F' (and therefore, non-analytic).*

Proof: By Lemma 15.3 we assume that F is a parametric fan such that Z_F has dimension ≥ 1 and the residue field k_F is finitely generated over $\mathbb{R}$. Thus, there is an algebraic set Z' with $\mathcal{K}(Z') = k_{F'}$, and we get a rational dominant map $\pi : Z' \to Z_F$ associated to the field extension $\mathcal{K}(Z_F) \subset \mathcal{K}(Z')$. Let $\tau_1, \tau_2 \in \mathrm{Spec}_r(\mathcal{K}(Z'))$ such that $F = F_{\tau_1,\tau_2}(x)$. By Lemma 16.4, we know that if $\gamma_i \in \mathrm{Spec}_r(\mathcal{K}(Z'))$ is close to τ_i, then $F' = F_{\gamma_1,\gamma_2}(x)$ is close to F. We will choose these γ_i so that they specialize to different points in Z_F. First, the condition that γ_i is close to τi is controled by an open semialgebraic set $U'_i \subset Z'$ with $\tau_i \in \widetilde{U'_i}$, and we look for a prime cone $\gamma_i \in \widetilde{U'_i}$. This can be done by choosing any non-singular point a'_i of Z' in U'_i and a maximal generization $\gamma_i \to a'_i$ (which always exists because a'_i is non-singular). Then, the restriction of γ_i to $\mathcal{K}(Z_F)$ specializes to $a_i = \pi(a'_i) \in Z_F$. Consequently, we are reduced to find two different points $a_i \in \pi(U'_i)$. But this is clearly possible, since the U'_i's are open, π is dominant, and Z_F has dimension ≥ 1. □

Note that the initial fan F of the proposition above might be analytic. For instance, this happens if it is centered at one point, despite the fact that its largest trivialization center has bigger dimension, see Examples 6.2. Thus, the result shows that to check basicness we can replace the family of algebraic fans with some center not reduced to a point by the family of 2pt-fans (even 2pt-parametric fans), or to put in another way, that any obstruction to basicness represented by a fan with some center bigger than one point is essentially non-analytic. The next result shows what happens with the parametric fans whose centers are all reduced to a point (which we could call strictly local). It turns out that they have a quite strong approximation property:

Proposition 16.6 *Let F be a parametric fan of X with* $\dim(Z_F) = 0$. *Then, for any fan F' close enough to F we have $Z_{F'} = Z_F$.*

Proof: Let $Y' \to Y$ be the blowing-up, so that $F = F_{\tau_1,\tau_2}(x) = (\alpha_j)$ is the parametric fan defined by the valuation ring V_m associated to a local regular ring $B = \mathcal{R}(Y')_{\mathfrak{q}}$, with regular system of parameters $x = (x_1, \dots, x_m)$, and two orderings τ_1, τ_2 of the residue field $\kappa(\mathfrak{q})$. Let the point $a \in X$ be the trivialization center of F, and set Z for the center of F in Y'. Now, let $F' = (\alpha'_j)$ be an approximation of F. We want to see that if F' is close enough to F then its biggest trivialization center on X is also the point a.

First of all, notice that from the definition of approximation, F' has also support Y, that is, F' is a fan of the field $\mathcal{K}(Y) = \mathcal{K}(Y')$. Now, by Lemma 16.3, F' is either 1pt or 2pt, and in particular, all its elements specialize to some point in X. Thus, it follows from Proposition 6.1, that all valuation rings compatible with F' are finite on Y, and, in particular, $W_{F'} \supset \mathcal{R}(Y)$. But since the map $Y' \to Y$ is a blowing-up, it is proper, which implies that $W_{F'} \supset \mathcal{R}(Y')$. We denote by $\mathfrak{q}'$ the center of $W_{F'}$ in $\mathcal{R}(Y')$. Our purpose is to prove that $\mathfrak{q}'$ lies over the maximal ideal $\mathfrak{m}$ of a in $\mathcal{R}(Y)$.

Now, the prime cones of F' specialize to two prime cones τ'_1, τ'_2 with support $\mathfrak{q}'$. We want to see that if the approximation is good enough, then some τ'_p (and then both) is in $\widetilde{Z}$, because that will imply $\mathfrak{q}' \supset \mathfrak{q}$, and so

$$\mathfrak{q}' \cap \mathcal{R}(Y) \supset \mathfrak{q} \cap \mathcal{R}(Y) = \mathfrak{m},$$

which will end the proof. Hence, let us suppose that neither τ_1 nor τ_2 belong to $\widetilde{Z}$ and deduce a contradiction.

First, we claim that if F' is close to F, and $\alpha_{j_1}, \alpha_{j_2}$ specialize to the same τ_p, then $\alpha'_{j_1}, \alpha'_{j_2}$ must specialize to the same τ'_p.

For, suppose for instance,

$$\alpha_{j_1}, \alpha_{j_2} \to \tau_1, \quad \alpha'_{j_1} \to \tau'_1, \quad \alpha'_{j_2} \to \tau'_2.$$

For the time being, we will work in the constructible set $M = \widetilde{(Y' \setminus Z)} \subset \mathrm{Spec}_r(\mathcal{R}(Y'))$, which is a normal space. In fact, given any two closed disjoint sets G_1, G_2 of M, there are two disjoint open constructible sets U_1 and U_2 of M such that $U_1 \supset G_1$ and $U_2 \supset G_2$ ([BCR, Prop. 7.1.24, p.117]). For $p = 1, 2$ we consider the set F_p of all specializations of the α_j's that specialize to and are different from τ_p. Clearly F_1 and F_2 are disjoint, and they are closed in M: F_p is the closure in M of the set of all α_j's that specialize to τ_p. Hence, we can apply normality with $G_p = F_p$, and get $U_p \supset F_p$. Now, we apply normality again, with $G_1 = F_1$ and $G_2 = M \setminus U_1$, and obtain $U_1' \supset F_1$, $U_2' \supset M \setminus U_1$. In conclusion, we have

$$F_1 \subset U = U_1' \subset H = M \setminus U_2' \subset D = U_1, \quad F_2 \subset U_2.$$

Then, as $\alpha_{j_p} \to \tau_1$, we have $\alpha_{j_p} \in U$. Hence, if α'_{j_p} is close to α_{j_p}, we get $\alpha'_{j_p} \in U \subset H$ (U is open). Now, since H is closed in M, $\tau_p' \in M$, and $\alpha'_{j_p} \to \tau_p$, it follows that $\tau_p' \in H \subset D$. Consequently, D being open, all generizations of τ_1' and τ_2' belong to D. Whence, no α_j is in $U_2 \subset M \setminus D$, which is a contradiction, since U_2 is an open neighborhood of F_2 and if $\alpha_j \in F_2$ and F' is close to F, then $\alpha_j' \in U_2$. Thus, the claim is proved.

To progress further, recall that, by the definition of parametric fans, for every parameter x_l there are $\alpha_{j_{1l}}, \alpha_{j_{2l}} \to \tau_1$ with

$$\alpha_{j_{1l}}(x_l) > 0, \quad \alpha_{j_{2l}}(x_l) < 0.$$

Hence, if F' is close to F, also $\alpha'_{j_{1l}}, \alpha'_{j_{2l}} \to \tau_1'$ and

$$\alpha'_{j_{1l}}(x_l) > 0, \quad \alpha_{j_{2l}}(x_l)' < 0.$$

so that $\tau_1'(x_l) = 0$. Consequently,

$$\tau_1' \in \{x_1 = \cdots = x_m = 0\} \subset \mathrm{Spec}_r(\mathcal{R}(Y')).$$

Now, since $x_1, \ldots, x_m$ generate $\mathfrak{q}\mathcal{R}(Y')_{\mathfrak{q}}$, the zero set Z of $\mathfrak{q}$ is an irreducible component of $\{x_1 = \cdots = x_m = 0\} \subset Y'$; we will denote by T the union of the other irreducible components. In this situation, the two semialgebraic sets $Z \setminus T$ and $T \setminus Z$ are closed in $Y' \setminus Z \cap T$, and, by semialgebraic normality, there are two disjoint open semialgebraic sets $U_1 \supset Z \setminus T$ and

$U_2 \supset T \setminus Z$. Then, since for any $\alpha_j \in F$ it is $\alpha_j \to \tau_p \in \widetilde{Z \setminus T}$, we get $\alpha_j \in \widetilde{U}_1 \subset \widetilde{Y \setminus U_2}$. Hence, if F' is close to F, $\alpha'_j \in \widetilde{Y \setminus U_2}$. Since the latter set is closed, we see that no specialization of α'_j belongs to $\widetilde{U}_2$. We have proved that no specialization of any $\alpha'_j \in F'$ belongs to $\widetilde{T \setminus Z}$, and so

$$\tau'_1 \in \{x_1 = \cdots = x_m = 0\} \setminus \left(\widetilde{T \setminus Z}\right) = \left(\widetilde{Z \cup T}\right) \setminus \left(\widetilde{T \setminus Z}\right) \subset \tilde{Z}.$$

We thus get a contradiction, since we started from the assumption that no τ'_p was in $\widetilde{Z}$. As was explained, this contradiction ends the proof of the proposition. □

The preceding result contains some interesting additional information, as showed by the following proposition, whose proof of part *b)* uses the notion of blowing down explained in the next section.

Proposition 16.7 *Let F be a parametric fan of X.*

a) If F has not trivialization centers in X then no fan F' close enough to F has trivialization centers in X.

b) For any fan F' close enough to F we have $Z_{F'} \subset Z_F$.

Proof : *a)* Let X^* be the Alexandroff compactification of X. The hypothesis on F means that the biggest trivialization center of F in X^* is the point at infinity. By Proposition 16.6, any F' close to F has also that point as biggest trivialization center, and, consequently, F' cannot have trivialization centers in X.

b) Let $X \to X'$ be the blowing-down of Z_F to a point $a' \in X'$ (Definition 17.1). Then F, as a fan of X', has $\{a'\}$ as the biggest trivialization center. By Proposition 16.6, any F' close enough to F has also $\{a'\}$ as the biggest trivialization center in X', and, back to X, $Z_{F'} \subset Z_F$. □

The following example shows that the converse of *a)* does not hold, showing at the same time that the inclusion assured by *b)* may be proper.

Example 16.8 Consider the following parametric fan $F = (\alpha_j)$ of $X = \mathbb{R}^2$: $Y' = Y = X$, $B = \mathbb{R}[\mathrm{x},\mathrm{y}]_{(\mathrm{y})}$, τ_1 and τ_2 are the two orderings of $\kappa_B = \mathbb{R}(\mathrm{x})$ compatible with the valuation ring $\mathbb{R}[1/\mathrm{x}]_{(1/\mathrm{x})}$ (*Figure 28*). Then we can construct fans F' close to F without trivialization centers in X. We will not give a formal proof of this fact, but *Figure 28* depicts an easy way to produce $F' = (\alpha'_j)$.

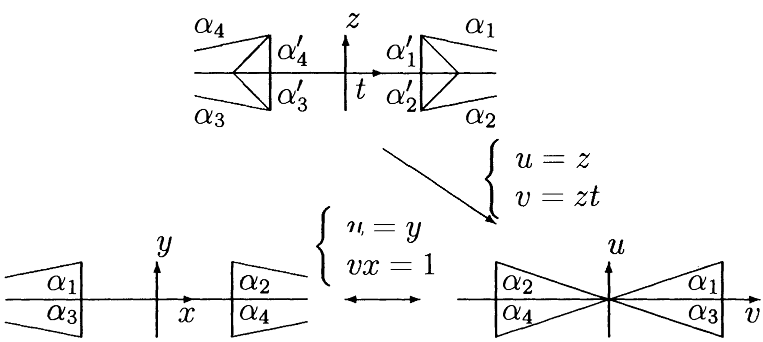


Figure 28

The study of parametric fans deserve a much more detailed analysis, in which we will not enter here. Notice that we could roughly summarize the last results, in geometric terms, as follows: Let $X \subset \mathbb{R}^n$ be an algebraic set and let $X^* = X \cup \infty$ be its Alexandroff compactification. Let $S \subset X$ be a semialgebraic set which is basic semianalytic at all points $a \in X$. Then S is basic semialgebraic if and only if it is basic semianalytic at ∞ and verifies the fan criterion 4.4 for all fans adherent to two points. For the very special case of $\dim(X) = 2$, this criterium for 2-pt fans can be translated geometrically in terms of some special boundary conditions for S ([AnRz4]). It is our feeling that this interpretation can also be done in general by means of semialgebraic stratifications, which is a challenging open problem.

17 Analyticity after birational blowing-down

We have already pointed out in the previous section that the analyticity of an algebraic fan is highly of non birational nature. Therefore it is a fairly natural question to consider the variations of analytic fans under changes of birational model. We start by fixing some definitions:

Definition 17.1 *Let $X \subset \mathbb{R}^p$ be a real algebraic set.*

a) A blowing-down of X is a regular mapping $X \to X'$ that collapses an algebraic subset $X_1 \subset X$ to a point $a \in X'$, and induces a biregular diffeomorphism $X \setminus X_1 \to X' \setminus \{a\}$.

b) A birational blowing-down of X is a composition $X \to X'$ of a birational regular map $X \to X_0$ followed by a blowing-down $X_0 \to X'$.

Note that blowings-down and birational blowings-down are all birational regular maps. A characteristic feature of real algebraic geometry is that an algebraic subset can always be collapsed to a point to obtain a blowing-down ([BCR, 3.5.5, p.69]). Note also that a birational blowing-down may very well induce a regular embedding $X \to X'$. This is actually a way to obtain the *(algebraic) Alexandroff compactification* $X \to X^*$, again something typical of real algebraic geometry ([BCR, 3.5.3, p.68]). The following result, which generalizes Example 14.4, makes the bridge between these notions and algebraic fans, showing that any algebraic fan can be rendered analytic in some suitable birational model.

Proposition 17.2 *Let $X \subset \mathbb{R}^p$ be a real algebraic set and F an algebraic fan of X. Then there is a birational blowing-down $X \to X'$ such that F is analytic at some point $a \in X'$.*

Proof: Let $Y \subset X$ be the support of F and let W_F be the largest valuation ring of $\mathcal{K}(Y)$ trivializing F. Let $X \to X^*$ be the Alexandroff compactification, and Y^* the Zariski closure of Y in X^*. Then $\mathcal{K}(Y) = \mathcal{K}(Y^*)$ and F can be seen as a fan of X^* with support Y^*, compatible with W_F. Since X^* is compact, W_F is finite on X^* (Propositions 5.5 and 7.1). Let $Z^* \subset Y^* \subset X^*$ be the center of W_F in X^*, $X^* \to X'$ a blowing-down that collapses Z^* to a point $a \in X'$ and Y' the image of Y^* by this blowing-down. After composition with $X \to X^*$ we obtain a birational blowing-down $X \to X'$ that restricts to another $Y \to Y'$. Then $\mathcal{K}(Y) = \mathcal{K}(Y')$ and can be seen as a fan of X' with support Y', compatible with W_F. Again, W_F is finite on X' and its center is now the point a. The situation can be summarized in the following diagram

$$\begin{array}{ccccccc}
\mathcal{R}(X') & \subset & \mathcal{R}(X^*) & \subset & \mathcal{R}(X) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{R}(Y') & \subset & \mathcal{R}(Y^*) & \subset & \mathcal{R}(Y) & \subset & \mathcal{K}(Y) \\
\cap & & \cap & & & & \cup \\
\mathcal{R}(Y')_{\mathfrak{m}} & \xrightarrow{\text{local}} & \mathcal{R}(Y^*)_{\mathfrak{q}^*} & & \xrightarrow{\quad\text{local}\quad} & & W \\
\| & & & & & & \downarrow \\
\mathcal{R}_{Y',a} & & & \xrightarrow{\qquad\quad \lambda_W \qquad\quad} & & & k_W
\end{array}$$

where $\mathfrak{q}^*$ is the center of W in $\mathcal{R}(Y^*)$ and $\mathfrak{m} = \mathfrak{q}^* \cap \mathcal{R}(Y')$ is the center of W in $\mathcal{R}(Y')$, which by construction is the maximal ideal of the point a. Therefore, by Remark 10.4 *c)*, the homomorphism $\lambda_W : \mathcal{R}_{Y',a} \to k_W$ amalgamates and, by Corollary 12.3, F is analytic at a. $\square$

This surprising result shows a way to characterize semialgebraic basicness in terms of semianalytic basicness, so coming back to our initial Problem I. We start with generic basicness.

Theorem 17.3 *Let $X \subset \mathbb{R}^p$ be an algebraic set and $S \subset X$ a semialgebraic subset and put $\partial_Z(S) = \overline{\overline{S} \setminus S^\circ}^{Z}$. Consider a birational blowing-down $X \to X'$ that collapses $\partial_Z(S)$ to a point $a \in X'$. Then the image S' of S is a semialgebraic subset of X' and the following assertions are equivalent:*

a) S is generically s-basic semialgebraic.

b) S' is generically s-basic semianalytic.

Proof: The implication $a) \Rightarrow b)$ is trivial, since the map $X \to X'$ is birational. To prove $b) \Rightarrow a)$ we argue by way of contradiction, supposing *b)* and that S is not generically s-basic semialgebraic. By Proposition 4.4 there is an algebraic fan F of X of heigth 0 with $\#(F) = 2^m, F \cap \tilde{S} \neq \emptyset$, and either $\#(F \cap \tilde{S})$ is not a power of 2 or $\#(F \cap \tilde{S}) = 2^n$ with $s < m - n < m$. Since the support Y of the fan F is not contained in $\partial_Z(S)$ (because $F \subset \tilde{Y}$ and $F \cap \tilde{S} \neq \emptyset$), our blowing-down $X \to X'$ restricts to another $Y \to Y'$ where Y' is the Zariski closure in X' of the image of Y. Now F is a fan of X' with support Y', and since the blowing-down is a birational map, the height of F in X' is also 0 (in other words, Y is an irreducible component of X and Y' one of X').

Our aim is to prove that F is analytic at $a \in X'$. Once this is known, $\#(F \cap \tilde{S}) = \#(F \cap \tilde{S}_a)$, and by Proposition 4.6, S' will not be generically s-basic semianalytic, which will contradict *b)*. Now, to show that F is analytic at a we will show that it is centered at a in X', or equivalently that its center in X collapses to a by the blowing-down.

Let X^* be the Alexandroff compactification of X, and put

$$T_1 = \partial_Z(S) \cup (X^* \setminus X).$$

Clearly T_1 is an algebraic set, and the blowing-down $X \to X'$ can be seen as the restriction of one $X^* \to X'$ that collapses T_1 to a. Let Z^* be the center of the valuation ring W_F in X^*, and let $\mathfrak{q}^*$ be its ideal in $\mathcal{R}(X^*)$. We claim that there are $\alpha, \alpha' \in F$ that specialize to the same ordering τ of $\mathcal{K}(Z^*)$ and such that $\alpha \in \tilde{S}$ and $\alpha' \notin \tilde{S}$.

Indeed, otherwise, F would specialize to two different orderings τ_1, τ_2 of the residue field k_F of W_F such that $F \cap \tilde{S}$ would consist of the prime cones of F specializing to τ_1, and $F \setminus \tilde{S}$ of the ones specializing to τ_2. In particular $F \cap \tilde{S}$ would be a fan with 2^n elements, $F \setminus \tilde{S}$ another with 2^l elements, and $2^n + 2^l = 2^m$. Hence, $n = l = m - 1$, so that $s < m - n = 1$, absurd. Thus the claim is proved.

Hence, $\alpha \in \tilde{S}$ and $\alpha \to \tau$ imply $\tau \in \overline{\tilde{S}}$. On the other hand, $\tau \notin \widetilde{S^\circ}$, since, $\widetilde{S^\circ}$ being open, $\alpha' \to \tau$ would imply $\alpha' \in \widetilde{S^\circ} \subset \tilde{S}$. Consequently, $\tau \in (\overline{S} \setminus S^\circ)\tilde{}$, and so, the support $\mathfrak{q}^*$ of τ contains all functions vanishing on

$\overline{S} \setminus S^{\circ}$, and in particular those vanishing on $T_1 = \partial_Z(S) \cup (X^* \setminus X)$. Hence, $Z^* \subset T_1$ and Z^* collapses to a as wanted. $\square$

A similar characterization holds in the non-generic case:

Theorem 17.4 *Let $X \subset \mathbb{R}^p$ be an algebraic set and $S \subset X$ an open semialgebraic set which does not meet its Zariski boundary $\partial_Z(S) = \overline{\overline{S} \setminus S}^Z$. Consider a birational blowing-down $X \to X'$ that collapses $\partial_Z(S)$ to a point $a \in X'$. Then the image S' of S is an open semialgebraic set of X' that does not meet its Zariski boundary and the following assertions are equivalent:*

a) S is s-basic semialgebraic.

b) S' is s-basic semianalytic.

Proof: First of all note that the restriction $X \setminus \partial_Z(S) \to X' \setminus \{a\}$ is a semialgebraic homeomorphism, so that S' is semialgebraic, and open and closed in $X' \setminus \{a\}$. Thus S' is open in X' and its Zariski boundary is the point $a \notin S'$.

a) $\Rightarrow$ *b)* We will show that S' is s-basic semialgebraic, which of course implies *b)*. By Proposition 3.2 we have to prove that for any irreducible algebraic set $Y' \subset X'$ the intersection $S' \cap Y'$ is generically s-basic. Then consider the preimage Y_1 of Y' by our blowing-down, and let Y be the irreducible component of Y_1 which is not contained in $\partial_Z(S)$. Then $X \to X'$ restricts to a birational map $Y \to Y'$ and the image of $S \cap Y$ is $S' \cap Y'$. Since we are assuming that S is s-basic semialgebraic, the intersection $S \cap Y$ is generically s-basic semialgebraic, and consequently so is $S' \cap Y'$.

b) $\Rightarrow$ *a)* Again by Proposition 3.2 we have to see that for any irreducible algebraic set $Y \subset X$, the intersection $S \cap Y$ is generically s-basic. Now, if that intersection is not empty (trivial case), the map $X \to X'$ induces a birational blowing-down $Y \to Y'$, where Y' is the Zariski closure in X' of the image of Y. This latter blowing-down collapses $\partial_Z(S) \cap Y$ to a point, and since $\partial_Z(S) \cap Y$ contains the Zariski boundary of $S \cap Y$ in Y, we can apply the preceding theorem and conclude that $S \cap Y$ is generically s-basic semialgebraic if its image in Y' is generically s-basic semianalytic. Now that image is $S' \cap Y'$, since the restriction to S of the blowing-down is bijective. Finally S' is generically s-basic semianalytic by hypothesis, and we are done.

□

Example 17.5 We apply the preceding ideas to Example 1.1 *c)*, which had no analytic obstruction to basicness. Indeed, to make analytic the obstruction for S to be basic, it is enough to consider the Alexandroff compactification

$$\varphi : \mathbb{R}^2 \to \mathbb{S}^2 \ : (x, y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

Then the semialgebraic set $S' = \varphi(S)$ is not basic semianalytic at the north pole $(0,0,1)$. To see this directly, we consider the stereographic projection

$$\mathbb{S}^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2 \ : (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)$$

to get a birational map

$$\phi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \ : (x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right).$$

Then

$$\phi(S_1) = S_1' \ : 3x'^2 + y'^2 - 4x'y' + (x'^2 + y'^2)^2 < 0$$

$$\phi(S_2) = S_2' \ : 3x'^2 + y'^2 + 4x'y' + (x'^2 + y'^2)^2 < 0$$

and $\phi(S) = S_1'^{+} \cup S_1'^{+} \cup S_2'^{+}$ is the set depicted in *Figure 29.*

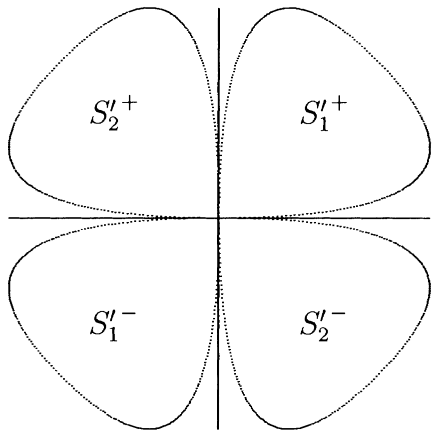


Figure 29

Now, that $\phi(S)$ is not basic semianalytic at $(0,0)$ follows easily by the argument shown in Example 1.1 *a)*.

The last two theorems point out the distinguished role of Zariski boundaries to locate obstructions to basicness: as the notation suggests, the set $\partial_Z(S)$ of Theorem 15.3 is the natural definition of the Zariski boundary of an arbitrary semialgebraic set S, not necessarily open. On the other hand, there is a notion of *generic Zariski boundary* which reflects more accurately generic basicness. This will be studied in detail in the forthcoming [AnRz4].

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