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Basicness of Semialgebraic Sets ⋆

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Abstract. This paper is concerned with the problem of deciding whether a semialgebraic set S of an algebraic variety X over $\mathbb{R}$ is basic. Furthermore, in such a case, we decide what is the sharp number of inequalities defining S. For that, it suffices to desingularize X, as well as the boundary of S, and then ask the same question for the trace of S on its boundary. In this way, after a finite number of blowing-ups, we lower the dimension of the data and by induction we get a finite decision procedure to solve this problem. Decidability of other known criteria is also analyzed.

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Introduction

A classical problem in real algebraic geometry is to ask whether a semialgebraic set S in a real algebraic variety X is basic, i.e., whether it is the set of solutions of a system of polynomial inequalities.

Several answers to this problem are well known. First of all, Bröcker (in [10], see also [15]) characterizes basicness in the real spectrum of the function field $\mathcal{K}(X)$. From this, he gets that S is basic if and only if $S \cap Y$ is generically basic in any irreducible algebraic subset $Y \subseteq X$. Note that this result holds over any real closed field. Working over the field $\mathbb{R}$, Andradas and Ruiz improved Bröcker's result in the sense that then it is enough to take any irreducible surface as Y (see [4]).

We need nothing more than Andradas and Ruiz's result to be able to decide, over $\mathbb{R}$, by a finite procedure whether a set is basic or not, as we prove in Section 2.

Nevertheless, a different approach is possible when working over $\mathbb{R}$. If we use one of the above-mentioned criteria to test basicness, we have to look inside S. But it is also possible to search relevant information about the basicness of the set S on its boundary. This is precisely our point of view. A nice result in this direction was

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given by Andradas and Ruiz [5], who found a 'universal obstruction' to basicness: if S is not generically basic, then there exists a birational model of S which is crossed by its Zariski boundary.

In this article, we fix instead a precise model $(\hat{X}, \hat{S})$, namely the one obtained by solving the singularities of X and the singularities of the Zariski boundary $\partial_z S$ of S, and we find geometric conditions (see 4.1) working on $\partial_z \hat{S}$ which are equivalent to the basicness of the set $S \setminus \partial_z S$. Note that the assumptions on $(\hat{X}, \hat{S})$ are not enough to guarantee the existence of the 'universal obstruction' of Andradas–Ruiz if S is not basic. We get a similar condition to test whether a basic set S can be described by exactly s inequalities (see 4.2).

Using these results and the fact that the resolution of singularities is algorithmic, as proved in [8] and [17], one can produce an algorithm to test basicness more in the spirit of computational algebraic geometry. We shall deal with this in the last section.

The decision procedure works over the field $\mathbb{R}$, and Theorems 4.1 and 4.2 are true only over archimedean real-closed fields, that is, subfields of $\mathbb{R}$. This is because there is no universal bound depending only on the complexity of the set S of the degrees of the polynomials describing it as a basic set (see 2.3).

1. Recall on Basicness

This section is devoted to a revision of some definitions and known results that we shall use in the sequel. See [3] for further reference.

Let $X \subseteq \mathbb{R}^N$ be an irreducible real algebraic variety of dimension n. Denote by $\mathcal{R}(X)$ the ring of regular functions on X. Let $S \subseteq X$ be a *semialgebraic set*; it is said to be *basic open* (resp. *basic closed*) if it has a description

$$S = \{x \in X : f_1(x) > 0, \dots, f_r(x) > 0\},$$
$$(\text{resp. } S = \{x \in X : f_1(x) \geqslant 0, \dots, f_r(x) \geqslant 0\}),$$

for some $f_1, \dots, f_r \in \mathcal{R}(X)$.

We shall say that S is *s-basic open* (resp. *s-basic closed*) if S is basic open (resp. closed) and can be described with s regular functions.

A semialgebraic set $S \subseteq X$ is *generically basic* (resp. *generically s-basic*) if there exists a Zariski closed set $C \subseteq X$ with $\dim(C) \leqslant n-1$ such that $S \setminus C$ is basic open (resp. s-basic open).

Remember that for a s-basic open set we may assume $s \leqslant \dim(S)$, as proved by Bröcker and Scheiderer (see [10] and [15]).

Given a semialgebraic set S we define its *Zariski boundary* $\partial_z S$ as the Zariski closure of the set $\partial(S) = \overline{S} \setminus \mathrm{Int}(S)$ and the associated regularly open set S^* as the set $\mathrm{Int}(\overline{\mathrm{Int}(S) \cap \mathrm{Reg}\, X})$. Note that if S is basic open, then $S \cap \partial_z S = \emptyset$.

The *real spectrum* $\mathrm{Spec}_r A$ of a commutative ring A with unit is the set of all pairs $\sigma = (\mathfrak{p}_\sigma, \leqslant_\sigma)$, where $\mathfrak{p}_\sigma$ is a prime ideal of A and $\leqslant_\sigma$ is an ordering in its residue field $k(\mathfrak{p}_\sigma)$. Thus, given $f \in A$, we write $f(\sigma) < 0, = 0$ or > 0 to mean that $f \bmod \mathfrak{p}_\sigma$ is $< 0, = 0$ or > 0 respectively in the ordered field $(k(\mathfrak{p}_\sigma), \leqslant_\sigma)$. Given $\sigma, \tau \in \operatorname{Spec}_r A$, we say that σ *specializes* to τ (or equivalently τ is a *generization* of σ) and write $\sigma \to \tau$ if $f(\tau) > 0$ implies $f(\sigma) > 0$.

The sets of the form $C = \{\sigma\colon f_1(\sigma) > 0, \dots, f_s(\sigma) > 0\}$ generate the so-called *Harrison topology* of $\operatorname{Spec}_r A$ and they are called *basic sets*.

We will now outline some general facts about fans in the real spectrum of a field. Let K be a real field, then $\operatorname{Spec}_r K$ is the space of orderings of K. We can see a given $\sigma \in \operatorname{Spec}_r(K)$ as a signature $\sigma\colon K^* \to \{-1, +1\}$ which maps $f \in K^*$ to $+1$ or -1 according to whether f is positive or negative with respect to the ordering σ. So we multiply orderings as signatures. A *fan* of K is a finite set $F \subseteq \operatorname{Spec}_r(K)$ such that for any three orderings $\sigma_1, \sigma_2, \sigma_3 \in F$, their product $\sigma_4 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ is a well-defined ordering again belonging to F. Subsets consisting of one or two orderings are *trivial* fans. In general, a fan has 2^k elements for some $k \geqslant 0$.

One can see $\operatorname{Spec}_r(K)$ as an abstract space of orderings, by identifying orders as characters on the group $K^*/\sum K^2$. In this setting the group G associated to the fan F is given by $(K^*/\sum K^2)/F^\perp$, where $F^\perp = \{\overline{g} \mid g \in K^* \text{ and } \sigma(g) = 1\ \forall \sigma \in F\}$.

Given a fan F we can find a valuation ring V of K such that

(a) each $\sigma_i \in F$ is compatible with V (i.e., the maximal ideal $\mathfrak{m}_V$ of V is σ_i-convex).

(b) F induces at most two orderings of the residue field k_V of V.

In this situation we say that F *trivializes along* V (see [3]).

Fans can be used to characterize basicness in $\operatorname{Spec}_r(K)$ (see [3]).

THEOREM 1.2. (a) *A constructible set $C \subseteq \operatorname{Spec}_r(K)$ is basic if and only if for every 4-element fan F, $\#(C \cap F) \neq 3$.*

(b) *There are s elements $f_1, \dots, f_s \in K$ such that $C = \{f_1 > 0, \dots, f_s > 0\}$ if and only if for every fan F such that $\#(F) = 2^l$ and $\#(F \cap C) = 1$ we have $l \leqslant s$.*

The *tilde operator* sends a semialgebraic set $S \subseteq X$ to the constructible set $\widetilde{S} \subseteq \operatorname{Spec}_r(\mathcal{R}(X))$ defined by any formula which also defines S. This map gives a bijection that preserves inclusions and topological operations (see [9]). The same symbol is used to define the map $S \to \widetilde{S} \cap \operatorname{Spec}_r(\mathcal{K}(X))$.

The tilde operators are used to reformulate many problems concerning X in terms of $\operatorname{Spec}_r(\mathcal{R}(X))$ or $\operatorname{Spec}_r(\mathcal{K}(X))$. For instance, Theorem 1.1 characterizes the generic basicness. When dealing with basic open or closed sets the following geometric criterion holds (see [3]):

THEOREM 1.2. *A semialgebraic $S \subset X$ is s-basic open (resp. closed) if and only if $S \cap \partial_z S = \emptyset$ (resp. S is closed) and for every irreducible subset $Y \subseteq X$ the intersection $S \cap Y$ is generically s-basic in Y.*

In fact, over the reals, in Theorem 1.2 one can reduce Y to be an irreducible surface. This was proved in [3] by using the so called algebroid fans.

A *birational model* of S is a semialgebraic set T in a real variety Y such that there is a birational map $\phi : Y \to X$ and $T = \phi^{-1}(S)$. Andradas and Ruiz also found the following characterization which involves all birational models of S (see [5]):

THEOREM 1.3. *A semialgebraic set $S \subseteq X$ is not generically basic if and only if there is a birational model T of S such that T is crossed by its generic Zariski boundary, that is, the intersection of the Zariski closure of $(\overline{T^*} \setminus T^*) \cap \mathrm{Reg}(Y)$ with T^* has codimension* 1.

Remarks 1.4. (1) Let A be a local regular ring of dimension d with residue field k and quotient field K. Take a system of parameters $x_1, \ldots, x_d$ for A. Then, there is a (discrete) valuation ring V of K dominating A with residue field k and value group $\mathbb{Z}^d$. Moreover, fixing $\tau \in \mathrm{Spec}_r(k)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{+1, -1\}^d$, there is an ordering σ of K which is compatible with V, induces τ in $\mathrm{Spec}_r k$, and such that $\sigma(x_j) = \varepsilon_j$ for all j.

Indeed, if $d = 1$, A is a rank 1 discrete valuation ring and by the Baer–Krull Theorem, (see [9]) we are done. If $d > 1$, we consider the rank 1 discrete valuation ring $A_{(x_1)}$ whose residue field is the quotient field of the $(d-1)$-dimensional regular local ring $A/(x_1)$, and by induction we are done.

(2) With the previous notations, fix an ordering τ of k and let F_τ be the fan of all orderings of K compatible with V inducing τ in $\mathrm{Spec}_r k$. Then, by the Baer–Krull Theorem the elements of F_τ are completely determined by the signs that they assign to the parameters $x_1, \ldots, x_d$.

Each element $\sigma \in F_\tau$ will be called a *generization* of τ to V (or to A) and F_τ will be called the *lifting* of τ to V.

2. Decidability of Basicness

In this section we show that from a model theoretical point of view the basicness problem is decidable. We give a ‘general recursive’ algorithm based on the surface criterium quoted after Theorem 1.2. References for the background needed for this section may be found in [12] or [14].

Since we are concerned now with algorithms, we assume throughout this section that X, as well as S, are defined over the field of real algebraic numbers $\mathbb{R}_0$. It is clear from Tarski’s Transfer Principle (see [9] that all the semialgebraic sets constructed in the sequel (like the irreducible components of the Zariski boundary of S) can be defined over $\mathbb{R}_0$ as well.

We now state the main result of this section.

THEOREM 2.1. *The basicness of semialgebraic sets on an affine variety defined over the field $\mathbb{R}_0$ is decidable.*

Proof. Decidability of basicness in dimension n is proved by recursively listing, on one hand, all pairs (X, S) where X is an $\mathbb{R}_0$-variety of dimension n and S is a basic open semialgebraic subset of X, and, on the other hand, those couples where S is open but not basic in X.

Thus, given a particular variety X_0 and an open semialgebraic subset S_0 of X_0, the decision about its basicness is reached by scrolling in turn both lists. The side in which (X_0, S_0) appears will tell us whether S_0 is basic in X_0 or not.

On the one hand, it is clear that we can recursively enumerate all the couples (X, S), where X is an affine variety over $\mathbb{R}_0$ and $S \subseteq X$ is a basic open semialgebraic set over $\mathbb{R}_0$. One way to do this is to delete from the list of all $(n+2)$-tuples $(X, S, p_1, \ldots, p_n)$, where X is an affine variety, S is an open semialgebraic set and $p_1, \ldots, p_n$ are polynomials over $\mathbb{R}_0$, the ones such that $\{p_1 > 0, \ldots, p_n > 0\} \neq S$.

Therefore it is enough to prove that the second list is also recursively enumerable. This second list is obtained from the list of all triples (X, S, Y), where $Y \subseteq X$ is an irreducible surface and either $S \cap \partial_z S \neq \emptyset$ or $S \cap Y$ is not generically basic. The first property is decidable by Tarski–Seidenberg principle, the second one can be decided by applying the results in [2] or [16] and using the fact that the resolution of singularities is algorithmic (see [8] or [17]). $\square$

Remark. Note that here, in order to get countable lists, we use in a essential way the fact that both the number of unions and the number of inequalities needed to describe an open semialgebraic set S are bounded by a recursive function depending only on the dimension (see [10] and [15]).

Remark 2.2. We can use Theorem 1.3, instead of the surface criterium to prove decidability since the birational models of X can also be recursively listed, as it is done in [1].

The following example is well known (see [3] Example VI 7.10). It shows that, despite we can decide basicness, there exists no uniform bound on the degrees of the polynomials describing basic sets of a given complexity.

EXAMPLE 2.3. Consider the family $S^{(n)}$ of basic open sets in $\mathbb{R}^2$ shown in Figure 1. All boundaries of the sets in the family have the same complexity. For each n there are polynomials $f_1^{(n)}, f_2^{(n)}$ such that $S^{(n)} = \{f_1^{(n)} > 0, f_2^{(n)} > 0\}$. Nevertheless, the degree of the polynomials $f_1^{(n)}, f_2^{(n)}$ goes to infinity when n goes to infinity.

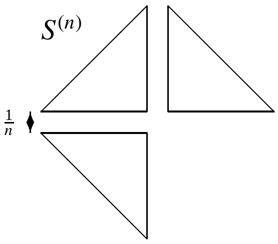

Figure 1.

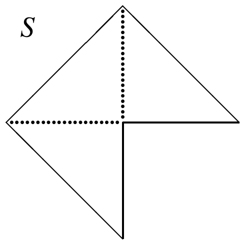

Figure 2.

Indeed, without loss of generality we may assume that the coefficients of $f_1^{(n)}$, $f_2^{(n)}$ have all absolute value $\leqslant 1$. Assume $\deg(f_i^{(n)}) \leqslant d$ for each i, n and consider the sequence of $2N(d)$-tuples

$$C = \left\{(a_{i,j}^{(n)}) : i = 1, 2, \quad j = 1, \ldots, N(d)\right\}_{n\in\mathbb{N}},$$

where $N(d)$ is the number of monomials of degree $\leqslant d$ in two variables and $\{a_{i,j}^{(n)}\}$ are the coefficients of $f_i^{(n)}$. Take a limit point $\overline{c}$ of the sequence C. We can identify $\overline{c}$ with a couple (f_1, f_2) of polynomials of degree at most d. Then it is easy to prove that the basic closed set $\{f_1 \geqslant 0, f_2 \geqslant 0\}$ is generically equal to the set $S = \overline{\bigcup S^{(n)}}$ which is not generically basic (see Figure 2). Contradiction.

3. Basicness and Generic Basicness

From ([3], V4) one gets the following proposition which relates basicness and generic basicness. Let X be a compact irreducible real algebraic variety and $S \subseteq X$ a semialgebraic set with $\dim(X) = \dim(S) = n$.

PROPOSITION 3.1. (1) *S is generically basic if and only if $S \setminus (\partial_z S \cup \mathrm{Sing}(X))$ is basic open.*

(2) *S is generically s-basic if and only if $S \setminus (\partial_z S \cup \mathrm{Sing}(X))$ is s-basic open.*

Remark 3.2. We can not avoid to remove $\mathrm{Sing}(X)$ in the statement of Proposition 3.1. Indeed, consider an irreducible real algebraic set X of dimension d with a 'tail' Y of lower dimension. Let be $S = S_1 \cup S_2$, where $\overline{S_1} \subseteq X \setminus Y$, S_1 is basic open and $S_2 \subseteq Y$ is open but not generically basic. Then, S is generically basic, but $S \setminus \partial_z S$ is not basic open.

COROLLARY 3.3. *Let X be a compact, irreducible real algebraic variety and let $S \subseteq X$ be a semialgebraic set with* $\dim(S) = \dim(X)$.

(1) *If $S \cap \partial_z S = \emptyset$, then S is s-basic open if and only if S is generically s-basic and $S \cap \mathrm{Sing}(X)$ is s-basic open.*
(2) *If S is closed, then S is s-basic closed if and only if S is generically s-basic and $S \cap (\partial_z S \cup \mathrm{Sing}(X))$ is s-basic closed.*

Proof. It follows from Proposition 3.1, Theorem 1.3 and the fact that basicness in a reducible algebraic set X is equivalent to basicness in all irreducible component of X. □

EXAMPLE 3.4. This example shows that Corollary 3.3.2 can not be improved in the sense that the hypothesis on $S \cap \partial_z S$ is not superfluous. Consider the closed semialgebraic set $S \subseteq \mathbb{R}^3$ given by

$$S = \{x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0\} \cup \{-y^2 \geqslant 0, \quad z \leqslant 0\}.$$

This set is not basic closed, since $S \cap \{y = 0\}$ is not generically basic. But it is generically basic.

4. A Geometric Criterion for Basicness

Let $X \subseteq \mathbb{R}^N$ be an irreducible compact non-singular real algebraic variety of dimension n. Consider a semialgebraic set $S \subseteq X$ with $\dim(S) = n$. In the following an irreducible component of the algebraic set $\partial_z S^*$ will be called a *wall of S* if it has dimension $n - 1$. We assume in this section that all the walls are non-singular and that their intersection is normal crossings (algebraically, we mean by this that the regular functions f_{ij}, g_i describing S are all normal crossings).

In the previous situation we claim the following.

THEOREM 4.1. *Assume that $S \cap \partial_z S = \emptyset$. Then S is not basic open if and only if there is a wall W of S such that either*

(i) $\dim(S^* \cap W) = n - 1$, *or*
(ii) $\overline{S} \cap W$ *is not generically basic in* W.

THEOREM 4.2. *Let S be a generically basic semialgebraic set in X which is not generically equal to X and such that $S \cap \partial_z S = \emptyset$. Then S is s-basic open if and only for each wall W of S the set $\overline{S} \cap W$ is generically $(s-1)$-basic in W.*

Remark 4.3. This result is motivated by Theorem 1.3. Namely, S is not generically basic if and only there is a birational model T which is *crossed* by its generic Zariski boundary. In general this birational model is constructed by blowing-up $\partial_z S$, but it is not canonical. For instance, if S is not generically basic, we could think that the birational model, obtained from S after we desingularize X and $\partial_z S$, is crossed by its generic Zariski boundary. This is true for two-dimensional semialgebraic sets (see [2]) and for many examples. But Figure 3 shows that this is not always true.

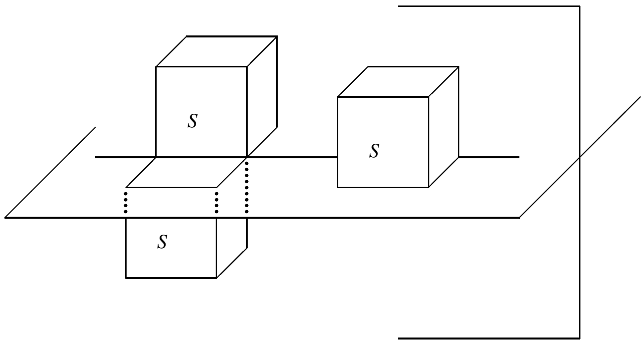

Figure 3.

The proof of Theorems 4.1, 4.2 is an easy consequence of the following Proposition 4.4 which was kindly suggested by the referee. Firstly let us introduce some notations.

Let $F \subset \mathrm{Spec}_r K(X)$ be a fan and $Z \subset X$ be its *center*, that is Z is the zero set of the ideal $(\mathfrak{m} \cap \mathcal{R}(X))$ where $\mathfrak{m}$ is the maximal ideal of the biggest valuation ring V of $K(X)$ such that F trivializes along V. So one has two orderings $\alpha, \beta \in \mathrm{Spec}_r K(Z)$ (possibly $\alpha = \beta$), such that every $\sigma \in F$ specializes either to α or to β.

Let $W_1, \ldots, W_r$ be the walls of S which contain Z. Since they are normal crossings, they correspond to regular parameters $x_1, \ldots, x_r$ of the regular local ring $\mathcal{R}(X)_{\mathcal{I}(Z)}$. Choose parameters $x_{r+1}, \ldots, x_d$, with $d = \dim X - \dim Z$, in order to get a full system. If $\alpha \neq \beta$ take a further element $y \in \mathcal{R}(X)$ such that its class in $K(Z)$ separates α and β. Denote again by V be the valuation ring of $K(Z)$, with value group $\mathbb{Z}^d$, which dominates $\mathcal{R}(X)_{\mathcal{I}(Z)}$. The lexicographic order of $\mathbb{Z}^d$ depends of course on the ordering of the parameters, but the following does not.

Let F'' be the pull back of the trivial fan $\{\alpha, \beta\}$ with respect to V and denote by T' the set

$$T' = \{\sigma \in \mathrm{Spec}_r\, K \mid \sigma \text{ specializes either to } \alpha \text{ or to } \beta\}.$$

Moreover for $\varepsilon_i = \pm 1$ and $i = 1, \ldots, d+1$ we define

$$T'(\varepsilon_1, \ldots, \varepsilon_{d+1}) = \{\sigma \in T' \mid \sigma(x_i) = \varepsilon_i, i = 1, \ldots, d \text{ and } \sigma(y) = \varepsilon_{d+1}\}$$

If $\alpha = \beta$ we omit the last condition and we forget y and ε_{d+1}. Then one has the following proposition.

PROPOSITION 4.4. *There is a subfan $F'' \subset F'$ and a natural number l such that for any sequence $\varepsilon_1, \dots, \varepsilon_{d+1}$ one has*

$$\#\left(F \cap T'(\varepsilon_1, \dots, \varepsilon_{d+1})\right) = \#\left(F'' \cap T'(\varepsilon_1, \dots, \varepsilon_{d+1})\right) \cdot 2^l.$$

Proof. Let G be the group of F'', that is $G = (K(X)^*/\sum K(X)^2)/F''^{\perp}$. So that G is an exponent two group generated by the classes $\overline{-1}, \overline{x_1}, \dots, \overline{x_d}, \overline{y}$.

Let U be the subgroup of G giving the sign 1 to each element in F and let $F'' = U^{\perp}$ be the corresponding subfan of F''. So we have $\#(F'') = \frac{1}{2}|G/U|$. By definition each $g \in G/U$, $g \neq \pm 1$, separates F'', and consequently it separates F. Therefore $\#(F) \geqslant \#(F'')$, which means $\#(F) = 2^l \cdot \#(F'')$ for a suitable l.

Now we prove, by induction on $\#(F'')$, that F'' and l are the subfan and the integer we are looking for.

If we suppose $\#(F'') = 1$, then $G/U = \{+1, -1\}$. Hence for every $\sigma \in F$ we have $\sigma(g) = F''(g)$ for all $g \in G$. Moreover, assume

$$F \cap T'(\varepsilon_1, \dots, \varepsilon_{d+1}) \neq \emptyset.$$

Therefore, $F'', F \subset T'(\varepsilon_1, \dots, \varepsilon_{d+1})$ which implies the thesis.

Now assume that $\#(F'') > 1$ and take $g \in G/U, g \neq \pm 1$. The fan F splits as $F = F_1 \cup F_2$ according to the sign of g, and correspondly F'' splits as $F'' = F_1'' \cup F_2''$. Then both F_1, F_1'' and F_2, F_2'' verify the hypothesis, hence the thesis by the induction hypothesis. So the thesis holds also for F, F''. □

Proof of Theorem 4.1. Assume that S is not basic open and let F be a 4-element fan in $\mathrm{Spec}_r K(X)$ such that $\#(F \cap \widetilde{S}) = 3$. Take $\alpha, \beta, x_1, \dots, x_d, y$ as above. Then both α and β are specializations of some elements in $\widetilde{S}$. Moreover, since you cannot leave $\widetilde{S}$ without crossing some wall among $W_1, \dots, W_r$, if $\widetilde{S} \cap T'(\varepsilon_1, \dots, \varepsilon_{d+1}) \neq \emptyset$ we get $T'(\varepsilon_1, \dots, \varepsilon_{d+1}) \subset \widetilde{S} \cap T'$.

Choose F'' as in Proposition 4.4 above. Thus, F has almost 4 elements and it is compatible with some discrete valuation ring $\mathcal{R}(M)_{(x_i)}$, for some $i = 1, \dots, r$. We may assume $i = 1$.

Let $\overline{F''}$ be the induced fan in $K(W_1)$. Then we have two possibilities:

(i) Each $\gamma \in \overline{F''}$ generizes twice in F'', in which case $\dim(S^* \cap \widetilde{W_1}) = n - 1$.

(ii) Each $\gamma \in \overline{F''}$ generizes once in F'', in which case $\#\overline{F''} \cap \overline{\widetilde{S}} \cap W_1 = 3$.

Conversely, (i) implies that S^* crosses its Zariski boundary, hence S is not basic open. On the other hand, if (ii) holds, take a four-elements fan $F_1 \subset \mathrm{Spec}_r K(W)$, such that $\#F_1 \cap (\widetilde{\overline{S} \cap W}) = 3$ and let F be its pull back with respect to the discrete valuation ring $\mathcal{R}(M)_{\mathcal{I}(W)}$. Then, an easy computation shows that $F \cap \widetilde{S}$ is not a fan, hence S is not basic open. □

Proof of Theorem 4.2. We assume that there is a fan $F \subset \mathrm{Spec}_r K(X)$ such that $\#(F \cap \widetilde{S}) > 2^s$. Then we choose F'' as in Proposition 4.4, and the parameter x_1, the valuation ring $V = \mathcal{R}(M)_{(x_1)}$ and the residue fan $\overline{F''}$ as in the proof of 4.1. Since S is basic open any $\overline{\gamma} \in \overline{F''}$ generizes uniquely to F''. Thus

$$\#(\overline{F''} \cap \widetilde{\overline{S} \cap W_1}) > 2^{s-1}.$$

Conversely, assume there is a wall W such that $\overline{S} \cap W$ is not generically $s-1$ basic and let $\overline{F}$ be a 2^s-element fan in $K(W)$ such that $\#(\overline{F} \cap \widetilde{\overline{S} \cap W}) = 1$.

Consider the discrete valuation ring $V = \mathcal{R}(M)_{\mathcal{I}(W)}$ and the 2^{s+1}–element fan F of $K(X)$ obtaining by lifting $\overline{F}$ to V.

Let σ_0 be the unique element in $\overline{F} \cap \widetilde{S_W}$. Then, only one of its generizations may belong to $\widetilde{S}$, since S is basic open. So S is not s-basic. □

Remark 4.5. If we drop the hypothesis on $\partial_z S$ in Theorems 4.1 and 4.2, only one implication is true. Namely:

(1) *Let $X \subset \mathbb{R}^N$ be a compact real algebraic set and let $S \subseteq X$ be a semialgebraic set. If S is generically basic (resp. (s)-basic), then*

  (i) $\dim(S^* \cap \partial_z S^*) < n-1$.

  (ii) *For any wall W of S such that $W \not\subseteq \mathrm{Sing}(X)$ the set $\overline{S} \cap W$ is generically basic (resp. $(s-1)$-basic) in W.*

  This is because, in the proof of the *only if* part of Theorems 4.1 and 4.2, we only use the fact that the local ring of W is regular of dimension 1.

(2) In the case of 1-basic sets (the so called principal sets), the statement of Theorem 4.2 becomes:

  Let $X \subset \mathbb{R}^N$ be a compact non-singular real algebraic set and let $S \subseteq X$ be a semialgebraic set such that $S \cap \partial_z S = \emptyset$. If S is generically basic, then S is 1-basic if and only if for any wall W of S the set $\overline{S} \cap W$ is generically equal to W.

This is because S is 1-basic open if and only if S and $X \setminus S$ are generically basic.

5. An Algorithmic Procedure

This section deals with a decision procedure for basicness based on the results obtained in Sections 3 and 4. The answer to the question whether S is basic or not can be given recursively, each time splitting the original problem into a finite number of similar ones in one dimension less.

When X is non-singular and $\partial_z S$ is normal crossings, Theorems 4.1 and 4.2 reduce the problem to the same question for the sets $\overline{S} \cap W$ for each wall W of S. For X and S arbitrary, we use Corollary 3.3 and the fact (proved by [8] and [17]) that the resolution of singularities is algorithmic.

Let $X \subset \mathbb{R}^N$ be a compact irreducible real algebraic variety and S a semialgebraic subset. The algorithm to test whether S is basic open (resp. closed) or not works as follows:

I. Algorithm for Generic Basicness:

(1) Desingularize X and put $\partial_z S$ in normal-crossings.
(2) Replace X and S by their strict transforms.
(3) Check whether S is generically basic as follows.

Build up a tree whose nodes are couples (Y, T) where Y is a non-singular compact variety and T is a closed semialgebraic set in Y with $\partial_z T$ at normal-crossings. The tree is constructed recursively in this way:

– Start with the node $(X, \overline{S})$.
– Given a node (Y, T) with $T \neq Y$, if $\dim(T^* \cap \partial_z T^*) = \dim(\partial_z T^*)$, then the branch stops; otherwise, for each wall W of T add an edge from (Y, T) to $(W, T \cap W)$.

The length of each branch is less than or equal to $n = \dim(S)$.

The algorithm runs over each branch of the tree in turn. Each branch stops either when $Y = T$, in which case the algorithm goes to the next branch, or when it arrives to a node (Y, T) with $\dim(T^* \cap \partial_z T^*) = \dim(\partial_z T^*)$ (i.e., a universal obstruction for T), in which case the algorithm stops and returns 'S is not generically basic'.

The set S is generically basic if and only if no universal obstruction appears in the tree. In this case, S is generically s-basic if and only if the longest branch has exactly $s + 1$ nodes.

II. Algorithm for Basicness:

(1) Start with the couple (X, S).
(2) For a given couple (Y, T) proceed as follows:

2.1. Verify, using **I**, whether T is generically basic.
2.2. If so, verify if $T \cap \partial_z T = \emptyset$ (resp. if T is closed).
2.3. If so, list the irreducible components Z of $\mathrm{Sing}(Y)$ (resp. of $\mathrm{Sing}(Y) \cup \partial_z T$).

(3) Repeat step 2 for all the new couples $(Z, T \cap Z)$.

Since for each loop from 3 to 2 the dimension decreases and for dimension 1 the problem has a trivial answer, the algorithm stops.

More precisely, the algorithm stops either when for a couple (Y, T) there is a negative answer for either step 2.1 or step 2.2, or it has tested all couples (Y, T) till dimension 1. Therefore, S is basic open (resp. basic closed) if and only if only the last case occurs. In this situation, we can compute the number of inequalities defining S as the maximum s such that T is generically s-basic in Y for any (Y, T) appearing in the procedure.

In order to have a true algorithm we need to compute the following objects:

- dimension of semialgebraic sets,
- interiors and closures of semialgebraic sets,
- Zariski boundary of semialgebraic sets,
- irreducible components of algebraic sets.

All of them are known to be computable, see for instance [7, 9, 11, 13].

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