

Multi-Trek Separation in Linear Structural Equation Models

Elina Robeva^{*} and Jean-Baptiste Seby[†]

^{*}*University of British Columbia*

[†]*Massachusetts Institute of Technology*

Abstract

Building on the theory of causal discovery from observational data, we study interactions between multiple (sets of) random variables in a linear structural equation model with non-Gaussian error terms. We give a correspondence between structure in the higher order cumulants and combinatorial structure in the causal graph. It has previously been shown that low rank of the covariance matrix corresponds to trek separation in the graph. Generalizing this criterion to multiple sets of vertices, we characterize when determinants of subtensors of the higher order cumulant tensors vanish. This criterion applies when hidden variables are present as well. For instance, it allows us to identify the presence of a hidden common cause of k of the observed variables.

1 Introduction

Although randomized experiments are the most commonly used method for causal inference, they are sometimes not feasible for practical or ethical reasons. Because of these constraints, scientists often need to learn the structure of the graph underlying the relationships between variables based on purely observational data. Suppose that $G = (V, \mathcal{D})$ is a directed acyclic graph (or DAG) with vertex set $V = \{1, \dots, p\}$ and edge set $\mathcal{D} \subseteq V \times V$. The graph G gives rise to a *linear structural equation model* (LSEM), which consists of joint distributions of a random vector $X = (X_1, \dots, X_p)$ in which the variable X_i associated to vertex $i \in V$ is a linear function of X_j , where j varies over the parent set $\text{pa}(i)$ of i (i.e., all vertices $j \in V$ such that $j \rightarrow i \in \mathcal{D}$), and a noise term ε_i ,

$$X_i = \sum_{j \in \text{pa}(i)} \lambda_{ji} X_j + \varepsilon_i, \quad i \in V. \quad (1)$$

If no hidden variables are present, we assume that the noise terms ε_i are mutually independent. To encode the presence of hidden variables, we allow dependencies between the ε_i variables, and graphically we depict this via *multi-directed edges* (see Figure 1a). These encode a hidden common cause of a few of the observed variables. We represent this more complicated hidden structure via a *mixed graph* $G = (V, \mathcal{D}, \mathcal{H})$, where \mathcal{H} is the set of multi-directed (hyper)edges (see Section 4).

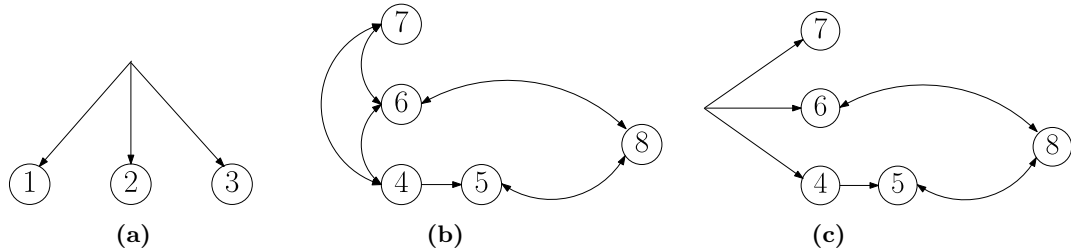


Figure 1: (a) A multi-directed edge between nodes 1, 2 and 3; (b) Mixed graph G_1 ; (c) Mixed graph G_2

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When the noise terms ε_i are Gaussian, then so are the X_i variables. In this setting, the linear structural equation model given by a graph G corresponds to the set of covariance matrices $\mathcal{M}^{(2)}(G)$ of a Gaussian distribution consistent with the graph G [Lauritzen, 1996]. This is precisely the set of covariance matrices that possess a certain *parametrization* arising from the structure of G . Furthermore, bidirected edges suffice to parametrize the model $\mathcal{M}^{(2)}(G)$ in the case of hidden variables. For example, the mixed graphs G_1 and G_2 in the figure above give rise to the same model in Zariski closure $\overline{\mathcal{M}^{(2)}(G_1)} = \overline{\mathcal{M}^{(2)}(G_2)}$. When the variables are non-Gaussian, we can depict the model using covariances as well as higher-order moments/cumulants of the random vector X . We denote by $\mathcal{M}^{(k)}(G)$ the set of cumulants up to order k consistent with the graph G . This set can also be parametrized using the graph (Definition 4), and provides a more refined description of the graph structure. For instance, the two graphs above give rise to different models $\overline{\mathcal{M}^{(3)}(G_1)} \neq \overline{\mathcal{M}^{(3)}(G_2)}$.

A parametrization of the model, however, may not always be sufficient. Statistical problems like model selection, model equivalence, and constraint based statistical inference often require an implicit description of the model in terms of (polynomial) equations which can be read off from the graph G , e.g., via a combinatorial criterion.

When G is a DAG and the variables are Gaussian, the implicit description of the model $\mathcal{M}^{(2)}(G)$ is given by the vanishing of specific subdeterminants of the covariance matrix which can be read off from the graph via *d-separation* and the more general *trek-separation* criteria [Sullivant et al., 2010]. In fact, the *trek-separation* helps describe the vanishing of all subdeterminants of the covariance matrix in any (not necessarily Gaussian) LSEM. It turns out that covariance information is only sufficient to identify the graph up to *Markov equivalence*. That is, if two graphs G and G' give rise to the same conditional independence relations, then they produce the same sets of covariance matrices $\mathcal{M}^{(2)}(G) = \mathcal{M}^{(2)}(G')$. Therefore, when the graph G is a DAG and the variables are Gaussian, we can only recover G up to Markov equivalence given observational data. Finding an implicit description of $\mathcal{M}^{(2)}(G)$ in the presence of hidden variables is more challenging, although there has been promising recent progress. In particular, [Yao and Evans, 2019] prove that the minimal generators for the vanishing ideal $\mathcal{I}(G)$ containing all the constraints for a Gaussian Acyclic Directed Mixed Graph G are in one-to-one correspondence with the pairs of non-adjacent vertices in the graph, and provide an algorithm to find all these generators.

When the variables are non-Gaussian, the graph G can be recovered uniquely from observational data. In particular, [Shimizu et al., 2006a] use Independent Component Analysis (ICA) [Comon, 1994] to estimate the graph structure via the Linear non-Gaussian Acyclic Model (LiNGAM). This framework and its derived versions DirectLiNGAM and PairwiseLiNGAM [Shimizu et al., 2006b] make it possible to distinguish graphs within Markov equivalence classes. Furthermore, [Wang and Drton, 2019] provide an algorithm that extends causal discovery of the causal structure in the high-dimensional setting based on higher-order moments, under a maximum in-degree condition.

In this paper, we also work under the framework of a non-Gaussian LSEM. Building on the trek rule [Sullivant et al., 2010], we define the *multi-trek rule* (Proposition 7) which gives a polynomial parametrization of the higher-order moments/cumulants, and enables us to study LSEMs (1) via their higher-order cumulant representation $\mathcal{M}^{(k)}(G)$ from the perspective of algebraic statistics [Sullivant, 2018] which has so far only been used for Gaussian and discrete graphical models. By analogy with the vanishing of subdeterminants in the covariance matrix, we give a necessary and sufficient combinatorial criterion, called *multi-trek separation*, for the vanishing of subdeterminants of the tensor $\mathcal{C}^{(k)}$ of k -th order cumulants (Theorem 15), which extends to the hidden variable case (Theorem 25). Our multi-trek separation criterion, for example, enables us to identify the presence of a hidden common cause of multiple observed variables.

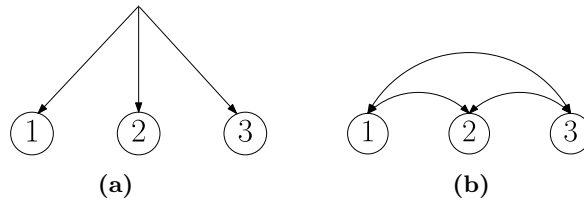


Figure 2: By Theorem 25, (a) $\mathcal{C}_{123}^{(3)} \neq 0$ (b) $\mathcal{C}_{123}^{(3)} = 0$

The rest of the paper is organized as follows. In Section 2, we define linear structural equation models (LSEMs) and their cumulant tensors. In Section 3, we introduce the notion of a multi-trek and we state our main theorem for DAG models that establishes a combinatorial criterion for the vanishing of subdeterminants of the k^{th} -order cumulant tensor. In Section 4 we consider the case of hidden variables. Graphically we encode the presence of such variables via multi-directed edges, and we show that our results generalize to this case. In Section 5, we conjecture that our multi-trek criterion is also equivalent to the vanishing of subdeterminants of higher-order moment (rather than cumulant) tensors. In section 6, we conclude and discuss directions for further research.

2 Background

In this section we provide the necessary background on linear structural equation models, and their higher-order cumulants.

2.1 Linear structural equation models

Let $G = (V, \mathcal{D})$ be a directed acyclic graph (DAG) with finite vertex set $V = \{1, \dots, p\}$ and edge set $\mathcal{D} \subseteq V \times V$. Here acyclic means that there are no directed cycles, i.e., no sequences of the form $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_s = i_0$, where $i_j \rightarrow i_{j+1} \in \mathcal{D}$. The edge set is always assumed to be free of self-loops, so $(i, i) \notin \mathcal{D}$ for all $i \in V$. For each vertex i , define its set of parents as $\text{pa}(i) = \{j \in V : (j, i) \in \mathcal{D}\}$. The graph G induces a statistical model, called a *linear structural equation model*, for the joint distribution of a collection of random variables $(X_i, i \in V)$, indexed by the graph's vertices. The model hypothesizes that each variable is a linear function of the parent variables and a noise term ε_i :

$$X_i = \lambda_{0i} + \sum_{j \in \text{pa}(i)} \lambda_{ji} X_j + \varepsilon_i, \quad i \in V. \quad (2)$$

The ε_i variables for $i \in V$, are independent and centered. The coefficients λ_{0i} and λ_{ji} are unknown real parameters that are assumed to be such that the system (2) admits a unique solution $X = (X_i : i \in V)$. Typically termed a system of structural equations, (2) specifies cause-effect relations whose straightforward interpretability explains the wide-spread use of the models [Spirtes et al., 2000, Pearl, 2009].

The random vector X that solves the system (2) may have an arbitrary mean depending on the choice of parameters λ_{0i} . Since the mean can easily be learned from data, and we are mainly concerned with learning the underlying graph structure, we disregard the offsets λ_{0i} , and the system (2) becomes

$$X = \Lambda^T X + \varepsilon. \quad (3)$$

Here $\Lambda = (\lambda_{ij}) \in \mathbb{R}^{\mathcal{D}}$, and $\mathbb{R}^{\mathcal{D}}$ is the set of $V \times V$ matrices Λ with support \mathcal{D} ,

$$\mathbb{R}^{\mathcal{D}} = \{\Lambda \in \mathbb{R}^{V \times V} : \lambda_{ij} = 0 \text{ if } i \rightarrow j \notin \mathcal{D}\}.$$

Since G is acyclic, the matrix $I - \Lambda$ is always invertible and the solution of the system (3) is:

$$X = (I - \Lambda)^{-T} \varepsilon. \quad (4)$$

2.2 Cumulants of linear structural equation models

Recall that the k -th *cumulant tensor* of a random vector $Z = (Z_1, \dots, Z_p)$ is the $p \times \dots \times p$ (k times) table with entry at position (i_1, \dots, i_k) given by

$$\text{cum}(Z_{i_1}, \dots, Z_{i_k}) = \sum_{(A_1, \dots, A_L)} (-1)^{L-1} (L-1)! \mathbb{E} \left[\prod_{j \in A_1} Z_j \right] \mathbb{E} \left[\prod_{j \in A_2} Z_j \right] \dots \mathbb{E} \left[\prod_{j \in A_L} Z_j \right],$$

where the sum is taken over all partitions (A_1, \dots, A_L) of the set $\{i_1, \dots, i_k\}$. If each of the variables Z_i is centered, i.e., has mean 0, then we can restrict to summing over partitions for which each A_i has size at least 2. For example, the first four cumulants are given as follows:

$$\begin{aligned} \text{cum}(Z_i) &= \mathbb{E}[Z_i] = 0, & \text{cum}(Z_{i_1}, Z_{i_2}) &= \mathbb{E}[Z_{i_1} Z_{i_2}], & \text{cum}(Z_{i_1}, Z_{i_2}, Z_{i_3}) &= \mathbb{E}[Z_{i_1} Z_{i_2} Z_{i_3}], \quad \text{and} \\ \text{cum}(Z_{i_1}, Z_{i_2}, Z_{i_3}, Z_{i_4}) &= \mathbb{E}[Z_{i_1} Z_{i_2} Z_{i_3} Z_{i_4}] - \mathbb{E}[Z_{i_1} Z_{i_2}] \mathbb{E}[Z_{i_3} Z_{i_4}] - \mathbb{E}[Z_{i_1} Z_{i_3}] \mathbb{E}[Z_{i_2} Z_{i_4}] - \mathbb{E}[Z_{i_1} Z_{i_4}] \mathbb{E}[Z_{i_2} Z_{i_3}]. \end{aligned}$$

Now, let $k \geq 2$, and let $\mathcal{E}^{(k)}$ and $\mathcal{C}^{(k)}$ be the k -th order cumulant tensors of the random vectors ε and X , respectively. The linear structural equation model, and, particularly, the expression (4) yield the following relationship between $\mathcal{C}^{(k)}$ and $\mathcal{E}^{(k)}$.

Lemma 1 ([Comon and Jutten, 2010, Chapter 5]). *The tensor $\mathcal{C}^{(k)}$ of k -th order cumulants of X equals*

$$\mathcal{C}^{(k)} = \mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}, \quad (5)$$

where $\mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}$ denotes the Tucker product of the order- k tensor $\mathcal{E}^{(k)}$ and the matrix $(I - \Lambda)^{-1}$ along each of its k dimensions. In other words,

$$(\mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k})_{i_1, \dots, i_k} = \sum_{j_1, \dots, j_k} \mathcal{E}_{j_1, \dots, j_k}^{(k)} ((I - \Lambda)^{-1})_{j_1, i_1} \cdots ((I - \Lambda)^{-1})_{j_k, i_k}.$$

One of the reasons this factorization is extremely useful is that when the entries of the noise vector ε are mutually independent, its cumulants $\mathcal{E}^{(k)}$ are *diagonal* tensors.

Lemma 2 ([Comon and Jutten, 2010, Chapter 5]). *If the variables Z_1, \dots, Z_p are independent, then the k -th order cumulant tensor of $Z = (Z_1, \dots, Z_p)$ is diagonal, i.e., the entry at position (i_1, \dots, i_k) is 0 unless i_1, \dots, i_k are all equal.*

Remark 3. (a) We originally considered the moments of X and ε , instead of their cumulants. Lemma 1 also applies to the factorization of the moments, however, the moment tensors of ε are no longer diagonal, i.e., Lemma 2 does not apply. We give further details of this study in Section 6.

(b) Lemma 2 is the reason cumulants are widely used for Independent Component Analysis (ICA) [Comon and Jutten, 2010]. Indeed, finding the CP-decomposition [Comon and Jutten, 2010, de Lathauwer, 2006] of the cumulant tensor $\mathcal{C}^{(k)}$ can recover the matrix $(I - \Lambda)^{-1}$. For an extended study of cumulants in ICA, we refer the reader to [Comon and Jutten, 2010].

Lemmas 1 and 2 provide a means to *parametrize* the set of all cumulant tensors of the distributions in a given graphical model.

Definition 4. Let $G = (V, \mathcal{D})$ be a DAG, and let $k \geq 2$ be an integer. The set

$$\mathcal{M}^{(2)}(G) = \{(I - \Lambda)^{-T} \mathcal{E}^{(2)} (I - \Lambda)^{-1} : \Lambda \in \mathbb{R}^{\mathcal{D}}, \mathcal{E}^{(2)} \succeq 0 \text{ diagonal}\}$$

consists of all covariance matrices of distributions in the graphical model given by G . For $k \geq 3$, define

$$\mathcal{M}^{(k)}(G) = \{(\mathcal{E}^{(2)} \bullet (I - \Lambda)^{-2}, \dots, \mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}) : \Lambda \in \mathbb{R}^{\mathcal{D}}, \mathcal{E}^{(i)} \text{ diagonal for } 2 \leq i \leq k, \text{ and } \mathcal{E}^{(2)} \succeq 0\}.$$

The set $\mathcal{M}^{(k)}(G)$ contains all cumulants up to order k of a distribution in the graphical model given by G .

When the error terms ε_i are Gaussian, the random vector X also follows a Gaussian distribution and all of its cumulants of order $k \geq 3$ equal 0. Therefore, the model equals $\mathcal{M}^{(2)}(G)$. In this case, different DAGs G_1 and G_2 that lie in the same *Markov equivalence class* can give rise to the same models $\mathcal{M}^{(2)}(G_1) = \mathcal{M}^{(2)}(G_2)$. An implicit description of $\mathcal{M}^{(2)}(G)$ is well-known when G is a DAG and is given by conditional independences [Harri Kiiveri, 1984, Luis David Garcia, 2005]. These correspond to *d-separations* in the graph G [Pearl, 1986].

When the noise terms ε_i are not Gaussian, the higher-order moments will not necessarily vanish, and we can obtain more information about the distribution from them. It turns out that $\mathcal{M}^{(k)}(G)$ uniquely identifies the DAG G [Shimizu et al., 2006a, Shimizu et al., 2006b, Wang and Drton, 2019]. An implicit description of $\mathcal{M}^{(k)}(G)$ is not known completely, although [Wang and Drton, 2019] discover enough of the defining equations to identify the DAG G .

3 Multi-trek separation for directed acyclic graphs

In this section we present our main result, a particular type of constraint on the model $\mathcal{M}^{(k)}(G)$ that corresponds to a combinatorial criterion in the graph G . Given data, one can check whether the constraint holds for the sample cumulant tensors in order to obtain information about the structure of the unknown DAG G . Our result generalizes to the case of hidden variables as shown in Section 4.

3.1 Multi-treks and the multi-trek rule

We begin by generalizing the notion of a *trek* [Sullivant et al., 2010].

Definition 5. A k -trek in a DAG G between k nodes v_1, \dots, v_k is an ordered collection of k directed paths (P_1, \dots, P_k) , where P_i has sink v_i , and P_1, \dots, P_k have the same source vertex, called the *top* of the k -trek and denoted by $\text{top}(P_1, \dots, P_k)$.

Note that a 2-trek is exactly the same as the usual notion of a trek [Sullivant et al., 2010].

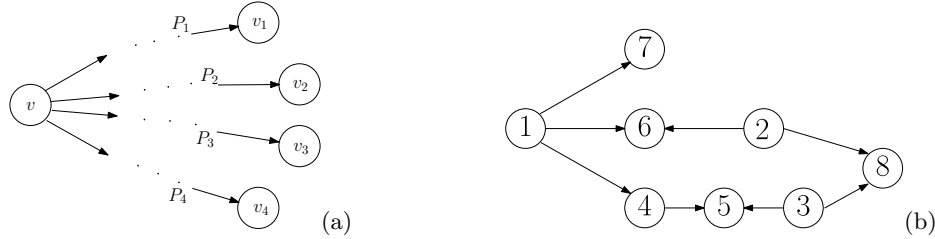


Figure 3: (a). An example of a 4-trek; (b) Example DAG.

Example 6. In the directed acyclic graph from Figure 3b,

$(2 \rightarrow 6, 2 \rightarrow 8)$ is a 2-trek between 6 and 8 with top 2;

$(1 \rightarrow 7, 1 \rightarrow 6, 1 \rightarrow 4 \rightarrow 5)$ is a 3-trek between 7, 6, and 5 with top 1;

$(1 \rightarrow 7, 1 \rightarrow 6, 1 \rightarrow 1, 1 \rightarrow 6)$ is a 4-trek between 7, 6, 1, and 6 with top 1.

We now generalize the *trek rule*, which originates in the work of [Wright, 1921, Wright, 1934] and relies on the observation that $(I - \Lambda)^{-1} = I + \Lambda + \Lambda^2 + \dots$. Since the graph G is acyclic, this formal sum is finite, and each of the entries of $(I - \Lambda)^{-1}$ can be expressed as

$$(I - \Lambda)_{ji}^{-1} = \sum_{P \in \mathcal{P}(j,i)} \lambda^P \quad (6)$$

where $\mathcal{P}(j, i)$ is the set of all directed paths from j to i , and λ^P is the product of the coefficients λ_{uv} along the edges $u \rightarrow v$ on a directed path P (e.g., if P is the path $1 \rightarrow 2 \rightarrow 4$, then λ^P is $\lambda_{12}\lambda_{24}$).

Proposition 7 (The multi-trek rule). *When the entries of the noise vector ε are independent, the entries of the k -th order cumulant tensor $\mathcal{C}^{(k)}$ of X can be expressed as a sum over k -trek monomials,*

$$\mathcal{C}_{i_1, \dots, i_k}^{(k)} = \sum_{(P_1, \dots, P_k) \in \mathcal{T}(i_1, \dots, i_k)} \mathcal{E}_{\text{top}(P_1, \dots, P_k), \dots, \text{top}(P_1, \dots, P_k)}^{(k)} \lambda^{P_1} \dots \lambda^{P_k}, \quad (7)$$

where $\mathcal{T}(i_1, \dots, i_k)$ is the set of all k -treks between i_1, \dots, i_k .

Proof. By Lemma 1, we can express the k -th cumulant tensor of X via the Tucker decomposition $\mathcal{C}^{(k)} = \mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}$. Furthermore, by Lemma 2, the k -th cumulant tensor $\mathcal{E}^{(k)}$ of ε is diagonal. Therefore,

$$\begin{aligned} \mathcal{C}_{i_1, \dots, i_k}^{(k)} &= \sum_{j_1, \dots, j_k} \mathcal{E}_{j_1, \dots, j_k}^{(k)} ((I - \Lambda)^{(-1)})_{j_1, i_1} \cdots ((I - \Lambda)^{(-1)})_{j_k, i_k} \\ &= \sum_j \mathcal{E}_{j, \dots, j}^{(k)} ((I - \Lambda)^{(-1)})_{j, i_1} \cdots ((I - \Lambda)^{(-1)})_{j, i_k}. \end{aligned}$$

Using equation (6), we obtain the result. \blacksquare

Example 8. Consider the DAG from Figure 3b. According to the multi-trek rule, we have, for instance, the following relationships between the cumulants $\mathcal{C}^{(k)}$ and $\mathcal{E}^{(k)}$ of X and ε .

$$\mathcal{C}_{4,5}^{(2)} = \mathcal{E}_{4,4}^{(2)} \lambda_{45} + \mathcal{E}_{1,1}^{(2)} \lambda_{14}^2 \lambda_{45}, \quad \mathcal{C}_{5,6,7}^{(3)} = \mathcal{E}_{1,1,1}^{(3)} \lambda_{14} \lambda_{45} \lambda_{16} \lambda_{17}, \quad \mathcal{C}_{5,6,8}^{(3)} = 0.$$

These follow because there are two 2-treks between 4 and 5: $(4, 4 \rightarrow 5)$ and $(1 \rightarrow 4, 1 \rightarrow 4 \rightarrow 5)$. There is one 3-trek between 5, 6, and 7: $(1 \rightarrow 4 \rightarrow 5, 1 \rightarrow 6, 1 \rightarrow 7)$. There are no tri-treks between 5, 6, and 8.

3.2 Multi-trek systems and determinants of higher-order cumulants

We now generalize the multi-trek rule, giving an expression of the *determinants* of subtensors of $\mathcal{C}^{(k)}$ in terms of *multi-trek systems*. We extend the notion of a matrix determinant to the case of tensors.

Definition 9. Let T be an order- k $n \times \dots \times n$ tensor. Then, its determinant is

$$\det(T) = \sum_{\sigma_1, \dots, \sigma_{k-1} \in \mathfrak{S}(n)} \text{sign}(\sigma_1) \cdots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p T_{i, \sigma_1(i), \dots, \sigma_{k-1}(i)}, \quad (8)$$

where $\mathfrak{S}(n)$ is the set of permutations of the set $\{1, \dots, n\}$.

Next, we introduce the notion of a multi-trek system.

Definition 10. Given a collection of k sets of nodes $S_1, \dots, S_k \subseteq V$ such that $\#S_1 = \dots = \#S_k = n$, a k -trek system T is a collection of n k -treks between S_1, \dots, S_k . We define the *top* of this k -trek system, $\text{top}(T)$, to be the union of the tops of the k -treks. We allow repeated elements in $\text{top}(T)$. A k -trek system T has *sided intersection* if there exist two k -treks (P_1, \dots, P_k) and (Q_1, \dots, Q_k) in T and a number $1 \leq i \leq k$ so that the directed paths P_i and Q_i have a common vertex. We denote by $\tilde{T}(S_1, \dots, S_k)$ the set of k -trek systems between S_1, \dots, S_k that have *no* sided intersection.

Example 11. In Figure 4 below, the two 3-treks between S_1, S_2, S_3 have a sided intersection along the paths leading to the set S_1 .

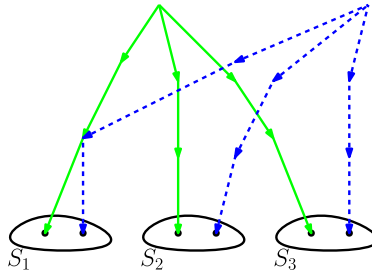


Figure 4

Given a collection of k sets of nodes $S_1, \dots, S_k \subseteq V$ such that $\#S_1 = \dots = \#S_k = n$, and an ordering of the nodes in each set S_i , a k -trek system T gives rise to a permutation of the nodes in each of S_2, \dots, S_k (if we keep the ordering of S_1 fixed). The *sign* of T is the product of the signs of those $(k-1)$ permutations.

Example 12. In Figure 5 below, $S_1 = \{v_1, v_2\}$, $S_2 = \{v_3, v_4\}$, $S_3 = \{v_5, v_6\}$. Assume that the initial ordering is $(v_1, v_2), (v_3, v_4), (v_5, v_6)$. The trek system in Figure 5a has two 3-treks, one between v_1, v_4, v_5 and one between v_2, v_3, v_6 . Therefore, $\text{sign}(T) = -1$. The trek system in Figure 5b has two 3-treks, one between v_1, v_4, v_6 and one between v_2, v_3, v_5 . Therefore, $\text{sign}(T) = +1$.

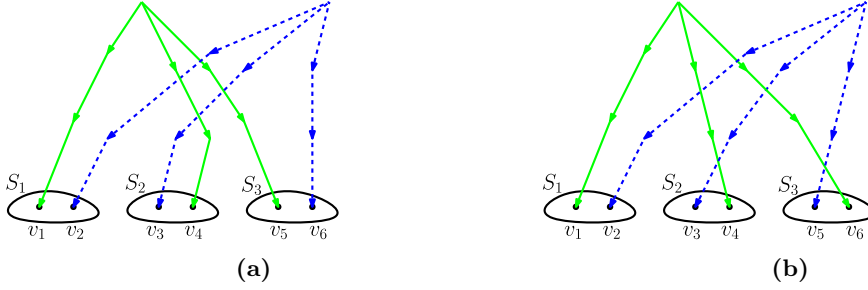


Figure 5

The determinant of the subtensor of k^{th} order cumulants indexed by the sets S_1, \dots, S_k can be expressed in terms of the trek systems involving S_1, \dots, S_k .

Proposition 13. Let $S_1, \dots, S_k \subseteq V$ be k sets of nodes such that $\#S_1 = \dots = \#S_k = n$. Then,

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{T \in \mathcal{T}(S_1, \dots, S_k)} \text{sign}(T) m_T, \quad (*)$$

where $\mathcal{T}(S_1, \dots, S_k)$ is the set of k -trek systems between S_1, \dots, S_k , and m_T is the trek-system monomial of the trek system $T = \{(P_1^{(1)}, \dots, P_k^{(1)}), \dots, (P_1^{(n)}, \dots, P_k^{(n)})\}$, defined as

$$m_T = \prod_{i=1}^n \mathcal{E}_{\text{top}((P_1^{(i)}, \dots, P_k^{(i)})), \dots, \text{top}((P_1^{(i)}, \dots, P_k^{(i)}))}^{(k)} \prod_{j=1}^k \lambda^{P_j^{(i)}}.$$

In fact, the sum in $(*)$ can be taken over treks $T \in \tilde{\mathcal{T}}(S_1, \dots, S_k)$ without sided intersections, i.e.,

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{T \in \tilde{\mathcal{T}}(S_1, \dots, S_k)} \text{sign}(T) m_T. \quad (9)$$

Example 14. Consider, once again, the DAG from Figure 3b. Then, we have that

$$\det \mathcal{C}_{46, 78}^{(2)} = \mathcal{E}_{1,1}^{(2)} \mathcal{E}_{2,2}^{(2)} \lambda_{14} \lambda_{17} \lambda_{26} \lambda_{28}.$$

since there is only one 2-trek system between 46 and 78 without sided intersection, namely $\{(1 \rightarrow 4, 1 \rightarrow 7), (2 \rightarrow 6, 2 \rightarrow 8)\}$. Similarly, we have

$$\det \mathcal{C}_{46, 58, 78}^{(3)} = \mathcal{E}_{1,1,1}^{(3)} \mathcal{E}_{2,2,2}^{(3)} \lambda_{14}^2 \lambda_{45} \lambda_{17} \lambda_{26} \lambda_{28}^2,$$

since there is only one 3-trek system between 46, 58, and 78, namely $\{(1 \rightarrow 4, 1 \rightarrow 4 \rightarrow 5, 1 \rightarrow 7), (2 \rightarrow 6, 2 \rightarrow 8, 2 \rightarrow 8)\}$.

3.3 Main result

Our main theorem shows that the non-existence of a trek system without sided intersection between k sets of vertices is equivalent to the vanishing of a specific subdeterminant of the k -th cumulant tensor $\mathcal{C}^{(k)}$.

Theorem 15. Let $G = (V, \mathcal{D})$ be a DAG, and let S_1, \dots, S_k be subsets of V with $\#S_1 = \dots = \#S_k$. Then,

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = 0$$

for every $\mathcal{C}^{(k)}$ from $\mathcal{M}^{(k)}(G)$ if and only if every system of k -treks between S_1, \dots, S_k has a sided intersection.

The proof of Theorem 15 can be found in Appendix B. The proof relies on an application of the Cauchy-Binet formula for tensors. This formula is of independent mathematical interest and is proven in Appendix A.

Example 16. Theorem 15 enables us to determine whether random variables have a common cause. Consider the graphs in Figures 6a and 6b below.

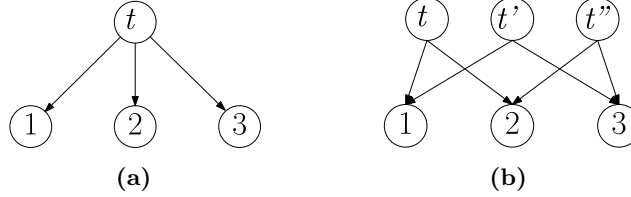


Figure 6

Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. In Figure 6a there is one tri-trek joining A, B and C , thus, $\det(\mathcal{C}_{ABC}^{(3)}) = \mathcal{C}_{123}^{(3)} \neq 0$. In Figure 6b there is no tri-trek joining A, B , and C and, a fortiori, no tri-trek without sided intersection. Therefore, $\det(\mathcal{C}_{ABC}^{(3)}) = 0$.

The seminal paper [Sullivant et al., 2010] shows that the vanishing of determinants of the covariance matrix of X is equivalent to a 2-trek separation criterion in the graph G . In the rest of this section, we illustrate that a generalization of this criterion to the case $k > 2$ only works in one direction.

Definition 17. The collection of sets (A_1, \dots, A_k) k -trek-separates S_1, \dots, S_k if for every k -trek with paths (P_1, \dots, P_k) between S_1, \dots, S_k , there exists $j \in \{1, \dots, k\}$ such that P_j contains a vertex from A_j .

Theorem 18 ([Sullivant et al., 2010, Theorem 2.8]). *The submatrix $\Sigma_{A,B}$ has rank less than or equal to r for all covariance matrices consistent with the graph G if and only if there exist subsets $C_A, C_B \subset V$ with $\#C_A + \#C_B \leq r$ such that (C_A, C_B) 2-trek-separates A from B . Consequently,*

$$\text{rk}(\Sigma_{A,B}) \leq \min\{\#C_A + \#C_B : (C_A, C_B) \text{ 2-trek-separates } A \text{ from } B\}$$

and equality holds for generic covariance matrices in the model $\mathcal{M}^{(2)}(G)$.

In Corollary 19, we show that when $k \geq 3$, k -trek-separation implies the vanishing of the corresponding cumulant tensor determinant (but not necessarily vice-versa).

Corollary 19. *Consider k sets of vertices S_1, \dots, S_k with $\#S_1 = \dots = \#S_k = n$. For all tensors $\mathcal{C}^{(k)}$ of k^{th} -order cumulants consistent with the graph G , the subtensor $\mathcal{C}_{S_1, \dots, S_k}^{(k)}$ has a null determinant if there exist subsets $A_1, \dots, A_k \subset V$ with $\#A_1 + \dots + \#A_k < n$ such that (A_1, \dots, A_k) k -trek separates S_1, \dots, S_k .*

Proof. Let us suppose that there exist A_1, \dots, A_k such that $\#A_1 + \dots + \#A_k < n$ and (A_1, \dots, A_k) k -trek-separates S_1, \dots, S_k . Suppose there exists a k -trek system with no sided intersection $T = (T_1, \dots, T_n)$ between S_1, \dots, S_k . Then, for every $i = 1, \dots, n$, there exists m_i such that the m_i -th component of T_i intersects A_{m_i} . Since $\#A_1 + \dots + \#A_k < n$, by the Pigeon-Hole Principle, there exist $i \neq j$ such that $m = m_i = m_j$, and the m -th components of T_i and T_j go through the same element $s \in A_m$. Therefore, T has a sided intersection. Contradiction! Thus, by Theorem 15, $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = 0$. ■

When $k \geq 3$, the reverse implication of Corollary 19 is not true, i.e., $\det(\mathcal{C}_{S_1, \dots, S_k}^{(k)}) = 0$ does not imply that there exists (A_1, \dots, A_k) such that $\#A_1 + \dots + \#A_k < n$ and (A_1, \dots, A_k) k -trek-separates (S_1, \dots, S_k) . Consider the graph in Figure 7 below. In this graph, $\#S_1 = \#S_2 = \#S_3 = 2$. There is no system of two 3-treks between S_1, S_2, S_3 without sided intersection. However, it is not possible to find three sets A_1, A_2, A_3 such that $\#A_1 + \#A_2 + \#A_3 < 2 = \#S_i$, i.e., there is no such set that 3-trek-separates S_1, S_2, S_3 . Intuitively, this happens because "max flow" and "min cut" are no longer equal, while the reason the statement holds when $k = 2$ is precisely because of the min-cut max flow (Menger's) Theorem.

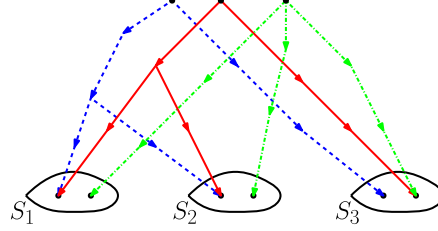


Figure 7

4 Hidden variables

We now consider the case where the structural equation model (3) involves some hidden (i.e., unobserved) variables. Alternatively, it is similar to consider that the noise variables ε_i 's are now correlated. We represent such a case with a mixed graph $G = (V, \mathcal{D}, \mathcal{H})$, where V is the set of vertices, $\mathcal{D} \subseteq V \times V$ is the set of directed edges, and \mathcal{H} is the set of *multidirected edges* (see Definition 20 below), which signify the dependencies between the ε variables. We assume that G does not contain any cycle nor loop.

Definition 20. A multidirected edge between nodes i_1, \dots, i_k is the union of k directed edges with the same source and with sinks i_1, \dots, i_k . The k -directed edges are merged at their source without an additional node. We call k the *order* of the multidirected edge.

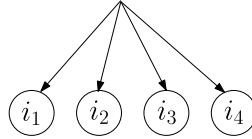


Figure 8: A multidirected edge between i_1, i_2, i_3, i_4 of order 4.

Remark 21. Note that this is a generalization of the notion of bidirected edges widely used in the literature.

As before, define $\mathcal{E}^{(k)}$ to be the tensor of k -th order cumulants of the variables ε_i 's. Since we do not assume the independence of the variables ε_i 's, then any entry of the tensor $\mathcal{E}^{(k)}$ may be non-zero. Specifically, $\mathcal{E}^{(k)}$ is a $\underbrace{V \times V \times \dots \times V}_k$ tensor for which the entry w_{i_1, \dots, i_k} is non-zero if there exists a multidirected edge between

a_1, \dots, a_l such that $\{i_1, \dots, i_k\} \subseteq \{a_1, \dots, a_l\}$. Note that some of the elements i_1, \dots, i_k could be equal.

We now define the notion of a k -trek in this setting.

Definition 22. A k -trek between vertices i_1, \dots, i_k in a mixed graph G is composed of k directed paths (P_1, \dots, P_k) where P_s goes from j_s to i_s , and the *tops* j_1, \dots, j_k are connected in one of the following ways.

- (a) Either the vertices j_1, \dots, j_k coincide to form the top of the trek; or
- (b) There exist an l -directed edge between a_1, \dots, a_l such that $\{j_1, \dots, j_k\} \subseteq \{a_1, \dots, a_l\}$.

Example 23. In the mixed graph from Figure 9a, $(1 \rightarrow 7, 1 \rightarrow 6, 1 \rightarrow 4 \rightarrow 5)$ is a 3-trek between 7, 6 and 5. The tops of the paths going to the nodes 7, 6 and 5 coincide with node 1. In the mixed graph from Figure 9b, $(7, 6, 4 \rightarrow 5)$ is a 3-trek between 7, 6 and 5. The tops of the paths are connected by a 3-directed edge between 4, 6, and 7.

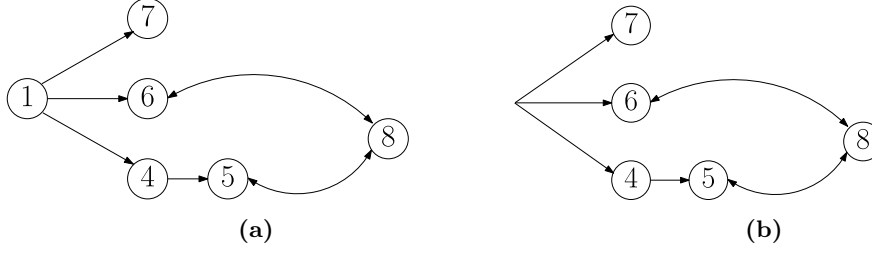


Figure 9

The tensor of k -th order cumulants of a mixed graphical model is then given by the equation (1), $\mathcal{C}^{(k)} = \mathcal{E}^{(k)}(I - \Lambda)^{-k}$. In a similar way to Proposition 7, every entry of the tensor $\mathcal{C}^{(k)}$ can be expressed as a sum of k -trek monomials.

Corollary 24. For a noise vector ε whose entries are dependent, the entries of the k -th order cumulant tensor $\mathcal{C}^{(k)}$ of X can be expressed as a sum over k -trek monomials,

$$\mathcal{C}_{i_1, \dots, i_k}^{(k)} = \sum_{(P_1, \dots, P_k) \in \mathcal{T}(i_1, \dots, i_k)} \mathcal{E}_{top(P_1), \dots, top(P_k)}^{(k)} \lambda^{P_1} \dots \lambda^{P_k}, \quad (10)$$

where $\mathcal{T}(i_1, \dots, i_k)$ is the set of all k -treks between i_1, \dots, i_k .

In a similar fashion to the directed acyclic case, we obtain the following result.

Theorem 25. Consider a mixed graph $G(V, \mathcal{D}, \mathcal{H})$. Let S_1, \dots, S_k be subsets of V with $\#S_1 = \dots = \#S_k$. Then,

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = 0$$

if and only if every system of k -treks between S_1, \dots, S_k has a sided intersection.

Theorem 25 is proven in Appendix C

Example 26. The following example shows that Theorem 25 enables us to determine whether random variables have a common cause.

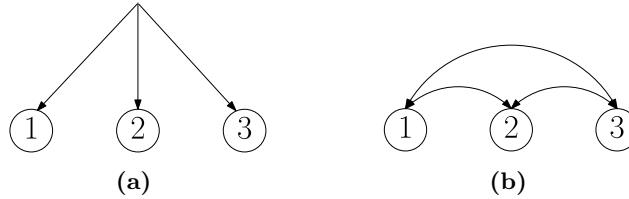


Figure 10

In the two graphs from Figure 10, let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. In Figure 10a, there is one tri-trek joining A, B , and C , hence $\det(\mathcal{C}_{ABC}^{(3)}) = \mathcal{C}_{123}^{(3)} \neq 0$. In Figure 10b, there is no tri-trek joining A, B and C and, a fortiori, no tri-trek without sided intersection. Therefore $\det(\mathcal{C}_{ABC}^{(3)}) = 0$.

5 Determinants of higher-order moments and multi-trek systems

At the start of this project, we focused on tensors of higher-order moments rather than cumulants. However, contrary to cumulant tensors (cf. Lemma 2), moment tensors of order greater than 3 are not diagonal when the variables are independent. Extending Theorem 15 to higher-order moment tensors is therefore not straightforward. Nevertheless, we conjecture that the result still holds. Before stating this conjecture, we translate our results obtained for cumulant tensors to moment tensors.

Let $\Phi^{(k)}$ be the tensor of k^{th} -order moments of ε . Then, the k^{th} -order moment tensor of X is given as follows.

Proposition 27. *The tensor $\mathcal{N}^{(k)}$ of k^{th} -order moments of the random vector X with mean $(0, \dots, 0)$ equals*

$$\mathcal{N}^{(k)} = \Phi^{(k)} \bullet (I - \Lambda)^{-k}. \quad (11)$$

Let $G = (V, \mathcal{D}, \mathcal{H})$ as in Section 4. In order to account for the non-zero off-diagonal entries in the tensor $\Phi^{(k)}$ of higher-order moments of ε , we need to adapt our definition of a k -trek as follows.

Definition 28. A k -split-trek in G between k nodes v_1, \dots, v_k is either:

- (a) an ordered collection of k directed paths (P_1, \dots, P_k) where P_i has sink v_i , and P_1, \dots, P_k have the same source; or
- (b) an ordered collection of k directed paths (P_1, \dots, P_k) where P_i has sink v_i , and P_1, \dots, P_k may have different sources, but each source must be shared by at least two paths.

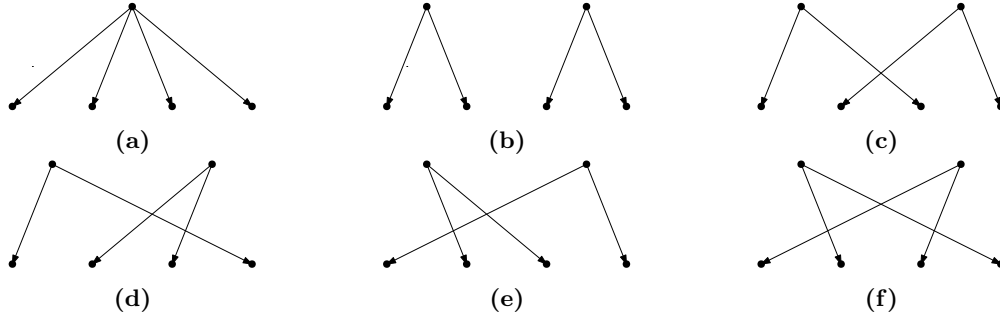


Figure 11: Types of k -split-treks for $k = 4$

Definition 29. The *top* of a k -split-trek (P_1, \dots, P_k) in G is either:

- (a) a node that is the source of each of the k paths P_1, \dots, P_k .
- (b) a set of j nodes where each of the j nodes is a source of ℓ_j of the paths P_1, \dots, P_k such that $\sum_j \ell_j = k$ and $\ell_j \geq 2$ for all j .

Example 30. In Figure 11, subfigure (a) illustrates the first type of a top, while subfigures (b-f) illustrate the second type.

Using equation (6), we rewrite the entries of $(I - \Lambda)^{-1}$ in equation (11) and thus express the entries of the tensor of k^{th} -order moments as follows.

Proposition 31. *We have that*

$$\mathcal{N}_{i_1, \dots, i_k}^{(k)} = \sum_{(P_1, \dots, P_k) \in \mathcal{S}(i_1, \dots, i_k)} \Phi_{top(P_1, \dots, P_k), \dots, top(P_1, \dots, P_k)}^{(k)} \lambda^{P_1} \dots \lambda^{P_k} \quad (12)$$

where $\mathcal{S}(i_1, \dots, i_k)$ is the set of all k -split-treks between i_1, \dots, i_k .

Similarly to the case of higher-order cumulants, the determinant of the subtensor of k^{th} -order moments indexed by the sets S_1, \dots, S_k can be rewritten in terms of the split-trek systems between S_1, \dots, S_k .

Proposition 32. *Let $S_1, \dots, S_k \subseteq V$ be k sets of nodes such that $\#S_1 = \dots = \#S_k = n$. Then,*

$$\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = \sum_{T \in \mathcal{S}(S_1, \dots, S_k)} \text{sign}(T) m_T,$$

where m_T is the split-trek-system monomial of the split-trek system $T = \{(P_1^{(1)}, \dots, P_k^{(1)}), \dots, (P_1^{(n)}, \dots, P_k^{(n)})\}$, defined as

$$m_T = \prod_{i=1}^n \Phi_{\text{top}((P_1^{(i)}, \dots, P_k^{(i)})), \dots, \text{top}((P_1^{(i)}, \dots, P_k^{(i)}))}^{(k)} \prod_{j=1}^k \lambda_j^{P_j^{(i)}}.$$

Furthermore, the sum can be taken over the set $\tilde{\mathcal{S}}(S_1, \dots, S_k)$ of k -split-trek-systems without sided intersection.

We prove an analog of Theorem 15 for the tensors of 3^{rd} -order moments ($k = 3$).

Theorem 33. *Let S_1, S_2, S_3 be subsets of V with $\#S_1 = \#S_2 = \#S_3$. Then,*

$$\det \mathcal{N}_{S_1, S_2, S_3}^{(3)} = 0$$

if and only if every system of 3-split-treks between S_1, S_2, S_3 has a sided intersection.

The proofs of Proposition 31 and 32 and Theorem 33 can be found in Appendix D. For higher-order moments ($k > 3$), we conjecture the following theorem by analogy with Theorem 15.

Conjecture 34. *Let S_1, \dots, S_k be subsets of V with $\#S_1 = \dots = \#S_k$. Then,*

$$\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = 0$$

if and only if every system of k -split-treks between S_1, \dots, S_k has a sided intersection.

Note that the *if* direction is straightforward since we can express the determinant as a sum of split-trek monomials of k -split-trek systems without sided intersection, as in Proposition 32. Provided Conjecture 34 is true, we can show the following relationship between moment tensors of different orders.

Proposition 35. *Consider $k \geq 4$ sets of vertices $S_1, \dots, S_k \subseteq V$ such that $\#S_1 = \dots = \#S_k = n$. Let us suppose that the tensor of k -th order moments $\mathcal{N}_{S_1, \dots, S_k}^{(k)}$ indexed by S_1, \dots, S_k has null determinant: $\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = 0$. Then, for any $2 \leq h \leq k - 2$ and any partition $\{1, \dots, k\} = \{i_1, \dots, i_h\} \cup \{j_1, \dots, j_{k-h}\}$, either the determinant of the h -th order moment tensor $\mathcal{N}_{S_{i_1}, \dots, S_{i_h}}^{(h)}$ is zero, or the determinant of the $(k-h)$ -th order moment tensor $\mathcal{N}_{S_{j_1}, \dots, S_{j_{k-h}}}^{(k-h)}$ is zero.*

Proof. Let us suppose that $\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = 0$, but there exists $2 \leq h \leq k - 2$ and a partition $\{1, \dots, k\} = \{i_1, \dots, i_h\} \cup \{j_1, \dots, j_{k-h}\}$ such that both $\det \mathcal{N}_{S_{i_1}, \dots, S_{i_h}}^{(h)} \neq 0$ and $\det \mathcal{N}_{S_{j_1}, \dots, S_{j_{k-h}}}^{(k-h)} \neq 0$. Then, by Conjecture 34 there exists an h -split-trek system T_1 with no sided intersection between S_{i_1}, \dots, S_{i_h} and a $(k-h)$ -split-trek system T_2 with no sided intersection between $S_{j_1}, \dots, S_{j_{k-h}}$. But then, combining together T_1 and T_2 , we get a valid k -split-trek system between S_1, \dots, S_k with no sided intersection, which contradicts the assumption $\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = 0$. Therefore, one of $\det \mathcal{N}_{S_{i_1}, \dots, S_{i_h}}^{(h)} = 0$ and $\det \mathcal{N}_{S_{j_1}, \dots, S_{j_{k-h}}}^{(k-h)} = 0$ is true. ■

Example 36. We illustrate Proposition 35 for the case $k = 4$ and $h = 2$. In the graph from Figure 12(a), $\det \mathcal{N}_{S_1, S_2, S_3, S_4}^{(4)} = 0$. In Figure 12(b), $\det \mathcal{N}_{S_1, S_2}^{(2)} = 0$ and in Figure 12(c), $\det \mathcal{N}_{S_3, S_4}^{(2)} \neq 0$.

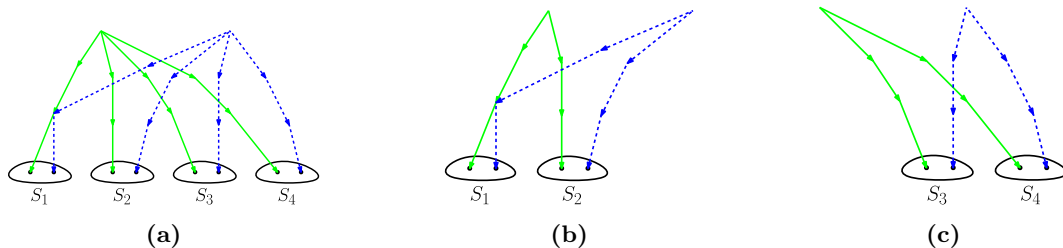


Figure 12

Proposition 35 would give a nice relationship between the vanishing of determinants of high-order moment tensors and low-order moment tensors. One of our initial hopes was that in the case of Gaussian random variables, the vanishing of high-order moment determinants would be able to explain constraints in the model that are not subdeterminants of the covariance matrix (sometimes called Verma constraints) [Verma and Pearl, 1991]. However, Proposition 35 implies that if a high-order moment determinant vanishes, then so does a lower-order one. On the other hand there are lots of Gaussian graphical models, for which there are no covariance determinants vanishing [Drton et al., 2018], thus the vanishing of determinants would not suffice to describe the model.

6 Conclusion

In this paper, we give implicit constraints on linear non-Gaussian structural equation models by providing a relationship between the vanishing of subdeterminants of the tensors of k -th order cumulants and a combinatorial criterion on the corresponding graph. Specifically, we show that the determinant of the subtensor of the k -th order cumulants for k sets of vertices with equal cardinality vanishes if and only if there is no system of k -treks between these sets without sided intersection. One of the main contributions of our work is the introduction of multi-directed edges in the hidden variable case, and our multi-trek criterion which allows us to, for example, detect the presence of a common cause of multiple vertices.

A few questions for further research remain. As shown in Example 16, Theorem 15 gives a criterion for checking if random variables have a common cause or not. It would be interesting to build a test statistic based on this criterion. Furthermore, our criterion can be coupled with existing algorithms to recover a mixed graph with multi-directed edges from observational data.

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A Proof of Cauchy-Binet for tensors

We prove a tensor version of the Cauchy-Binet Theorem [Broida and Williamson, 1989] for the determinant of the product AB , where A is a matrix and B is a tensor of k -th order. We then apply this proposition to the Tucker decomposition of tensors $\mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}$ in the proof of Proposition 13.

Proposition 37. *Suppose that A is a $\underbrace{p \times n \times p \times \dots \times p}_k$ tensor of order k and B is a $n \times p$ matrix, I and J are subsets of $\{1, \dots, p\}$. Then $\det(AB) = \sum_{I \subseteq [n], \#I=p} \det(A_I) \det(B_I)$, where the sum is over subsets I of $\{1, \dots, n\}$ such that $\#I = p$, and A_I denotes the subtensor of A given by $A_{[p], I, [p], \dots, [p]}$, and B_I denotes the submatrix of B given by $B_{I, [p]}$.*

Proof. Note that the entry $c_{i_1 \dots i_k}$ of the product $C = AB$ is given by:

$$c_{i_1 \dots i_k} = \sum_l A_{i_1 l i_3 \dots i_k} b_{l i_2}. \quad (13)$$

Let’s adopt the following notation:

- F is the set of functions with domain $[n] := \{1, \dots, n\}$ and range $[n] := \{1, \dots, n\}$
- G is the set of injective functions that map p ($p < n$) elements in $[n]$ to p elements in $[n]$
- H is the set of strictly increasing functions that map p ($p < n$) elements in $[n]$ to p elements in $[n]$
- \mathfrak{S} is the set of permutations of the elements $\{1, \dots, p\}$

Using Definition 9 for the tensor determinant, we have:

$$\begin{aligned}
\det(AB) &= \sum_{\sigma_2, \dots, \sigma_{k-1}} \text{sign}(\sigma_2) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p (AB)_{i\sigma_2(i) \dots \sigma_{k-1}(i)} \\
&= \sum_{\sigma_2, \dots, \sigma_{k-1}} \text{sign}(\sigma_2) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p \sum_l A_{il\sigma_3(i) \dots \sigma_{k-1}(i)} B_{l\sigma_2(i)} \\
&= \sum_{\sigma_2, \dots, \sigma_{k-1}} \text{sign}(\sigma_2) \dots \text{sign}(\sigma_{k-1}) \sum_{f \in F} \prod_{i=1}^p A_{if(i)\sigma_3(i) \dots \sigma_{k-1}(i)} B_{f(i)\sigma_2(i)} \\
&= \sum_{f \in F} \left(\sum_{\sigma_3, \dots, \sigma_{k-1}} \text{sign}(\sigma_3) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p A_{if(i)\sigma_3(i) \dots \sigma_{k-1}(i)} \right) \left(\sum_{\sigma_2} \text{sign}(\sigma_2) \prod_{i=1}^p B_{f(i)\sigma_2(i)} \right) \\
&= \sum_{f \in F} \left(\sum_{\sigma_3, \dots, \sigma_{k-1}} \text{sign}(\sigma_3) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p A_{if(i)\sigma_3(i) \dots \sigma_{k-1}(i)} \right) \det(B_f) \\
&= \sum_{f \in G} \left(\sum_{\sigma_3, \dots, \sigma_{k-1}} \text{sign}(\sigma_3) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p A_{if(i)\sigma_3(i) \dots \sigma_{k-1}(i)} \right) \det(B_f) \text{ because } \det(B_f) = 0 \text{ for } f \notin G \\
&= \sum_{h \in H} \sum_{\gamma \in \mathfrak{S}} \left(\sum_{\sigma_3, \dots, \sigma_{k-1}} \text{sign}(\sigma_3) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p A_{ih(\gamma(i))\sigma_3(i) \dots \sigma_{k-1}(i)} \right) \det(B_{h(\gamma)})
\end{aligned} \tag{14}$$

Since $A_{ih(\gamma(i))\sigma_3(i) \dots \sigma_{k-1}(i)} = (A_{i\gamma(i)\sigma_3(i) \dots \sigma_{k-1}(i)})_h$ where $(A_{i\gamma(i)\sigma_3(i) \dots \sigma_{k-1}(i)})_h$ is the submatrix of A with columns selected by h , and $\det(B_{h(\gamma)}) = \text{sign}(\gamma) \det(B_h)$, we have:

$$\begin{aligned}
\det(AB) &= \sum_{h \in H} \left(\sum_{\gamma \in \mathfrak{S}} \sum_{\sigma_3, \dots, \sigma_{k-1}} \text{sign}(\gamma) \text{sign}(\sigma_3) \dots \text{sign}(\sigma_{k-1}) \prod_{i=1}^p (A_{i\gamma(i)\sigma_3(i) \dots \sigma_{k-1}(i)})_h \right) \det(B_h) \\
&= \sum_{h \in H} \det(A_h) \det(B_h).
\end{aligned} \tag{15}$$

■

B Proof of Theorem 15

The starting point of the proof of Theorem 15 is Lemma 1 that gives the relationship between the k -th order cumulant tensors of the random vectors ε and X , $\mathcal{C}^{(k)} = \mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k}$. We first apply the determinant operator to each side of this equation, using the tensor version of the Cauchy-Binet theorem proved above.

Proposition 38. *Consider k subsets of vertices $S_1, \dots, S_k \subset V$ with $\#S_1 = \dots = \#S_k$. Then, the determinant of the tensor of k^{th} -order cumulants can be written as:*

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{R_1, \dots, R_k} \det \mathcal{E}_{R_1, \dots, R_k}^{(k)} \det (I - \Lambda)_{R_1, S_1}^{-1} \dots \det (I - \Lambda)_{R_k, S_k}^{-1} \tag{16}$$

Proof. We apply the tensor version of the Cauchy-Binet Theorem stated in Proposition 37 k times to the Tucker product in equation (5) and we obtain equation (16). ■

We can now prove Proposition 13 which gives a combinatorial expression for the determinant of the k -th order cumulant tensor of X in terms of the DAG G .

Proof of Proposition 13. By Proposition 38, we have:

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{R_1, \dots, R_k} \det \Phi_{R_1, \dots, R_k} \det(I - \Lambda)_{R_1, S_1}^{-1} \dots \det(I - \Lambda)_{R_k, S_k}^{-1} \quad (17)$$

Additionally, from Proposition 7, we can write

$$\mathcal{C}_{i_1, \dots, i_k}^{(k)} = \sum_{T \in \mathcal{T}(i_1, \dots, i_k)} m_T, \quad (18)$$

where m_T is the k -trek monomial defined by $m_T = \phi_{\text{top}(P_1, \dots, P_k)} \lambda^{P_1} \dots \lambda^{P_k}$, where $\lambda^{P_j} = \prod_{k \rightarrow l \in P_j} \lambda_{kl}$.

Assuming that $\#R_1 = \dots = \#R_k = \#S_1 = \dots = \#S_k = n$ and using the Leibniz expansion formula for determinants, we then get:

$$\begin{aligned} \det \mathcal{C}_{S_1, \dots, S_k}^{(k)} &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}, \dots, \sigma_{k-1} \in \mathfrak{S}_{n_{k-1}}} \left(\sum_{\substack{T_1 \in \mathcal{T}(s_1, s_{\sigma_1(1)}, \dots, s_{\sigma_{k-1}(1)}) \\ \vdots \\ T_n \in \mathcal{T}(s_n, s_{\sigma_1(n)}, \dots, s_{\sigma_{k-1}(n)})}} \dots \text{sign}(\sigma_1) \dots \text{sign}(\sigma_{k-1}) m_{T_1} \dots m_{T_n} \right) \\ &= \sum_{\substack{\sigma_i \in \mathfrak{S}_{n_i} \\ i \in \{1, \dots, k-1\}}} \left(\sum_{\substack{T_j \in \mathcal{T}(s_j, s_{\sigma_1(j)}, \dots, s_{\sigma_{k-1}(j)}) \\ j \in \{1, \dots, n\}}} \prod_{l=1}^{k-1} \text{sign}(\sigma_l) \prod_{s=1}^n m_{T_s} \right) \\ &= \sum_{T \in \mathcal{T}(S_1, \dots, S_k)} \text{sign}(T) m_T, \end{aligned} \quad (19)$$

where \mathfrak{S}_{n_i} is the set of permutations of the nodes in S_{i+1} , T runs over all k -trek systems between S_1, \dots, S_k and $\text{sign}(T) = \text{sign}(\sigma_1) \dots \text{sign}(\sigma_{k-1})$. In this expression, we have $m_T = \prod_{x=1}^n m_{T_x}$, where $m_{T_x} = \Phi_{\text{top}(P_{s_1}, \dots, P_{s_k})} \lambda^{P_{s_1}} \dots \lambda^{P_{s_k}}$ is the trek monomial corresponding to the trek T_x . ■

In order to now prove the equation (9)

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{T \in \tilde{\mathcal{T}}(S_1, \dots, S_k)} \text{sign}(T) m_T,$$

we need to show that $m_T = 0$ when T is a trek-system between S_1, \dots, S_k with a sided intersection. A nice tool to proof this is given by the Gessel-Viennot-Lindstrom Lemma. This Lemma is actually a cornerstone for our paper since we use it again in the proof of our main Theorem 15 itself.

Lemma 39 ([Gessel and Viennot, 1985][Lindström, 1973]). *Suppose G is a directed acyclic graph with vertex set $[p] = \{1, \dots, p\}$. Let R and S be subsets of $[p]$ with $\#R = \#S = n$. Then,*

$$\det (I - \Lambda)_{R, S}^{-1} = \sum_{P \in N(R, S)} (-1)^P \lambda^P,$$

where $N(R, S)$ is the set of all collections of nonintersecting systems of n directed paths in G from R to S , and $(-1)^P$ is the sign of the induced permutation of elements from R to S . In particular, $\det (I - \Lambda)_{R, S}^{-1}$ is identically 0 if and only if every system of n directed paths from R to S has two paths which share a vertex.

Therefore, only the terms that involve trek-systems without sided intersection remain in equation (19), which completes the proof of equation (9).

We need a last intermediary result before proving Theorem 15.

Lemma 40. *Consider k subsets of vertices $S_1, \dots, S_k \subseteq V$ with $\#S_1 = \dots = \#S_k = n$. Then, $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)}$ is identically 0 if and only if for any set $R \subseteq V$ such that $\#R = n$, there exists $i \in \{1, \dots, k\}$ with $\det (I - \Lambda)_{R, S_i}^{-1} = 0$.*

Proof. Let us suppose that $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)}$ is identically 0. Using equation (1) and the Cauchy-Binet theorem, we compute $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)}$. We obtain equation (16):

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{R_1, \dots, R_k} \det \mathcal{E}_{R_1, \dots, R_k}^{(k)} \det(I - \Lambda)_{R_1, S_1}^{-1} \cdots \det(I - \Lambda)_{R_k, S_k}^{-1},$$

where the sum runs over subsets R_1, \dots, R_k of V of cardinality $\#R_1 = \dots = R_k = \#S_1 = \dots = \#S_k = n$. However, since $\mathcal{E}^{(k)}$ is a diagonal tensor, $\det \mathcal{E}_{R_1, \dots, R_k}^{(k)} = 0$ unless $R_1 = R_2 = \dots = R_k = R$. In this case, denoting $\det \mathcal{E}_{R_1, \dots, R_k}^{(k)} = \det \mathcal{E}_R^{(k)}$ we get:

$$\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{R \subseteq V} \det \mathcal{E}_R^{(k)} \det(I - \Lambda)_{R, S_1}^{-1} \cdots \det(I - \Lambda)_{R, S_k}^{-1} \quad (20)$$

Each monomial $\det \mathcal{E}_R^{(k)}$ appears only once, therefore for any set R satisfying $\#R = \#S_1 = \dots = \#S_k = n$, there exists $i \in \{1, \dots, k\}$ such that $\det(I - \Lambda)_{R, S_i}^{-1} = 0$, which proves the if-direction.

Let us now suppose that for any set R satisfying $\#R = \#S_1 = \dots = \#S_k = n$, there exists $i \in \{1, \dots, k\}$ such that $\det(I - \Lambda)_{R, S_i}^{-1} = 0$. Then from the expression (16), we conclude that $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = 0$. ■

Equipped with the Gessel-Viennot-Lindstrom Lemma (Lemma 39) and Lemma 40, we are now ready to prove our main theorem.

Proof of Theorem 15. Let us first suppose that $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = 0$ and let T be a k -trek system between S_1, \dots, S_k .

- If all elements of the multiset $\text{top}(T)$ are distinct, then Lemma 40 implies that there exists an integer $i \in [k]$ such that $\det(I - \Lambda)_{\text{top}(T), S_i}^{-1} = 0$. By Lemma 39, the path system from $\text{top}(T)$ to S_i has a sided intersection, therefore T has a sided intersection.
- If $\text{top}(T)$ has repeated elements, then at least two k -treks in T intersect, namely at their top, hence T has a sided intersection.

Conversely, let's suppose that every k -trek system (T) between S_1, \dots, S_k has a sided intersection. Let's consider any set $R \subseteq V$ that satisfies $\#R = \#S_1 = \dots = \#S_k$.

- If R forms the top of a k -trek system that ends at S_1, \dots, S_k , then there exists at least one integer $i \in [k]$ such that there is a sided intersection in the path system between R and S_i . By the Gessel-Viennot-Lindstrom Lemma 39, $\det(I - \Lambda)_{R, S_i}^{-1} = 0$, i.e., $\det(I - \Lambda)_{R, S_1}^{-1} \cdots \det(I - \Lambda)_{R, S_k}^{-1} = 0$ and therefore $\det \mathcal{E}_R^{(k)} \det(I - \Lambda)_{R, S_1}^{-1} \cdots \det(I - \Lambda)_{R, S_k}^{-1} = 0$.
- Alternatively, if R does not form the top of a k -trek system that ends at S_1, \dots, S_k , then there is no k -path system from R to S_1, \dots, S_k , i.e., there is no path system between R and at least one of S_1, \dots, S_k . This implies that at least one of $\det(I - \Lambda)_{R, S_i}^{-1} = 0$ and therefore $\det \mathcal{E}_R^{(k)} \det(I - \Lambda)_{R, S_1}^{-1} \cdots \det(I - \Lambda)_{R, S_k}^{-1} = 0$.

Thus, $\det \mathcal{C}_{S_1, \dots, S_k}^{(k)} = \sum_{R \subseteq V} \det \mathcal{E}_R^{(k)} \det(I - \Lambda)_{R, S_1}^{-1} \cdots \det(I - \Lambda)_{R, S_k}^{-1} = 0$. ■

C Proof of Theorem 25

Theorem 25 extends Theorem 15 to the case of mixed graphs. We use a common argument in the graphical models literature that enables us to convert the multidirected case to the directed case. We replace every multidirected edge joining i_1, \dots, i_k with a vertex v and a directed edge between v and any of the vertices i_1, \dots, i_k . We call this graph \tilde{G} , also known as the *canonical DAG* associated to G .

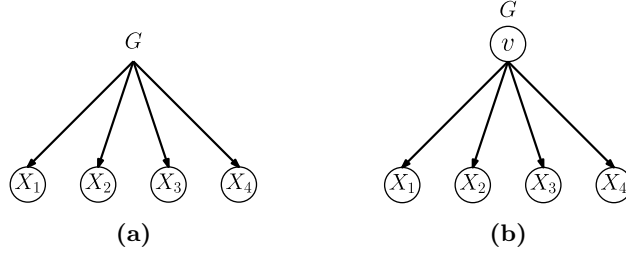


Figure 13: Mixed graph G in (a) and its corresponding canonical DAG \tilde{G} in (b)

Definition 41. The linear structural equation model $X_j = \sum_{i \in \text{pa}(j)} \lambda_{ij} X_i + \varepsilon_j$ given by a mixed graph $G = (V, \mathcal{D}, \mathcal{H})$, where $V = [p]$, is the family of distributions on \mathbb{R}^p with tensor of k^{th} -order cumulants in the set:

$$\mathcal{M}^{(k)}(G) = \{\mathcal{E}^{(k)} \bullet (I - \Lambda)^{-k} : \Lambda = (\lambda_{ij}) \in \mathbb{R}^{\mathcal{D}}, \mathcal{E}^{(k)} \in (\mathbb{R}^{\otimes k})^{\mathcal{H}}\} \quad (21)$$

where $(\mathbb{R}^{\otimes k})^{\mathcal{H}}$ denotes the set of $p \times \dots \times p$ (k -times) cumulant tensors of ε , which are zero at all entries (i_1, \dots, i_k) unless $i_1 = \dots = i_k$ or there exists a multi-directed edge $(j_1, \dots, j_\ell) \in \mathcal{H}$ such that $\{i_1, \dots, i_k\} \subseteq \{j_1, \dots, j_\ell\}$

Proposition 42. Let $S_1, \dots, S_k \subset V$ be k sets of vertices such that $\#S_1 = \dots = \#S_k$. Then the determinant of $\mathcal{C}_{S_1, \dots, S_k}^{(k)}$ is zero for all cumulant tensors $\mathcal{C} \in \mathcal{M}^{(k)}(G)$ if and only if the determinant of $\tilde{\mathcal{C}}_{S_1, \dots, S_k}^{(k)}$ is zero for all cumulant tensors $\tilde{\mathcal{C}} \in \mathcal{M}^{(k)}(\tilde{G})$. In other words, if there is no k -trek system without sided intersection between S_1, \dots, S_k in G , then there is no k -trek system without sided intersection between S_1, \dots, S_k in \tilde{G} , and vice-versa.

Proof. Let $\mathcal{M}^{(k)}(G)$ and $\mathcal{M}^{(k)}(\tilde{G})$ be the sets of k^{th} -order cumulant tensors of G and \tilde{G} , respectively. We will prove that $\mathcal{M}^{(k)}(G)$ and $\mathcal{M}^{(k)}(\tilde{G})$ have the same Zariski closure, i.e., an equation vanishes on $\mathcal{M}^{(k)}(G)$ if and only if it vanishes on $\mathcal{M}^{(k)}(\tilde{G})$. To do this, let us show that the two parametrizations give the same family of tensors near the identity tensor. Note that there exist distributions of independent variables X_1, \dots, X_p for which the k^{th} -order cumulant is the identity tensor. As shown in Proposition 31, $\mathcal{C}_{i_1, \dots, i_k}^{(k)}$ can be expressed as a sum of the trek monomials of all k -treks in $\mathcal{T}(i_1, \dots, i_k)$ between i_1, \dots, i_k in G as $\mathcal{C}_{i_1, \dots, i_k}^{(k)} = \sum_{(P_1, \dots, P_k) \in \mathcal{T}(i_1, \dots, i_k)} \phi_{\text{top}(P_1, \dots, P_k)} \lambda^{P_1} \dots \lambda^{P_k}$. Similarly, $\tilde{\mathcal{C}}_{i_1, \dots, i_k}^{(k)}$ is the sum of the trek monomials of all k -treks in $\tilde{\mathcal{T}}(i_1, \dots, i_k)$ between i_1, \dots, i_k in \tilde{G} , i.e., $\tilde{\mathcal{C}}_{i_1, \dots, i_k}^{(k)} = \sum_{(\tilde{P}_1, \dots, \tilde{P}_k) \in \tilde{\mathcal{T}}(i_1, \dots, i_k)} \phi_{\text{top}(\tilde{P}_1, \dots, \tilde{P}_k)} \lambda^{\tilde{P}_1} \dots \lambda^{\tilde{P}_k}$.

Now, let's set

$$\mathcal{E}_{i_1, \dots, i_k}^{(k)} = \tilde{\mathcal{E}}_{v, \dots, v}^{(k)} \tilde{\lambda}_{v, i_1} \dots \tilde{\lambda}_{v, i_k} \quad (22)$$

if there is a multi-directed edge between i_1, \dots, i_k in G , i.e., there is a directed edge from the vertex v to each of the nodes $i_l, l \in \{1, \dots, k\}$ in \tilde{G} , and let

$$\mathcal{E}_{i, \dots, i}^{(k)} = \tilde{\mathcal{E}}_{i, \dots, i}^{(k)} + \sum_{w \in \mathcal{H}} \tilde{\mathcal{E}}_{v, \dots, v}^{(k)} \tilde{\lambda}_{v, i}^k \quad (23)$$

for each $i \in [p]$. As we initially assumed we are near the identity tensor $\mathcal{I}^{(k)}$, we can switch from one parametrization to the other as follows. First, note that $\mathcal{I}^{(k)} \in \mathcal{M}^{(k)}(G)$ and $\mathcal{I}^{(k)} \in \mathcal{M}^{(k)}(\tilde{G})$. Then, given $\tilde{\mathcal{E}}^{(k)}$ and $\tilde{\lambda}$, we can find $\mathcal{E}^{(k)}$ from equations (22) and (23), and $\lambda_{ij} = \tilde{\lambda}_{ij}$ for $i \rightarrow j$, i.e., given $\tilde{\mathcal{C}}^{(k)} \in \mathcal{M}^{(k)}(\tilde{G})$, using equation (22) and equation (23), we get that the corresponding $\mathcal{C}^{(k)} \in \mathcal{M}^{(k)}(G)$, therefore $\mathcal{M}^{(k)}(\tilde{G}) \subseteq \mathcal{M}^{(k)}(G)$. Conversely, given $\mathcal{E}_{i_1, \dots, i_k}^{(k)}$ and λ_{ij} small enough, we can choose $\tilde{\mathcal{E}}_{v, \dots, v}^{(k)} = \varepsilon > 0$, $\tilde{\lambda}_{ij} = \lambda_{ij}$ for $i \rightarrow j$ in \mathcal{D} , and $\tilde{\lambda}_{v, i_l} = \sqrt[k]{\frac{|\mathcal{E}_{i_1, \dots, i_k}^{(k)}|}{\varepsilon}}$ for $l \in \{1, \dots, k\}$. Therefore, equation (22) is satisfied. Since $\mathcal{E}_{i_1, \dots, i_k}^{(k)}$

is small and $\mathcal{E}_{i,\dots,i}^{(k)}$ is near 1, we can find $\tilde{\mathcal{E}}_{i,\dots,i}^{(k)} > 0$ such that equation (23) is satisfied and such that $\tilde{\mathcal{E}}^{(k)}$ is a diagonal tensor. This shows that if $\mathcal{C}^{(k)} \in \mathcal{M}^{(k)}(G)$ is in a neighborhood of $\mathcal{I}^{(k)}$, then we can find the corresponding $\tilde{\mathcal{C}}^{(k)} \in \mathcal{M}^{(k)}(\tilde{G})$ as well. Note that this correspondence between $\mathcal{C}^{(k)}$ and $\tilde{\mathcal{C}}^{(k)}$ is a bijection. Thus $\mathcal{M}^{(k)}(G)$ and $\mathcal{M}^{(k)}(\tilde{G})$ are equal in an open neighborhood and so they have the same Zariski closure, i.e., $\overline{\mathcal{M}^{(k)}(G)} = \overline{\mathcal{M}^{(k)}(\tilde{G})}$, where $\overline{\mathcal{M}^{(k)}(G)}$ denotes the Zariski closure of $\mathcal{M}^{(k)}(G)$. Therefore the determinant of $\tilde{\mathcal{C}}_{S_1,\dots,S_k}^{(k)}$ vanishes on $\mathcal{M}^{(k)}(\tilde{G})$ if and only if the determinant of $\mathcal{C}_{S_1,\dots,S_k}^{(k)}$ vanishes on $\mathcal{M}^{(k)}(G)$. This is equivalent to saying that there is no system of k -treks without sided intersection between S_1, \dots, S_k in \tilde{G} if and only if there is no system of k -treks without sided intersection between S_1, \dots, S_k in G . ■

Going back to the proof of Theorem 25, Proposition 42 enables us to reduce the multidirected setting to a directed acyclic graph for which we proved the Theorem 15.

D Proofs of the results in Section 5

In this section, we prove our results in the case of higher-order moment tensors. Proofs are analogous to the proofs for higher-order cumulant tensors.

Proof of Proposition 31. We have that $\mathcal{N}^{(k)} = \Phi^{(k)} \bullet (I - \Lambda)^{-k}$. The entries of $\Phi^{(k)}$ are non-zero if and only if they are of the form $\mathbb{E}[\varepsilon_1^{x_1} \dots \varepsilon_k^{x_k}]$ where x_1, \dots, x_k are integers either equal to 0, or strictly greater than 1. Such entries correspond to the cases when there is a k -split-trek between i_1, \dots, i_k as defined in Definition 28. Furthermore, we have $(I - \Lambda)_{ij}^{-1} = \sum_{P \in \mathcal{P}(ij)} \lambda^P$ by equation (6), and replacing this expression in equation (11), we obtain equation (12), which completes the proof. ■

Proof of Proposition 32. By applying the tensor version of the Cauchy-Binet Theorem k times to equation (11), we get:

$$\det \mathcal{N}_{S_1, \dots, S_k}^{(k)} = \sum_{R_1, \dots, R_k} \det \Phi_{R_1, \dots, R_k}^{(k)} \det(I - \Lambda)_{R_1, S_1}^{-1} \dots \det(I - \Lambda)_{R_k, S_k}^{-1} \quad (24)$$

Additionally, from equation (24), we can write

$$\mathcal{N}_{i_1, \dots, i_k}^{(k)} = \sum_{T \in \mathcal{S}(i_1, \dots, i_k)} \text{sign}(T) m_T, \quad (25)$$

where m_T is the k -split-trek monomial defined by $m_T = \phi_{\text{top}(P_1, \dots, P_k)} \lambda^{P_1} \dots \lambda^{P_k}$.

Assuming that $\#R_1 = \dots = \#R_k = \#S_1 = \dots = \#S_k = n$, we then get:

$$\begin{aligned} \det \mathcal{N}_{S_1, \dots, S_k}^{(k)} &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}, \dots, \sigma_{k-1} \in \mathfrak{S}_{n_{k-1}}} \left(\sum_{\substack{T_1 \in \mathcal{S}(s_1, s_{\sigma_1(1)}, \dots, s_{\sigma_{k-1}(1)}) \\ \vdots \\ T_n \in \mathcal{S}(s_n, s_{\sigma_1(n)}, \dots, s_{\sigma_{k-1}(n)})}} \text{sign}(\sigma_1) \dots \text{sign}(\sigma_{k-1}) m_{T_1} \dots m_{T_n} \right) \\ &= \sum_{\substack{\sigma_i \in \mathfrak{S}_{n_i} \\ i \in \{1, \dots, k-1\}}} \left(\sum_{\substack{T_j \in \mathcal{S}(s_j, s_{\sigma_1(j)}, \dots, s_{\sigma_{k-1}(j)}) \\ j \in \{1, \dots, n\}}} \prod_{l=1}^{k-1} \text{sign}(\sigma_l) \prod_{s=1}^n m_{T_s} \right) \\ &= \sum_{T \in \mathcal{S}(S_1, \dots, S_k)} \text{sign}(T) m_T \end{aligned}$$

where \mathfrak{S}_{n_i} is the set of permutations of the nodes in S_{i+1} , T runs over all k -split-trek systems between S_1, \dots, S_k and $\text{sign}(T) = \prod_{k-1} \text{sign}(\sigma_1) \dots \text{sign}(\sigma_{k-1})$. In this expression, we have $m_T = \prod_{x=1}^n m_{T_x}$, where

$m_{T_x} = \Phi_{top(P_{s_1}, \dots, P_{s_k})} \lambda^{P_{s_1}} \dots \lambda^{P_{s_k}}$, i.e., m_T is the product of the monomials of the n k -split-treks that form the k -split-trek-system T .

Similarly to the proof of Proposition 13, we can use the Gessel-Viennot-Lindstrom Lemma to show that the sum in the expression of $\det \mathcal{N}_{S_1, \dots, S_k}^{(k)}$ can be taken over the set $\tilde{\mathcal{S}}(S_1, \dots, S_k)$ of k -split-trek systems between S_1, \dots, S_k without sided intersection. ■

In order to prove Theorem 33, notice that in equation (24), $\Phi^{(k)}$ is diagonal for $k = 3$. Furthermore, a 3-trek and a 3-split-trek are the same (and similarly for a 3-trek system and a 3-split-trek-system). Then the proof of Theorem 33 follows the same reasoning as for Theorem 15.

Notice finally that for $k > 3$, $\Phi^{(k)}$ is not diagonal, and for that reason, we cannot easily extend Theorem 33 to higher-order moments.