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Methods

Mixed-Projection Conic Optimization: A New Paradigm for Modeling Rank Constraints

Dimitris Bertsimas,[a] Ryan Cory-Wright,[b] Jean Pauphilet[c]

[a] Sloan School of Management and Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02142; [b] Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02142; [c] London Business School, London NW1 4SA, United Kingdom

Contact: dbertsim@mit.edu, https://orcid.org/0000-0002-1985-1003 (DB); ryancw@mit.edu, https://orcid.org/0000-0002-4485-0619 (RC-W); jpauphilet@london.edu, https://orcid.org/0000-0001-6352-0984 (JP)

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Abstract. We propose a framework for modeling and solving low-rank optimization problems to certifiable optimality. We introduce symmetric projection matrices that satisfy $\boldsymbol{Y}^2 = \boldsymbol{Y}$, the matrix analog of binary variables that satisfy $z^2 = z$, to model rank constraints. By leveraging regularization and strong duality, we prove that this modeling paradigm yields convex optimization problems over the nonconvex set of orthogonal projection matrices. Furthermore, we design outer-approximation algorithms to solve low-rank problems to certifiable optimality, compute lower bounds via their semidefinite relaxations, and provide near optimal solutions through rounding and local search techniques. We implement these numerical ingredients and, to our knowledge, for the first time solve low-rank optimization problems to certifiable optimality. Our algorithms also supply certifiably near-optimal solutions for larger problem sizes and outperform existing heuristics by deriving an alternative to the popular nuclear norm relaxation. Using currently available spatial branch-and-bound codes, not tailored to projection matrices, we can scale our exact (respectively, near-exact) algorithms to matrices with up to 30 (600) rows/columns. All in all, our framework, which we name *mixed-projection conic optimization*, solves low-rank problems to certifiable optimality in a tractable and unified fashion.

Supplemental Material: The online appendix is available at https://doi.org/10.1287/opre.2021.2182.

Keywords: rank minimization • semidefinite optimization • global optimization • discrete optimization • outer approximation • regularization • perspective relaxation • matrix completion • nuclear norm

1. Introduction

Many central problems in optimization, machine learning, and control theory are equivalent to optimizing a low-rank matrix over a convex set. For instance, low-rank constraints model notions of minimal complexity, low dimensionality, or orthogonality in a system. However, although rank constraints offer unparalleled modeling flexibility, no generic code currently solves these problems to certifiable optimality at even moderate sizes. This state of affairs has led influential works on low-rank optimization (Candès and Plan 2010, Recht et al. 2010) to characterize low-rank optimization as intractable and advocate convex relaxations or heuristics that do not enjoy assumption-free optimality guarantees.

The manner in which rank-constrained optimization is regarded today is reminiscent of how mixed-integer conic optimization (MICO), which can model NP-complete problems, was originally considered. After decades of research effort, however, algorithms and software for MICO are now widely available (see, e.g., Bonami et al. 2008, Coey et al. 2020) and solve large instances of disparate nonconvex problems to certifiable optimality. Unfortunately, rank constraints cannot be represented using mixed-integer convex optimization (Lubin et al. 2021, lemma 4.1) and do not benefit from these advances.

In this work, we characterize the complexity of rank-constrained optimization and propose a new framework to model them, which we term *mixed-projection conic optimization* (MPCO). Our proposal generalizes MICO by replacing binary variables z that satisfy $z^2 = z$ with symmetric orthogonal projection matrices $\boldsymbol{Y}$ that satisfy $\boldsymbol{Y}^2 = \boldsymbol{Y}$ and offers the following contributions: First, it supplies certificates of (near) optimality for low-rank problems. Second, it demonstrates that some of the best ideas in MICO, such as decomposition methods, cutting planes, relaxations, and random rounding schemes, admit straightforward extensions to MPCO. Finally, we implement a near-optimal rounding strategy and a globally optimal cutting-plane algorithm that improve upon the state of the art for matrix completion and sensor location problems. We hope that MPCO gives rise to exciting new challenges for the optimization community to tackle.

1.1. Scope of the Framework

Formally, we consider the problem

$$\begin{aligned} \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \quad & \langle \boldsymbol{C}, \boldsymbol{X}\rangle + \lambda \cdot \text{Rank}(\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B},\ \text{Rank}(\boldsymbol{X}) \le k,\ \boldsymbol{X} \in \mathcal{K}, \end{aligned} \tag{1}$$

where λ (respectively, k) prices (bounds) the rank of $\boldsymbol{X}$; $(\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{R}^{\ell\times n} \times \mathbb{R}^{\ell\times m}$ defines an affine subspace; and $\mathcal{K}$ is a proper cone in the sense of Boyd and Vandenberghe (2004), that is, closed, convex, solid, and pointed. Observe that Problem (1) offers significant modeling flexibility as it allows arbitrary conic constraints on $\boldsymbol{X}$. As a result, linear, convex quadratic, semidefinite, exponential, and power constraints and objectives can be captured by letting $\mathcal{K}$ be an appropriate product of the nonnegative orthant and the second order, semidefinite, exponential, and power cones.

We now introduce our notation and present some central problems from the optimization and machine learning literature that admit low-rank formulations and fall within our framework.

Notation: We let non-boldface characters such as b, denote scalars; lowercase boldface characters such as $\boldsymbol{x}$, denote vectors; uppercase boldface characters such as $\boldsymbol{X}$, denote matrices; and calligraphic uppercase characters, such as $\mathcal{Z}$, denote sets. We let $[n]$ denote the set of running indices $\{1, \ldots, n\}$. We let $\boldsymbol{e}$ denote a vector of all ones, $\boldsymbol{0}$ denote a vector of all zeros, and $\mathbb{I}$ denote the identity matrix.

We also use an assortment of matrix operators. We let $\sigma_i(\boldsymbol{X})$ denote the ith largest singular value of a matrix $\boldsymbol{X}$, $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product between two vectors or matrices of the same dimension, $\boldsymbol{X}^\dagger$ denote the Moore–Penrose pseudoinverse of a matrix $\boldsymbol{X}$, $\|\cdot\|_F$ denote the Frobenius norm of a matrix, $\|\cdot\|_\sigma$ denote the spectral norm of a matrix, and $\|\cdot\|_*$ denote the nuclear norm of a matrix; see Horn and Johnson (1985) for a general theory of matrix operators.

Finally, we use a variety of convex cones. We let S^n denote the $n \times n$ cone of symmetric matrices, and S^n_+ denote the $n \times n$ positive semidefinite (PSD) cone.

1.1.1. Low-Rank Matrix Completion.

Given a subsample $(A_{i,j} : (i,j) \in \mathcal{I} \subseteq [n] \times [m])$ of a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, the matrix completion problem is to recover the entire matrix by seeking a rank-k matrix $\boldsymbol{X}$ that approximately fits the observed values. This problem admits the formulation

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \quad \frac{1}{2}\sum_{(i,j)\in\mathcal{I}} \left(X_{i,j} - A_{i,j}\right)^2 \quad \text{s.t.} \quad \text{Rank}(\boldsymbol{X}) \le k. \tag{2}$$

Because there are $(n+m)k$ degrees of freedom in a singular value decomposition of a rank-k matrix $\boldsymbol{X} \in \mathbb{R}^{n\times m}$, Problem (2) is not well defined unless $|\mathcal{I}| \ge (n+m)k$.

1.1.2. Minimum Dimension Euclidean Embedding.

Given a set of pairwise distances $d_{i,j}$, the Euclidean distance embedding problem is to determine the lowest dimensional space in which the distances can be embedded such that the distances correspond to Euclidean distances. A set of distances $d_{i,j}$ can be embedded in a Euclidean space of dimension k if and only if there exists some Gram matrix $\boldsymbol{G} \succeq \boldsymbol{0}$ of rank k such that $d^2_{i,j} = G_{i,i} + G_{j,j} - 2G_{i,j}$ on all pairs (i, j), where $d_{i,j}$ is supplied (Blekherman et al. 2012, theorem 2.49). Denoting $D_{i,j} = d^2_{i,j}$, this is equivalent to

$$\min_{\boldsymbol{G}\in S^n_+} \text{Rank}(\boldsymbol{G}) \text{ s.t. } \text{Diag}(\boldsymbol{G})\boldsymbol{e}^\top + \boldsymbol{e}\text{Diag}(\boldsymbol{G})^\top - 2\boldsymbol{G} = \boldsymbol{D}, \tag{3}$$

where the equality is implicitly imposed only for pairs (i, j), where $d_{i,j}$ is supplied.

1.1.3. Quadratically Constrained Optimization.

A quadratically constrained quadratic optimization problem (QCQO) seeks an $\boldsymbol{x} \in \mathbb{R}^n$ that solves

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \boldsymbol{x}^\top \boldsymbol{Q}_0 \boldsymbol{x} + \boldsymbol{q}_0^\top \boldsymbol{x} \ \text{ s.t. } \ \boldsymbol{x}^\top \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^\top \boldsymbol{x} \le r_i \quad \forall i \in [m], \tag{4}$$

where $\boldsymbol{Q}_0$, $\boldsymbol{Q}_i$, $\boldsymbol{q}_0$ $\boldsymbol{q}_i$, r_i are given problem data. We assume that $\boldsymbol{Q}_0, \boldsymbol{Q}_i$ are symmetric matrices but do not assume that they are positive semidefinite. Therefore, this problem is nonconvex and encompasses binary quadratic optimization (Goemans and Williamson 1995) and alternating current optimal power flow problems (Lavaei and Low 2011). The fundamental difficulty in Problem (4) is the nonconvexity of the outer product $\boldsymbol{x}\boldsymbol{x}^\top$, which can be modeled by a rank-one matrix $\boldsymbol{X}$:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{X}\in S^n} \ & \langle \boldsymbol{Q}_0, \boldsymbol{X}\rangle + \langle \boldsymbol{q}_0, \boldsymbol{x}\rangle \\ \text{s.t.} \ & \langle \boldsymbol{Q}_i, \boldsymbol{X}\rangle + \langle \boldsymbol{q}_i, \boldsymbol{x}\rangle \le r_i \quad \forall i \in [m], \\ & \text{Rank}\begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} = 1. \end{aligned} \tag{5}$$

We have established that QCQOs are rank-constrained problems. The converse is also true: rank-constrained problems with linear, second-order cone, or semidefinite constraints are QCQOs. Indeed, the constraint $\text{Rank}(\boldsymbol{X}) \le k$ is equivalent to requiring that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top$: $\boldsymbol{U} \in \mathbb{R}^{n\times k}$, $\boldsymbol{V} \in \mathbb{R}^{m\times k}$, that is, imposing $m \times n$ nonconvex quadratic equalities. As modern solvers, such as Gurobi, can now solve nonconvex QCQOs to global optimality, this QCQO formulation can be used to solve low-rank problems although it is not particularly scalable; we expand on this point in Online Appendix EC.5.2.

1.2. Background and Literature Review

Our work arises at the intersection of three complementary areas of the low-rank optimization literature: (a) global optimization algorithms for nonconvex quadratically constrained problems; (b) the interplay of

convex relaxations and their dual side, randomized rounding methods; and (c) heuristics that provide high-quality solutions to nonconvex problems in an efficient fashion.

1.2.1. Global Optimization Techniques.

1.2.1.1. Branch-and-Bound. A broad class of global optimization algorithms have been proposed for QCQOs since McCormick (1976) observed that convex envelopes of nonconvex regions supply globally valid lower bounds. This gives rise to a numerical strategy in which one recursively partitions the QCQO's feasible region into subregions, constructs convex envelopes for each subregion, and uses these envelopes to construct iteratively improving lower bounds. This approach is known as spatial branch-and-bound; see Lee and Zou (2014) for a scheme that decomposes a matrix into a sparse matrix plus a low-rank matrix, Kocuk et al. (2016) for a modern implementation in alternating current optimal power flow, and Bertsimas et al. (2017) for an exact branch-and-bound approach to low-rank factor analysis.

1.2.1.2. Branch-and-Cut. In a complementary direction, several branch-and-cut methods (Audet et al. 2000, Linderoth 2005) are proposed for solving nonconvex QCQOs by borrowing decomposition schemes from the mixed-integer nonlinear optimization (MINLO) literature (Duran and Grossmann 1986). Although often efficient in practice, a common theme in these methods is that the more efficient decomposition schemes used for MINLOs cannot be applied out of the box because they may fail to converge to a globally optimal solution (see Grossmann 2002 for a counterexample). As a result, nonconvex problems need to be preprocessed in an expensive fashion. This preprocessing step inhibits the use of global optimization methods for low-rank problems; indeed, we are not aware of any works that apply branch-and-cut techniques to solve low-rank problems to certifiable optimality.

1.2.1.3. Complementarity. In an opposite direction, several authors propose applying general nonlinear optimization techniques to address low-rank problems since Ding et al. (2014) observed that a low-rank constraint is equivalent to a complementarity constraint over the positive semidefinite cone and, thus, can be addressed by general techniques for mathematical programs with equilibrium constraints (see Luo et al. 1996 for a general theory). Among others, Bai et al. (2016) invoke the complementarity observation to design a completely positive reformulation of low-rank semidefinite optimization (SDO) problems, and Bi et al. (2020) develop a multistage convex relaxation of the complementarity constraint. We are not aware of any works that use these ideas to solve low-rank problems exactly when $n \geq 10$.

1.2.1.4. Algebraic. By taking an algebraic view of rank constraints, several algebraic geometry techniques are proposed for addressing low-rank SDOs. Among others, d'Aspremont (2003) proposes reformulating low-rank constraints as systems of polynomial equations that can be addressed via the sum-of-squares hierarchy (Lasserre 2001). More recently, Naldi (2018) proposes a semialgebraic reformulation of rank-constrained SDOs that can be optimized over via Gröbner basis computation (Cox et al. 2013). Unfortunately, algebraic approaches do not scale well in practice. Indeed, as observed by Recht et al. (2010), it seems unlikely that algebraic approaches can solve low-rank SDOs when $n > 10$.

1.2.2. Convex Relaxations and Rounding Methods.

1.2.2.1. Convex Relaxations. A number of authors has studied convex relaxations of low-rank problems since Fazel (2002) observed that the nuclear norm of a matrix is the convex envelope of a rank constraint on the set of matrices with spectral norm at most M, i.e.,

$$\begin{aligned}&\mathrm{Conv}(\{\boldsymbol{X} \in \mathbb{R}^{n\times m} : \|\boldsymbol{X}\|_\sigma \leq M, \mathrm{Rank}(\boldsymbol{X}) \leq k\})\\ &\quad = \{\boldsymbol{X} \in \mathbb{R}^{n\times m} : \|\boldsymbol{X}\|_\sigma \leq M,\ \|\boldsymbol{X}\|_* \leq kM\}. \qquad (6)\end{aligned}$$

Because the epigraph of a nuclear norm is semidefinite representable, this gives rise to semidefinite relaxations of low-rank problems that can be computed in polynomial time.

1.2.2.2. Rounding Methods. A complementary line of work aims to supply certifiably near-optimal solutions to low-rank problems by rounding their semidefinite relaxations. Initiated by Goemans and Williamson (1995), in the context of binary quadratic optimization, who established that randomly rounding an SDO relaxation supplies a 0.878-approximation, it has evolved into a successful framework for solving rank-one optimization problems; see Nemirovski et al. (1999) for a unified approach in the rank-one case. However, this line of work has a key drawback. Namely, existing rounding methods do not address rank-k problems, such as matrix completion, because of the analytic difficulty of constructing a rounding mechanism that preserves both feasibility and near-optimality in the rank-k case.

1.2.3. Heuristic Methods.

Because of the computational difficulty of solving Problem (1) to certifiable optimality and the analytic difficulty of deriving a high-quality

randomized rounding procedure, a variety of heuristic methods are proposed for solving Problem (1), originating with methods for solving low-rank linear matrix inequalities in the optimal control literature (Boyd et al. 1994).

Although slow and somewhat ad hoc in their original implementations, heuristic methods were moved front-and-center by the works of Fazel (2002) and Burer and Monteiro (2003, 2005). Fazel (2002) observes that low-rank positive semidefinite matrices lie on the boundary of the PSD cone and uses this observation to justify a "log-det" heuristic, in which a rank minimization objective is replaced with the function $\log\det(\boldsymbol{X}+\delta\mathbb{I})$. Burer and Monteiro (2003, 2005) propose implicitly modeling a rank constraint $\text{Rank}(\boldsymbol{X}) \leq k$ by applying the nonlinear reformulation $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top$, where $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n\times k}$ and eliminating $\boldsymbol{X}$, to obtain a problem that is nonconvex in $(\boldsymbol{U}, \boldsymbol{V})$. Although originally solved using augmented Lagrangian techniques, subsequent implementations of the Burer–Monterio (BM) heuristic typically use alternating minimization (Jain et al. 2013), successive over-relaxations (Wen et al. 2012), (stochastic) gradient descent (Recht and Ré 2013, Zheng and Lafferty 2015, Tu et al. 2016), and manifold methods (Boumal et al. 2016, 2020). This popularity is driven by the fact that, under particular assumptions, the problem has no spurious local optima (see Bhojanapalli et al. 2016, Boumal et al. 2016, Ge et al. 2016, Cifuentes 2019), and the Burer–Monterio approach recovers a globally optimal solution; see Udell et al. (2016) and Nguyen et al. (2019) for reviews of heuristic approaches.

1.3. Contributions and Structure

The key contribution of the paper is to propose using orthogonal projection matrices that satisfy $\boldsymbol{Y}^2 = \boldsymbol{Y}$ to model low-rank constraints via the nonlinear equation $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$. Under this lens, low-rank problems admit reformulations as optimization problems in which some decision variables comprise a projection matrix. We term this family of problems MPCO in reference to mixed-integer optimization. By leveraging regularization and strong duality, we derive tractable formulations and algorithms to solve MPCO problems and their convex relaxations. Using a generic spatial branch-and-bound code, we are already able to solve low-rank optimization problems exactly for matrices with 30 rows and columns and find near-exact solutions for matrices with up to 600 rows and columns. To our knowledge, our approach is the first mathematical framework that solves low-rank optimization problems to certifiable (near) optimality.

We note that the idea of using projection matrices to model low-rank constraints is not entirely new as Peng and Xia (2005) reformulate k-means clustering as a semidefinite optimization problem over the set of orthogonal projection matrices. Our proposal differs in two ways: (1) we consider optimizing over projection matrices directly, and Peng and Xia (2005) use projection matrices as a vehicle to derive semidefinite relaxations; (2) we use projection matrices to solve low-rank optimization problems that do not admit mixed-integer reformulations, and k-means does (Grötschel and Wakabayashi 1989).

The rest of the paper is laid out as follows:

In Section 2, we show that projection matrices are a natural generalization of binary vectors to matrices. Inspired by a common tactic in cardinality-constrained optimization, namely, introducing binary variables to encode the support of the decision vector, we propose introducing a projection matrix to encode the image of the decision matrix and thereby model rank. We also investigate the complexity of low-rank optimization problems and show that rank minimization is in PSPACE.

In Section 3, we derive the MPCO formulations of the aforementioned rank optimization problems. By introducing a constraint on the spectral norm of $\boldsymbol{X}$ or a penalty on its Frobenius norm—the matrix analogs of big-M constraints and perspective formulations (Günlük and Linderoth 2012), respectively—we leverage strong duality, reformulate Problem (1) as a saddle-point problem, and prove the resulting optimization problem admits a convex objective.

We propose numerical algorithms to solve these MPCO problems to provable (near) optimality in Section 4 by extending some of the most successful techniques from MICO. First, we propose an outer-approximation (OA) scheme for solving Problem (1) exactly. Then, we obtain valid lower bounds from solving its convex relaxations and propose an alternating minimization algorithm to do so. In addition, we prove that a singular value decomposition (SVD) followed by greedily rounding the eigenvalues provides certifiably near-optimal solutions in polynomial time.

In Section 5, we implement and numerically evaluate our proposed algorithms. On examples from matrix completion and sensor location, we demonstrate that methods proposed in this paper solve instances of Problem (1) to certifiable optimality in minutes for n in the tens. To our knowledge, our work is the first to demonstrate that moderately sized rank-constrained problems can be solved to provable optimality in a tractable fashion. For n in the hundreds, our proposal scales and provides in minutes solutions of higher quality than existing heuristics, such as nuclear norm minimization.

2. From Cardinality to Rank: A Unifying Nonlinear Perspective

Low-rank constraints $\text{Rank}(\boldsymbol{X}) \le k$ are a natural generalization of cardinality constraints $\|\boldsymbol{x}\|_0 \le k$ from vectors to matrices. Indeed, if $\boldsymbol{X}$ is a diagonal matrix, then $\text{Rank}(\boldsymbol{X}) \le k$ if and only if $\|\boldsymbol{X}\|_0 \le k$, and more generally, $\text{Rank}(\boldsymbol{X}) \le k$ if and only if $\|\sigma(\boldsymbol{X})\|_0 \le k$, where $\sigma(\boldsymbol{X})$ is the vector of singular values of $\boldsymbol{X}$. However, although cardinality and rank constraints are intimately linked, they are addressed using different algorithms. Namely, we can solve cardinality-constrained problems with 100,000s of variables to optimality (Bertsimas et al. 2020), and low-rank problems are dramatically harder and have not yet been solved to certifiable optimality for $n > 10$ (Naldi 2018).

In our opinion, the difference between the community's understanding of cardinality and rank constraints has arisen because of two barriers. First, as we show in this section, rank constraints belong to a harder complexity class. Second, cardinality constraints can be represented using binary variables, and rank constraints cannot (Lubin et al. 2021, corollary 4.1). This presents a challenge for researchers, who have developed scalable methods for cardinality constraints by exploiting advances in MICO but cannot use these advances to address rank constraints. In this section, we question these barriers by characterizing the complexity of low-rank problems and proposing a new framework for modeling rank.

2.1. Complexity of Rank-Constrained Optimization

Existing studies of Problem (1) typically claim that it is intractable and support this claim by proving it is NP-hard by reduction from an NP-complete problem such as Boolean linear programming (see, e.g., Vandenberghe and Boyd 1996, section 7.3). In our opinion, this argument needs to be revisited for there is no evidence that Problem (1) is even in NP. Indeed, Problem (1) cannot be represented using mixed-integer convex optimization (Lubin et al. 2021, corollary 4.1), while all 21 of Karp's NP-complete problems can, and the best-known algorithms for Problem (1) run in EXPTIME (Chistov and Grigoriev 1984, Naldi 2018).

In this paper, we provide a more complete characterization of the complexity of Problem (1) than is currently available in the literature. First, we demonstrate that it belongs to a different class than NP. In particular, we prove that it is *existential theory of the reals* hard ($\exists\mathbb{R}$-hard; see Renegar 1992 for a general theory), that is, as hard as any polynomial optimization problem, which implies that, if NP $\subsetneq \exists\mathbb{R}$, Problem (1) is strictly harder than NP-complete problems. Second, we prove that Problem (1) is actually in $\exists\mathbb{R}$. Although Candès and Plan (2010) observe that Problem (1) is EXPTIME, we show it belongs to $\exists\mathbb{R} \subseteq$ PSPACE $\subseteq$ EXPTIME, thus improving upon the bound stated by Recht et al. (2010) and Candès and Plan (2010), among others. Moreover, it seems unlikely to us that this bound can be further improved without settling fundamental questions in complexity theory (e.g., characterizing NP versus $\exists\mathbb{R}$ versus PSPACE versus EXPTIME). Formally, our main complexity-theoretic results are as follows; all other statements and proofs are deferred to Online Appendix EC.1.

Theorem 1. *Problem* (1) *is* $\exists\mathbb{R}$*-hard.*

Theorem 2. *Let* $\mathcal{K} = S^n_+$*. Then,* Problem (1) *is in* $\exists\mathbb{R}$ *and, hence,* $\exists\mathbb{R}$*-complete.*

2.2. Projection Matrices for Modeling Rank

As previously discussed, rank constraints can be seen as a generalization to the matrix case of cardinality constraints. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, the cardinality constraint $\|\boldsymbol{x}\|_0 \le k$ ensures that at most k coordinates of $\boldsymbol{x}$ are nonzero and can be modeled by introducing a vector of binary variables because

$$\|\boldsymbol{x}\|_0 \le k \Leftrightarrow \exists \boldsymbol{z} \in \{0,1\}^n : \boldsymbol{e}^\top \boldsymbol{z} \le k, \boldsymbol{x} = \boldsymbol{z} \circ \boldsymbol{x}, \tag{7}$$

where $\boldsymbol{z} \circ \boldsymbol{x}$ denotes the component-wise product of $\boldsymbol{z}$ and $\boldsymbol{x}$. Actually, nonlinear constraints of the form "$\boldsymbol{x} = \boldsymbol{z} \circ \boldsymbol{x}$," where $\boldsymbol{z}$ is binary and $\boldsymbol{x}$ is continuous, occur in a variety of mixed-integer optimization problems far beyond cardinality-constrained optimization. Bertsimas et al. (2021) observe that such nonlinear constraints "$\boldsymbol{x} = \boldsymbol{z} \circ \boldsymbol{x}$" actually lead to tractable optimization problems provided that the overall objective is appropriately regularized. In particular, big-M constraints (Glover 1975) and perspective reformulations (Günlük and Linderoth 2012) can be seen as appropriate regularizers. By building upon this observation and the work of several other authors (Fischetti et al. 2017, Bertsimas et al. 2020), they successfully solve cardinality-constrained problems at scale via a combination of branch-and-cut, randomized rounding, and heuristic methods.

Unfortunately, rank constraints cannot be modeled using integer variables. Therefore, we now propose a new framework to model rank in optimization problems. Instead of a binary vector $\boldsymbol{z}$ to encode the support of $\boldsymbol{x}$, we introduce a projection matrix $\boldsymbol{Y}$ to capture the column space of $\boldsymbol{X}$ and obtain a similar nonlinear reformulation.

Definition 1. A matrix $\boldsymbol{Y} \in \mathbb{R}^{n \times n}$ is a projection matrix if it satisfies the equality $\boldsymbol{Y}^2 = \boldsymbol{Y}$. In addition, if $\boldsymbol{Y}$ is symmetric, $\boldsymbol{Y}$ is an orthogonal projection matrix.

As symmetric matrices, orthogonal projection matrices are diagonalizable, and their eigenvalues satisfy

$\lambda_i^2 = \lambda_i$, that is, are binary. As a result, the Moore–Penrose pseudoinverse of an orthogonal projection $\boldsymbol{Y}$ is $\boldsymbol{Y}$ itself ($\boldsymbol{Y} = \boldsymbol{Y}^\dagger$). In addition, because its eigenvalues are binary, the trace of $\boldsymbol{Y}$ equals the number of nonzero eigenvalues, that is, $\text{Rank}(\boldsymbol{Y}) = \text{tr}(\boldsymbol{Y})$. We are now in a position to link projection matrices and rank constraints.

Proposition 1. *For any $\boldsymbol{X} \in \mathbb{R}^{n\times m}$, $\text{Rank}(\boldsymbol{X}) \le k \Leftrightarrow \exists \boldsymbol{Y} \in \mathcal{Y}_n : \text{tr}(\boldsymbol{Y}) \le k,\ \boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$, where $\mathcal{Y}_n := \{\boldsymbol{P} \in S^n : \boldsymbol{P}^2 = \boldsymbol{P}\}$ is the set of $n \times n$ orthogonal projection matrices.*

Proof of Proposition 1. We prove the two implications successively.

- Let $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ with $\boldsymbol{U} \in \mathbb{R}^{n\times k}$, $\boldsymbol{\Sigma} \in \mathbb{R}^{k\times k}$, $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ be a singular value decomposition of $\boldsymbol{X}$ and define $\boldsymbol{Y} = \boldsymbol{U}(\boldsymbol{U}^\top\boldsymbol{U})^{-1}\boldsymbol{U}^\top = \boldsymbol{U}\boldsymbol{U}^\top$. By construction, $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$ because $\boldsymbol{U}^\top\boldsymbol{U} = \mathbb{I}$. Moreover, $\text{tr}(\boldsymbol{Y}) = \text{rank}(\boldsymbol{Y}) = \text{rank}(\boldsymbol{X}) \le k$.
- Because $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$, $\text{rank}(\boldsymbol{X}) \le \text{rank}(\boldsymbol{Y}) = \text{tr}(\boldsymbol{Y}) \le k$.

Remark 1. In Proposition 1, $\boldsymbol{Y}$ is the orthogonal projection onto the image or column space of $\boldsymbol{X}$. Alternatively, one could model the rank constraint via a matrix $\boldsymbol{Y}' \in \mathcal{Y}_m$ such that $\text{tr}(\boldsymbol{Y}') \le k$ and $\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Y}'$. In this case, $\boldsymbol{Y}'$ encodes the projection onto the row space of $\boldsymbol{X}$. In practice, one could introduce both $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ and obtain tighter formulations at the price of introducing additional notation. We explore this idea in Online Appendix EC.2.

Proposition 1 suggests that projection matrices are to rank constraints what binary variables are to cardinality constraints. Indeed, similarities between the two are evident: binary variables z are idempotent scalars that solve $z^2 = z$, and projection matrices $\boldsymbol{Y}$ are idempotent matrices that solve $\boldsymbol{Y}^2 = \boldsymbol{Y}$. Also, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are diagonal, Proposition 1 recovers cardinality-constrained optimization.

Over the past decades, extensive efforts have been devoted to improving the scalability of mixed-integer optimization. We believe that similar achievements can be obtained for rank-constrained problems by adapting techniques from MICO to MPCO. In this direction, Table 1 establishes a dictionary linking cardinality and rank constraints and demonstrates that many of the techniques developed for binary convex optimization admit generalizations to MPCO, including the main results from our recent work (Bertsimas et al. 2021). Note that we have not yet established most of the connections claimed in Table 1; this is the focus of the next two sections of the paper.

3. Regularization and a Saddle-Point Reformulation

In this section, we prove that (9) can be reformulated as a saddle-point mixed-projection problem by leveraging regularization terms analogous to the big-M and ridge regularization techniques from MICO and derive their semidefinite relaxations as summarized in Table 1.

Throughout this paper, we let $\mathcal{Y}_n := \{\boldsymbol{P} \in S^n : \boldsymbol{P}^2 = \boldsymbol{P}\}$ denote the set of $n \times n$ orthogonal projection matrices and $\mathcal{Y}_n^k := \{\boldsymbol{P} \in S^n : \boldsymbol{P}^2 = \boldsymbol{P}, \text{tr}(\boldsymbol{P}) \le k\}$ denote projection matrices with rank at most k. Although $\mathcal{Y}_n$ and $\mathcal{Y}_n^k$ do not commonly appear in the optimization literature, their convex hulls are well studied as we now remind the reader by restating Overton and Womersley (1992, theorem 3):

Lemma 1. *Let $\mathcal{Y}_n$ denote the $n \times n$ orthogonal projection matrices and $\mathcal{Y}_n^k$ denote the low-rank orthogonal projection matrices. Then, $\text{Conv}(\mathcal{Y}_n) = \{\boldsymbol{P} : 0 \preceq \boldsymbol{P} \preceq \mathbb{I}\}$ and $\text{Conv}(\mathcal{Y}_n^k) = \{\boldsymbol{P} : 0 \preceq \boldsymbol{P} \preceq \mathbb{I}, \text{tr}(\boldsymbol{P}) \le k\}$. Moreover, the extreme points of $\text{Conv}(\mathcal{Y}_n)$ are $\mathcal{Y}_n$, and the extreme points of $\text{Conv}(\mathcal{Y}_n^k)$ are $\mathcal{Y}_n^k$.*

3.1. A Regularization Assumption

By invoking Proposition 1, we rewrite Problem (1) as the following mixed-projection conic problem:

$$\begin{aligned} \min_{\boldsymbol{Y}\in\mathcal{Y}_n^k} \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \ & \langle \boldsymbol{C}, \boldsymbol{X}\rangle + \lambda \cdot \text{tr}(\boldsymbol{Y}) \\ \text{s.t.} \ & \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B},\ \boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X},\ \boldsymbol{X} \in \mathcal{K}. \end{aligned} \tag{8}$$

Observe that Problem (8) has a two-stage structure that involves first selecting a low-rank projection matrix $\boldsymbol{Y}$ and second selecting a matrix $\boldsymbol{X}$ under the constraint $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$. Moreover, selecting an optimal $\boldsymbol{X}$ given $\boldsymbol{Y}$ is *easy* because it involves solving a conic optimization problem under the linear constraint $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$, and selecting an optimal $\boldsymbol{Y}$ is *hard* because $\mathcal{Y}_n^k$ is a nonconvex set. Therefore, our modeling framework isolates the hardness of Problem (8) in $\mathcal{Y}_n^k$.

To cope with the nonlinear constraints $\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X}$ in a tractable fashion, we augment the objective function in (8) with a regularization term. Namely, we consider

$$\begin{aligned} \min_{\boldsymbol{Y}\in\mathcal{Y}_n^k} \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \ & \langle \boldsymbol{C}, \boldsymbol{X}\rangle + \Omega(\boldsymbol{X}) + \lambda \cdot \text{tr}(\boldsymbol{Y}) \\ \text{s.t.} \ & \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B},\ \boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X},\ \boldsymbol{X} \in \mathcal{K}. \end{aligned} \tag{9}$$

Here, the regularization term $\Omega(\boldsymbol{X})$ satisfies the following assumption:

Assumption 1. *In Problem (9), the regularization term $\Omega(\boldsymbol{X})$ is one of*

- *A spectral norm penalty, $\Omega(\boldsymbol{X}) = 0$, if $\|\boldsymbol{X}\|_\sigma \le M$ and $\Omega(\boldsymbol{X}) = +\infty$ otherwise.*
- *A Frobenius norm penalty, $\Omega(\boldsymbol{X}) = 1/2\gamma \|\boldsymbol{X}\|_F^2$.*

As we demonstrate in Section 3.2, Assumption 1 is crucial for developing efficient low-rank algorithms for the regularizer drives the convexity (see Theorem 3) and smoothness (see Lemma 3) of the problem and also makes subgradients readily accessible (Table 2). The idea of leveraging regularization to optimize nonsmooth

Table 1. Analogy Between Mixed-Integer Conic and Mixed-Projection Conic Optimization

Framework	Bertsimas et al. (2021)	This paper
Parsimony concept	Cardinality	Rank
Nonconvex outer set	Binaries	Orthogonal projection matrices
Strongly convex regularizer	ℓ_2^2 penalty	Frobenius norm squared
Boundedness regularizer	ℓ_∞ norm	Spectral norm
Nonlinear formulation	$\boldsymbol{x} = \boldsymbol{x} \circ \boldsymbol{z};\ \boldsymbol{z} \in \{0,1\}^n$	$\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X},\ \boldsymbol{Y} \in \mathcal{Y}_n$
Big-M formulation	$-M\boldsymbol{z} \le \boldsymbol{x} \le M\boldsymbol{z}$	$\begin{pmatrix} M\boldsymbol{Y} & \boldsymbol{X} \\ \boldsymbol{X}^\top & M\mathbb{I} \end{pmatrix} \succeq \boldsymbol{0}$
Perspective formulation	$\begin{pmatrix} \theta_i & x_i \\ x_i & z_i \end{pmatrix} \succeq \boldsymbol{0}$	$\begin{pmatrix} \boldsymbol{\theta} & \boldsymbol{X} \\ \boldsymbol{X}^\top & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{0}$
Convex relaxation complexity	Linear/second-order cone	Semidefinite
Greedy rounding mechanism	Coordinate-wise	Singular value decomposition

functions is reminiscent of the smoothing technique (Nesterov 2005, 2007).

The two regularizers are matrix analogues of the big-M constraints (constraints on the ℓ_∞ norm of the continuous variables) and ridge regularization (penalty on the ℓ_2^2 norm) for vectors. In MICO, such regularization terms can efficiently cope with nonlinear constraints between continuous and binary variables (Bertsimas et al. 2021) and motivate our current approach. Practically speaking, regularization can be a natural component of the original Problem (8); otherwise, we advocate for introducing it artificially for it leads to tractable algorithms with moderate impact on the resulting solution. For instance, if M is large enough so that the optimal solution to Problem (8), $\boldsymbol{X}^\star$, satisfies $\|\boldsymbol{X}^\star\|_\sigma \le M$, Problems (9) and (8) are equivalent. In Online Section EC.5.1, we develop a disciplined technique for computing such an M. With the Frobenius norm penalty, the gap between the objectives of Problems (9) and (8) is at most $(\gamma/2)\|\boldsymbol{X}^\star\|_F^2$, which can certainly be bounded whenever $\mathrm{tr}(\boldsymbol{X})$ is bounded as often occurs in practice.

For ease of notation, we let

$$g(\boldsymbol{X}) = \langle \boldsymbol{C}, \boldsymbol{X}\rangle + \begin{cases} 0, & \text{if } \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B},\ \boldsymbol{X} \in \mathcal{K}, \\ +\infty, & \text{otherwise,} \end{cases}$$

denote the unregularized second-stage cost for a given $\boldsymbol{X}$. Therefore, Problem (9) can be written as

$$\min_{\boldsymbol{Y} \in \mathcal{Y}_n^k} f(\boldsymbol{Y}) + \lambda \cdot \mathrm{tr}(\boldsymbol{Y}), \tag{10}$$

$$\text{where } f(\boldsymbol{Y}) := \min_{\boldsymbol{X} \in \mathbb{R}^{n\times m}} g(\boldsymbol{X}) + \Omega(\boldsymbol{X}) \quad \text{s.t.} \quad \boldsymbol{X} = \boldsymbol{Y}\boldsymbol{X} \tag{11}$$

yields a best choice of $\boldsymbol{X}$ given $\boldsymbol{Y}$. As we establish in this section, this turns out to be a computationally useful reformulation for f is convex in $\boldsymbol{Y}$ (see Theorem 3) and Lipschitz continuous (see Lemma 3), and therefore, the nonconvexity in the problem has been isolated within the set $\mathcal{Y}_n^k$.

Observe that both regularizers are coercive (i.e., "blow up" to $+\infty$ as $\|\boldsymbol{X}\| \to \infty$) and, therefore, render all unbounded solutions infeasible and ensure the compactness of the level sets of $\boldsymbol{X} \mapsto g(\boldsymbol{X}) + \Omega(\boldsymbol{X})$. This alleviates two of the major issues with conic duality (Ben-Tal and Nemirovski 2001, theorem 2.4.1). First, regularization ensures that optimal solutions to conic problems are attained (see Blekherman et al. 2012, example 2.27, for a regularization-free counterexample). Second, regularization ensures that infeasibility of a conic system is *certifiable*,[1] that is, there is either a feasible solution or a certificate of infeasibility. In general, a conic system could be infeasible but asymptotically feasible, that is,

$$\nexists \boldsymbol{X} : \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}, \boldsymbol{X} \in \mathcal{K} \text{ but}$$

$$\exists \{\boldsymbol{X}_t\}_{t=1}^\infty : \boldsymbol{X}_t \in \mathcal{K} \quad \forall\, t \text{ with } \|\boldsymbol{A}\boldsymbol{X}_t - \boldsymbol{B}\| \to 0.$$

Here, the regularization term ensures that the set of feasible $\boldsymbol{X}$ (with objective at most $\theta_0 \in \mathbb{R}$) is a closed

Table 2. Regularization Penalties and Conjugates as Defined in Lemma 2

Penalty	$\Omega(\boldsymbol{X})$	$\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$	$\frac{\partial}{\partial Y_{i,j}}\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$
Spectral norm ($\boldsymbol{X}$ rectangular)	$\begin{cases} 0, & \text{if } \|\boldsymbol{X}\|_\sigma \le M, \\ +\infty, & \text{otherwise,} \end{cases}$	$\frac{M}{2}\langle \boldsymbol{Y}, \boldsymbol{V}_{11}\rangle + \frac{M}{2}\langle \boldsymbol{I}_m, \boldsymbol{V}_{22}\rangle$ s.t. $\begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^\top & \boldsymbol{V}_{22} \end{pmatrix} \succeq \boldsymbol{0}$,	$\frac{M}{2}V_{11,i,j}$.
Spectral norm ($\boldsymbol{X}$ symmetric)	$\begin{cases} 0, & \text{if } \|\boldsymbol{X}\|_\sigma \le M, \\ +\infty, & \text{otherwise,} \end{cases}$	$M\langle \boldsymbol{Y}, \boldsymbol{V}_{11} + \boldsymbol{V}_{22}\rangle$ s.t. $\boldsymbol{\alpha} = \boldsymbol{V}_{11} - \boldsymbol{V}_{22}$, $\boldsymbol{V}_{11}, \boldsymbol{V}_{2,2} \succeq \boldsymbol{0}$,	$M(V_{11} + V_{22})_{i,j}$.
Frobenius norm	$\frac{1}{2\gamma}\|\boldsymbol{X}\|_F^2$	$\frac{\gamma}{2}\langle \boldsymbol{\alpha}, \boldsymbol{Y}\boldsymbol{\alpha}\rangle$	$\frac{\gamma}{2}\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\rangle$

convex compact set. Therefore, $f(\boldsymbol{Y})$ cannot generate an asymptotically feasible problem.

Finally, the two regularization functions in Assumption 1 satisfy a nontrivial property that turns out to be crucial in both proving that $f(\boldsymbol{Y})$ is convex and deriving our overall algorithmic strategy.

Lemma 2. *Consider a regularization function $\Omega(\boldsymbol{X})$ satisfying Assumption 1. Then, there exists a Fenchel conjugate $\Omega^\star$ (see, e.g., Boyd and Vandenberghe 2004, chapter 3.3.1) such that, for any projection matrix $\boldsymbol{Y} \in \mathcal{Y}_n$ and any matrix $\boldsymbol{\alpha}$, we have*

$$\min_{\boldsymbol{X}} \{\Omega(\boldsymbol{YX}) + \langle \boldsymbol{\alpha}, \boldsymbol{YX}\rangle\} = \max_{\boldsymbol{V}_{11},\boldsymbol{V}_{22}} \; -\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}),$$

and $\Omega^\star$ is linear in $\boldsymbol{Y}$ (see Table 2 for its explicit definition).

Proof of Lemma 2. We start with the Frobenius regularization case, $\Omega(\boldsymbol{X}) = (\gamma/2)\|\boldsymbol{X}\|_F$ and $\min_{\boldsymbol{X}} \{\Omega(\boldsymbol{Y}\,\boldsymbol{X}) + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle\} = (\gamma/2)\,\|\,\boldsymbol{Y}\,\boldsymbol{X}\|_F + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle$. Any solution to the minimization problem satisfies the first-order condition $\frac{1}{\gamma}\boldsymbol{Y}\,\boldsymbol{X} + \boldsymbol{Y}\boldsymbol{\alpha} = 0$. Hence, because $\boldsymbol{Y}^2 = \boldsymbol{Y}$, $\boldsymbol{X}^\star = -\gamma\boldsymbol{Y}\boldsymbol{\alpha}$ satisfies the first-order condition, and the optimal objective value is $-\Omega^\star(\alpha, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}) = -(\gamma/2)\langle \boldsymbol{\alpha}, \boldsymbol{Y}\boldsymbol{\alpha}\rangle$.

The spectral case is technically more challenging and detailed proofs are deferred to Online Appendix EC.2. In the rectangular case, Online Lemma EC.2.2 with $\boldsymbol{Y}' = \mathbb{I}_m$ yields

$$\min_{\boldsymbol{X}} \{\Omega(\boldsymbol{Y}\,\boldsymbol{X}) + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle\} = \max_{\boldsymbol{V}_{11},\boldsymbol{V}_{22}} \; -\frac{M}{2}\langle \boldsymbol{Y}, \boldsymbol{V}_{11}\rangle - \frac{M}{2}\langle \boldsymbol{I}_m, \boldsymbol{V}_{22}\rangle \quad \text{s.t.} \begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^\top & \boldsymbol{V}_{22} \end{pmatrix} \succeq \boldsymbol{0}.$$

In the symmetric case, Online Lemma EC.2.3 states that

$$\min_{\boldsymbol{X}} \{\Omega(\boldsymbol{Y}\,\boldsymbol{X}) + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle\} = \max_{\boldsymbol{V}_{11},\boldsymbol{V}_{22} \succeq \boldsymbol{0}} \; -M\langle \boldsymbol{Y}, \boldsymbol{V}_{11} + \boldsymbol{V}_{22}\rangle \text{ s.t. } \boldsymbol{\alpha} = \boldsymbol{V}_{11} - \boldsymbol{V}_{22}.$$

3.2. A Saddle-Point Reformulation

We now reformulate Problem (9) as a saddle-point problem. This reformulation is significant for two reasons. First, as shown in the proof of Theorem 3, it leverages the nonlinear constraint $\boldsymbol{X} = \boldsymbol{Y}\,\boldsymbol{X}$ by introducing a new matrix of variables $\boldsymbol{V} \in \mathbb{R}^{n\times m}$ such that $\boldsymbol{V} = \boldsymbol{Y}\,\boldsymbol{X}$, giving

$$f(\boldsymbol{Y}) = \min_{\boldsymbol{V},\boldsymbol{X}} \{g(\boldsymbol{V}) + \Omega(\boldsymbol{Y}\,\boldsymbol{X}) : \boldsymbol{V} = \boldsymbol{Y}\,\boldsymbol{X}\},$$

a substitution reminiscent of the Douglas–Rachford splitting technique for composite convex optimization problems (Douglas and Rachford 1956, Eckstein and Bertsekas 1992). Second, it proves that the regularizer $\Omega(\boldsymbol{X})$ drives the convexity and smoothness of $f(\boldsymbol{Y})$. To derive the problem's dual, we require the following:

Assumption 2. *For each Subproblem (11) generated by $f(\boldsymbol{Y})$, where $\boldsymbol{Y} \in \mathcal{Y}_n^k$, either the optimization problem is infeasible, or strong duality holds.*

Assumption 2 holds under Slater's constraint qualification (Boyd and Vandenberghe 2004, section 5.2.3). By invoking Assumption 2, the following theorem reformulates (10) as a saddle-point problem:

Theorem 3. *Suppose that Assumption 2 holds and $\Omega(\cdot)$ is either the spectral or Frobenius regularizer. Then, the following two optimization problems are equivalent:*

$$f(\boldsymbol{Y}) := \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \; g(\boldsymbol{X}) + \Omega(\boldsymbol{X}) \quad \text{s.t.} \quad \boldsymbol{X} = \boldsymbol{Y}\,\boldsymbol{X}, \tag{12}$$

$$= \max_{\boldsymbol{\alpha},\boldsymbol{V}_{11},\boldsymbol{V}_{22}} \; h(\boldsymbol{\alpha}) - \Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}), \tag{13}$$

where $h(\boldsymbol{\alpha}) := \max_{\boldsymbol{\Pi}: \boldsymbol{C}-\boldsymbol{\alpha}-\boldsymbol{A}^\top\boldsymbol{\Pi}\in\mathcal{K}^\star} \langle \boldsymbol{b}, \boldsymbol{\Pi}\rangle$, $\mathcal{K}^\star := \{\boldsymbol{W} : \langle \boldsymbol{W}, \boldsymbol{X}\rangle \geq 0,\ \forall\, \boldsymbol{X} \in \mathcal{K}\}$ denotes the dual cone to $\mathcal{K}$, and $\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$ is defined in Table 2.

Proof of Theorem 3. Let us fix $\boldsymbol{Y} \in \mathcal{Y}_n^k$, and suppose that strong duality holds for the inner minimization problem that defines $f(\boldsymbol{Y})$. To progress, we introduce a matrix $\boldsymbol{V} \in \mathbb{R}^{n\times m}$ such that $\boldsymbol{V} = \boldsymbol{Y}\,\boldsymbol{X}$ and obtain the relaxation

$$\min_{\boldsymbol{X},\boldsymbol{V}} \; g(\boldsymbol{V}) + \Omega(\boldsymbol{Y}\,\boldsymbol{X}) \quad \text{s.t.} \quad \boldsymbol{V} = \boldsymbol{Y}\,\boldsymbol{X}. \tag{14}$$

Let us verify that this relaxation is a valid substitution, that is, that Problems (12) and (14) have the same optimal objective, $f(\boldsymbol{Y})$. If $\boldsymbol{X}$ is feasible for (12), then $(\boldsymbol{V} = \boldsymbol{X}, \boldsymbol{X})$ is obviously feasible for (14) with same objective value. Similarly, let $(\boldsymbol{V}, \boldsymbol{X})$ be feasible for (14). $\boldsymbol{Y}\,\boldsymbol{V} = \boldsymbol{Y}^2\boldsymbol{X} = \boldsymbol{Y}\,\boldsymbol{X} = \boldsymbol{V}$ because $\boldsymbol{Y}^2 = \boldsymbol{Y}$. Hence, $\boldsymbol{V}$ is feasible for (12) with same objective value.

Now, let $\boldsymbol{\alpha}$ denote the dual variables associated with the coupling constraints $\boldsymbol{V} = \boldsymbol{Y}\,\boldsymbol{X}$. The minimization problem is then equivalent to its dual problem, which is given by

$$f(\boldsymbol{Y}) = \max_{\boldsymbol{\alpha}} \; h(\boldsymbol{\alpha}) + \min_{\boldsymbol{X}} \; [\Omega(\boldsymbol{Y}\,\boldsymbol{X}) + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle],$$

where $h(\boldsymbol{\alpha}) := \inf_{\boldsymbol{V}} g(\boldsymbol{V}) - \langle \boldsymbol{V}, \boldsymbol{\alpha}\rangle$ is, up to a sign, the Fenchel conjugate of g. By a standard application of Fenchel duality, it follows that

$$h(\boldsymbol{\alpha}) = \max_{\boldsymbol{\Pi}} \langle \boldsymbol{b}, \boldsymbol{\Pi}\rangle + \begin{cases} 0, & \text{if } \boldsymbol{C} - \boldsymbol{\alpha} - \boldsymbol{A}^\top\boldsymbol{\Pi} \in \mathcal{K}^\star, \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, from Lemma 2, we have $\min_{\boldsymbol{X}}\{\Omega(\boldsymbol{Y}\,\boldsymbol{X}) + \langle \boldsymbol{\alpha}, \boldsymbol{Y}\,\boldsymbol{X}\rangle\} = \max_{\boldsymbol{V}_{11},\boldsymbol{V}_{22}} -\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$, which concludes the proof. Alternatively, under either penalty, if the inner minimization problem defining $f(\boldsymbol{Y})$ is infeasible, then its dual problem is unbounded by weak duality.[2] □

Remark 2. In the unregularized case, that is, $\Omega(\boldsymbol{X}) = 0$, we can derive a similar reformulation:

$$\min_{\boldsymbol{Y}\in\mathcal{Y}_n^k} \max_{\boldsymbol{\alpha}\in\mathbb{R}^{n\times m}} h(\boldsymbol{\alpha}) + \lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \text{ s.t. } \boldsymbol{Y}\boldsymbol{\alpha} = \boldsymbol{0}. \tag{15}$$

Under this lens, regularization of the primal problem is equivalent to a relaxation in the dual formulation: the hard constraint $\boldsymbol{Y}\boldsymbol{\alpha} = \boldsymbol{0}$ is penalized by $-\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$.

Remark 3. By Theorem 3 and Lemma 2, $f(\boldsymbol{Y})$ is convex as the point-wise maximum of functions that are linear in $\boldsymbol{Y}$.

By Theorem 3, when we evaluate $f(\hat{\boldsymbol{Y}})$, one of two alternatives occurs. The first is that we have $f(\hat{\boldsymbol{Y}}) < +\infty$ and there is some optimal $(\boldsymbol{\alpha}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$. In this case, we construct the lower approximation

$$f(\boldsymbol{Y}) \geq f(\hat{\boldsymbol{Y}}) + \langle \boldsymbol{H}, \boldsymbol{Y} - \hat{\boldsymbol{Y}}\rangle,$$

where $H_{i,j} = \frac{\partial}{\partial Y_{i,j}}\Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22})$ (see Table 2 for closed-form expression of the partial derivatives, which follow readily from Danskin's theorem; see, e.g., Bertsekas 2016, proposition B.22). The second alternative is that $f(\hat{\boldsymbol{Y}}) = +\infty$, in which case, by the conic duality theorem (see Ben-Tal and Nemirovski 2001, chapter 2), there exists some $(\boldsymbol{\alpha}, \boldsymbol{\Pi})$ such that

$$\boldsymbol{C} - \boldsymbol{\alpha} - \boldsymbol{A}^\top\boldsymbol{\Pi} \in \mathcal{K}^\star \text{ and } \langle \boldsymbol{b}, \boldsymbol{\Pi}\rangle > \langle -\boldsymbol{H}, \hat{\boldsymbol{Y}}\rangle. \tag{16}$$

Under this alternative, we can separate $\hat{\boldsymbol{Y}}$ from the set of feasible $\boldsymbol{Y}$s by imposing the cut $0 \geq \langle \boldsymbol{b}, \boldsymbol{\Pi}\rangle + \langle \boldsymbol{H}, \boldsymbol{Y}\rangle$. Under either alternative, we obtain a globally valid first-order under-estimator of the form

$$zf(\boldsymbol{Y}) \geq h + \langle \boldsymbol{H}, \boldsymbol{Y} - \hat{\boldsymbol{Y}}\rangle, \tag{17}$$

where z, h are defined as

$$z = \begin{cases} 1, & \text{if } f(\hat{\boldsymbol{Y}}) < +\infty, \\ 0, & \text{if } f(\hat{\boldsymbol{Y}}) = +\infty, \end{cases} \quad \text{and}$$

$$h = \begin{cases} f(\hat{\boldsymbol{Y}}), & \text{if } f(\hat{\boldsymbol{Y}}) < +\infty, \\ \langle \boldsymbol{b}, \boldsymbol{\Pi}\rangle + \langle \boldsymbol{H}, \hat{\boldsymbol{Y}}\rangle, & \text{if } f(\hat{\boldsymbol{Y}}) = +\infty. \end{cases} \tag{18}$$

Accordingly, a valid strategy for minimizing $f(\boldsymbol{Y})$ is to iteratively minimize and refine a piecewise linear under-estimator of f, in which each linear piece is of the form (17). Indeed, as we see in Section 4, this strategy gives rise to the global optimization algorithm known as outer approximation.

3.2.1. Smoothness. We now demonstrate that $f(\boldsymbol{Y})$ is smooth in the sense of Lipschitz continuity under a boundedness assumption on the size of the dual variables, which is a crucial property for ensuring the convergence of our global optimization methods and bounding the quality of our semidefinite relaxation and greedy rounding methods. Formally, the following result follows directly from Theorem 3.

Lemma 3. *Let $\boldsymbol{Y}, \boldsymbol{Y}' \in \mathrm{Conv}(\mathcal{Y}_n^k)$ be on the convex hull of the orthogonal projection matrices. Then,*

$$f(\boldsymbol{Y}) - f(\boldsymbol{Y}') \leq \Omega^\star(\boldsymbol{\alpha}^\star(\boldsymbol{Y}), \boldsymbol{Y}' - \boldsymbol{Y}, \boldsymbol{V}_{11}^\star(\boldsymbol{Y}), \boldsymbol{V}_{22}^\star(\boldsymbol{Y})).$$

Moreover, suppose $\boldsymbol{\alpha}^\star(\boldsymbol{Y}), \boldsymbol{V}_{11}^\star(\boldsymbol{Y}), \boldsymbol{V}_{22}^\star(\boldsymbol{Y})$ can be bounded independently from $\boldsymbol{Y}$, that is, $\|\boldsymbol{\alpha}^\star(\boldsymbol{Y})\|_\sigma \leq L_1$, $\|\boldsymbol{V}_{11}^\star(\boldsymbol{Y})\|_\sigma \leq L_2$, $\|\boldsymbol{V}_{22}^\star(\boldsymbol{Y})\|_\sigma \leq L_2$. Then, under spectral regularization, we have

$$f(\boldsymbol{Y}) - f(\boldsymbol{Y}') \leq M\langle \boldsymbol{V}_{11}^\star(\boldsymbol{Y}), \boldsymbol{Y}' - \boldsymbol{Y}\rangle \leq ML_2\|\boldsymbol{Y}' - \boldsymbol{Y}\|_*, \tag{19}$$

and under Frobenius regularization, we have

$$f(\boldsymbol{Y}) - f(\boldsymbol{Y}') \leq \frac{\gamma}{2}\langle \boldsymbol{\alpha}^{\star\top}(\boldsymbol{Y})\boldsymbol{\alpha}^\star(\boldsymbol{Y}), \boldsymbol{Y}' - \boldsymbol{Y}\rangle \leq \frac{\gamma}{2}L_1^2\|\boldsymbol{Y}' - \boldsymbol{Y}\|_*, \tag{20}$$

where the bounds involving L_1, L_2 follow from Holder's inequality, that is, $|\langle \boldsymbol{X}, \boldsymbol{Y}\rangle| \leq \|\boldsymbol{X}\|_\sigma \|\boldsymbol{Y}\|_*$.

Remark 4. Online Section EC.5.1 develops disciplined techniques for computing an M such that the constraint $\|\boldsymbol{X}\|_\sigma \leq M$ in the primal does not alter the optimal objective. The same technique, applied to the dual, yields explicit bounds on L_1. Moreover, because there exists an optimal pair $(\boldsymbol{V}_{11}, \boldsymbol{V}_{22})$ that is an explicit function of an optimal $\boldsymbol{\alpha}$, this translates into explicit bounds on L_2.

3.3. Semidefinite Relaxations

To lower bound (10)'s objective, we invoke Lemma 1 to relax the nonconvex constraint $\boldsymbol{Y} \in \mathcal{Y}_n^k$ to

$$\boldsymbol{Y} \in \mathrm{Conv}\left(\mathcal{Y}_n^k\right) = \{\boldsymbol{Y} \in S^n : \boldsymbol{0} \preceq \boldsymbol{Y} \preceq \mathbb{I}, \mathrm{tr}(\boldsymbol{Y}) \leq k\}.$$

This yields the saddle-point problem

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}\left(\mathcal{Y}_n^k\right)} \max_{\boldsymbol{\alpha}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}\in S^m} h(\boldsymbol{\alpha}) - \Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}) + \lambda\cdot\mathrm{tr}(\boldsymbol{Y}). \tag{21}$$

Problem (21) can, in turn, be reformulated as an SDO. Indeed, under Assumption 2, we obtain a semidefinite formulation by taking the dual of Problem (21) with respect to $\boldsymbol{\alpha}$. Formally, we have the following results (proofs deferred to Online Appendices EC.3.1 and EC.3.2, respectively):

Lemma 4. *Suppose that Assumption 2 holds. Then, strong duality holds between*

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}\left(\mathcal{Y}_n^k\right)} \max_{\boldsymbol{\alpha}\in\mathbb{R}^{n\times m}} h(\boldsymbol{\alpha}) - \frac{\gamma}{2}\langle \boldsymbol{\alpha}, \boldsymbol{Y}\boldsymbol{\alpha}\rangle + \lambda\cdot\mathrm{tr}(\boldsymbol{Y}), \tag{22}$$

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}\left(\mathcal{Y}_n^k\right)} \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m},\boldsymbol{\theta}\in S^n} \quad g(\boldsymbol{X})+\frac{1}{2\gamma}\mathrm{tr}(\boldsymbol{\theta})+\lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \quad \text{s.t.} \quad \begin{pmatrix}\boldsymbol{\theta} & \boldsymbol{X}\\ \boldsymbol{X}^\top & \boldsymbol{Y}\end{pmatrix}\succeq \mathbf{0}. \tag{23}$$

Lemma 5. *Suppose that Assumption 2 holds. Then, strong duality holds between*

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}\left(\mathcal{Y}_n^k\right)} \max_{\boldsymbol{\alpha}\in S^n,\boldsymbol{V}_{11},\boldsymbol{V}_{22}\succeq\mathbf{0}} h(\boldsymbol{\alpha})-M\langle\boldsymbol{Y},\boldsymbol{V}_{11}+\boldsymbol{V}_{22}\rangle+\lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \quad \text{s.t.} \quad \boldsymbol{\alpha}=\boldsymbol{V}_{11}-\boldsymbol{V}_{22}, \tag{24}$$

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}\left(\mathcal{Y}_n^k\right)} \min_{\boldsymbol{X}\in S^n} \quad g(\boldsymbol{X})+\lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \quad \text{s.t.} -M\boldsymbol{Y}\preceq\boldsymbol{X}\preceq M\boldsymbol{Y}. \tag{25}$$

We now offer some remarks on these bidual problems:

- We can derive a more general version of Lemma 5 without the symmetry assumption on $\boldsymbol{X}$ in much the same manner via the Schur complement lemma.
- Problem (23)'s formulation generalizes the perspective relaxation from vectors to matrices. This suggests that (23) is an efficient formulation for addressing rank constraints as perspective formulations efficiently address cardinality-constrained problems with conic quadratic (Günlük and Linderoth 2012) or power cone (Aktürk et al. 2009) objectives; indeed, they provide a theoretical basis for scalable algorithms for sparse regression (Bertsimas et al. 2020, Hazimeh et al. 2021), sparse portfolio selection (Zheng et al. 2014, Bertsimas and Cory-Wright 2022), and network design (Fischetti et al. 2017) problems among others.

3.4. Penalty Interpretations of Relaxations

In this section, we consider instances in which rank is penalized in the objective only and interpret the preceding convex relaxations as penalty functions in the tradition of Fazel (2002) and Recht et al. (2010).

With a spectral regularizer, the convex relaxation is equivalent to using the popular nuclear norm penalty (see Online Lemmas EC.1 and EC.2). However, Zhang et al. (2013) show that the nuclear norm cannot encourage low-rank solutions for problems with constraints $\boldsymbol{X}\succeq\mathbf{0},\mathrm{tr}(\boldsymbol{X})=k$, such as sparse PCA (d'Aspremont et al. 2007) or k-means clustering (Peng and Wei 2007). In the presence of the Frobenius penalty, Lemma 6 exhibits an alternative to the nuclear norm penalty that can encourage low-rank solutions in these situations. Our result generalizes the *reverse Huber penalty* of Pilanci et al. (2015) and Dong et al. (2015) from cardinality to rank objectives (proof deferred to Online Appendix EC.3.3).

Lemma 6. *Suppose that Assumption 2 holds. Then, the following problems are equivalent:*

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}(\mathcal{Y}_n)} \min_{\boldsymbol{X}\in\mathbb{R}^{n\times m},\boldsymbol{\theta}\in S^n} \quad g(\boldsymbol{X})+\frac{1}{2\gamma}\mathrm{tr}(\boldsymbol{\theta})+\lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \quad \text{s.t.} \quad \begin{pmatrix}\boldsymbol{\theta} & \boldsymbol{X}\\ \boldsymbol{X}^\top & \boldsymbol{Y}\end{pmatrix}\succeq\mathbf{0}, \tag{26}$$

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} \quad g(\boldsymbol{X})+\sum_{i=1}^{n}\min\left(\sqrt{\frac{2\lambda}{\gamma}}\sigma_i(\boldsymbol{X}),\lambda+\frac{\sigma_i(\boldsymbol{X})^2}{2\gamma}\right). \tag{27}$$

Remark 5. Since

$$\min_{0\le\theta\le 1}\left[\lambda\theta+\frac{t^2}{\theta}\right]=\begin{cases}2\sqrt{\lambda}\,|t|, & \text{if } |t|\le\sqrt{\lambda},\\ t^2+\lambda, & \text{otherwise,}\end{cases}$$

the proof of Lemma 6 reveals that Problems (26) and (27) are equivalent to minimizing

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times m},\boldsymbol{\theta}\in\mathbb{R}^n:\,\mathbf{0}\le\boldsymbol{\theta}\le\boldsymbol{e}} \quad g(\boldsymbol{X})+\sum_{i=1}^{n}\left(\lambda\theta_i+\frac{\sigma_i(\boldsymbol{X})^2}{2\gamma\theta_i}\right), \tag{28}$$

which applies the smooth penalty $t\mapsto\lambda\theta+t^2/(2\gamma\theta):0\le\theta\le 1$ to model the nonconvex cost $t\mapsto\lambda\|t\|_0+t^2/(2\gamma)$ incurred by each singular value of $\boldsymbol{X}$. Indeed, this smooth penalty is precisely the convex envelope of the nonconvex cost function (see, e.g., Günlük and Linderoth 2012). Compared with other penalties for low-rank problems (Fan and Li 2001, Zhang 2010), this generalized Huber penalty is convex, amenable to efficient alternating minimization procedures (see Section 4.2.2), and could be of independent interest to the statistical learning community.

4. Efficient Algorithmic Approaches

In this section, we present an efficient numerical approach to solve Problem (1) and its convex relaxations. The backbone is an outer-approximation strategy, embedded within a nonconvex QCQO branch-and-bound procedure to solve the problem exactly. We also propose rounding heuristics to find good feasible solutions and semidefinite-free methods for optimizing over (1)'s convex relaxations.

The primary motivations for developing an outer-approximation procedure and solving a mixed-projection problem as saddle-point problems are twofold. First, we are not aware of any exact solvers that address mixed-projection problems with semidefinite constraints. Instead, a decomposition strategy, such as outer approximation, can be readily implemented using a conjunction of Gurobi (to solve nonconvex quadratically constrained master problems) and Mosek (to solve conic subproblems). Second,

decomposition schemes for mixed-integer semidefinite problems typically outperform one-shot strategies (Belotti et al. 2013), so we expect—and observe in Section 5.3—a similar comparison for mixed-projection optimization, hence connecting the frameworks in both theory (see Table 1) and practice.

4.1. A Globally Optimal Cutting-Plane Method

The analysis in the previous section reveals that evaluating $f(\boldsymbol{Y})$ yields a globally valid first-order under-estimator of f. Therefore, a numerically efficient strategy for minimizing $f(\boldsymbol{Y})$ is to iteratively minimize and refine a piecewise linear under-estimator of $f(\boldsymbol{Y})$. This strategy is known as (OA) and is originally proposed by Duran and Grossmann (1986). OA iteratively constructs under-estimators of the following form at each iterate $t+1$:

$$f_{t+1}(\boldsymbol{Y}) = \max_{1\le i\le t} \left\{ f(\boldsymbol{Y}_i) + \langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}_i \rangle \right\}. \tag{29}$$

By iteratively minimizing $f_{t+1}(\boldsymbol{Y})$, we obtain a nondecreasing sequence of under-estimators $f_t(\boldsymbol{Y}_t)$ and nonincreasing sequence of over-estimators $\min_{i\in[t]} f(\boldsymbol{Y}_i)$ that converge to an ϵ-optimal solution within a finite number of iterations; see Section 3.2 for details on cut generation. Indeed, because $\mathrm{Conv}(\mathcal{Y}_n^k)$ is a compact set and f is an L-Lipschitz continuous function in $\boldsymbol{Y}$, OA never visits a ball of radius $\frac{\epsilon}{L}$ twice.

We formalize this numerical procedure in Algorithm 1 and state its convergence properties (proof of convergence deferred to Online Appendix EC.3.4).

Algorithm 1 (An Outer-Approximation Method for Problem (10))

Require: Initial solution $\boldsymbol{Y}_1$

$t \leftarrow 1$

repeat

Compute $\boldsymbol{Y}_{t+1}, \theta_{t+1}$ solution of

$$\min_{\boldsymbol{Y}\in\mathcal{Y}_n^k,\theta} \ \theta + \lambda\cdot\mathrm{tr}(\boldsymbol{Y})$$
$$\text{s.t.} \ \ z_i\theta \ge h_i + \langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}_i\rangle \quad \forall\, i\in[t].$$

Compute $f(\boldsymbol{Y}_{t+1})$, $\boldsymbol{H}_{t+1}$, z_{t+1}, d_{t+1}

until $f(\boldsymbol{Y}_t) - \theta_t \le \varepsilon$

return $\boldsymbol{Y}_t$

Theorem 4. *Suppose that Assumptions 1 and 2 hold and that there exists some Lipschitz constant L such that, for any feasible $\boldsymbol{Y}, \boldsymbol{Y}' \in \mathrm{Conv}(\mathcal{Y}_n^k)$, we have $|f(\boldsymbol{Y}) - f(\boldsymbol{Y}')| \le L\|\boldsymbol{Y}-\boldsymbol{Y}'\|_F$, and for any feasibility cut $\langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}_i\rangle + h_i \le 0$, we have $|\langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}'\rangle| \le L\|\boldsymbol{Y}-\boldsymbol{Y}'\|_F$. Let $\boldsymbol{Y}_t \in \mathcal{Y}_n^k$ be a feasible solution returned by the t^{th} iterate of Algorithm 1, where*

$$t \ge \left(\frac{Lk}{\epsilon} + 1\right)^{n^2}.$$

Then, $\boldsymbol{Y}_t$ is an ϵ-optimal and ϵ-feasible solution to Problem (9). Moreover, suppose that we set $\epsilon \to 0$. Then, any limit point of $\{\boldsymbol{Y}_t\}_{t=1}^{\infty}$ solves (9).

In Algorithm 1, we need to repeatedly solve optimization problems of the form

$$\min_{\boldsymbol{Y}\in\mathcal{Y}_n^k,\theta} \ \theta + \lambda\cdot\mathrm{tr}(\boldsymbol{Y})$$
$$\text{s.t.} \ \ z_i\theta \ge h_i + \langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}_i\rangle \quad \forall\, i\in[t], \tag{30}$$

which requires a tractable representation of $\mathcal{Y}_n^k$. Fortunately, Gurobi 9.0 contains a globally optimal spatial branch-and-bound method for general QCQOs that recursively partitions the feasible region into boxes and invokes McCormick inequalities to obtain valid upper and lower bounds on each box; see Achterberg and Towle (2020) for a discussion of Gurobi's bilinear solver and Belotti et al. (2013) for a general theory of spatial branch-and-bound. Here, we represent $\boldsymbol{Y}$ by introducing a matrix $\boldsymbol{U}\in\mathbb{R}^{n\times k}$ and requiring that $\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{U}^\top$ and $\boldsymbol{U}^\top\boldsymbol{U} = \mathbb{I}$. This allows Algorithm 1 to be implemented by iteratively solving a sequence of QCQOs and conic optimization problems. Moreover, to decrease the amount of branching required in each iteration of Algorithm 1, we strengthen the formulation by imposing second-order cone relaxations of the valid PSD constraint $\boldsymbol{Y} \succeq \boldsymbol{U}\boldsymbol{U}^\top$. First, we require that the 2×2 minors in $\boldsymbol{Y}$ are nonnegative, that is, $Y_{i,j}^2 \le Y_{i,i}Y_{j,j}\ \forall\, i,j\in[n]$, as proposed in Ahmadi and Majumdar (2019) and Bertsimas and Cory-Wright (2020). Second, we require that the on-diagonal entries of $\boldsymbol{Y} \succeq \boldsymbol{U}\boldsymbol{U}^\top$ are nonnegative, that is, $Y_{i,i} \ge \sum_{t=1}^k U_{i,t}^2\ \forall\, i\in[n]$. Finally, we follow Atamtürk and Gomez (2019, proposition 5) in taking a second-order cone approximation of the 2×2 minors in $\boldsymbol{Y} \succeq \boldsymbol{U}\boldsymbol{U}^\top$, that is, $0 \ge \|\boldsymbol{U}_i \pm \boldsymbol{U}_j\|_2^2 \pm 2Y_{i,j} - Y_{i,i} - Y_{j,j},\ \forall\, i,j\in[n]$. All told, we have[3]

$$\begin{aligned}
\min_{\boldsymbol{Y}\in S^n,\boldsymbol{U}\in\mathbb{R}^{n\times k},\theta} \ & \theta + \lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \\
\text{s.t.} \ \ & z_i\theta \ge h_i + \langle \boldsymbol{H}_i, \boldsymbol{Y} - \boldsymbol{Y}_i\rangle \quad \forall\, i\in[t], \\
& \boldsymbol{Y} = \boldsymbol{U}\boldsymbol{U}^\top, \boldsymbol{U}^\top\boldsymbol{U} = \mathbb{I}, \\
& Y_{i,i}Y_{j,j} \ge Y_{i,j}^2 \ \ \forall\, i,j\in[n], \\
& Y_{i,i} \ge \sum_{t=1}^k U_{i,t}^2 \ \ \forall\, i\in[n], \mathrm{tr}(\boldsymbol{Y}) = k, \\
& 0 \ge \|\boldsymbol{U}_i + \boldsymbol{U}_j\|_2^2 - 2Y_{i,j} - Y_{i,i} - Y_{j,j}, \\
& 0 \ge \|\boldsymbol{U}_i - \boldsymbol{U}_j\|_2^2 + 2Y_{i,j} - Y_{i,i} - Y_{j,j} \quad \forall\, i,j\in[n].
\end{aligned} \tag{31}$$

Finally, for a given $(\boldsymbol{Y}, \boldsymbol{U})$, we strengthen the formulation further by imposing second-order cone cuts of the form $\langle \boldsymbol{Y} - \boldsymbol{U}\boldsymbol{U}^\top, \boldsymbol{u}\boldsymbol{u}^\top \rangle \geq 0$, where $\boldsymbol{u}$ is the most negative eigenvector of $\boldsymbol{Y} - \boldsymbol{U}\boldsymbol{U}^\top$ as proposed by Sherali and Fraticelli (2002).

At each iteration, a linear optimization problem over the set of orthogonal projection matrices is solved, hence building a new branch-and-bound tree each time. We refer to this implementation as a "multitree" method. Although inefficient if implemented naively, multitree methods benefit from gradually tightening the numerical tolerance of the solver as the number of cuts increases. One can also integrate the entire procedure within a single branch-and-cut tree using lazy callbacks as proposed in the context of MICO by Quesada and Grossmann (1992). Henceforth, we refer to this implementation as a "single-tree" method. However, the benefit from using multitree over single tree is not straightforward for it depends on how the method is engineered. We benchmark both implementations in Section 5.3 and compare them to a simple benchmark described in Online Appendix EC.5.2.

4.2. Lower Bounds via Semidefinite Relaxations

To certify optimality, high-quality lower bounds are of interest and can be obtained by relaxing the nonconvex constraint $\boldsymbol{Y} \in \mathcal{Y}_n^k$ to $\boldsymbol{Y} \in \text{Conv}(\mathcal{Y}_n^k)$ as discussed in Section 3.3. In addition to a valid lower bound, the optimal solution to the relaxation $\boldsymbol{Y}^\star$ is a natural candidate for a random rounding strategy. We explore such rounding strategies in detail in the next section.

The convex relaxation yields the optimization problem (21), which can be solved using a cutting-plane method (see Section 4.2.1), an alternating minimization method (see Section 4.2.2), or reformulated as an SDO and solved as such. Because Algorithm 1 is also an outer-approximation scheme, solving the convex relaxation via a cutting-plane method has the additional benefit of producing valid linear lower approximations of $f(\boldsymbol{Y})$ with which to initialize Algorithm 1.

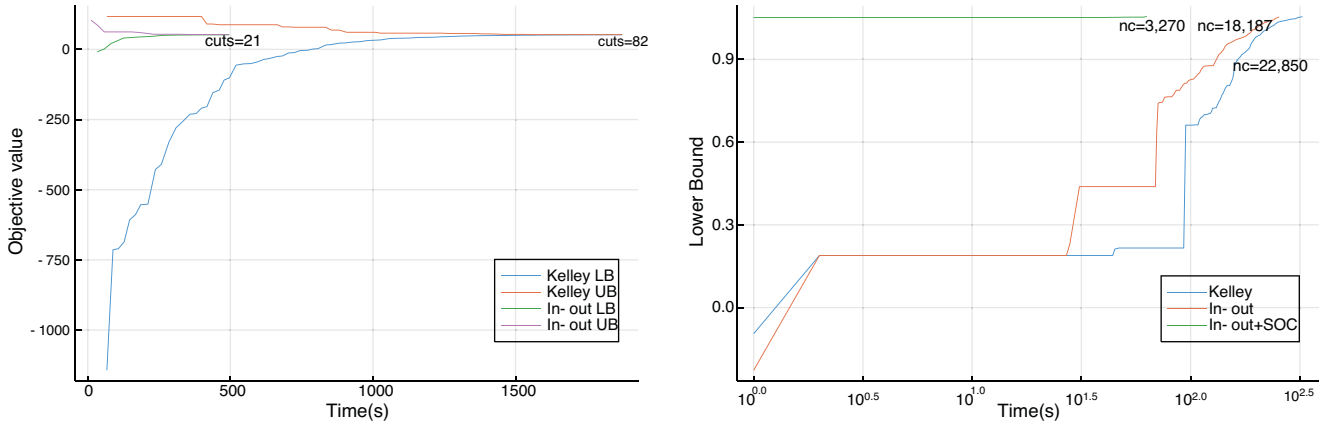
4.2.1. Cutting-Planes for Improving the Root Node.

As mentioned previously, Problem (21) can be solved by a cutting-plane method such as Kelley's (1960) algorithm, which is a continuous analog of Algorithm 1 that solves Problem (10) over $\text{Conv}(\mathcal{Y}_n^k)$ instead. The main benefit of such a method is that the cuts generated are valid for both $\text{Conv}(\mathcal{Y}_n^k)$ and $\mathcal{Y}_n^k$, and therefore, can be used to initialize Algorithm 1 and ensure that its *initial* lower bound is equal to the semidefinite relaxation. This approach often accelerates the convergence of decomposition schemes by orders of magnitude in the context of MICO (Fischetti et al. 2017). We present pseudocode in Online Appendix EC.5.3

The left panel of Figure 1 illustrates the convergence of Kelley's method and the in-out method for solving the semidefinite relaxation of a noiseless matrix completion problem.[4] Note that, in our plot of the in-out method on the continuous relaxation, we omit the time required to first solve the SDO relaxation; this is negligible (38.4 seconds) compared with the time required for either approach to solve the relaxation using cutting planes. Observe that the in-out method's lower bound is both initially better and converges substantially faster to the optimal solution than Kelley's method. This justifies our use of the in-out method over Kelley's method in numerical experiments.

Once the relaxation is solved, the generated cuts are used to initialize Algorithm 1. The right panel of Figure 1 displays the convergence profile of the lower bound of Algorithm 1 initialized with cuts from Kelley's or the in-out method (with a limit of 100 cuts). We use a single-tree implementation of Algorithm 1[5]

Figure 1. (Color online) Convergence Behavior of Kelley's Method and the In-Out Method for Solving the Semidefinite Relaxation of a Synthetic Matrix Completion Instance in Which $n = 100$ (Left), and Lower Bounds Generated by a Single-Tree Implementation of Algorithm 1 for a Synthetic Matrix Completion Instance in Which $n = 10$ (Right)



and again a noiseless matrix completion setting.[6] We also consider the impact of using the second-order cone (SOC) inequalities $Y_{i,j}^2 \leq Y_{i,i}Y_{j,j}$ in the master problem formulation. Using the in-out method and imposing the SOC inequalities are both vitally important for obtaining high-quality lower bounds from Algorithm 1. Accordingly, we make use of both ingredients in our numerical experiments.

4.2.2. Solving the Semidefinite Relaxation at Scale.

Instead of solving the convex relaxation as a saddle-point problem (21) via Kelley's method, we can leverage its formulation as a convex SDO in $(\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$ (Section 3.3) and solve it as such. However, modern IPM codes such as Mosek 9.0 cannot optimize over the Frobenius/nuclear norm penalties when $n > 200$ on a standard laptop, so we need to explore more scalable alternatives. As scalable alternatives for the nuclear norm have been studied (see Recht et al. 2010), we focus on the Frobenius penalty.

We propose solving (23) via an alternating minimization (AM) scheme in $(\boldsymbol{Y}, \boldsymbol{X})$ (see Beck and Teboulle 2009 for a modern implementation). Note that, because the objective in (23) is jointly convex in $(\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$, the problem remains jointly convex in $(\boldsymbol{Y}, \boldsymbol{X})$ after partial minimization with respect to $\boldsymbol{\theta}$. Hence, AM converges to an optimal solution to the semidefinite relaxation under standard convergence conditions for block coordinate descent techniques for convex programs (see, e.g., Bertsekas 2016, section 3.7). Our choice for AM is motivated by the fact that optimizing over $\boldsymbol{X}$ for a fixed $\boldsymbol{Y}_t$ is straightforward as stated in the following result.

Lemma 7. *Let us fix $\boldsymbol{X}_t$ and consider the following optimization problem:*

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}(\mathcal{Y}_n^k)} \min_{\boldsymbol{\theta}\in S^n} \quad g(\boldsymbol{X}_t) + \frac{1}{2\gamma}\mathrm{tr}(\boldsymbol{\theta}) + \lambda\cdot\mathrm{tr}(\boldsymbol{Y}) \quad \text{s.t.} \quad \begin{pmatrix} \boldsymbol{\theta} & \boldsymbol{X}_t \\ \boldsymbol{X}_t^\top & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{0}. \tag{32}$$

Then, an optimal choice of $\boldsymbol{\theta}$ is given by $\boldsymbol{\theta}^\star = \boldsymbol{X}_t^\top(\boldsymbol{Y}^\star)^\dagger\boldsymbol{X}_t$, where $\boldsymbol{Y}^\star = \sum_{i=1}^n \rho_i^\star \boldsymbol{u}_i\boldsymbol{u}_i^\top$, $\boldsymbol{X}_t = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ is an SVD of $\boldsymbol{X}_t$ and $\boldsymbol{\rho}^\star$ is an optimal solution to the following second-order cone problem:

$$\min_{\boldsymbol{\rho}\in[0,1]^n:\boldsymbol{e}^\top\boldsymbol{\rho}\leq k} \lambda\cdot\boldsymbol{e}^\top\boldsymbol{\rho} + \sum_{i=1}^n \frac{\sigma(\boldsymbol{X}_t)^2}{2\gamma\rho_i}. \tag{33}$$

By specializing Beck and Teboulle (2009)'s implementation of AM to the Frobenius norm penalty, we obtain an efficient numerical strategy for obtaining an optimal solution to (23), which we present in Algorithm 2. We discuss some enhancements and implementation details in Online Appendix EC.5.4.

Algorithm 2 (Accelerated Alternating Minimization)

Require: Initial solution $\boldsymbol{X}_1, \tau_1 \leftarrow 1$

$t \leftarrow 1, T_{\max}$

repeat

Compute $\boldsymbol{W}^{t+1}$ solution of
$\mathrm{argmin}_{\boldsymbol{Y}\in\mathrm{Conv}(\mathcal{Y}_n^k)}\ g(\boldsymbol{X}_t) + \frac{1}{2\gamma}\langle \boldsymbol{X}_t\boldsymbol{X}_t^\top, \boldsymbol{Y}^\dagger\rangle$.
Set $\boldsymbol{Y}^{t+1} = \boldsymbol{W}^t + \frac{\tau_{t-1}}{\tau_{t+1}}(\boldsymbol{W}_t - \boldsymbol{W}_{t-1})$.

Compute $\boldsymbol{V}^{t+1}$ solution of
$\mathrm{argmin}_{\boldsymbol{X}\in\mathbb{R}^{n\times m}} g(\boldsymbol{X}) + \frac{1}{2\gamma}\langle \boldsymbol{X}\boldsymbol{X}^\top, \boldsymbol{Y}_t^\dagger\rangle$.
Set $\boldsymbol{X}^{t+1} = \boldsymbol{V}^t + \frac{\tau_{t-1}}{\tau_{t+1}}(\boldsymbol{V}_t - \boldsymbol{V}_{t-1})$.
Set $\tau_{t+1} = \frac{1+\sqrt{1+4\tau_t^2}}{2}$.
If $t \mod 20 = 0$, compute dual bound at $\boldsymbol{Y}^{t+1}$ via Equation (35).

$t \leftarrow t+1$

until $t > T_{\max}$ or duality gap $\leq \epsilon$
return $\boldsymbol{X}_t, \boldsymbol{Y}_t$.

To confirm that Algorithm 2 has indeed converged (at least approximately) to an optimal solution, we require a dual certificate. As optimizing over the set of dual variables $\boldsymbol{\alpha}$ for a fixed $\boldsymbol{Y}_t$ does not supply such a bound, we now invoke strong duality to derive a globally valid lower bound. Formally, we have the following result (proof deferred to Online Appendix EC.3.6).

Lemma 8. *Suppose that Assumption 2 holds. Then, strong duality holds between*

$$\min_{\boldsymbol{Y}\in\mathrm{Conv}(\mathcal{Y}_n^k)} \max_{\boldsymbol{\alpha}\in\mathbb{R}^{n\times m}} \quad h(\boldsymbol{\alpha}) - \frac{\gamma}{2}\sum_{i=1}^n\sum_{j=1}^n Y_{i,j}\langle\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\rangle, \tag{34}$$

$$\max_{\boldsymbol{\alpha}\in\mathbb{R}^{n\times m}, \boldsymbol{U}\succeq\boldsymbol{0}, t\geq 0} \quad h(\boldsymbol{\alpha}) - \mathrm{tr}(\boldsymbol{U}) - kt \quad \text{s.t.} \quad \boldsymbol{U} + \mathbb{I}t \succeq \frac{\gamma}{2}\boldsymbol{\alpha}\boldsymbol{\alpha}^\top. \tag{35}$$

Note that Lemma 8 supplies a valid dual bound for a given $\boldsymbol{Y}_t$ by fixing $\boldsymbol{Y}_t$, partially maximizing for $\boldsymbol{\alpha}$ in (34), and evaluating this $\boldsymbol{\alpha}$'s objective value in (35).

Lemma 8 demonstrates that Problem (1)'s semidefinite relaxation is equivalent to maximizing the dual conjugate $h(\boldsymbol{\alpha})$, minus the k largest eigenvalues of $\frac{\gamma}{2}\boldsymbol{\alpha}\boldsymbol{\alpha}^\top$. Moreover, as proven in the special case of sparse regression by Bertsimas et al. (2020), one can show that, if the kth and $k+1$th largest eigenvalues of $\boldsymbol{\alpha}\boldsymbol{\alpha}^\top$ in a solution to (34) are distinct, then Problem (34)'s lower bound is tight.

Despite superficial similarities, we should emphasize the difference between Algorithm 2 and the BM

heuristic method discussed in the introduction. BM decomposes $\boldsymbol{X}$ into $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top$ and iteratively optimizes over $\boldsymbol{U}$ and $\boldsymbol{V}$. Although the problem is usually convex in $\boldsymbol{U}$ for a fixed $\boldsymbol{V}$ (and vice versa), it is not jointly convex, and BM is only guaranteed to converge to a stationary point of the original problem. In our setting, we decompose $\boldsymbol{X}$ into $\boldsymbol{X} = \boldsymbol{Y}\,\boldsymbol{X}$, leading to a semidefinite relaxation that is jointly convex in $(\boldsymbol{Y}, \boldsymbol{X})$. In short, BM returns a stationary solution to the original problem, and Algorithm 2 solves its semidefinite relaxation exactly.

4.3. Upper Bounds via Greedy Rounding

We now propose a greedy rounding method for rounding any $\boldsymbol{Y}^\star \in \text{Conv}(\mathcal{Y}_n)$, for example, an optimal $\boldsymbol{Y}$ in a semidefinite relaxation of Problem (9), to obtain certifiably near-optimal solutions quickly. Approximately solving low-rank problems by rounding the solution to their SDO relaxations has received vivid attention since they were first proposed by Goemans and Williamson (1995). Our analysis is, however, more general than typically conducted as it involves rounding the projection matrix $\boldsymbol{Y}$ rather than $\boldsymbol{X}$ and, therefore, is able to generalize to the rank-k case for $k > 1$, which has historically been challenging.

Observe that, for any feasible $\boldsymbol{Y} \in \text{Conv}(\mathcal{Y}_n)$, $\lambda_i(\boldsymbol{Y}) \in [0,1]$ for each eigenvalue of $\boldsymbol{Y}$, and $\boldsymbol{Y}$ is a projection matrix if and only if its eigenvalues are binary. Combining this observation with the Lipschitz continuity of $f(\boldsymbol{Y})$ suggests that high-quality feasible projection matrices can be found in the neighborhood of a solution to the semidefinite relaxation (21), $\boldsymbol{Y}^\star$, and a good method for obtaining them is to greedily round its eigenvalues. Let $\boldsymbol{Y}^\star = \boldsymbol{U}\Lambda^\star\boldsymbol{U}^\top$ be a singular value decomposition of $\boldsymbol{Y}^\star$ such that $\boldsymbol{\Lambda}$ is a diagonal matrix with diagonal entries $\boldsymbol{\Lambda}_{i,i}$ and $\boldsymbol{\Lambda}_{greedy}$ be a diagonal matrix obtained from rounding up (to one) k of the highest diagonal coefficients of $\boldsymbol{\Lambda}^\star$ and rounding down (to zero) the $n-k$ others. We then let $\boldsymbol{Y}_{greedy} = \boldsymbol{U}\boldsymbol{\Lambda}_{greedy}\boldsymbol{U}^\top$. We now provide guarantees on the quality of the greedily rounded solution (proof deferred to Online Appendix EC.3.7).

Theorem 5. *Let $\boldsymbol{Y}^\star$ denote a solution to the semidefinite relaxation (21), $\boldsymbol{Y}^\star = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top$ be a singular value decomposition of $\boldsymbol{Y}^\star$, $\mathcal{R}$ denote the indices of strictly fractional diagonal entries in $\boldsymbol{\Lambda}$, and $\boldsymbol{\alpha}^\star(\boldsymbol{Y})$ denote an optimal choice of $\boldsymbol{\alpha}$ for a given $\boldsymbol{Y}$, that is,*

$$\alpha^\star(\boldsymbol{Y}) \in \arg\max_{\boldsymbol{\alpha}} \left\{ \max_{V_{11}V_{22}} h(\boldsymbol{\alpha}) - \Omega^\star(\boldsymbol{\alpha}, \boldsymbol{Y}, \boldsymbol{V}_{11}, \boldsymbol{V}_{22}) \right\}.$$

Suppose that, for any $\boldsymbol{Y} \in \mathcal{Y}_n^k$, we have $\sigma_{\max}(\boldsymbol{\alpha}^\star(\boldsymbol{Y})) \le L$. Then, any valid rounding of $\boldsymbol{Y}^\star$ that preserves the relaxation's eigenbasis, that is, $\boldsymbol{Y}_{rounded} = \boldsymbol{U}\boldsymbol{\Lambda}_{rounded}\boldsymbol{U}^\top$, where $\boldsymbol{Y}^\star = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top$ and $\boldsymbol{\Lambda}_{rounded}$ is a diagonal matrix with binary diagonal entries $\boldsymbol{\Lambda}_{i,i}^{rounded}$ such that $1 \le \text{tr}(\boldsymbol{\Lambda}_{rounded}) \le k$ satisfies

$$\begin{aligned} &f(\boldsymbol{Y}_{rounded}) - f(\boldsymbol{Y}^\star) \le \frac{\gamma}{2} L^2 \,|\mathcal{R}| \max_{\boldsymbol{\beta}\ge\boldsymbol{0}:\|\boldsymbol{\beta}\|_1\le 1} \\ &\sum_{i\in\mathcal{R}} (\boldsymbol{\Lambda}_{i,i}^\star - \boldsymbol{\Lambda}_{i,i}^{rounded})\beta_i \le \frac{\gamma}{2} L^2 \,|\mathcal{R}| \max_{i\in\mathcal{R}} \left\{ \boldsymbol{\Lambda}_{i,i}^\star - \boldsymbol{\Lambda}_{i,i}^{rounded} \right\}, \end{aligned} \tag{36}$$

under the Frobenius penalty and

$$\begin{aligned} &f(\boldsymbol{Y}_{rounded}) - f(\boldsymbol{Y}^\star) \le ML \,|\mathcal{R}| \max_{\boldsymbol{\beta}\ge\boldsymbol{0}:\|\boldsymbol{\beta}\|_1\le 1} \\ &\sum_{i\in\mathcal{R}} \left(\boldsymbol{\Lambda}_{i,i}^\star - \boldsymbol{\Lambda}_{i,i}^{rounded} \right)\beta_i \le ML \,|\mathcal{R}| \max_{i\in\mathcal{R}} \left\{ \boldsymbol{\Lambda}_{i,i}^\star - \boldsymbol{\Lambda}_{i,i}^{rounded} \right\}, \end{aligned} \tag{37}$$

for the spectral penalty. Moreover, let $\boldsymbol{Y}_{greedy} = \boldsymbol{U}\boldsymbol{\Lambda}_{greedy}\boldsymbol{U}^\top$ be an instance of $\boldsymbol{Y}_{rounded}$ obtained by setting $\boldsymbol{\Lambda}_{i,i} = 1$ for k of the highest diagonal coefficients in $\boldsymbol{\Lambda}^\star$. Then, the preceding bounds imply that $0 \le f(\boldsymbol{Y}_{greedy}) - f(\boldsymbol{Y}^\star) \le \epsilon$, where $\epsilon = ML\min(|\mathcal{R}|, n-k)$ for the spectral penalty and $\epsilon = (\gamma/2)\min(|\mathcal{R}|, n-k)L^2$ for the Frobenius penalty.

This result calls for multiple remarks:

- When the relaxation gap $f(\boldsymbol{Y}_{\text{greedy}}) - f(\boldsymbol{Y}^\star) = 0$ and the optimal solution to the relaxation, $\boldsymbol{Y}^\star$, is unique, $|\mathcal{R}| = 0$. This justifies retaining $\mathcal{R}$ in the bound rather than replacing it with n.
- The techniques introduced in Online Section EC.5.1 for computing an M so that an optimal solution $\boldsymbol{X}^\star$ obeys $\|\boldsymbol{X}^\star\|_\sigma \le M$ also apply to computing an explicit L such that $\|\boldsymbol{\alpha}^\star\|_\sigma \le L$ in the preceding bound.
- The rounding technique is robust because it minimizes the worst-case Lipschitz upper bound under the assumption $\sigma_{\max}(\boldsymbol{\alpha}^\star) \le L$ (i.e., we have no information about which coordinate has the largest Lipschitz upper bound). For instance, under Frobenius regularization, the bound is

$$f(\boldsymbol{Y}_{rounded}) - f(\boldsymbol{Y}^\star) \le \frac{\gamma}{2} L^2 \,|\mathcal{R}| \max_{i\in\mathcal{R}} \left\{ \boldsymbol{\Lambda}_{i,i}^\star - \boldsymbol{\Lambda}_{i,i}^{rounded} \right\}, \tag{38}$$

which is minimized over $\boldsymbol{\Lambda}^{rounded} : \text{tr}(\boldsymbol{\Lambda}^{rounded}) \le k$ by solving $\min_{\lambda\in\mathcal{S}_n^k} (\gamma L^2/2)\,|\mathcal{R}| \max_{i\in\mathcal{R}} \{\boldsymbol{\Lambda}_{i,i}^\star - \lambda_i\}$, that is, rounding greedily. This interpretation suggests that greedy rounding never performs too badly.

To improve the greedily rounded solution, we implement a local search strategy that obtains even higher quality warm starts, namely, a variant of the popular BM heuristic (Burer and Monteiro 2003) that seeks low-rank solutions $\boldsymbol{X}$ by applying a nonlinear factorization $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top$, where $\boldsymbol{U} \in \mathbb{R}^{n\times l}, \boldsymbol{V} \in \mathbb{R}^{m\times k}$ and iteratively optimizing over $\boldsymbol{U}$ for a fixed $\boldsymbol{V}$ ($\boldsymbol{V}$ for a fixed $\boldsymbol{U}$) until convergence to a local optima occurs.

This strategy improves our greedily rounded solution because we initially set $\boldsymbol{U}$ to be the square root of $\boldsymbol{Y}_{greedy}$ and optimize over $\boldsymbol{V}$; recall that, if $\boldsymbol{Y}$ is a projection matrix, we have $\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{U}^\top$ and $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ for some singular value decomposition $\boldsymbol{U}, \boldsymbol{\Sigma}, \boldsymbol{V}^\top$.

5. Numerical Experiments

In this section, we evaluate the algorithms derived in the previous section, implemented in Julia 1.3 using JuMP.jl 0.20.1, Gurobi 9.0.1 to solve the QCQO master problems,[7] and Mosek 9.1 to solve the conic subproblems/continuous relaxations. Except where indicated otherwise, all experiments were performed on an Intel Xeon E5—2690 v4 2.6 GHz CPU core using 32 GB RAM. Our code is available at github.com/ryancorywright/MixedProjectionSoftware.

We evaluate the different ingredients of our numerical strategy on a matrix completion example: in Section 5.1, we solve the semidefinite relaxation by implementing Algorithm 2 and demonstrate its increased scalability over Mosek's interior point method (IPM). From the solution of the relaxation, our rounding and local search heuristics then provide near-optimal solutions that outperform state-of-the-art heuristic methods (Section 5.2). We implement Algorithm 1, benchmark its performance, and for the first time, solve low-rank matrix completion to certifiable optimality in Section 5.3. In Section 5.4, we explore the role played by regularization: increasing the amount of regularization in Problem (1) decreases the relative gap, the problem's complexity, and the amount of time required to solve the problem to optimality. We solve sensor location problems to certifiable optimality in Online Appendix EC.6.2.

5.1. Scalability of the Convex Relaxations

In this section, we explore the relative scalability of Mosek's IPM and Algorithm 2.

We consider convex relaxations of matrix completion problems. As in Candès and Plan (2010), we generate two matrices $\boldsymbol{M}_L, \boldsymbol{M}_R \in \mathbb{R}^{n\times r}$ with independent and identically distributed $\mathcal{N}(0,1)$ entries and attempt to recover the matrix $\boldsymbol{M} = \boldsymbol{M}_L\boldsymbol{M}_R^\top$ given a proportion p of its observations. Here, we fix $p = 0.25$ and $k = r = 5$, vary n, and set $\gamma = 20/p$ so that the relative importance of $\|\boldsymbol{X}\|_F^2$ and $\sum_{(i,j)\in\Omega}(X_{i,j} - A_{i,j})^2$ remains constant with p.

We solve the continuous relaxation

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times n}, \boldsymbol{Y}\in \mathrm{Conv}(\mathcal{Y}_n^k), \boldsymbol{\theta}\in S^n} \frac{1}{2\gamma}\mathrm{tr}(\boldsymbol{\theta}) + \sum_{(i,j)\in\Omega}(X_{i,j} - A_{i,j})^2 \quad \text{s.t.} \quad \begin{pmatrix} \boldsymbol{\theta} & \boldsymbol{X} \\ \boldsymbol{X}^\top & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{0}. \tag{39}$$

Table 3 reports the time required by Algorithm 2 to obtain a solution with a relative duality gap of 0.1%. To evaluate numerical stability, we also report the relative mean squared error (MSE) of the greedily rounded solution; experiments in which $n \le 250$ were run on a standard MacBook pro with 16GB RAM, and larger experiments were run on the previously described cluster with 100 GB RAM.

Our results demonstrate the efficiency of Algorithm 2: the relative MSE is comparable to Mosek's, but computational time does not explode with n. Because it does not require solving any SDOs and avoids the computational burden of performing the Newton step in an IPM, Algorithm 2 scales beyond $n = 600$ (1,440,000 decision variables) compared with $n = 200$ for IPMs (80,000 decision variables).

5.2. Numerical Evaluation of Greedy Rounding

In this section, we compare the greedy rounding method with state-of-the-art heuristic methods and demonstrate that, by combining greedy rounding with a local search heuristic, our approach outperforms other methods and, therefore, should be considered as a viable and efficient warm start for Algorithm 1.

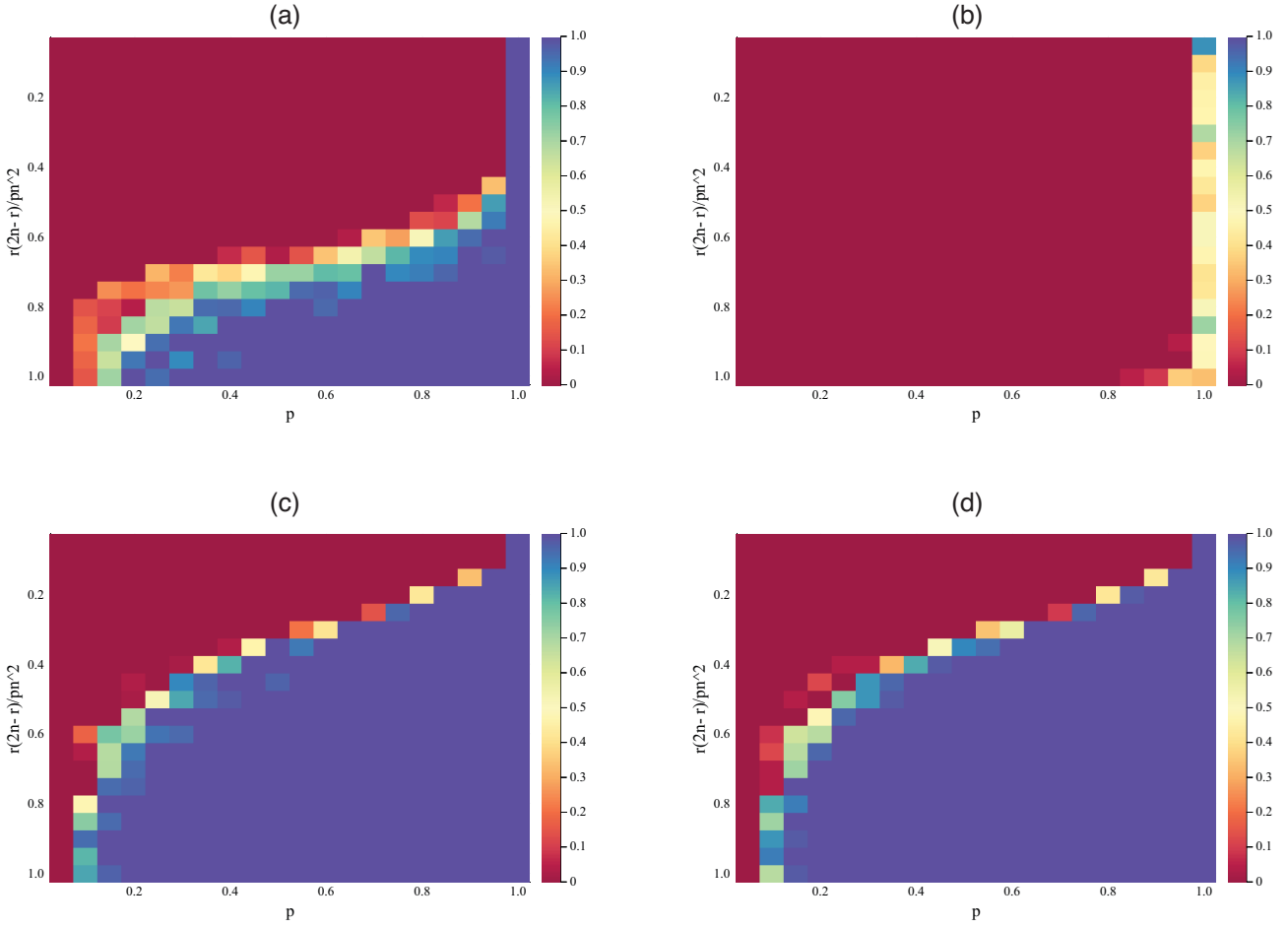
We consider the previous matrix completion problems and assess the ability to recover the low-rank matrix $\boldsymbol{M}$ (up to a relative MSE of 1%) for a varying fraction of observed entries p and rank r with $n = 100$ fixed. Note that, other than the inclusion of a

Table 3. Scalability of Convex Relaxations Averaged over Five Matrices

n	Mosek		Algorithm 2		n	Algorithm 2	
	Relative MSE	Time (s)	Relative MSE	Time (s)		Relative MSE	Time (s)
50	0.429	2.28	0.438	17.28	350	0.058	6,970
100	0.138	47.20	0.139	79.01	400	0.056	8,096
150	0.082	336.1	0.081	228.7	450	0.055	26,350
200	0.0675	1,906	0.067	841.7	500	0.054	28,920
250	—	—	0.062	1,419	550	0.0536	39,060
300	—	—	0.059	2,897	600	0.0525	38,470

Notes. Problem is regularized with Frobenius norm and $\gamma = 20/p$. Dashes indicate an instance could not be solved with the supplied memory budget.

Figure 2. (Color online) Proportion Matrices Recovered with ≤ 1% Relative MSE (Higher Is Better) for Different Values of p (x-Axis) and $r(2n-r)/pn^2 \propto 1/n$ (y-Axis), Averaged over 25 Rank-r Matrices



Notes. (a) Greedy rounding. (b) Nuclear norm. (c) SVD + local improvement. (d) Greedy + local improvement.

Frobenius regularization term, this is the same experimental setup considered by Candès and Recht (2009) and Recht et al. (2010), among others.

We compare the performance of four methods: the greedy rounding method, both with and without the local improvement heuristic from Burer and Monteiro (2003), against the local improvement heuristic alone (with a thresholded-SVD initialization point) and the nuclear norm approach. Specifically, the greedy rounding method takes the solution of the previous convex relaxation with $\gamma = 500/p$ and rounds its singular values to generate a feasible solution $\boldsymbol{Y}_{greedy}$. For the local improvement heuristic, we solve

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times n},\boldsymbol{U},\boldsymbol{V}\in\mathbb{R}^{n\times k}} \frac{1}{2\gamma}\|\boldsymbol{X}\|_2^2 + \sum_{(i,j)\in\Omega}(X_{i,j}-A_{i,j})^2$$
$$\text{s.t.} \quad \boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top,$$

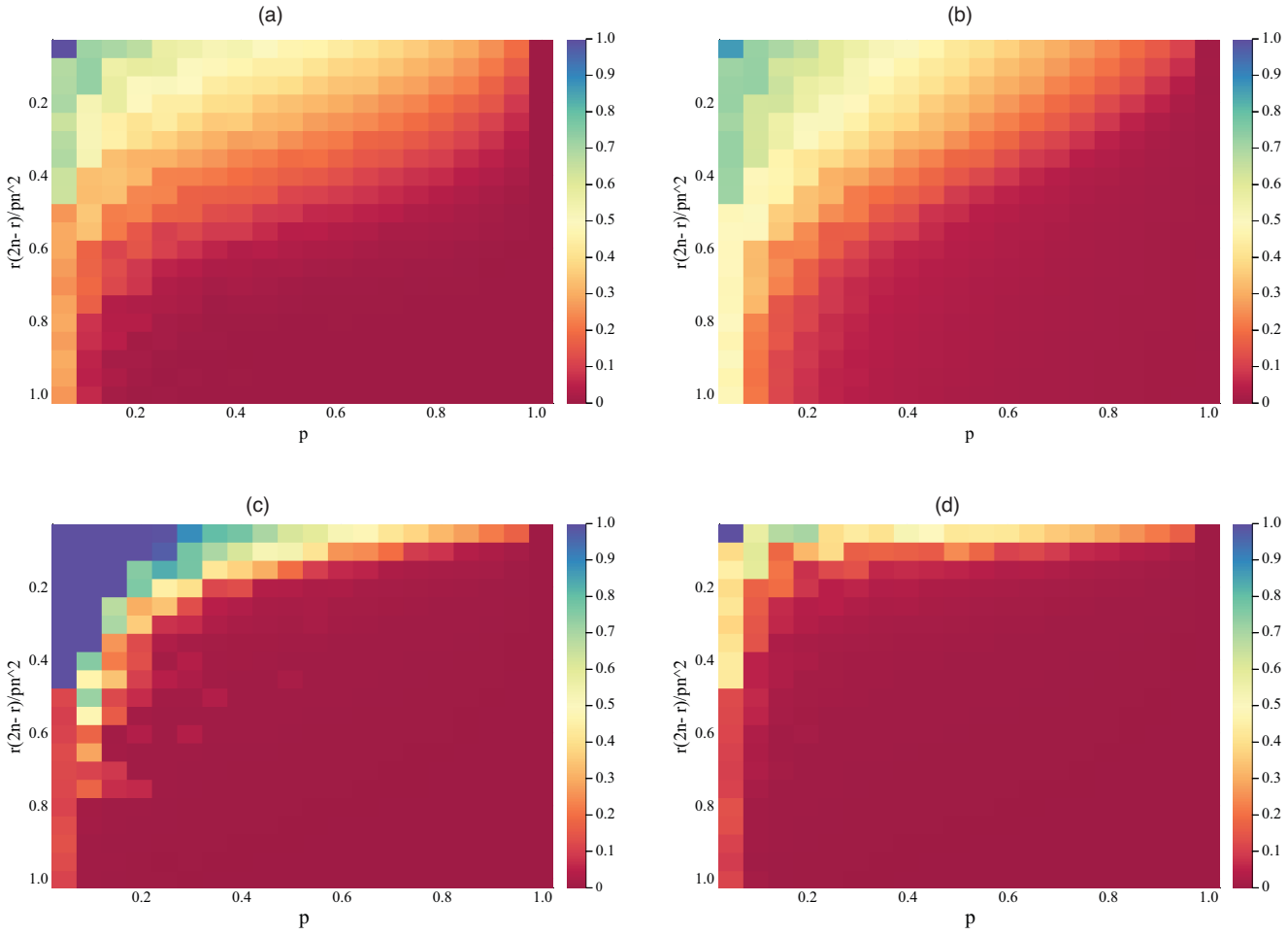
for $\gamma = 500/p$ and $k = r$ and iteratively optimize over $\boldsymbol{U}$ and $\boldsymbol{V}$ using Mosek. We provide an initial value for $\boldsymbol{U}$ by either taking the first k left-singular vectors of a matrix $\boldsymbol{A}$ in which unobserved entries are replaced by zero or taking the square root of $\boldsymbol{Y}_{greedy}$. For the nuclear norm regularization strategy, because our observations are noiseless, we solve

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times n}} \|\boldsymbol{X}\|_*$$
$$\text{s.t. } X_{i,j} = A_{i,j} \quad \forall (i,j)\in\Omega.$$

Figure 2 depicts the proportion of times the matrix was recovered exactly, and Figure 3 depicts the relative average MSE over all instances. As in Candès and Recht (2009) and Recht et al. (2010), we vary p between zero and one and consider all ranks r such that $r(2n-r) \le pn^2$.

From this set of experiments, we make several observations: First, greedy rounding and the local improvement heuristic outperform nuclear norm minimization in terms of both average relative MSE and amount of data required to recover the matrix.

Figure 3. (Color online) Average Relative MSE (Lower Is Better) Averaged over 25 Rank-r Matrices



Notes. We cap the relative MSE at 1.0. (a) Greedy rounding. (b) Nuclear norm. (c) SVD + local improvement. (d) Greedy + local improvement.

Second, the local improvement heuristic improves upon greedy rounding. In terms of its ability to recover the underlying matrix exactly, it performs equally well with either initialization strategy. However, initialization with the greedy rounding supplies dramatically lower average MSEs in instances in which no

Table 4. Scalability of Algorithm 1 vs. Gurobi for Solving Rank-1 Matrix Completion Problems to Certifiable Optimality, Averaged over the Same 20 Random Matrices per Row

			Gurobi (direct)			Algorithm 1 (single-tree)				Algorithm 1 (multitree)				
n	p	γ	Time (s)	Nodes	Gap	Time (s)	Nodes	Gap	Cuts	Time (s)	Nodes	Nodes, t	Gap	Cuts
10	0.1	$20/p$	> 3,600	313,700	0.0301	10,310	40,060	0.0004	23,460	252.3	630.2	10,099	0.0019	2.95
10	0.2	$20/p$	> 3,600	299,200	0.0854	19,440	28,430	0.0229	19,370	1,672	2,277	28,895	0.0104	11.0
10	0.3	$20/p$	> 3,600	274,500	0.1167	20,368	25,480	0.0433	20,290	2,319	2,684	30,906	0.0317	15.4
10	0.1	$100/p$	> 3,600	281,800	0.0068	18,580	62,680	0.0015	42,200	239.5	405	7,499	0.0003	3.20
10	0.2	$100/p$	> 3,600	271,300	0.0178	27,990	39,330	0.0492	31,060	1,269	1,931	21,420	0.0042	8.40
10	0.3	$100/p$	3,239	237,000	0.0178	25,750	29,350	0.0434	23,390	2,472	2,134	32,196	0.0098	19.6
20	0.1	$20/p$	> 3,600	80,760	0.8915	> 30,000	13,110	0.741	13,070	2,917	413.2	4,835	0.0166	18.5
20	0.2	$20/p$	> 3,600	65,310	4.094	> 30,000	7,023	0.1816	7,008	3,512	143.0	1,746	0.247	20.0
20	0.3	$20/p$	> 3,600	64,850	4.745	28,700	6,914	0.1066	6,828	3,287	204.9	2,081	0.253	19.6
20	0.1	$100/p$	> 3,600	60,830	0.428	> 30,000	13,790	0.7714	13,799	3,013	508.0	6,152	0.0072	17.6
20	0.2	$100/p$	> 3,600	43,850	1.421	> 30,000	6,395	0.0543	6,395	3,106	80.9	1,027	0.0903	17.6
20	0.3	$100/p$	> 3,600	55,150	2.810	29,530	6,538	0.0271	6,510	2,910	62.7	744.2	0.1368	17.0

Note. In multitree, "Nodes" denotes the number of nodes expanded in the final branch-and-cut tree, and "Nodes, t" denotes the number of nodes expanded over all trees for the multitree implementation.

approach recovers the true matrix exactly. This suggests that initialization strategies for the Burer–Monterio heuristic should be revisited and greedy rounding considered as a viable and more accurate alternative than selecting a random feasible point.

5.3. Benchmarking Algorithm 1 on Synthetic Matrix Completion Problems

We now benchmark Algorithm 1 on matrix completion problems in which $n \in \{10, 20, 30\}$.

We first compare the two different implementations of Algorithm 1, single- and multitree, with solving the problem directly as a QCQO in Gurobi (Online Appendix EC.5.2). In Algorithm 1, the lower bounds are warm-started with 200 cuts from the in-out method, and greedy rounding with local search improvement is used for the upper bounds; if a single-tree instance fails to find a feasible solution (because of numerical instability in Gurobi), we return the gap between the warm start and the semidefinite relaxation. At the tth iteration, we impose a time limit of $10t$ seconds for generating the new cut so as to increase numerical precision as the solver progresses. We also impose a limit of 20 cuts for the multitree approach, a time limit of 30,000 seconds for the single-tree approach,[8] a time limit of 3,600 seconds for Gurobi, and an optimality gap of 1%.[9] Average runtime, number of nodes, and optimality gap are reported in Table 4.

We observe that Algorithm 1 drastically improves upon Gurobi in terms of both computational time (reduced by up to an order of magnitude) and accuracy (absolute gap reduced by around an order of magnitude). Multitree dominates single-tree and Gurobi in terms of runtime and the quality of the solution found although single-tree occasionally has a smaller gap at termination. Moreover, multitree consistently finds high-quality feasible solutions earlier than single-tree and accepts our warm-start more consistently, which suggests it may scale better to high-dimensional settings.

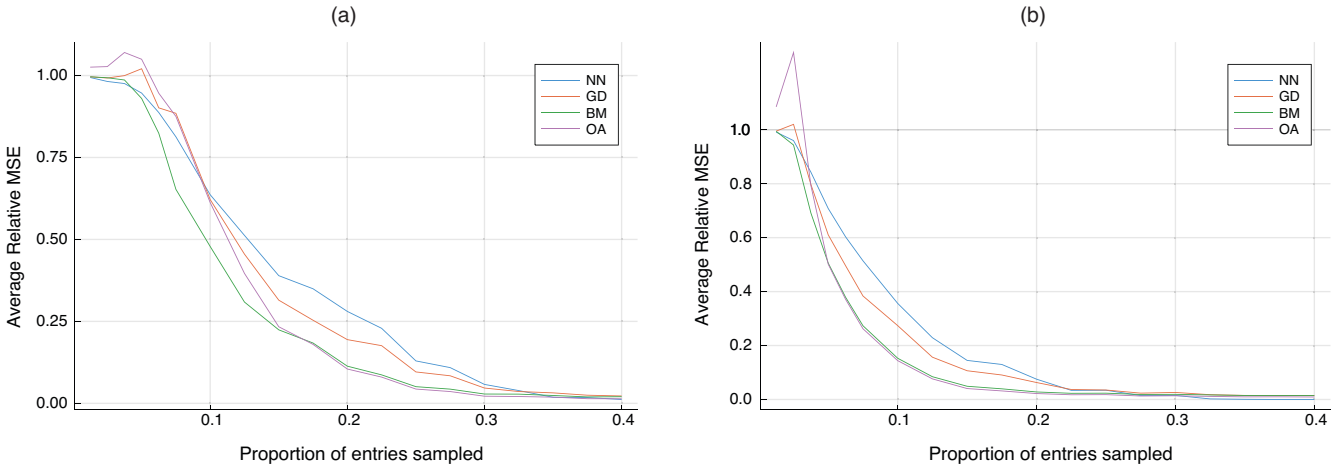
Next, we evaluate the performance of the multitree implementation of Algorithm 1 on a more extensive test set, including instances in which $\text{Rank}(\boldsymbol{M}) > 1$, in Table 5. Note that when $r = 1$, we use the same experimental setup (although we impose a time limit of $30t$ seconds or 7200 seconds if there has been no improvement for two consecutive iterations and a cut limit of 50 cuts when $n > 20$), and when $r > 1$, we increase the time limit per iteration to 300t seconds (or 7200 seconds if there has been no improvement for two consecutive iterations) and allow up to 100 PSD cuts per iteration to be added at the root node via a user cut callback in order to strengthen the approximation of the PSD constraint $\boldsymbol{Y} \succeq \boldsymbol{0}$. We observe that the problem's complexity increases with the rank although not too excessively. Moreover, when $r > 1$, the bound gap is actually smaller when $\gamma = 100/p$ than when $\gamma = 20/p$. We believe this is because Gurobi cannot represent the SDO constraint $\boldsymbol{Y} \succeq \boldsymbol{0}$, and its SOC approximation is inexact (even with PSD cuts), and in some cases, refining this approximation is actually harder than refining our approximation of $g(\boldsymbol{X})$.

Note that the main bottleneck inhibiting solving matrix completion problems in which $n \geq 50$ is Gurobi itself as the nonconvex solver takes increasing amounts of time to process warm starts (sometimes in the 100s or 1000s of seconds) when n increases. We

Table 5. Scalability of Algorithm 1 (Multitree) for Solving Low-Rank Matrix Completion Problems to Certifiable Optimality Averaged over 20 Random Matrices per Row

			Rank-1				Rank-2				Rank-3			
n	p	γ	Time (s)	Nodes	Gap	Cuts	Time (s)	Nodes	Gap	Cuts	Time (s)	Nodes	Gap	Cuts
10	0.1	$20/p$	182.1	9,755	0.0005	2.56	24,220	35,670	0.0034	5.78	37,780	39,870	0.0071	9.28
10	0.2	$20/p$	3,508	21,060	0.0026	10.8	209,900	108,000	0.0252	35.3	135,260	35,870	0.031	26.2
10	0.3	$20/p$	5,488	30,970	0.0039	13.1	302,200	70,500	0.0866	50.0	302,100	31,870	0.0197	50.0
10	0.1	$100/p$	656.5	28,870	0.0001	2.14	676.1	25,493	0.0009	1.83	842.7	20,700	0.0024	1.79
10	0.2	$100/p$	1,107	10,010	0.0009	4.29	2,065	42,490	0.0019	5.61	57,530	36,910	0.0124	10.7
10	0.3	$100/p$	3,364	48,730	0.0022	6.30	272,300	33,150	0.0195	44.7	249,700	35,530	0.0499	42.2
20	0.1	$20/p$	2,017	4,756	0.0061	8.20	253,900	8,030	0.0279	42.7	255,400	3,015	0.0309	43.2
20	0.2	$20/p$	6,369	6,636	0.0136	15.0	298,700	3,342	0.549	50.0	295,500	236.5	0.879	50.0
20	0.3	$20/p$	6,687	4,187	0.0082	18.4	296,500	3,175	1.123	50.0	291,100	41.35	2.147	50.0
20	0.1	$100/p$	1,266	8,792	0.0087	8.35	211,700	6,860	0.0073	34.24	171,900	2,350	0.0131	29.8
20	0.2	$100/p$	1,220	2,710	0.0104	7.80	302,800	2,426	0.123	50.0	298,800	221.4	0.123	50.0
20	0.3	$100/p$	1,272	1,837	0.0064	3.14	299,000	2,518	0.264	50.0	293,500	43.0	0.659	50.0
30	0.1	$20/p$	300,300	2,735	0.0905	50.0	304,300	164.0	0.790	50.0	303,100	1.10	0.365	50.0
30	0.2	$20/p$	298,700	1,511	0.136	50.0	301,700	9.62	3.105	50.0	302,600	1.00	5.581	50.0
30	0.3	$20/p$	183,800	1,743	0.0476	36.9	303,000	1.63	5.232	50.0	305,000	0.70	14.60	50.0
30	0.1	$100/p$	305,600	2,262	0.0273	50.0	302,800	97.40	0.0973	50.0	305,000	1.90	0.0967	50.0
30	0.2	$100/p$	246,300	3,285	0.0315	43.6	304,300	6.17	0.697	50.0	302,600	1.00	1.419	50.0
30	0.3	$100/p$	25,970	11,020	0.0089	17.1	304,000	1.00	0.923	50.0	304,700	1.00	3.221	50.0

Figure 4. (Color online) Average Relative MSE for Nuclear Norm (NN), Greedy Rounding (GD), Burer–Monterio (BM), and Outer Approximation (OA) When Imputing a Rank-1 $n \times n$ Matrix



Notes. All results are averaged over 25 matrices. (a) $n = 25$. (b) $n = 50$.

believe this may be because of the way Gurobi translates orthogonal projection matrices to a piecewise linear formulation. Encouragingly, this suggests that our approach may successfully scale to 100×100 matrices as Gurobi improves their solver.

Finally, we compare the solution from the exact formulation (9) solved using Algorithm 1 (multitree) with the initial warm start we propose and two state-of-the-art heuristics, namely, nuclear norm minimization and the Burer–Monterio approach as in Section 5.2. Here, we take $n \in \{25, 50\}$, $r = 1$, p ranging from 0 to 0.4, and $\gamma = 100/p$. Figure 4 depicts the average relative MSE over the entire matrix averaged over 25 random instances per value of p. When $p \geq 0.2$, the exact method supplies an out-of-sample relative MSE around 0.6% lower than Burer–Monterio.[10]

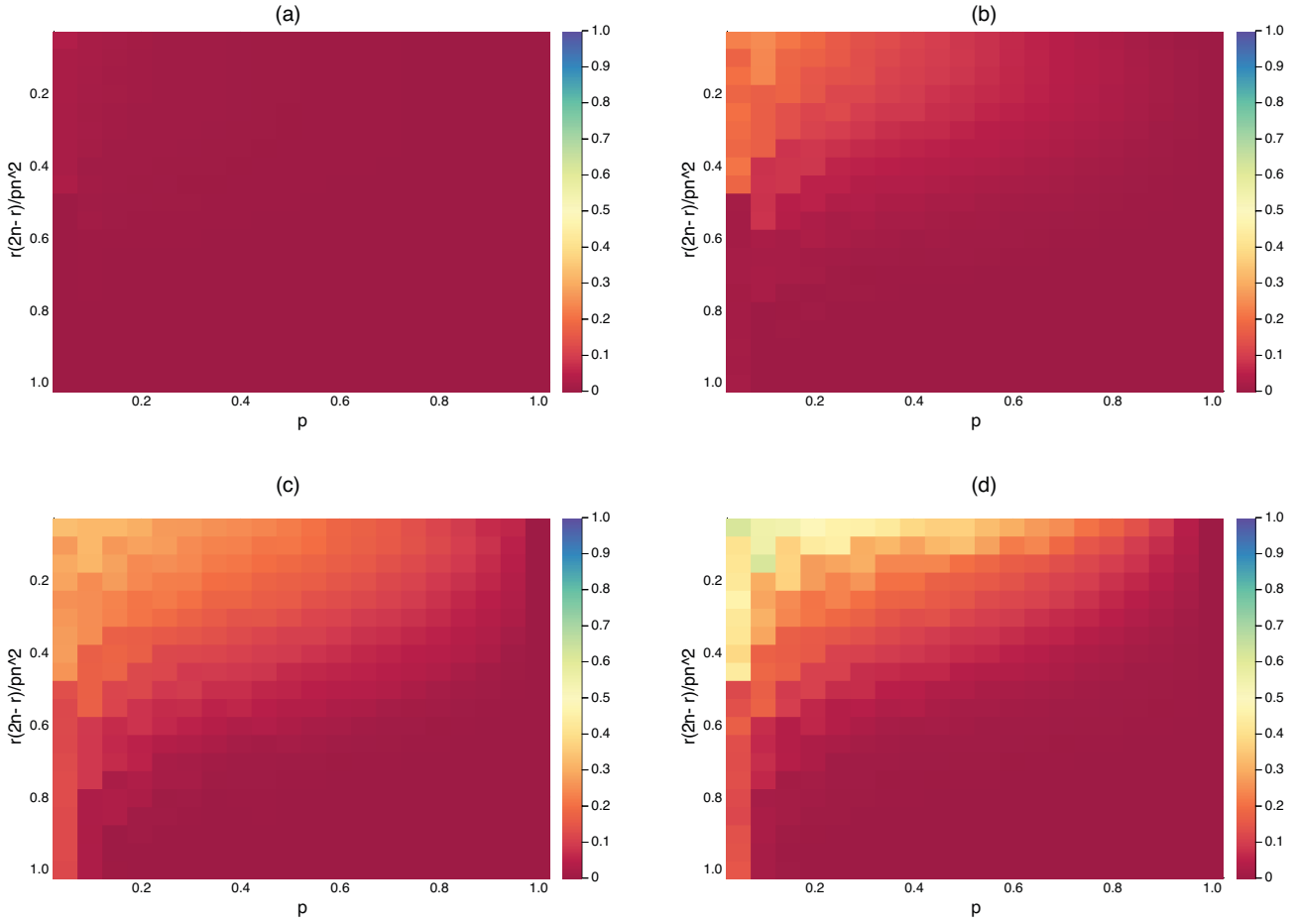
5.4. Exploring the Impact of Regularization on Problem Complexity

We now examine the impact of the regularization term $1/2\gamma \|\boldsymbol{X}\|_F^2$ on the problem complexity as captured by the relative in-sample duality gap between the semidefinite relaxation and the objective value of the greedy solution with local search improvement. We generate the problem data as in previous experiments and display results for four values of γ in Figure 5. Observe that, as γ increases, both the duality gap and the problem's complexity increase. This observation confirms similar results on the impact of regularization in mixed-integer conic optimization problems (cf. Bertsimas and Cory-Wright 2022, Bertsimas et al. 2021). Additionally, when $\gamma = 500/p$ in Figure 5(d), the region in which the in-sample duality gap is zero corresponds to recovering the underlying matrix with high probability, and a strictly positive duality gap corresponds to instances with partial recovery only (see Figure 2(d)). This suggests a deep connection between relaxation tightness and statistical recovery.

Although the relative in-sample semidefinite relaxation gap is a theoretical measure of problem difficulty, it does not indicate how fast Algorithm 1 converges in practice. In this direction, we solve the 20 synthetic matrix completion problems considered in Table 4, in which $n \in \{10, 20\}$, $r = 1$, and $p \in \{0.2, 0.3\}$ for 20 different values of $\gamma \in [10^0, 10^4]$ (distributed uniformly on a log-scale), and compare the relative in-sample semidefinite gap (greedily rounded solution versus semidefinite bound) with Algorithm 1's runtimes in Figure 6 for the single-tree (left panel) and multitree (right panel) implementation. Results are averaged over 20 random synthetic instances per value of γ. We observe that the relaxation gap does correlate with runtime for single tree. Yet the relationship between the relaxation gap and runtime is less straightforward for multitree as it depends on how Gurobi balances cut generation and node expansion and the conditioning of the problem.

The regularizer γ also impacts the bias term $1/(2\gamma) \|\boldsymbol{X}\|_F^2$ added to the objective function and, hence, the suboptimality of the solution. To further illustrate the impact of the regularizer γ on solve times and the trade-off between tractability and suboptimality, Figure 7 reports the average runtime and MSE for the previously solved instances as a function of γ. Figure 7 illustrates how γ balances tractability (runtime, top row) and optimality of the solution (MSE, bottom row). Also, single-tree (left panel) is one order of magnitude slower than multitree (right panel), generates substantially more cuts (see Online Figure EC.1) and is also more numerically instable when γ increases, largely because of the difficulty of combining a nonconvex master problem and lazy constraint callbacks. Echoing our findings in the previous section, this suggests that, although, in MICO, single

Figure 5. (Color online) Average Relative In-Sample Bound Gap (%) Averaged over 25 Rank-r Matrices



Notes. (a) $\gamma = 0.5/p$. (b) $\gamma = 5/p$. (c) $\gamma = 50/p$. (d) $\gamma = 500/p$.

tree typically outperforms multitree, at the current state of technology, multitree appears more efficient for matrix completion problems that have nonconvex master problems. However, as the algorithmic implementations of nonconvex QCQO solvers mature, this conclusion should be revisited.

Figure 6. (Color online) Average Runtime Against Relative Semidefinite Relaxation Gap for Algorithm 1 Single Tree (Left) and Multitree (Right) over 20 Synthetic Matrix Completion Instances per Data Point for Which $p \in \{0.2, 0.3\}$, $r = 1$, $n \in \{10, 20\}$

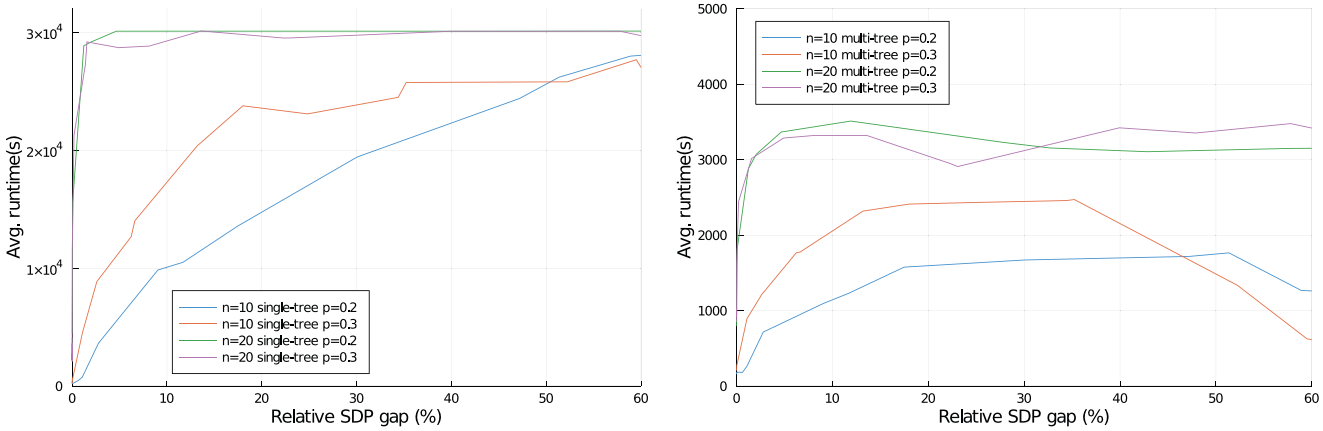
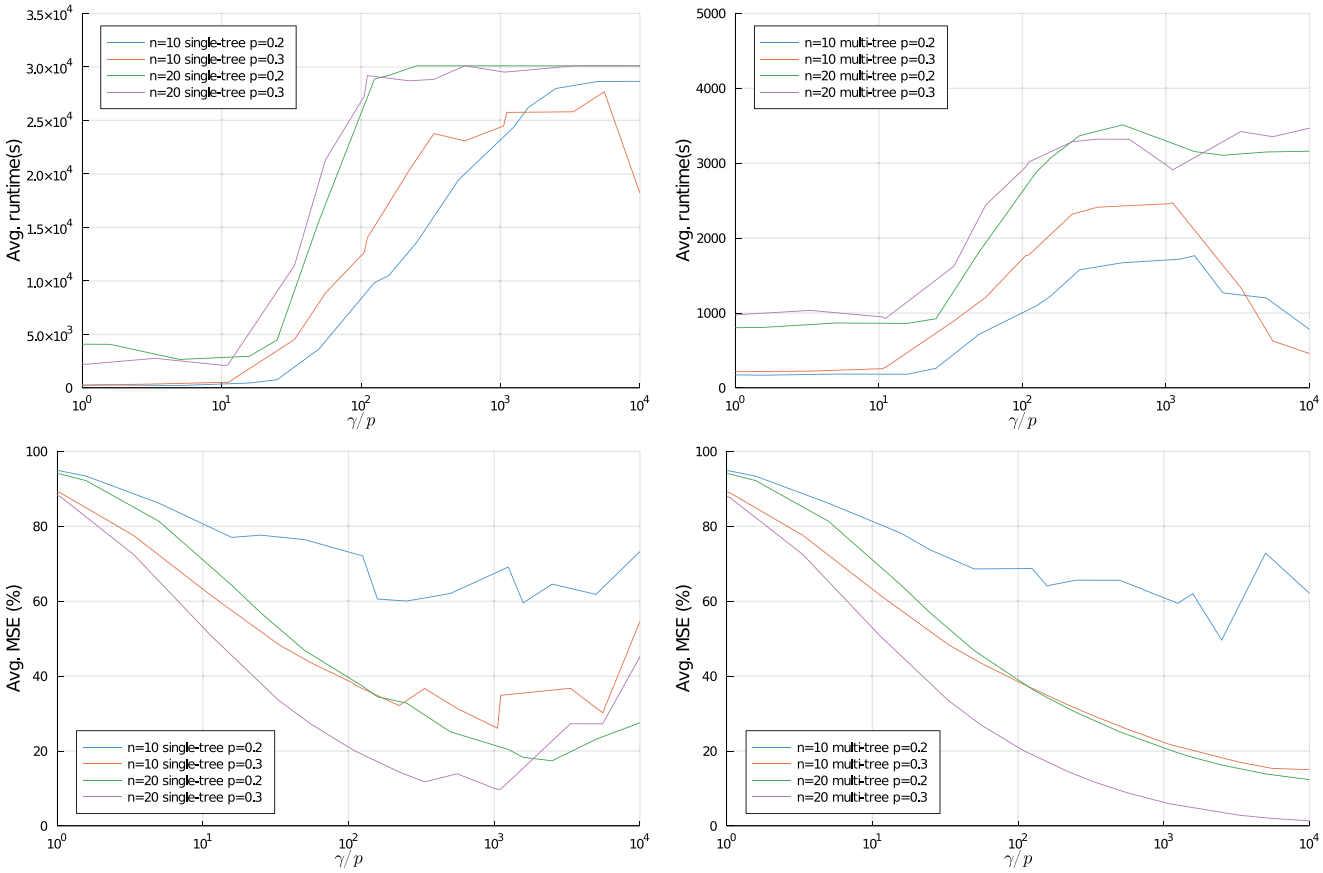


Figure 7. (Color online) Average Runtime (Top) and MSE (Bottom) vs. γ for Algorithm 1 Single Tree (Left) and Multitree (Right) Implementations over 20 Synthetic Matrix Completion Instances in Which $p \in \{0.2, 0.3\}$, $r = 1$, and $n \in \{10, 20\}$



Note. The same random seeds were used to generate random matrices completed by single tree and multitree.

5.5. Summary of Findings from Numerical Experiments

Our main findings from the numerical experiments in this section are as follows:

- Algorithm 2 successfully solves convex relaxations of low-rank problems in which $n = 100$s in a faster and more scalable fashion than interior point codes, such as Mosek (Section 5.1).
- Based on Section 5.2, an effective strategy for obtaining good solutions to low-rank problems is the following: solve the convex relaxation using Algorithm 2, greedily round its solution, and polish this greedily rounded solution by starting a local search method, such as Burer and Monteiro (2003, 2005).
- As demonstrated in Section 5.4, increasing the amount of regularization in a low-rank problem decreases the duality gap between a low-rank problem and its convex relaxation, hence making the problem easier in a practical sense (although not necessarily in a complexity-theoretic sense).
- In Section 5.3 and Online Section EC.6.2, Algorithm 1 scales to problems in which n is in the tens, that is, up to 1,000 decision variables, in hours. The main bottleneck is that we solve our master problems using Gurobi, a generic QCQO solver that translates the orthogonal projection matrix constraint into many piecewise linear constraints. This suggests that a custom branch-and-bound solver that natively handles orthogonal projection matrices constitutes a promising area for future work.

6. Conclusion

In this paper, we introduce mixed-projection conic optimization, a new framework for modeling rank-constrained optimization problems that, for the first time, solves low-rank problems to certifiable optimality at moderate problem sizes. We also provide a characterization of the complexity of rank constraints and propose new convex relaxations and rounding methods that lead to viable and more accurate solutions than those obtained via existing techniques, such as the log-det or nuclear norm heuristics. Inspired by the collective successes achieved in mixed-integer optimization, we hope that MPCO constitutes an exciting new research direction for the optimization community. For

instance, we believe that custom branch-and-bound solvers that explicitly model orthogonal projection matrices and dedicated semidefinite codes for the relaxations could further enhance the numerical scalability of the MPCO framework.

Acknowledgments

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Endnotes

[1] Unless the conic dual is also infeasible, this case is unimportant for our purposes because it only arises when the original problem is itself infeasible for any $\boldsymbol{Y}$, which can be checked a priori.

[2] Weak duality implies that the dual problem is either unfeasible or unbounded. Because the feasible set of the maximization problem does not depend on Y, it is always feasible unless the original problem is itself infeasible. Therefore, we assume without loss of generality that it is unbounded.

[3] It should be noted that this formulation is rather complicated because nonconvex QCQO solvers, such as Gurobi, currently do not model PSD constraints. If they did, we would supplant the second-order cone constraints with $\boldsymbol{Y} \succeq \boldsymbol{u}\boldsymbol{u}^\top$ and thereby obtain a simpler master problem.

[4] The data-generation process is detailed in Section 5.2. Here, $n = 100$, $p = 0.25$, $r = 1$, and $\gamma = 20/p$.

[5] We warm start the upper bound with greedy rounding and the Burer–Monterio local improvement heuristic described in Section 4.3. To mitigate against numerical instability, we opted to be conservative with our parameters and, therefore, turned Gurobi's heuristics off, set FuncPieceError and FuncPieceLength to their minimum possible values (10^{-5} and 10^{-6}), set the MIP gap to 1%, and the time limit for each solve to one hour.

[6] Here, $n = 10$, $p = 0.25$, $r = 1$, and $\gamma = 5/p$.

[7] We remark that Gurobi solves the nonconvex QCQO master problems by translating them to piecewise linear optimization problems. Because rank constraints are not MICO representable, this introduces some error. To mitigate against this error, we set the Gurobi parameters FuncPieceError and FuncPieceLength to their minimum possible values (10^{-6} and 10^{-5}, respectively). Additionally, we set NonConvex to 2 and otherwise use default Gurobi/Mosek parameters.

[8] We require a larger time limit than 3,600 seconds because Gurobi often fails to find any feasible solutions within this time limit because of the numerical difficulties inherent in integrating lazy constraint callbacks and a nonconvex master problem.

[9] We report the absolute gap between the better of Gurobi's lower bound and the semidefinite lower bound and compare with the objective value that we evaluate directly; this is sometimes $1-2\%$ even when Gurobi reports that it has found an optimal solution because of numerical instability in Gurobi. Note that we report the absolute, rather than relative, gap because the relative gap depends on the quality of Gurobi's approximation of $\mathcal{Y}_n^k$, which is controlled by the parameter FuncPieceError and cannot be set lower than 10^{-6}; also note that the objective values are on the order of 0.5–5.0 for the problems reported in Table 4.

[10] Because we ran all methods on the same random instances, this difference is statistically significant with a p-value of 2×10^{-51} (2×10^{-129}) that the relative MSE is lower for the exact method when $n = 25$ ($n = 50$).

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Dimitris Bertsimas is the Boeing Professor of Operations Research and associate dean of business analytics at the Massachusetts Institute of Technology. He is the recipient of numerous research and teaching awards, including the von Neumann theory prize and the INFORMS president award in 2019.

Ryan Cory-Wright is a PhD candidate at the Massachusetts Institute of Technology, advised by Dimitris Bertsimas. His research interests include optimization, machine learning, and statistics and their applications to energy systems. He is the recipient of several research awards, including the INFORMS George Nicholson Prize.

Jean Pauphilet is an assistant professor in the management science and operations group at London Business School. His research interests include optimization, machine learning, and statistics and their application to healthcare operations. He is the recipient of several research awards, including the INFORMS George Nicholson Prize.