Algebraic Boundaries of Convex Semi-Algebraic Sets

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Introduction

The algebraic boundary of a convex set is the smallest algebraic variety containing its euclidean boundary. For a polytope it is the hyperplane arrangement defined by its facets, which has been studied extensively in the theory of polytopes and linear programming. The algebraic boundary of a convex set which is not a polytope has recently been considered in other special cases, most notably the convex hull of a variety. This class includes prominent families such as the moment matrices of probability distributions and the highly symmetric orbitopes. It does not include examples such as hyperbolicity cones and spectrahedra, which have received attention from applications of semi-definite programming in polynomial optimisation. We want to consider the class of all sets for which the algebraic boundary is an algebraic hypersurface: convex semi-algebraic sets with non-empty interior.

With the currently available methods, it is hard to compute invariants of the algebraic boundary because it is often reducible and of high degree. As an extreme example, consider the algebraic boundary of a polytope with k facets: It is a hypersurface of degree k with k irreducible components and examples of polytopes with a large number of facets come as examples of high complexity in linear programming. So our goal is a theoretical and geometric understanding of algebraic boundaries, in particular relating to notions in convex geometry. Again, we look to the theory of polytopes for inspiration.

A theoretically convenient way to study facets of polytopes is to introduce the dual polytope: The facets of the polytope correspond to the vertices of the dual polytope. Given a vertex ℓ of the dual polytope, the corresponding irreducible component of the algebraic boundary is the affine hyperplane of all points x such that $\ell(x) = -1$. We use the established theory of dual convex sets for general compact convex sets and the duality theory of projective algebraic varieties to get analogous statements for general convex semi-algebraic sets. These provide powerful tools which lead to a close connection between algebraic families of extreme points of the dual convex body and irreducible components of the algebraic boundary. One of our results in this direction is the following generalisation of the duality for polytopes:

Theorem (Theorem 2.4.5). *Let $C \subset \mathbb{R}^n$ be a compact convex semi-algebraic set. Let Z be an irreducible component of the Zariski closure of the set of extreme points of its dual convex body. Then the variety dual to Z is an irreducible component of the algebraic boundary of C.*

In the general case, we do not get every irreducible component of the algebraic boundary of C in this way. We study the exceptional cases and give a complete semi-algebraic characterisation of the exceptional families of extreme points, see Theorem 2.4.9.

We study these phenomena for two concrete classes of examples. The first is the class of $SO(2)$-orbitopes which are convex hulls of rational curves in even-dimensional spaces with high symmetry. The results on the algebraic boundary of such convex bodies are of synthetic flavor. In particular, higher secant varieties to curves will play an important role in this case, see Theorems 3.1.14, 3.1.21 and 3.1.26.

Secondly, we study the sums of squares cone of forms of fixed degree in three variables corresponding to plane projective curves. Here, we see the dual point of view: Our results are about semi-algebraic families of extreme rays of the dual cone, which is the cone of positive semi-definite moment matrices, see Theorems 3.2.22 and 3.2.30. This cone appears naturally in several contexts: It is a special spectrahedral cone and it is central in the study of the truncated moment problem, which is at the heart of moment relaxation methods used in polynomial optimisation. A main tool is a connection to Gorenstein ideals and the Cayley-Bacharach Theorem.

Our presentation is organised as follows: We will review basic facts from convex geometry (faces, extreme points and duality) and real algebraic geometry (semi-algebraic sets and duality of projective varieties) needed for our work in Chapter 1 (Preliminaries).

In Chapter 2, we introduce the algebraic boundary of a convex semi-algebraic set and explore first connections to convex duality. In Section 2.2, we study applications of the algebraic boundary to a question of semi-algebraic nature, namely: Can a closed convex set be defined by simultaneous polynomial inequalities? Starting in Section 2.3, we aim to describe the algebraic boundary of a convex semi-algebraic set. First, we consider the case where the convex set is the convex hull of a real algebraic set. This extends work by Ranestad and Sturmfels in [33] and [32]. It is first treated from a synthetic point of view, extending the results of Ranestad and Sturmfels on convex hulls of space curves to higher dimensional varieties. We then also present the dual description given in [32] in a slightly more general version. In the last section of the chapter, we treat the general case. We use both duality theories for a convex semi-algebraic set without further assumptions on the set of extreme points.

In Chapter 3, we present the application of the abstract results to two classes of examples. In the first Section 3.1, we study $SO(2)$-orbitopes, i.e. convex hulls of orbits under linear actions of $SO(2)$. This is the simplest non-discrete case of an orbitope and already non-trivial from the point of view of convex algebraic geometry: Convex duality relates the study of these bodies with Laurent-polynomials that are non-negative on the unit circle in the complex plane. In general, a description of the algebraic boundary of an $SO(2)$-orbitope is lacking. We treat several special cases, namely the centrally symmetric Barvinok-Novik orbitopes (used in [2] to prove the existence of centrally symmetric polytopes with many faces), 4-dimensional $SO(2)$-orbitopes and another special case, where a concrete factorisation of Laurent polynomials can be found as a certificate of non-negativity. In Section 3.2, we will present results from an ongoing project with Blekherman concerning the extreme rays of the cone $\Sigma_{2d}^{\vee}$ dual to the cone of sums of squares of ternary forms of fixed degree $2d$. Our approach is constructive and we find the Zariski closure of the set of extreme rays, which turns out to be an irreducible variety of codimension 10. The main tools are the Cayley-Bacharach Theorem for plane curves and the Buchsbaum-Eisenbud Structure Theorem for height 3 Gorenstein algebras.

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1. Preliminaries

In this chapter, we want to fix terminology and notation and shortly review the relevant results from convex and real algebraic geometry that we will later use. We give proofs of basic results that are hard to find in the literature and give references to proofs we omit.

1.1. Convexity

In convexity, we mainly want to discuss the notion of faces, convex duality and normal cones. We refer to the textbooks by Barvinok [1], Rockafellar [35], Ziegler [43] and Grünbaum [21]. Throughout this section, let V be a finite-dimensional vector space over the real numbers $\mathbb{R}$ endowed with the euclidean topology.

1.1.1. Convex Bodies and Cones

Definition 1.1.1. (a) A subset $C \subset V$ is called *convex* if for all $x, y \in C$, the line segment between x and y is contained in C, i.e. for all $\lambda \in [0,1] \subset \mathbb{R}$ the point $\lambda x + (1-\lambda)y \in C$.
(b) A convex set $C \subset V$ is called a *convex body* if it is compact and has non-empty interior.

Example 1.1.2. Given any subset $D \subset V$, we denote the smallest convex set containing D by $\operatorname{conv}(D)$. It consists of all *convex combinations* of points in D, i.e. elements of the form

$$\sum_{i=1}^{r} \lambda_i x_i$$

where the coefficients λ_i are non-negative real numbers, the points x_i are in D and $\sum_{i=1}^{r} \lambda_i = 1$. Prominent examples are *polytopes* which are convex hulls of finite sets.

One important class of convex sets in convex algebraic geometry is the class of spectrahedra and their projections, see e.g. Blekherman-Parrilo-Thomas [9].

Definition 1.1.3. A *spectrahedron* is the preimage of the cone of positive semi-definite matrices under an affine linear map from $\mathbb{R}^n$ into the space of real symmetric matrices of some size d.

The definition of a face is not consistent in the literature. We take the more general definition that differs from the one given in Barvinok's textbook:

Definition 1.1.4. (a) A subset $F \subset C$ of a convex set C is called a *face* of C if F is convex and if for all $x, y \in C$, we can deduce from $\frac{1}{2}x + \frac{1}{2}y \in F$ that the points x and y lie in F.
(b) We call the faces $\varnothing$ and C of C the *trivial faces* of C. A face F of C is called *proper* if $F \neq C$.
(c) A point $x \in C$ is called an *extreme point* of C if the singleton $\{x\}$ is a face of C. We write $\operatorname{Ex}(C)$ for the set of extreme points of C.

Example 1.1.5. (a) The only extreme point of the positive orthant is 0.
(b) The set of extreme points of the unit ball in $\mathbb{R}^n$ with respect to the euclidean norm is its boundary, i.e. the unit sphere. In the case of the ℓ_1-norm on $\mathbb{R}^n$, the unit ball is a polytope, namely the cross polytope whose set of extreme point consists of all permutations of $(\pm 1, 0, \ldots, 0)$. The unit ball of the ℓ_∞-norm is the cube $\operatorname{conv}(\{\pm 1\}^n)$ having the 2^n extreme points $(\pm 1, \ldots, \pm 1)$.
(c) If the convex set C is the convex hull of D, then every extreme point of C lies in D.

Due to the following theorem by Krein and Milman, the extreme points of a compact convex set determine it uniquely.

Theorem 1.1.6 (Krein-Milman, cf. Barvinok [1], Theorem II.3.3). *Let $C \subset V$ be a compact convex set. Then C is the convex hull of its extreme points.*

Definition 1.1.7. A subset $C \subset V$ is called a *cone* if it is convex and if for all $\lambda \in \mathbb{R}_{\geq 0}$ and all $x \in C$, the point λx lies in C.

Remark 1.1.8. Analogous to the convex hull of a set, we call the smallest cone containing a given set D as the conic hull and denote it by $\operatorname{co}(D)$. It is the set of all conic combinations of elements of D, i.e.

$$\operatorname{co}(D) = \{\lambda_1 x_1 + \ldots + \lambda_r x_r : r \in \mathbb{N}, \lambda_i \geq 0, x_i \in D\}.$$

Definition 1.1.9. (a) A cone is called *pointed* if it does not contain a line.
(b) A *basis* of a cone $C \subset V$ is the intersection $C \cap H$ with an affine hyperplane $H \subset V$ not containing the origin such that C is the conic hull of $C \cap H$.

Proposition 1.1.10. *A cone $C \subset V$ admits a compact basis if and only if it is pointed and closed.*

For a proof, see Chapter II.8 of Barvinok's book [1].

By the subsequently described construction of homogenisation, every convex body $C \subset V$ gives rise to a pointed, closed cone with non-empty interior in $\mathbb{R} \times V$.

Construction 1.1.11. Let $C \subset V$ be a convex body. Embed V into $\mathbb{R} \times V$ at height 1, i.e. consider the image of the map

$$\Phi : \begin{cases} V & \to & \mathbb{R} \times V \\ x & \mapsto & (1, x) \end{cases}$$

and let $\widehat{C}$ be the conic hull of $\Phi(C)$. Then $\widehat{C}$ is a closed, pointed cone with non-empty interior in $\mathbb{R} \times V$. It is easily verified that $F \mapsto \widehat{F}$ is a bijection between the faces of C and the non-empty faces of $\widehat{C}$. More precisely, this bijection is an isomorphism of face lattices.

Of particular interest are the non-trivial faces of minimal dimension of a cone.

Definition 1.1.12. (a) We call the faces $\varnothing$, C and $\{0\}$, in case it is a face, the *trivial faces* of C. We call a face F of C *proper* if $F \neq C$. For cones, the empty face is usually disregarded.
(b) A face of C of the form $\mathbb{R}_{\geq 0} x$ is called an *extreme ray* of C. We write $\operatorname{Exr}(C)$ for the union of all extreme rays of C.

Now, the Krein-Milman Theorem translates as follows into the language of cones.

Theorem 1.1.13 (cf. Barvinok [1], Chapter II.8). *Every closed, pointed cone is the convex hull of its extreme rays.*

Definition 1.1.14. Let $C \subset V$ be a convex set and $F \subset C$ a face of C. We call a point $x \in F$ a *relative interior point* of F if x lies in the topological interior of F as a subset of its affine span. The *relative interior* of F is the set of all relative interior points of F.

Remark 1.1.15. (a) The relative interior of C is the usual interior if C has non-empty interior.
(b) If F is a non-empty face of C, then its relative interior is non-empty.
(c) The relative interior of a face is dense in it.
These statements are discussed in Barvinok [1], section II.2. More precisely, statements (a) and (b) follow from [1], Theorem 2.4, statement (c) follows from [1], Lemma 2.2.

1.1.2. Convex Duality and Normal Cones

Convex duality is essentially the study of all half-spaces containing a given convex set. That these define the convex set uniquely (up to closure) as their intersection is a corollary to the Separation Theorem.

Theorem 1.1.16 (cf. Barvinok [1], Theorem II.1.6). *Let $C \subset V$ be an open convex set and let $u \in V \setminus C$ be a point. Then there is a linear functional $\ell \in V^*$ with $\ell(x) > \ell(u)$ for all $x \in C$.*

This motivates the study of faces given by separating hyperplanes.

Definition 1.1.17. A face F of C is called *exposed* if there is a linear functional $\ell \in V^*$ such that

$$F = \{x \in C\colon \forall y \in C \ \ell(y) \geq \ell(x)\}.$$

In this situation, we say that the face F is exposed by the linear functional ℓ. We call the affine hyperplane $\{x \in V\colon \ell(x) = \ell(y)\}$ for some $y \in F$ a *supporting hyperplane* of F. Note that this hyperplane does not depend on the choice of $y \in F$.

Example 1.1.18. (a) In the case of polytopes, every face is exposed.
(b) The same is true for the unit ball in $\mathbb{R}^n$.
(c) Every face of a spectrahedron is exposed, cf. Barvinok [1], Section II.12, or Ramana-Goldman [31], Corollary 1. This result is more general than (a) and (b).
(d) Consider the convex subset C of $\mathbb{R}^2$ defined by the following inequalities

$$C = \{(x, y) \in \mathbb{R}^2\colon -1 \leq x \leq 1, -1 \leq y \leq 1\} \cup \{(x, y) \in \mathbb{R}^n\colon (x-1)^2 + y^2 - 1 \geq 0\},$$

see Figure 1.1. It can be checked, that $(1, \pm 1)$ are extreme points of C, but they are not exposed since the lines $y = 1$ and $y = -1$ are tangent to their intersection points $(1, 1)$ and $(1, -1)$ respectively with the circle.

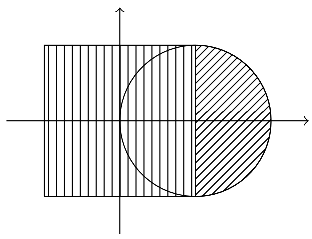

Figure 1.1.: The subset C of $\mathbb{R}^2$ of example 1.1.18(d)

Remark 1.1.19. (a) Note that if F is the set of minimisers in C of a linear functional, then F is a face of C as defined in 1.1.4(a).
(b) Let C be a convex body. If we separate a point x in the boundary of C from the interior of C, which is convex (cf. Barvinok [1], Corollary II.2.3), we get a linear functional $\ell \in V^*$ such that $\ell(y) \geq \ell(x)$ for all $y \in C$. In particular, every boundary point of C is contained in an exposed face of C. Moreover, every face that is maximal with respect to inclusion among proper faces of C is exposed.

Extreme points are in general not exposed, but a general extreme point is exposed.

Theorem 1.1.20 (Straszewicz, cf. Rockafellar [35], Theorem 18.6). *Let $C \subset V$ be a closed convex set. Then the set of exposed points of C is a dense subset of the set of extreme points of C. In particular, every extreme point is the limit of a sequence of exposed points.*

We turn to the local study of supporting hyperplanes at a face of a convex set, which is formally done by introducing normal cones.

Definition 1.1.21. The *normal cone* of a face F of C, denoted by $N_C(F)$, is the set of all linear functionals on V minimising on F, i.e.

$$N_C(F) = \{\ell \in V^* \colon \forall y \in C \ \forall x \in F \ \ell(y) \geq \ell(x)\}.$$

Remark 1.1.22. The normal cone is the set of all (inward) normal vectors of supporting hyperplanes to C at F.

Example 1.1.23. (a) Given a polytope $P \subset \mathbb{R}^n$, the normal cone of a face $F \subset P$ is the cone generated by the inner normal vectors of all facets containing F, cf. [43], Lecture 2.3.
(b) Consider the unit ball $\{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$ with respect to the euclidean norm. The normal cone to a boundary point x is the one-dimensional cone $-\mathbb{R}_{\geq 0}x$.

Proposition 1.1.24. *Let $C \subset V$ be a convex set. Let F, F_1 and F_2 be exposed faces of C.*
(a) The linear functionals exposing F are exactly the relative interior points of $N_C(F)$.
(b) The inclusion $F_1 \subset F_2$ holds if and only if $N_C(F_2) \subset N_C(F_1)$.
(c) The intersection of the normal cones $N_C(F_1)$ and $N_C(F_2)$ is the normal cone of the inclusion minimal exposed face $F_1 \vee F_2$ containing both F_1 and F_2 (possibly $F = C$).
(d) The normal cone of the face $F_1 \cap F_2$ contains the cone generated by the normal cones to F_1 and F_2. If $F_1 \cap F_2$ is exposed, there is a linear functional exposing it in $\mathrm{co}(N_C(F_1) \cup N_C(F_2))$.

Proof. Let ℓ be a linear functional exposing F, i.e. $\ell(y) \geq \ell(x)$ for all $y \in C$ and $x \in F$ with strict inequality if $y \notin F$. Let ℓ' be a relative interior point of $N_C(F)$. If $\ell'(y) = \ell'(x)$ for a point $y \in C \setminus F$ and all $x \in F$, then for every $\varepsilon > 0$ and $x \in F$ we would have

$$(\ell' - \varepsilon\ell)(y) = \ell'(x) - \varepsilon\ell(y) < (\ell' - \varepsilon\ell)(x)$$

which contradicts the assumption that ℓ' is a relative interior point. Basically the same computation shows that ℓ is a relative interior point of $N_C(F)$.
The claim in (b) is obvious, because F_1 is exposed. To prove (c), note that the normal cone of the inclusion minimal exposed face $F_1 \vee F_2$ containing both F_1 and F_2 is contained in $N_C(F_1) \cap N_C(F_2)$ by (b). By assumption, $F_1 \vee F_2$ is exposed. Now one checks that every linear functional exposing $F_1 \vee F_2$ lies in the relative interior of $N_C(F_1) \cap N_C(F_2)$. In (d), the first statement is obvious. For the second, let ℓ_i be a linear functional exposing F_i ($i = 1, 2$), then $\ell_1 + \ell_2$ exposes $F_1 \cap F_2$. □

Remark 1.1.25. The assumption on F_1 and F_2 being exposed in Proposition 1.1.24 cannot be dropped, cf. Example 1.1.36.

Proposition 1.1.26. *Let C be a closed convex set. A face of C is exposed if and only if it is the intersection of all inclusion-maximal proper faces containing it. In this case, it is the intersection of finitely many of them.*

Proof. Let F be an exposed face of C. Obviously, F is contained in the intersection of all inclusion maximal proper faces containing F. Let F' be such an inclusion maximal proper face and let $y \in F' \setminus F$. We prove that there exists an inclusion-maximal proper face F'' contaning F but not y. There is an exposed extreme ray $\mathbb{R}_{\geq 0}\ell$ of the normal cone $N_C(F)$ such that $\ell(y) > \ell(x)$ for all $x \in F$ by Straszewicz's Theorem, because F is exposed. Let F'' be the face supported by ℓ. Then it contains F and is a proper inclusion-maximal face by the extremality of ℓ in the normal cone of F, cf. Proposition 1.1.24(b).
Conversely, let F be the intersection of proper inclusion-maximal faces, say

$$F = \bigcap_{i \in I} F_i.$$

Let C_F be the cone generated by the normal cones $N_C(F_i)$ and let ℓ be a point in the relative interior of C_F. Then ℓ exposes F. In this case if ℓ is a conic combination of $\ell_{i_1}, \ldots, \ell_{i_r}$, where $\mathbb{R}_{\geq 0}\ell_{i_j}$ is an extreme ray of $N_C(F_{i_j})$, then F is the intersection of the corresponding faces F_{i_j}. □

The set of all supporting hyperplanes is a convex set in the dual vector space if we do the following normalisation of the inward normal vectors.

Definition 1.1.27. Let $D \subset V$ be a set. Define the *dual convex set* as

$$D^\vee = \{\ell \in V^* : \forall y \in D \; \ell(y) \geq -1\}.$$

We call the dual convex set of a compact convex set the *dual convex body* or *polar* and the dual convex set of a convex cone the *dual cone*. If we want to notationally clarify that we are dealing with a convex body, we denote the polar by D^o.

Remark 1.1.28. Since a linear functional is bounded from below on a convex cone if and only if it only takes non-negative values, the definition of the dual convex set is for cones equivalent to the standard definition in the literature, i.e.

$$C^\vee = \{\ell \in V^*: \forall x \in C\ \ell(x) \geq 0\}.$$

With this notation, the Biduality Theorem in convexity can be stated as follows.

Theorem 1.1.29 (cf. Barvinok [1], Theorem IV.1.2). *Let $D \subset V$ be a set. Then*

$$(D^\vee)^\vee = \operatorname{cl}(\operatorname{conv}(D) \cup \{0\}).$$

Remark 1.1.30. The two most important special cases are:
(a) If $C \subset V$ is a convex body containing the origin, then $C = (C^\vee)^\vee$. Note that the polar is not bounded if the origin lies in the boundary of C.
(b) If $C \subset V$ is a closed convex cone, then $C = (C^\vee)^\vee$.

Definition 1.1.31. Let $C \subset V$ be a convex set. Given a face F of C, we call the set of all linear functionals minimising on F over C the *dual face* of F and denote it by $F^\vee$.

Remark 1.1.32. (a) The dual face is an exposed face of the dual convex set.
(b) If C is a convex body containing the origin in its interior, then the definition of the dual face to $F \subset C$ reads explicitly

$$\{\ell \in C^\vee: \forall x \in F\ \ell(x) = -1\}.$$

In the case of a convex cone $C \subset V$, it reads

$$\{\ell \in C^\vee: \forall x \in F\ \ell(x) = 0\}.$$

Remark 1.1.33. Let C be a convex body containing the origin in its interior.
(a) The dual face to F is an affine section and compact basis of the normal cone $N_C(F)$.
(b) If F is an exposed face of C, then we also have biduality for this face relative to C, i.e. the dual face to $F^\vee \subset C^o$ is $F \subset C$.

Of particular interest are the faces dual to extreme points and inclusion maximal proper faces.

Remark 1.1.34. The dual face to an exposed extreme point of C is maximal with respect to inclusion among the proper faces of $C^\vee$, by Proposition 1.1.24(b). In this case, the dual face $F_\ell = \{x \in C: \ell(x) = -1\}$ has a 1-dimensional normal cone, namely $N_C(F_\ell) = \mathbb{R}_+\ell$ by part (b) of the preceding Remark 1.1.33. Unfortunately, the converse is not true in general, as Example 1.1.36 shows: the dual face to an inclusion maximal face of C does not need to be an extreme point of $C^\vee$. Moreover, if this is the case, all extreme points of the dual face are non-exposed.

Remark 1.1.35. Let $C \subset V$ be a closed convex cone.
(a) The dual cone is the set of all inward normal vectors of supporting hyperplanes to C.
(b) The dual cone is the union of the normal cones $N_C(F)$ over all faces F of C.

Example 1.1.36. (a) Let C be the convex hull of the real points of the two circles $1 = (x+1)^2 + y^2$ and $1 = (x-1)^2 + y^2$ of radius 1 around $(-1,0)$ and $(1,0)$, respectively, in the plane. Then the four points $(\pm 1, \pm 1)$ are non-exposed extreme points of C. The dual convex body is given as the intersection of two parabolas, namely

$$C^\vee = \{(a,b) \in \mathbb{R}^2 \colon a \geq \frac{1}{2}b^2 - \frac{1}{2}, a \leq -\frac{1}{2}b^2 + \frac{1}{2}\}.$$

The inclusion maximal faces $(0, \pm 1)$ of $C^\vee$ dualise to the edges between $(\pm 1, -1)$ and $(\pm 1, 1)$ of C, which are evidently not extreme points. On the other hand, since all faces of $C^\vee$ are exposed, every inclusion maximal face of C dualises to an extreme point of $C^\vee$.
(b) Let C be the convex set in the plane defined by the inequalities $y \geq (x+1)^2 - 3/2$, $y \geq (x-1)^2 - 3/2$ and $y \leq 1$. Consider the extreme point $x = (0, -1/2)$ of C. This face is inclusion maximal, but the dual face is not an extreme point. Indeed, the normal cone $N_C(x)$ has dimension 2. It is the conic hull of the vectors $(-2,1)$ and $(2,1)$, the normal vectors to the tangent lines to the curves defined by $y - (x+1)^2 + 3/2$ and $y - (x-1)^2 + 3/2$, which meet transversallly in x.
We will later see that this is a non-generic phenomenon.
Note that this set is spectrahedral, namely

$$C = \{(x,y) \in \mathbb{R}^2 \colon y \leq 1, \begin{pmatrix} 1 & x+1 \\ x+1 & y+\frac{3}{2} \end{pmatrix} \geq 0, \begin{pmatrix} 1 & x-1 \\ x-1 & y+\frac{3}{2} \end{pmatrix} \geq 0\}$$

Again, essentially the same statements are true for convex cones. We continue the above construction of homogenisation of a convex body, cf. Construction 1.1.11.

Construction 1.1.37. As before, write Φ for the map

$$\Phi \colon \begin{cases} V & \to & \mathbb{R} \times V \\ x & \mapsto & (1, x) \end{cases}.$$

Every face of the cone $\widehat{C}$ is the cone over $\Phi(F)$ for a face F of C. This face is exposed if and only if F is an exposed face of C. In fact, the normal cone $N_C(F)$ is naturally isomorphic to the normal cone $N_{\widehat{C}}(\widehat{F})$, where $\widehat{F} = \mathrm{co}(\Phi(F))$, for every face F of C, via the dual map $\Phi^t \colon \mathbb{R} \times V^* \to V^*$. The dual cone to $\widehat{C} \subset \mathbb{R} \times V^*$ is the cone over the dual convex body $C^\vee \subset V^*$ embedded into $\mathbb{R} \times V^*$ at height 1 via

$$\Psi \colon \begin{cases} V^* & \to & \mathbb{R} \times V^* \\ \ell & \mapsto & (1, \ell) \end{cases}.$$

Remark 1.1.38. Since faces of a closed and pointed cone and their normal cones correspond nicely to faces of a convex body and their normal cones (cf. Construction 1.1.37 for technically precise statements), we get the same properties mentioned above for convex bodies for closed and pointed convex cones, most notably analogous statements to Propsitions 1.1.26 and 1.1.24 with essentially the same counter examples in general (cf. e.g. Example 1.1.36).

1.2. Real Algebraic Geometry

In this section, we will review some important definitions and results from (real) algebraic geometry, e.g. semi-algebraic subsets of real varieties and the duality theory for irreducible projective varieties. The results of the first part can mostly be found in the textbooks by Bochnak-Coste-Roy [10], Basu-Pollack-Roy [3] or Knebusch-Scheiderer [27]. For the second part, we refer to the books by Gelfand-Kapranov-Zelevinsky [19], Tevelev [42] or Harris [22].

1.2.1. Real Varieties

In this part, we want to collect some basic facts about real varieties and semi-algebraic subsets thereof. Mostly, results will be given without proofs; nearly all results can be found in Bochnak-Coste-Roy [10] or Basu-Pollack-Roy [3].

Note that our definition of an affine real is different from the one in Bochnak-Coste-Roy [10].

Definition 1.2.1. A *real affine variety* is the common zero set in $\mathbb{C}^n$ of finitely many real polynomials, i.e. a set of the form

$$\mathcal{V}(f_1, \dots, f_r) = \{x \in \mathbb{C}^n : f_1(x) = 0, \dots, f_r(x) = 0\}$$

where $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$.
The real *Zariski topology* on $\mathbb{C}^n$ is the topology that has the affine real varieties as closed sets. When we take the Zariski closure in what follows, we always mean with respect to the real Zariski topology. As usual, we will denote $\mathbb{C}^n$ equipped with the real Zariski topology by $\mathbb{A}^n$. We write $\mathbb{R}[X]$ for the coordinate ring of the affine real variety $X \subset \mathbb{A}^n$, i.e.

$$\mathbb{R}[X] = \mathbb{R}[x_1, \dots, x_n]/\mathcal{I}(X)$$

where $\mathcal{I}(X)$ is the ideal of polynomials vanishing identically on X. An real affine variety X is called *irreducible* if it cannot be written as the union of two proper subvarieties. Equivalently, X is irreducible if $\mathcal{I}(X)$ is a prime ideal.
We denote the set of real points of X by $X(\mathbb{R}) = X \cap \mathbb{R}^n$.

Definition 1.2.2. (a) The *projective space* $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is the set of all 1-dimensional subspaces of $\mathbb{C}^{n+1}$. We represent the line $\{\lambda x : \lambda \in \mathbb{C}\}$ by $[x]$, where $x \in \mathbb{C}^{n+1}$ is non-zero. A point $[x] \in \mathbb{P}^n$ is called *real* if there is a $v \in \mathbb{R}^{n+1}$ with $[x] = [\{\lambda v : \lambda \in \mathbb{C}\}]$. (b) A homogeneous polynomial $p \in \mathbb{R}[x_0, \dots, x_n]$ is said to be zero on $[x] \in \mathbb{P}^n$ if $p(x) = 0$. Note that p vanishes identically on $\{\lambda x : \lambda \in \mathbb{C}\}$ by homogeneity. A *real projective variety* is the common zero set in $\mathbb{P}^n$ of finitely many real homogeneous polynomials in $n+1$, i.e. a set of the form

$$\mathcal{V}_+(f_1, \dots, f_r) = \{[x] \in \mathbb{P}^n : f_1(x) = 0, \dots, f_r(x) = 0\}.$$

We equip $\mathbb{P}^n$ with the real Zariski topology, i.e. as above the topology that has the real projective varieties as closed sets. A real projective variety is called *irreducible* if it cannot be written as the union of two proper subvarieties.

It will sometimes be more convenient and conceptually accurate to work without choosing coordinates: Let V be a $\mathbb{C}$-vector space of dimension n with an antiholomorphic involution $\sigma\colon V \to V$, i.e. an $\mathbb{R}$-linear map such that $\sigma \circ \sigma$ is the identity and $\sigma(ix) = -i\sigma(x)$ for all $x \in V$. We call those points that are fixed under σ the real points of V and denote the set of all real points by $V(\mathbb{R})$ - this is an $\mathbb{R}$-vector space of dimenion n. As usual, we write $\mathbb{P}(V)$ for the projective space of V, i.e. the space $(V \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by scalar multiplication on $V \setminus \{0\}$. A point $p \in \mathbb{P}(V)$ is called real if it has a real representative, i.e. if there is a $v \in V(\mathbb{R})$ with $p = [v]$.

Definition 1.2.3. (a) Let $X \subset \mathbb{A}^n$ be an affine variety and let its vanishing ideal be generated by $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n]$. Let $p = (p_1, \ldots, p_n)$ be a point on X. The *tangent space* to X at p, denoted as T_pX, is the affine linear space defined by the equations

$$(x_1 - p_1)\frac{\partial f_j}{\partial x_1}(p) + \ldots + (x_n - p_n)\frac{\partial f_j}{\partial x_n}(p) = 0$$

for $j = 1, \ldots, r$. The point p is called *regular* if the tangent space T_pX has the same dimension as X. The variety X is called *smooth* if all its points are regular.
(b) Let $X \subset \mathbb{P}^n$ be a projective variety and let its vanishing ideal be generated by homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{R}[x_0, \ldots, x_n]$. Let $p = (p_0 : \ldots : p_n)$ be a point on X. The *tangent space* T_pX to X at p is the projective linear space defined by the equations

$$x_0\frac{\partial f_j}{\partial x_0}(p) + \ldots + x_n\frac{\partial f_j}{\partial x_n}(p) = 0$$

for $j = 1, \ldots, r$. The point p is called *regular* if the tangent space T_pX has the same dimension as X. Again, X is called *smooth* if all its points are regular.

The tangent space to X at a regular point will be the embedded tangent space for us. It does not depend on the choice of the generators for the vanishing ideal, see e.g. Harris [22], Lecture 14.

Definition 1.2.4. (a) A semi-algebraic subset of an affine real variety $X \subset \mathbb{A}^n$ is a finite boolean combination (complements, unions and intersections) of sets of the form

$$U_X(f) = \{x \in X(\mathbb{R}) : f(x) > 0\}$$

where $f \in \mathbb{R}[X]$ is a regular function on X.
(b) A subset $S \subset X(\mathbb{R})$ of a projective variety $X \subset \mathbb{P}^n$ is semi-algebraic if its intersection with the canonical affine charts $\mathrm{D}_+(x_j) = \mathbb{P}^n \setminus \mathcal{V}_+(x_j)$ is a semi-algebraic subset of the affine variety $X \cap \mathrm{D}_+(x_j)$ for every $j = 0, \ldots, n$.

Remark 1.2.5. A subset $S \subset X(\mathbb{R})$ of a projective variety $X \subset \mathbb{P}^n$ is semi-algebraic if and only if the intersection with every affine subvariety is semi-algebraic, cf. Plaumann [29], Proposition A.3.

Definition 1.2.6. A real point $p \in X(\mathbb{R})$ is called a *central point* of X if it lies in the euclidean closure of the set of regular real points.

Example 1.2.7. (a) Every regular real point of X is a central point of X.
(b) Let $X = \mathcal{V}(x^2 + y^2) \subset \mathbb{A}^2$, then X has only one real point, namely $X(\mathbb{R}) = \{(0,0)\}$. The origin is an isolated singularity of X and not a central point.
(c) The Cartan umbrella is a surface in $\mathbb{A}^3 = \{(x,y,z)\}$ defined by $z(x^2 + y^2) - x^3 = 0$. It is irreducible and contains the z-axis. The only central point on the z-axis is the origin. See Bochnak-Coste-Roy [10], Example 3.1.2 for a picture and more examples of surfaces with non-central points.

Theorem 1.2.8. *Let $X \subset \mathbb{A}^n$ be an irreducible real affine variety.*
(a) The real points $X(\mathbb{R})$ are Zariski dense in X if and only if X has a central real point.
(b) A point $p \in X(\mathbb{R})$ is central if and only if $B(p,\varepsilon) \cap X(\mathbb{R})$ is Zariski dense in X for every $\varepsilon > 0$. In other words, p is a central point if and only if $X(\mathbb{R})$ has full local dimension at p.

This is a summary of several results in Bochnak-Coste-Roy: Part (b) is [10], Proposition 7.6.2. Part (a) follows from part (b) by the definition of local dimension, cf. [10], 2.8.10-2.8.13.

We will later use the following observation about central points under morphisms, which is a corollary to generic smoothness of morphisms.

Proposition 1.2.9. *Let $f: X \to Y$ be a dominant morphism of irreducible real algebraic varieties. Then the image of a central point of X under f is a central point of Y.*

Proof. By generic smoothness, cf. Hartshorne [23], Corollary III.10.7, there is an open subset $U \subset Y_{\mathrm{reg}}$ such that $f: f^{-1}U \to U$ is smooth. Concretely, $f(p)$ is a regular point of Y for every $p \in (f^{-1}U)(\mathbb{R})$. Let p be a central point of X, then every neighbourhood $B(p,\varepsilon)$ intersects $f^{-1}U$ in a non-empty open set. Therefore, $B(f(p),\varepsilon) \cap U$ is non-empty for every $\varepsilon > 0$, which proves the claim. □

1.2.2. Duality of Algebraic Varieties

In this section, we review the duality theory in the context of algebraic geometry. It generalises the duality theory of linear subspaces in linear algebra to irreducible algebraic varieties. It is best stated for projective varieties.

Throughout this section, let V be a $\mathbb{C}$-vector space of dimension $n+1$ with an antiholomorphic involution σ as in the previous section. We denote the dual space of V by $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V(\mathbb{R}),\mathbb{R}) \otimes \mathbb{C}$. If we choose coordinates, we always take $V = \mathbb{C}^{n+1}$ with componentwise complex conjugation as the antiholomorphic involution σ and write $\mathbb{P}^n$ for $\mathbb{P}(V)$ in this case.

Definition 1.2.10. The projective space $\mathbb{P}(V^*)$ of the dual vector space is called the *dual projective space*. We will also denote it by $(\mathbb{P}(V))^*$.

Every point of $\mathbb{P}(V)$ corresponds to a hyperplane of $(\mathbb{P}(V))^*$ by the canonical isomorphism of V with its bidual space. In the following, we often think of a hyperplane in $\mathbb{P}(V)$ as a point of $(\mathbb{P}(V))^*$. Notationally, given a hyperplane $H \subset \mathbb{P}(V)$, we denote the corresponding point in the dual space by $[H]$ or q if the affine cone over H is the kernel of q, i.e. $\widehat{H} = \{v \in V : q(v) = 0\}$. Given a linear subspace $L \subset \mathbb{P}(V)$, we write $L^\perp \subset (\mathbb{P}(V))^*$ for the subspace of all hyperplanes containing L, i.e. $L^\perp = \{[q] \in \mathbb{P}(V^*) : \widehat{L} \subset \ker(q)\}$.

Definition 1.2.11. Let $X \subset \mathbb{P}(V)$ be a projective variety.
(a) We call a hyperplane *tangent* to X at the regular point $p \in X_{\text{reg}}$ if it contains the embedded tangent space to X at p.
(b) The Zariski closure of the set of all hyperplanes in $\mathbb{P}(V)$ that are tangent to X at some regular point of X as a subset of $(\mathbb{P}(V))^*$ is called the *dual projective variety* and denoted by X^*.
(c) We define the *conormal variety* $\text{CN}(\text{X})$ of X to be the Zariski closure of the set of all tuples $(p, [H]) \in \mathbb{P}(V) \times (\mathbb{P}(V))^*$ where $p \in X_{\text{reg}}$ is a regular point of X and $H \subset \mathbb{P}(V)$ is a hyperplane tangent to X at p.

Remark 1.2.12. (a) If $X \subset \mathbb{P}(V)$ is irreducible, its conormal variety is also irreducible and has dimension $n-1$, where $n+1 = \dim(V)$. Indeed, the projection to the first factor makes the dense subset

$$\text{CN}_0(X) = \{(p, [H]) \in \text{CN}(\text{X}) : p \in X_{\text{reg}}, \text{T}_p X \subset H\}$$

into a projective bundle over X_{reg}, namely the conormal bundle, whose fibre over $p \in X_{\text{reg}}$ is the projective linear space of all hyperplanes containing the embedded tangent space $\text{T}_p X$ and thus has dimension $n - \dim(X) - 1$, cf. Tevelev [42], section 1.2. In particular, a tuple $(p, [H]) \in \text{CN}(\text{X})$ is a regular point of $\text{CN}(\text{X})$ if $p \in X_{\text{reg}}$.
(b) Let $X \subset \mathbb{P}(V)$ be irreducible, let $U \subset X$ be a Zariski-dense subset of X and denote by $\text{CN}_0(U)$ the set of all tuples $(x, [H]) \in \text{CN}(\text{X})$ where $x \in U$ is a regular point of X and $H \subset \mathbb{P}(V)$ a hyperplane tangent to X at x. Then $\text{CN}_0(U)$ is dense in $\text{CN}(\text{X})$: First note that $\text{CN}_0(U)$ is a bundle over the irreducible topological space $U \cap X_{\text{reg}}$ of rank $n - \dim(X) - 1$, namely the conoraml bundle restricted to $U \cap X_{\text{reg}}$. Therefore, $\text{cl}(\text{CN}_0(U)) \subset \text{CN}(\text{X})$ is a subvariety of dimension $n-1$ and by irreducibility of $\text{CN}(\text{X})$, equality follows. Since the projection onto the second factor $\pi_2(\text{CN}(\text{X})) = X^*$ of the conormal variety is the dual variety to X, we conclude by continuity of the projection that the set of all hyperplanes $H \subset \mathbb{P}(V)$ tangent to X at a point $x \in U$ is dense in X^*. In particular, if the real points of X are Zariski-dense in X, then the real points of X^* are Zariski-dense in X^*, too. See Shafarevich [39], Chapter VI for more information on the conormal bundle.
(c) If $X \subset \mathbb{A}^n$ is an irreducible affine variety and $\overline{X}$ denotes its projective closure with respect to the embedding of $\mathbb{A}^n$ into $\mathbb{P}^n$ via the map $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n)$, then the union of all hyperplanes in $\mathbb{P}^n$ that are tangent to X at some regular point of X is dense in the dual projective variety to $\overline{X}$. Therefore, we usually just write X^* and mean the dual projective variety to the projective closure of X with respect to this embedding.

Theorem 1.2.13 (Biduality Theorem, cf. Gelfand-Kapranov-Zelevinsky [19], Theorem I.1.1 or Harris [22], Theorem 15.24)**.** *Let $X \subset \mathbb{P}(V)$ be an irreducible projective variety. Then the conormal varieties of X and X^* are equal as subsets of $\mathbb{P}(V) \times (\mathbb{P}(V))^*$; in particular*

$$(X^*)^* = X.$$

Example 1.2.14. (a) Let A be an invertible symmetric $(n+1) \times (n+1)$ matrix and $f = x^t A x$ be the corresponding non-degenerate quadratic form. Then the tangent space at a point P to $\mathcal{V}(f) \subset \mathbb{P}^n$ is

$$\text{T}_P \mathcal{V}(f) = \{x \in \mathbb{P}^n : x^t A P = 0\}.$$

So the coordinates of the dual hyperplane $[H]$ in $(\mathbb{P}^n)^*$ are AP and therefore the dual variety

$$\mathcal{V}(f)^* = \{AP: P^tAP = 0\} = \{[H] \in (\mathbb{P}^n)^*: [H]^tA^{-1}[H] = 0\}$$

is the quadric hypersurface corresponding to the inverse matrix.
(b) Let $U = \operatorname{span}(e_i: i \in I) \subset \mathbb{C}^n$ be a coordinate subspace where $I \subset \{1,\dots,n\}$. Consider the Veronese embedding ν_2 of order 2 of $\mathbb{C}^n$ into the space of symmetric $n \times n$ matrices $\operatorname{Sym}_n(\mathbb{C})$, i.e. $(x_1,\dots,x_n) \mapsto (x_ix_j)_{1\le i,j\le n}$. We want to compute the dual variety to $\nu_2(U) \subset \operatorname{Sym}_n(\mathbb{C})$. The tangent space to $\nu_2(U)$ at xx^t is

$$\mathrm{T}_{xx^t}\nu_2(U) = \{xy^t + yx^t: y \in U\}.$$

A matrix $Z \in \operatorname{Sym}_n(\mathbb{C})$ is perpendicular to this linear space with respect to the standard linear pairing $(M,N) \mapsto \mathrm{tr}(\mathrm{M^tN})$ if and only if $\mathrm{tr}(\mathrm{Z}(\mathrm{xy^t}+\mathrm{yx^t})) = 0$, i.e. if and only if $Zx \in U^\perp$. Since $U = \operatorname{span}(e_i: i \in I)$, this means that the submatrix $Z_{[I]}$ of Z obtained by deleting the rows and columns indexed by elements of I has a non-trivial kernel, so the corresponding minor $\det(Z_{[I]})$ vanishes. Indeed, the hypersurface $\mathcal{V}(\det(Z_{[I]}))$ is the dual variety to $\nu_2(U) \subset \operatorname{Sym}_n(\mathbb{C})$.
By the Biduality Theorem 1.2.13, the dual variety of a hypersurface $X \subset \operatorname{Sym}_n(\mathbb{C})$ defined by a minor of a symmetric matrix is the Veronese embedding $\nu_2(U)$ of the coordinate subspace $U \subset \mathbb{C}^n$ spanned by the basis vectors indexed by the deleted rows of the defining minor.

Remark 1.2.15. We expect the dual variety of a given projective variety $X \subset \mathbb{P}(V)$ to be a hypersurface in $(\mathbb{P}(V))^*$. However if the dual variety is of smaller dimension, say $n-1-k$, then X is ruled by k-spaces, i.e. there is a Zariski-dense subset of X which is the union of k-dimensional linear spaces. Namely if $[H] \in X^*$ is a regular point, then $\mathrm{T}_{[H]}X^*$ has dimension $n-k-1$ and therefore $(\mathrm{T}_{[H]}X^*)^\perp \subset \mathbb{P}(V)$ has dimension k. The union $\bigcup\{(\mathrm{T}_{[H]}X^*)^\perp: [H] \in X^*_{\mathrm{reg}}\}$ is dense in X by the Biduality Theorem 1.2.13.

Remark 1.2.16. Suppose that the real points of $X \subset \mathbb{P}^n$ are dense in X. Then the same is true for its dual variety, as we saw above (Remark 1.2.12). Let $[H] \in X^*$ be a general real point. By the Biduality Theorem, H is tangent to X at a smooth real point.

For plane curves, the duality theory is classical, most famously by results of Plücker, the so-called Plücker formulas, see Tevelev [42], section 1.3, specifically Theorem 1.18. These formulas relate the number and nature of singular points of the primal and dual curve in an elegant way. For higher dimensions, the singularities of the dual variety are subtle and important also in the context of algebraic boundaries of convex hulls of varieties, as we will see later, cf. Section 2.3.2.

We will review the partial characterisation of the tangent space to the conormal variety as presented in Harris [22], Lecture 16. It can be used to prove the Biduality Theorem 1.2.13.

It will prove convenient to make the following identification of the tangent space to $\mathbb{P}(V)$ at a point p.

Proposition 1.2.17. *The tangent space* $\mathrm{T}_p\mathbb{P}(V)$ *to* $\mathbb{P}(V)$ *at* $p \in \mathbb{P}(V)$ *is canonically isomorphic to the vector space* $\operatorname{Hom}_{\mathbb{C}}(p, V/p)$*, where we consider* $p \subset V$ *as a* 1*-dimensional subspace.*

Proof. Identify V with $\mathbb{C}^{n+1}$ and let $f, g \in \mathbb{C}[X_0, \ldots, X_n]$ be homogeneous polynomials of the same degree and $g(p) \neq 0$. Denote by $\partial_v h$ the directional derivative of any rational function h with respect to $v \in V$. Then by homogeneity of f and g and the product rule, we have for all $u \in p$, $u \neq 0$ and $a \in \mathbb{C}^*$

$$\partial_v \frac{f}{g}(au) = \frac{1}{a}\partial_v \frac{f}{g}(u)$$

By Euler's identity, $\partial_u \frac{f}{g}(u) = 0$ and therefore $\partial_v \frac{f}{g}(u)$ is well defined for all $v \in V/p$. Since $\partial_{av}\frac{f}{g}(au) = \frac{1}{a}\partial_{av}\frac{f}{g}(u) = \partial_v \frac{f}{g}(u)$ for all $a \in \mathbb{C}^*$, the value $\partial_{\ell(u)}\frac{f}{g}(u) \in \mathbb{C}$ is well defined for all linear maps $\ell \in \operatorname{Hom}(p, V/p)$ and independent of the choice of $u \in p$. For fixed $\ell \in \operatorname{Hom}(p, V/p)$ and $u \in p$, $u \neq 0$, the map

$$\begin{cases} \mathcal{O}_{\mathbb{P}(V)\times \operatorname{Spec}(\mathbb{C}),p} \supset \mathfrak{m}_p & \to & \mathbb{C} \\ \frac{f}{g} & \mapsto & \partial_{\ell(u)}\frac{f}{g}(u) \end{cases}$$

vanishes on $\mathfrak{m}_p^2$ by the Leibniz rule and it is $\mathbb{C}$-linear, i.e. in the Zariski tangent space $(\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ to $\mathbb{P}(V)$ at p. The canonical isomorphism we are looking for is given by the map

$$\begin{cases} \operatorname{Hom}(p, V/p) & \to & (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \\ \ell & \mapsto & (\frac{f}{g} \mapsto \partial_{\ell(u)}\frac{f}{g}(u)) \end{cases}$$

This map is injective. Since both vector spaces have the same dimension n, it is an isomorphism. □

Using this identification, we can describe the tangent space of the incidence correspondence

$$\Sigma = \{(x, [H]) \in \mathbb{P}(V) \times (\mathbb{P}(V))^* : x \in H\}$$

at a point (p, q).

Proposition 1.2.18. *Let $p = \mathbb{C}u \subset V$, $q = \mathbb{C}w \subset V^*$ be 1-dimensional subspaces and assume $w(u) = 0$. The tangent space $\mathrm{T}_{(p,q)}\Sigma$ to the incidence correspondence at (p,q) is canonicallly isomorphic to the vector space*

$$\{(\alpha, \beta) \in \operatorname{Hom}(p, V/p) \oplus \operatorname{Hom}(q, V^*/q) : \langle u, \beta(w)\rangle + \langle \alpha(u), w\rangle = 0\}$$

Proof. We prove this statement by a computation in coordinates. Let $\mathbb{P}(V) = \{(x_0 : \ldots : x_n)\}$ and $(\mathbb{P}(V))^* = \{(y_0 : \ldots : y_n)\}$ and assume $p = \mathbb{C}e_n$ and $q = \mathbb{C}e_0^*$. The incidence correspondence $\Sigma \subset \mathbb{P}(V) \times (\mathbb{P}(V))^*$ is defined by the equation $\sum_{i=0}^n x_i y_i = 0$. Let $\alpha \in \operatorname{Hom}(p, V/p)$ and $\beta \in \operatorname{Hom}(q, V^*/q)$ and write $\alpha(e_n) = \sum_{i=0}^{n-1} a_i e_i + p$ and $\beta(e_0^*) = \sum_{j=1}^n b_j e_j^* + q$. After choosing the affine neighbourhoods $\mathrm{D}_+(x_n) \subset \mathbb{P}(V)$ and $\mathrm{D}_+(y_0) \subset (\mathbb{P}(V))^*$ of p and q respectively, these homomorphisms correspond to the points $(a_0 : \ldots : a_{n-1} : 1) \in \mathbb{P}(V)$ and $(1 : b_1 : \ldots : b_n) \in (\mathbb{P}(V))^*$ under the isomporphism of tangent spaces given in Proposition 1.2.17 because $\alpha \mapsto (\partial_{\alpha(u)}\frac{x_i}{x_n}(u) : i = 0, \ldots, n-1)$ and $\beta \mapsto (\partial_{\beta(w)}\frac{y_j}{y_0}(w) : j = 0, \ldots, n)$. Now

$$(a_0 : \ldots : a_{n-1} : 1;\ 1 : b_1 : \ldots : b_n) \in \mathrm{T}_{p,q}\Sigma$$

is equivalent to $a_0 + b_n = 0$, i.e. $\langle \alpha(e_n), e_0^*\rangle + \langle e_n, \beta(e_0^*)\rangle = 0$ as claimed. □

Using this proposition, one can give a proof of the projective biduality theorem (cf. Harris [22], Lecture 16, Example 16.20). Another application is the following useful statement.

Corollary 1.2.19. *Let $X \subset \mathbb{P}(V)$ be a projective variety and $p \in X_{\text{reg}}$. Let $H \subset \mathbb{P}(V)$ be a hyperplane tangent to X at p. Then the image of the tangent space $\mathrm{T}_{(p,[H])}\mathrm{CN}(\mathrm{X}) \subset \mathbb{P}(V) \times (\mathbb{P}(V))^*$ under the second projection to $(\mathbb{P}(V))^*$ is contained in the hyperplane dual to p.*

Proof. Let $H = \{x \in \mathbb{P}(V) : w(x) = 0\}$ for $w \in V^*$. Since H contains the embedded tangent space to X at p, w vanishes identically on the affine cone $\mathrm{T}_p\widehat{X}$ of the embedded tangent space to X at p and therefore $\langle \alpha(u), w \rangle = 0$ for all $\alpha \in \mathrm{Hom}(p, \mathrm{T}_p\widehat{X}/p)$ and $u \in p$, $u \neq 0$. The conormal variety $\mathrm{CN}(\mathrm{X})$ is contained in the incidence correspondence Σ, so $\mathrm{T}_{(p,[H])}\mathrm{CN}(\mathrm{X}) \subset \mathrm{T}_{(p,[H])}\Sigma$. By Proposition 1.2.18, we conclude from $\langle \alpha(u), w \rangle = 0$ that $\langle u, \beta(w) \rangle = 0$ for all $\beta \in \mathrm{Hom}(\mathbb{C}w, V^*/(\mathbb{C}w))$ such that there is an $\alpha \in \mathrm{Hom}(p, V/p)$ with $(\alpha, \beta) \in \mathrm{T}_{(p,q)}\mathrm{CN}(\mathrm{X})$. □

2. The Algebraic Boundary of a Convex Semi-algebraic Set

In this chapter, we prove abstract results about the algebraic boundary of a convex semi-algebraic set. In the first part, we collect basic facts about it and its relation to notions from convex duality. In the second part, we study its implications for the question whether or not a closed semi-algebraic set can be defined by finitely many simultaneous polynomial inequalities. In the third part, we take a look at the special case of the convex hull of a variety X and use the algebraic duality theory to construct varieties in terms of X that certainly contain the algebraic boundary of the convex hull. In the final part, we consider our most general case of a convex semi-algebraic set. We will restrict ourselves to closed and pointed cones or compact sets with non-empty interior. The main reason is the convex duality theory, which will be one of the main tools in the proofs.

2.1. Algebraic Boundaries of Convex Semi-algebraic Sets

Definition 2.1.1. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. The *algebraic boundary* of S, denoted as $\partial_a S$, is the $\mathbb{R}$-Zariski closure in $\mathbb{A}^n$ of the euclidean boundary of S.

We first want to establish that the algebraic boundary of a convex body is a hypersurface.

Definition 2.1.2. A subset of $\mathbb{R}^n$ is called *regular* if it is contained in the closure of its interior.

Remark 2.1.3. Every convex semi-algebraic set with non-empty interior is regular and the complement of a convex semi-algebraic set is also regular.

Lemma 2.1.4. *Let $\emptyset \neq S \subset \mathbb{R}^n$ be a regular semi-algebraic set and suppose that its complement $\mathbb{R}^n \setminus S$ is also regular and non-empty. The algebraic boundary of S is an affine variety such that every irreducible component has codimension 1, i.e. $\partial_a S$ is a hypersurface.*

Proof. By Bochnak-Coste-Roy [10], Proposition 2.8.13, $\dim(\partial S) \leq n-1$. Conversely, we prove that every point in the boundary ∂S of S has local dimension $n-1$ in ∂S: Let $x \in \partial S$ be a point and take $\varepsilon > 0$. Then $\operatorname{int}(S) \cap \mathrm{B}(x,\varepsilon)$ and $\operatorname{int}(\mathbb{R}^n \setminus S) \cap \mathrm{B}(x,\varepsilon)$ are non-empty, because both S and $\mathbb{R}^n \setminus S$ are regular. Applying [10], Lemma 4.5.2, yields that

$$\dim(\partial S \cap \mathrm{B}(x,\varepsilon)) = \dim(\mathrm{B}(x,\varepsilon) \setminus (\operatorname{int}(S) \cup (\mathbb{R}^n \setminus \overline{S}))) \geq n-1$$

Therefore, all irreducible components of $\partial_a S = \mathrm{cl}_{Zar}(\partial S)$ have dimension $n-1$. □

Example 2.1.5. The assumption of S being regular cannot be dropped in the above lemma. Write $h := x^2 + y^2 + z^2 - 1 \in \mathbb{R}[x,y,z]$. Let S be the union of the unit ball with the first coordinate axis, i.e. $S = \{(x,y,z) \in \mathbb{R}^3 : y^2 h(x,y,z) \le 0, z^2 h(x,yz) \le 0\}$. The algebraic boundary of S is the union of the sphere $\mathcal{V}(h)$ and the line $\mathcal{V}(y,z)$, which is a variety of codimension 1 with a lower dimensional irreducible component.

Corollary 2.1.6. *Let $C \subset \mathbb{R}^n$ be a convex body. Its algebraic boundary is a hypersurface.* □

This property characterises the compact semi-algebraic convex sets.

Proposition 2.1.7. *A compact convex set with non-empty interior is semi-algebraic if and only if its algebraic boundary is a (algebraic) hypersurface.*

Proof. The converse follows from standard results in semi-algebraic geometry. Namely if the algebraic boundary $\partial_a C$ is an algebraic hypersurface, its complement $\mathbb{R}^n \smallsetminus (\partial_a C)(\mathbb{R})$ is a semi-algebraic set and the closed convex set C is the closure of the union of finitely many of its connected components. This is semi-algebraic by Bochnak-Coste-Roy [10], Proposition 2.2.2 and Theorem 2.4.5. □

By the construction of homogenisation in convexity introduced in Chapter 1, Section 1.1, the algebraic boundary of a pointed and closed convex cone relates to the algebraic boundary of a compact base via the notion of affine cones in algebraic geometry.

Remark 2.1.8. Let $C \subset \mathbb{R}^n$ be a compact semi-algebraic convex set and let $\widehat{C} \subset \mathbb{R} \times \mathbb{R}^n$ be the cone over C as defined in Construction 1.1.11. Since a point $(1,x)$ lies in the boundary of $\widehat{C}$ if and only if x is a boundary point of C, the affine cone $\{(\lambda, \lambda x) : \lambda \in \mathbb{C}, x \in \partial_a C\}$ over the algebraic boundary of C is a constructible subset of the algebraic boundary of $\widehat{C}$. More precisely, we mean that $\partial_a \widehat{C} = \widehat{X}$, where X is the projective closure of $\partial_a C$ with respect to the embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n)$.

Corollary 2.1.9. *Let $C \subset \mathbb{R}^{n+1}$ be a semi-algebraic pointed closed convex cone. Its algebraic boundary is a hypersurface in $\mathbb{A}^{n+1}$ and an algebraic cone. In particular, it is the affine cone over its projectivisation in $\mathbb{P}^n$, i.e.*

$$\widehat{\mathbb{P}\partial_a C} = \partial_a C. \qquad \square$$

Proposition 2.1.10. *Let $C \subset \mathbb{R}^n$ be a semi-algebraic convex body and set $S := \partial C^\vee \cap (\partial_a C^\vee)_{\mathrm{reg}}$. For every $\ell \in S$, the face supported by ℓ is a point. The set S is an open and dense (in the euclidean topology) semi-algebraic subset of the set $\partial C^\vee$ of all supporting hyperplanes to C.*

Proof. If ev_x is a supporting hyperplane to $C^\vee$ at ℓ, then $\ell(x) = -1$ and $C^\vee$ lies in one halfspace defined by ev_x. Therefore, ev_x defines the unique tangent hyperplane to $\partial_a C^\vee$ at ℓ. Now we show that x is an extreme point of C, exposed by ℓ. Suppose $x = \frac{1}{2}(y+z)$ with $y, z \in C$, then $\ell(y) = -1$ and $\ell(z) = -1$. Since y and z are, by the same argument as above, also inward normal vectors to the tangent hyperplane $\mathrm{T}_\ell \partial_a C^\vee$, we conclude $x = y = z$. □

The same statement is true for convex cones:

Corollary 2.1.11. *Let $C \subset \mathbb{R}^{n+1}$ be a semi-algebraic closed and pointed convex cone with non-empty interior and set $S := \partial C^\vee \cap (\partial_a C^\vee)_{\mathrm{reg}}$. For every $\ell \in S$, the face supported by ℓ is an extreme ray of C. The set S is open and dense (in the euclidean topology) semi-algebraic subset of $\partial C^\vee$.* □

Example 2.1.12. (a) In the case of polytopes, the set S of regular points of the algebraic boundary is exactly the set of linear functionals exposing extreme points. Indeed, the algebraic boundary of a polytope is a union of (affine) hyperplanes, namely the affine span of its facets and a point on the boundary of a polytope is a regular point of the algebraic boundary if and only if it lies in the relative interior of a facet, cf. Barvinok [1], Theorem VI.1.3.
(b) In general, a linear functional $\ell \in \partial C^\vee$ exposing an extreme point of C does not need to be a regular point of the algebraic boundary of $C^\vee$ as example 1.1.36(b) shows. Indeed, the linear functionals $(-2,1)$ and $(2,1)$ are intersection points of a line and a quadric in the algebraic boundary of $C^\vee$, but they both expose the extreme point $(0,-1/2)$ of C.

The extreme points (resp. rays) of a convex set play an important role for duality. They will also be essential in a description of the algebraic boundary using the algebraic duality theory. So we fix the following notation:

Definition 2.1.13. (a) Let $C \subset \mathbb{R}^n$ be a convex semi-algebraic set. We denote by $\mathrm{Ex}_a(C)$ the $\mathbb{R}$-Zariski closure of the union of all extreme points of C in $\mathbb{A}^n$.
(b) Let $C \subset \mathbb{R}^{n+1}$ be a semi-algebraic convex cone. We write $\mathrm{Exr}_a(C)$ for the $\mathbb{R}$-Zariski closure of the union of all extreme rays of C in $\mathbb{A}^{n+1}$.

Remark 2.1.14. (a) Note that the union of all extreme points of a convex semi-algebraic set is a semi-algebraic set by quantifier elimination because the definition is expressible as a first order formula in the language of ordered rings, cf. Bochnak-Coste-Roy [10], Proposition 2.2.4. Therefore, its Zariski closure is an algebraic variety whose dimension is equal to the dimension of $\mathrm{Ex}(C)$ as a semi-algebraic set, cf. Theorem 1.2.8. Of course, the same is true for convex cones and the Zariski closure of the union of all extreme rays.
(b) Note that $\mathrm{Exr}_a(C)$ is an algebraic cone. In particular, we have

$$\mathrm{Exr}_a(C) = \mathbb{P}\widehat{\mathrm{Exr}_a(C)}.$$

Lemma 2.1.15. *Let $C \subset \mathbb{R}^n$ be a semi-algebraic convex body with $0 \in \mathrm{int}(C)$. For a general extreme point $x \in \mathrm{Ex}_a(C)$ there is a supporting hyperplane $\ell_0 \in \partial C^\vee$ exposing the face x and a semi-algebraic neighbourhood U of ℓ_0 in $\partial C^\vee$ such that every $\ell \in U$ supports C in an extreme point x_ℓ and all x_ℓ lie on the same irreducible component of $\mathrm{Ex}_a(C)$ as x.*

By general we mean in this context that the statement is true for all points in a dense semi-algebraic subset of $\mathrm{Ex}_a(C)$.

Proof. By Straszewicz's Theorem 1.1.20 and the Curve Selection Lemma from semi-algebraic geometry Bochnak-Coste-Roy [10], Theorem 2.5.5, a general extreme point is exposed. Let $y \in \mathrm{Ex}(C)$ be an exposed extreme point contained in a unique irreducible component Z of $\mathrm{Ex}_a(C)$ and denote by ℓ_y an exposing linear functional. Let $Z_1, \ldots, Z_r$ be the irreducible components of $\mathrm{Ex}_a(C)$ labelled such that $Z = Z_1$. Since the sets $Z_i \cap \partial C \subset C$ are closed, they are compact.

Now ℓ_y is strictly greater than -1 on $Z_i \cap \partial C$ for $i > 1$ and therefore, there is a neighbourhood U in $\partial C^\vee$ of ℓ_y such that every $\ell \in U$ is still strictly greater than -1 on $Z_i \cap \partial C$. The intersection of this neighbourhood with the semi-algebraic set S of linear functionals exposing extreme points, which is open and dense in the euclidean topology by Proposition 2.1.10, is non-empty and open in $\partial C^\vee$. Pick ℓ_0 from this open set, then the extreme point x exposed by ℓ_0 has the claimed properties. □

Example 2.1.16. (a) Again, the above lemma has a simple geometric meaning in the case of polytopes: Every extreme point of the polytope is exposed exactly by the relative interior points of the facet of the dual polytope dual to it, again by Barvinok [1], Theorem VI.1.3.
(b) In Example 1.1.36(b) , the dual convex body looks essentially the same and the only linear functional not exposing an extreme point is the dual face to the edge of C, i.e. the intersection point $(0, -1)$ of the quadrics in the boundary.

By homogenisation, we can draw the same conclusions for closed and pointed convex cones.

Corollary 2.1.17. *Let $C \subset \mathbb{R}^{n+1}$ be a semi-algebraic closed and pointed convex cone with non-empty interior. Let $F_0 \subset C$ be an extreme ray of C such that the line $[F_0]$ is a general point of $\mathbb{P}\mathrm{Exr}_a(C)$. Let Z be the irreducible component of $\mathbb{P}\mathrm{Exr}_a(C)$ with $[F_0] \in Z$. Then there is a supporting hyperplane $\ell_0 \in \partial C^\vee$ exposing F_0 and a semi-algebraic neighbourhood U of ℓ_0 in $\partial C^\vee$ such that every $\ell \in U$ supports C in an extreme ray F_ℓ of C contained in the regular locus of Z, i.e. $[F_\ell] \in Z_{\mathrm{reg}}$.* □

The above notion of general now translates simply into the projective notion, i.e. the statement is true for points in a dense semi-algebraic subset of the semi-algebraic set of extreme rays as a subset of $\mathbb{P}\mathrm{Exr}_a(C) \subset \mathbb{P}^n$.

2.2. Basic Closed Convex Semi-algebraic Sets

We want to study the relation between the algebraic boundary of a closed semi-algebraic set and the property that it can be defined by finitely many simultaneous polynomial inequalities in the special case of convex semi-algebraic sets.

Definition 2.2.1. A semi-algebraic set $S \subset \mathbb{R}^n$ is called *basic closed* if there are polynomials $g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$ such that

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}.$$

Remark 2.2.2. The Finiteness Theorem in semi-algebraic geometry states that every closed semi-algebraic set is a finite union of basic closed sets, see Bochnak-Coste-Roy [10], Theorem 2.7.2. In general, you cannot avoid the union, see e.g. Example 2.2.6.

Let us recall the definition of a central point of a real variety.

Definition 2.2.3. Let X be a variety. A real point $x \in X(\mathbb{R})$ of X is called a *central point* of X if it has full local dimension in the set of real points, i.e.

$$\dim_x(X(\mathbb{R})) = \dim(X)$$

See Section 1.2.1 for basic facts about central points.

Lemma 2.2.4. *Let $\emptyset \neq S \subset \mathbb{R}^n$ be a regular semi-algebraic set and suppose that its complement $\mathbb{R}^n \setminus S$ is also regular and non-empty. If the interior of S intersects the algebraic boundary of S in a central point, then S is not basic closed.*

Proof. Assume that S is basic closed, i.e. there are polynomials $g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$ such that $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$. Since S is regular, in every point of the boundary of S, at least one polynomial $g_k \in \{g_1, \ldots, g_r\}$ must change sign. Let $h \in \mathbb{R}[X_1, \ldots, X_n]$ be a polynomial defining an irreducible component of $\partial_a S$ intersecting the interior of S in a central point. There is a Zariski dense subset $M \subset \mathcal{V}(h)_{\text{reg}}(\mathbb{R})$ that is contained in ∂S. Since $M = \bigcup_{k=1}^r \mathcal{V}(g_k) \cap M$, we get

$$\mathcal{V}(h) = \overline{M}^{Zar} \subset \bigcup_{k=1}^{r} \mathcal{V}(g_k) \cap \overline{M}^{Zar}.$$

It follows from the irreducibility of $\mathcal{V}(h)$ that $\mathcal{V}(h) \subset \mathcal{V}(g_j)$ for some g_j that changes sign along M. Consequently, g_j changes sign in every point of the set $\mathcal{V}(h)_{\text{reg}} \cap \text{int}(S) \neq \emptyset$, which is a contradiction to $S \subset \{x \in \mathbb{R}^n : g_j(x) \geq 0\}$. $\square$

The assumption in the previous lemma on S being regular cannot be dropped, as the following example shows.

Example 2.2.5. Let $g := x^2 + y^2 - 1 \in \mathbb{R}[x, y]$ and let $S = \{(x, y) \in \mathbb{R}^2 : y^2 g(x, y) \leq 0\}$ be the union of the closed disc $\{(x, y) \in \mathbb{R}^2 : g(x, y) \leq 0\}$ with the line defined by $y = 0$. The set S is basic closed and its algebraic boundary has two components, namely the circle $\mathcal{V}(g)$ and the line $\mathcal{V}(y)$. The origin is a regular point of this hypersurface.

The assumption in the above lemma that the algebraic boundary does not intersect the interior of the convex set in a central point is not the only obstruction to being basic closed.

Example 2.2.6. We consider the convex hull of two circles in orthogonal planes in 3-space, namely $C = \text{conv}(X(\mathbb{R}))$, where $X = \mathcal{V}(x, (y - \frac{1}{2})^2 + z^2 - 1) \cup \mathcal{V}(z, x^2 + y^2 - 1)$. Then C is a compact convex semi-algebraic set containing the origin in its interior.

First note that C is not basic closed because the hyperplane section $C \cap \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$ shown in Figure 2.1 is not basic closed. The circle $\mathcal{V}(x^2 + y^2 - 1)$ intersects the interior of the hyperplane section and we can apply Lemma 2.2.4 to show that C is not basic closed.

We will now argue that the algebraic boundary of C does not intersect the interior of C in a regular point: We will compute the algebraic boundary in Example 2.3.10(b). It consists of the so-called edge surface which is the closure of the union of all stationary bisecant lines, cf. Ranestad-Sturmfels [33] and Section 2.3.1 below. A stationary bisecant line is a line between two regular points P and Q on X whose tangent lines lie in an affine plane. The union of all stationary bisecant lines is a constructible subset of the edge surface. We show that there is no stationary bisecant line to X that intersects the interior of C and therefore, the interior of C contains no central point of the edge surface. Let $P, Q \in X(\mathbb{R})$ be two points such that the embedded tangent lines $\mathrm{T}_P X$ and $\mathrm{T}_Q X$ lie in an affine plane $H \subset \mathbb{A}^3$. We can assume that P

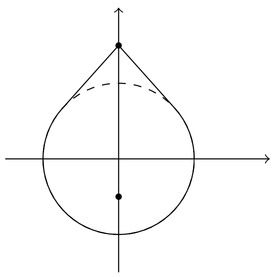

Figure 2.1.: The hyperplane section $C \cap \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$.

and Q lie on different irreducible components of X because the circles are contained in two orthogonal planes, which are not irreducible components of the algebraic boundary of C. Denote by X_P the circle containing P and analogously X_Q, i.e. $X = X_P \cup X_Q$. Then the plane does not contain any of the circles and therefore $H \cap X = \{P, Q\}$. This means that X_P is contained in one half-space defined by H as is X_Q. Since $\operatorname{conv}(X_P(\mathbb{R})) \cap X_Q \neq \emptyset$, it is the same half-space and H is a supporting hyperplane to C exposing the edge of C joining P and Q. This argument shows that no stationary bisecant line joining two points that are not on the same circle intersects the interior of C, as claimed.

Even in the plane, the situation seems to be more complicated then it appears.

Example 2.2.7. Let $f_\varepsilon = (y - x^3)(y - 2x^3) - \varepsilon yx$. Then f_ε is an irreducible sextic for every $\varepsilon > 0$. Let C be the connected component of the complement of $\mathcal{V}(f)$ containing $(-1, 1)$ intersected with the box $\{(x, y) : -1 \le x \le 1, 0 \le y \le 2\}$. This is a convex semi-algebraic set. For small ε, it looks qualitatively like the set shown in Figure 2.2. Solving $f_\varepsilon = 0$ for y gives

$$y = \frac{3x^3 + \varepsilon x}{2} + \frac{1}{2} x \sqrt{x^4 + 6\varepsilon x^2 + \varepsilon^2},$$

for the upper branch. This is a convex function for $x > 0$, which can be checked by computing the second derivative. The set C is indeed basic closed although it is not locally around the

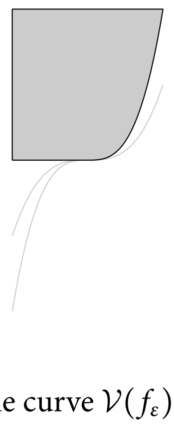

Figure 2.2.: The convex set C. The curve $\mathcal{V}(f_\varepsilon)$ is shaded in grey (qualitatively).

origin defined by the inequality $f_\varepsilon \geq 0$ and the linear inequality $y \geq 0$. The two branches of the sextic $\mathcal{V}(f_\varepsilon)$ need to be separated by an inequality, e.g. by $y > 1/2(3x^3 + \varepsilon x)$.

Being basic closed as a semi-algebraic set does not seem to relate closely to the notions of exposed/non-exposed faces and duality in convex geometry as we will see in the following two examples.

Example 2.2.8. Let $B = \operatorname{conv}\{(-1,0),(-1,1),(1,1),(1,0)\}$ be a box in the plane and consider the following four closed semi-algebraic convex sets: Set

$$\begin{aligned} C_1 &= B \cap \{(x,y) \in \mathbb{R}^2 : y \geq (x + \frac{1}{2})^2 - 1\} \\ C_2 &= (B \cap \{(x,y) \in \mathbb{R}^2 : y \geq x^2 - \frac{1}{2}\}) \cup \operatorname{conv}\{(-1,0),(-1,1),(0,1),(0,0)\} \\ C_3 &= B \cap \{(x,y) \in \mathbb{R}^2 : y \geq x^3\} \\ C_4 &= (B \cap \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}) \cup \operatorname{conv}\{(-1,0),(-1,1),(0,1),(0,0)\}. \end{aligned}$$

Then C_1 and C_3 are basic closed, C_2 and C_4 are not (by Lemma 2.2.4) and C_1 and C_2 have only exposed faces and C_3 and C_4 both have a non-exposed face, namely the origin.

Example 2.2.9 (cf. Example 1.1.36(a)). Let $C = \{(x,y) \in \mathbb{R}^2 : 1 - 2x \geq y^2, 1 + 2x \geq y^2\}$ be the intersection of two parabolas, which is a convex body. The polar is the convex hull of the union of two circles $C^o = \operatorname{conv}(X_1(\mathbb{R}) \cup X_2(\mathbb{R}))$ where $X_1 = \{(x,y) : (x+1)^2 + y^2 = 1\}$ and $X_2 = \{(x,y) : (x-1)^2 + y^2 = 1\}$. By definition, C is basic closed. By Lemma 2.2.4, its polar is not.

2.3. The Convex Hull of a Variety

The setup throughout this section is the following: Given an affine variety $X \subset \mathbb{A}^n$ with the property that its real points are compact in the euclidean topology on $\mathbb{R}^n$ and Zariski-dense in X, we consider the compact convex semi-algebraic set $C = \operatorname{conv}(X(\mathbb{R})) \subset \mathbb{R}^n$. One class of examples that satisfy our assumptions made here are orbitopes: An orbitope is the convex hull of the orbit of a point under the linear action of a compact linear real algebraic group. The closure of such an orbit is an affine variety satisfying the assumptions, cf. Appendix A.

2.3.1. Synthetic Description of the Algebraic Boundary

We investigate the algebraic boundary of C from a synthetic point of view, i.e. we want to describe it using constructions involving only X. The results of this section generalise the description of the algebraic boundary of the convex hull of a general space curve given by Ranestad and Sturmfels in their paper [33]. As a convex body in 3-space, it has 0-dimensional faces, which are points on the curve, and 1- and 2-dimensional faces. Indeed, they prove that the algebraic boundary consists for a general curve of the irreducible edge surface (the Zariski closure of the edges) and tritangent planes (the Zariski closure of the 2-dimensional faces). They further give a formula for the degree of the edge surface and the number of tritangent planes in the general case. We extend their notion of a tritangent plane to general varieties with potentially high codimension and correspondingly higher tangency following the notion of secant varieties.

It will often be convenient to take the projective closure of an affine variety $Z \subset \mathbb{A}^n$ with respect to the embedding $\mathbb{A}^n \to \mathbb{P}^n$, $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n)$. We will denote it by $\overline{Z}$ without explicit mention of the embedding.

Definition 2.3.1. Let $X \subset \mathbb{P}^n$ be a projective variety and let $k \in \mathbb{N}$. We denote the smallest projective linear space containing $x_0, \ldots, x_k \in \mathbb{P}^n$ by $\langle x_0, \ldots, x_k \rangle$.
(a) A *secant k-plane* to X is a k-dimensional plane spanned by points of X.
(b) The Zariski closure of the union of all secant k-planes is called the *k-th secant variety* to X and denoted by $S_k X$.
(c) A *special secant k-plane* or *special k-secant* to X is a k-dimensional linear subspace of $\mathbb{P}^n$ spanned by $k+1$ regular points $x_0, \ldots, x_k$ of X such that there exists a hyperplane $H \subset \mathbb{P}^n$ with $\mathrm{T}_{x_i} X \subset H$ for $i = 0, \ldots, k$. We define the *variety of special secant k-planes* or *k-th special secant variety* to X, denoted by $S_{[k]}X$, to be the closure of the union of all special secant k-planes to X, i.e.

$$S_{[k]}X = \mathrm{cl}\left(\bigcup \{\langle x_0, \ldots, x_k \rangle : x_0, \ldots, x_k \in X_{\mathrm{reg}}, \mathrm{T}_{x_i} X \subset H \text{ for a hyperplane } H \subset \mathbb{P}^n\}\right).$$

These definitions generalise those given by Ranestad and Sturmfels in [33] for space curves: They call the first special secant variety the variety of stationary bisecant lines and the second special secant variety the variety of tritangent planes.

Proposition 2.3.2. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety.*
(a) The k-th secant variety to X is irreducible for all $k \in \mathbb{N}$.
(b) (Terracini's Lemma) For a general point $y \in \langle x_0, \ldots, x_k \rangle$ of the k-th secant variety, the tangent space is

$$\mathrm{T}_y(S_k X) = \langle \mathrm{T}_{x_0} X, \ldots, \mathrm{T}_{x_k} X \rangle.$$

(c) If $S_k X \neq \mathbb{P}^n$, then the k-th special secant variety is equal to $S_k X$, i.e.

$$S_{[k]}X = S_k X.$$

In other words, all secant k-planes are special. In particular, we have $S_{[k]}X = S_k X$ for $k(\dim(X)+1) + \dim(X) < n$.
(d) For $k \geq n$, the k-th special secant variety is empty.

Proof. (a) The k-th secant variety to X is the closure of the image of the rational map from the $(k+1)$-fold product of the affine cone $\widehat{X}$ to $\mathbb{P}^n$ that takes $(v_0, \ldots, v_k)$ to $\sum_{i=0}^{k} v_i$. This implies the irreducibility of $S_k X$. By generic smoothness, part (b) (Terracini's Lemma) follows, cf. Flenner-O'Carroll-Vogel [17], Proposition 4.3.2. Part (c) follows from Terracini's Lemma because the tangency condition in the definition of a special secant k-plane is redundant in this case. The lower bound for k follows from a count of dimensions using the map described at the beginning of the proof. And (d) is the fact that the only n-secant in $\mathbb{P}^n$ is $\mathbb{P}^n$ itself, which is not contained in a hyperplane. □

The k-th special secant variety to an irreducible variety need not be irreducible.

Example 2.3.3. (a) Consider the first secant variety and the first special secant variety to the rational normal curve $X = \{(s^3 : s^2t : s^1t^2 : t^3) : (s : t) \in \mathbb{P}^1\}$ in $\mathbb{P}^3$. It is well known that the first secant variety to the curve X will fill $\mathbb{P}^3$ (cf. Harris [22], Exercise 8.6). Yet, the first special secant variety to X is empty. In order to prove this, we check that the tangent lines to any two points $P, Q \in X$ are not coplanar. So consider the points $P = (a_1^3 : a_1^2b_1 : a_1b_1^2 : b_1^3)$ and $Q = (a_2^3 : a_2^2b_2 : a_2b_2^2 : b_2^3)$ on X and check that the radical of the ideal generated by the determinant of the matrix

$$M = \begin{pmatrix} 3a_1^2 & 2a_1b_1 & b_1^2 & 0 \\ 0 & a_1^2 & 2a_1b_1 & 3b_1^2 \\ 3a_2^2 & 2a_2b_2 & b_2^2 & 0 \\ 0 & a_2^2 & 2a_2b_2 & 3b_2^2 \end{pmatrix}$$

is the same as the radical of the ideal of $2x2$-minors of the matrix

$$N = \begin{pmatrix} a_1^3 & a_1^2b_1 & a_1b_1^2 & b_1^3 \\ a_2^3 & a_2^2b_2 & a_2b_2^2 & b_2^3 \end{pmatrix}$$

which turns out to be the ideal generated by $a_1b_2 - a_2b_1$, implying that the points P and Q are equal ($\det(M) = 9(a_1b_2 - a_2b_1)^4$).
(b) Now consider the rational curve $X = \{(s^4 : s^3t : s^2t^2 : t^4) \in \mathbb{P}^3 : (s : t) \in \mathbb{P}^1\}$, which is a projection of the rational normal curve in $\mathbb{P}^4$ to $\mathbb{P}^3$. Then again, the first secant variety to X will fill $\mathbb{P}^3$ because the curve is non-degenerate. Yet, in this case, the first special secant variety is a hypersurface. It is by definition the closure of the set of all lines spanned by two points P and Q on X such that the tangent lines T_PX and T_QX are not skew but in fact coplanar. To compute its equation, consider two points $P = (a_1^4 : a_1^3b_1 : a_1^2b_1^2 : b_1^4)$ and $Q = (a_2^4 : a_2^3b_2 : a_2^2b_2^2 : b_2^4)$ on X. The condition that the tangent lines to X at these points are coplanar is the same as saying that the determinant of the matrix

$$M = \begin{pmatrix} 4a_1^3 & 3a_1^2b_1 & 2a_1b_1^2 & 0 \\ 0 & a_1^3 & 2a_1^2b_1 & 4b_1^3 \\ 4a_2^3 & 3a_2^2b_2 & 2a_2b_2^2 & 0 \\ 0 & a_2^3 & 2a_2^2b_2 & 4b_2^3 \end{pmatrix}$$

vanishes. So eliminating the variables u, v, a_1, b_1, a_2 and b_2 from the system of equations $\det(M) = 0$, $w = ua_1^4 + va_2^4$, $x = ua_1^3a_2 + va_2^3b_2$, $y = ua_1^2b_1^2 + va_2^2b_2^2$ and $z = ub_1^4 + vb_2^4$ gives the equation of the first special secant variety to X in the coordinates $(w : x : y : z)$ on $\mathbb{P}^3$. The result is the following quadric

$$S_{[1]}(X) = \mathcal{V}_+(y^2 - wz).$$

`Macaulay2`-code ([20]) for this computation can be found in B.1. The curve is actually the intersection of two quadratic cones, namely

$$X = \mathcal{V}_+(y^2 - wz) \cap \mathcal{V}_+(x^2 - wy)$$

where the second cone is the cone over the curve from its singular point $(0 : 0 : 0 : 1)$, cf. Figure 2.3. The first special secant variety captures only the cone $\mathcal{V}_+(y^2 - wz)$ (which is the green cone shown in Figure 2.3). It does not capture $\mathcal{V}_+(x^2 - wy)$, because all lines on this surface go through the singular point of X.

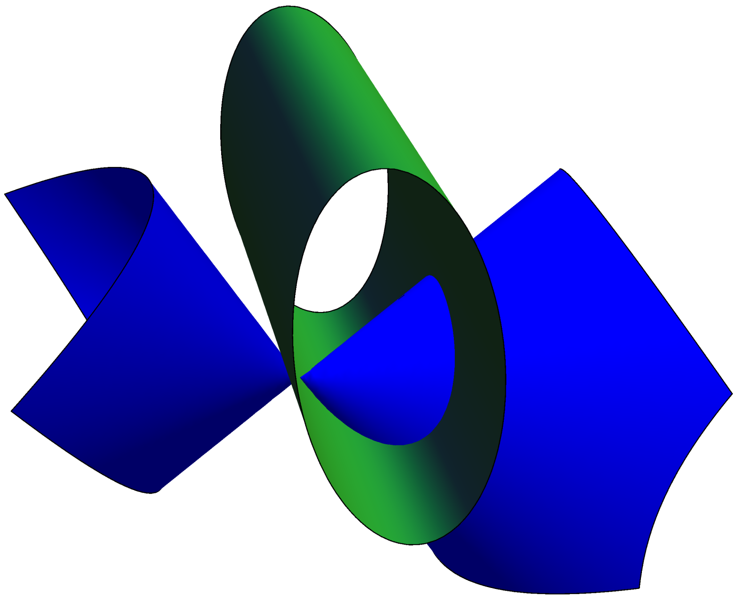

Figure 2.3.: The curve X from Example 2.3.3(b) as the intersection of two quadratic cones on the affine chart $w + y + z = 1$ on $\mathbb{P}^3 = \{(w : x : y : z)\}$.

The tangency condition in the definition of the special secant varieties to X implies, that the codimension of these varieties is at least 1, by a similar argument as in the proof of Terracini's Lemma.

Proposition 2.3.4. *Let $X \subset \mathbb{P}^n$ be a projective variety. For every $k \in \mathbb{N}$, the dimension of $S_{[k]}X$ is at most $n - 1$. More precisely, the tangent space to a general point $x \in X$ which lies on the special k-secant $\langle x_0, \ldots, x_k \rangle$ is contained in the span of the tangent spaces $\langle \mathrm{T}_{x_0}X, \ldots, \mathrm{T}_{x_k}X \rangle$ to X at the x_i.*

Proof. As for the usual k-th secant variety, consider the map

$$f \colon \widehat{X} \times \ldots \times \widehat{X} \to \mathbb{A}^{n+1}, (x_0, \ldots, x_k) \mapsto x_0 + \ldots + x_k$$

and now restrict it to the subvariety $\widehat{Y} = \{(x_0, \ldots, x_k) \colon \mathrm{T}_{x_i}\widehat{X} \subset H \text{ for a hyperplane } H \subset \widehat{\mathbb{P}^n}\}$, which is an algebraic cone. By definition, the special secant variety $S_{[k]}X$ is the closure of $f(\widehat{Y})$. By generic smoothness, cf. Hartshorne [23], proof of III.10.7, the differential of f is surjective over a generic point $x \in S_{[k]}X$ lying on the special k-secant $\langle x_0, \ldots, x_k \rangle$. Therefore

$$\widehat{T}_x(S_{[k]}X) = \mathrm{d}f(\mathrm{T}_{(x_0,\ldots,x_k)}\widehat{Y}) \subset \mathrm{span}(\widehat{T}_{x_0}X \cup \ldots \cup \widehat{T}_{x_k}X) \subset \widehat{H},$$

which shows both claims. □

Before we come to the statement and proof of the main result of this section, we take a short look at varieties ruled by subspaces.

Definition 2.3.5. A projective variety $Y \subset \mathbb{P}^n$ is said to be *ruled by k-spaces* if Y is the closure of a union of k-dimensional linear subspaces of $\mathbb{P}^n$.

Proposition 2.3.6. *Let $X \subset \mathbb{P}^n$ be an irreducible variety and suppose that there is a dense subset which is the union of subspaces of $\mathbb{P}^n$ of dimension > 0. Then there exists a $k \in \mathbb{N}$ and a Zariski open subset U of X such that every point in U lies on a k-dimensional linear space contained in X and not in a higher dimensional linear space contained in X. In particular, X is ruled by k-spaces.*

Proof. Denote by $\mathrm{F}_k(X) \subset \mathbb{G}(k,n)$ the variety of all k-dimensional subspaces contained in X as a subset of the Grassmannian, cf. Harris [22], Example 6.19. Then the union

$$\bigcup_{\Lambda \in \mathrm{F}_k(X)} \Lambda$$

of all k-planes contained in X is a constructible and closed subset of X because it is the image of the projective variety $\{(\Lambda, x) \colon \Lambda \in \mathrm{F}_k(X), x \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n$ under the projection to the second factor (cf. Harris [22], Example 6.12 and Example 6.19), . Therefore it contains an open subset if it is dense, by Chevalley's theorem (cf. Hartshorne [23], Exercises II.3.19). Now take the maximal $k \in \{1, \ldots, \dim(X)\}$ such that this union is a dense subset. □

We can now give a list of possible candidates for the irreducible components of the algebraic boundary of C in terms of the special secant varieties to X. The following theorem generalises the results of Ranestad and Sturmfels on the algebraic boundary of space curves in [33] to affine varieties of any dimension and codimension.

Theorem 2.3.7. *Assume that all real points of X are regular. Let $\overline{X}$ be the projective closure of X with respect to the embedding $\mathbb{A}^n \to \mathbb{P}^n$, $x \mapsto (1 : x)$ and let r be the smallest integer k such that* $\dim(S_k\overline{X}) \geq n-1$. *Then the projective closure of every irreducible component of $\partial_a C$ is an irreducible component of $\overline{X}$ or $S_{[k]}\overline{X}$ for some $k \in \{r, \ldots, n-1\}$. In particular,*

$$\overline{\partial_a C} \subset \overline{X} \cup \bigcup_{k=r}^{n-1} S_{[k]}\overline{X}.$$

Differently put, every irreducible component of $\partial_a C$ is an irreducible component of X or isomorphic to an irreducible component of $S_{[k]}\overline{X} \cap D_+(x_0)$ under the embedding $\mathbb{A}^n \to \mathbb{P}^n$, $x \mapsto (1 : x)$.

A lower bound for r in terms of the dimension of X is $r \geq \frac{n-1-\dim(X)}{\dim(X)+1}$, cf. Remark 2.3.2.

Proof. Let $Y \subset \partial_a C$ be an irreducible component. Then $Y \subset X$ or Y is a ruled hypersurface. Indeed, by definition of the algebraic boundary, the intersection of Y with the euclidean boundary of C is Zariski dense in Y. The euclidean boundary of C is covered by faces of C and the Zariski closure of a face is its affine span. Therefore, the set

$$\bigcup_{F \subset C \cap Y \text{ face of } C} \operatorname{affspan}(F)$$

is Zariski dense in Y. If the set of extreme points of C in $C \cap Y$ is Zarsiki-dense in Y, then $Y \subset X$ and therefore, it is an irreducible component of X. Otherwise, Proposition 2.3.6 implies that there is a $k \in \{1, \dots, n-1\}$ such that a general point on $\overline{Y}$ lies in a k-dimensional linear space contained in $\overline{Y}$ and not in a higher dimensional one. This in turn implies, that a general point $x \in Y \cap \partial C$ is contained in an inclusion maximal face $F \subset C \cap Y$ of C of dimension k, which is therefore exposed, cf. Remark 1.1.19. In order to prove, that $\overline{Y}$ is contained in $S_{[k]}\overline{X}$, take a linear polynomial $\ell \in \mathbb{R}[x_1, \dots, x_n]$ that defines the affine hyperplane strictly separating F from C, i.e. $\ell(y) = 0$ for all $y \in F$ and $\ell(y) > 0$ for all $y \in C \setminus F$. Since $X(\mathbb{R}) \subset C$, we conclude that this linear polynomial vanishes on the tangent space of all points $y \in F \cap X(\mathbb{R})$ (note that all of these points are regular points of X). Since the dimension of F is k and the set of extreme points of F is contained in $X(\mathbb{R})$, we find affinely independent points $y_0, \dots, y_k \in F \cap X$ such that ℓ vanishes on $\mathrm{T}_{y_i}X$ for $i = 0, \dots, k$. This proves $\overline{Y} \subset S_{[k]}\overline{X}$. Since the dimension of Y is $n-1$ and the dimension of $S_{[k]}\overline{X}$ is at most $n-1$ (cf. Proposition 2.3.4), we conclude $\dim S_{[k]}\overline{X} = n-1$; so $\overline{Y}$ is an irreducible component of $S_{[k]}\overline{X}$. □

Remark 2.3.8. For the algebraic boundary of C, we are only interested in irreducible components of $S_{[k]}X$ of codimension 1. In this case, looking at the proof of Proposition 2.3.4, we see that the tangent space at a general point $y \in S_{[k]}X$ lying on the secant $\langle y_0, \dots, y_k\rangle$ spanned by regular points of X is actually equal to the span of the tangent spaces to X at the points y_i, i.e.

$$\mathrm{T}_y(S_{[k]}X) = \langle \mathrm{T}_{y_0}X, \dots, \mathrm{T}_{y_k}X\rangle.$$

The assumption that the real points of X are regular cannot be dropped:

Example 2.3.9. Consider the rational curve $X = \{(s^4 : s^3t : s^2t^2 : t^4) \in \mathbb{P}^3 : (s:t) \in \mathbb{P}^1\}$ discussed in Example 2.3.3(b) above. The convex hull of its real points in the affine chart $w + y + z = 1$ is a compact convex set. Its algebraic boundary is the union of two quadratic cones whose projective closures are $\mathcal{V}(y^2 - wz)$ and $\mathcal{V}(x^2 - wy)$. But the latter is not an irreducible component of the special secant variety $S_{[1]}X$ because it is the cone over the curve from its singular point.

Example 2.3.10. We consider the convex hull C of two circles of radius 1 in 3-space, namely $X = \mathcal{V}(x, (y-\frac{1}{2})^2 + z^2 - 1) \cup \mathcal{V}(z, x^2 + y^2 - 1) \subset \mathbb{A}^3 = \{(x, y, z)\}$. Then C has no 2-dimensional faces, because there are no hyperplanes in $\mathbb{R}^3$ that are tangent to 3 different points of $X(\mathbb{R})$ and do not contain one of the circles. In other words, the second special secant variety contributes no irreducible components to the algebraic boundary of C. Therefore, the algebraic boundary of C actually is the first special secant variety (or edge surface), which turns out to be an irreducible surface of degree 8 with (dehomogenised) equation

$$\begin{aligned}
& 240x^8 + 608x^6y^2 + 240x^4y^4 - 384x^2y^6 - 256y^8 - 840x^6z^2 - 696x^4y^2z^2 + 192x^2y^4z^2 \\
- \;& 384y^6z^2 + 1215x^4z^4 - 696x^2y^2z^4 + 240y^4z^4 - 840x^2z^6 + 608y^2z^6 + 240z^8 + 896x^6y \\
+ \;& 2304x^4y^3 + 1920x^2y^5 + 512y^7 - 1152x^4yz^2 - 192x^2y^3z^2 - 768y^5z^2 + 1848x^2yz^4 - 2784y^3z^4 \\
- \;& 1504yz^6 - 832x^6 - 1312x^4y^2 + 160x^2y^4 + 640y^6 + 984x^4z^2 + 4144x^2y^2z^2 + 3520y^4z^2 \\
+ \;& 234x^2z^4 + 2504y^2z^4 - 232z^6 - 2176x^4y - 3584x^2y^3 - 1408y^5 - 2048x^2yz^2 - 576y^3z^2 \\
+ \;& 1640yz^4 + 800x^4 + 288x^2y^2 - 656y^4 + 424x^2z^2 - 2808y^2z^2 - 313z^4 + 1664x^2y + 1280y^3 \\
+ \;& 128yz^2 - 64x^2 + 416y^2 + 456z^2 - 384y - 144,
\end{aligned}$$

which can be computed by a computer algebra system. In Figure 2.4, you find a picture of this beautiful edge surface.

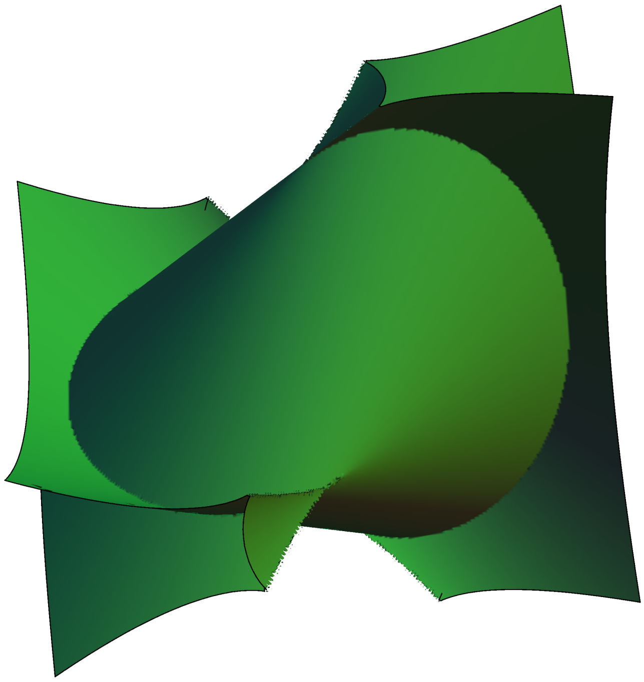

Figure 2.4.: The algebraic boundary of the convex hull of two circles in 3-space.

The question which irreducible components of codimension 1 of a given k-th special secant variety to X are contained in the algebraic boundary of $\mathrm{conv}(X(\mathbb{R}))$ is semi-algebraic in nature. In the case that X is a curve in an even-dimensional space, this semi-algebraic condition can be made very explicit for the secant variety of appropriate order. To be able to state the condition concisely, first observe the following.

Remark 2.3.11. Given an affine variety $X \subset \mathbb{A}^n$, consider the subset M of the k-fold product of $X(\mathbb{R})$ consisting of all tuples $(x_1, \ldots, x_k)$ such that the affine span of the points $x_1, \ldots, x_k$ intersected with the convex hull C of $X(\mathbb{R})$ is a face of C. This is a semi-algebraic subset of $\prod_{j=1}^k X$ because the property of a subset $F \subset C$ being a face as well as being in the affine span of $x_1, \ldots, x_k$ can be stated in terms of a first order formula in the language of ordered rings.

Theorem 2.3.12. *Let $X \subset \mathbb{A}^{2r}$ be an irreducible curve and assume that the real points $X(\mathbb{R})$ of X are Zariski-dense in X. Let C be the convex hull of $X(\mathbb{R}) \subset \mathbb{R}^{2r}$ and suppose that the interior of C is non-empty. Let $M \subset X \times X \times \ldots \times X$ be the semi-algebraic subset of the r-fold product of X defined as the set of all r-tuples of real points whose affine span intersects C in a face of C. Then the $(r-1)$-th secant variety to $\overline{X}$ is an irreducible component of the projective closure of the algebraic boundary of C if and only if the dimension of M is r.*

Proof. The $(r-1)$-th secant variety $S_{r-1}\overline{X}$ to the projective closure of X is a hypersurface (cf. Lange [28]), because it follows from the assumption that C has non-empty interior that the

curve is not contained in any hyperplane. In particular, the morphism $f\colon \Delta \times X \times \ldots \times X \to \mathbb{A}^n$, $(\lambda, x_1, \ldots, x_r) \mapsto \lambda_1 x_1 + \ldots + \lambda_r x_r$, where $\Delta \subset \mathbb{A}^r$ is the affine hyperplane defined by $x_1 + \ldots + x_r = 1$, is generically finite, because its image is dense in $S_{r-1}\overline{X}$. Note that $S_{r-1}\overline{X}$ is irreducible as the secant variety to an irreducible curve, cf. Proposition 2.3.2. We write $S_{r-1}X$ for the affine variety isomorphic to $S_{r-1}\overline{X} \cap \mathrm{D}_+(x_0)$ via the embedding $\mathbb{A}^{2r} \to \mathbb{P}^{2r}$, $x \mapsto (1 : x)$. The irreducible hypersurface $S_{r-1}X$ is contained in the algebraic boundary of C if and only if the intersection $S_{r-1}X \cap \partial C$ has dimension $2r-1$ as a semi-algebraic set.
Set $M_0 := M \smallsetminus V(\mathbb{R})$ where $V \subset X \times \ldots \times X$ is the subvariety of all r-tuples of points on X which are affinely dependent. If it is non-empty, it is a semi-algebraic set of dimension $\dim(M)$. Consider the map

$$\Phi\colon \begin{cases} \Delta_{r-1} \times M_0 & \to \quad \mathbb{R}^{2r} \\ ((\lambda_1, \ldots, \lambda_r), (x_1, \ldots, x_r)) & \mapsto \quad \sum_{i=1}^r \lambda_i x_i \end{cases}$$

This is a semi-algebraic map and the image under Φ of $\Delta_{r-1} \times M_0$ is contained in the intersection $S_{r-1}X \cap \partial C$ by definition of M_0. We claim that $\dim(\Phi(\Delta_{r-1} \times M_0)) = 2r-1$ if and only if $\dim(S_{r-1}X \cap \partial C) = 2r-1$:
Let $U \subset \bigcup\{\operatorname{affspan}(x_1, \ldots, x_r)\colon x_i \in X_{\mathrm{reg}}, \dim(\operatorname{affspan}(x_1, \ldots, x_r)) = r-1\}$ be a Zariski-open subset of $S_{r-1}X$, which exists by Chevalley's Theorem (see Hartshorne [23], Exercise II.3.19). Then $\dim(S_{r-1}X \cap \partial C) = 2r-1$ if and only if $\dim(U \cap \partial C) = 2r-1$. Since $U \cap \partial C \subset \operatorname{im}(\Phi)$, we conclude that $\dim((S_{r-1}X) \cap \partial C) = 2r-1$ implies $\dim(\operatorname{im}(\Phi)) = 2r-1$. The converse of the claimed equivalence is trivial, because $\Phi(\Delta_{r-1} \times M_0) \subset S_{r-1}(X) \cap \partial C$.
From the claim, it follows that if $S_{r-1}X \subset \partial_a C$, the dimension of M_0 is r by a count of dimensions in the source of Φ and Bochnak-Coste-Roy [10], Theorem 2.8.8.
Conversely, assume that the dimension of M_0 is r. Denote by $\mathrm{Gr}(\Phi)$ the graph of the map Φ in $(\Delta_{r-1} \times M_0) \times \mathbb{R}^{2r}$ and by π_2 the projection of this product to the second factor $\mathbb{R}^{2r}$. The fibre of a generic real point in $S_{r-1}X$ under this projection is finite, because a general point on this secant variety lies on only finitely many secant $(r-1)$-planes to X. This implies that the image of Φ, which is the same as $\pi_2(\mathrm{Gr}(\Phi))$, is locally homeomorphic to the graph of Φ. This can be seen by a cylindrical decomposition of the semi-algebraic set $\mathrm{Gr}(\Phi)$ adapted to the projection π_2 (cf. Basu-Pollack-Roy [3], Chapter 5.1): Over every open cell of the decomposition of $S_{r-1}X(\mathbb{R})$ into semi-algebraic sets, there are only graphs and no bands, so the projection π_2 is a local homeomorphism of $\mathrm{Gr}(\Phi)$ with the image of Φ. Since the graph of Φ is in turn homeomorphic to the source of Φ, it follows that the dimension of $\Phi(\Delta_{r-1} \times M_0) \subset S_{r-1}X \cap \partial C$ is $r + r - 1 = 2r - 1$. □

Remark 2.3.13. In the general case, i.e. for special secant varieties or higher dimensional varieties $X \subset \mathbb{P}^n$, the situation becomes more difficult mainly because the morphism

$$\begin{cases} Y & \to \quad S_{[k]}X \\ (x_0, \ldots, x_k) & \mapsto \quad \sum_{i=0}^k x_i \end{cases},$$

where $Y \subset \widehat{X} \times \ldots \times \widehat{X}$ is the subvariety of all tuples $(x_0, \ldots, x_k)$ such that the tangent spaces $\mathrm{T}_{x_i}X$ are contained in a hyperplane H, is in general not generically finite.
Given an irreducible component Z of $S_{[k]}X$ of codimension 1, suppose that the semialgebraic

set $M \cap f^{-1}(Z)$ is full-dimensional in an irreducible component of the fibre, then the same argument as in the above proof implies that Z is an irreducible component of the algebraic boundary of C. Note that $M \cap f^{-1}(Z)$ might well be lower-dimensional even if the image $f(M) \cap Z$ is dense in Z.

We will later use this more explicit corollary to the above theorem in applications to SO(2)-orbitopes, cf. Section 3.1.

Corollary 2.3.14. *Let $X \subset \mathbb{A}^{2r}$ be an irreducible curve and assume that the real points of X are Zariski-dense in X. Set $C := \operatorname{conv}(X(\mathbb{R})) \subset \mathbb{R}^{2r}$ and suppose that C has non-empty interior. Then the $(r-1)$-th secant variety to $\overline{X}$ is an irreducible component of the projective closure of the algebraic boundary of C if and only if there are r real points $x_1, \ldots, x_r \in X(\mathbb{R})$ of X and semi-algebraic neighbourhoods $U_j \subset X(\mathbb{R})$ of x_j for $j = 1, \ldots, r$ such that for all $(y_1, \ldots, y_r) \in U_1 \times \ldots \times U_r$, the set* $\operatorname{affspan}(y_1, \ldots, y_r) \cap C$ *is a face of C.*

Proof. That the $(r-1)$-th secant variety to X is an irreducible component of $\partial_a C$ means that M as in the above notation has dimension r. The euclidean topology of $X(\mathbb{R}) \times \ldots \times X(\mathbb{R})$ is the product topology. So M contains a set of the form $U_1 \times \ldots \times U_r$ for open semi-algebraic sets $U_j \subset X$ if and only if it has dimension r. □

For the study of the question whether or not the convex hull of a variety is a basic closed semi-algebraic set, the following observation about central points on secant varieties is useful:

Proposition 2.3.15. *Let $X \subset \mathbb{P}^n$ be an irreducible variety. Take $x_0, \ldots, x_k \in X_{\text{reg}}(\mathbb{R})$ to be regular real points of X that are projectively independent. Then every real point $y \in \langle x_0, \ldots, x_k \rangle$ is a central point of the k-th secant variety:*

$$\dim_y(S_k X(\mathbb{R})) = \dim(S_k X)$$

In particular, if $X(\mathbb{R})$ is Zariski-dense in X, the union of all k-dimensional real projective spaces spanned by $k+1$ real points of X is a Zariski-dense subset of $S_k(X)$.

Proof. The morphism $f\colon H \times X_{\text{reg}} \times \ldots \times X_{\text{reg}} \to S_k X$, $((\lambda_0, \ldots, \lambda_k), z_0, \ldots, z_k) \mapsto \lambda_0 z_0 + \ldots + \lambda_k z_k$, where $H = \{(\lambda_0, \ldots, \lambda_k) \in \mathbb{A}^{k+1} : \lambda_0 + \ldots + \lambda_k = 1\}$, is defined over the reals and the points $((\lambda_0, \ldots, \lambda_k), x_0, \ldots, x_k) \in \mathbb{A}^{k+1} \times X \times \ldots \times X$ are regular for all $\lambda \in \mathbb{A}^{k+1}$ by the assumption $x_i \in X_{\text{reg}}$. Any $y \in \langle x_0, \ldots, x_k \rangle$ is $f(\lambda, x_0, \ldots, x_k)$ for a unique $\lambda \in \mathbb{A}^{k+1}$. As the image of a central point under a real morphism it is therefore central in $\operatorname{im}(f)$ and in particular central in $S_k X$, cf. Proposition 1.2.9. □

2.3.2. Dual Description of the Algebraic Boundary

In this part, we will use the algebraic duality theory (cf. section 1.2.2) to give a description of an algebraic variety containing the algebraic boundary of C, again in terms of tangency to X. The basic idea behind the tangency conditions are the same as in the synthetic description of the preceding section.

This time, we follow the paper [32] by Ranestad and Sturmfels, where much the same result is proved, cf. [32], Theorem 1.1. Our result will be slightly more general: We will have to assume regularity only for the real points of X and can do without the technical tangency assumption stating that only finitely many hyperplanes are tangent to X at infinitely many points.

We consider the following nested chain of subvarieties of the dual variety. Note, that the indices are shifted by 1 compared to Ranestad-Sturmfels [32].

Definition 2.3.16. Let $X \subset \mathbb{P}^n$ be a projective variety. For $k \in \mathbb{N}$, denote by $X^{[k]}$ the Zariski closure in $(\mathbb{P}^n)^*$ of the set of all hyperplanes in $\mathbb{P}^n$ which are tangent to X at $k+1$ regular, projectively independent points. Write $X^{[0]} = X^*$.

For small values of k, these subsets of the dual variety are closely related to secant varieties to X by Terracini's Lemma.

Proposition 2.3.17. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety and and let $k \in \mathbb{N}$ be an integer such that $S_k X \neq \mathbb{P}^n$. Then $(S_k X)^* = X^{[k]}$. In particular, $X^{[k]}$ is irreducible.*

Proof. Terracini's Lemma states that the embedded tangent space to a general point y of the secant variety $S_k X$ on the general secant $\langle x_0, \dots, x_k \rangle$ is the linear space spanned by the embedded tangent spaces to X at the points $x_0, \dots, x_k$, cf. Flenner-O'Carroll-Vogel [17], Proposition 4.3.2. That means, that a hyperplane in $\mathbb{P}^n$ is tangent to a general point $y \in S_k X$ if and only if it contains the tangent spaces $\mathrm{T}_{x_i,X}$ to $k+1$ points $x_0, \dots, x_k$ on X. The irreducibility follows from the irreducibility of $S_k X$, cf. Remark 1.2.12(a). □

Remark 2.3.18. (a) As we will see in the applications to SO(2)-orbtiopes in section 3.1, the varieties $X^{[k]}$ need not be irreducible. We will give an example of an irreducible smooth curve $X \subset \mathbb{P}^4$ such that $X^{[2]}$ is not irreducible, see Remark 3.1.18. More precisely, we will see that the dual variety to $X^{[2]}$ has two irreducible components.
(b) We compute $X^{[1]}$ for the curve $X = \{(s^4 : s^3 t : s^2 t^2 : t^4) : (s : t) \in \mathbb{P}^2\}$ in 3-space considered in Example 2.3.3(b): There, we computed that $S_{[1]} X = \mathcal{V}_+(y^2 - wz) \subset \mathbb{P}^3 = \{(w : x : y : z)\}$. Using `Macaulay2` [20], we can also compute $X^{[1]}$ (code for this computation can be found B.1). Actually, $X^{[1]} = \mathcal{V}_+(y^2 - wz)^* = \mathcal{V}_+(X, Y^2 - 4WZ) \subset (\mathbb{P}^3)^* = \{(W : X : Y : Z)\}$.

Proposition 2.3.19. *[cf. Ranestad-Sturmfels [32], Lemma 3.1] The dimension of $X^{[k]}$ is at most $n - 1 - k$.*

Proof. Consider the projection $\pi \colon \mathrm{CN(X)} \to X^*$ of the conormal variety onto the dual variety. Fix an irreducible component $Z \subset X^{[k]}$. For every irreducible component $Y \subset \pi^{-1}(Z)$ such that $\pi(Y) = Z$, the restriction $\pi|_Y \colon Y \to Z$ of the projection is generically smooth, in particular $\pi|_Y(\mathrm{T}_{(x,[H])} Y) = \mathrm{T}_{[H]} Z$ for a generic point $(x, [H]) \in Y$, cf. Hartshorne [23], Lemma III.10.5. Let $[H] \in Z$ be a generic point. By definition of $X^{[k]}$, there are $k+1$ projectively independent points $x_0, \dots, x_k \in X_{\mathrm{reg}}$ such that $T_{x_j} X \subset H$. In other words, $(x_j, [H]) \in \pi^{-1}(Z)$ for all $j = 0, \dots, k$. For every $j \in \{0, \dots, k\}$ there is an irreducible component Y_j of $\pi^{-1}(Z)$ such that $\pi(Y_j) = Z$ and $(x_j, [H]) \in Y_j$ because $[H]$ is a general point of Z. As argued above, generic smoothness implies $T_{[H]} Z = \pi(T_{(x_j,[H])} Y_j)$. Since x_j is a regular point of X, the point $(x_j, [H])$ is regular on CN(X), cf. Remark 1.2.12(a). By Corollary 1.2.19, we know

$$\mathrm{T}_{[H]} Z \subset \bigcap_{j=0}^{k} \pi(\mathrm{T}_{(x_j,[H])} \mathrm{CN(X)}) \subset \bigcap_{j=0}^{k} x_j^\perp = (\langle x_0, \dots, x_k \rangle)^\perp.$$

Since $x_0, \dots, x_k$ are projectively independent, we conclude that $\mathrm{T}_{[H]}Z$ has at most dimension $n-k-1$. □

Remark 2.3.20. (a) The above proof shows that the dual variety to an irreducible component of $X^{[k]}$ of dimension $n-1-k$ is an irreducible subvariety of $\mathrm{S}_{[\mathrm{k}]}\mathrm{X}$, because in this case, we get equality in the above inclusion of tangent spaces, i.e. for a generic point $[H]$ on this irreducible component, we have

$$(\mathrm{T}_{[H]}X^{[k]})^{\perp} = (\bigcap \mathrm{d}\,\pi(\mathrm{T}_{(x_i,[H])}\mathrm{CN}(\mathrm{X})))^{\perp} = \langle x_0, \dots, x_k \rangle.$$

(b) Note that the dual variety to an irreducible component $Z \subset X^{[k]}$ of maximal dimension $n-1-k$ need not be a hypersurface: Consider the quadratic Veronese embedding $\nu_2 \colon \mathbb{P}^3 \to \mathbb{P}^9$,

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0, x_1, x_2, x_3)^t (x_0, x_1, x_2, x_3) \in \mathrm{Sym}_4(\mathbb{R}),$$

and let X be its image. The dual variety to X is defined by the determinant of the generic symmetric 4×4-matrix. Its singular locus is the set of all symmetric matrices of rank at most 2 which has dimension 6. In fact $(X^*)_{\mathrm{sing}} = X^{[1]} = X^{[2]}$: Let A be a symmetric matrix of rank 2. It is tangent to X at a point $\nu_2(z)$ if and only if z is in the kernel of A because the tangent space to X at $\nu_2(z)$ is $\{zy^t + yz^t \colon y \in \mathbb{P}^3\}$, cf. Example 1.2.14(b). Therefore, it is tangent to X along a curve, namely $\nu_2(\ker(A))$, which spans a projective 2-space. So $X^{[2]}$ is irreducible of maximal dimension $n-1-k = 6$. The dual variety to $X^{[k]}$ is again the variety of rank 2 matrices in the primal space. In particular, it has dimension 6.

Remark 2.3.21. Note that for $k = n-1$, Proposition 2.3.19 states $\dim(X^{[n-1]}) \le 0$, i.e. there are at most finitely many hyperplanes that are tangent to $X \subset \mathbb{P}^n$ at n projectively independent regular points.

We now come to the main result of this section, which is a generalisation of a result due to Ranestad and Sturmfels.

Theorem 2.3.22 (cf. Ranestad-Sturmfels [32], Theorem 1.1)**.** *Let $X \subset \mathbb{A}^n$ be a variety such that all real points of X are regular and $X(\mathbb{R}) \subset \mathbb{R}^n$ is compact with respect to the euclidean topology. Let $C = \mathrm{conv}(X(\mathbb{R}))$ be the convex hull of the real points of X and suppose it has non-empty interior. Denote by r the smallest value of k such that $S_k\overline{X}$ has codimension at most 1. The projective closure of the algebraic boundary of C is contained in the union*

$$\overline{\partial_a C} \subset \overline{X} \cup \bigcup_{k=r}^{n-1} (\overline{X}^{[k]})^*.$$

More precisely, every irreducible component of $\overline{\partial_a C}$ is an irreducible component of $\overline{X}$ or $(\overline{X}^{[k]})^*$ for some k.

Proof. First observe that supporting hyperplane of a face F of C of dimension k lies in $\overline{X}^{[k]}$: Such a face is Zariski dense in the span of $(k+1)$ affinely independent points of X, because the set of extreme points of C is contained in $X(\mathbb{R})$. The supporting hyperplane contains the

tangent space to X at these regular real points because X is contained in one half space defined by this hyperplane.
Consider the $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ via the embedding $x \mapsto (1:x)$. Let $Y \subset \mathbb{P}^n$ be an irreducible hypersurface that intersects the euclidean boundary ∂C of C in a semi-algebraic set of dimension $n-1$, i.e. a dense subset of Y. Then $Y \cap \partial C$ is covered by faces of C and therefore, by Proposition 2.3.6, there is a maximal $k \in \mathbb{N}$ such that the union of k-dimensional faces of C contained in Y is dense in Y. Since Y locally lies in one half space defined by a tangent hyperplane, say H, at a general point $x \in Y \cap \partial C$, so does ∂C. By the convexity of C, this is true for all of C, i.e. C lies in one halfspace defined by H. Therefore, H is the supporting hyperplane of a k-dimensional face. Thus, by our observation at the beginning of the proof, $[H]$ is a point of $\overline{X}^{[k]}$. By Remark 1.2.12(b), we conclude $Y^* \subset \overline{X}^{[k]}$. Since Y is ruled by k-planes, the dual variety to Y has dimension $n-k-1$ (cf. Tevelev [42], Theorem 1.18). Since this is an upper bound for the dimension of $\overline{X}^{[k]}$ by Lemma 2.3.19, we conclude that Y is an irreducible component of $\overline{X}^{[k]}$. □

Remark 2.3.23. We have seen in the proof of the above theorem, that only those irreducible components of $\overline{X}^{[k]}$ with dimension $n-k-1$ can contribute to the algebraic boundary of C, i.e. if $Z \subset \overline{X}^{[k]}$ is an irreducible component such that $Z^* \subset \overline{\partial_a C}$, then $\dim(Z) = n-k-1$. In this case, the union of all k-dimensional faces of C contained in $Z^* \cap \partial C$ is Zariski dense in Z^*. In particular, Z^* is an irreducible component of $S_{[k]}\overline{X}$ and Z is an irreducible component of $(S_{[k]}\overline{X})^*$ with a regular real point.

The assumption that the real points of X be regular cannot be dropped as the following example shows.

Example 2.3.24. (a) In Example 2.3.3 we consider the rational curve $X = \{(s^4 : s^3t : s^2t^2 : t^4) \in \mathbb{P}^3 \colon (s:t) \in \mathbb{P}^1\}$ that is singular at $(0:0:0:1)$. The algebraic boundary of the convex hull is the union of two quadratic cones. Again, the dual description does not pick up the cone over the curve from the singular point, i.e. $\mathcal{V}(x^2 - wy)$, see Remark 2.3.18.
(b) Let $f = (x^2+y^2-1)^2 - z^3(1-z)$ and consider $X = \mathcal{V}(f) \subset \mathbb{A}^3 = \{(x,y,z)\}$. The z-coordinate of all real points of X is between 0 and 1. For $z=0$ and $z=1$, the real points form a circle of radius 1 in the (x,y)-plane around $(0,0,0)$ and $(0,0,1)$ respectively. For values of z in $(0,1)$, the real points consist of two circles around $(0,0,z)$, one with radius $1+\sqrt{z^3(1-z)}$ and the other with radius $1-\sqrt{z^3(1-z)}$. The extreme points of the convex hull of $X(\mathbb{R})$ are therefore all the points (x,y,z) with $z \in [0,1]$ and $x^2+y^2 = 1+\sqrt{z^3(1-z)}$ and there are two 2-dimensional faces, namely in the hyperplanes $\{z=1\}$ and $\{z=0\}$. So the algebraic boundary of $\operatorname{conv}(X(\mathbb{R}))$ is $\mathcal{V}(f) \cup \mathcal{V}(1-z) \cup \mathcal{V}(z)$. But the circle in the hyperplane $\{z=0\}$ consists only of singuular points of $\mathcal{V}(f)$. Therefore the irreducible component $\mathcal{V}(z)$ of the algebraic boundary of the convex hull of X is not an irreducible component of $(\overline{X}^{[k]})^*$ for any $k \in \mathbb{N}$.

2.4. The General Case

We consider compact semi-algebraic sets with nonempty interior with the goal to give a semi-algebraic description of the irreducible components of their algebraic boundary in terms of the duality theories in convex and algebraic geometry. We no longer assume that the set of extreme points is real algebraic. Since the definition of an extreme point is expressible as a first order formula in the language of ordered rings, the set of extreme points of a convex semi-algebraic set is also semi-algebraic. Its Zariski closure is consequently an algebraic variety of the same dimension. We will see that the geometry of the set of extreme points in their Zariski closure governs the algebraic boundary of the polar.

We want to start this section with an easy example in the plane, which showcases the following results.

Example 2.4.1. Let $C \subset \mathbb{R}^2$ be the convex set defined by the inequalities $x^2 + y^2 - 1 \geq 0$ and $x \leq 3/5$, see Figure 2.5. The dual convex body is the convex hull of the set $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 1 \geq 0, x \geq -3/5\}$ and the point $(-5/3, 0)$ (it cannot be defined by simultaneous polynomial inequalities, i.e. it is not a basic closed semi-algebraic set). Its algebraic boundary has three components, namely the circle and the two lines $y = 3/4x + 5/4$ and $y = -3/4x - 5/4$. The set of extreme points of C is $\{(x,y) : x^2 + y^2 - 1 = 0, x \leq 3/5\}$.

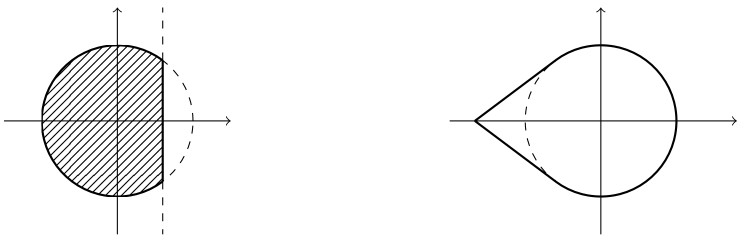

Figure 2.5.: A circle cut by a halfspace and its dual convex body.

It is technically more convenient to talk about closed pointed semi-algebraic cones, which is for our purposes equivalent to compact closed semi-algebraic sets by homogenisation in convex geometry, cf. Construction 1.1.11.

Proposition 2.4.2. *Let $C \subset \mathbb{R}^{n+1}$ be a pointed closed semi-algebraic cone with non-empty interior. Then the dual variety to the algebraic boundary of C is contained in the $\mathbb{R}$-Zariski closure of the extreme rays of the dual cone, i.e.*

$$(\mathbb{P}\partial_a C)^* \subset \mathbb{P}\mathrm{Exr}_a(C^\vee)$$

Proof. Let $Y \subset \mathbb{P}\partial_a C$ be an irreducible component of the algebraic boundary of C. Let $x \in \widehat{Y} \cap \partial C$ be a general point. Let $H \subset \mathbb{R}^{n+1}$ be a supporting hyperplane to C at x. Since C lies in one half-space defined by H, so does $\widehat{Y}$ locally around x. Therefore, H is the tangent hyperplane

$T_x\widehat{Y}$ and the intersection with H and $\widehat{Y}$ is the span of the face exposed by H, i.e. $H \cap C$. Now the tangent hyperplane to $\widehat{Y}$ at x is unique, because $\widehat{Y}$ has codimension 1. Since every supporting hyperplane to C at x is tangent to $\widehat{Y}$, this means that the dual face to $H \cap C$ is an extreme ray of the dual cone. Since $\widehat{Y} \cap C$ is Zariski dense in $\widehat{Y}$, the hyperplanes tangent to $\widehat{Y}$ at points $x \in \widehat{Y} \cap C$ are dense in the dual variety to Y. □

Following this Proposition, it is natural to ask if the dual variety to the algebraic boundary of C is equal to the $\mathbb{R}$-Zariski closure of the extreme rays of the dual cone. In the following, we will elaborate on this question.

Remark 2.4.3. Let $Z \subset \mathrm{Exr}_a(C)$ be an irreducible component. Then the dual variety to $\mathbb{P}Z \subset \mathbb{P}^n$ is a hypersurface in $(\mathbb{P}^n)^*$, which follows from Remark 1.2.15 because $\mathbb{P}Z$ cannot contain a dense subset of projective linear spaces of dimension ≥ 1.

Theorem 2.4.4. *Let $C \subset \mathbb{R}^{n+1}$ be a pointed closed semi-algebraic cone with non-empty interior. The dual variety to the locus of extreme rays of C is contained in the algebraic boundary of the dual cone $C^\vee$, i.e.*

$$(\mathbb{P}\mathrm{Exr}_a(C))^* \subset \mathbb{P}\partial_a C^\vee.$$

More precisely, the dual variety to every irreducible component of $\mathbb{P}\mathrm{Exr}_a(C)$ is an irreducible component of $\mathbb{P}\partial_a C$.

Proof. Let $Z \subset \mathbb{P}\mathrm{Exr}_a(C)$ be an irreducible component of the locus of extreme rays of C. By Corollary 2.1.17, a general extreme ray $[F_0] \in Z \cap (\mathbb{P}\mathrm{Exr}(C))$ is exposed by $\ell_0 \in \partial C^\vee$ and there is a semi-algebraic neighbourhood U of ℓ_0 in $\partial C^\vee$ such that every $\ell \in U$ exposes an extreme ray F_ℓ of C such that $[F_\ell] \in Z_{\mathrm{reg}}$. The hyperplane $\mathbb{P}\ker(\ell)$ is tangent to Z at $[F_\ell]$ because Z is locally contained in C; so $\mathbb{P}U$ is a semi-algebraic subset of Z^* of full dimension and the claim follows. □

In the Introduction, we gave an affine version of the preceding theorem that follows from it via homogenisation.

Theorem 2.4.5. *Let $C \subset \mathbb{R}^n$ be a compact convex semi-algebraic set with $0 \in \mathrm{int}(C)$. Let Z be an irreducible component of the Zariski closure of the set of extreme points of its dual convex body. Then the variety dual to Z is an irreducible component of the algebraic boundary of C. More precisely, the dual variety to the projective closure $\overline{Z}$ of Z with respect to the embedding $\mathbb{A}^n \to (\mathbb{P}^n)^*$, $x \mapsto (1 : x)$ is an irreducible component of the projective closure of $\partial_a C$ with respect to $\mathbb{A}^n \to \mathbb{P}^n$, $x \mapsto (1 : x)$.*

Proof. We homogenise the convex body and its dual convex body as described in Constructions 1.1.11 to get cones $\widehat{C}$ and $\widehat{C^o} = (\widehat{C})^\vee$. The projective closure $\overline{Z}$ of the irreducible component $Z \subset \mathrm{Ex}_a(C^o)$ with respect to the embedding $\mathbb{A}^n \to (\mathbb{P}^n)^*$, $x \mapsto (1 : x)$ is an irreducible component of $\mathbb{P}\mathrm{Exr}_a(\widehat{C}^\vee)$. By the above Theorem 2.4.4, the dual variety to $\overline{Z}$ is an irreducible component of $\mathbb{P}\partial_a\widehat{C}$, which is the projective closure of an irreducible component of the algebraic boundary of C with respect to the embedding $\mathbb{A}^n \to \mathbb{P}^n$, $x \mapsto (1 : x)$. □

Corollary 2.4.6 (to Theorem 2.4.4). *Let $C \subset \mathbb{R}^{n+1}$ be a pointed, closed semi-algebraic cone with non-empty interior. We have $(\mathbb{P}\partial_a C)^* = \mathbb{P}\mathrm{Exr}_a(C^\vee)$.* □

Remark 2.4.7. It does not follow from the biduality theorems in both theories that $(\mathbb{P}\mathrm{Exr}_a(C^\vee))^* = \mathbb{P}\partial_a C$ simply because the biduality theorem in the algebraic context does not in general apply to this situation, since the varieties in question tend to be reducible. In fact, the mentioned equality does not hold in general, as we have seen in Example 2.4.1. There, $\mathrm{Ex}_a(C) = \mathcal{V}(x^2 + y^2 - 1) = \mathrm{Ex}_a(C)^* \subsetneq \partial_a C^o$.

The following statement gives a complete semi-algebraic characterisation of the irreducible subvarieties $Y \subset \mathrm{Exr}_a(C)$ with the property that Y^* is an irreducible component of the algebraic boundary of $C^\vee$.

Theorem 2.4.8. *Let $C \subset \mathbb{R}^{n+1}$ be a pointed closed semi-algebraic cone with non-empty interior. Let Z be an irreducible algebraic cone contained in $\mathrm{Exr}_a(C)$ and suppose $Z \cap \mathrm{Exr}(C)$ is Zariski dense in Z. Then the dual variety to $\mathbb{P}Z$ is an irreducible component of $\mathbb{P}\partial_a C^\vee$ if and only if the dimension of the normal cone to a general point $x \in Z \cap \mathrm{Exr}(C)$ is equal to the codimension of Z, i.e.*

$$\dim(Z) + \dim(N_C(\widehat{\{x\}})) = n + 1.$$

Conversely, if Y is an irreducible component of the algebraic boundary of $C^\vee$, then the dual variety to $\mathbb{P}Y$ is an irreducible subvariety of $\mathbb{P}\mathrm{Exr}_a(C)$, the set $(\mathbb{P}Y)^* \cap \mathrm{Exr}(C)$ is Zariski dense in $(\mathbb{P}Y)^*$ and the above condition on the normal cone is satisfied at a general extreme ray for the affine cone over $(\mathbb{P}Y)^*$.

Proof. Consider the semi-algebraic set $\Sigma \subset \partial C \times \partial C^\vee \subset \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^*$ defined as

$$\Sigma = \{(x, \ell) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^* : x \in Z_{\mathrm{reg}} \cap \mathrm{Exr}(C), \ell \in C^\vee, \ell(x) = 0\}$$

This is the set of all tuples (x, ℓ), where x spans an extreme ray of C and is a regular point of Z and ℓ is a supporting hyperplane to C at x, i.e. the fibre of the projection π_1 onto the first factor over a point x is the normal cone $N_C(\mathbb{R}_+ x)$. Since a supporting hyperplane to C at x is tangent to Z at x, this bihomogeneous semi-algebraic incidence correspondence is naturally contained in the conormal variety $\mathrm{CN}(\mathbb{P}\mathrm{Z}) \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$ of the projectivisation of Z. Now the image $\pi_2(\Sigma)$ is Zariski dense in $\mathbb{P}Z^*$ if and only if $\mathbb{P}Z^*$ is an irreducible component of the projectivisation of the algebraic boundary of $C^\vee$. Indeed, $\pi_2(\Sigma) \subset \mathbb{P}Z^* \cap \mathbb{P}\partial C^\vee$ and so if it is dense in $\mathbb{P}Z^*$, we immediately get that $\mathbb{P}Z^* \subset \mathbb{P}\partial_a C^\vee$ is an irreducible component, because $\mathbb{P}Z^*$ is a hypersurface (cf. Remark 2.4.3(b)). Conversely, we have seen in the proof of the above proposition that if $\mathbb{P}Z^* \subset \mathbb{P}\partial_a C^\vee$ is an irreducible compnent, the unique tangent hyperplane to a general point of $\mathbb{P}Z^* \cap \mathbb{P}\partial C^\vee$ spans an extreme ray of C, i.e. a general point of $\mathbb{P}Z^* \cap \mathbb{P}\partial C^\vee$ is contained in $\pi_2(\Sigma)$.

This says, that Σ is dense in $\mathrm{CN}(\mathbb{P}\mathrm{Z})$, i.e. $\dim(\Sigma) = \dim(\mathrm{CN}(\mathbb{P}\mathrm{Z})) + 2 = n + 1$ if and only if $\mathbb{P}Z^*$ is an irreducible component of $\mathbb{P}\partial_a C^\vee$.

On the other hand, counting dimensions of Σ as the sum of the dimensions of Z and the dimension of the fibre over a general in $Z_{\mathrm{reg}} \cap \mathrm{Exr}(C)$, we see that $\dim(\Sigma) = n + 1$ if and only if the claimed equality of dimensions

$$\dim(Z) + \dim(N_C(\{x\})) = n + 1$$

holds. The second part of the statement follows from the first by applying Proposition 2.4.2. □

In this case, the corresponding affine statement is the following. We take projective closures with respect to the same embeddings as in the affine version Theorem 2.4.5 of Theorem 2.4.4 above.

Theorem 2.4.9. *Let $C \subset \mathbb{R}^n$ be a compact convex semi-algebraic set with $0 \in \operatorname{int}(C)$. Let Z be an irreducible subvariety of $\overline{\operatorname{Ex}_a(C)}$ and suppose $Z \cap \operatorname{Ex}(C)$ is dense in Z. Then the dual variety to $\overline{Z}$ is an irreducible component of $\overline{\partial_a C^o}$ if and only if*

$$\dim(Z) + \dim(N_C(\{x\})) = n$$

for a general extreme point $x \in Z \cap \operatorname{Ex}(C)$. Conversely, if Y is an irreducible component of $\partial_a C^o$, then the dual variety to $\overline{Y}$ is an irreducible subvariety of $\overline{\operatorname{Ex}_a(C)}$, the set $\overline{Y}^* \cap \operatorname{Ex}(C)$ is dense in $\overline{Y}^*$ and the condition on the normal cone is satisfied at a general extreme point.

Proof. Again, the proof is simply by homogenising as above. Note that the dimension of the normal cone does not change when homogenising, cf. Construction 1.1.37. □

In the following affine examples we will drop the technical precision of taking projective closures and talk about the dual variety to an affine variety to make them more readable.

Example 2.4.10. Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\} \subset \mathbb{R}^n$ be a convex basic closed semi-algebraic set with non-empty interior defined by $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$. Then the algebraic boundary $\partial_a C$ is contained in the variety $\mathcal{V}(g_1) \cup \ldots \cup \mathcal{V}(g_r) = \mathcal{V}(p_1) \cup \ldots \mathcal{V}(p_s)$, where $p_1, \ldots, p_s$ are the irreducible factors of the polynomials $g_1, \ldots, g_r$. The irreducible hypersurface $\mathcal{V}(p_i)$ is an irreducible component of $\partial_a C$ if and only if $\mathcal{V}(p_i) \cap \partial C$ is a semi-algebraic set of codimension 1. By the above Theorem 2.4.9, we can equivalently check the following conditions on the dual varieties X_i to the projective closure $\overline{\mathcal{V}(p_i)}$:

- The extreme points of the polar are dense in X_i via $\mathbb{R}^n \to (\mathbb{P}^n)^*$, $x \mapsto (1 : x)$.
- A general extreme point of the polar in X_i exposes a face of dimension $\operatorname{codim}(X_i) - 1$.

We consider the convex set shown in Figure 2.6, whose algebraic boundary is the cubic curve $X = \mathcal{V}(y^2 - (x+1)(x-1)^2)$, with different descriptions as a basic closed semi-algebraic set. The dual convex body is the convex hull of a quartic curve. Its algebraic boundary is

$$\mathcal{V}(4x^4 + 32y^4 + 13x^2y^2 - 4x^3 + 18xy^2 - 27y^2) \cup \mathcal{V}(x+1).$$

The line $\mathcal{V}(x+1)$ is a bitangent to the quartic. We define C using the cubic inequality and additionally either one linear inequality or the two tangents to the branches of X in $(1, 0)$

$$\begin{aligned} C &= \{(x,y) \in \mathbb{R}^2 : y^2 - (x+1)(x-1)^2 \leq 0, x \leq 1\} \\ &= \{(x,y) \in \mathbb{R}^2 : y^2 - (x+1)(x-1)^2 \leq 0, y \geq \sqrt{2}(x-1), y \leq -\sqrt{2}(x-1)\}, \end{aligned}$$

and we see both conditions in action. First, the dual variety to the affine line $x = 1$ is $(-1, 0)$, which is not an extreme point of C^o. The first condition mentioned above shows, that the line $\mathcal{V}(x-1)$ corresponding to the second inequality in the first description is not an irreducible component of $\partial_a C$. In the second description, the dual variety to the affine line $y = \sqrt{2}(x-1)$

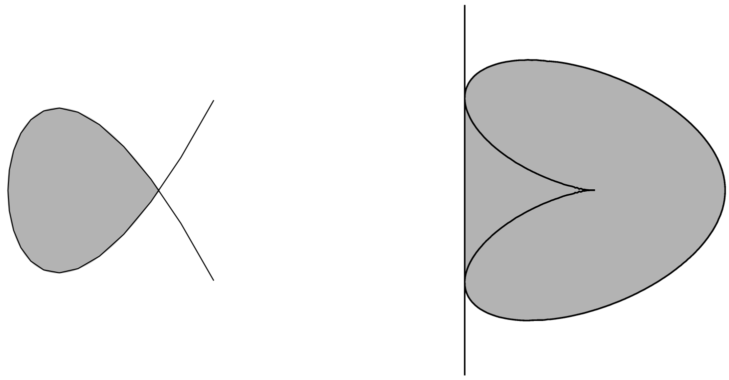

Figure 2.6.: A basic closed semi-algebraic set in the plane on the left and its polar on the right.

is the point $P = (-1, \frac{1}{\sqrt{2}})$, which is an extreme point of C^o. The normal cone $N_{C^o}(\{P\})$ is 1-dimensional, because the supporting hyperplane is uniquely determined - it is the bitangent to the quartic. So by the second condition above, the line $\mathcal{V}(y - \sqrt{2}(x-1))$ is not an irreducible component of $\partial_a C$.

Corollary 2.4.11. *[to Theorem 2.4.9] Let $C \subset \mathbb{R}^n$ be a compact semi-algebraic set with $0 \in \mathrm{int}(C)$. Let $Y \subset \partial_a C^\vee$ be an irreducible component such that $\overline{Y}^* \subset \overline{\mathrm{Ex}_a(C)}$ is not an irreducible component. If $\overline{Y}^*$ is contained in a bigger irreducible subvariety $Z \subset \overline{\mathrm{Ex}_a(C)}$ such that $Z \cap \mathrm{Ex}(C)$ is dense in Z, then*

- *$\overline{Y}^* \subset Z_{\mathrm{sing}}$ or*
- *$\overline{Y}^*$ is contained in the algebraic boundary of the semi-algebraic subset $Z \cap \mathrm{Ex}(C)$ of Z.*

Proof. Let $Z \subset \overline{\mathrm{Ex}_a(C)}$ be an irreducible subvariety. If $\ell \in (\mathbb{R}^n)^*$ defines a supporting hyperplane to an extreme point $x \in \mathrm{Ex}(C)$ that is an interior point of the semi-algebraic set $\mathrm{Ex}(C) \cap Z$ as a subset of Z and $(1 : x) \in Z_{\mathrm{reg}}$, then the variety Z lies locally in one of the half spaces defined by ℓ and therefore ℓ is tangent to Z at $(1 : x)$. In particular, the dimension of the normal cone $N_C(\{x\})$ is bounded by the local codimension of Z at $(1 : x)$. Now if $\overline{Y}^*$ is strictly contained in Z, it cannot contain $(1 : x)$ by Theorem 2.4.9 because $\dim(\overline{Y}^*) < \dim(Z)$. □

The set $Z \cap \mathrm{Ex}(C)$ in the above corollary does not need to be a regular semi-algebraic set. So the second condition can also occur in the following way.

Example 2.4.12. Consider the convex hull C of the half ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1, x \ge 0\}$ and the circle $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1, z = 0\}$. The Zariski closure of the extreme points of C is the sphere S^2. The dual variety of X is an irreducible component of the algebraic boundary of the polar. Every point of X is a regular point of the sphere and X is contained in the algebraic boundary of $\mathrm{Ex}(C) \cap S^2 \subset S^2$, because the semi-algebraic set $S^2 \cap \mathrm{Ex}(C)$ does not have local dimension 2 at the extreme points $(x, y, 0) \in X \cap \mathrm{Ex}(C)$ where $x < 0$. The algebraic boundary of the polar has three irreducible components, namely the sphere S^2 and the dual varieties to the two irreducible components X and $\mathcal{V}(y^2 + z^2 - 1, x)$ of $\partial_a(Z \cap \mathrm{Ex}(C)) \subset Z$.

The following examples show how the statement of the corollary can be used to determine the algebraic boundary in concrete cases.

Example 2.4.13. (a) Consider the spectrahedron $P = \{(x, y, z) \in \mathbb{R}^3 : Q(x, y, z) \geq 0\}$ where Q is the symmetric matrix

$$Q = \begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix},$$

studied by Rostalski and Sturmfels in [36], Section 1.1 and called pillow. The Zariski closure of the set of extreme points of P is defined by the equation $\det(Q) = 0$, where

$$\det(Q) = x^2(y-z)^2 - 2x^2 - y^2 - z^2 + 1.$$

The algebraic boundary of the dual convex body P^o is the hypersurface

$$\begin{aligned} \partial_a P^o = \quad & \mathcal{V}(b^2 + 2bc + c^2 - a^2b^2 - a^2c^2 - b^4 - 2b^2c^2 - 2bc^3 - c^4 - 2b^3c) \cup \\ & \mathcal{V}(2 - a^2 + 2ab - b^2 + 2bc - c^2 - 2ac) \cup \mathcal{V}(2 - a^2 - 2ab - b^2 + 2bc - c^2 + 2ac), \end{aligned}$$

computed in Rostalski-Sturmfels [36], Section 1.1, cf. Equations 1.7 and 1.8. The first quartic is the dual variety to the quartic $\mathcal{V}(\det(Q))$. The two quadric hypersurfaces are products of linear forms over $\mathbb{R}$ and they are the dual varieties to the four corners of the pillow, namely $\frac{1}{\sqrt{2}}(1, 1, -1)$, $\frac{1}{\sqrt{2}}(-1, -1, 1)$, $\frac{1}{\sqrt{2}}(1, -1, 1)$ and $\frac{1}{\sqrt{2}}(-1, 1, -1)$. These four points are extreme points of P and singular points of $\mathcal{V}(\det(Q))$.
(b) We want to return to Example 2.3.10 and take the dual point of view: Recall that we consider the convex hull C of the variety $X = \mathcal{V}(x, (y - \frac{1}{2})^2 + z^2 - 1) \cup \mathcal{V}(z, x^2 + y^2 - 1) \subset \mathbb{A}^3 = \{(x, y, z)\}$, cf. Figure 2.4. The dual variety to the variety X is the union of two cylinders, namely $X^* = \mathcal{V}(X^2 + Y^2 - 1) \cup \mathcal{V}(3Y^2 + 4Z^2 - 4Y - 4) \subset \mathbb{A}^3 = \{(X, Y, Z)\}$. The algebraic boundary of the polar C^o consists of these two cylinders and the extreme points of C^o are the real points of their intersection. This curve is actually dual to the edge surface. In Figure 2.7, you find a picture of the algebraic boundary of the polar C^o.

Another interesting consequence of Theorem 2.4.9 concerns the semi-algebraic set $\mathrm{Ex}(C)$.

Corollary 2.4.14. *Let $C \subset \mathbb{R}^n$ be a compact convex set with $0 \in \mathrm{int}(C)$. Every extreme point x of C is a central point of the dual variety of at least one irreducible component of $\overline{\partial_a C^o}$.*

Proof. By Straszewicz's Theorem 1.1.20, it suffices to prove, that the statement holds for exposed extreme points because every extreme point is the limit of an exposed one. So let x be an exposed extreme point of C and let $F_x = \{\ell \in C^\vee : \ell(x) = -1\}$ be the dual face. Remark 1.1.34 says that the normal cone $N_{C^\vee}(F_x) = \mathbb{R}_+ x$ is 1-dimensional. Fix a relative interior point $\ell \in F_x$. Let Y be an irreducible component of $\partial_a C^\vee$ on which ℓ is a central point and let $(\ell_j)_{j \in \mathbb{N}} \subset Y_{\mathrm{reg}}(\mathbb{R})$ be a sequence of regular real points converging to ℓ in the euclidean topology. There is a unique (up to scaling) linear functional minimising in ℓ_j over $C^\vee$, namely $y_j \in \partial C$ with $\ell_j(y_j) = -1$ and $\alpha_j(y_j) = -1$ for all $\alpha \in T_{\ell_j} Y$. Since (y_j) is a sequence in a compact set, there exists a converging

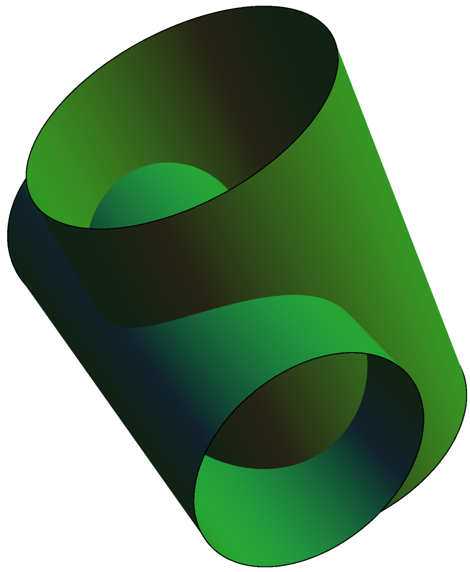

Figure 2.7.: The algebraic boundary of the dual convex body.

subsequence; without loss of generality, we assume that $(y_j)_{j\in\mathbb{N}}$ converges and we call the limit y. Note that y represents a central point of $\overline{Y}^*$. We know that $y \in \partial C$ and

$$\ell(y) = \lim_{j\to\infty} \ell_j(y) = \lim_{j\to\infty} \ell_j(\lim_{k\to\infty} y_k) = -1,$$

so y exposes the face F_x of $C^\vee$ and is therefore equal to x by $N_{C^\vee}(F_x) = \mathbb{R}_+ x$. □

We take a short look at implications of this corollary to hyperbolicity cones.

Example 2.4.15. A homogeneous polynomial $p \in \mathbb{R}[x_0, \ldots, x_n]$ of degree d is called hyperbolic with respect to $e \in \mathbb{R}^{n+1}$ if $p(e) \neq 0$ and the univariate polynomial $p(x + te) \in \mathbb{R}[t]$ has only real roots for every $x \in \mathbb{R}^{n+1}$. We consider the set

$$C_p(e) = \{x \in \mathbb{R}^{n+1} \colon \text{all roots of } p(te - x) \text{ are non-negative}\},$$

which is called the hyperbolicity cone of p (with respect to e). It turns out to be a convex cone, cf. [34]. Assume that all non-zero points in the boundary of $C_p(e)$ are regular points of $\mathcal{V}(p)$. Then by Corollary 2.4.11 the algebraic boundary of the dual convex cone is the dual variety to $\mathcal{V}(q)$ where q is the unique irreducible factor of p which vanishes on ∂C_p.

The assumption on the hyperbolicity cone being smooth is essential: Consider the hyperbolicity cone of $p = y^2 z - (x+z)(x-z)^2 \in \mathbb{R}[x,y,z]$ with respect to $(0,0,1)$. The cubic $\mathcal{V}(p) \subset \mathbb{R}^3$ is singular along the line $\mathbb{R}(1,0,1)$ and the algebraic boundary of the dual convex cone has an additional irreducible component, namely the hyperplane dual to this line because the normal cone has dimension 2 at this extreme ray, see Figure 2.6.

Let now $C_p(e)$ be any hyperbolicity cone and decompose $\partial_a C_p(e) = X_1 \cup \ldots \cup X_r$ into its irreducible components $X_1, \ldots, X_r$. The dual cone $C_p(e)^\vee$ is the conic hull of the regular real points of the dual varieties of the irreducible components X_i up to closure, i.e.

$$C_p(e)^\vee = \mathrm{cl}(\mathrm{co}((X_1^*)_{\mathrm{reg}}(\mathbb{R}) \cup \ldots \cup (X_r^*)_{\mathrm{reg}}(\mathbb{R}))).$$

Indeed, the right hand side contains every central point of every variety X_i^* and by Corollary 2.4.14, this gives one inclusion. Conversely, let ℓ be a general real point of X_i^* for any i. Then ℓ is tangent to X_i in a regular real point of $\partial_a C_p(e)$ and by hyperbolicity of p, the linear functional has constant sign on the hyperbolicity cone $C_p(e)$ because every line through the hyperbolicity cone intersects every regular real point of $\partial_a C_p(e)$ with multiplicity 1, cf. Plaumann-Vinzant [30], Lemma 2.4.

Example 2.4.16 (Copositive Matrices). Let $\mathrm{CoP}_n \subset \mathrm{Sym}_n(\mathbb{R})$ ($n \geq 2$) be the cone of copositive matrices, i.e. the set of all matrices A such that $x^t A x \geq 0$ for all vectors $x \in \mathbb{R}^n_+$ in the positive orthant. By definition, the dual cone is the cone of completely positive matrices, i.e. the conic hull of the set $\{xx^t : x \in \mathbb{R}^n_+\}$. The completely positive matrices are strictly contained in the cone of positive semi-definite matrices and the cone of copositive matrices is strictly larger than the cone of positive semi-definite matrices, because the latter is self-dual with respect to the standard inner product $(A, B) \mapsto \mathrm{tr}(\mathrm{A}^t\mathrm{B})$ on symmetric matrices. We want to characterise the algebraic boundary of CoP_n.

The Zariski closure of the set of extreme rays of the cone of completely positive matrices are the rank 1 symmetric matrices, i.e. the Veronese embedding $\nu_2(\mathbb{C}^n) = \{xx^t : x \in \mathbb{C}^n\}$, which is irreducible and smooth. So its dual variety is an irreducible component of the algebraic boundary of the cone of copositive matrices by Theorem 2.4.4, namely the determinant. But there are more: The relative boundary of the extreme rays inside the set of rank 1 matrices is the image of the coordinate hyperplanes under the Veronese map ν_2. These are n irreducible subvarieties and by a theorem of Dickinson [13], Theorem 5.5, the dimension of a face exposed by a general extreme point has codimension $n - 1$. Therefore, by Theorem 2.4.8, the images $\nu_2(H_i)$ of the coordinate hyperplanes $H_i = \{x \in \mathbb{C}^n : x_i = 0\}$ dualise to irreducible components of the algebraic boundary of CoP_n. In turn, the relative boundary of extreme rays of the form xx^t for $x \in H_i$ inside the variety $\nu_2(H_i)$ is the image of all coordinate subspaces of codimension 1 in H_i. Again by Dickinson's theorem, the dimension condition of Theorem 2.4.8 is satisfied. We conclude that the irreducible varieties $\nu_2(U_I)$ are dual to an irreducible component of the algebraic boundary of CoP_n for all coordinate subspaces $U_I \subset \mathbb{C}^n$ of dimension at least 1. There are $2^n - 1$ such subspaces indexed by subsets $I \subset \{1, \ldots, n\}$ of cardinality at most $n - 1$, where $U_I = \{x \in \mathbb{C}^n : \forall\ i \in I\ x_i = 0\}$.

The dual variety to $\nu_2(U_I)$ is the hypersurface defined by the determinant of the submatrix obtained after deleting the rows and columns with indices from the set I, cf. Example 1.2.14(b). So the degree of the algebraic boundary of CoP_n is

$$\deg(\partial_a \mathrm{CoP}_n) = \sum_{j=0}^{n-1} \binom{n}{j}(n - j) = \sum_{j=1}^{n} \binom{n}{j} j = \frac{\mathrm{d}}{\mathrm{dt}}(1 + t)^n|_{t=1} = n2^{n-1}$$

If the set of extreme points is a smooth real algebraic set, the algebraic boundary of the polar is the variety dual to the set of extreme points:

Example 2.4.17. Let $X \subset \mathbb{A}^n$ be an affine variety such that $X(\mathbb{R})$ is compact and every real point is a regular point of X. Let $C = \operatorname{conv}(X(\mathbb{R}))$ and assume $\operatorname{int}(C) \neq \emptyset$, cf. Section 2.3. Then the algebraic boundary of the polar C^o is the dual variety to X by Theorem 2.4.5 and Corollary 2.4.11.

In particular, this applies to orbitopes: Let G be a real algebraic group with $G(\mathbb{R})$ compact and $G \to \mathrm{GL}(V)$ be a rational representation on a finite dimensional real vector space V. Let $C \subset V$ be the convex hull of an orbit $G(\mathbb{R}).w$ in V. Then C is an orbitope by definition. The set of extreme points is exactly the orbit $G(\mathbb{R}).w$, cf. Sanyal-Sottile-Sturmfels [37], Proposition 2.2. The Zariski closure of $G(\mathbb{R}).w$ is the complex orbit, i.e. $\{g.w \colon g \in G_{\mathbb{C}}\} \subset V_{\mathbb{C}}$ by Birkes [5], Corollary 5.3. In particular, it is a smooth affine variety and the set of extreme points is a real algebraic set by Theorem A.1.

3. Applications

We now give applications of the descriptions of the algebraic boundary of Chapter 2 to two different classes of convex semi-algebraic sets. First, we consider the easiest non-discrete cases of orbitopes, which is already intricate; it is a special case of the convex hull of a smooth variety, the setup considered in Section 2.3. Second, we consider cones of sums of squares of ternary forms. Their sets of extreme rays are not real algebraic, so we apply results from Section 2.4.

3.1. SO(2)-Orbitopes

The group $\mathrm{SO}(2)$ is the compact real form of the torus $\mathbb{C}^*$. In this sense, it is the easiest non-discrete real algebraic group. It is 1-dimensional, commutative and reductive and all its irreducible non-trivial representations are 2-dimensional. They are parametrised by the integers. Given a linear action of $\mathrm{SO}(2)$ on V, we consider the convex hull of the orbit of a vector $v \in V$, which is called the $\mathrm{SO}(2)$-orbitope of v.

We are mainly interested in a suitable secant variety to the Zariski closure of the orbit of v, which is a hypersurface, and its containment in the algebraic boundary of the orbitope.

3.1.1. Setup and basic facts

Definition 3.1.1. A *representation* of $\mathrm{SO}(2)$ is a pair (ρ, V) of a finite-dimensional real vector space V and a homomorphism $\rho\colon \mathrm{SO}(2) \to \mathrm{GL(V)}$ of real algebraic groups. This means, after choosing a basis of V and thereby identifying $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{R})$, that ρ is a group homomorphism defined by polynomials with real coefficients.
The dimension of the vector space V is called the *dimension* of the representation (ρ, V). The representation (ρ, V) is called *irreducible* if V has no non-trivial invariant subspace.

3.1.2. We fix the following notation for representations of $\mathrm{SO}(2)$: For $j \in \mathbb{Z}$, $j \neq 0$, write

$$\rho_j : \left\{ \begin{array}{ccc} \mathrm{SO}(2) & \to & \mathrm{GL}_2(\mathbb{R}) \\ A = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} & \mapsto & A^j = \begin{pmatrix} \cos(j\vartheta) & -\sin(j\vartheta) \\ \sin(j\vartheta) & \cos(j\vartheta) \end{pmatrix} \end{array} \right.$$

Denote by ρ_0 the trivial representation of $\mathrm{SO}(2)$, i.e. the representation $(\rho, \mathbb{R})$, where ρ is the constant group homomorphism. The set $\{\rho_j \colon j \in \mathbb{Z}\}$ is the family of all irreducible representations of $\mathrm{SO}(2)$ (up to linear isomorphism commuting with the group action on V via ρ). Since $\mathrm{SO}(2)$ is a compact group, every finite-dimensional representation of $\mathrm{SO}(2)$ is the sum of some of these irreducible representations. In particular, any representation of $\mathrm{SO}(2)$ that does not contain the trivial representation is even-dimensional.

Remark 3.1.3. (a) If we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \mapsto x + iy$, then SO(2) gets identified with the unit circle $S^1 \subset \mathbb{C}$ in the complex plane by sending a rotation matrix as above to $\exp(i\vartheta)$. The represenation ρ_j becomes multiplication by the exponential, i.e. for all $z \in \mathbb{C}$ and $\vartheta \in [0, 2\pi)$ we have

$$\rho_j(\exp(i\vartheta))z = \exp(i\vartheta)^j z = \exp(ij\vartheta)z.$$

(b) The complexification, i.e. the tensor product with $\mathbb{C}$ over $\mathbb{R}$ of SO(2) is

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\} =: \mathrm{SO}(2, \mathbb{C})$$

The complexification $\rho_j \otimes \mathbb{C}$ of the representation ρ_j acts on $\mathbb{C}^2 = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$ via the same expression, i.e. $(\rho_j \otimes \mathbb{C})(A) = A^j \in \mathrm{GL}_2(\mathbb{C})$ for all $A \in \mathrm{SO}(2, \mathbb{C})$.
This representation is isomorphic to a representation of the algebraic torus $\mathbb{C}^\times$: Every matrix in $\mathrm{SO}(2, \mathbb{C})$ is diagonalisable with diagonal form

$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix}$$

Since this change of coordinates simultaneously diagonalises $\mathrm{SO}(2, \mathbb{C})$, the group is conjugate in $\mathrm{GL}_2(\mathbb{C})$ to the subgroup

$$\left\{ \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\}$$

of $\mathrm{GL}_2(\mathbb{C})$ which is isomorphic to $\mathbb{C}^\times$. The base change in $\mathbb{C}^2$ that corresponds to the conjugation of $\mathrm{SO}(2, \mathbb{C})$ to this torus subgroup gives an isomorphism of the representation $\rho_j \otimes \mathbb{C}$ with the representation

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^2 &\to \mathbb{C}^2 \\ (z, (x, y)) &\mapsto (z^j x, z^{-j} y) \end{aligned}$$

of $\mathbb{C}^\times$. Here, the real form of the torus $\mathbb{C}^\times$ isomorphic to SO(2) is the unit circle.

Definition 3.1.4. Let (ρ, V) be a representation of SO(2). Take $w \in V$. The convex hull of the orbit of w by the action of SO(2) on V, i.e. the set

$$\mathrm{conv}(\rho(\mathrm{SO}(2))\mathrm{w}) = \mathrm{conv}(\{\rho(\mathrm{A})\mathrm{w} \colon \mathrm{A} \in \mathrm{SO}(2)\})$$

is called the SO(2)-*orbitope* of w with respect to (ρ, V).

Remark 3.1.5. Fix a representation (ρ, V) of SO(2).
(a) If there is a vector $w \in V$ such that the SO(2)-orbitope of w with respect to (ρ, V) has non-empty interior then the representation (ρ, V) must be multiplicity-free and must not contain the trivial representation as an irreducible factor.
(b) Any two SO(2)-orbitopes with respect to (ρ, V) and with non-empty interior are linearly isomorphic. Further, we can assume up to linear SO(2)-automorphism that the indices $j_1, \ldots, j_r$ of the irreducible components of the representation $\rho = \rho_{j_1} \oplus \ldots \oplus \rho_{j_r}$ are relatively prime. See Sanyal-Sottile-Sturmfels [37], Lemma 5.1 for a proof of these statements.

The following proposition is a special case of Sanyal-Sottile-Sturmfels [37], Proposition 2.2.

Proposition 3.1.6. *Let* (ρ, V) *be a representation of* $\mathrm{SO}(2)$ *and let* $C := \mathrm{conv}(\rho(\mathrm{SO}(2))\mathrm{w}) \subset \mathrm{V}$ *be an* $\mathrm{SO}(2)$*-orbitope. Then every point of the orbit of which C is the convex hull is an exposed point of C. In particular, the orbit is the set of extreme points of C.*

Example 3.1.7. The convex hull of the orbit of $(1,0,1,0,\ldots,1,0) \in \mathbb{R}^{2n}$ under the representation $\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_n$ of $\mathrm{SO}(2)$, for some $n \in \mathbb{N}$, is called the *universal* $\mathrm{SO}(2)$*-orbitope* of dimension $2n$. We will denote it by C_n. Explicitly, it is the convex hull of the trigonometric curve

$$\{(\cos(\vartheta), \sin(\vartheta), \cos(2\vartheta), \sin(2\vartheta), \ldots, \cos(n\vartheta), \sin(n\vartheta)) \in \mathbb{R}^{2n} : \vartheta \in [0, 2\pi)\}$$

Every $\mathrm{SO}(2)$-orbitope is the projection of a universal $\mathrm{SO}(2)$-orbitope. Sanyal, Sottile and Sturmfels proved (cf. [37], Theorem 5.2) that the universal $\mathrm{SO}(2)$-orbitope C_n is isomophic to the spectrahedron of positive semi-definite hermitian Toeplitz matrices of size $(n+1) \times (n+1)$ via the linear map

$$\begin{cases} \mathbb{R}^{2n} & \to & \mathrm{M_{n+1}}(\mathbb{C}) \\ (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) & \mapsto & \begin{pmatrix} 1 & x_1 + iy_1 & \ldots & x_n + iy_n \\ x_1 - iy_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 + iy_1 \\ x_n - iy_n & \ldots & x_1 - iy_1 & 1 \end{pmatrix} \end{cases}$$

It follows from this theorem that C_n is an n-neighbourly, simplicial convex set. The maximal dimension of a face of C_n is $n-1$.

We now want to collect basic facts about an $\mathrm{SO}(2)$-orbit from the point of view of real algebraic geometry.

Proposition 3.1.8. *Let* $\rho : \mathrm{SO}(2) \to \mathrm{GL_n}(\mathbb{R})$ *be a representation of* $\mathrm{SO}(2)$*. Let* O_w *be the orbit of* $w \in \mathbb{R}^n$*,* $w \neq 0$*. Denote by X the Zariski closure of* O_w *in* $\mathbb{A}^n$*. The variety X is a rational curve and the real points of X are exactly the orbit* O_w*, i.e.* $X(\mathbb{R}) = O_w$*.*

Proof. The variety $\mathrm{SO}(2)$ is isomorphic to the curve $\{(x,y) \in \mathbb{A}^2 : x^2 + y^2 = 1\}$ which is rational over the reals via stereographic projection. Now the Zariski closure X of O_w is the closure of the image of the rational curve $\mathrm{SO}(2)$ under a morphism of real algebraic varieties and therefore also a rational curve. The statement that $X(\mathbb{R})$ is exactly the orbit follows from a general result for linear real algebraic groups, see Appendix A, Theorem A.1. In the case of $\mathrm{SO}(2)$, it can also be proven elementarily by birationality of $\mathrm{SO}(2,\mathbb{C})$ and X using that $X = \rho(\mathrm{SO}(2,\mathbb{C}))\mathrm{w}$ is smooth (cf. the proof of the following Proposition 3.1.10) and $\mathrm{SO}(2) = \mathrm{SO}(2,\mathbb{C})(\mathbb{R})$ is compact. □

Definition 3.1.9. We call the curve $X = \mathrm{cl}_{Zar}(O_w) \subset \mathbb{A}^n$ of the preceding proposition the *curve associated with* the $\mathrm{SO}(2)$-orbitope $\mathrm{conv}(O_w)$. We denote the projective closure of X with respect to the embedding $\mathbb{A}^n \to \mathbb{P}^n$, $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n)$ by $\bar{X}$.

Proposition 3.1.10. *Let* $\rho_{j_1} \oplus \rho_{j_2} \oplus \ldots \oplus \rho_{j_r}$ *be a representation of* $\mathrm{SO}(2)$*. Let C be the* $\mathrm{SO}(2)$*-orbitope of* $w \in \mathbb{R}^{2r}$ *in this representation and assume that C has non-empty interior. Denote by d*

the greatest common divisor of $j_1, \dots, j_r$ and by $j = \max\{|j_1|, \dots, |j_r|\}$. Denote by $\bar{X}$ the projective curve associated with the orbitope C.
(a) The curve $\bar{X}$ is non-singular if and only if $j - d \in \{|j_1|, \dots, |j_r|\}$.
(b) The degree of the curve $\bar{X}$ is $2\frac{j}{d}$.

Proof. Setting $j_i' = |\frac{j_i}{d}|$, we assume that the j_i are relatively prime and $0 < j_1 < j_2 < \dots < j_r = j$. We complexify the situation as explained in Remark 3.1.3 and get the following parametrisation of the complex orbit of w after a complex change of coordinates:

$$\mathbb{C}^\times \to \mathbb{C}^{2r}, z \mapsto (z^{j_1}, z^{-j_1}, z^{j_2}, z^{-j_2}, \dots, z^{j_r}, z^{-j_r})$$

(a) This map extends to a morphism $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^{2r}$ which is given by the equation $\varphi(1 : z) = (z^{j_r} : z^{j_r+j_1} : z^{j_r-j_1} : \dots : z^{2j_r} : 1)$ on the affine chart $\mathbb{P}^1 \setminus \{(0 : 1)\}$ and $\varphi(s : 1) = (s^{j_r} : s^{j_r-j_1} : s^{j_r+j_1} : \dots : 1 : s^{2j_r})$ on $\mathbb{P}^1 \setminus \{(1 : 0)\}$. This morphism is injective: If $y, z \in \mathbb{C}^\times$ with $(y^{j_r}, y^{j_r+j_1}, y^{j_r-j_1}, \dots, y^{2j_r}) = (z^{j_r}, z^{j_r+j_1}, z^{j_r-j_1}, \dots, z^{2j_r})$, then $(y/z)^{j_r} = 1$ and therefore, $(y/z)^{j_i} = 1$ for $i = 1, \dots, r$. Since the j_i are relatively prime, it follows that $(y/z) \in \mathrm{U}(j_1) \cap \dots \cap \mathrm{U}(j_r) = \{1\}$, where $\mathrm{U}(n)$ denotes the group of the n-th roots of unity.
The curve $\bar{X}$ is the image of this injective morphism φ. If $\bar{X}$ is smooth, then φ must be an isomorphism, because the inverse rational map extends to a morphism on the non-singular curve $\bar{X}$ (Fulton [18], Chapter 7, Corollary 1). In particular, if $\bar{X}$ is smooth, φ is an isomorphism of the structure sheaves and therefore, the differential is an isomorphism. This means that $\bar{X}$ is smooth if and only if the derivative of φ is non-zero at every point.
The derivative of φ is obviously non-zero at every point except for $(1 : 0)$ and $(0 : 1)$. It is non-zero at these points if and only if $j_r - 1 = j_{r-1}$, because only then $z^{j_r - j_{r-1}} = z$ and the gradient does not vanish.
(b) As for the degree, if we take a hyperplane $\mathcal{V}(a_0x_0 + a_1x_1 + b_1y_1 + \dots + a_rx_r + b_ry_r) \subset \mathbb{P}^{2r}$ and intersect it with the image of $\varphi|_{\mathbb{P}^1 \setminus \{(0:1)\}}$, we get the identity

$$a_0z^{j_r} + a_1z^{j_r+j_1} + b_1z^{j_r-j_1} + a_2z^{j_r+j_2} + b_2z^{j_r-j_2} + \dots + a_rz^{2j_r} + b_r = 0$$

For a general choice of the hyperplane, this is a polynomial of degree $2j_r$ and therefore, it will have $2j_r = 2\frac{j}{d}$ roots in $\mathbb{C}$. $\square$

Remark 3.1.11. In fact, tracing all changes of coordinates, one can show that the points at infinity $\bar{X} \setminus X$ are always complex, i.e. $\bar{X}(\mathbb{R}) = X(\mathbb{R})$ is the $\mathrm{SO}(2)$-orbit which is dense in X.

Example 3.1.12. The (projective closure of the) curve associated with the universal $\mathrm{SO}(2)$-orbitope C_n is the rational normal curve of degree $2n$ in $\mathbb{P}^{2n}$. Over $\mathbb{C}$, we have seen this in the proof of Proposition 3.1.10. Over $\mathbb{R}$, it is parametrised by the Chebyshev polynomials of the first kind over the circle $\{(x, y) \in \mathbb{A}^2 \colon x^2 + y^2 = 1\}$.

For the universal $\mathrm{SO}(2)$-orbitopes, the result [37], Theorem 5.2, of Sanyal, Sottile and Sturmfels gives a complete description of the algebraic boundary (see also Example 3.1.7).

Example 3.1.13. The algebraic boundary of the universal SO(2)-orbitope of dimension $2n$ is defined by the vanishing of the determinant

$$\det\begin{pmatrix} 1 & x_1+iy_1 & \dots & x_n+iy_n \\ x_1-iy_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1+iy_1 \\ x_n-iy_n & \dots & x_1-iy_1 & 1 \end{pmatrix}.$$

It has real coefficients and is the (dehomogenisation of the) equation of the $(n-1)$th secant variety to the curve $\bar{X}_n$ associated with C_n.
More generally, for $k < n$, the k-th secant variety to the curve $\bar{X}_n$ is defined by the $(k+2)\times(k+2)$ minors of that matrix. The union of all k-dimensional faces of C_n is Zariski dense in the k-th secant variety to $\bar{X}_n$.

3.1.2. Four-dimensional SO(2)-orbitopes

Smilansky completely characterised the face lattice of an arbitrary 4-dimensional SO(2)-orbitope in his paper [40]. Let $p, q \in \mathbb{N}$ be relatively prime integers with $p < q$, let $\rho = \rho_p \oplus \rho_q$ be the corresponding 4-dimensional representation of SO(2). Denote by C_{pq} the convex hull of the orbit of $(1,0,1,0)$. He proved that C_{pq} always has a 2-dimensional family of edges. Furthermore, if $q \geq 3$, there is a 1-dimensional family of 2-dimensional q-gons among the faces of C_{pq}. The vertices of these q-gons correspond to the q-th roots of unity. Analogously, if $p \geq 3$, there is also a 2-dimensional family of p-gons among the faces of C_{pq}. This theorem has two immediate consequences that we want to emphasise:
(1) A 4-dimensional SO(2)-orbitope C_{pq} is simplicial (i.e. all faces are simplices) if and only if $(p,q) = (1,2)$ or $(p,q) = (1,3)$.
(2) A four-dimensional SO(2)-orbitope has non-exposed faces (also called facelets) if and only if $(p,q) \neq (1,2)$. Combining this with the theorem of Sanyal, Sottile and Sturmfels about universal orbitopes (cf. [37], Theorem 5.2) and the fact that every face of a spectrahedron is exposed (cf. Ramana-Goldman [31], Corollary 1), it is immediate that a 4-dimensional SO(2)-orbitope is a spectrahedron if and only if it is universal. An even stronger statement is true (cf. Corollary 3.1.15).

We investigate the algebraic boundary of 4-dimensional SO(2)-orbitopes.

Theorem 3.1.14. *Let p and q be relatively prime integers, $q > p$. Choose the coordinates $\mathbb{R}^4 \subset \mathbb{A}^4 = \{(w,x,y,z)\}$ and denote by X_{pq} the curve associated with C_{pq}. The algebraic boundary of C_{pq} is*

$$\partial_a C_{pq} = \begin{cases} S_1(X_{pq}) & \text{if } p=1, q=2 \\ S_1(X_{pq}) \cup \mathcal{V}(y^2+z^2-1) & \text{if } p \in \{1,2\}, q \geq 3 \\ S_1(X_{pq}) \cup \mathcal{V}(w^2+x^2-1) \cup \mathcal{V}(y^2+z^2-1) & \text{if } p \geq 3 \end{cases}$$

Proof. The fact that the secant variety to the curve X_{pq} associated with C_{pq} is a component of the algebraic boundary of C_{pq} follows from Theorem 2.3.12 and the list of 1-dimensional faces of C_{pq} because there is always a 2-dimensional family of edges.

The case of the universal 4-dimensional orbitope, i.e. $p = 1$, $q = 2$, follows from Sanyal-Sottile-Sturmfels [37], Theorem 5.2 (cf. Example 3.1.13).
Next, consider the case $p = 1$ or $p = 2$ and $q \geq 3$. Then the boundary of C_{pq} consists of a 2-dimensional family of edges and a 1-dimensional family of regular q-gons. The union of the q-gons is a semi-algebraic set of dimension 3: Consider the semi-algebraic map

$$\begin{cases} (0,1) \times \mathrm{relint}(\Delta_2) \to \mathbb{R}^4 \\ (t,(\lambda_0,\lambda_1,\lambda_2)) \mapsto \lambda_0 z(t) + \lambda_1 z(t+\frac{1}{q}) + \lambda_2 z(t+\frac{2}{q}) \end{cases}$$

which is injective, because three vertices of a regular q-gon are affinely independent and the relative interiors of the q-gons in the boundary of C_{pq} are disjoint. By Bochnak-Coste-Roy [10], Theorem 2.8.8, it follows that the image has dimension 3. To calculate the Zariski closure of this set, note that the last two components of the vectors $z(t)$, $z(t+\frac{1}{q})$ and $z(t+\frac{2}{q})$ are equal, and therefore the same is true for every element in the convex hull of these three points. This implies that the image is contained in the hypersurface $\mathcal{V}(y^2+z^2-1)$, which is irreducible. Therefore, the Zariski closure of the image is this hypersurface. This shows $S_1(X_{pq}) \cup \mathcal{V}(y^2+z^2-1) \subset \partial_a C_{pq}$ and since every face of C_{pq} is contained in this variety, there are no further components in this case.
The case $p \geq 3$ is completely analogous to the last case. The new component $\mathcal{V}(w^2+x^2-1)$ is the Zariski closure of the regular p-gons that lie in the boundary of C_{pq}. □

Corollary 3.1.15. *Let C be a 4-dimensional $\mathrm{SO}(2)$-orbitope. The following are equivalent:*
(a) C is linearly isomorphic to the universal $\mathrm{SO}(2)$-orbitope C_2.
(b) C is a spectrahedron.
(c) C is a basic closed semi-algebraic set.

Proof. The implication from (a) to (b) is Sanyal-Sottile-Sturmfels [37], Theorem 5.2, (b) to (c) is linear algebra: A spectrahedron $\{(x_1,\dots,x_n) \in \mathbb{R}^n : A_0 + x_1A_1 + \dots + x_nA_n \geq 0\}$ can be defined in terms of polynomial inequalities by simultaneous sign conditions on the minors of the matrix inequality. We prove the implication from (c) to (a) by contraposition:
Let C be a 4-dimensional SO(2) orbitope which is not linearly isomorphic to the universal orbitope. Then the algebraic boundary consists of at least two components, one of which is the secant variety to the curve X associated with the orbitope C. Smilansky's list of all faces shows that there is a line segment joining two points $X_{\mathrm{reg}}(\mathbb{R})$ of the orbit associated with C that intersects the interior of C. This point has full local dimension in the real points of the secant variety $S_1(X)(\mathbb{R})$ to X by Corollary 2.3.15. By Lemma 2.2.4 we conclude that C is not basic closed. □

Remark 3.1.16. The degree of the algebraic boundary of a 4-dimensional $\mathrm{SO}(2)$-orbitope can be computed if the curve associated with it is smooth, or more precisely its projective closure: If $X \subset \mathbb{P}^n$ is a smooth curve of degree d and genus g, $n \geq 4$, then the secant variety to X has degree

$$\deg(S_1(X)) = \frac{(d-1)(d-2)}{2} - g$$

(see Flenner-O'Carroll-Vogel [17], section 8.2, page 259). So, if C_{pq} is a 4-dimensional SO(2)-orbitope and $p = q-1$, then the (projective closure of the) curve X_{pq} associated with it is smooth, has degree $2q$ and genus 0 (cf. Proposition 3.1.10). In this case, the algebraic boundary of C_{pq} has degree

$$\deg(\partial_a C_{pq}) = \frac{(2q-1)(2q-2)}{2} + 4$$

for $p \geq 3$ and degree 12 for $p = 2$, $q = 3$.

Example 3.1.17. We explicitly compute the algebraic boundary of the 4-dimensional Barvinok-Novik orbitope $B_4 = C_{13}$ - we will introduce the family of Barvinok-Novik orbitopes in section 3.1.3. This means that we have to compute the equation of the secant variety to the curve X_{13} associated with C_{13}. We will use the ideal defining the secant variety to the curve associated with the universal SO(2)-orbitope C_3, which is given by the 3×3 minors of the linear matrix inequality defining C_3, cf. Example 3.1.13.
The union of all lines joinig two general real points of X_{13} is a Zariski-dense subset of the secant variety $S_1(\bar{X}_{13})$ because the real points of X_{13} are by definition Zariski-dense in X_{13} (cf. Corollary 2.3.15). The projection from $\mathbb{R}^6$ to $\mathbb{R}^4$ that projects C_3 onto C_{13} gives a bijection if restricted to the union of all lines joining two real points of the curve X_3 associated with C_3 because it is a bijection if restricted to the orbit $X_{3_{\mathrm{reg}}}(\mathbb{R})$. Therefore, the secant variety $S_1(X_{13})$ is the image of the secant variety $S_1(X_3)$ under this projection. This leads to an elimination problem that can be solved using e.g. the computer algebra system `Macaulay2` [20] (code can be found in B.2). In the coordinates $\mathbb{A}^4 = \{w, x, y, z\}$, the equation of the secant variety is the following polynomial f of degree 8 and 47 terms:

$$\begin{aligned} f = &-36w^4x^2y^2 + 24w^2x^4y^2 - 4x^6y^2 + 24w^5xyz - 80w^3x^3yz + 24wx^5yz - 4w^6z^2 \\ &+ 24w^4x^2z^2 - 36w^2x^4z^2 + 4w^6 + 12w^4x^2 + 12w^2x^4 + 4x^6 - 12w^5y + 24w^3x^2y \\ &+ 36wx^4y + 12w^4y^2 + 24w^2x^2y^2 + 12x^4y^2 - 4w^3y^3 + 12wx^2y^3 - 36w^4xz \\ &- 24w^2x^3z + 12x^5z - 12w^2xy^2z + 4x^3y^2z + 12w^4z^2 + 24w^2x^2z^2 + 12x^4z^2 \\ &- 4w^3yz^2 + 12wx^2yz^2 - 12w^2xz^3 + 4x^3z^3 - 3w^4 - 6w^2x^2 - 3x^4 + 8w^3y - 24wx^2y \\ &- 6w^2y^2 - 6x^2y^2 + y^4 + 24w^2xz - 8x^3z - 6w^2z^2 - 6x^2z^2 + 2y^2z^2 + z^4 \end{aligned}$$

Remark 3.1.18. Consider the curve $\overline{X}_{34}$ associated with the 4-dimensional SO(2)-orbitope C_{34}. It is a smooth curve of degree 8 by Proposition 3.1.10. The algebraic boundary of C_{34} has three irreducible components. Besides the secant variety, we have the two quadrics $\mathcal{V}(w^2+x^2-1)$ and $\mathcal{V}(y^2+z^2-1)$, cf. 3.1.14. Both these quadrics lie in $S_{[2]}(X_{34})$ because they are the Zariski closure of the union of 3-gons resp. 4-gons in the boundary of C. These faces are 2-dimensional and spanned by three affinely independent points. In particular, the dual varieties to both quadrics are irreducible components of $(X_{34})^{[2]}$.

Of course, the same argument shows that $X_{p,p+1}$ is a smooth curve of degree $2p+2$ in $\mathbb{P}^4$ with the property that $(X_{p,p+1})^{[2]}$ has at least two irreducible components.

3.1.3. Barvinok-Novik orbitopes

Barvinok-Novik orbitopes are a special family of centrally symmetric $SO(2)$-orbitopes that are an essential tool in the construction of centrally symmetric polytopes with a large number of faces, cf. Barvinok-Novik [2].

Definition 3.1.19. For any odd integer $n \in \mathbb{N}$, we consider the direct sum of representations $\rho = \rho_1 \oplus \rho_3 \oplus \ldots \oplus \rho_n$ of $SO(2)$ indexed by all odd integers from 1 to n. The $(n+1)$-dimensional *Barvinok-Novik orbitope*, denoted by B_{n+1}, is the convex hull of the orbit of $(1,0,1,0,\ldots,1,0) \in (\mathbb{R}^2)^{\frac{n+1}{2}}$ under the representation ρ. Explicitly, it is the convex hull of the symmetric trigonometric moment curve

$$\{(\cos(\vartheta), \sin(\vartheta), \cos(3\vartheta), \sin(3\vartheta), \ldots, \cos(n\vartheta), \sin(n\vartheta)) \colon \vartheta \in [0, 2\pi]\}.$$

Proposition 3.1.20. *Every Barvinok-Novik orbitope is a simplicial compact convex set.*

Proof. Let $n \geq 3$ be an odd integer. We will prove that any $n+1$ points on the orbit

$$\{(z, z^3, \ldots, z^n) \colon z \in \mathbb{C}^*, z\overline{z} = 1\}$$

are $\mathbb{R}$-affinely linearly independent. Take pairwise distinct points $z_0, \ldots, z_n \in \mathbb{C}^*$ with $z_j \overline{z_j} = 1$. The corresponding points $(z_j, z_j^3, \ldots, z_j^n)$ on the orbit are $\mathbb{R}$-affinely linearly independent if and only if we can conclude from the equations

$$\begin{array}{rcl} \sum_{j=0}^n a_j & = & 0 \\ \sum_{j=0}^n a_j \begin{pmatrix} z_j \\ z_j^3 \\ \vdots \\ z_j^n \end{pmatrix} & = & 0 \end{array}$$

and $a_j \in \mathbb{R}$ ($j = 0, \ldots, n$) that all the coefficients a_j are zero. This is true if the $(n+2) \times (n+1)$-matrix

$$\begin{pmatrix} \overline{z_0}^n & \overline{z_1}^n & \ldots & \overline{z_n}^n \\ \vdots & \vdots & & \vdots \\ \overline{z_0}^3 & \overline{z_1}^3 & \ldots & \overline{z_n}^3 \\ \overline{z_0} & \overline{z_1} & \ldots & \overline{z_n} \\ 1 & 1 & \ldots & 1 \\ z_0 & z_1 & \ldots & z_n \\ z_0^3 & z_1^3 & \ldots & z_n^3 \\ \vdots & \vdots & & \vdots \\ z_0^n & z_1^n & \ldots & z_n^n \end{pmatrix}$$

has full rank $n+1$ over $\mathbb{C}$. By using Vandermonde's rule, we prove that the determinant of the matrix obtained from the one above by deleting the row of ones does not vanish. To see this, we first rescale the j-th column of the new $(n+1)\times(n+1)$ matrix, i.e. the column with z_{j-1}, by z_{j-1}^{-1}. Using the identity $z_j^{-1}\overline{z_j} = z_j^{-2}$, we get the matrix

$$\begin{pmatrix} z_0^{-2}\overline{z_0}^{n-1} & z_1^{-2}\overline{z_1}^{n-1} & \dots & z_n^{-2}\overline{z_n}^{n-1} \\ \vdots & \vdots & & \vdots \\ z_0^{-2}\overline{z_0}^{2} & z_1^{-2}\overline{z_1}^{2} & \dots & z_1^{-2}\overline{z_n}^{2} \\ z_0^{-2} & z_1^{-2} & \dots & z_n^{-2} \\ 1 & 1 & \dots & 1 \\ z_0^{2} & z_1^{2} & \dots & z_n^{2} \\ \vdots & \vdots & & \vdots \\ z_0^{n-1} & z_1^{n-1} & \dots & z_n^{n-1} \end{pmatrix}$$

This gives a Vandermonde matrix if we rescale the j-th column by z_{j-1}^{n+1} and substitute $y_j := z_j^2$. We conclude that the determinant of this matrix does not vanish. This proves the claim. □

We have already seen that the 4-dimensional Barvinok-Novik orbitope B_4 is not basic closed (cf. Corollary 3.1.15). We proved this by calculating the algebraic boundary of B_4. By the description of its faces, we saw that the algebraic boundary of this convex set intersects the interior in regular points. The component on which these points lie is the secant variety to the curve associated with B_4. We will now generalise this to higher dimensions by examining higher secant varieties.

The essential ingredient from convex geometry is the result [2], Theorem 1.2 by Barvinok and Novik, stating that Barvinok-Novik orbitopes are locally $(n-1)/2$-neighbourly, i.e. the convex hull of any $(n-1)/2$ points on the trigonometric moment curve form a face if these points lie sufficiently close together.

Theorem 3.1.21. *Let $n\in\mathbb{N}$ be an odd integer greater than 2. Denote by X_{n+1} the curve associated with the $(n+1)$-dimensional Barvinok-Novik orbitope B_{n+1}. The $\frac{n-1}{2}$-th secant variety to $\bar{X}_{n+1}$ is an irreducible component of the algebraic boundary of B_{n+1}.*

Proof. Set $k := \frac{n-1}{2}$. Firstly, the origin is an interior point of the Barvinok-Novik orbitope B_{n+1} because it is an interior point of all universal SO(2)-orbitopes (cf. Sanyal-Sottile-Sturmfels [37], Theorem 5.2) and B_{n+1} is a linear projection of C_n. Therefore, X_{n+1} is not contained in any hyperplane. So by Lange [28], the dimension of $S_k(X_{n+1})$ equals $2k+1 = 2\frac{n-1}{2}+1 = n$. Because it is the secant variety to an irreducible curve, it is irreducible.
To see that it is a component of the algebraic boundary of B_{n+1}, observe that the result Barvinok-Novik [2], Theorem 1.2, states that the semi-algebraic set M in Theorem 2.3.12 has non-empty interior. □

Corollary 3.1.22. *No Barvinok-Novik orbitope is a basic closed semi-algebraic set.*

Proof. The $\frac{n-1}{2} =: k$-th secant variety to the curve X_{n+1} associated with B_{n+1} is a component of the algebraic boundary. The origin lies on this component because it lies on the line joining $(1,1,1,\dots,1)\in X_{n+1}(\mathbb{R})$ and $(-1,-1,-1,\dots,-1)\in X_{n+1}(\mathbb{R})$. It is a central point of $S_k(X_{n+1})$ by Corollary 2.3.15, i.e. in every euclidean neighbourhood of the origin there is a regular point of $S_k(X_{n+1})$. By Lemma 2.2.4, this implies that the Barvinok-Novik orbitope is not basic closed. □

In the special case of B_4, we look into this argument more concretely by considering a fortunately chosen slice of the convex set.

Example 3.1.23. We intersect B_4 with the subspace $W := \{(0,x,0,z) \in \mathbb{R}^4 : x,z \in \mathbb{R}\}$. The polynomials defining the irreducible components of $\partial_a B_4$ restricted to this subspace factor $0^2 + z^2 - 1 = (z+1)(z-1)$ and $f(0,x,0,z) = (x+z)^3(4x^3 - 3x + z)$ (cf. figure 3.1). The polynomial $4x^3 - 3x + z$ is part of the algebraic boundary of the convex and semi-algebraic set $W \cap B_4$ but the origin is an interior point of $W \cap B_4$ and a regular point of the hypersurface $\mathcal{V}(4x^3 - 3x + z)$. Using Lemma 2.2.4, we can conclude from this that B_4 is not basic closed.

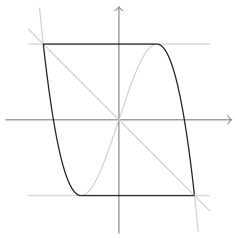

Figure 3.1.: The intersection of B_4 with the two-dimensional coordinate subspace W is the set enclosed by the black lines.

3.1.4. Another Class

In this section, we show that the appropriate secant variety is an irreducible component of the algebraic boundary for another class of SO(2)-orbitopes, which are close to universal SO(2)-orbitopes in the following sense: For any finite subset $A \subset \mathbb{N}$, we write C_A for the SO(2)-orbitope

$$C_A = \operatorname{conv}(\rho(\mathrm{SO}(2))\mathrm{w})$$

where $w = (1,0,1,0,\dots,1,0) \in \mathbb{R}^{2|A|}$ and $\rho = \oplus_{i \in A} \rho_i$. Denote by $[n] := \{1,\dots,n\}$ the set of numbers from 1 up to n. We investigate SO(2)-orbitopes of the form C_A where $A = [n+1] \setminus \{j\}$ for a fixed $j \in [n]$.

To prove the announced result, we will relate exposed faces of SO(2)-orbitopes with Laurent polynomials as suggested by Barvinok and Novik [2].

Construction 3.1.24. An exposed face F of an SO(2)-orbitope $C_A \subset \mathbb{R}^{2r} = \{(x_1, y_1, \dots, x_r, y_r\}$, where $r = |A|$, is given by a linear polynomial

$$\ell = a_0 + a_1 x_1 + b_1 y_1 + \dots + a_r x_r + b_r y_r$$

by the conditions $\ell(x) = 0$ for all $x \in F$ and $\ell(x) > 0$ for all $x \in C \setminus F$. Or equivalently, we can restrict our attention to the set of extreme points of C, i.e. the orbit of which we take the convex hull, by saying $\ell(x) = 0$ for all $x \in F \cap \rho(\mathrm{SO}(2))\mathrm{w}$ and $\ell(x) > 0$ for all $x \in \rho(\mathrm{SO}(2))\mathrm{w} \setminus \mathrm{F}$. By making the complex change of coordinates described in Remark 3.1.3 that diagonalises all elements in SO(2) simultaneously and applying its dual to the homogeneous part of degree 1 of ℓ, we get the polynomial

$$a_0 + \frac{a_1 - ib_1}{2}(x_1 + iy_1) + \frac{a_1 + ib_1}{2}(x_1 - iy_1) + \ldots + \frac{a_r - ib_r}{2}(x_r + iy_r) + \frac{a_r + ib_r}{2}(x_r - iy_r)$$

In these new coordinates, we can think of the above polynomial restricted to the orbit $\rho(\mathrm{SO}(2))w$ as a Laurent polynomial in one complex variable z, namely

$$f_\ell = \sum_{k\in A} c_k z^k + a_0 + \sum_{k\in A} \overline{c_k} z^{-k} \in \mathbb{C}[z, z^{-1}]$$

where $c_k = \frac{a_k - ib_k}{2}$. Conversely, every Laurent polynomial $f = \sum_{k=-m}^{m} c_k z^k \in \mathbb{C}[z, z^{-1}]$, where $m = \max(A)$, that satisfies $c_k = 0$ for all $k \notin A$, $c_k = \overline{c_{-k}}$ for all $k \in [m]$ and is non-negative as a function on S^1 gives rise to a supporting hyperplane to C by inverting the change of coordinates.

So finding supporting hyperplanes to an SO(2)-orbitope that has r extreme points is equivalent to finding a Laurent polynomial in the real subspace of Laurent polynomials

$$\{\sum_{k=-m}^{m} c_k z^k \colon c_k = 0 \text{ for all } k \notin A, c_k = \overline{c_{-k}} \text{ for all } k \in [m]\} \subset \mathbb{C}[z, z^{-1}]$$

that is non-negative as a function on S^1. The idea is to factor such polynomials into polynomials of low degree satisfying the assumption. Here is our primary candidate for this section:

Remark 3.1.25. A Laurent polynomial of the form $f = 2 + cz + \overline{c}z^{-1} \in \mathbb{C}[z, z^{-1}]$ is non-negative on S^1 if and only if $|c| \le 1$ and it has a zero on S^1 if and only if $|c| = 1$. This zero is then a double zero at $-\overline{c}$.

Theorem 3.1.26. *Let $n \ge 3$ be an integer and let $A = [n+1] \setminus \{j\}$ for some $j \in [n]$. If $j > \sqrt{\frac{n+1}{2}}$, the $(n-1)$-th secant variety to X_A is an irreducible component of the algebraic boundary of C_A. More precisely, in this case, the convex hull of any n points on the curve that are sufficiently close together, will be an exposed face of C_A (this property is known as locally n-neighbourly).*

Proof. We look for exposed faces of C_A spanned by n points. As explained in Construction 3.1.24, we can find them by finding a Laurent polynomial that takes real values and is non-negative on S^1 with the additional property that the coefficient of z^j is zero. By Remark 3.1.25, we know that the Laurent polynomial

$$f = (2 + z + z^{-1})^n (2 + cz + \overline{c}z^{-1}) = \sum_{k=-n-1}^{n+1} c_k z^k$$

takes only real values on S^1 and is non-negative on it if and only if $|c| \le 1$. We need to impose the condition $c_j = 0$ for it to correspond to a supporting hyperplane to C_A. This condition is a linear condition in c and therefore uniquely determines it. We will compute it now.
First note that f is also equal to

$$f = \frac{1}{z^n}(z+1)^{2n}(2 + cz + \overline{c}z^{-1}) = \sum_{k=-n}^{n} \binom{2n}{n+k} z^k (2 + cz + \overline{c}z^{-1}).$$

Therefore the coefficient of z^j in f is

$$c_j = 2\binom{2n}{n+j} + \binom{2n}{n+j-1}c + \binom{2n}{n+j+1}\overline{c}.$$

Writing $c = a + ib$ with $a, b \in \mathbb{R}$, we get two linear equations over the reals and compute

$$\begin{aligned} a &= -\frac{2\binom{2n}{n+j}}{\binom{2n}{n+j-1} + \binom{2n}{n+j+1}} \\ b &= 0 \end{aligned}$$

We now want that $|c| < 1$, i.e. $|a| < 1$ which is equivalent to

$$2\binom{2n}{n+j} < \binom{2n}{n+j-1} + \binom{2n}{n+j+1}$$

Writing out the binomial coefficients and clearing denominators shows that this is equivalent to

$$\begin{aligned} 2(n-j+1)(n+j+1) &< (n+j)(n+j+1) + (n-j)(n+j+1) \Leftrightarrow \\ 2n^2 + 4n + 2 - 2j^2 &< 2n^2 + 2n + 2j^2 \Leftrightarrow \\ j &> \sqrt{\frac{n+1}{2}}. \end{aligned}$$

Since the condition $|c| < 1$ on c is open, we can now perturb the coefficients of $(2 + z + z^{-1})$ slightly and we still satisfy the condition that the coefficient of z^j in f is 0 by multiplying with $2 + cz + \overline{c}z^{-1}$ for a $c \in \mathbb{C}$ with $|c| < 1$. This shows that C_A is locally n-neighbourly for $j > \sqrt{\frac{n+1}{2}}$, as claimed. □

Remark 3.1.27. In the above proof, we considered the very special factorisation

$$f = (2 + z + z^{-1})^n(2 + cz + \overline{c}z^{-1})$$

that showed that C_A is locally n-neighbourly. Of course, we could try other factorisations with different factors. Yet already in the simple case

$$f = (2 - z - z^{-1})(2 + z + z^{-1})^{n-1}(2 + cz + \overline{c}z^{-1})$$

the computations become complicated and explicit conditions on j as above are hard to extract. For the alternative factorisation, we have to check the inequality

$$\frac{2(n-j+1)(n+j+1)\,(4j^2 - 2j - 4n + 2)}{5j^4 + (6n-3)j^3 + (8n^2 - 19n - 5)j^2 - 3(2n-1)(n+1)^2 j + n(3n^3 - n^2 + n + 5)} < 1$$

To determine the sign of the denominator is the first problem because it indeed depends on j. Knowing the sign, we then get a polynomial inequality for j of degree 4 that can, of course, be solved symbolically. The expressions in this iterated roots will be rational functions in n of high degree with large coefficients.

3.2. Cones of Sums of Squares of Ternary Forms

The cone of sums of squares intersected with the vector space of homogeneous polynomials of a given even degree is a closed and pointed convex cone with non-empty interior in this finite-dimensional space. Its dual cone is a spectrahedron and we study its extreme rays for ternary forms. The main tool is the Cayley-Bacharach Theorem for plain curves and the point of view of Gorenstein ideals developped by Blekherman.

We denote by $\mathbb{R}[\underline{x}] = \mathbb{R}[x, y, z]$ the polynomial ring over the reals generated by 3 variables. We consider it with the standard total degree grading and denote by $\mathbb{R}[\underline{x}]_m$ the $\mathbb{R}$-vector space of homogeneous polynomials of degree m. Note that $\mathbb{R}[\underline{x}]_m$ has dimension $\binom{m+2}{2}$.

Let $\ell \in \mathbb{R}[\underline{x}]_m^*$ be a linear functional on ternary forms of degree m. To ℓ and every pair of positive integers $u, v \in \mathbb{N}$ with $u + v = m$, we associate the bilinear form

$$B_{\ell,u,v}\colon \begin{cases} \mathbb{R}[\underline{x}]_u \times \mathbb{R}[\underline{x}]_v & \to & \mathbb{R} \\ (p, q) & \mapsto & \ell(pq) \end{cases}.$$

The representing matrices of these bilinear forms with respect to the monomial basis are usually called the *Catalecticant matrices* of ℓ. In the special case of $m = 2d$ and $u = v = d$, it is often called the *generalised Hankel matrix*, *moment matrix* or *middle Catalecticant* associated with ℓ. It is a real symmetric matrix that we denote as B_ℓ. This symmetric matrix has rank 1 if and only if the linear functional ℓ is a real point evaluation, see e.g. Blekherman [7], Lemma 2.4.

The dual convex cone $\Sigma_{2d}^\vee$ to the cone of sums of squares $\Sigma_{2d} \subset \mathbb{R}[\underline{x}]_{2d}$ is the set of all linear functionals ℓ such that the associated bilinear form B_ℓ is positive semi-definite. It certainly contains the convex cone generated by real point evaluations, i.e. positive semi-definite rank 1 moment matrices, which is dual to the cone of non-negative polynomials of degree $2d$. Since the sums of squares cone is strictly contained in the cone of non-negative polynomials for $d \geq 3$, the cone $\Sigma_{2d}^\vee$ has extreme rays $\mathbb{R}_+\ell$ such that B_ℓ has rank > 1. Our goal is to study these extreme rays of higher rank and their Zariski closure. Then our main results from Section 2.4 have applications to the algebraic boundary of the cone of sums of squares.

We build on results by Blekherman discussed in [7] and [6]. An important tool is the Gorenstein ideal associated with ℓ.

3.2.1. Gorenstein Ideals

Definition 3.2.1. Let $\ell \in \mathbb{R}[\underline{x}]_m^*$ be a linear functional. We call the homogeneous ideal $I(\ell)$ of $\mathbb{R}[\underline{x}]$ generated by

$$\{p \in \mathbb{R}[\underline{x}]_k \colon k > m \text{ or } \ell(pq) = 0 \text{ for all } q \in \mathbb{R}[\underline{x}]_{m-k}\}$$

the *Gorenstein ideal with socle* ℓ. We call m the *socle degree* of the ideal.

These ideals were studied extensively in the literature, cf. Iarrobino-Kanev [24]. Our definition is probably the most direct for 0-dimensional Gorenstein ideals, cf. [15], Theorem 21.6.

Remark 3.2.2. The degree u part of the ideal is the left-kernel of the bilinear form $B_{\ell,u,v}$ for $u \leq m$. In particular, the Hilbert function of a Gorenstein ideal I with even socle degree $2d$ is symmetric around d, i.e. $\mathrm{Hilb}(I, i) = \mathrm{Hilb}(I, 2d - i)$ for all $0 \leq i \leq 2d$.

We can consider the set of all Gorenstein ideals with a fixed socle degree m as a projective space by identifying an ideal with its socle, which is uniquely determined by the ideal up to scaling. In this projective space, we consider the set $\mathrm{Gor}(T)$ of all Gorenstein ideals with a given Hilbert function T.

Proposition 3.2.3. *The set* $\mathrm{Gor}(T)$ *of all Gorenstein ideals with socle degree m and Hilbert function T is a quasiprojective subvariety of the projective space of all Gorenstein ideals with socle degree m.*

Proof. The condition to have a given Hilbert function can be expressed as rank conditions on the Catalecticant matrices, namely

$$\mathrm{rank}(B_{\ell,u,v}) = T(u).$$

□

Definition 3.2.4. We call a Hilbert function T *permissible* if there is a Gorenstein ideal $I \subset \mathbb{R}[\underline{x}]$ with Hilbert function T.

Using the Buchsbaum-Eisenbud Structure Theorem for height 3 Gorenstein ideals (cf. Buchsbaum-Eisenbud [12]), Diesel proved the following.

Theorem 3.2.5 (cf. Diesel [14], Theorem 1.1 and 2.7). *For every permissible Hilbert function T, the variety* $\mathrm{Gor}(T)$ *is an irreducible unirational variety.*

We will use the fact that $\mathrm{Gor}(T)$ is unirational to determine the dimension of $\mathrm{Gor}(T)$ for special Hilbert functions T. In order to do this, we need the more precise information on the unirationality of $\mathrm{Gor}(T)$ given by Diesel. The information we need is spread out over the paper Diesel [14]. We will give a short summary with references.

Remark 3.2.6. Diesel proves that for a given permissible Hilbert function T there is a minimal set (with respect to inclusion) $D_{min} = (Q, P)$ of degrees of generators $Q = \{q_1, \ldots, q_u\}$ and relations $P = \{p_1, \ldots, p_u\}$ for a Gorenstein ideal with Hilbert function T. We assume $q_1 \leq q_2 \leq \ldots \leq q_u$ and $p_1 \geq p_2 \geq \ldots \geq p_u$. The set $\mathrm{Gor}_{D_{min}}$ of all Gorenstein ideals with generators of degree as specified by Q is a dense subset of $\mathrm{Gor}(T)$, see the proof of [14], Theorem 2.7 and Theorem 3.8. Given D_{min}, we consider the affine space $\mathbb{A}^{h(E_M)}$ of skew-symmetric matrices with entries in $\mathbb{R}[\underline{x}]$ where the (i, j)-th entry is homogeneous of degree $p_j - q_i$ $(i \neq j)$ and the rational map $\pi\colon \mathbb{A}^{h(E_M)} \dashrightarrow \mathrm{Gor}_{D_{min}}$ that takes a matrix to the Gorenstein ideal generated by its Pfaffians. This statement uses the Buchsbaum-Eisenbud Structure Theorem, cf. [14], p. 367 and p. 369. Given a Hilbert function T, the set D_{min} of degrees of generators and relations for T is determined in a combinatorial way: Given the socle degree m and the minimal degree k of a generator of the ideal, there is a one-to-one correspondence between permissible Hilbert functions of order k and self-complementary partitions of $2k$ by $m - 2k + 2$ blocks, cf. [14], Proposition 3.9. These partitions give the maximum number of generators, which is $2k + 1$, cf. [14], Theorem 3.3. To refine these sequences to D_{min}, we iteratively delete pairs (q_i, q_j) from Q and (p_i, p_j) from P whenever they satisfy $r_i + r_j = p_i + p_j - q_i - q_j = 0$, cf. [14], p. 380.

We are particularly interested in Gorenstein ideals with socle in even degree $2d$ with the property that the middle Catalecticant has corank 4, i.e. rank $\binom{d+2}{2} - 4$. The proof of the following statement is analogous to the proof of Diesel [14], Theorem 4.4.

Lemma 3.2.7. *Let $d \geq 4$ be an integer. The projective variety X_{-4} of moment matrices of corank at least 4, i.e. of rank at most $\binom{d+2}{2} - 4$, is irreducible of codimension 10 in the linear space of moment matrices. In particular, it is defined by the symmetric $(\binom{d+2}{2} - 3)$-minors of the generic moment matrix.*

Proof. Let $N = \binom{d+2}{2}$. The quasiprojective variety S_{-4} of symmetric $N \times N$-matrices of rank $N-4$ has codimension 10 in the projective space of the vector space of symmetric $N \times N$-matrices. Therefore the intersection X_{-4} of S_{-4} with the subspace of moment matrices has codimension at most 10 in this linear space. We will show, that it has codimension exactly 10 by counting dimensions of the possible $\mathrm{Gor}(T)$ using their unirationality.

There are only two possible Hilbert functions for a Gorenstein ideal I with socle degree $2d$ and $\mathrm{Hilb}(I, d) = \binom{d+2}{2} - 4$ by their symmetry, namely

$$T_1 = \left(1, 3, 6, \ldots, \binom{d+1}{2}, \binom{d+2}{2} - 4, \ldots\right),$$

which corresponds to the case of four generators in degree d and no generators of lower degree, and

$$T_2 = \left(1, 3, 6, \ldots, \binom{d+1}{2} - 1, \binom{d+2}{2} - 4, \ldots\right),$$

which corresponds to the case of one generator of degree $d-1$ and one generator of degree d. More precisely, these two Hilbert functions correspond to the self-complementary partitions of $2 \times 2d$ resp. $4 \times (2d-2)$ blocks shown in Figure 3.2 by the correspondence explained in Diesel [14], section 3.4, in paricular Proposition 3.9.

We first consider T_1. The sequence of degrees of the generators for the minimal set D_{min} is in this case different for $d = 4$ and $d \geq 5$, namely $(4, 4, 4, 4, 6)$ for $d = 4$ and $(d, d, d, d, d+1, \ldots, d+1)$ with $(2d-9)$ many generators of degree $d+1$ for $d \geq 5$, cf. Remark 3.2.6. Since $q_i + p_i = 2d + 3$, the degree matrices are

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 1 \\ & 0 & 3 & 3 & 1 \\ & & 0 & 3 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 3 & 3 & 2 & \cdots & \cdots & 2 \\ & 0 & 3 & 3 & \vdots & & & \vdots \\ & & 0 & 3 & \vdots & & & \vdots \\ & & & 0 & 2 & \cdots & \cdots & 2 \\ & & & & 0 & 1 & \cdots & 1 \\ & & & & & 0 & \ddots & \vdots \\ & & & & & & \ddots & 1 \\ & & & & & & & 0 \end{pmatrix}$$

where the right one is of size $(2d-5) \times (2d-5)$. Every entry of the matrix can be generically chosen among the forms of the indicated degree and its Pfaffians will generate a Gorenstein ideal

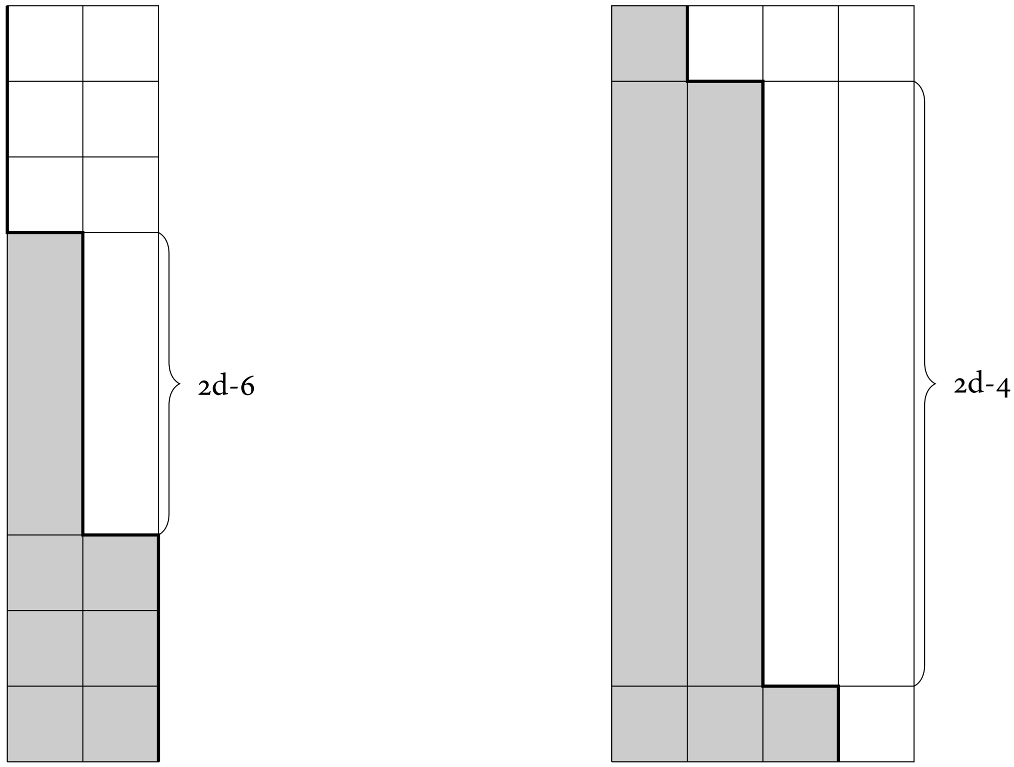

Figure 3.2.: The partition on the right of $2d \times 2$ blocks corresponds to T_1, the partition on the left of $(2d-2) \times 4$ blocks to T_2.

with Hilbert function T_1. Therefore, for $d = 4$, we have $h(E_M) = 6\dim(\mathbb{R}[\underline{x}]_3) + 4\dim(\mathbb{R}[\underline{x}]_1) = 72$ and for $d \geq 5$ we have

$$\begin{aligned} h(E_M) &= 6\dim(\mathbb{R}[\underline{x}]_3) + 4(2d-9)\dim(\mathbb{R}[\underline{x}]_2) \\ &\quad + \binom{2d-9}{2}\dim(\mathbb{R}[\underline{x}]_1) \\ &= 6d^2 - 9d - 21. \end{aligned}$$

This is an overcount of the dimension of $\mathrm{Gor}_{D_{min}}$ because for every choice of generators of a given ideal we get a matrix with these generators as Pfaffians. So for $d = 4$, we choose a basis of a 4-dimensional subspace of forms of degree 4 and one generator of degree 6 from a $\dim(\mathbb{R}[\underline{x}]_6) - T_1(2) = 22$-dimensional space. Therefore we overcount the dimension of $\mathrm{Gor}_{D_{min}}$ by at least $4^2 + 22 = 38$ and the dimension of $\mathrm{Gor}(T)$ is at most 34. Since $\dim(\mathbb{P}(\mathbb{R}[\underline{x}]_8^*)) = 44$, its codimension is at least 10. For $d \geq 5$, we choose a basis of a 4-dimensional subspace of forms of degree d and $2d-9$ linearly independent generators from a space of dimension $\dim(\mathbb{R}[\underline{x}]_{d+1}) - T_1(d-1) = 2d+3$. The overcount in this case is at least $4^2 + (2d-9)(2d+3)$ and the dimension of $\mathrm{Gor}_{D_{min}}$ is at most $2d^2 + 3d - 10$. The projective dimension of the space of moment matrices is $\dim(\mathbb{R}[\underline{x}]_{2d}^*) - 1 = 2d^2 + 3d$, which again implies that the codimension of $\mathrm{Gor}(T_1)$ is at least 10. From the fact that it can be at most 10, it follows that it is exactly 10.

We now repeat the count for the Hilbert function T_2. In this case, $D_{min} = \{Q_{min}, P_{min}\} =$

$\{(d-1, d, d+1, d+1, \ldots, d+1), (d+4, d+3, d+2, d+2, \ldots, d+2)\}$ with $(2d-5)$ times the entry $d+1$ in Q_{min} and $d+2$ in P_{min}, cf. Figure 3.2. Therefore, the degree matrix is

$$\begin{pmatrix} 0 & 4 & 3 & \cdots & \cdots & 3 \\ & 0 & 2 & \cdots & \cdots & 2 \\ & & 0 & 1 & \cdots & 1 \\ & & & 0 & \ddots & \vdots \\ & & & & & 1 \\ & & & & & 0 \end{pmatrix}$$

which is of size $(2d-3) \times (2d-3)$. We compute

$$\begin{aligned} h(E_M) &= \dim(\mathbb{R}[\underline{x}]_4) + (2d-5)\dim(\mathbb{R}[\underline{x}]_3) + (2d-5)\dim(\mathbb{R}[\underline{x}]_2) \\ &\quad + \binom{2d-5}{2}\dim(\mathbb{R}[\underline{x}]_1) \\ &= 6d^2 - d - 20. \end{aligned}$$

Here we choose one generator of degree $d-1$, one generator of degree d from a 4-dimensional space and $(2d-5)$ generators from a $\dim(\mathbb{R}[\underline{x}]_{d+1}) - T_2(d-1) = (2d+4)$-dimensional space. Therefore the dimension of $\mathrm{Gor}(T_2)$ is at most $6d^2 - d - 20 - 4 - (2d-5)(2d+4) = 2d^2 + d - 4$. The codimension is at least $2d + 4 \geq 12$. So $\mathrm{Gor}(T_2)$ cannot be an irreducible component of X_{-4} and we conclude that $\mathrm{Gor}(T_2) \subset \mathrm{cl}(\mathrm{Gor}(T_1))$. In summary, $\mathrm{Gor}(T_1)$ is a dense subset of X_{-4} and X_{-4} is irreducible (cf. Diesel [14], Theorem 2.7) and has the expected codimension 10 in the space of moment matrices. □

The tangent space to the quasiprojective variety $\mathrm{Gor}(T)$ for a permissible Hilbert function T at a Gorenstein ideal I can be described in terms of the ideal. We identify $\mathbb{R}[\underline{x}]_m$ with its dual space by using the apolar bilinear form, i.e. we identify a monomial $x^\alpha \in \mathbb{R}[\underline{x}]_m$ with the linear form $p \mapsto \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha} p$ that takes a polynomial $p = \sum p_\beta x^\beta$ to p_α. Using this identification, we can state a characterisation of the tangent space to $\mathrm{Gor}(T)$ at an ideal I in terms of this ideal.

Theorem 3.2.8 (Iarrobino-Kanev [24], Theorem 3.9 and 4.21). *Let T be a permissible Hilbert function. The quasiprojective variety* $\mathrm{Gor}(T)$ *is smooth. Let $\ell \in \mathbb{R}[\underline{x}]_m^*$ be a linear functional such that the corresponding Gorenstein ideal $I = I(\ell)$ has Hilbert function T. Then the tangent space to* $\mathrm{Gor}(T)$ *at ℓ is*

$$((I^2)_m)^\perp \subset \mathbb{R}[\underline{x}]_m.$$

3.2.2. Extreme Rays of Maximal Rank and Positive Gorenstein Ideals

In this section, we recapitulate bounds on the rank of moment matrices of extreme rays of $\Sigma_{2d}^\vee$ which are not point evaluations. The lower bound and its tightness are proved in Blekherman [6], Theorem 2.1. We constructively establish tightness of the upper bound. We show that the Zariski closure of the set of extreme rays is the variety of moment matrices of corank at least 4, which is irreducible; in particular, it is (at least set-theoretically) defined by the symmetric $r \times r$ minors of the generic moment matrix, where $r = \binom{d+2}{2} - 3$.

One of the main results of Blekherman is a characterisation of extreme rays of $\Sigma_{2d}^\vee$ by the associated Gorenstein ideals.

Proposition 3.2.9 (Blekherman [7], Lemma 2.2; Blekherman [6], Proposition 4.2). *(a) A linear functional $\ell \in \mathbb{R}[\underline{x}]_{2d}^*$ spans an extreme ray of $\Sigma_{2d}^\vee$ if and only if the bilinear form B_ℓ is positive semi-definite and the degree d part $I(\ell)_d$ of the Gorenstein ideal $I(\ell)$ is maximal with respect to inclusion over all Gorenstein ideals with socle degree $2d$.*
(b) Let I be a Gorenstein ideal with socle degree $2d$. Then I_d is maximal with respect to inclusion over all Gorenstein ideals with socle degree $2d$ if and only if the degree $2d$ part of the ideal generated by I_d is a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$. In this case, it is equal to I_{2d}.

Lower bounds on the ranks for extreme rays were established by Blekherman.

Theorem 3.2.10 (Blekherman [6], Theorem 2.1). *Let $d \geq 3$ and $\ell \in \Sigma_{2d}^\vee$ and suppose $\mathbb{R}_+\ell$ is an extreme ray. Then the rank r of B_ℓ is 1, in which case ℓ is a point evaluation, or its rank is at least $3d-2$. These bounds are tight and extreme rays $\Sigma_{2d}^\vee$ of rank $3d-2$ can be explicitly constructed.*

From Blekherman's work, we can easily deduce an upper bound.

Theorem 3.2.11. *Let $\ell \in \Sigma_{2d}^\vee$, $d \geq 4$ and suppose $\mathbb{R}_+\ell$ is an extreme ray. The rank of B_ℓ is at most $\binom{d+2}{2} - 4$, i.e. the corank is at least 4.*

Proof. Since $\mathbb{R}_+\ell$ is an extreme ray, we know that the degree $2d$ part of the ideal generated by $I(\ell)_d$ is a hyperplane in the space of forms of degree $2d$. The dimension of the space $\mathbb{R}[\underline{x}]_d I(\ell)_d$ is bounded by $\dim(\mathbb{R}[\underline{x}]_d)\dim(I(\ell)_d) = \binom{d+2}{2}\operatorname{corank}(B_\ell)$. In case $\operatorname{corank}(B_\ell) \leq 3$ and $d \geq 5$, this bound is smaller than the dimension $\binom{2d+2}{2} - 1$ of a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$. The case $\operatorname{corank}(B_\ell) \leq 3$ and $d = 4$ needs a more precise count: Suppose that $\operatorname{corank}(B_\ell) = 3$ and the kernel of B_ℓ is generated by f_1, f_2, f_3. Then the dimension of the space $\mathbb{R}[\underline{x}]_4 I(\ell)_4$ is bounded by $3\dim(\mathbb{R}[\underline{x}]_4) - 3 = 42 < 45 - 1 = \dim(\mathbb{R}[\underline{x}]_8) - 1$ because there are the 3 obvious relations, namely $f_i f_j - f_j f_i = 0$ for $i \neq j$. □

Remark 3.2.12. The upper bound in the case $d = 3$ is corank 3, which agrees with the lower bound.

A main tool in this section is the Cayley-Bacharach Theorem.

Theorem 3.2.13 (Cayley-Bacharach, cf. Eisenbud-Green-Harris [16], CB5). *Let $X_1, X_2 \subset \mathbb{P}^2$ be plane curves defined over $\mathbb{R}$ of degree d and e intersecting in $d \cdot e$ points. Set $s = d + e - 3$ and decompose $X_1 \cap X_2 = \Gamma_1 \cup \Gamma_2$ into two disjoint sets defined over $\mathbb{R}$. Then for all $k \leq s$, the following equality holds*

$$\dim(\mathcal{I}(\Gamma_1)_k) - \dim(\mathcal{I}(X_1 \cap X_2)_k) = |\Gamma_2| - \dim \operatorname{span}\{\operatorname{Re} \operatorname{ev}_x, \operatorname{Im} \operatorname{ev}_x \in \mathbb{R}[\underline{x}]_{s-k}^* : x \in \Gamma_2\}.$$

The left hand side is the dimension of the space of forms of degree k vanishing on Γ_1 modulo the subspace of forms vanishing in every point of $X_1 \cap X_2$. The right hand side is the linear defect of point evaluations on forms of dual degree $s-k$ at points of Γ_2.

Probably the most famous instance of this theorem is the following application to the complete intersection of two cubic curves, stated here for a totally real intersection.

Example 3.2.14. Suppose $X_1, X_2 \subset \mathbb{P}^2$ are plane cubic curves intersecting in 9 points. Then $d = e = 3$ and so $s = 3$. Pick $\Gamma_2 = \{P\}$ for any intersection point P and put $\Gamma_1 = (X_1 \cap X_2) \setminus \{P\}$. Let us consider $k = 3$ and compute the right hand side of the Cayley-Bacharach equality: Since $\dim \operatorname{span}\{\mathrm{ev}_x \in \mathbb{R}[\underline{x}]_0^* : x \in \Gamma_2\} = 1$, we conclude

$$\dim(\mathcal{I}(\Gamma_1)_3) - \dim(\mathcal{I}(X_1 \cap X_2)_3) = 0,$$

which means that every cubic form that vanishes in the 8 points of Γ_1 also vanishes at the ninth point P of the intersection. In other words, the point evaluation $\mathrm{ev}_P \in \mathbb{R}[\underline{x}]_3^*$ lies in the subspace U_{Γ_1} spanned by the eight point evaluations $\{\mathrm{ev}_x \in \mathbb{R}[\underline{x}]_3^* : x \in \Gamma_1\}$. The annihilator of U_{Γ_1} is the 2-dimensional subspace of $\mathbb{R}[\underline{x}]_3$ spanned by the defining equations of X_1 and X_2. Since this is true for any point $P \in X_1 \cap X_2$, we conclude, that there is a unique linear relation among the point evaluations $\{\mathrm{ev}_x \in \mathbb{R}[\underline{x}]_3^* : x \in X_1 \cap X_2\}$ and all coefficients of this relation are non-zero.

Using the Cayley-Bacharach Theorem, we will first show that there are extreme rays of corank 4 under the following constraint on the degree.

Constraint 3.2.15. Let $d \geq 4$. There is a unique conic C going through the following six points in the plane: $(0,0), (1,0), (0,1), (d-1, d-1), (d-2, d-1), (d-1, d-2)$; its equation is given by $C = \mathcal{V}(x^2 + y^2 - \frac{2(d-2)}{d-1}xy - x - y)$. From now on, we assume that this conic does not go through any other integer point. The only exceptional cases in the interval $\{4, 5, \ldots, 100\}$ are: $9, 19, 21, 29, 33, 34, 36, 40, 49, 51, 57, 61, 73, 78, 79, 81, 89, 99$.

Proposition 3.2.16. *Set* $L_1 = \prod_{j=0}^{d-1}(x - jz)$ *and* $L_2 = \prod_{j=0}^{d-1}(y - jz)$ *and let* $\Gamma = \mathcal{V}(L_1, L_2) = \{(j : k : 1) : j, k = 0, \ldots, d-1\}$ *be the intersection of their zero sets in* $\mathbb{P}^2$*. Split these points into*

$$\Gamma_2 = \{(x : y : 1) : x + y = 2\} \cup \bigcup_{j=1}^{d-4} \{(x : y : 1) : x + y = d + j\}$$

and $\Gamma_1 = \Gamma \setminus \Gamma_2$*. Then there is a unique linear relation* $\sum_{v \in \Gamma_1} u_v \mathrm{ev}_v = 0$ *among the point evaluations on forms of degree* d *at points of* Γ_1 *and all coefficients* $u_v \in \mathbb{R}$ *in this relation are non-zero. The set of all forms of degree* d *vanishing on* Γ_1 *is a 3-dimenional space spanned by* L_1, L_2 *and a form* p *which is non-zero at any point of* Γ_2*.*

See Figure 3.3 for the case $d = 5$ and Figure 3.4 for the case $d = 9$.

Proof. First observe that there is a unique (up to scaling) form of degree $d-3$ vanishing on Γ_2, namely $(x+y-2z)\prod_{j=1}^{d-4}(x+y-(d+j)z)$, the product of diagonals defining Γ_2: Indeed, suppose f is a form of degree $d-3$ vanishing on Γ_2, then it intersects the line $x + y = d+1$ in $d-2$ integer points. Therefore it vanishes identically on it and we can divide f by this linear polynomial and get a form of degree $d-4$ vanishing on $d-3$ points on the line $x + y = d + 2$. Inductively, we conclude that f is (again up to scaling) the claimed product of linear forms. Therefore, by the Cayley-Bacharach Theorem, the space of forms of degree d vanishing on Γ_1 is 3-dimensional, so it is spanned by L_1, L_2 and a third form p. We will explicitly construct this form: Let p be the product of the linear forms $x+y-jz$ for $j=3,\dots,d$ and of the ellipse $\mathcal{V}(x^2+y^2-\frac{2(d-2)}{(d-1)}xy-x-y)$ passing through the six points $(0,0)$, $(1,0)$, $(0,1)$, $(d-2,d-1)$, $(d-1,d-1)$ and $(d-1,d-2)$. By construction, p vanishes on Γ_1, is of degree d and does not vanish on all of Γ. Therefore $\{L_1,L_2,p\}$ is a basis of the space of forms of degree d vanishing on Γ_1. By assumption on d, the form p does not vanish on any point of Γ other than the six mentioned above.

Note that $|\Gamma_1| = \binom{d+2}{2}-2$, because $|\Gamma_1| = d^2-|\Gamma_2| = d^2-(3+\sum_{j=1}^{d-4}(d-1-j)) = d^2-\binom{d-1}{2}$. So the fact that the space of forms of degree d vanishing on Γ_1 is 3-dimensional implies that there is a unique linear relation among the point evalutaions on forms of degree d at points of Γ_1. To see that all coefficients u_v in the relation $\sum_{v\in\Gamma_1}u_v\mathrm{ev}_v=0$ are non-zero, note that the unique form f of degree $d-3$ vanishing on Γ_2 does not vanish on any point of Γ_1. Therefore, there is no form of degree $d-3$ vanishing on $\Gamma_2\cup\{v_0\}$ for any $v_0\in\Gamma_1$ and Cayley-Bachrach implies that the point evaluations $\{\mathrm{ev}_v\colon v\in\Gamma_1\}\setminus\{\mathrm{ev}_{v_0}\}$ are linearly independent. □

Lemma 3.2.17. *There is an extreme ray $\mathbb{R}_+\ell$ of $\Sigma_{2d}^\vee$ such that B_ℓ has corank 4. The Hilbert function of the ideal $I(\ell)$ is $\mathrm{Hilb}(I(\ell),j)=\binom{j+2}{2}=\mathrm{Hilb}(I(\ell),2d-j)$ for $0\le j<d$ and $\mathrm{Hilb}(I(\ell),d)=\binom{d+2}{2}-4$.*

Proof. Let L_1,L_2,p be as in Proposition 3.2.16 and consider the splitting $\mathcal{V}(L_1,L_2)=\Gamma=\Gamma_1\cup\Gamma_2$ of the points defined there. Pick a point $P\in\Gamma_1$ and set $\Lambda=\Gamma_1\setminus\{P\}$. We claim that the linear functional

$$\ell=\sum_{v\in\Lambda}\mathrm{ev}_v-\frac{u_P^2}{\sum_{v\in\Lambda}u_v^2}\mathrm{ev}_P,$$

where u_v are the coefficients of the Cayley-Bacharach relation as in Proposition 3.2.16, is an extreme ray of $\Sigma_{2d}^\vee$ and that the corresponding middle Catalecticant B_ℓ has corank 4.

First note that B_ℓ is positive semi-definite because

$$\begin{aligned}\ell(f^2) &= \sum_{v\in\Lambda}f(v)^2-\frac{u_P^2}{\sum_{v\in\Lambda}u_v^2}f(P)^2\\ &= \sum_{v\in\Lambda}f(v)^2-\frac{u_P^2}{\sum_{v\in\Lambda}u_v^2}\frac{1}{u_P^2}\left(\sum_{v\in\Lambda}u_vf(v)\right)^2\\ &= \|(f(v)_{v\in\Lambda}\|^2-\left|\left\langle\frac{1}{\|(u_v)_{v\in\Lambda}\|}(u_v)_{v\in\Lambda},(f(v))_{v\in\Lambda}\right\rangle\right|^2\\ &\ge 0\end{aligned}$$

by the Cauchy-Schwarz inequality for all polynomials $f\in\mathbb{R}[\underline{x}]_d$. More precisely, $\ell(f^2)$ is zero for a form f not identically zero on Γ if and only if $f(v)=\alpha u_v$ for all $v\in\Lambda$ and some $\alpha\in\mathbb{R}^*$. Therefore, the degeneration space of the middle Catalecticant is spanned by L_1,L_2,p and the form uniquely determined (modulo L_1,L_2,p) by $f(v)=u_v$ for all $v\in\Lambda$; it has dimension 4 as desired. Indeed, the form f is uniquely determined because $\{\mathrm{ev}_x\in(\mathbb{R}[\underline{x}]_d/\mathrm{span}(L_1,L_2,p))^*\colon x\in\Lambda\}$ is a basis.

We now prove extremality of ℓ in $\Sigma_{2d}^\vee$ by checking the characterisation that $I(\ell)_d$ generates a hyperplane in the vector space of forms of degree $2d$, cf. Blekherman [6], Proposition 4.2. As a first step, we show that $\langle L_1,L_2,p\rangle_{2d-3}$ has codimension $|\mathcal{V}(L_1,L_2,p)|=|\Gamma_1|$. So suppose

$a_1L_1 + a_2L_2 + bp = 0$, where $a_1, a_2, b \in \mathbb{R}[\underline{x}]_{d-3}$ are forms of degree $d-3$. By evaluating at points of Γ_2, we conclude that b is the uniquely determined form of degree $d-3$ vanishing on Γ_2, cf. proof of Proposition 3.2.16. Since L_1 and L_2 are coprime, this is a unique syzygy and we conclude

$$\dim(\langle L_1, L_2, p\rangle_{2d-3}) = 3\binom{d-1}{2} - 1 = \frac{3}{2}(d^2 - 3d + 2) - 1,$$

which means codimension $|\Gamma_1| = d^2 - \binom{d-1}{2}$ in $\mathbb{R}[\underline{x}]_{2d-3}$. In particular, the codimension of $\langle L_1, L_2, p\rangle_{2d}$ in $\mathbb{R}[\underline{x}]_{2d}$ is also $|\Gamma_1|$ because the point evaluations $\{\mathrm{ev}_v : v \in \Gamma_1\}$ are linearly independent on forms of degree $2d-3$ and consequently also on forms of degree $2d$.

Now suppose $a_1L_1 + a_2L_2 + bp + cf = 0$ for forms $a_1, a_2, b, c \in \mathbb{R}[\underline{x}]_d$ of degree d. Evaluation at points of Γ_1 implies that c lies in the span of L_1, L_2, p. So we have three syzygies and the codimension of $\langle L_1, L_2, p, f\rangle_{2d}$ is $|\Gamma_1| - \binom{d+2}{2} + 3 = 1$, as desired. □

Example 3.2.18. We follow the construction in Proposition 3.2.16 and Lemma 3.2.17 in the case $d = 5$. Then $\Gamma = \mathcal{V}(L_1, L_2)$ consists of the 25 points $(i : j : 1) \in \mathbb{P}^2$ where $i, j = 0, \ldots, 4$, see Figure 3.3. The six points on the two lines $x + y = 2$ and $x + y = 6$ are the points of Γ_2. Indeed, the point evaluations at the 19 points of $\Gamma_1 = \Gamma \setminus \Gamma_2$ on forms of degree 5 satisfy a unique linear relation, namely

$$\begin{pmatrix} -1 & 3 & & -5 & 3 \\ 3 & -16 & 18 & & -5 \\ & 18 & -36 & 18 & \\ -5 & & 18 & -16 & 3 \\ 3 & -5 & & 3 & -1 \end{pmatrix},$$

where the (i,j)-th entry of this matrix is the coefficient of the point evaluation at $(5-i : j-1 : 1)$ in the linear relation, i.e. optically, it is the coefficient corresponding to the points as seen in Figure 3.3. The 21×21 moment matrix can be exactly computed in Mathematica, e.g. using the code given in the appendix, cf. Computation B.3.

The fact that the conic vanishes in additional integer points on the $d \times d$ grid defined by the products of linear forms L_1 and L_2 in Proposition 3.2.16 destroys the extremality of the constructed linear functional because we get additional syzygies among the generators of the corresponding Gorenstein ideal. In order to deal with this problem, we will make a perturbation to our point arrangement. First, we want to observe the following fact, which motivates why we should be able to get around this obsatcle by perturbation:

Remark 3.2.19. Consider the setup in Propsition 3.2.16 and suppose the conic C vanishes in additional integer points in the $d \times d$ integer grid $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Pick such a point $P \in \Gamma$. Then every form of degree d vanishing on Γ_1 will also vanish at P because L_1, L_2 and the third form p, which is the product of lines and the conic, form a basis of this space. By the Theorem of Cayley-Bacharach applied to $\Gamma = \Gamma_1' \cup \Gamma_2'$ for $\Gamma_1' = \Gamma_1 \cup \{P\}$ and $\Gamma_2' = \Gamma_2 \setminus \{P\}$, there is a unique linear relation among the point evaluations at points of Γ_2' on forms of degree $d-3$. In particular, the coefficient of the point evaluation at P in the unique linear relation among point

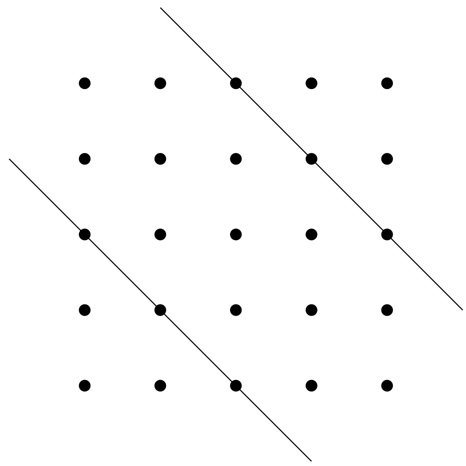

Figure 3.3.: The construction of an extreme ray of $\Sigma_{2d}^{\vee}$ of corank 4 for $d = 5$.

evaluations at Γ_2 on forms of degree $d-3$ is zero. The converse is also true by Cayley-Bacharach, so we have:
The conic C vanishes in a point in $P \in \Gamma_2$ if and only if the coefficient of the point evaluation at P in the unique linear relation among $\{\mathrm{ev}_v \in \mathbb{R}[\underline{x}]_{d-3}^* : v \in \Gamma_2\}$ is zero. This seems to be a non-generic property and we will indeed show that we can make all coefficients in the linear relation among these point evaluations non-zero by a careful perturbation of L_1 and L_2.

We now drop the assumptions on d made in 3.2.15 and prove Lemma 3.2.17 for all $d \geq 4$:

Lemma 3.2.20. *For any $d \geq 4$, there is an extreme ray $\mathbb{R}_+\ell$ of $\Sigma_{2d}^{\vee}$ such that B_ℓ has corank 4. The Hilbert function of the ideal $I(\ell)$ is* $\mathrm{Hilb}(I(\ell), j) = \binom{j+2}{2}$ *for $0 \leq j < d$ and* $\mathrm{Hilb}(I(\ell), d) = \binom{d+2}{2} - 4$.

Proof. We start as above with the products of linear forms $L_1 = \prod_{j=0}^{d-1}(x - jz)$ and $L_2 = \prod_{j=0}^{d-1}(y - jz)$ and denote by Γ the complete intersection $\mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Split Γ into

$$\Gamma_2 = \{(x:y:1) : x + y = 2\} \cup \bigcup_{j=1}^{d-4} \{(x:y:1) : x + y = d + j\}$$

and $\Gamma_1 = \Gamma \setminus \Gamma_2$. Then the space of forms of degree d vanishing on Γ_1 has dimension 3. Let p be the uniquely determined form of degree d such that L_1, L_2, p is a basis of this space. By Cayley-Bacharach, we know that there is a unique relation among the point evaluations $\{\mathrm{ev}_x \in \mathbb{R}[\underline{x}]_{d-3}^* : x \in \Gamma_2\}$, say

$$\sum_{x \in \Gamma_2} w_x \,\mathrm{ev}_x = 0.$$

Note that by the preceding Remark 3.2.19, the coefficient of $\mathrm{ev}_{(1:1:1)}$ is non-zero. Set $\Gamma_1' = \Gamma_1 \cup \{(1:1:1)\}$ and $\Gamma_2' = \Gamma_2' \setminus \{(1:1:1)\}$. Then the point evaluations $\{\mathrm{ev}_x \in \mathbb{R}[\underline{x}]_{d-3}^* : x \in \Gamma_2'\}$ are linearly independent and span a hyperplane H in $\mathbb{R}[\underline{x}]_{d-3}^*$. So there is a unique form q of degree $d-3$ vanishing on Γ_2', namely the one vanishing on all of Γ_2, i.e. $q = (x+y-2z)\prod_{j=1}^{d-4}(x+y-(d+j)z)$.

We will now perturb the point $(1:1:1)$ along the line $x+y=2$, see Figure 3.4 for a visualisation in case $d=9$: Let $v_t := (t, 2-t)$. Of course, $q(v_t) = 0$ for every $t \in \mathbb{R}$, i.e. the point evaluation $\mathrm{ev}_{v_t} \in \mathbb{R}[\underline{x}]_{d-3}^*$ lies in the hyperplane spanned by the point evaluations at Γ_2'; write

$$\mathrm{ev}_{v_t} = \sum_{x\in\Gamma_2'} \alpha_x(t)\,\mathrm{ev}_x,$$

where the coefficients $\alpha_x(t)$ are rational functions of the parameter t.

Suppose there is a point $P \in \Gamma_2'$ such that $\alpha_P(t) = 0$ for all $t \in \mathbb{R}$. Then $\mathrm{ev}_{v_t} \in \mathrm{span}(\mathrm{ev}_v : v \in \Gamma_2' \smallsetminus \{P\})$. Dually this means, that there is a form f_P of degree $d-3$, uniquely determined modulo q, such that $f_P(P) = 1$, $f_P(v) = 0$ for all $v \in \Gamma_2' \smallsetminus \{P\}$ and consequently $f_P(v_t) = 0$ for all $t \in \mathbb{R}$. Such a form cannot exist: Since v_t ranges over the whole line defined by $x+y=2$, the form f_P vanishes identically on this line; so we can factor it out. Furthermore, f_P vanishes identically on every diagonal defining Γ_2 to the left of P, i.e. $f_P(x, j-x) = 0$ for all $d < j < P_1 + P_2$ because it has too many zeros on these lines from Γ_2'.

Now $\Gamma_2' \cap \{x+y = P_1+P_2\}$ consists of $2d-1-P_1-P_2$ many points. We have already established P_1+P_2-d linear factors of f_P, so the remaining cofactor has degree $2d-P_1-P_2-3$. Therefore, f_P vanishes identically on this line, which is a contradiction because it contains P.

So there is an $\varepsilon > 0$ such that for all $t \in (1-\varepsilon, 1)$, all coefficients of the linear relation

$$\mathrm{ev}_{v_t} = \sum_{v\in\Gamma_2'} \alpha_v(t)\,\mathrm{ev}_v$$

are non-zero. Pick a t_0 in this interval and consider the totally real complete intersection $\Gamma = \mathcal{V}(L_1') \cap \mathcal{V}(L_2')$ for $L_1 = x(x-t_0z)\prod_{j=2}^{d-1}(x-jz)$ and $L_2 = y(y-(2-t_0)z)\prod_{j=2}^{d-1}(x-jz)$ and argue as above: we split the points into Γ_1 and Γ_2, where Γ_2 is the same union of diagonals as above. The Theorem of Cayley-Bacharach then implies the existence of a form of degree d vanishing on Γ_1 and not identically on Γ. In fact, by Remark 3.2.19, this form does not vanish in any point of Γ_2, so we can now complete the proof as in Lemma 3.2.17. □

Remark 3.2.21. In particular, the union of all extreme rays of $\Sigma_{2d}^\vee$ need not be closed, e.g. for $d=9$, extremality fails in our original construction but a perturbation gives an extreme ray.

Theorem 3.2.22. *For any $d \geq 4$, the Zariski closure of the set of extreme rays of $\Sigma_{2d}^\vee$ is the variety of moment matrices of corank at least 4. It is irreducible and has codimension 10.*

Proof. We have shown in the proof of Lemma 3.2.7 that the quasi-projective variety $\mathrm{Gor}(T)$ of all Gorenstein ideals with Hilbert function $T(j) = \binom{j+2}{2}$ for $0 \leq j < d$ and $T(d) = \binom{d+2}{2} - 4$ is dense in X_{-4}. It is also smooth, cf. Theorem 3.2.8 or Iarrobino-Kanev [24], Theorem 4.21. We have shown in Lemma 3.2.17 that there is an extreme ray $\mathbb{R}_+\ell_0$ of $\Sigma_{2d}^\vee$ with $I(\ell_0) \in \mathrm{Gor}(T)$. We will now show that every linear functional in an open neighbourhood of ℓ_0 in $\mathrm{Gor}(T)$ spans an extreme ray of $\Sigma_{2d}^\vee$. Since $I(\ell) \in \mathrm{Gor}(T)$ implies that the corank of the middle Catalecticant B_ℓ is 4, there is an open neighbourhood of ℓ_0 such that B_ℓ is positive semi-definite for all ℓ in this neighbourhood, because the eigenvalues of a symmetric matrix depend continuously

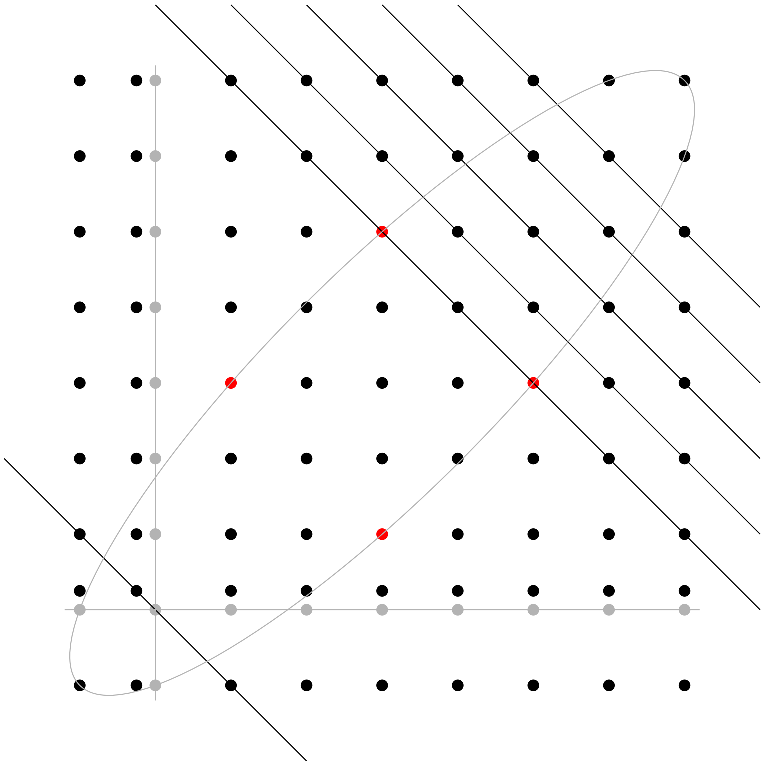

Figure 3.4.: A picture of the perturbation for general $d \geq 4$ shown for the first critical case $d = 9$: The black points and the four red points are the perturbed point configuration for which our construction works. The four red points are the additional points through which the grey ellipse goes.

on its entries. Therefore, a linear functional ℓ in this neighbourhood spans an extreme ray of $\Sigma_{2d}^{\vee}$ if and only if $I(\ell)_d$ generates a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$, i.e. $\langle I(\ell)_d \rangle_{2d} = I(\ell)_{2d}$. By Gauss' algorithm (column echelon form), we can write a basis (b_1, b_2, b_3, b_4) of the kernel of B_ℓ in terms of rational functions in the entries of B_ℓ. We consider the linear map

$$\mathbb{R}[\underline{x}]_d^4 \to \mathbb{R}[\underline{x}]_{2d}, (f_1, f_2, f_3, f_4) \mapsto f_1 b_1 + f_2 b_2 + f_3 b_3 + f_4 b_4.$$

The rank of this map is at least $\binom{2d+2}{2} - 1$ (i.e. the image is a hyperplane) because $\ell \in \mathrm{Gor}(T)$. The image is a hyperplane for $\ell = \ell_0$. So the same is true for every ℓ in a neighbourhood of ℓ_0 in $\mathrm{Gor}(T)$, which shows that these ℓ are extreme rays of $\Sigma_{2d}^{\vee}$. □

Remark 3.2.23. In the proof of the above Theorem, we see that if T is a Hilbert function occurring for a Gorenstein ideal corresponding to an extreme ray of $\Sigma_{2d}^{\vee}$, then there is an open subset of extreme rays in a connected component of $\mathrm{Gor}(T)(\mathbb{R})$ because $\mathrm{Gor}(T)$ is smooth. As we remarked above, it might not be the entire connected component.

Our construction of an extreme ray of maximal rank also gives a base-point free special linear system with a totally real representative on a smooth curve of degree $d \geq 4$, which might be interesting in itself.

Proposition 3.2.24. *Let $d \geq 4$. There is a smooth real curve $X \subset \mathbb{P}^2$ of degree d and an effective divisor D of degree $g = \binom{d-1}{2}$ supported on $X(\mathbb{R})$ such that $|D|$ has dimension* 1 *and is base-point free.*

Proof. Start with a complete intersection $\mathcal{V}(L_1) \cap \mathcal{V}(L_2)$ of linear forms and a choice of $\binom{d-1}{2}$ points $\Gamma_2 \subset \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$ such that there is a unique curve of degree $d-3$ passing through these points. Moreover, assume that all coefficients in the linear relation among the point evaluations $\{\mathrm{ev}_v \in \mathbb{R}[\underline{x}]^*_{d-3} : v \in \Gamma_2\}$ are non-zero. This situation is established in the proof of Lemma 3.2.20. By Bertini's Theorem [4], Theorem 6.2.11 or [25], Théorème 6.6.2, there is a smooth curve $\mathcal{V}(f)$ of degree d passing through Γ_2 such that f is a small perturbation of L_1; more precisely, we want $\Gamma = \mathcal{V}(f) \cap \mathcal{V}(L_2)$ to be a totally real transversal intersection. Then the complete linear system $|\Gamma_2| \subset \mathrm{Div}(\mathcal{V}(f))$ is cut out by forms of degree d, i.e.

$$|\Gamma_2| = \{C.\mathcal{V}(f) - (\Gamma - \Gamma_2) \geq 0 : C \subset \mathbb{P}^2 \text{ of degree } d\},$$

cf. Eisenbud-Green-Harris [16], Corollary 5 (to Brill-Noether's Restsatz). We have argued in Remark 3.2.19 that this linear system is base-point free. We compute its dimension with the help of the Cayley-Bacharach Theorem, more precisely [16], Corollary 6:

$$1 = |\Gamma_2| - (\ell((d-3)H) - \ell((d-3)H - \Gamma_2)) = g - (g - \ell((d-3)H - \Gamma_2)),$$

which implies

$$\ell(\Gamma_2) = \deg(\Gamma_2) + 1 - g + \ell((d-3)H - \Gamma_2) = 2.$$

□

Remark 3.2.25. Conversely, given such a linear system on a smooth curve $X \subset \mathbb{P}^2$, we can apply the construction in the proof of Lemma 3.2.17 to construct an extreme ray of $\Sigma_{2d}^\vee$ of maximal rank, at least if there is a totally real transveral intersection $C \cap X$ with $C.X - D \geq 0$. The fact, that the linear system has dimension 1 gives the unique linear relation among the point evaluations at $C.X - D$ on forms of degree d. Extremality then follows from the fact that $|D|$ is base-point free by the count of dimensions as in the proof of Lemma 3.2.17.

3.2.3. The case $d = 5$

For $d = 3$, a complete characterisation of extreme rays of $\Sigma_6^\vee$ was given by Blekherman in [7]. It led to a complete description of the algebraic boundary of the sums of squares cone Σ_6 by Blekherman, Hauenstein, Ottem, Ranestad and Sturmfels, cf. [8]. For $d = 4$, there are only two possible ranks for extreme rays of $\Sigma_8^\vee$, namely 10 and 11; in particular, we know how to construct one of each rank. It is possible to prove, similarly to the cases below, that both these ranks give rise to irreducible components of the algebraic boundary of Σ_8 by projective duality.

So the first interesting case from this point of view is $d = 5$: In fact, we can construct an extreme ray of $\Sigma_{10}^\vee$ of every rank in the interval $\{13, \ldots, 17\}$ between the lower and upper bound using the Cayley-Bacharach Theorem. Moreover, using the results of Section 2.4, we can prove by projective duality that there is an irreducible component of the algebraic boundary of Σ_{10} for every one of these ranks; in particular, $\partial_a \Sigma_{10}$ has at least 6 irreducible components.

The construction given in the preceding section for extreme rays of maximal rank $\binom{d+2}{2} - 4$ leads to an extreme ray $\mathbb{R}_+ \ell$ of $\Sigma_{10}^\vee$ such that the Hilbert function of the corresponding Gorenstein ideal $I(\ell)$ is

$$T_{17} = (1, 3, 6, 10, 15, 17, 15, 10, 6, 3, 1).$$

By Theorem 3.2.22, the Zariski closure of the set of extreme rays of $\Sigma_{10}^\vee$ is $\mathrm{cl}(\mathrm{Gor}(T_{17}))$, a unirational variety of codimension 10 in $\mathbb{P}^{65}$. So Theorem 2.4.4 implies that its dual variety is an irreducible component of the algebraic boundary of Σ_{10}.

We now work our way up beginning with the lowest rank 13, following the construction in Blekherman [6]:

Proposition 3.2.26. *There is an extreme ray $\mathbb{R}_+ \ell$ of $\Sigma_{10}^\vee$ of rank* 13*. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{13} = (1, 3, 6, 9, 12, 13, 12, 9, 6, 3, 1)$$

and the variety dual to $\mathrm{cl}(\mathrm{Gor}(T_{13}))$ *is an irreducible component of the algebraic boundary of* Σ_{10}.

Proof. Let $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$ and $L_2 = y(y-z)(y-2z)$ and $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. There is a unique linear relation among $\{\mathrm{ev}_v \in \mathbb{R}[\underline{x}]_5^* : v \in \Gamma\}$, say $\sum_{v\in\Gamma} u_v \, \mathrm{ev}_v = 0$, and all coefficients in this relation are non-zero. The linear functional

$$\ell = \sum_{v \in \Gamma \setminus \{P\}} ev_v - \frac{u_P^2}{\sum_{v\in\Gamma\setminus\{P\}} u_v^2} \mathrm{ev}_P$$

is positive semi-definite of rank 13 for any $P \in \Gamma$ by the Cauchy-Schwarz inequality, cf. proof of Lemma 3.2.17. By a Hilbert function computation using `Macaulay2` [20], we verify, that the degree 5 part of the corresponding Gorenstein ideal $I(\ell)$ generates a hyperplane in degree 10. To prove, that the dual variety to $\mathrm{Gor}(T_{13})$ is an irreducible component of $\partial_a \Sigma_{10}$, note that the condition in Theorem 2.4.8 is equivalent to

$$(T_\ell \, \mathrm{Gor}(T_{13}))^\perp = (I(\ell)_5)^2$$

because the face of Σ_{10} supported by ℓ is the set of sums of squares of polynomials in $I(\ell)_5$, which spans the vector space $(I(\ell)_5)^2$. By the description of the tangent space to $\mathrm{Gor}(T_{13})$ at ℓ (cf. Theorem 3.2.8), this is equivalent to

$$(I(\ell)^2)_{10} = (I(\ell)_5)^2,$$

which we also check using `Macaulay2` [20]. □

Proposition 3.2.27. *There is an extreme ray $\mathbb{R}_+ \ell$ of $\Sigma_{10}^\vee$ of rank* 14*. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{14} = (1, 3, 6, 10, 13, 14, 13, 10, 6, 3, 1)$$

and the variety dual to $\mathrm{cl}(\mathrm{Gor}(T_{14}))$ *is an irreducible component of the algebraic boundary of* Σ_{10}.

Proof. In this case, take $L_1 = x(x-z)(x-2z)(x-3z)$ and $L_2 = y(y-z)(y-2z)(y-3z)$ and set $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. There is a unique linear relation among $\{\mathrm{ev}_v \in \mathbb{R}[\underline{x}]_5^* : v \in \Gamma\}$, say $\sum_{v\in\Gamma} u_v \,\mathrm{ev}_v = 0$, and all its coefficients are non-zero. As above, the linear functional

$$\ell = \sum_{v\in\Gamma\smallsetminus\{P\}} ev_v - \frac{u_P^2}{\sum_{v\in\Gamma\smallsetminus\{P\}} u_v^2}\,\mathrm{ev}_P$$

is positive semi-definite of rank 14 for any $P \in \Gamma$. Again, using `Macaulay2` [20], we verify, that the degree 5 part of the corresponding Gorenstein ideal $I(\ell)$ generates a hyperplane in degree 10 and that

$$(I(\ell)^2)_{10} = (I(\ell)_5)^2.$$

□

Proposition 3.2.28. *There is an extreme ray $\mathbb{R}_+\ell$ of $\Sigma_{10}^\vee$ of rank 15. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{15} = (1,3,6,10,14,15,14,10,6,3,1)$$

and the variety dual to $\mathrm{cl}(\mathrm{Gor}(T_{15}))$ *is an irreducible component of the algebraic boundary of* Σ_{10}.

Proof. In this case, we start with a complete intersection of a quartic and a quintic, $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$, $L_2 = y(y-z)(y-2z)(y-3z)$ and $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Choose $\Gamma_2 = \{(0:2:1),(1:1:1),(2:0:1)\}$ and set $\Gamma_1 = \Gamma \smallsetminus \Gamma_2$. By Cayley-Bacharach gives a unique linear relation among the 17 points of Γ_1 with non-zero coefficients. Using `Macaulay2` [20], we complete the proof as above. □

Proposition 3.2.29. *There is an extreme ray $\mathbb{R}_+\ell$ of $\Sigma_{10}^\vee$ of rank 16. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{16} = (1,3,6,10,14,16,14,10,6,3,1)$$

and the variety dual to $\mathrm{cl}(\mathrm{Gor}(T_{16}))$ *is an irreducible component of the algebraic boundary of* Σ_{10}.

Proof. Again, choose $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$, $L_2 = y(y-z)(y-2z)(y-3z)$ and $\Gamma = \mathcal{V}(L_1)\cap\mathcal{V}(L_2)$. This time, $\Gamma_2 = \{(0:1:1),(1:0:1)\}$ and $\Gamma_1 = \Gamma \smallsetminus \Gamma_2$ do the job: Cayley-Bacharach gives a unique linear relation among the 18 points of Γ_1 with non-zero coefficients. Using `Macaulay2` [20], we complete the proof as above. □

In summary, we proved the following statement.

Theorem 3.2.30. *For every $r \in \{13,\dots,17\}$, there is an extreme ray $\mathbb{R}_+\ell_r$ of $\Sigma_{10}^\vee$ such that the rank of the moment matrix B_{ℓ_r} is r. The Hilbert function T_r of $I(\ell_r)$ is*

- $T_{13} = (1,3,6,9,12,13,12,9,6,3,1)$
- $T_{14} = (1,3,6,10,13,14,13,10,6,3,1)$

- $T_{15} = (1,3,6,10,14,15,14,10,6,3,1)$
- $T_{16} = (1,3,6,10,14,16,14,10,6,3,1)$
- $T_{17} = (1,3,6,10,15,17,15,10,6,3,1)$

The dual varieties to $\mathrm{Gor}(T_r)$ *are irreducible components of the algebraic boundary of the sums of squares cone* Σ_{10} *for all* $r \in \{13,\dots,17\}$.

For general $d > 5$, our constructive method using the Cayley-Bacharach Theorem cannot construct an extreme ray of every rank in the interval $\{3d-2,\dots,\binom{d+2}{2}\}$ given by the lower and upper bound. The first failure occurs for $d = 6$ and rank $17 \in \{16,\dots,24\}$. It is at this point still unclear if there is an extreme ray of $\Sigma_{12}^{\vee}$ of rank 17.

Remark 3.2.31. Our construction starts with a totally real intersection of two curves X_1 and X_2 with $\deg(X_1)+\deg(X_2) \geq d+3$; we then need 19 intersection points such that the corresponding point evaluations on forms of degree 6 satisfy a unique linear relation in which all coefficients are non-zero. This configuration would lead to a positive linear functional such that the moment matrix has the desired rank 17 (of course we would still need to prove extremality). We will see that this is not possible:

The following tuples are permissible choices for the degrees of the curves $(3,6)$, $(4,5)$, $(4,6)$, $(5,5)$, $(5,6)$ and $(6,6)$. For $(\deg(X_1),\deg(X_2)) = (3,6)$, the transversal intersection has only 18 points. In the case $(4,5)$, there is a unique linear relation among point evaluations at the 20 intersection points such that all coefficients are non-zero; in particular, whatever point we remove, the remaining 19 point evaluations are linearly independent on forms of degree 6. In the cases $(4,6)$, $(5,5)$ and $(5,6)$, we cannot have the desired number of points on a curve of dual degree $s-d$: For example, in order to apply the duality of the Cayley-Bacharach Theorem to the 24 intersection points in the case $(4,6)$, we would need to have 5 of the intersection points on a line, which intersects the quartic in only 4 points. The last case $(6,6)$ is more subtle: We would like to find exactly 17 intersection points on a cubic, which is impossible, because there is a unique linear relation among the corresponding point evaluations on forms of degree 6 on the complete intersection of a cubic and a sextic, cf. Eisenbud-Green-Harris [16], Theorem CB4.

A. Orbits of Real Algebraic Groups

An orbitope is by definition the convex hull of the orbit of a vector under the linear action of a compact real algebraic group, cf. Sanyal-Sottile-Sturmfels [37]. We will show that such an orbit is the set of real points of a smooth algebraic variety; so orbitopes are a class of examples for the setup considered in section 2.3, i.e. the convex hull of the set of real points, assumed to be compact, of an affine real algebraic variety (with no real singularities).

Theorem A.1. *Let G be a linear real algebraic group with the property that $G(\mathbb{R})$ is compact in the euclidean topology and let $\rho\colon G \to \mathrm{GL}(V)$ be a rational representation of G. For every $v \in V$, the orbit $G.v = \{gv\colon g \in G(\mathbb{C})\} \subset V \otimes \mathbb{C}$ is an affine variety defined over $\mathbb{R}$. The real points of $G.v$ are the images of real points of G, i.e.*

$$(G.v)(\mathbb{R}) = G(\mathbb{R})v = \{gv\colon g \in G(\mathbb{R})\}.$$

A *linear algebraic group* is an algebraic group with the property that the underlying algebraic variety is affine. A *rational representation* of a real algebraic group G is a morphism $G \to \mathrm{GL}(V)$ of real algebraic groups, where V is a finite-dimensional real vector space. It induces a linear action of this group on the complexification $V \otimes \mathbb{C}$ of V, considered as a real algebraic variety with coordinate ring $\mathrm{Sym}(V^*)$, namely the restriction of the natural action of $\mathrm{GL}(V)$ on $V \otimes \mathbb{C}$.

The first part of the assertion of Theorem A.1 is a special case of a theorem of Birkes, cf. [5], Corollary 5.3, because every linear real algebraic group with $G(\mathbb{R})$ compact is reductive, cf. Springer [41], Lemma 2.1.4, Proposition 2.2.4 and Knapp [26], Corollary IV.4.7.

We will now prove the second part using Galois cohomology, following Serre [38]. We will need non-abelian Galois cohomology groups for the Galois group $\mathfrak{g}$ of $\mathbb{C}/\mathbb{R}$, i.e. $\mathfrak{g} \cong \mathbb{Z}/2\mathbb{Z}$. In general, when dealing with profinite groups, one needs to worry about continuity. In our special case $\mathfrak{g} = \mathbb{Z}/2\mathbb{Z}$, this is not an issue and we will omit the continuity assumptions.

A $\mathfrak{g}$-set is a set E with a left action of $\mathfrak{g}$, i.e. for all $s, t \in \mathfrak{g}$ and $x \in E$ the equality $(st)x = s(tx)$ holds and $ex = x$, where e denotes the unit element of $\mathfrak{g}$. A $\mathfrak{g}$-group A is a group with a left action of $\mathfrak{g}$ such that $s(xy) = (sx)(sy)$ for all $s \in \mathfrak{g}$ and $x, y \in A$.

For a $\mathfrak{g}$-set E, we define the 0-th cohomology group, denoted by $H^0(\mathbb{R}, E)$, to be the fixed points of E under the action of $\mathfrak{g}$. For a $\mathfrak{g}$-group A, the first cohomology group, denoted by $H^1(\mathbb{R}, A)$, is the following quotient: A cocycle is a map $\varphi\colon \mathfrak{g} \to A$ with the property that $\varphi(st) = \varphi(s)s\varphi(t)$, i.e. a group homomorphism up to a twist by the action of $\mathfrak{g}$ on A. We call two cocycles φ and φ' equivalent if there is a $b \in A$ such that for all $s \in \mathfrak{g}$ we have $\varphi'(s) = b^{-1}\varphi(s)sb$, i.e. equality up to a twisted conjugation by elements of A. The first cohomology group is the set of residue classes of cocycles modulo the given equivalence relation.

In general, the first cohomology group of a non-abelian $\mathfrak{g}$-group is not a group, merely a *pointed set*, i.e. a set X together with a point $p \in X$, which is the residue class of the constant map $\varphi\colon \mathfrak{g} \to A$, $\varphi(s) = 1$. It makes sense to talk of exactness for a sequence of pointed sets: If

$\delta: (X, p) \to (Y, q)$ is a map of pointed sets, the kernel of δ is $\{x \in X: \delta(x) = q\}$. In this sense, we get a cohomology sequence, cf. Serre [38], Chapter I.5: If

$$0 \to A \to B \to B/A \to 0$$

is an exact sequence of $\mathfrak{g}$-sets where A and B are $\mathfrak{g}$-groups, then the cohomology sequence

$$0 \to H^0(\mathbb{R}, A) \to H^0(\mathbb{R}, B) \to H^0(\mathbb{R}, B/A) \to H^1(\mathbb{R}, A) \to H^1(\mathbb{R}, B)$$

is exact (as a sequence of pointed sets), cf. Serre [38], Chapter I, Proposition 36.

In our setup, the first cohomology group can be identified:

Theorem A.2 (Serre [38], Chapter III.4.5, Theorem 6). *Let G be a linear real algebraic group with $G(\mathbb{R})$ compact. Then the first cohomology group $H^1(\mathbb{R}, G)$ is (as a pointed set) isomorphic to the set of 2-torsion points of $G(\mathbb{R})$ modulo $G(\mathbb{R})$-conjugation together with the residue class of the unit element.*

Serre's theorem says that $H^1(\mathbb{R}, G)$ is isomorphic to $H^1(\mathbb{R}, G(\mathbb{R}))$. This is the set of 2-torsion points of $G(\mathbb{R})$ modulo conjugation by definition because $\mathfrak{g}$ acts trivially on $G(\mathbb{R})$.

Corollary A.3. *Let G be a linear real algebraic group with $G(\mathbb{R})$ compact and let H be a closed real subgroup. The real points of the $\mathfrak{g}$-set G/H are the image of real points of G, i.e. the residue class map $G \to G/H$ restricted to real points is surjective.*

Proof. From the short exact sequence of $\mathfrak{g}$-sets

$$0 \to H \to G \to G/H \to 0$$

we get the exact cohomology sequence (of pointed sets)

$$0 \to H^0(\mathbb{R}, H) \to H^0(\mathbb{R}, G) \to H^0(\mathbb{R}, G/H) \to H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G).$$

The kernel of the last map in this sequence is trivial by Theorem A.2 (note that this means that the only element of $H^1(\mathbb{R}, H)$ that gets mapped to 1 is 1, which is trivial - it does not imply injectivity of the map). By exactness of the sequence, surjectivity of the residue class map $H^0(\mathbb{R}, G) \to H^0(\mathbb{R}, G/H)$ follows. $\square$

We are ready to give a proof of Theorem A.1.

Proof of Theorem A.1. Given a rational representation $G \to \mathrm{GL}(V)$ of the linear real algebraic group G and $v \in V$, we let H be the stabiliser of v in G. By continuity of the left action, this is a closed subgroup of G, defined over $\mathbb{R}$. Therefore, the orbit $G.v$ is isomorphic as a real algebraic variety to the quotient G/H, cf. Borel [11], Chapter II, Theorem 6.8. The statement now follows from Corollary A.3. $\square$

B. Programme code

Computation B.1 (Example 2.3.3 and Remark 2.3.18). We are given the rational space curve $X = \{(s^4 : s^3t : s^2t^2 : t^4) : (s : t) \in \mathbb{P}^2\} \subset \mathbb{P}^3$ and we want to compute the first special secant variety $S_{[1]}X$ and the variety $X^{[1]}$. The first computation is explained in Example 2.3.3. The following Macaulay2 [20] code will do the computation.

```
R = QQ[s1,t1,s2,t2,u,v,w,x,y,z];
M = matrix{
     {4*s1^3,3*s1^2*t1,2*s1*t1^2,0},
     {0,s1^3,2*s1^2*t1,4*t1^3},
     {4*s2^3,3*s2^2*t2,2*s2*t2^2,0},
     {0,s2^3,2*s2^2*t2,4*t2^3}};
N1 = matrix{{4*s1^3,3*s1^2*t1,2*s1*t1^2,0},
     {0,s1^3,2*s1^2*t1,4*t1^3}};
N2 = matrix{{4*s2^3,3*s2^2*t2,2*s2*t2^2,0},
     {0,s2^3,2*s2^2*t2,4*t2^3}};
Sing = intersect(saturate(minors(2,N1),ideal(s1,t1)),
                 saturate(minors(2,N2),ideal(s2,t2)));
fooGSec1 = saturate(ideal(det(M),w-u*s1^4-v*s2^4,x-u*s1^3*t1-v*s2^3*t2,
     y-u*s1^2*t1^2-v*s2^2*t2^2,z-u*t1^4-v*t2^4),
     intersect(Sing,ideal(s1*t2-s2*t1)));
GSec1 = eliminate(fooGSec1,{u,v,s1,t1,s2,t2})
```

The first saturation in the computation of Sing is necessary to get rid of the irrelevant ideal (i.e. to make sure that $(s1 : t1)$ and $(s2 : t2)$ are points of $\mathbb{P}^1$). The second saturation in the computation of fooGSec1 is necessary to make sure that $(s1 : t1) \neq (s2 : t2)$.
In order to compute $X^{[1]}$, recall the definition

$$X^{[1]} = \mathrm{cl}\left(\{[H] \subset (\mathbb{P}^3)^* : \mathrm{T}_x X, \mathrm{T}_y X \subset H \text{ for linearly independent points } x, y \in X_{\mathrm{reg}}\}\right).$$

The curve is given as $\mathcal{V}(f, g)$, where $f = x^2 - wy$ and $g = y^2 - wz$. We are looking for linear functions $(W : X : Y : Z) \in (\mathbb{P}^3)^*$ with $(W : X : Y : Z) = \lambda_1 f'(p) + \lambda_2 g'(p) = \mu_1 f'(q) + \mu_2 g'(q)$ with $p, q \in X$, $p \neq q$, because this means, that $(W : X : Y : Z)$ is tangent to X at p and q. During the computation, we will have to saturate, to make sure that p and q are non-singular points of X and $p \neq q$. The following Macaulay2 [20] code will do this computation:

```
R = QQ[s1,s2,t1,t2,W,X,Y,Z];
fooM1 = matrix{{-s1^2*t1^2,-t1^4},
     {2*s1^3*t1,0},
```

```
     {-s1^4,2*s1^2*t1^2},
     {0,-s1^4}};
M1 = matrix{{-s1^2*t1^2, -t1^4,W},
     {2*s1^3*t1,0,X},
     {-s1^4,2*s1^2*t1^2,Y},
     {0,-s1^4,Z}};
fooM2 = matrix{{-s2^2*t2^2,-t2^4},
     {2*s2^3*t2,0},
     {-s2^4,2*s2^2*t2^2}
     {0,-s2^4}};
M2 = matrix{{-s2^2*t2^2,-t2^4,W},
     {2*s2^3*t2,0,X},
     {-s2^4,2*s2^2*t2^2,Y},
     {0,-s2^4,Z}};
Minors = saturate(minors(3,M1),minors(2,fooM1))+
         saturate(minors(3,M2),minors(2,fooM2));
fooXup = saturate(Minors,ideal(s1*t2-s2*t1));
Xup = eliminate(fooXup,{s1,s2,t1,t2})
```

Computation B.2. We want to compute the algebraic boundary of the 4-dimensional Barvinok-Novik orbitope as explained in Example 3.1.17. This is an elimination of the two variables x_2 and y_2 from the ideal of 3×3 minors of the following matrix:

$$\begin{pmatrix} 1 & w+ix & x_2+iy_2 & y+iz \\ w-ix & 1 & w+ix & x_2+iy_2 \\ x_2-iy_2 & w-ix & 1 & w+ix \\ y-iz & x_2-iy_2 & w-ix & 1 \end{pmatrix}$$

The following `Macaulay2` [20] code will do the job. Since we want to do exact computations, we need to avoid dealing with the complex numbers. We introduce `a` in the following code to substitute a root of -1, which is the only complex number, we need.

```
R = QQ[a,w,x,y,z,x2,y2]; S = R/(a^2+1);
M = matrix{{1, w+a*x, x2+a*y2, y+a*z},
     {w-a*x, 1, w+a*x, x2+a*y2},
     {x2-a*y2, w-a*x, 1, w+a*x},
     {y-a*z, x2-a*y2, w-a*x, 1}};
fooSec = minors(3,M);
Sec = eliminate(lift(fooSec,R),{lift(x2,R),lift(y2,R)});
(mingens Sec)_(0,1)
```

Computation B.3 (Example 3.2.18). We compute the 21×21 Hankel matrix of an extreme ray of $\Sigma_{10}^{\vee}$ using Mathematica. We set up the monomial basis `m` and the totally real complete intersection of 25 points, where `q1` $= L_1$ and `q2` $= L_2$. The two lines using the `Select`-command remove the points on the two diagonals $x + y = 2$ and $x + y = 6$, so $\Gamma_1 =$ `Peval`. With the

Drop-command, we remove one of the points from the list. The matrix H is the general moment matrix and Q is the moment matrix of the linear functional $\sum_{v\in\Gamma_1\setminus\texttt{Peval[[-1]]}} \mathrm{ev}_v$ and Qp the moment matrix of the point evaluation at Peval[[-1]]. So Hankel is the moment matrix of the extreme ray that we construct in Example 3.2.18.

```
d = 5;
m = MonomialList[(x+y+z)^d];
q1 = x (x - z) (x - 2 z) (x - 3 z) (x - 4 z);
q2 = y (y - z) (y - 2 z) (y - 3 z) (y - 4 z);
Pevalall = Solve[{q1 ==  0, q2 ==  0, z ==  1}, {x, y, z}];
Pevalfoo = Select[Pevalall, ({y + x - 2 z} /. #) != {0} &];
Peval = Select[Pevalfoo, ({y + x - 6 z} /. #) != {0} &];
Peval0 = Drop[Peval, -1];

H = Transpose[{m}].{m};
P = Peval0;
Q = Sum[H /. P[[i]], {i, 1, Length[P]}];
Qp = H /. Peval[[-1]];
l = Norm[CB]^2;
Hankel = Q - 1/l (rel[[1]][[-1]])^2 Qp;
```

Zusammenfassung auf Deutsch

Der algebraische Rand einer konvexen Menge ist die kleinste algebraische Varietät, die ihren euklidischen Rand enthält. Für ein Polytop ist er das Hyperebenen-Arrangement, welches durch die Facetten definiert wird und das in der Theorie von Polytopen und der linearen Programmierung wohl studiert ist. Der algebraische Rand einer konvexen Menge, welche kein Polytop ist, wurde kürzlich in anderen Spezialfällen studiert, vor allem im Fall der konvexen Hülle einer reellen Varietät. Diese Klasse umfasst bekannte Familien wie die Momentenmatrizen von Wahrscheinlichkeitsverteilungen und die symmetrischen Orbitope. Sie umfasst Hyperbolizitätskegel und Spektraeder, welche durch Anwendungen in der polynomialen Optimierung ins Rampenlicht geraten sind, nicht. In dieser Arbeit wollen wir die Klasse aller Mengen, für die der algebraische Rand eine algebraische Hyperfläche ist, studieren: konvexe semi-algebraische Mengen mit nichtleerem Inneren.

Mit den aktuell zur Verfügung stehenden Methoden ist es schwierig, Invarianten des algebraischen Randes auszurechnen, weil er oft reduzibel und von hohem Grad ist. Als ein extremes Beispiel sei der algebraische Rand eines Polytops mit k Facetten genannt: Er ist eine Hyperfläche vom Grad k mit k irreduziblen Komponenten. Beispiele von Polytopen mit einer großen Anzahl an Facetten sind als Beispiele von hoher Komplexität in der linearen Programmierung bekannt. Deshalb ist es unser Ziel, in dieser Arbeit ein gutes theoretisches und geometrisches Verständnis des algebraischen Randes zu erlangen, vor allem im Hinblick auf Begriffe aus der Konvexgeometrie. Die Theorie von Polytopen dient uns als Inspiration.

Eine theoretisch elegante Art und Weise, Facetten von Polytopen zu studieren, ist das duale Polytop: Die Facetten des Polytops entsprechen den Ecken des dualen Polytops. Die irreduzible Komponente des algebraischen Randes eines Polytops, die einer Ecke ℓ des dualen Polytops entspricht, ist die affine Hyperebene aller Punkte x mit $\ell(x) = -1$. Wir verwenden die etablierte Dualitätstheorie für kompakte konvexe Mengen und projektive algebraische Varietäten, um analoge Aussagen im allgemeinen Fall zu beweisen. Sie stellen gute Methoden zur Verfügung, die enge Verbindungen zwischen algebraischen Familien von Extremalpunkten des dualen konvexen Körpers und irreduziblen Komponenten des algebraischen Randes herstellen. Eines unserer Hauptresultate in dieser Richtung ist die folgende Verallgemeinerung der Dualität für Polytope:

Theorem (Theorem 2.4.4). *Sei $C \subset \mathbb{R}^n$ eine kompakte, konvexe, semi-algebraische Menge. Sei Z eine irreduzible Komponente des Zariski-Abschlußes der Extremalpunkte des dualen konvexen Körpers. Dann ist die duale Varietät zu Z eine irreduzible Komponente des algebraischen Randes von C.*

Im allgemeinen ist nicht jede irreduzible Komponente des algebraischen Randes von C von dieser Form. Wir studieren die Ausnahmefälle und geben eine vollständige semi-algebraische

Charakterisierung der besonderen algebraischen Familien von Extremalpunkten der dualen konvexen Menge, vgl. Theorem 2.4.8.

Wir studieren diese Phänomene an zwei konkreten Klassen von Beispielen. Zuerst wenden wir uns SO(2)-Orbitopen, welche die konvexe Hülle einer rationalen Kurve in Räumen gerader Dimension mit hoher Symmetrie sind, zu. Die Ergebnisse über deren algebraische Ränder sind von synthetischer Natur. Eine besondere Rolle spielen höhere Sekantenvarietäten an Kurven, vgl. Theoreme 3.1.14, 3.1.21 und 3.1.26.

Danach studieren wir Kegel von Quadratsummen von Formen in drei Variablen von festem Grad. In diesem Fall nehmen wir den dualen Standpunkt ein: Unsere Ergebnisse beschreiben semi-algebraische Familien von Extremalstrahlen des dualen Kegels, welcher der Kegel der positiv semi-definiten Momentenmatrizen ist, vgl. Theoreme 3.2.22 und 3.2.30. Diese Kegel kommen in verschiedenen Kontexten natürlich vor: Sie sind spezielle spektraedrische Kegel und zentral im Studium von trunkierten Momentenproblemen, welche für Relaxierungsmethoden in der polynomialen Optimierung wichtig sind. Die wesentliche Methode in diesem Abschnitt ist eine Verbindung zur Theorie von Gorenstein Idealen und dem Satz von Cayley-Bacharach für ebene Kurven.

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