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A Least-Squares Method for Computing Balance Corrections

To compute final correction masses for multispeed, multiplane balancing of rotating machinery, a least-squares computing procedure has been developed. This procedure uses plain least squares to minimize the rms residual vibration of selected points on the machinery foundation, and then uses weighted least squares to reduce the maximum residual vibration. The computations have been programmed for a digital computer.

Introduction

When a rotating machine such as a marine turbine is designed to operate over a range of speeds and loads, the unbalance forces transmitted to the foundation change with speed and load. Unless means are provided for continuously rebalancing the rotor during operating conditions, the final balance of the machine must be a compromise to achieve the minimum unbalance force transmission over the range of speeds and loads to be encountered during operation of the equipment. The least-squares method to be described in this paper is a mathematical procedure for effecting this compromise. The calculations required for this procedure have been embodied in a digital-computer program which is now routinely used in balancing marine propulsion turbine-gear assemblies.

The conventional procedure [1][1] for final balancing of a rotor at assembly or in the field is as follows:

1 Install vibration pickups at the points whose vibration is to be minimized. If smooth running of the rotor itself is the primary concern, the pickups may be located on the rotor bearing caps; if transmission of noise through the machine foundation is the primary concern, as it often is in marine propulsion machinery, the pickups may be located at the four corners of the foundation. The pickups may be either displacement, velocity, or acceleration pickups; their output must be amplified and filtered to give the amplitude and phase of the vibration at rotor frequency at each pickup location. The phase angle is measured relative to a rotating reference vector, as discussed later in the paper. The vibration amplitude and phase angle together can be conveniently expressed and plotted as a complex vector.

2 Take the *zero-rotor vibration readings*, i.e., the rotor-frequency vibration amplitudes and phase angles of all the pickups for the original condition of the rotor.

3 Take the *trial-mass data.* A rotor designed for final balancing at assembly or in the field will have two or more "balance planes" in which correction masses can be attached. To take the trial-mass data, a trial correction mass is inserted in one balance plane, and the resulting rotor-frequency vibration amplitudes and phase angles of all the pickups are measured. This procedure is repeated for each of the other balance planes, and the resulting collection of pickup vibration readings constitutes the "trial-mass data."

4 Calculate the *response coefficients*, i.e., the effect at each pickup location of a unit correction mass in each balance plane. For each balance plane, this calculation is performed by vectorially subtracting the zero-rotor vibration readings from the corresponding trial-mass readings and dividing the result by the magnitude of the trial mass used. This calculation presupposes that the effects of correction masses are linear.

5 Use the response coefficients to choose a set of *correction masses* in the various balance planes that will as nearly as possible cancel out the zero-rotor vibrations. In all but the simplest cases, this choice is greatly facilitated by using a computer solution, as discussed in the remainder of the paper.

6 Install the correction masses and take a new set of pickup vibration readings. These may be termed modified zero-rotor vibration readings. Due to inaccuracies in the data and the inexactness of the balancing process itself, it is often necessary or desirable to attempt a further improvement in balance. For this further improvement, it is not necessary to repeat the trial-mass data; the previously computed response coefficients can be used with the modified zero-rotor readings to choose a set of additional correction masses.

While the above discussion was limited for the sake of simplicity to a single rotor at a single speed and load, the same steps are applicable for the balancing of a set of synchronized rotors operating over a range of speeds and loads, which is the problem considered in this paper.

Mathematical Background

Mathematically the problem of balancing rotating machinery may be stated as follows: Given N balance planes (locations for correction masses) and $M = K \times L$ vibration readings, taken at L different pickup locations at each of K different operating conditions of speed and load, find the optimum magnitudes and angular locations of correction masses in the N balance planes to minimize the amplitudes of the M vibration readings.

When two or more shafts on the same foundation (for example, two turbine shafts coupled by one-to-one gearing) rotate at the same frequency, N is taken as the total number of balance planes in all the shafts. When two (or more) rotors (for example, a turbine shaft and an intermediate gear shaft) rotate at different frequencies, the vibration readings at the two (or more) frequencies do not affect each other, and the problem can be treated as two (or more) entirely separate balancing problems.

When M is equal to or less than N, the amplitudes of the M vibration readings can *all* theoretically be reduced to zero by means of suitable correction masses found by solving a set of simultaneous equations (see Appendix). When M is greater than N, however, the amplitudes of the M vibration readings generally *cannot even theoretically* all be reduced to zero simultaneously, and the N correction masses must be chosen by an optimum compromise to make the M vibration amplitudes all simultaneously as small as possible. The least-squares method of calculation provides a straightforward mathematical procedure for effecting this compromise. This procedure is carried through in successive iterations, as described below.

Plain Least Squares (First Iteration)

The input data for this calculation are the zero-rotor data (the vibration readings for the unbalanced rotor with no correction masses) and the trial-mass data (the vibration readings for the

[1] Numbers in brackets designate References at end of paper.

Contributed by the Machine Design Division and presented at the Winter Annual Meeting, Philadelphia, Pa., November 17–22, 1963, of The American Society of Mechanical Engineers. Manuscript received at ASME Headquarters, September 30, 1963. Paper No. 63—WA-295.

rotor with trial masses added one at a time in each balance plane). These data are used to calculate the response coefficients, i.e., the effect on each vibration reading of a unit correction mass in each balance plane. The procedure for calculating the response coefficients is indicated in the Appendix.

Using the response coefficients and the zero-rotor data, the correction masses which minimize the sum of the squares of the M vibration amplitudes are computed. The calculation for "minimization of the sum of the squares"—usually abbreviated as "least squares" [2]—can be reduced to the solution of a set of simultaneous linear equations. This mathematical derivation is shown in the Appendix.

In the first iteration, this "least squares" calculation is carried out to give the magnitudes and angular locations of the correction masses and the amplitudes and phase angles of the predicted residual vibration readings. The sum of the squares of the residual amplitudes and the rms residual amplitude are also computed.

Weighted Least Squares (Second and Third Iterations)

When the sum of the squares of the M residual vibration amplitudes is minimized, it is possible that one or two of the residual amplitudes may still be considerably larger than the others. If the goal of the balancing procedure is to minimize the *maximum* residual amplitude, then the plain least-squares solution is not the optimum.

There is no direct procedure for minimizing the maximum residual amplitude when $M - N > 1$. However, an iterative procedure can be used [3] which converges to a minimizing solution for the maximum residual amplitude in most practical cases.

In the weighted least-squares procedure, each of the simultaneous linear equations of the original least-squares solution is weighted by the amplitude of the corresponding residual vibration. Thus the vibration readings which had large residuals are weighted more heavily than those which had small residuals. Mathematical details of this procedure are given in the Appendix. This weighting leads to a solution for the magnitudes and angular locations of correction masses which in general reduce the larger residual amplitudes at the expense of increasing the smaller residual amplitudes. The second iteration computes these new correction masses, along with the amplitudes and phase angles of the corresponding residual vibration readings. The sum of the squares of the residual amplitudes and the rms residual amplitude, which is also computed, must necessarily be larger than the corresponding values for first iteration, since the first iteration computed the correction masses which make these values a minimum.

In the third iteration, the simultaneous equations of the second iteration are weighted by the corresponding residual amplitudes of the second iteration. The resulting correction masses and residual vibrations are computed, along with the sum of the squares of the residual amplitudes and the rms residual amplitude, which again must be larger than the corresponding values for the first iteration.

This iterative procedure could be continued indefinitely, and previous investigators [3] found that in all practical cases they tried, it converges to a minimization of the maximum error, although counter-examples can be artificially concocted. The results of the three iterations, however, should make possible a satisfactory choice of correction masses for most balancing problems.

Choice of Correction Masses

A rule for which set of correction masses to choose is as follows: The first iteration should ordinarily be used unless one or two residuals are considerably higher than the others. In those cases where a few residuals are so much higher than the others that an increase in the other residuals is acceptable for the sake of reducing the high residuals, the results of the second or third iteration may be used instead. In all cases, the rms residual of the first iteration is a lower limit to all possible values of the maximum residual vibration amplitude. The rms residual of any other calculation must necessarily be larger than the rms residual of the first iteration, and the maximum residual of any calculation must necessarily be larger than the rms residual of the same calculation. If the rms residual of the first iteration is already greater than the maximum allowable residual vibration amplitude, then it is mathematically impossible to balance the machine within the allowable limits by means of correction masses in the balance planes used for the trial masses.

Meaning of "Vibration Amplitude"

The term "vibration amplitude," as used here, can be either a displacement, a velocity, or an acceleration amplitude. In the turbine-balancing problem for which this calculation procedure was developed, the measured amplitudes are velocity amplitudes, and the residual amplitudes which it is desired to minimize are also velocity amplitudes. However, the procedure would work equally well with either displacement amplitudes or acceleration amplitudes; if the input amplitudes were displacements, it would minimize the residual displacement amplitudes, and if the input amplitudes were accelerations, it would minimize the residual acceleration amplitudes. When balancing over a range of speeds the optimum balance correction depends, of course, on whether displacement, velocity, or acceleration amplitudes are being minimized.

Compensation for Load Effects

In the zero-rotor and trial-mass data, the calculation procedure cannot distinguish between vibration due to unbalance and vibration due to other once-per-revolution events such as cyclic load variation. The computed solution, therefore, gives the optimum correction masses to compensate for load effects as well as unbalance.

Selection of Balance Planes

The least-squares computation procedure is of no help in selecting the balance planes; all it can do is to compute optimum correction masses for a given set of balance planes.

The selection of balance planes was discussed in a previous paper [4]. In their conclusions the authors of that paper state, "The number of balance weights and measurement points that should be used for any flexible-shaft balancing is not optional, but equals the number of critical speeds below the maximum balancing speed." In the Appendix of the paper they point out that when the number of measurement points exceeds the number of balance weights (i.e., when $M > N$ in the notation of this paper), a least-squares procedure may be used to choose optimum weights.

When the rotor is relatively rigid, as is usual for turbines, the two rigid-body modes of the shaft—the static-unbalance mode and the dynamic-unbalance mode—correspond to the two lowest modes of a flexible rotor, and the lowest flexible mode of the relatively rigid rotor corresponds to the third mode of the flexible rotor [5]. When the first critical speed of a rotor is in the operating-speed range, there should, therefore, be three balance planes per rotor. Thus, for two rotors geared together, $N = 6$.

The most effective locations for balance planes can be determined by experimental response measurements, as well as by calculation of mode shapes.

Input Data and Results

Input Data

The input data for the least-squares computation consist of the zero-rotor data and trial-mass data for N balance planes at K operating speeds and loads and at L pickup locations. Provision should be made for using modified zero-rotor data (obtained after a set of correction masses have been applied) as an additional input, so that the trial-mass measurements do not need to be repeated. When modified zero-rotor data are supplied, the initial zero-rotor data and trial-mass data are used to compute the response coefficients, which are then used with the modified zero-rotor data to compute the correction masses.

In the zero-rotor data and trial-mass data, the phase angles of the vibrations, relative to a rotating reference vector, may be measured in either a "moving-clock" or a "moving-mark" coordinate system, Fig. 1, using stroboscopic illumination triggered by the vibration itself. However, the angles of the trial masses in the computer solution must be interpreted accordingly, as discussed later in this section.

The radii at which the trial masses are located need not be specified, since any correction mass in each balance plane would be at the same radius as the trial mass. Also, the angular locations of the trial masses need not be specified, since the computed solution gives the location of the correction mass in each plane as an angle measured from the trial mass, rather than from an arbitrary reference vector.

Response Coefficients

The first step in the computations is to compute the response coefficients unless these are already available from a previous computation. The response coefficients for each balance plane give the vibration at each pickup location due to a unit mass (i.e., a mass of one ounce if the trial masses are given in ounces) at the location of the trial mass.

The magnitudes of the response coefficients have the same units as the magnitudes of the zero-rotor and trial-mass data, and the angles of the response coefficients are to be interpreted in the same way as the angles of the zero-rotor and trial-mass data.

The response coefficients represent the characteristics of the rotating machine in combination with the attenuation and phase shift of the vibration pickups used to obtain the vibration readings. Thus, even if two rotating machines were identical, their response coefficients would be different if different pickups were used.

Correction Masses: Magnitude

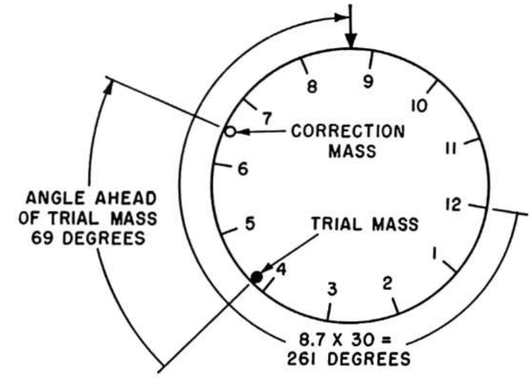
The magnitudes of the correction masses are in the same units as the trial masses—ordinarily in ounces. The correction mass in each plane is understood to be applied at the same radius as the trial mass in that plane.

Correction Masses: Angle Ahead of Trial Mass

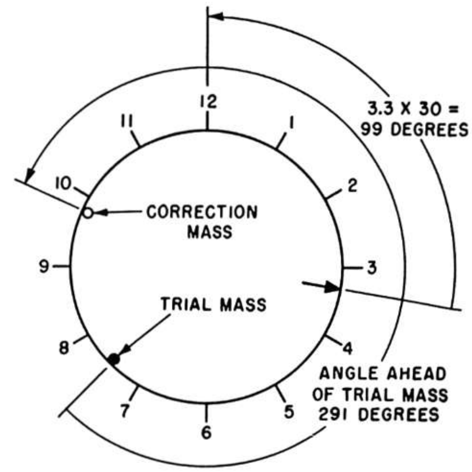
The location of each correction mass is given in degrees as "angle ahead of trial mass." The word "ahead" must be interpreted as *ahead in degrees*, or *ahead in hours on the clock face* used for angular measurement; it has *nothing to do with the direction of rotation of the shaft.* The interpretation of "ahead" for both moving-clock and moving-mark reference systems is illustrated in Fig. 1.

For a moving-clock reference system, using stroboscopic illumination triggered by the vibration itself, phase angles of vibrations are measured from a rotating reference vector in the shaft (marked 12 o'clock) to a fixed mark on the machine housing. In the diagram in Fig. 1(*a*), the angle would be recorded as 8.7 hr, or $8.7 \times 30 = 261$ deg. Suppose that the trial mass is located at 4.2 o'clock or $4.2 \times 30 = 126$ deg, as measured from the 12 o'clock reference vector. Suppose the computer gives "angle ahead of trial mass" as 69 deg; then the correction mass should be located at $126 + 69 = 195$ deg, or $4.2 + 2.3 = 6.5$ o'clock.

For a moving-mark reference system, phase angles are measured from a fixed mark (labeled 12 o'clock) on the machine housing to a rotating mark on the shaft. In the diagram of Fig. 1(*b*), the angle would be recorded as 3.3 hr, or $3.3 \times 30 = 99$ deg. Suppose that the trial mass is still located 126 deg clockwise from the rotating mark. If "angle ahead of trial mass" were given by the computer as 69 deg for the moving-clock reference system, it would be computed as $360 - 69 = 291$ deg when the computer input is in the moving-mark reference system. This angle of 291



(a) Moving-clock system: Clock face on rotor rotates past reference mark on housing



(b) Moving-mark system: Reference mark on rotor rotates past clock face on housing

Fig. 1 Interpretation of "Angle Ahead of Trial Weight"

deg must be measured from the trial mass *in a counterclockwise direction*, i.e., in the same direction as the 99 deg is measured from the reference mark on the shaft to the fixed mark on the ground.

Residual Vibrations

For each iteration, the sum of squared residuals, rms residual, and magnitudes of the individual residual vibrations are computed. The units of the residual vibration amplitudes are the same as the units of the vibration amplitudes in the input.

The angles of the residual vibrations are given in degrees measured from "zero," i.e., from the same rotating reference vector in the shaft that was used to measure the angles of the original vibrations.

The magnitudes and angles of the residual vibrations are a mathematical prediction of the vibration readings that would be obtained experimentally if the correction masses were applied in the balance planes.

Sample Problem

The following sample problem, for which the numerical values are simple enough to permit an arithmetical check, is both an illustration and a check of the computation procedure. The algebraic notation used for this problem is that of the Appendix.

In this problem we consider a single rotor with two balance planes, operating at a single speed and load condition. However, we desire to minimize the maximum vibration at three pickup locations on the machine frame. Thus, for this problem $N = 2$,

$K = 1$, $L = 3$, and $M = K \times L = 3$. The response coefficients, as computed from a previous balancing trial, are given in Table 1. Carrying out the steps indicated in the Appendix, we obtain Table 2.

Table 1 Response coefficients

	Balance plane 1	Balance plane 2
Location 1	$\alpha_{11} = 3.00 \angle 0$	$\alpha_{12} = 2.00 \angle 180°$
Location 2	$\alpha_{21} = 5.00 \angle 0$	$\alpha_{22} = 2.00 \angle 180°$
Location 3	$\alpha_{31} = 5.00 \angle 0$	$\alpha_{32} = 3.00 \angle 180°$

The measured zero-rotor data are:

Location 1	$A_1 = 1.00 \angle 0$
Location 2	$A_2 = 1.00 \angle 180°$
Location 3	$A_3 = 0 \angle 0$

Table 2

First iteration

Balance plane	Correction mass (magnitude)	Correction mass, angle ahead of trial mass (deg)
1	$W_1 = 0.81$	0
2	$W_2 = 1.48$	0

Residual vibrations

Location	
1	$\epsilon_1 = 0.476 \angle 0$
2	$\epsilon_2 = 0.0952 \angle 0$
3	$\epsilon_3 = 0.381 \angle 180°$

Sum of squared residuals $S = 0.381$ Rms residual $R = 0.356$

Second iteration

Balance plane	Correction mass (magnitude)	Correction mass, angle ahead of trial mass (deg)
1	$W_1' = 1.00$	0
2	$W_2' = 1.80$	0

Residual vibrations

Location	
1	$\epsilon_1' = 0.400 \angle 0$
2	$\epsilon_2' = 0.400 \angle 0$
3	$\epsilon_3' = 0.400 \angle 180°$

Sum of squared residuals $S' = 0.480$ Rms residual $R' = 0.400$

Third Iteration

For this sample problem, the results are the same as for the second iteration, since the ϵ' weighting values used in the third iteration are all equal.

The first iteration shows an rms residual of 0.356 units, with a maximum residual amplitude of 0.456. Since the maximum residual is only 27 percent higher than the rms residual, only a small reduction in the maximum residual can be expected from the second iteration.

The second iteration shows that the maximum residual has been reduced to 0.400, but the rms residual has been increased to 0.400. Because $M - N = 1$ in this example, the second iteration gives an exact solution for minimizing the maximum error, and the third iteration gives exactly the same result [3]. When $M - N > 1$, the second and third iterations usually converge [3] toward a solution in which the maximum error is minimized, but do not actually reach this exact solution.

References

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APPENDIX

Mathematical Derivation of Least-Squares Procedure

Given: N balance planes (locations for correction masses)
M vibration readings at K different conditions of speed and load
L different locations
where $M = K \times L$

To find: Optimum correction masses in the N balance planes to minimize the M vibration readings.

By *plain least squares*, we can minimize the rms value of the M vibration readings; this is a standard statistical procedure [2].

By successive iterations using *weighted least squares*, it may be possible to minimize the maximum of the M vibration readings [3].

Computer inputs are the following experimentally measured values:

1 Zero-rotor data: $A_1, A_2, \ldots\ldots, A_M$, which are complex numbers giving amplitudes and phases of original vibrations at the M locations and operating conditions. Phase angles are measured relative to a rotating reference vector [5].

2 Magnitudes $U_1, U_2, \ldots, U_N$ of trial masses in the N balance planes.

3 Trial-mass data: B_{mn} $\quad m = 1, \ldots, M$; $\quad n = 1, \ldots, N$

which are complex numbers giving amplitudes and phases of vibrations when trial masses are added. B_{mn} is the vibration of the mth location and speed when a trial mass of magnitude U_n is placed in the nth balance plane.

Preliminary computations

As a preliminary step, the following computations are made:

1 Real and imaginary parts (A_{mx} and A_{my}) of A_m

2 Real and imaginary parts (B_{mnx} and B_{mny}) of B_{mn}

3 Real and imaginary parts of the *response coefficients* α_{mn}, defined by

$$\begin{aligned} \alpha_{mnx} &= (B_{mnx} - A_{mx}) \div U_n \\ \alpha_{mny} &= (B_{mny} - A_{my}) \div U_n \end{aligned} \quad (1)$$

Theory of plain least squares computation

If M were equal to N, we could compute exact values of $W_1, \ldots, W_N$, which would reduce vibration to zero at the M locations and speeds.

W_n is defined as a complex number with

magnitude = desired correction mass in nth balance plane
phase angle = angular location of desired correction mass *relative to trial mass*

When $M > N$, we cannot in general reduce all residual vibrations to zero but we can minimize the sum of their squares.

Let ϵ_m be residual vibration (complex number giving magnitude and phase) at mth location and speed.

Then

$$\epsilon_m = A_m + \alpha_{m1}W_1 + \ldots + \alpha_{mN}W_N = A_m + \sum_{n=1}^{N} \alpha_{mn}W_n \quad (2)$$

Splitting this into real and imaginary parts,

$$\epsilon_{mx} = A_{mx} + \alpha_{m1x}W_{1x} - \alpha_{m1y}W_{1y} + \ldots$$
$$= A_{mx} + \sum_n (\alpha_{mnx}W_{nx} - \alpha_{mny}W_{ny})$$
$$\epsilon_{my} = A_{my} + \alpha_{m1y}W_{1x} + \alpha_{m1x}W_{1y} + \ldots \tag{3}$$
$$= A_{my} + \sum_n (\alpha_{mny}W_{nx} + \alpha_{mnx}W_{ny})$$

Let

$$S = \sum_m |\epsilon_m|^2 = \sum_m (\epsilon_{mx}{}^2 + \epsilon_{my}{}^2) \tag{4}$$

Choose $W_{1x}, W_{1y}, \ldots$ to minimize S

This requires

$$\frac{\partial S}{\partial W_{1x}} = \frac{\partial S}{\partial W_{1y}} = \ldots . = \frac{\partial S}{\partial W_{Nx}} = \frac{\partial S}{\partial W_{Ny}} = 0 \tag{5}$$

This requires

$$\sum_m \left\{\alpha_{mnx}\left[A_{mx} + \sum_n (\alpha_{mnx}W_{nx} - \alpha_{mny}W_{ny})\right] + \alpha_{mny}\left[A_{my} + \sum_n (\alpha_{mny}W_{nx} + \alpha_{mnx}W_{ny})\right]\right\} = 0$$
$$\sum_m \left\{-\alpha_{mny}\left[A_{mx} + \sum_n (\alpha_{mnx}W_{nx} - \alpha_{mny}W_{ny})\right] + \alpha_{mnx}\left[A_{my} + \sum_n (\alpha_{mny}W_{nx} + \alpha_{mnx}W_{ny})\right]\right\} = 0 \tag{6}$$

for all n.

This gives $2N$ linear equations which can be solved for the W_{nx} and W_{ny}.

In matrix notation

$$[\alpha]^T[\alpha][W] + [\alpha]^T[A] = 0 \tag{7}$$

and the matrix solution for $[W]$ is

$$[W] = -\{[\alpha]^T[\alpha]\}^{-1}[\alpha]^T[A] \tag{8}$$

Here

$[W]$ is the column matrix $\begin{bmatrix} W_{1x} \\ W_{1y} \\ \cdot \\ \cdot \\ \cdot \\ W_{Nx} \\ W_{Ny} \end{bmatrix}$

$[A]$ is the column matrix $\begin{bmatrix} A_{1x} \\ A_{1y} \\ \cdot \\ \cdot \\ \cdot \\ A_{Mx} \\ A_{My} \end{bmatrix}$

$[\alpha]$ is the rectangular matrix

$$\begin{bmatrix} \alpha_{11x} & -\alpha_{11y} & \cdots & \alpha_{1Nx} & -\alpha_{1Ny} \\ \alpha_{11y} & \alpha_{11x} & \cdots & \alpha_{1Ny} & \alpha_{1Nx} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha_{M1x} & -\alpha_{M1y} & \cdots & \alpha_{MNx} & -\alpha_{MNy} \\ \alpha_{M1y} & \alpha_{M1x} & \cdots & \alpha_{MNy} & \alpha_{MNx} \end{bmatrix}$$

$[\alpha]^T$ is the transpose of $[\alpha]$

First Iteration: Computation procedure for plain least squares

1 Form the matrices $[A]$ and $[\alpha]$; print out $[\alpha]$.

2 Solve for $[W] = -\{[\alpha^T][\alpha]\}^{-1}[\alpha^T][A]$ (8)

3 Compute amplitudes and phases (in degrees) of $W_1, \ldots, W_N$ and print out (in print out, label these "correction mass magnitude" and "correction mass, angle ahead of trial mass, degrees").

4 Compute $[\epsilon] = [\alpha][W] + [A]$ (9)

where $[\epsilon]$ is the column matrix $\begin{bmatrix} \epsilon_{1x} \\ \epsilon_{1y} \\ \cdot \\ \cdot \end{bmatrix}$

5 Compute amplitudes (and phases in degrees) of $\epsilon_1, \ldots, \epsilon_M$ and print out (in print out, label these as "residual vibration: magnitude" and "residual vibration: angle from zero degrees").

6 Compute $S = \sum_m \epsilon_m{}^2$ and $R = \sqrt{\frac{S}{M}}$ and print out. (In print out label S as "sum of squared residuals"; label R as "rms residual").

Note: By comparing R to $|\epsilon_m|_{\max}$, we can see whether $|\epsilon_m|_{\max}$ could probably be reduced by using weighted least squares as described below.

If $|\epsilon_m|_{\max}$ is only slightly greater than R, then probably no improvement is possible.

Note: After preliminary computation of $[\alpha]$ using original values of $[A]$, program should be capable of accepting *revised* values of $[A]$ to use with original $[\alpha]$ in this and subsequent iterations.

Second iteration: Computation procedure for weighted least squares

1 Form the matrix $[\alpha]^{T'}$ which is defined as the $[\alpha]^T$ matrix with the first two columns multiplied by $\left|\frac{\epsilon_1}{R}\right|$, 3d and 4th columns multiplied by $\left|\frac{\epsilon_2}{R}\right|$, etc.

2 Use $[\alpha]^{T'}$ in place of $[\alpha]^T$ to compute $[W']$ in place of $[W]$, i.e., $[W'] = -\{[\alpha]^{T'}[\alpha]\}^{-1}[\alpha]^{T'}[A]$ (10)

3 Compute amplitudes and phases (in degrees) of $W_1', \ldots, W_N'$ and print out

4 Compute $[\epsilon'] = [\alpha][W'] + [A]$ (11)

5 Compute amplitudes of phases (in degrees) of $\epsilon_1', \ldots, \epsilon_M'$ and print out

6 Compute $S' = \sum_m |\epsilon_m'|^2$ and $R' = \sqrt{\frac{S'}{M}}$ and print out (12)

By applying this procedure in successive iterations, using residual errors of each iteration as multipliers of $[\alpha]$ for the next iteration, solution often converges [3] to give minimum value of maximum residual vibration, i.e., minimum value of $|\epsilon_m|_{\max}$. However, this convergence is not guaranteed; hence, result should be checked by inspection at each stage.

Third and final iteration: Further refinement of weighted least-squares computation

1 Form the matrix $[\alpha]^{T''}$ which is defined as the $[\alpha]^{T'}$ matrix with first two columns multiplied by $\left|\frac{\epsilon_1'}{R'}\right|$, third and fourth columns multiplied by $\left|\frac{\epsilon_2'}{R'}\right|$, etc.

2 Use $[\alpha]^{T''}$ to compute $[W''] = -\{[\alpha]^{T''}[\alpha]\}^{-1}[\alpha]^{T''}[A]$ (13)

3 Compute amplitudes and phases (in degrees) of $W_1'', \ldots, W_N''$ and print out

4 Compute $[\epsilon''] = [\alpha][W''] + [A]$ (14)

5 Compute amplitudes and phases (in degrees) of $\epsilon_1'' \ldots, \epsilon_M''$ and print out

6 Compute $S'' = \sum_m |\epsilon_m''|^2$ and $R'' = \sqrt{\frac{S''}{M}}$ and print out