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Robust digital tracking with perturbation estimation via the Euler operator

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Robust digital tracking with perturbation estimation via the Euler operator

ADDISU TESFAYE† and MASAYOSHI TOMIZUKA†

A model reference control (MRC) method is presented for linear, time-invariant systems with unknown dynamics. The class of systems investigated can be described by $\dot{x}(t) = (A + \Delta A)\, x(t) + (B + \Delta B)\, u(t) + d(t)$ where ΔA, ΔB and $d(t)$ are unknown dynamics and unexpected disturbances, respectively. The design method is described in the discrete time form using the Euler operator, which approaches the Laplace operator as the sampling interval, T, approaches zero. The control is constructed, based on the estimation of system perturbations using time delay control (TDC), under the following two assumptions: (1) the dynamics stemming from the perturbations are considerably slower than the discretization frequency, $1/T$; and (2) the control has access to all the plant states. The controller is tested through simulation and experiment. The theoretical and experimental results indicate that the proposed method has potential applications for controlling servo-mechanisms subjected to unknown and unexpected disturbances.

1. Introduction

Digital controllers for continuous time plants have traditionally been designed and analysed using difference equations in the time domain or z transfer functions in the frequency domain. This approach results in a different appearance of the digital control laws compared with those developed in the continuous-time domain for the same plant. Furthermore, the zero-order hold equivalent of the continuous time plant often: (1) involves unstable sampling zeros upon discritization (Åström *et al.* 1984, Goodwin *et al.* 1986); and (2) exhibits differences in the limiting properties as the sampling interval $T \mapsto 0$ compared with those of continuous-time systems (Goodwin *et al.* 1986). The Euler operator (ε operator) also called the delta operator (δ operator) by Middleton and Goodwin (1986, 1990), bypasses some of these limitations. The operator gives a reasonable approximation of the Laplace operator when the sampling interval is sufficiently small. The stability region defined by the ε operator is given as $|\varepsilon + 1/T| < 1/T$, which is a circle centred at $-1/T$ with radius $1/T$. Thus, as $T \mapsto 0$ the stability region defined by the ε operator approaches that using the s operator. In this paper, we define $\delta = (q-1)/T$ and $\varepsilon = (z-1)/T$ i.e. counterparts of the operators q and z in the time and frequency domains respectively.

In addition to the challenge of developing reliable controllers in the discrete time domain, there is an increasing demand for robust controllers that operate in environments where large system parameter variations and unexpected disturbances are possible. Several control techniques are available for such purposes, amongst which is the perturbation estimation scheme, also known as the time delay control (TDC) algorithm (Youcef-Toumi and Ito 1990, Youcef-Toumi and Wu 1992, Tesfaye and Tomizuka 1993). Such controllers attempt to estimate a function representing the

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effect of uncertainties via the mechanism of time delay. The gathered information is subsequently used to cancel the unknown dynamics and the unexpected disturbances simultaneously, and to let the plant output follow a reference model and/or desired trajectory.

Youcef-Toumi and Ito (1990) and Youcef-Toumi and Wu (1992) have developed the time delay control (TDC) in the continuous time domain. In this paper we will develop the counterpart of the TDC in the discrete time domain. The resulting controller is tested through simulation and experiment on a single-axis robot arm driven by a 14 inch NSK direct drive motor.

2. Problem statement

The controlled plant is a single-input single-output (SISO) linear, time-invariant, minimum-phase continuous time system, and is given by

$$\left.\begin{aligned} \dot{x}(t) &= A_{\rm c}\,x(t) + B_{\rm c}\,u(t) + F_{\rm c}\,d(t) \\ y(t) &= [1 \quad 0 \quad \dots \quad 0]\,x(t) \end{aligned}\right\} \tag{1}$$

where $x \in R^n$ is the plant state vector, $u(t) \in R$ is the control input, $d(t) \in R$ is an unknown disturbance and $y(t) \in R$ is the plant output. The pair $(A_{\rm c}, B_{\rm c})$ are assumed to be controllable, and we assume that the parameter uncertainties satisfy the so-called matching conditions. This means if we denote $A_{\rm c} = A_{\rm nc} + \Delta A_{\rm c}$ and $B_{\rm c} = B_{\rm nc} + \Delta B_{\rm c}$, where the pair $(A_{\rm nc}, B_{\rm nc})$ represent the nominal system, we can write $\Delta A_{\rm c} = B_{\rm nc}\,\Delta_{\rm a}$, $\Delta B_{\rm c} = B_{\rm nc}\,\Delta_{\rm b}$. Also, $F_{\rm c} = B_{\rm nc}\,\Delta_{\rm f}$ where $\Delta_{\rm a} \in \Re^{1\times n}$, $\Delta_{\rm b} \in \Re$ and $\Delta_{\rm f} \in \Re$. Without loss of generality we express the above system in the controllable canonical form with

$$\left.\begin{aligned} A_{\rm c} &= A_{\rm nc} + \Delta A_{\rm c} \\ &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ -- & -- & -- & \dots & -- \\ -(a_{1{\rm c}} + \Delta a_{1{\rm c}}) & -(a_{2{\rm c}} + \Delta a_{2{\rm c}}) & -- & \dots & -(a_{n{\rm c}} + \Delta a_{n{\rm c}}) \end{bmatrix} \\ B_{\rm c}^{\rm T} &= B_{\rm nc}^{\rm T} + \Delta B_{\rm c}^{\rm T} = [0 \quad 0 \quad -- \quad \dots \quad b_{\rm c} + \Delta b_{\rm c}] \\ F_{\rm c}^{\rm T} &= [\;0 \quad 0 \quad -- \quad \dots \quad 1] \end{aligned}\right\} \tag{2}$$

This characterization implies that both the nominal and the uncertain systems are controllable and the uncertainties reside within the space spanned by the nominal control input channel, $B_{\rm nc}$. When this plant is preceded by a zero-order hold, under digital control, the discrete time model using the δ operator is given by

$$\left.\begin{aligned} \delta x(k) &= A_{\rm d}\,x(k) + B_{\rm d}\,u(k) + F_{\rm d}\,d(k) \\ y(k) &= [1 \quad 0 \quad \dots \quad 0]\,x(k) \\ A_{\rm d} &= \frac{{\rm e}^{A_{\rm c}T} - I}{T}; \quad B_{\rm d} = \frac{1}{T}\int_0^T {\rm e}^{A_{\rm c}\tau} B_{\rm c}\,{\rm d}\tau \\ F_{\rm d} &= \frac{1}{T}\int_0^T {\rm e}^{A_{\rm c}\tau} F_{\rm c}\,{\rm d}\tau; \qquad \delta = \frac{q-1}{T} \end{aligned}\right\} \tag{3}$$

where q is the traditional shift operator used in difference equations. Because of the exponential terms arising from the discretization process it is evident that the uncertainties will no longer be constrained to lie within the range space of the nominal control input channel, B_{dn}

$$B_{dn} = \frac{1}{T}\int_0^T e^{A_{nc}\tau} B_{nc}\,d\tau$$

To enforce the matching conditions being preserved upon discretization, we ask for a suitable choice of sampling time by means of the assumption below.

Small sampling time assumption: *Let the sampling interval, T, which transforms the continuous time dynamical system given by* (1) *into a discrete time system, be small where a sampling time T is considered small if any function that is expanded in powers of T, can be approximated with some degree of accuracy by keeping terms up to and including order T.*

By making use of this assumption we obtain the following (approximate) discrete time model of the continuous time system given by (1)

$$\left.\begin{aligned} \delta x(k) &= Ax(k) + B\{(1+b)\,u(k) + E(k)\} \\ A &= \frac{e^{A_{nc}T} - I}{T}; \qquad B = \frac{1}{T}\int_0^T e^{A_{nc}\tau} B_{nc}\,d\tau \\ y(k) &= [1 \quad 0 \quad \dots \quad 0]\,x(k) \end{aligned}\right\} \tag{4}$$

where $E(k) = \Delta A x(k) + w(k)$, $w(k) = (1 + \Delta_a B_{nc}\frac{T}{2})\Delta_f d(k)$ and the system parameters have the following values: $\Delta A = \Delta_a\{I + (A_{nc} + B_{nc}\Delta_a)\frac{T}{2}\}$, $b = \Delta_a B_{nc}(1 + \Delta_b)\frac{T}{2} + \Delta_b$. See Appendix A for the derivation.

Remark 1: Note that the pair (A, B) in (4) does not involve any uncertainty, while the pair (A_d, B_d) in (3) does involve uncertainty. □

Remark 2: The above expression is approximate because uncertainty terms of order T^2 and higher in the sampling time have been neglected. □

Remark 3: If there were no uncertainties, (4) equals the discretized version of the nominal continuous time system of (1), i.e. no approximation is necessary. □

Remark 4: If the perturbation due to the uncertainties exists and if the sampling time $T \to 0$, then the discretized uncertain system given by (4) approaches the uncertain continuous time system. This is an advantage obtained using the ε operator, which could not be attained if the conventional z operator was used. □

It is of interest to determine the error introduced by using the small sampling time assumption. Appendix B presents this analysis and we see that an upper bound on the error is of the order of T^2. Therefore, the accuracy of the discrete time representation increases with smaller sampling times.

3. Derivation of control law

3.1. *Error dynamics and structural constraint*

Define the reference model that generates the desired dynamics as a linear time-invariant system given by

$$\left.\begin{aligned} \delta x_m(k) &= A_m x_m(k) + B_m r(k) \\ y_m(k) &= [1 \quad 0 \quad \dots \quad 0]\,x_m(k) \end{aligned}\right\} \tag{5}$$

where

$$A_{\mathrm{m}} = \begin{bmatrix} 0 & I_q \\ f_{\mathrm{m}} & \end{bmatrix}; \quad B_{\mathrm{m}} = \begin{bmatrix} 0 \\ b_{\mathrm{m}} \end{bmatrix}$$

$x_{\mathrm{m}} \in R^n$ is the model state vector, $r \in R$ is the command input vector to the model, I_q is an $(n-1) \times (n-1)$ matrix, 0 and f_{m} are $(n-1) \times 1$ and $1 \times n$ vectors respectively and $b_{\mathrm{m}} \in R$. The error, e, is defined as the difference between the states of the model and the plant

$$e = x_{\mathrm{m}} - x \tag{6}$$

By combining (4), (5) and (6), an equation that governs the error dynamics is obtained

$$\delta e(k) = A_{\mathrm{m}} e(k) + \{B_{\mathrm{m}} r(k) - (A - A_{\mathrm{m}}) x(k) - B[(1+b) u(k) + E(k)]\} \tag{7}$$

The control objective is to force the error to vanish with the desired dynamics

$$\delta e(k) = A_{\mathrm{e}} e(k) \tag{8}$$

where A_{e} is an $n \times n$ error system matrix, which defines some desired dynamics. If a control u exists such that the following equation is satisfied

$$B_{\mathrm{m}} r - (A - A_{\mathrm{m}}) x - B[(1+b) u + E] = Ke \tag{9}$$

then substituting (9) into (7) leads to

$$\delta e(k) = (A_{\mathrm{m}} + K) e(k) = A_{\mathrm{e}} e(k) \tag{10}$$

and the error system matrix, A_{e} can be arbitrarily assigned through proper choice of the error feedback gain matrix K. The control, u, should be determined such that (9) is satisfied. However, (9) cannot always be satisfied because, in general, the number of controls is smaller than the number of states. Consequently, the control that gives the best approximate solution to (9) is determined as

$$u(k) = \frac{1}{(1+\hat{b})}[B^{+}\{B_{\mathrm{m}} r(k) - (A - A_{\mathrm{m}}) x(k) - Ke(k) - BE(k)\}] \tag{11}$$

where $B^{+} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ and is a particular pseudo-inverse matrix. It should be noted that the control law in (11) is not implementable because of the presence of the term $E(k)$. This problem will be resolved in the next section by applying TDC. Substituting (11) into (4), incorporating (5), and through some algebraic manipulation, we obtain the following equation

$$\delta e(k) = (A_{\mathrm{m}} + K) e(k) + \left\{I - \left(\frac{1+b}{1+\hat{b}}\right) BB^{+}\right\}\left\{A_{\mathrm{m}} x(k) - Ax(k) + B_{\mathrm{m}} r(k) - Ke(k)\right\}$$
$$+ \left\{\left(\frac{1+b}{1+\hat{b}}\right) - 1\right\} BE(k) \tag{12}$$

In order to obtain the desired error dynamics given by (10) the following structural constraint conditions need to be satisfied: (*a*) $(1+b)/(1+\hat{b}) = 1 \Rightarrow b = \hat{b}$, and (*b*)

$\{I - BB^{+}\}\{A_{\mathrm{m}} x(k) - Ax(k) + B_{\mathrm{m}} r(k) - Ke(k)\} = 0$. Suppose condition (*a*) is satisfied. To see if condition (*b*) can be satisfied the error feedback gain matrix, K, in (9) is selected to have the following structure

$$K = \begin{bmatrix} 0 \\ --- \\ k_{\mathrm{r}} \end{bmatrix}; \quad k_{\mathrm{r}} = [-k_{\mathrm{r}1} \quad -k_{\mathrm{r}2} \quad \dots \quad -k_{\mathrm{r}n}] \tag{13}$$

where 0 is an $(n-1) \times n$ vector and k_{r} is a $1 \times n$ vector. Making a transformation, $x \mapsto \bar{z}$, to the controllable canonical form the system matrices A and B of (4) have the following representations in the transformed coordinates $\bar{z}$

$$Ax \mapsto \begin{bmatrix} \bar{z}_{\mathrm{p}} \\ f\bar{z} \end{bmatrix}; \quad B \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

where $\bar{z}_{\mathrm{p}} = [\bar{z}_2 \quad \bar{z}_3 \quad \dots \quad \bar{z}_n]^{\mathrm{T}}$ and 0 are $1 \times n$ and $(n-1) \times 1$ vectors respectively. From this

$$(B^{\mathrm{T}}B)^{-1} = 1; \quad [I - BB^{+}] = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and finally, structural constraint condition (*b*) becomes

$$\begin{aligned} &[I - BB^{+}][A_{\mathrm{m}} x(k) - Ax(k) + B_{\mathrm{m}} r(k) - Ke(k)] \\ &= \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_{\mathrm{p}} - \bar{z}_{\mathrm{p}} \\ b_{\mathrm{m}} r(k) - k_{\mathrm{r}} e(k) + (f_{\mathrm{m}} - f)\, \bar{z}(k) \end{bmatrix} = 0 \end{aligned}$$

Thus, if $b = \hat{b}$ (condition (*a*)) and the error feedback matrix, K, is selected as in (13), the structural constraint condition (*b*) is always satisfied and the desired error dynamics given by (10) can be attained. However, further considerations are required because: (1) b cannot always be guaranteed to be equal to $\hat{b}$; and (2) the matched discrete time equation given by (4) is an approximation using the small sampling time assumption. The next sections will address these issues.

3.2. *Perturbation estimation and control formulation*

It is of interest to determine the required control action, u, that will force the plant to follow the reference model given by (5) in the face of the unknown perturbation $E(k)$. $E(k)$ is known to contain the effects of unknown dynamics, ΔA and ΔB and also the unexpected disturbance, d. At this point, to estimate the perturbations, we use a technique similar to the time delay control (TDC) (Youcef-Toumi and Ito 1990, Youcef-Toumi and Wu 1992, Tesfaye and Tomizuka 1993). To obtain an estimate of the uncertainties manifested by the term E, TDC assumes that the value of the function $E(t)$ remains very close to that at time $t - L$ in the past for a small time delay L, i.e. $E(t) \cong E(t - L)$. In the discrete time case this would be equivalent to making the assumption that the value of $E(k)$ at time $t = kT$ can be considered to be close to the value of $E(k-1)$ at time $t = (k-1)\,T$, $E(kT) \cong E((k-1)\,T)$. Based on this argument, we make an equivalent assumption that the dynamics stemming from the perturbations manifested by the function $E(k)$ are considerably slower than the sampling frequency,

$1/T$ (for example, an order of magnitude smaller). If the above assumption holds, then the value of $E(k)$ at time $t = kT$ can be considered to be close to the value at time $t = (k-1)T$. Estimating $E(k)$ from $E(k-1)$ from (4) we have

$$\left.\begin{aligned} E(k) &\approx E(k-1) \\ BE(k) &\approx BE(k-1) = \delta x(k-1) - Ax(k-1) - B(1+b)\,u(k-1) \end{aligned}\right\} \tag{15}$$

Substituting the above expression into the control equation (11) one obtains

$$\begin{aligned} u(k) = \frac{B^+}{(1+b)}\{&B_m r(k) - (A - A_m)\,x(k) - Ke(k) - \delta x(k-1) \\ &+ Ax(k-1) + B(1+b)\,u(k-1)\} \end{aligned} \tag{16}$$

Since b is unavailable we replace b by $\hat{b}$ in the above equation to obtain the final control law as

$$\begin{aligned} u(k) = \frac{B^+}{(1+\hat{b})}\{&B_m r(k) - (A - A_m)\,x(k) - Ke(k) - \delta x(k-1) \\ &+ Ax(k-1) + B(1+\hat{b})\,u(k-1)\} \end{aligned} \tag{17}$$

In this equation, each term has the following interpretation: (1) B^+, a pseudo inverse matrix attempts to cancel B; (2) the term $-Ax(k) - \delta x(k-1) + Ax(k-1) + B(1+\hat{b})\,u(k-1)$ attempts to cancel undesired known dynamics $Ax(k)$ and unknown dynamics $E(k)$; (3) the term $A_m x(k) + B_m r(k)$ inserts desired dynamics; and (4) the error feedback term, $-Ke(k)$, adjusts the error dynamics.

The issue of the possible mismatch between b and $\hat{b}$ will be postponed until later. For the moment, assuming that such a mismatch does not exist, we see that the above controller observes the states and the input of the system at time $t = (k-1)T$, one step into the past, and determines the control action that should be commanded at time $t = kT$. In other words the controller estimates the unknown perturbation by evaluating a function at every time step, $t = kT$.

4. Frequency domain analysis

In this section, we will show that the control given by (17) achieves model matching. We will also address the issue of the possible mismatch between b and $\hat{b}$. First, note that the ε transform of a delayed quantity, $U(t-\tau)$, where $\tau = NT$ for $N \in Z^+$, is given by $\varepsilon[U(t-\tau)] = (1+T\varepsilon)^{-N} U(\varepsilon)$.

Transforming to the controllable canonical form, $x \mapsto \bar{z}$, we denote from (5), (13) and (14) the following

$$\left.\begin{aligned} f &= [-a_1 \quad -a_2 \quad \dots \quad -a_n]; \quad f_m = [-a_{1m} \quad -a_{2m} \quad \dots \quad -a_{nm}] \\ k_r &= [-k_{r1} \quad -k_{r2} \quad \dots \quad -k_{rn}] \end{aligned}\right\} \tag{18}$$

In general, (4) will have $(n-1)$ zeros. Therefore, the transfer function between the output y and the input u (from (4) with the external disturbance set to zero) can be represented by

$$\left.\begin{aligned} G(\varepsilon) &= \frac{y(\varepsilon)}{u(\varepsilon)} = \frac{(1+b)\,c(\varepsilon)}{P(\varepsilon)} \\ c(\varepsilon) &= c_1 + c_2\varepsilon + \dots + c_n\varepsilon^{n-1}; \quad P(\varepsilon) = \varepsilon^n + a_n\varepsilon^{n-1} + \dots + a_1 \end{aligned}\right\} \tag{19}$$

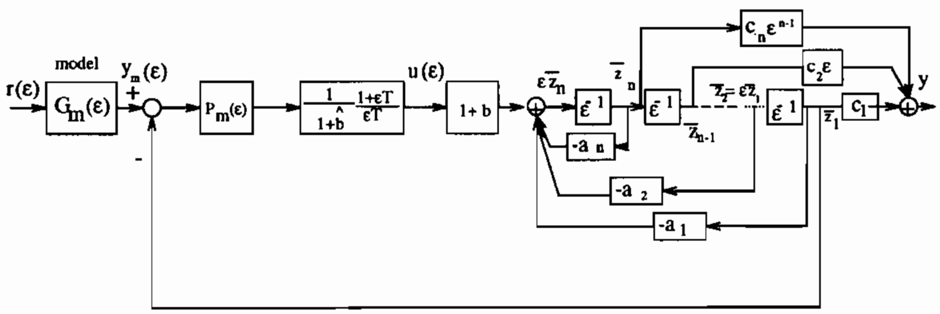

Figure 1. Block diagram of model, controller and plant.

where $P(\varepsilon)$ represents the characteristic polynomial. The model equation given by (5) transforms into

$$r(\varepsilon) = \frac{P_{\mathrm{m}}(\varepsilon)}{b_{\mathrm{m}}} y_{\mathrm{m}}(\varepsilon) \tag{20}$$

The ε transform of the control given by (17) is given by

$$u(\varepsilon) = \frac{1}{1+\hat{b}}\{b_{\mathrm{m}} r(\varepsilon) + P_{\mathrm{kr}} y_{\mathrm{m}}(\varepsilon) - P_{\mathrm{pmk}} \bar{z}_1(\varepsilon)\} + \frac{1}{1+\varepsilon T} u(\varepsilon) \tag{21}$$

$$P_{\mathrm{kr}} = k_{nr}\varepsilon^{n-1} + k_{(n-1)\mathrm{r}}\varepsilon^{n-2} + \cdots + k_{1\mathrm{r}}$$

$$\begin{aligned} P_{\mathrm{pmk}} &= \frac{1}{1+\varepsilon T}\{[1 - Ta_n]\varepsilon^n + [(a_{n\mathrm{m}} + k_{nr})(1+\varepsilon T) - Ta_{n-1}] \\ &\quad + [(a_{(n-1)\mathrm{m}} + k_{(n-1)\mathrm{r}})(1+\varepsilon T) - Ta_{n-2}]\varepsilon^{n-2} \\ &\quad + \cdots + [(a_{1\mathrm{m}} + k_{1\mathrm{r}})(1+\varepsilon T)]\} \\ &\approx \varepsilon^n + (a_{n\mathrm{m}} + k_{nr})\varepsilon^{n-1} + (a_{(n-1)\mathrm{m}} + k_{(n-1)\mathrm{r}})\varepsilon^{n-2} \\ &\quad + \cdots + (a_{1\mathrm{m}} + k_{1\mathrm{r}}) \end{aligned} \tag{22}$$

where $\bar{z}_1$ expresses the position in the controllable canonical form (see Appendix C for the derivation).

Suppose that the desired dynamics are governed by

$$\varepsilon e(\varepsilon) = A_{\mathrm{m}} e(\varepsilon); \quad P_{\mathrm{m}}(\varepsilon) = \varepsilon^n + a_{n\mathrm{m}}\varepsilon^{n-1} + \cdots + a_{1\mathrm{m}} \tag{23}$$

which simplifies the algebra by making all the error feedback gains zero, $k_{1\mathrm{r}} = k_{2\mathrm{r}} = \cdots = k_{nr} = 0$. Combining (21), (22) and (23) leads to

$$u(\varepsilon) = \frac{1}{1+\hat{b}} \frac{1+\varepsilon T}{\varepsilon T}\{P_{\mathrm{m}}(\varepsilon) y_{\mathrm{m}}(\varepsilon) - P_{\mathrm{m}}(\varepsilon)\bar{z}_1(\varepsilon)\} \tag{24}$$

A block diagram for the whole system can then be obtained as shown in Fig. 1. A reduced block diagram is shown in Fig. 2. Since the middle block in Fig. 2 can be regarded as almost unity, the whole system behaves as the reference model whose numerator is multiplied by the zeros of the plant.

Based on the above analysis, a perturbation estimation controller system based on time delay, in the discrete time domain, can be interpreted as one where the command r is filtered by the reference model into y_{m} and where the error between y_{m} and a prefiltered output $y'(y = cy')$, $y - y'$ is forced to zero by the high gain integrator $1/(T\varepsilon)$

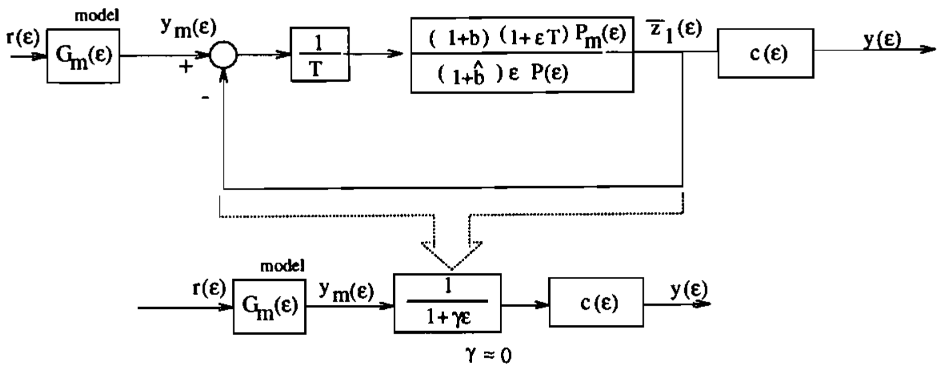

Figure 2. Reduced block diagram.

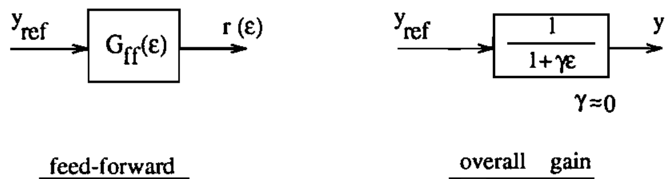

Figure 3. Feedforward controller and overall gain transfer diagram.

with pole/zero cancellation. Also, if the discrepancy between the actual value of b and its estimated value is not too large then this mismatch is easily absorbed by the high gain, $1/T$. At this point, to complete the design of the tracking controller we have to consider two situations that are dependent on the zeros of $c(\varepsilon)$.

Case 1. The zeros of $c(\varepsilon)$ are stable, i.e. they lie within the region described by $|\varepsilon + \frac{1}{T}| < \frac{1}{T}$.

Case 2. Some of the zeros of $c(\varepsilon)$ are unstable, i.e. they lie within the region described by $|\varepsilon + \frac{1}{T}| \geqslant \frac{1}{T}$.

First, notice that the transfer function of the reference model, G_m, can be chosen to be stable and minimum phase. Therefore, for Case 1, the following feedforward controller will satisfy the tracking requirements

$$r(\varepsilon) = G_{ff}(\varepsilon)\, y_{ref}(\varepsilon)$$
$$G_{ff}(\varepsilon) = (c(\varepsilon)\, G_m(\varepsilon))^{-1}$$

where $y_{ref}(k)$ defines the desired or reference output. In this case the transfer function from y_{ref} to y can be regarded as almost unity (see Fig. 3).

For Case 2, it is known that, using the ε operator, the unstable zeros that are introduced by sampling are easily distinguishable from the zeros that correspond to the continuous time system if the sampling time, $T \mapsto 0$ (Middleton and Goodwin 1990). These sampling zeros can then be neglected so that the minimum phase property of the continuous time system can be preserved, if T is selected to be sufficiently small (Goodwin *et al.* 1986). Since the zeros are continuous functions of the continuous-time system parameters, such a T always exists. Therefore, for Case 2, we can neglect the sampling zeros provided T is sufficiently small. In this case the feedforward controller is given by

$$r(\varepsilon) = G_m^{-1}(\varepsilon)\, y_{ref}(\varepsilon)$$

5. Simulation and experiment

The proposed controller is simulated on the following model of a single-axis robot arm driven by a 14 inch NSK direct drive motor (Model No. RS1410). The transfer function between the joint angle, θ, and the torque command input is given in the s domain by

$$G(s) = \frac{k/J}{s[s+(b/J)]}$$

where $b = 1{\cdot}4$ N m s, $k = 39$ N m V^{-1} and $J \in [0{\cdot}83,\ 2{\cdot}95]$ kg m^2. The inertia change represents a 355% variation, which should test the robustness of the proposed controller. In the ε domain the same transfer function is given by

$$G(\varepsilon, T) = \frac{b_1 \varepsilon + b_0}{\varepsilon(\varepsilon + a_1)}$$

In the limit, as the sampling interval, T, tends to zero, it is known that the properties of discrete-time systems approach those of the corresponding continuous-time system if the Euler operator (ε operator) is used. This is verified by this example since

$$b_1 = \frac{kJ}{b^2}\left\{\frac{b}{J} - \frac{1-\exp(-bT/J)}{T}\right\} \mapsto 0 \quad \text{as} \quad T \mapsto 0$$

$$b_0 = \frac{k}{b}\left\{\frac{1-\exp(-bT/J)}{T}\right\} \quad \mapsto \frac{k}{J} \quad \text{as} \quad T \mapsto 0$$

$$a_1 = \left\{\frac{1-\exp(-bT/J)}{T}\right\} \quad \mapsto \frac{b}{J} \quad \text{as} \quad T \mapsto 0$$

We note that the zero introduced by the sampling process would be non-existent if $T \mapsto 0$. The zero introduced by sampling is given by

$$z_\varepsilon = \frac{-a(1-e^{-aT})}{aT-(1-e^{-aT})} \quad \text{where} \quad a = \frac{b}{J} \tag{25}$$

Hence

$$\begin{aligned}
\left|z_\varepsilon + \frac{1}{T}\right| &= \left|\frac{-aT(1-e^{-aT}) + aT - (1-e^{-aT})}{T[aT-(1-e^{-aT})]}\right| \\
&= \left|\frac{aTe^{-aT} - (1-e^{-aT})}{T[aT-(1-e^{-aT})]}\right| \\
&< \left|\frac{aT-(1-e^{-aT})}{T[aT-(1-e^{-aT})]}\right| \quad \text{for} \quad T \neq 0, \quad a > 0 \\
&= \frac{1}{T}
\end{aligned} \tag{26}$$

Since the zero is contained within the region $|\varepsilon + (1/T)| < 1/T$ we conclude the system is minimum phase for all positive values of the parameters b, J, k.

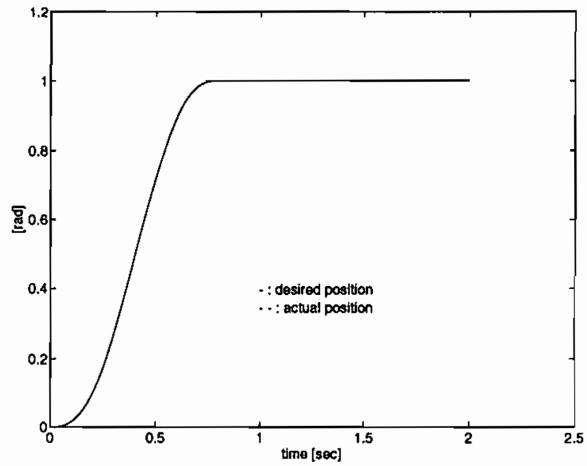

Figure 4. Position response, $J = 0{\cdot}83$ kg m^2 (simulation).

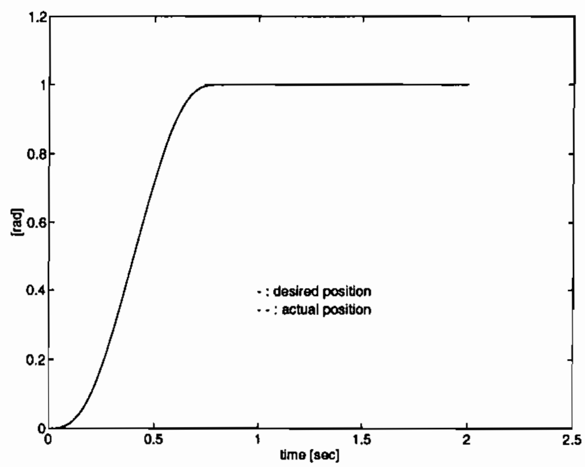

Figure 5. Position response, $J = 2{\cdot}95$ kg m^2 (simulation).

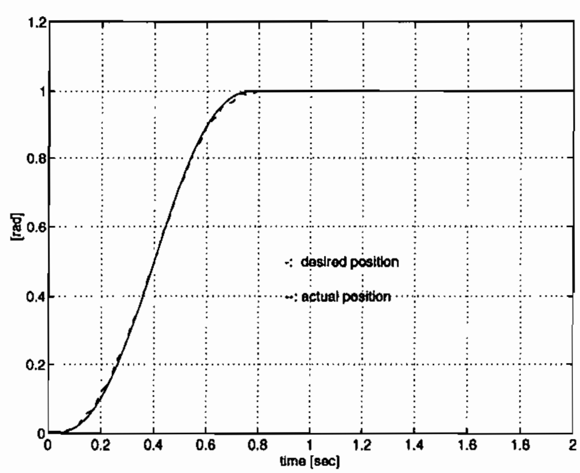

Figure 6. Position response, $J = 0{\cdot}83$ kg m^2 (experiment).

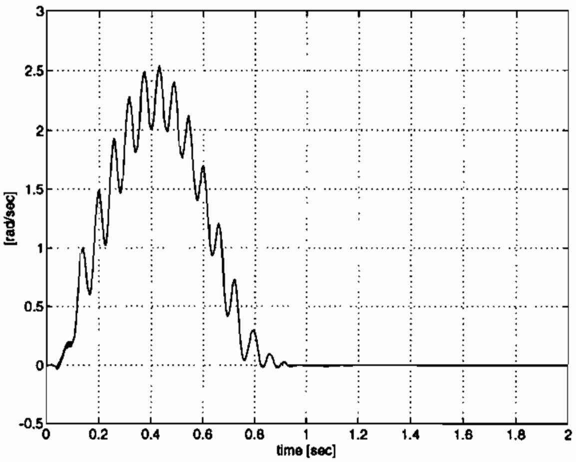

Figure 7. Velocity response, $J = 0{\cdot}83$ kg m^2 (experiment).

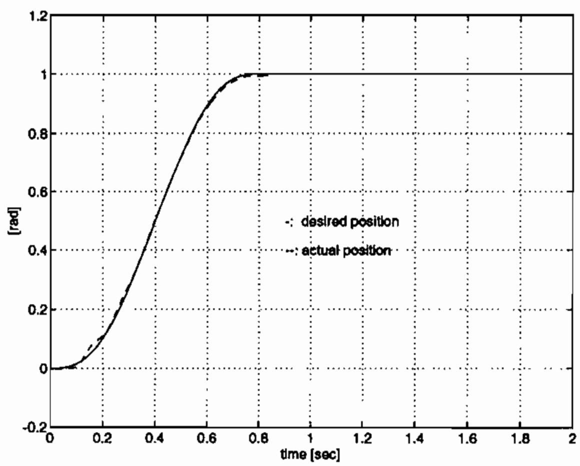

Figure 8. Position response, $J = 2{\cdot}95$ kg m^2 (experiment).

Figures 4 and 5 show simulation plots of the desired and actual paths for the cases $J = 0{\cdot}83$ kg m^2 and $J = 2{\cdot}95$ kg m^2 respectively. In order to avoid high frequency excitation, the desired trajectory was selected to be a fifth-order polynomial input, which is smooth up to the third derivative at the beginning and end points. Sampling time in both instances was selected to be $T = 2$ ms. It is noteworthy that the desired path is fairly fast and is representative of the actual speeds required by robot arm servo applications. Figures 4 and 5 show results that are obtained by computer simulation. Figures 6 through 9 show results that are obtained from experiment. Looking at the plots, it is evident that the proposed control gives good tracking. The small tracking error that is obtained under the inertia variation demonstrates the robustness of the proposed controller. The high-frequency components observed in the velocity response (Figs 7 and 9) is attributed to the influence of unmodelled friction and to numerical differentiation of the position to determine the velocity. The high gain nature of the control action can excite unmodelled dynamics. The influence of the high gain nature of the controller is observed if one compares Fig. 7 for the smaller inertia

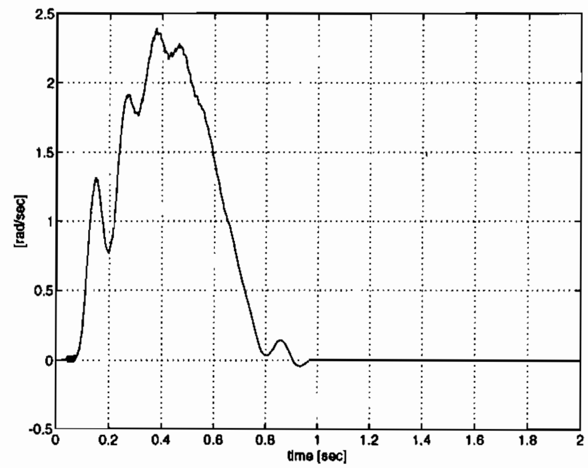

Figure 9. Velocity response, $J = 2{\cdot}95$ kg m^2 (experiment).

with Fig. 9 for the larger inertia. This can be reduced by filtering the control to remove high-frequency components and/or by identification of the friction and incorporating a friction compensator. A friction compensator and the time delay control are not mutually exclusive and can co-exist in the overall system.

6. Conclusions

A robust controller capable of tracking prescribed trajectories, in the presence of unknown parameters and unexpected disturbances, was proposed and tested through both simulation and experiment. The analysis has been presented in the discrete time form using the Euler operator with the assumption of a reasonably small sampling time. The strategy in dealing with the unknown system perturbations is to delay time by one sampling interval to estimate their effects. The information gathered is then used to cancel these effects. In this respect the control, which has been formulated, is consistent with the time delay control (TCD). Simulation and experimental results show that the controller exhibits reasonably good asymptotic tracking when the underlying system is perturbed by unknown plant parameters.

Appendix A: Derivation of (4)

Consider the continuous-time system description given by (1). For notational simplicity denote

$$A_c = A_{nc} + B_{nc}\Delta_a \tag{A 1}$$

$$B_c = B_{nc} + B_{nc}\Delta_a \tag{A 2}$$

$$\Delta A_c = B_{nc}\Delta_a \tag{A 3}$$

$$\Delta B_c = B_{nc}\Delta_b \tag{A 4}$$

$$F_c = B_{nc}\Delta_f \tag{A 5}$$

If (1) is discretized using a zero-order hold for which $u(t) = u(k)$ for $kT \leqslant t \leqslant (k+1)T$

and is expressed in 'delta' form (3) is obtained. Equation (3) can be rewritten in succinct form as

$$\delta x(k) = (G_n + \Delta G)\, x(k) + Hu(k) + Fd(k) \tag{A 6}$$

where $G_n = (e^{A_{nc}T} - I)/T$ and $\Delta G = (e^{A_c T} - I)/T - (e^{A_{nc}T} - I)/T$.

$$\begin{aligned} H &= \frac{1}{T}\int_0^T e^{A_c \tau} B_{nc}(1+\Delta_b)\, d\tau \\ &= \frac{1}{T}\int_0^T (e^{A_c \tau} - e^{A_{nc}\tau} + e^{A_{nc}\tau})\, B_{nc}(1+\Delta_b)\, d\tau \\ &= (H_n + \Delta H)(1+\Delta_b) \end{aligned} \tag{A 7}$$

$$H_n \frac{1}{T}\int_0^T e^{A_{nc}\tau} B_{nc}\, d\tau = \frac{1}{T}\left(T + \frac{1}{2!}A_{nc}T^2 + \frac{1}{3!}A_{nc}^2 T^3 + \cdots\right) B_{nc} \tag{A 8}$$

$$\begin{aligned} \Delta H &= \frac{1}{T}\int_0^T (e^{A_c\tau} - e^{A_{nc}\tau})\, B_{nc}\, d\tau \\ &= \frac{1}{T}\int_0^T \left[(A_c - A_{nc})\,\tau + \frac{1}{2!}(A_c^2 - A_{nc}^2)\,\tau^2 + \frac{1}{3!}(A_c^3 - A_{nc}^3)\,\tau^3 + \cdots\right] B_{nc}\, d\tau \\ &= \frac{1}{T}\left\{B_{nc}\, T\Delta_a B_{nc}\frac{T}{2!} + (A_c^2 - A_{nc}^2)\, B_{nc}\frac{T^3}{3!} + (A_c^3 - A_{nc}^3)\, B_{nc}\frac{T^4}{4!} + \cdots\right\} \end{aligned} \tag{A 9}$$

This equation and (A 8) yield

$$\begin{aligned} \Delta H = \frac{1}{T}\Bigg\{ & H_n \Delta_a B_{nc}\frac{T^2}{2!} - \frac{1}{2}\left(A_{nc} B_{nc} \Delta_a \frac{T^3}{2!} + A_{nc}^2 B_{nc}\Delta_a \frac{T^4}{3!} + \cdots\right) B_{nc} \\ & + (A_c^2 - A_{nc}^2)\, B_{nc}\frac{T^3}{3!} + (A_c^3 - A_{nc}^3)\, B_{nc}\frac{T^4}{4!} + \cdots\Bigg\} \end{aligned}$$

By observing this equation, if the condition imposed by the small sampling time assumption holds then we drop all terms of second order or higher in the sampling time T and approximate ΔH by

$$\Delta H \approx H_n \Delta_a B_{nc}\frac{T}{2} \tag{A 10}$$

We can do a similar analysis with respect to ΔG as follows:

$$\begin{aligned} \Delta G &= \frac{e^{A_c T} - I}{T} - \frac{e^{A_{nc}T} - I}{T} \\ &= \frac{1}{T}\left\{(A_c - A_{nc})\, T + \frac{1}{2!}(A_c^2 - A_{nc}^2)\, T^2 + \frac{1}{3!}(A_c^3 - A_{nc}^3)\, T^3 + \cdots\right\} \\ &= \frac{1}{T}\left\{\left(B_{nc}T + \frac{1}{2!}A_{nc}B_{nc}T^2 + \cdots\right)\Delta_a + B_{nc}T\left(\Delta_a A_{nc}\frac{T}{2} + \Delta_a B_{nc}\Delta_a\frac{T}{2}\right) + \cdots\right\} \end{aligned} \tag{A 11}$$

This equation with (A 8), taking into account the small sampling time assumption, yields

$$\Delta G \approx H_{\mathrm{n}} \Delta_{\mathrm{a}} \left\{ I + (A_{\mathrm{nc}} + B_{\mathrm{nc}} \Delta_{\mathrm{a}}) \frac{T}{2} \right\} \tag{A 12}$$

By employing the same technique to the coefficient of $d(k)$ and by invoking the assumption of small sampling time we arrive at the following:

$$\delta x(k) = G_{\mathrm{n}} x(k) + H_{\mathrm{n}} \Delta_{\mathrm{a}} \left\{ I + (A_{\mathrm{nc}} + B_{\mathrm{nc}} \Delta_{\mathrm{a}}) \frac{T}{2} \right\} x(k)$$
$$+ H_{\mathrm{n}} \left(I + \Delta_{\mathrm{a}} B_{\mathrm{nc}} \frac{T}{2} \right) (1 + \Delta_{\mathrm{b}}) u(k) + H_{\mathrm{n}} \left(I + \Delta_{\mathrm{a}} B_{\mathrm{nc}} \frac{T}{2} \right) \Delta_{\mathrm{f}} d(k) \tag{A 13}$$

It is readily obvious from this last equation that the uncertainties in the control input channel and the coefficient of the disturbance are matched with respect to H_{n}. Let $A = G_{\mathrm{n}}$, $\Delta A = \Delta_{\mathrm{a}}\{I + (A_{\mathrm{nc}} + B_{\mathrm{nc}} \Delta_{\mathrm{a}}) \frac{T}{2}\}$, $B = H_{\mathrm{n}}$, $b = \Delta_{\mathrm{a}} B_{\mathrm{nc}}(1+\Delta_{\mathrm{b}}) \frac{T}{2} + \Delta_{\mathrm{b}}$, $w(k) = (I + \Delta_{\mathrm{a}} B_{\mathrm{nc}} \frac{T}{2}) \Delta_{\mathrm{f}} d(k)$. Then (A 13) and (4) are identical.

Appendix B: Error analysis

The continuous-time plant dynamic equation considered is given by

$$\left.\begin{aligned} \dot{x}(t) &= (A_{\mathrm{nc}} + \Delta A_{\mathrm{c}}) x(t) + (B_{\mathrm{nc}} + \Delta B_{\mathrm{c}}) u(t) \\ \bar{y}(t) &= c x(t) \\ A_{\mathrm{c}} &= A_{\mathrm{nc}} + \Delta A_{\mathrm{c}}, \quad B_{\mathrm{c}} = B_{\mathrm{nc}} + \Delta B_{\mathrm{c}} \\ \Delta A_{\mathrm{c}} &= B_{\mathrm{nc}} \Delta_{\mathrm{a}}; \quad \Delta B_{\mathrm{c}} = B_{\mathrm{nc}} \Delta_{\mathrm{b}} \end{aligned}\right\} \tag{B 1}$$

where $\bar{y}$ is the output and all variables and symbols are as defined with (1) and the disturbance term is assumed not to exist. We seek to analyse the error that is introduced by invoking the small sampling time assumption, i.e. by using the approximate equations given by (4). We have, from (4) with the external disturbance set to zero

$$\delta x(k) = (A + H_{\mathrm{n}} \Delta A) x(k) + B(1+b) u(k)$$
$$y(k) = c x(k)$$

where

$$A = (\mathrm{e}^{A_{\mathrm{nc}} T} - I)/T \quad \text{and} \quad B = \frac{1}{T} \int_0^T \mathrm{e}^{A_{\mathrm{nc}} \tau} B_{\mathrm{nc}} \, \mathrm{d}\tau$$

express the nominal matrices in discrete-time after discretizing using a zero-order hold and $\Delta A = \Delta_{\mathrm{a}}\{I + (A_{\mathrm{nc}} + B_{\mathrm{nc}} \Delta_{\mathrm{a}}) \frac{T}{2}\}$, $b = \Delta_{\mathrm{a}} B_{\mathrm{nc}}(1+\Delta_{\mathrm{b}}) \frac{T}{2} + \Delta_{\mathrm{b}}$. The solution of this equation for $x(0) = 0$ is given by

$$y(k) = \sum_{j=0}^{k-1} c\{I + (A + H_{\mathrm{n}} \Delta A) T\}^{k-1-j} B(1+b) T u(j) \tag{B 2}$$

Assume the initial condition of the continuous-time system given by (B 1) is $x(0) = 0$. Subject to the input $u(j)$, $\bar{y}(k)$ is obtained by

$$\begin{aligned}
\bar{y}(k) &= c\int_0^{kT} e^{A_c(kT-\tau)} B_c u(\tau)\, d\tau \\
&= \sum_{j=0}^{k-1} c\int_{jT}^{(j+1)T} e^{A_c(kT-\tau)} B_c u(j)\, d\tau \\
&= \sum_{j=0}^{k-1} c(e^{A_c T})^{k-1-j}\int_0^T e^{A_c\tau}\, d\tau\, B_c u(j) \qquad \text{(B 3)}
\end{aligned}$$

For a finite sampling interval, T, we need to consider the deviation between the true output, $(\bar{y}(k))$, of the continuous-time system given by (B 3) and the output of the approximate system, $y(k)$, given by (B 2) over the duration of one sampling interval. This is because the time delay controller is a state-based controller and the states of the approximate system and the true system are equalized at the end of each sampling interval. For ease of analysis we can consider the interval $0 \leqslant t \leqslant T$, i.e. $j = 0$, $k = 1$ and $\|u(0, T)\| \leqslant U_M$. We have

$$\begin{aligned}
\|\bar{y}(k) - y(k)\| &\leqslant \left\| c\left\{ \int_0^T e^{A_c\tau}\, d\tau\, B_{nc}(1+\Delta_b) \right.\right. \\
&\quad \left.\left. - \int_0^T e^{A_{nc}\tau}\, d\tau\, B_{nc}\left(I + \Delta_a B_{nc}\frac{T}{2}\right)(1+\Delta_b)\right\}\right\| U_m \\
&\leqslant \left\| c\left\{ B_{nc}\Delta_a\frac{T^2}{2} + \left(A_c^2\frac{T^3}{3!} + A_c^3\frac{T^4}{4!} + A_c^4\frac{T^5}{5!} + \cdots\right)\right.\right. \\
&\quad - \left(A_{nc}^2\frac{T^3}{3!} + A_{nc}^3\frac{T^4}{4!} + \cdots\right) - \left(TI + A_{nc}\frac{T^2}{2!} + A_{nc}^2\frac{T^3}{3!}\right. \\
&\quad \left.\left.\left. + A_{nc}^3\frac{T^4}{4!} + \cdots\right)\Delta_a B_{nc}\frac{T}{2}\right\} B_{nc}(1+\Delta_b)\right\| U_M \\
&\leqslant \left\| c\left\{\left(A_c^2\frac{T^3}{3!} + A_c^3\frac{T^4}{4!} + A_c^4\frac{T^5}{5!} + \cdots\right) B_{nc}(1+\Delta_b)\right.\right. \\
&\quad - \left(A_{nc}^2\frac{T^3}{3!} + A_{nc}^3\frac{T^4}{4!} + \cdots\right) B_{nc}(1+\Delta_b) - \left(A_{nc}\frac{T^2}{2!} + A_{nc}^2\frac{T^3}{3!}\right. \\
&\quad \left.\left.\left. + A_{nc}^3\frac{T^4}{4!} + \cdots\right) B_{nc}\Delta_a B_{nc}(1+\Delta_b)\frac{T}{2}\right\}\right\| U_M \\
&= \left\| c\left\{\left(A_c^2\frac{T^3}{3!} + A_c^3\frac{T^4}{4!} + A_c^4\frac{T^5}{5!} + \cdots\right) - \left(A_{nc}^2\frac{T^3}{3!} + A_{nc}^3\frac{T^4}{4!} + \cdots\right)\right.\right. \\
&\quad \left.\left. - \left(A_{nc}\frac{T^2}{2!} + A_{nc}^2\frac{T^3}{3!} + A_{nc}^3\frac{T^4}{4!} + \cdots\right)\Delta_a B_{nc}\frac{T}{2}\right\} B_{nc}(1+\Delta_b)\right\| U_M \\
&\leqslant \|c\|\left\{\int_0^T \|e^{A_c\tau} - (I + A_c\tau)\|\, d\tau + \int_0^T \|e^{A_{nc}\tau} - (I + A_{nc}\tau)\|\, d\tau\right. \\
&\quad \left. + \int_0^T \|e^{A_{nc}T} - I\|\, d\tau \|\Delta_a B_{nc}\|\frac{T}{2}\right\} \|B_c\|\, U_M \qquad \text{(B 4)}
\end{aligned}$$

Consider a truncation of a power series (Middleton and Goodwin 1990),

$$E(AT) = \sum_{k=0}^{N} \frac{(AT)^k}{k!} \tag{B 5}$$

If $\|AT\| < N+1$ then

$$\|e^{AT} - E(AT)\| \leqslant \frac{\|AT\|^{N+1}}{(N+1)!} \left\{ \frac{1}{1 - \dfrac{\|AT\|}{(N+1)}} \right\} \tag{B 6}$$

where $\|\cdot\|$ denotes the Frobenius norm. Applying this result to (B 4) we have

$$\begin{aligned} \|\bar{y}(k) - y(k)\| \leqslant \|c\| \Bigg\{ & \int_0^T \frac{\|A_c\|^2 \tau_2}{2!} \left\{ \frac{1}{1 - \dfrac{\|A_c\| \tau}{2}} \right\} d\tau \\ & + \int_0^T \frac{\|A_{nc}\|^2 \tau^2}{2} \left\{ \frac{1}{1 - \dfrac{\|A_{nc}\| \tau}{2}} \, d\tau \right\} \\ & + \int_0^T \|A_{nc}\| \tau \left\{ \frac{1}{1 - \|A_{nc}\| \tau} \right\} d\tau \, \| \Delta_a B_{nc} \frac{T}{2} \Bigg\} \|B_c\| \, U_M \end{aligned} \tag{B 7}$$

Integrating and making the approximation

$$\ln \left(1 - \frac{\|A\| T}{2} \right) \approx \left(- \frac{\|A\| T}{2} - \frac{\|A\|^2 T^2}{4} \right) \tag{B 8}$$

$$\ln (1 - \|A\| T) \approx \left(- \|A\| T - \frac{\|A\|^2 T^2}{2} \right) \tag{B 9}$$

where $\ln(\cdot)$ represents the natural logarithmic function we get

$$\|\bar{y}(k) - y(k)\| \leqslant \frac{T^2}{2} \|c\| \left\{ \|A_c\| + \|A_{nc}\| + \|A_{nc}\| \, \|\Delta_a B_{nc}\| \frac{T}{2} \right\} \|B_c\| \, U_M \tag{B 10}$$

From this equation we conclude that the maximum error between the true output, $\bar{y}(k)$, and the approximate output, $y(k)$, over one sampling interval is dependent on the square of the sampling interval as well as on the magnitude of the uncertain terms. We can make the error small by reducing the sampling time. In this case, reducing the sampling interval by half translates into a quarter reduction of the output error.

Appendix C: Algebraic manipulation leading to (21) and (22)

Under a controllable-canonical transformation ($x \to \bar{z}$, $x_m \to \bar{z}_m$), (17) becomes

$$\begin{aligned} u(k) = \frac{1}{1+\hat{b}} \{ & b_m r(k) + (f_m + k_r) \bar{z}(k) - k_r \bar{z}_m(k) - \delta \bar{z}_n(k-1) \\ & + f(\bar{z}(k-1) - \bar{z}(k)) \} + u(k-1) \end{aligned} \tag{C 1}$$

where $e(k) = \bar{z}_m(k) - \bar{z}(k)$. Next, performing an ε-transformation we have

$$u(\varepsilon) = \frac{1}{1+\hat{b}}\{b_m r(\varepsilon) + P_{kr} y_m(\varepsilon) - P_{mk} \bar{z}_1(\varepsilon)\} + \frac{1}{1+\varepsilon T} u(\varepsilon) + \frac{1}{1+\hat{b}} Q_m \bar{z}_1(\varepsilon) \tag{C 2}$$

where

$$P_{kr} = k_{nr} \varepsilon^{n-1} + k_{(n-1)r} \varepsilon^{n-2} + \cdots + k_{1r} \tag{C 3}$$

$$P_{mk} = \frac{\varepsilon^n}{1+\varepsilon T} + (a_{nm} + k_{nr}) \varepsilon^{n-1} + \cdots + (a_{1m} + k_{1r}) \tag{C 4}$$

$$Q_m = \frac{\varepsilon T}{1+\varepsilon T}(a_n \varepsilon^{n-1} + \cdots + a_1) \tag{C 5}$$

We can consider the effects of P_{mk} and Q_m as follows:

$$\begin{aligned} P_{pmk} = P_{mk} - Q_m &= \frac{\varepsilon^n}{1+\varepsilon T} + \left(a_{nm} + k_{nr} - \frac{\varepsilon T}{1+\varepsilon T} a_n\right) \varepsilon^{n-1} \\ &\quad + \left(a_{(n-1)m} + k_{(n-1)r} - \frac{\varepsilon T}{1+\varepsilon T} a_{(n-1)}\right) \varepsilon^{n-2} + \cdots + \left(a_{1m} + k_{1r} - \frac{\varepsilon T}{1+\varepsilon T} a_1\right) \\ &= \frac{1}{1+\varepsilon T}\{[1 - Ta_n] \varepsilon^n + [(a_{nm} + k_{nr})(1+\varepsilon T) - Ta_{n-1}] \varepsilon^{n-1} \\ &\quad + [(a_{(n-1)m} + k_{(n-1)r})(1+\varepsilon T) - Ta_{(n-2)}] \varepsilon^{n-2} + \cdots + [(a_{1m} + k_{1r})(1+\varepsilon T)]\} \end{aligned} \tag{C 6}$$

For a small sampling time, T, we can consider the previous equation as a perturbation polynomial and approximate it by

$$\begin{aligned} P_{pmk} = P_{mk} + Q_m &\approx \frac{1}{1+\varepsilon T}\{\varepsilon^n + (a_{nm} + k_{nr})(1+\varepsilon T) \varepsilon^{n-1} \\ &\quad + (a_{(n-1)m} + k_{(n-1)r})(1+\varepsilon T) \varepsilon^{n-2} + \cdots + (a_{1m} + k_{1r})(1+\varepsilon T)\} \\ &\approx \frac{1}{1+\varepsilon T} \varepsilon^n + (a_{nm} + k_{nr}) \varepsilon^{n-1} + [(a_{(n-1)m} + k_{(n-1)r}) \varepsilon^{n-2} + \cdots + (a_{1m} + k_{1r})] \end{aligned} \tag{C 7}$$

Now $\varepsilon = 0$ corresponds to $z = 1$ (or $s = 0$). $1 + \varepsilon T = 0$ corresponds to $z = 0$ (or $s = -\infty$). Thus, it is reasonable to approximate $\varepsilon^n/(1+\varepsilon T) \approx \varepsilon^n$ in the low frequency region. Combining the last expression with the expression for $u(\varepsilon)$ we have

$$u(\varepsilon) \approx \frac{1+\varepsilon T}{\varepsilon T} \frac{1}{1+\hat{b}}\{b_m r(\varepsilon) + P_{kr} y_m(\varepsilon) - P_{pmk} \bar{z}_1(\varepsilon)\} \tag{C 8}$$

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