Uncertainty and Disturbance Estimation

February 16, 2014

1 Introduction

Design of conventional controllers to stabilise an unstable system or improve the performance of a stable systems has been in vogue for many decades. The controllers designed by *classical* methods do not yield acceptable results when the plant undergoes various changes in the due course of operation and/or acted upon by disturbances. Hence there is a necessity to design or augment the conventional controllers which can tackle 'uncertainties' and 'disturbances'. This is the idea behind design of Robust Control Strategies. Although many robust control strategies are in place for both linear as well as nonlinear systems, Uncertainty and Disturbance Estimation (UDE) proposed in 2004 is an elegant, systematic strategy in the design of robust control systems.

The primary idea behind this technique is to 'estimate' the uncertainty and disturbance, in an integrated manner, then cancel their effects. Another notable feature of this technique is that it does not require any knowledge of the uncertainty and/or the disturbance; such as their magnitude or bounds. The estimation is done dynamically and compensated. Most of the other famous robust control strategy such as 'sliding mode control' require the upper and lower bounds of the uncertainty/disturbance.

2 Robust Control using the strategy of UDE

2.1 Concept of State Feedback Control

Consider the following second order system defined in the 'controllable canonical' form;

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1 x_1 + a_2 x_2 + bu \end{aligned} \tag{1}$$

In the above equations a_1, a_2 are the parameters or coefficients in System matrix and b is the coefficient in the Input Matrix. x_1 and x_2 are the state variables with u as the control.

If we define the control (u) to have the following form, **assuming that** x_1 **and** x_2 **are measurable and available**;

$$u = \frac{1}{b}[-a_1 x_1 - a_2 x_2] \tag{2}$$

By substituting u in (1), we get

$$\dot{x}_2 = \ddot{x}_1 = 0 \tag{3}$$

If we take x_1 as the output, the expression $\ddot{x}_1 = 0$ does not guarantee x_1 or the output to be stable. The reason being the second derivative of x_1 is zero, which indicates the first derivative is 'constant' and finally x_1 is varying. The ultimate aim in control engineering is that the 'output'; here it is x_1, should come to zero from any non-zero initial condition or maintain a constant value (Regulation Problem) or it should track a desired reference trajectory (Tracking Problem).

Therefore to ensure that x_1 tends to zero 'asymptotically', we need to augment or redefine the control u. Now we shall redefine the control u to have the following form;

$$u = \frac{1}{b}[u_a + \nu] \tag{4}$$

with $u_a = -a_1 x_1 - a_2 x_2$ and ν is the control which would drive the system to the 'desired' dynamics. u_a will be henceforth known as 'nominal' control and ν as the 'outer loop' control. Let us assume ν to have the following form;

$$\nu = -k_1 x_2 - k_0 x_1 \tag{5}$$

where k_1 and k_0 are the user defined feedback gains, which can be chosen during the design. By substituting the expressions of u_a and ν in (1) would result in;

$$\dot{x}_2 = -k_1 x_2 - k_0 x_1$$

Since $\dot{x}_1 = x_2$, the above equation can be expressed as

$$\ddot{x}_1 = -k_1 \dot{x}_1 - k_0 x_1 \tag{6}$$

or

$$\ddot{x}_1 + k_1 \dot{x}_1 + k_0 x_1 = 0 \tag{7}$$

This second order dynamics of x_1 can be made stable by proper choice of k_1 and k_0. The necessary and sufficient condition for a second order dynamics to be stable (for both regulation and tracking) is that $k_1, k_0 > 0$. However the actual values of desired k_1 and k_0 depend upon the design specifications like settling time, rise time, peak overshoot etc. They can also be chosen as per requirement of 'pole placement' using Ackerman's formula.

The activities discussed till now is the classical control using feedback gains. The controls u_a and ν once chosen cannot be altered. This means the design is frozen assuming a_1 and a_2 do not change. This assumptions are not generally valid in real-time situations or applications; since a_1 and a_2 are physical parameters, they tend to change in due course of time like spring stiffness, hydraulic pressure due to leakage, consumption of fuel, change in Centre of Gravity in aerospace vehicles etc. The effects of these changes in a_1 and a_2 manifest themselves in poor or degraded performance and even instability. However these effects can be mitigated to some extent by proper choice of k_1 and k_0. ν is also PD controller (k_0 being the proportional gain and k_1 is the derivative gain). Industries generally use PID controllers to take care of uncertainties and disturbances; which is not a viable solution.

2.2 Concept of UDE control

Let us assume that there are uncertainties in a_1, a_2 and b. In addition there is some disturbance d acting on the system. The system is now expressed as;

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + (b + \Delta b)u + d \end{aligned} \tag{8}$$

Where Δa_1, Δa_2 and Δb are the uncertainties in a_1, a_2 and b, respectively. d is disturbance which may not be measurable. Combining the uncertainties and disturbance, we can express the dynamics as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1x_1 + a_2x_2 + bu + D \end{aligned} \tag{9}$$

with $D = \Delta a_1x_1 + \Delta a_2x_2 + \Delta bu + d$, known as 'lumped' uncertainty. Now, if we apply the control given by (4), the resulting dynamics would be;

$$\ddot{x}_1 + k_1\dot{x}_1 + k_0x_1 = D \tag{10}$$

The dynamics of (10) is indeed stable for $k_1, k_0 > 0$; however due to the presence of D in the RHS of (10), which may or may not be zero, results in 'steady-state errors', which cannot assure the desired regulation or tracking from the plant. Hence we have to resort to a technique to make D as zero. One way is to estimate D and use it in the control law and cancel its effects. The technique of UDE follows this strategy.

Let us assume $\hat{D}$ be the estimate D and related to it through the following expression;

$$\hat{D} = G_f(s)D \tag{11}$$

The idea is that if we pass D through a filter and take $\hat{D}$ as its estimate. The first-order filter $G_f(s)$ has the form

$$G_f(s) = \frac{1}{1 + s\tau} \tag{12}$$

where τ is the filter time constant; whose bandwidth is sufficient enough to pass D through it. This means we are able to estimate D by passing it through the filter. Now we shall redefine the control (u) once again as

$$u = \frac{1}{b}[u_a + u_d + \nu] \tag{13}$$

with u_d as the part which takes care of D. We are now left with the design of u_d. Since $\hat{D}$ is the estimate of D, we can define $u_d = -\hat{D}$. Now applying the control given by (13) with the definition of u_d, would result in the dynamics as

$$\ddot{x}_1 + k_1\dot{x}_1 + k_0 x_1 = D - \hat{D} \tag{14}$$

If we can ensure $\hat{D}$ to be correct estimate of D, then RHS of (14) would become zero, which is our ultimate aim. The estimation of D has to be done fast with the available resources, namely the states x_1 and x_2. Since $\hat{D}$ and u_d are also related, we should be able to find an expression for u_d in terms of the states.

To this end, let us revisit the system dynamics with uncertainties and disturbance given in (9). From the second line of this equation we can write

$$D = \dot{x}_2 - a_1 x_1 - a_2 x_2 - bu \tag{15}$$

Substituting for u from (13), results in

$$D = \dot{x}_2 - u_d - \nu \tag{16}$$

Since $\hat{D} = G_f(s)D$, we write

$$\hat{D} = G_f(s)[\dot{x}_2 - u_d - \nu] \tag{17}$$

Also $u_d = -\hat{D}$, then

$$-u_d = G_f(s)[\dot{x}_2 - u_d - \nu] \tag{18}$$

Solving for u_d and noting that $\frac{G_f(s)}{(1-G_f(s))} = \frac{1}{s\tau}$, we can get the final form of u_d as

$$\begin{aligned} u_d &= -\frac{G_f(s)}{(1 - G_f(s))}[\dot{x}_2 - \nu] \\ &= -\frac{1}{s\tau}[\dot{x}_2 - \nu] \\ &= -\frac{1}{\tau}\left[x_2 - \int \nu dt\right] \end{aligned} \tag{19}$$

We can see from (19) that we are able to find an expression for u_d and in turn $\hat{D}$. Now the control u using the expressions for u_a, ν and u_d would ensure robustness for the plant dynamics for uncertainties and disturbances. One can also note that none of the expressions for the control had the limits of D, which goes on to prove that this technique is dynamic. Integrating the expression for ν would give rise to a dynamic PID controller.

2.3 Stability Analysis

Design of the control law is not the end of the road. We have to do Stability analysis of overall Closed loop system. This analysis is important to understand the conditions imposed on the values of feedback gains (k_1, k_0) and time constant τ, for the desired performance. The stability analysis also gives an insight into other factors that affect the system performance.

The stability analysis, generally consists to two parts. Firstly, we have to derive the system dynamics by substituting the 'designed' control. Secondly, the 'disturbance estimation' error dynamics have to be analysed. Proceeding further, if we substitute the control u given by (13) with the expressions for u_a and ν; using $u_d = -\hat{D}$ into the plant dynamics of (9),we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_0 x_1 - k_1 x_2 + D - \hat{D} \end{aligned} \tag{20}$$

Let us define the disturbance estimation error as $\tilde{D} = D - \hat{D}$; (20) takes the following form;

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_0 x_1 - k_1 x_2 + \tilde{D} \end{aligned} \tag{21}$$

2.3.1 Disturbance Estimation Error Dynamics

Using the relation between D and $\hat{D}$ which is $\hat{D} = G_f(s)D$, we can derive the following;

$$\begin{aligned} \hat{D} &= \frac{1}{1+\tau s} D \\ \hat{D} + \tau\dot{\hat{D}} &= D \\ \tau\dot{\hat{D}} &= D - \hat{D} \\ \dot{\hat{D}} &= \frac{1}{\tau}\tilde{D} \end{aligned} \tag{22}$$

Now

$$\begin{aligned} \tilde{D} &= D - \hat{D} \\ \dot{\tilde{D}} &= \dot{D} - \dot{\hat{D}} \end{aligned} \tag{23}$$

Substituting (22) in (23) we get the disturbance estimation error dynamics as

$$\dot{\tilde{D}} = \dot{D} - \frac{1}{\tau}\tilde{D} \tag{24}$$

2.4 Stability Analysis - contd

Combining (21) and (24) we write the overall closed loop dynamics as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_0 x_1 - k_1 x_2 + \tilde{D} \\ \dot{\tilde{D}} &= \dot{D} - \frac{1}{\tau}\tilde{D} \end{aligned} \tag{25}$$

This can also be written in the matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\tilde{D}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_0 & -k_1 & 1 \\ 0 & 0 & (-\frac{1}{\tau}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \tilde{D} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{D} \tag{26}$$

The dynamics given above is stable if and and only if the eigen values of the System matrix is in the LHP of the s - plane, i.e., $|\lambda I - A| = 0$ where A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -k_0 & -k_1 & 1 \\ 0 & 0 & (-\frac{1}{\tau}) \end{bmatrix}$$

Solving the expression $|\lambda I - A| = 0$ gives rise to a situation

$$\left(\lambda + \frac{1}{\tau}\right)\left(\lambda^2 + k_1\lambda + k_0\right) = 0 \tag{27}$$

This gives the condition for the stability as k_1, k_0 and $\tau > 0$. One more important inference from (26) is that the disturbance estimation error is also governed by $\dot{D}$. If $\dot{D}$ is very small or rate of change of disturbance (including the uncertainty) is negligible, then asymptotic stability is assured for $k_1, k_0 \& \tau > 0$. If not, i.e $\left|\dot{D}\right|$ has a definite value, then Bounded-Input-Bounded-Output (BIBO) stability is guaranteed. In other words small steady state errors may be present.

This technique can applied to linear as well as nonlinear problems. It would be prudent, if we can formulate a problem to the type shown in (1) or (9).