International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Robust control of discretized continuous systems using the theory of sliding modes

ADDISU TESFAYE [a] & MASAYOSHI TOMIZUKA [a]
[a] Department of Mechanical Engineering , University of California at Berkeley , Berkeley, CA, 94720, U.S.A.
Published online: 24 Feb 2007.

INT. J. CONTROL, 1995, VOL. 62, NO. 1, 209–226

Robust control of discretized continuous systems using the theory of sliding modes

ADDISU TESFAYE† and MASAYOSHI TOMIZUKA†

The idea of sliding mode control (SMC) as a robust control technique is utilized to control systems where the dynamics can be described by $\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + d(t)$ where ΔA, ΔB and $d(t)$ characterize unknown plant parameters and unexpected disturbances, respectively. The analysis is described in the discrete form using the Euler operator. The proposed controller regards the influence of unknown disturbances and parameter uncertainties as an equivalent disturbance and generates a control function (estimate) to cancel their influence by the mechanism of time delay. The states of the uncertain plant are steered from an arbitrary initial state to a stable surface, $s = 0$, and the plant output is subsequently regulated. One feature of the control is that the sliding conditions are satisfied without the discontinuous control action used in classical SMC. It therefore retains the positive features of SMC without the disadvantage of control chattering. The controller has been studied through simulations and experiment on an NSK direct drive robot arm driven by a DC motor. The results confirm that the control is robust to disturbances and parameter variations.

1. Introduction

Sliding mode control (SMC) is a robust control method which can handle both linear and nonlinear systems hampered by parametric uncertainties and unexpected disturbances (Slotine and Sastry 1983, Utkin 1977). It is based on the concept of steering the states of a system to a specified stable manifold and using high-speed switching to maintain the subsequent motion on this manifold. Due to the requirement for instantaneous control action, SMC has been developed in the continuous time domain to a great extent.

With advances in computer technology, control algorithms are frequently implemented on digital computers. Associated with this trend is the necessity to investigate and design such controllers in the discrete time domain. In discrete time, where the control input is computed at discrete instants of time and instantaneous switching cannot be enforced, a non-ideal (quasi) sliding regime will appear which is different from that in continuous-time systems due to non-ideal behaviour of the analogue components (Utkin 1977, Sarpturk *et al.* 1987). The extent to which the states approach the prescribed manifold in discrete time depends on the sampling time and on the uncertainties in the system (Sarpturk *et al.* 1987, Furuta and Morisada 1989, Furuta 1990, Jabbari and Tomizuka 1992). In spite of this limitation, it is possible to guarantee asymptotic stability of uncertain sampled-data systems using the theory of sliding modes. Attempts have been made to develop discrete counterparts to the continuous-time SMC, with various degrees of success (Jabbari and Tomizuka

Received in final revised form 19 May 1994. Communicated by Professor P. A. Ioannou.
† Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA 94720, U.S.A.

1992, Manela 1985). Compared with previous attempts, the motivation of this paper stems from three important considerations:

(1) to address the issue of uncertainty transformation from the continuous to the discrete time domain for sampled-data systems. This issue has not been adequately addressed in the control literature. To cite an example, the input matrix in the discrete time model is usually assumed to satisfy the matching conditions. However, because of the exponential terms arising from discretization this assumption is not always true, even if the continuous time input matrix does satisfy them;
(2) to develop a systematic methodology by which a stable manifold is prescribed or designed. It is reasonable to expect that the sliding surface selection should be dependent on the dynamics of the system under investigation and on the choice of sampling time. This will affect the transient behaviour of the system dynamics as the states are steered to the sliding surface, and becomes important when a non-ideal (quasi) sliding regime develops;
(3) to reduce unwanted control chattering by developing a smooth (continuous) control action; simultaneously, to develop a controller which does not require knowledge on the bounds of parameter uncertainties and disturbances. In this way the control input can be realized with smaller gains and will retain the positive features of SMC without the disadvantage of control chatter.

The control law proposed is based on the concept of time delay control (TDC) (Tesfaye and Tomizuka 1993, Youcef-Toumi and Ito 1990). The main feature of the control is that uncertain dynamics of the system are estimated through observation of the system response. This information is used by the controller to cancel the undesired dynamics and to regulate the plant output.

The design analysis is presented in terms of an operator called the Euler (ε) operator or, as termed by Middleton and Goodwin (1986, 1990), the delta (δ) operator. One major advantage of using these operators is that the dichotomy between results obtained using continuous and discrete time control laws are resolved, especially with regard to the limiting properties as $T \to 0$ (Middleton and Goodwin 1990, Goodwin *et al.* 1986). This paper defines $\delta = (q-1)/T$ and $\varepsilon = (z-1)/T$, i.e. counterparts of the shift operators q and z in the time and frequency domains, respectively.

2. Problem statement

The continuous plant dynamic equations considered are of the following form:

$$\left.\begin{aligned} \dot{x}(t) &= \mathbf{A}_c x(t) + \mathbf{B}_c u(t) + \mathbf{F}_c d(t) \\ \mathbf{A}_c &= \mathbf{A}_{nc} + \Delta\mathbf{A}_c, \quad \mathbf{B}_c = \mathbf{B}_{nc} + \Delta\mathbf{B}_c \end{aligned}\right\} \tag{1}$$

The subscript 'nc' denotes the nominal part of the uncertain continuous-time system, that is, SISO systems are treated with structured addition type uncertainties. Here $x(t) \in \Re^n$ is the plant state vector, $u(t) \in \Re$ is the control input, $d(t) \in \Re$ is an external disturbance vector. $\Delta\mathbf{A}_c$ and $\Delta\mathbf{B}_c$ are the uncertainties in the system matrix and input channel respectively. In what follows it is assumed that the nominal pair $(\mathbf{A}_{nc}, \mathbf{B}_{nc})$ is stabilizable and the uncertainties satisfy the *matching conditions*, i.e. $\Delta\mathbf{A}_c = \mathbf{B}_{nc}\Delta_a$, $\Delta\mathbf{B}_c = \mathbf{B}_{nc}\Delta_b$ and $\mathbf{F}_c = \mathbf{B}_{nc}\Delta_f$.

When the above continuous-time plant is preceded by a zero-order hold, under digital control, the discrete time model using the δ operator is given by

$$\begin{aligned}\delta x(k) &= \mathbf{A}_d x(k) + \mathbf{B}_d u(k) + \mathbf{F}_d d(k)\\ \mathbf{A}_d &= \frac{e^{\mathbf{A}_c T} - I}{T}, \quad \mathbf{B}_d = \frac{1}{T}\int_0^T e^{\mathbf{A}_c\tau}\mathbf{B}_c\,d\tau \\ \mathbf{F}_d &= \frac{1}{T}\int_0^T e^{\mathbf{A}_c\tau}\mathbf{F}_c\,d\tau, \quad \delta = \frac{q-1}{T}\end{aligned} \tag{2}$$

where q is the traditional shift operator used in difference equations. Due to the exponential terms arising from the discretization process it is evident that the uncertainties will no longer be constrained to lie within the range space of the nominal control input channel B_{dn}. To enforce the matching conditions in the discrete time model, a suitable sampling time is chosen by means of the assumption below.

Assumption 1—Small sampling time: Let the sampling interval T, which transforms the continuous time dynamical system given by (1) into a discrete system, be small where a sampling time T is considered small if any function which is expanded in powers of T can be approximated with some degree of accuracy by keeping only terms up to and including order T^2. □

Making use of this assumption, one obtains the following discrete representation of the continuous-time system given by (1):

$$\delta x(k) = \mathbf{A}x(k) + \mathbf{B}\{(1 + b)u(k) + \bar{E}(k)\} \tag{3}$$

where $\bar{E}(k) = \Delta\mathbf{A}x(k) + w(k)$, $w(k) = (1 + \Delta_a\mathbf{B}_{nc}T/2)\Delta_f d(k)$ (see Appendix A for the derivation). In the above equation, the system parameters have the following values:

$$\mathbf{A} = \frac{e^{\mathbf{A}_{nc}T} - I}{T}, \quad \mathbf{B} = \frac{1}{T}\int_0^T e^{\mathbf{A}_{nc}\tau}\mathbf{B}_{nc}\,d\tau, \quad \Delta\mathbf{A} = \Delta_a\left\{I + (\mathbf{A}_{nc} + \mathbf{B}_{nc}\Delta_a)\frac{T}{2}\right\}$$

$$b = \Delta_a\mathbf{B}_{nc}(1 + \Delta_b)\frac{T}{2} + \Delta_b$$

Comparing the three models given by (1)–(3):

Remark 1: A in (3) does not involve any uncertainty, while A_d in (2) does. □

Remark 2: Both (2) and (3) are valid discrete-time models of the same continuous-time system. □

Remark 3: Equation (3) can be considered as an approximation of (2) wherein uncertainty terms of order T^3 and higher in the sampling time have been neglected. □

Remark 4: If there are no uncertainties, (3) is the discretized version of the nominal continuous-time system of (1). □

Remark 5: $\lim_{T\to 0}\Delta\mathbf{A} = \Delta_a$, $\lim_{T\to 0} b = \Delta_b$, $\lim_{T\to 0}(1 + \Delta_a\mathbf{B}_{nc}(T/2))\Delta_f = \Delta_f$, $\lim_{T\to 0}\mathbf{A} = \mathbf{A}_{nc}$, and $\lim_{T\to 0}\mathbf{B} = \mathbf{B}_{nc}$. This means that as $T \to 0$, the discretized uncertain system given by (3) approaches the uncertain continuous-time system of (1). The latter is an advantage obtained using the ε operator which could not be attained if the conventional z operator was used. □

If the discrete-time model (3) is regarded as an approximation of the continuous-time model (1), an upper bound on the error or maximum deviation between the outputs of the two systems, introduced by using the small sampling time assumption (Assumption 1), can be computed and is given in Appendix B. Its upper bound is of the order of T^2, so the accuracy with which (3) matches (1) increases with smaller sampling times.

3. Control formulation

3.1. *Linear feedback control stabilizing the nominal system*

To generalize the analysis, consider the case when the matrix $\mathbf{A}$ in (3) is not necessarily stable. Then, by assumption of stabilizability, there exists a linear feedback gain matrix $\mathbf{K}$ such that $\sigma(\mathbf{A}+\mathbf{BK}) \subset \Lambda$ ($\sigma(\cdot)$ denotes spectrum), where Λ represents the stability region which is a circle centred at $-1/T$ with radius $1/T$. Under this feedback system (3) can be rewritten as

$$\begin{aligned}\delta \boldsymbol{x}(k) &= \mathbf{A}\boldsymbol{x}(k) + \mathbf{B}\{(1+b)(\mathbf{K}\boldsymbol{x}(k) + \boldsymbol{u}^{\mathrm{td}}(k)) + \bar{E}(k)\} \\ \boldsymbol{u}(k) &= \mathbf{K}\boldsymbol{x}(k) + \boldsymbol{u}^{\mathrm{td}}(k)\end{aligned} \tag{4}$$

where $\boldsymbol{u}^{\mathrm{td}}$ denotes the time delay control. The fundamental idea in this approach is to augment the linear feedback $\mathbf{K}\boldsymbol{x}(k)$, which stabilizes the nominal linear system, by a time delay controller $\boldsymbol{u}^{\mathrm{td}}(k)$ which counteracts the uncertainty. With this formulation it is possible to rewrite (4) as

$$\begin{aligned}\delta \boldsymbol{x}(k) &= (\mathbf{A}+\mathbf{BK})\boldsymbol{x}(k) + \mathbf{B}\{(1+b)\boldsymbol{u}^{\mathrm{td}}(k) + \boldsymbol{E}_x(k)] \\ \boldsymbol{E}_x(k) &= b\mathbf{K}\boldsymbol{x}(k) + \bar{\boldsymbol{E}}(k)\end{aligned} \tag{5}$$

The equation of the sliding manifold s_x satisfies $s_x = \{\boldsymbol{x}(k): \boldsymbol{c}_x\boldsymbol{x}(k) = 0\}$, where $\boldsymbol{c}_x$ is an $(1 \times n)$ vector. At this point we make the additional requirement that $\boldsymbol{c}_x\mathbf{B} \neq 0$, which is easy to satisfy by a proper choice of $\boldsymbol{c}_x$. First, a canonical transformation is performed which will prove useful in the analysis below. Let $L_1 \in \Re^{(n-1)\times n}$ be such that $\ker \mathbf{L}_1 = \operatorname{im}\mathbf{B}$(ker and im stand for null space and range space, respectively). Define $\mathbf{L}_2 \triangleq (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}$. Next denote the non-singular transformation matrix by

$$\mathbf{L} \triangleq \begin{bmatrix}\mathbf{L}_1 \\ \mathbf{L}_2\end{bmatrix}$$

with inverse

$$\mathbf{L}^{-1} = \mathbf{R} = [\mathbf{R}_1 \quad \mathbf{B}]$$

As a result of this transformation, $y(k) = \mathbf{L}\boldsymbol{x}(k)$, we have

$$\begin{bmatrix}\delta\bar{y}(k) \\ \delta y_n(k)\end{bmatrix} = \begin{bmatrix}\mathbf{L}_1(\mathbf{A}+\mathbf{BK})\mathbf{R}_1 & \mathbf{L}_1(\mathbf{A}+\mathbf{BK})\mathbf{B} \\ \mathbf{L}_2(\mathbf{A}+\mathbf{BK})|\mathbf{R}_1 & \mathbf{L}_2(\mathbf{A}+\mathbf{BK})\mathbf{B}\end{bmatrix}\begin{bmatrix}\bar{y}(k) \\ y_n(k)\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\{(1+b)\boldsymbol{u}^{\mathrm{td}}(k) + \boldsymbol{E}_y(k)\} \tag{6}$$

where $y_n \in \Re$ and the sliding condition is $c_x \mathbf{R} y(k) = 0$, which we can be rewritten as

$$s_y = \bar{c}_y \bar{y} + c_{ny} y_n = 0 \tag{7}$$

It can easily be shown that the condition $c_x \mathbf{B}$ be non-zero implies that c_{ny} in (7) must also be non-zero. This follows from the fact that $|c_x \mathbf{B}| \neq 0 \Leftrightarrow |c_y \mathbf{LB}| = |c_{ny}| \neq 0$. Hence

$$y_n(k) = c_{ny}^{-1}(s_y(k) - \bar{c}_y \bar{y}(k)) \tag{8}$$

$$y_n(k) = -\bar{S}\bar{y}(k) \quad \text{if } s_y(k) \equiv 0 \tag{9}$$

$$\bar{S} = c_{ny}^{-1} \bar{c}_y$$

In fact $c_{ny} = 1$ will always be chosen. In this case $s_y = \bar{S}\bar{y} + y_n \Rightarrow s_x = (\mathbf{L}_2 + \bar{S}\mathbf{L}_1)x = c_x x$.

The following lemma (Ryan 1990) defines and proves that the sliding surface, s_x, defined by $s_x = c_x x = 0$ is invariant under $(\mathbf{A} + \mathbf{BK})$, i.e. if $x(k) \in s_x$ then $(\mathbf{A} + \mathbf{BK})x(k) \in s_x$. The notations and results of the above canonical transformation will be utilized.

Lemma 1: *Let $\bar{\mathbf{A}} \in \Re$ be such that $\sigma(\bar{\mathbf{A}}) \subset \mathbf{\Lambda}$, and define*

$$\mathbf{K} \triangleq \bar{\mathbf{A}} c_x - c_x \mathbf{A} \tag{10}$$

Then:

(1) $\sigma(\mathbf{A} + \mathbf{BK}) \subset \mathbf{\Lambda}$;

(2) $s_x = c_x x = 0 \subset \Re^n$ *is an* $(\mathbf{A} + \mathbf{BK})$ *invariant subspace.*

Proof: First form a second transformation starting from the transformed state y. Let $z(k) = \mathbf{T}y(k) = \mathbf{TL}x(k)$, where $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} I_{(n-1)\times(n-1)} & 0 \\ \bar{S} & I_{1\times 1} \end{bmatrix} \tag{11}$$

In the z-system of state variables

$$\mathbf{TL}(\mathbf{A} + \mathbf{B} \quad \mathbf{K})\mathbf{L}^{-1}\mathbf{T}^{-1} = \begin{bmatrix} I & 0 \\ \bar{S} & I \end{bmatrix} \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix} (\mathbf{A} + \mathbf{BK})[\mathbf{R}_1 \quad \mathbf{B}] \begin{bmatrix} I & 0 \\ -\bar{S} & I \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{L}_1\mathbf{A}(\mathbf{R}_1 - \mathbf{B}\bar{S}) & \mathbf{L}_1\mathbf{AB} \\ 0 & \bar{\mathbf{A}} \end{bmatrix} \tag{12}$$

Therefore, by selecting $\bar{S}$ such that the eigenvalues of $\mathbf{L}_1\mathbf{A}(\mathbf{R}_1 - \mathbf{B}\bar{S})$ are contained inside the stability circle centred at $-1/T$ with radius $1/T$, $\sigma(\mathbf{A} + \mathbf{BK}) = \sigma(\mathbf{TL}(\mathbf{A} + \mathbf{BK})\mathbf{L}^{-1}\mathbf{T}^{-1}) = \sigma[\mathbf{L}_1\mathbf{A}(\mathbf{R}_1 - \mathbf{B}\bar{S})] \cup \sigma(\bar{\mathbf{A}}) = \mathbf{\Lambda} \cup \sigma(\bar{\mathbf{A}}) \subset \mathbf{\Lambda}$. This establishes part (1) of Lemma 1. Now, noting that $c_x \mathbf{BK} = (\bar{S}\mathbf{L}_1 + \mathbf{L}_2)\mathbf{BK} = \bar{\mathbf{A}} c_x - c_x \mathbf{A}$, it may be concluded that $c_x(\mathbf{A} + \mathbf{BK})x = 0 \; \forall x \in s =$ kernel of c_x, thereby proving part (2). □

Note that Lemma 1 not only proves the invariance of the sliding surface with respect to $(\mathbf{A} + \mathbf{BK})$, but also gives a systematic procedure for determining a stabilizing feedback matrix $\mathbf{K}$ for the nominal system. Having stabilized the nominal system, we next focus on driving the states to the surface $s = 0$. This is accomplished by TDC.

3.2. *Time delay control (TDC)*

Start with (6) with $E_y(k) = b\mathbf{K}y(k) + \bar{E}(y, k)$, $\bar{y} = [y_1\ y_2\ \cdots\ y_{n-1}]$, $s_y = \bar{S}\bar{y}(k) + y_n(k) = 0$. Next apply the transformation given by (11) where $z(k) = \mathbf{T}y(k) = \mathbf{TL}x(k)$. The relation expressed by this transformation implies that $\bar{z} = [z_1\ z_2\ \cdots\ z_{n-1}] = \bar{y}$, $z_n = \bar{S}\bar{y} + y_n$. Notice that points at which $z_n = 0$ and $s = s_y = 0$ are equivalent. This allows (6) to be expressed as

$$\begin{bmatrix} \delta\bar{y}(k) \\ \delta s(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \bar{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \bar{y}(k) \\ s(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \{(1 + b)u^{\mathrm{td}}(k) + E(k)\} \tag{13}$$

where $\mathbf{A}_{11} = \mathbf{L}_1\mathbf{A}(\mathbf{R}_1 - \mathbf{B}\bar{S})$ and $\bar{\mathbf{A}}$ are stable, $\mathbf{A}_{12} = \mathbf{L}_1\mathbf{AB}$, $E(k) = \eta_1\bar{y}(k) + \eta_2 s(k) + w(k)$, $\eta_1 = bk_1 + \Delta\mathbf{A}_1$, $\eta_2 = bk_2 + \Delta\mathbf{A}_2$. In these expressions the linear feedback gain matrix $\mathbf{K}$ and the uncertain term $\Delta\mathbf{A}$ have been partitioned as $\mathbf{K} = [k_1\ k_2]$ and $\Delta\mathbf{A} = [\Delta\mathbf{A}_1\ \Delta\mathbf{A}_2]$, respectively. The control objective is to force the plant states to the surface $s = 0$. The bottom row of the above equation expresses the s-dynamics

$$\delta s(k) = \bar{\mathbf{A}}s(k) + (1 + b)u^{\mathrm{td}}(k) + E(k) \tag{14}$$

If a control u^{td} exists such that the following equation is satisfied:

$$(1 + b)u^{\mathrm{td}}(k) + E(k) = 0 \tag{15}$$

then substitution of (15) into (14) leads to $\delta s(k) = \bar{\mathbf{A}}s(k)$ and the s-dynamics are forced to vanish asymptotically. However, further considerations are necessary since the perturbation $E(k)$ in (15) is unknown.

It is of interest to determine the required control action that will steer the plant states to the surface $s = 0$ in the face of $E(k)$, which is known to contain the effects of unknown dynamics and unexpected disturbances. An approximate solution to this problem is

$$u^{\mathrm{td}}(k) = \frac{-1}{1 + \hat{b}} E(k - 1) \tag{16}$$

This procedure of delaying time t by a small amount to obtain an estimate of the system perturbations at time $t - L$, where L is a small time delay, is based on the concept of TDC (Youcef-Toumi and Ito 1990). Previous work with TDC dwells in the continuous time domain. Recently, Tesfaye and Tomizuka (1993) extended the idea to discrete time models of continuous time systems, which are expressed in the Euler operator. In this case an assumption is made that the dynamics stemming from the perturbations, manifested by the function $E(k)$, are considerably slower than the sampling frequency $1/T$ (for example, an order of magnitude smaller). If the above assumption holds, then the value of $E(k)$ at time $t = kT$ can be considered to be close to the value at time $t = (k - 1)T$. The value of $E(k - 1)$, one step into the past, can be obtained from (14). If this is done and the result is substituted in (16),

$$u^{\mathrm{td}}(k) = \frac{1}{1 + \hat{b}}[-\delta s(k - 1) + \bar{\mathbf{A}}s(k - 1) + (1 + b)u^{\mathrm{td}}(k - 1)] \tag{17}$$

Since b is unavailable it is replaced by $\hat{b}$ in (17) to obtain the final control equation as

$$u^{\mathrm{td}}(k) = \frac{1}{1 + \hat{b}}\{-\delta s(k - 1) + \bar{\mathbf{A}}s(k - 1)\} + u^{\mathrm{td}}(k - 1) \tag{18}$$

The issue of the possible mismatch between b and $\hat{b}$ will be postponed until later.

Noting that the ε-transform of a delayed quantity $U(t - \tau)$, where $\tau = NT$ for $N \in Z^+$, is given by $\varepsilon[U(t-\tau)] = (1 + T\varepsilon)^{-N} U(\varepsilon)$, the following ε-transform is obtained for the control input given by (18)

$$\begin{aligned} \boldsymbol{u}^{\mathrm{td}}(\varepsilon) &= \frac{1+\varepsilon T}{\varepsilon T}\frac{1}{1+\hat{b}}\left\{\frac{-\varepsilon}{1+\varepsilon T} + \frac{\bar{\mathbf{A}}}{1+\varepsilon T}\right\} s(\varepsilon) \\ &= \frac{-(\varepsilon - \bar{\mathbf{A}})}{(1+\hat{b})\varepsilon T} s(\varepsilon) \end{aligned} \tag{19}$$

Substituting the expression for $\boldsymbol{E}(k)$ defined in (13) into (14) gives the following expression for the s-dynamics:

$$\begin{aligned} \delta s(k) &= \bar{\mathbf{A}} s(k) + (1+b)\boldsymbol{u}^{\mathrm{td}}(k) + \boldsymbol{E}(k) \\ &= (\bar{\mathbf{A}} + \eta_2) s(k) + (1+b)\boldsymbol{u}^{\mathrm{td}}(k) + \eta_1 \bar{y}(k) + w(k) \end{aligned}$$

Note that $w(k)$ is not a function of state. Taking an ε-transform between s and the control $\boldsymbol{u}^{\mathrm{td}}$ gives

$$s(\varepsilon) = \{\varepsilon - (\bar{\mathbf{A}} + \eta_2) - \eta_1(\varepsilon I - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}\}^{-1}(1+b)\boldsymbol{u}^{\mathrm{td}}(\varepsilon)$$

Denote $p = \bar{\mathbf{A}} + \eta_2 + \eta_1(\varepsilon I - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}$. A block diagram for the whole system can then be obtained as shown in Fig. 1. This diagram can first be reduced to the one shown in Fig. 2 and, if the sampling time T is small, in other words if the effective controller gain $1/T$ is sufficiently large, it can further be reduced to that shown in Fig. 3. This means pole/zero cancellation occurs (the zero is stable and fixed at $\varepsilon = \bar{\mathbf{A}}$), which makes the s-dynamics behave as a first-order system with a large bandwidth and command input 0. Notice that if the discrepancy

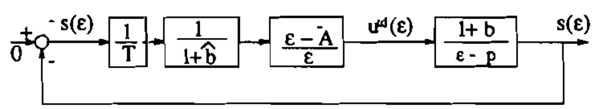


Figure 1. System block diagram.

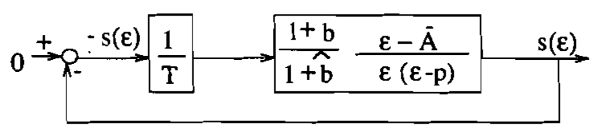


Figure 2. Reduced system block diagram, version 1.

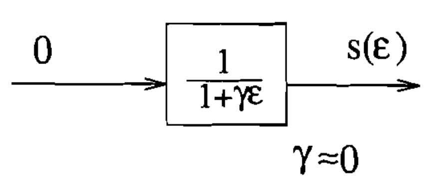


Figure 3. Reduced system block diagram, version 2.

between the actual value of b and its estimated value $\hat{b}$ is not too large then the mismatch is easily absorbed by the high gain $1/T$. This resolves the issue of the possible mismatch between b and $\hat{b}$.

From the above analysis, the proposed controller can be interpreted as one which steers the states to the surface $s = 0$ by the high-gain integrator $1/T\varepsilon$ with pole/zero cancellation. From the stability viewpoint the convergence of s to zero implies the dynamics of the system given by (13) behaves asymptotically as

$$\delta\bar{y}(k) = \mathbf{A}_{11}\bar{y}(k), \quad \delta s(k) = E(k) - \frac{1+b}{1+\hat{b}}E(k-1)$$

where

$$E(k) = \eta_1\bar{y}(k) + w(k)$$

Since the eigenvalues, $\sigma(\mathbf{A}_{11})$, are stable, if the state-independent disturbance $w(k)$ is zero, then $\bar{y} \to 0$ as $t = kT \to \infty$. Next, utilizing the definition $\delta = (z-1)/T$, it is obvious that both $s(k+1)$ and $\bar{y}(k+1)$ converge to zero. This shows that sliding motion is achieved asymptotically and, further, that the stability of the uncertain system is assured.

Assume, for simplicity, perfect knowledge of b, i.e. assume $b = \hat{b}$, then the effectiveness of the TDC in cancelling system uncertainties and external disturbances (characterized by $E(k)$) becomes explicit if the control equation given by (16) is substituted into the equation for the s-dynamics (14) to obtain $\delta s(k) = \bar{\mathbf{A}}s(k) + (E(k) - E(k-1))$. Taking ε transforms of this equation obtains $s(\varepsilon)/E(\varepsilon) = (\varepsilon - \bar{\mathbf{A}})^{-1}\varepsilon T/(1 + \varepsilon T)$. Hence the influence of the uncertain terms characterized by $E(k)$ becomes small for small T.

4. Stability analysis

It is possible to extend the above formulation and address stability issues. Substituting (16) into (13) and using the expression for $E(k)$,

$$\begin{bmatrix} \delta\bar{y}(k) \\ \delta s(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \bar{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \bar{y}(k) \\ s(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left\{ \left(1 - \frac{1+b}{1+\hat{b}}q^{-1}\right)(\eta_1\bar{y}(k) + \eta_2 s(k)) + \left(w(k) - \frac{1+b}{1+\hat{b}}w(k-1)\right) \right\}$$

where q represents the traditional shift operator. To simplify the algebra consider the state-independent disturbance $w(k)$ to be zero. Also, rewrite the above equation in the difference equation form by making use of the relation $\delta = (q-1)/T$, which gives

$$\bar{y}(k+1) = (I + \mathbf{A}_{11}T)\bar{y}(k) + T\mathbf{A}_{12}s(k) \tag{20}$$

$$s(k+1) = (1 + \bar{\mathbf{A}}T)s(k) + \left(1 - \frac{1+b}{1+\hat{b}}q^{-1}\right)(T\eta_1\bar{y}(k) + T\eta_2 s(k)) \tag{21}$$

The eigenvalues $\sigma(I + \mathbf{A}_{11}T)$ and $\sigma(1 + \bar{\mathbf{A}}T)$ as presented in the above analysis are stable. The interconnection of the feedback system expressed by (20) and (21) appears in Fig. 4. The stability of such an interconnection can be

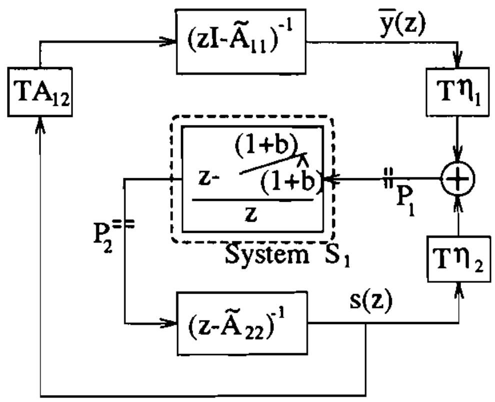


Figure 4. Feedback interconnection.

investigated by application of the small-gain theorem. Defined the open-loop stable subsystem S_1 (see Fig. 4) with transfer function

$$S_1 = \frac{P_2}{P_1} = \left.\frac{z - \dfrac{1+b}{1+\hat{b}}}{z}\right|_{z=e^{j\omega T}} \tag{22}$$

and the open-loop stable subsystem S_2 with transfer function

$$S_2 = \frac{P_1}{P_2} = (T^2\eta_1(zI - \tilde{\mathbf{A}}_{11})^{-1}\mathbf{A}_{12} + T\eta_2)(z - \tilde{\mathbf{A}}_{22})^{-1}|_{z=e^{j\omega T}} \tag{23}$$

$$\tilde{\mathbf{A}}_{11} = I + \mathbf{A}_{11}T, \quad \tilde{\mathbf{A}}_{22} = 1 + \bar{\mathbf{A}}T$$

By the small-gain theorem, the feedback loop characterized by (20) and (21) is stable provided the following condition is satisfied:

$$\|S_1S_2(e^{j\omega T})\|_\infty < 1 \tag{24}$$

Assume that the sign of b, whether positive or negative, is fixed and known. This requirement might be motivated by a desire to prevent the control input from being reversed by the uncertainty in the input matrix. Note that b will have the same sign as Δb for sufficiently small T. Next design $\hat{b}$ such that $(1 + b)/(1 + \hat{b}) \leq 1$. Then the condition given by (24) can be satisfied by means of loop-shaping the open-loop gain $\|S_1S_2(e^{j\omega T})\|$ over the frequency range of interest. A conservative estimate is given by the following condition on S_1:

$$\|S_1(e^{j\omega T})\|_\infty = \left\|\frac{e^{j\omega T} - 1}{e^{j\omega T}}\right\|_\infty < 2 \tag{25}$$

If this condition is satisfied, the condition expressed by (24) is satisfied if

$$\|S_2(e^{j\omega T})\|_\infty < \frac{1}{2} \tag{26}$$

or

$$\|T\eta_1(zI - \tilde{\mathbf{A}}_{11})^{-1}\mathbf{A}_{12} + \eta_2)(z - \tilde{\mathbf{A}}_{22})^{-1}\|_\infty < \frac{1}{2T} \tag{27}$$

One way of satisfying the above condition is to choose a suitably small sampling time T to increase the magnitude of the term on the right-hand side of the inequality. Another way is by judicious shaping of the magnitude response of the term on the left-hand side of the inequality such that the condition given by (27) is satisfied over the whole frequency range of interest. This is made feasible by the fact that there is freedom in choosing $\tilde{\mathbf{A}}_{11} = I + \mathbf{A}_{11}T$ and $\tilde{\mathbf{A}}_{22} = 1 + \bar{\mathbf{A}}T$.

Remark 6: If norm bounds on $\mathbf{A}_{12}$, η_1 and η_2 can be determined, the frequency response of S_2 can be shaped such that the condition given by (27) is satisfied. □

Remark 7: In general, if $\tilde{\mathbf{A}}_{11}$ and $\tilde{\mathbf{A}}_{22}$ are chosen such that their singular values/poles are near the point (0,1) on the real axis and stable, then the magnitude of S_2 will quickly drop as the frequency increases. This would guarantee that the condition imposed by the small-gain theorem is satisfied at high frequency. At low frequency the term characterized by S_1 gives a low overall gain for a choice of $b \approx \hat{b}$, favouring that the small-gain condition is met. Physically, the presence of S_1 weakens the coupling between subsystems S_1 and S_2 at low frequency. Ideally, for this to be achieved it is preferable that $(1 + b)/(1 + \hat{b}) \approx 1$. This means that the value of $\hat{b}$ chosen by the designer must satisfy a certain condition with respect to the actual b. For best performance, the ratio $b/\hat{b}$ should be 1 since the coupling between subsystems S_1 and S_2 is weakened, which also translates into a controller with reduced gain. □

5. Simulation and experiment

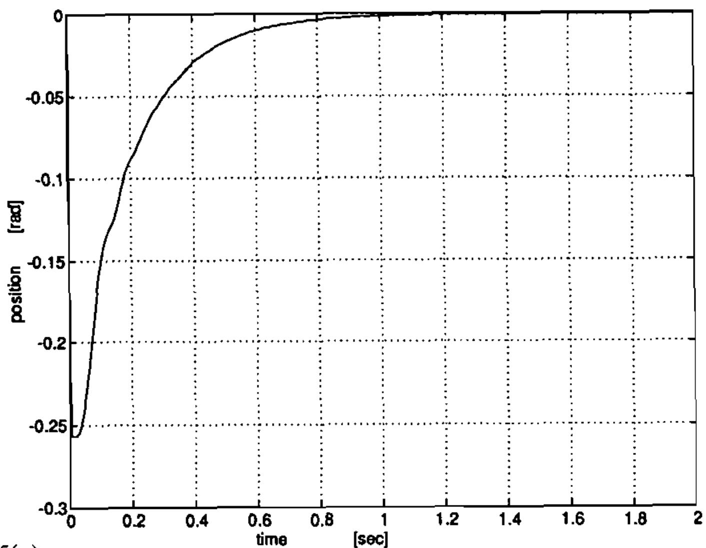
The proposed controller was studied on the following model of a single-axis robot arm driven by an NSK direct drive motor (model no. RS1410).

$$\left.\begin{aligned} &\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{F}_c d(t) \\ &\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{k}{J} \end{bmatrix}; \quad \mathbf{F}_c = \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix} \end{aligned}\right\} \tag{28}$$

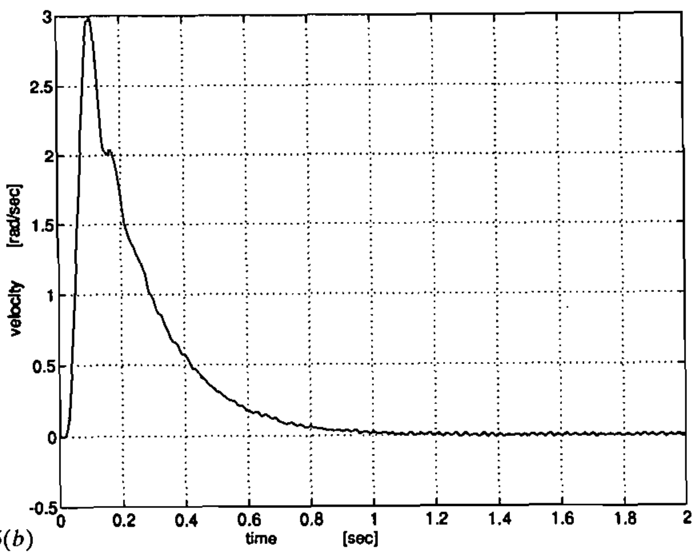
where $k = 39\ \mathrm{N\,m\,V^{-1}}$ and $J \in [0{\cdot}83, 2{\cdot}95]\ \mathrm{kg\,m^2}$. The inertia change represents a 355 % variation. In both simulation and experimental studies the sampling time was chosen as $T = 2$ ms and the initial conditions were $x(0) = -0{\cdot}245$ rad and $\dot{x}(0) = 0\ \mathrm{rad\,s^{-1}}$. Initially, the external disturbance d was set to zero. In this case, simulation results (not included here) exhibited a smooth response in which sliding motion was achieved quickly for all ranges of the parameter values considered. Figure 5(*a*) and (*b*) show experimental results of the position and velocity response for $J = 0{\cdot}83\ \mathrm{kg\,m^2}$; (*c*) depicts the approach of the plant state to the sliding surface (shown dotted) and the subsequent motion in and around the sliding surface (phase plot). Figure 6(*a*)–(*c*) shows analogous results obtained from experiment for $J = 2{\cdot}95\ \mathrm{kg\,m^2}$. Notice that the system achieves sliding motion but with some chatter. This discrepancy between simulation and experimental results is attributed to friction effects and unmodelled actuator dynamics. Figure 7 shows what happens when a step disturbance of magnitude $d = 60$ N m is simulated starting at time equal to 0·08 s. Sliding motion had been attained prior to application of the disturbance. It is evident that the TDC exhibits robust characteristics as the disturbance was rejected quickly.

6. Conclusion

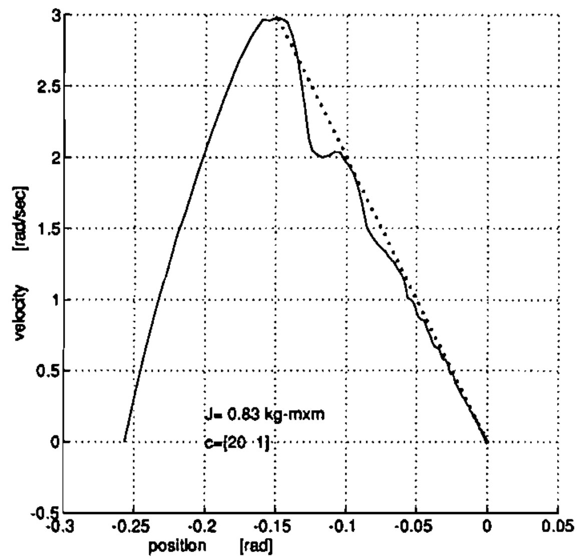
A robust discrete-time sliding control using the ε operator has been proposed and simulated. The sliding surface is not chosen arbitrarily but reflects assignment of poles based on the nominal parameters of the system; consequently, it is also dependent on the sampling time. The controller was designed assuming a reasonably small sampling interval in addition to assumptions that are needed for continuous-time sliding control, namely matching conditions. Knowledge of the bounds of parameter variations, however, is not required. The strategy in dealing with the unknown system perturbations is to delay time by one sampling interval to estimate the effects. The gathered information is then used to cancel them. In this respect the control that has been formulated is consistent with time delay control.



5(*a*)



5(*b*)



(c)

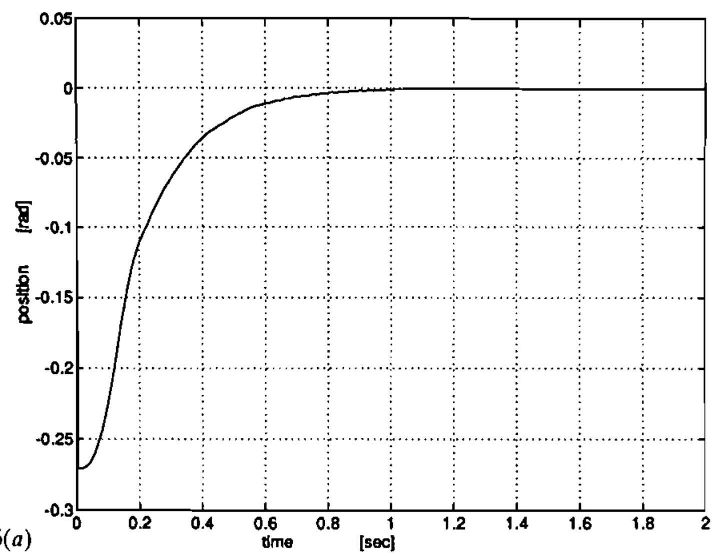
Figure 5. (a) Position response, $J = 0{\cdot}83\ \mathrm{kg\,m^2}$; (b) velocity response; (c) phase plot.

Appendix A

Derivation of (3)

Consider the discretization of the system of (1) using a zero-order hold for which $u(t) = u(k)$ for $kT \leqslant t \leqslant (k+1)T$. Invoking the matching conditions assumption the unknowns $\Delta\mathbf{A}_c$ and $\Delta\mathbf{B}_c$ can be expressed as follows: $\Delta\mathbf{A}_c = \mathbf{B}_{nc}\Delta_a$ and $\Delta\mathbf{B}_c = \mathbf{B}_{nc}\Delta_b$, respectively. Also, $\mathbf{F}_c$ can be written as $\mathbf{F}_c = \mathbf{B}_{nc}\Delta_f$ where Δ_f is a known quantity.

$$\delta x(k) = (\mathbf{G}_n + \Delta\mathbf{G})x(k) + \mathbf{H}u(k) + \mathbf{F}d(k) \tag{A 1}$$



6(a)

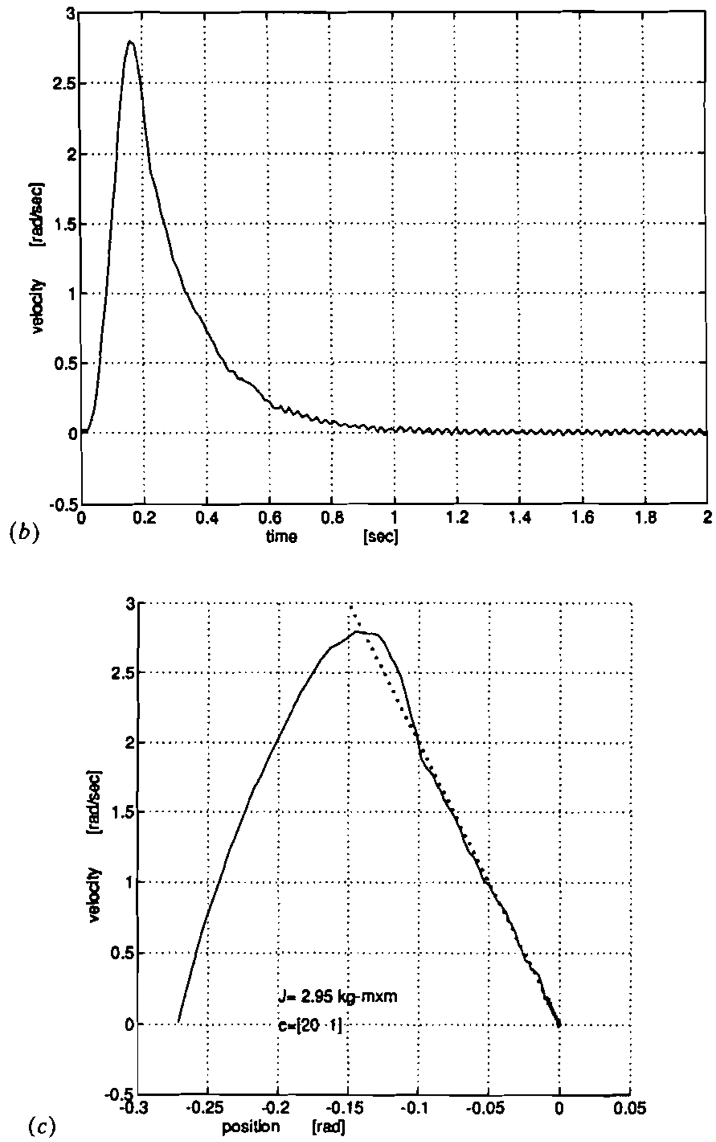


Figure 6. (*a*) Position response, $J = 2{\cdot}95\ \mathrm{kg\,m^2}$; (*b*) velocity response; (*c*) phase plot.

where

$$\mathbf{G}_n = \frac{e^{\mathbf{A}_{nc}T} - I}{T}, \quad \Delta\mathbf{G} = \frac{e^{\mathbf{A}_c T} - I}{T} - \frac{e^{\mathbf{A}_{nc}T} - I}{T} \quad \text{and } \mathbf{F} = \frac{1}{T}\int_0^T e^{\mathbf{A}_c\tau}\mathbf{F}_c\,d\tau$$

$$\begin{aligned}\mathbf{H} &= \frac{1}{T}\int_0^T e^{\mathbf{A}_c\tau}\mathbf{B}_{nc}(1 + \Delta_b)\,d\tau \\ &= \frac{1}{T}\int_0^T (e^{\mathbf{A}_c\tau} - e^{\mathbf{A}_{nc}\tau} + e^{\mathbf{A}_{nc}\tau})\mathbf{B}_{nc}(1 + \Delta_b)\,d\tau \\ &= (H_n + \Delta H)(1 + \Delta_b) \qquad (A\,2)\end{aligned}$$

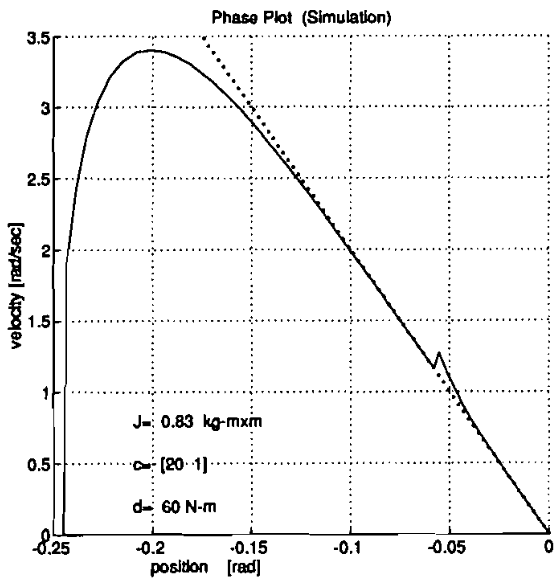


Figure 7. Phase plot (simulation).

$$\mathbf{H}_n = \frac{1}{T}\int_0^T e^{\mathbf{A}_{nc}\tau}\mathbf{B}_{nc}\,d\tau = \frac{1}{T}\left(T + \frac{1}{2!}\mathbf{A}_{nc}T^2 + \frac{1}{3!}\mathbf{A}_{nc}^2T^3 + \cdots\right)\mathbf{B}_{nc} \qquad \text{(A 3)}$$

$$\begin{aligned}\Delta\mathbf{H} &= \frac{1}{T}\int_0^T (e^{\mathbf{A}_c\tau} - e^{\mathbf{A}_{nc}\tau})\mathbf{B}_{nc}\,d\tau \\ &= \frac{1}{T}\int_0^T\left[(\mathbf{A}_c - \mathbf{A}_{nc})\tau + \frac{1}{2!}(\mathbf{A}_c^2 - \mathbf{A}_{nc}^2)\tau^2 + \frac{1}{3!}(\mathbf{A}_c^3 - \mathbf{A}_{nc}^3)\tau^3 + \cdots\right]\mathbf{B}_{nc}\,d\tau \\ &= \frac{1}{T}\left\{\mathbf{B}_{nc}T\Delta_a\mathbf{B}_{nc}\frac{T}{2!} + (\mathbf{A}_c^2 - \mathbf{A}_{nc}^2)\mathbf{B}_{nc}\frac{T^3}{3!} + (\mathbf{A}_c^3 - \mathbf{A}_{nc}^3)\mathbf{B}_{nc}\frac{T^4}{4!} + \cdots\right\}\end{aligned} \qquad \text{(A 4)}$$

Equations (A 3) and (A 4) yield

$$\begin{aligned}\Delta\mathbf{H} = \frac{1}{T}\Bigg\{&\mathbf{H}_n\Delta_a\mathbf{B}_{nc}\frac{T}{2!} - \frac{1}{2}\left(\mathbf{A}_{nc}\mathbf{B}_{nc}\Delta_a\frac{T^3}{2!} + \mathbf{A}_{nc}^2\mathbf{B}_{nc}\Delta_a\frac{T^4}{3!} + \cdots\right)\mathbf{B}_{nc} \\ &+ (\mathbf{A}_c^2 - \mathbf{A}_n^2)\mathbf{B}_{nc}\frac{T^3}{3!} + (\mathbf{A}_c^3 - \mathbf{A}_{nc}^3)\mathbf{B}_{nc}\frac{T^4}{4!} + \cdots\Bigg\}\end{aligned}$$

By observing this equation, if the condition of the small sampling time assumption holds then all terms of third-order or higher in the sampling time T are dropped and $\Delta\mathbf{H}$ is approximated by

$$\Delta\mathbf{H} \approx \mathbf{H}_n\Delta_a\mathbf{B}_{nc}\frac{T}{2} \qquad \text{(A 5)}$$

A similar analysis can be done with respect to $\Delta\mathbf{G}$:

$$\Delta\mathbf{G} = \frac{e^{\mathbf{A}_cT} - \mathbf{I}}{T} - \frac{e^{\mathbf{A}_{nc}T} - \mathbf{I}}{T}$$

$$= \frac{1}{T}\left\{(\mathbf{A}_c - \mathbf{A}_{nc})T + \frac{1}{2!}(\mathbf{A}_c^2 - \mathbf{A}_{nc}^2)T^2 + \frac{1}{3!}(\mathbf{A}_c^3 - \mathbf{A}_{nc}^3)T^3 + \cdots\right\}$$

$$= \frac{1}{T}\left\{\left(\mathbf{B}_{nc}T + \frac{1}{2!}\mathbf{A}_{nc}\mathbf{B}_{nc}T^2 + \cdots\right)\Delta_a + \mathbf{B}_{nc}T\left(\Delta_a\mathbf{A}_{nc}\frac{T}{2} + \Delta_a\mathbf{B}_{nc}\Delta_a\frac{T}{2}\right) + \cdots\right\}$$

This equation with (A 3), taking into account the small sampling time assumption, yields $\Delta\mathbf{G} \approx \mathbf{H}_n\Delta_a\{I + (\mathbf{A}_{nc} + \mathbf{B}_{nc}\Delta_a)(T/2)\}$. By employing the same technique to the coefficient of $\boldsymbol{d}(k)$ and by invoking the assumption of small sampling time gives

$$\begin{aligned}\delta\boldsymbol{x}(k) = \mathbf{G}_n\boldsymbol{x}(k) &+ \mathbf{H}_n\Delta_a\left\{I + (\mathbf{A}_{nc} + \mathbf{B}_{nc}\Delta_a)\frac{T}{2}\right\}\boldsymbol{x}(k) \\ &+ \mathbf{H}_n\left(1 + \Delta_a\mathbf{B}_{nc}\frac{T}{2}\right)(1 + \Delta_b)\boldsymbol{u}(k) \\ &+ \mathbf{H}_n\left(1 + \Delta_a\mathbf{B}_{nc}\frac{T}{2}\right)\Delta_f\boldsymbol{d}(k)\end{aligned} \quad \text{(A 6)}$$

It is obvious from (A 6) that the uncertainties in the control input channel and the coefficient of the disturbance are matched with respect to $\mathbf{H}_n$. Let $\mathbf{A} = \mathbf{G}_n$, $\Delta\mathbf{A} = \Delta_a\{I + (\mathbf{A}_{nc} + \mathbf{B}_{nc}\Delta_a)(T/2)\}$, $\mathbf{B} = \mathbf{H}_n$, $b = \Delta_a\mathbf{B}_{nc}(1 + \Delta_b)(T/2) + \Delta_b$, $\boldsymbol{w}(k) = (1 + \Delta_a\mathbf{B}_{nc}(T/2))\Delta_f\boldsymbol{d}(k)$. Then (A 6) and (3) are identical.

Appendix B

Error analysis

Here we seek to analyse the error that is introduced by invoking the small sampling time assumption, i.e. by using (3). Assuming there is no external disturbance,

$$\delta\boldsymbol{x}(k, T) = (\mathbf{A} + \Delta\mathbf{A})\boldsymbol{x}(k, T) + (\mathbf{B} + \Delta\mathbf{B})\boldsymbol{u}(k, T)$$
$$\boldsymbol{y}(k, T) = \boldsymbol{c}\boldsymbol{x}(k, T)$$

where $\mathbf{A} = \mathbf{G}_n$, $\Delta\mathbf{A} = \mathbf{H}_n\Delta_a\{I + (\mathbf{A}_{nc} + \mathbf{B}_{nc}\Delta_a)(T/2)\}$, $\mathbf{B} = \mathbf{H}_n$, $\Delta\mathbf{B} = \mathbf{H}_n\{\Delta_a\mathbf{B}_{nc}(1 + \Delta_b)(T/2) + \Delta_b\}$. $\mathbf{G}_n = (e^{\mathbf{A}_{nc}} - I)/T$ and $\mathbf{H}_n = (1/T)\int_0^T e^{\mathbf{A}_{nc}\tau}\mathbf{B}_{nc}\,d\tau$ express the nominal matrices in discrete time after discretization using a zero-order hold. The solution of this equation for $\boldsymbol{x}(0) = 0$ is given by

$$\boldsymbol{y}(k, T) = \sum_{j=0}^{k-1}\boldsymbol{c}\{I + (\mathbf{A} + \Delta\mathbf{A})T\}^{k-1-j}(\mathbf{B} + \Delta\mathbf{B})T\boldsymbol{u}(j, T) \quad \text{(B 1)}$$

The continuous plant dynamic equation considered is given by

$$\dot{\boldsymbol{x}}(t) = (\mathbf{A}_{nc} + \Delta\mathbf{A}_c)\boldsymbol{x}(t) + (\mathbf{B}_{nc} + \Delta\mathbf{B}_c)\boldsymbol{u}(t)$$
$$\Delta\mathbf{A}_c = \mathbf{B}_{nc}\Delta_a; \quad \Delta\mathbf{B}_c = \mathbf{B}_{nc}\Delta_b$$

Let $\bar{\boldsymbol{y}}(k, T)$ be the output of this continuous-time system subject to the input $\boldsymbol{u}(j, T)$ and assume the initial condition starts at zero. In this case $\bar{\boldsymbol{y}}(k, T)$ is obtained by

$$\begin{aligned}\bar{y}(k, T) &= \boldsymbol{c}\int_0^{kT} \mathrm{e}^{\mathbf{A}_c(kT-\tau)}\mathbf{B}_c \boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ &= \sum_{j=0}^{k-1} \boldsymbol{c}\int_{jT}^{(j+1)T} \mathrm{e}^{\mathbf{A}_c(kT-\tau)}\mathbf{B}_c \boldsymbol{u}(j, T)\,\mathrm{d}\tau \\ &= \sum_{j=0}^{k-1} \boldsymbol{c}(\mathrm{e}^{\mathbf{A}_c T})^{k-1-j}\int_0^{T} \mathrm{e}^{\mathbf{A}_c \tau}\,\mathrm{d}\tau \mathbf{B}_c \boldsymbol{u}(j, T) \qquad \text{(B 2)}\end{aligned}$$

For a finite sampling interval T we need to consider the deviation between the true output, ($\bar{y}$ (k, T)), of the continuous-time system given by (B 2) and the output of the approximate system, $y(k, T)$, given by (B 1), over the duration of only one sampling interval. This is because the time delay controller is a state-based controller and the sate of the approximate system and the true system are equalized at the end of each sampling interval. For ease of analysis, consider the interval $0 \leq t \leq T$, i.e. $j = 0$, $k = 1$ and $\|\boldsymbol{u}(0, T)\| \leq U_M$:

$$\begin{aligned}
&\|\bar{y}(k, T) - y(k, T)\| \\
&\leq \left\|\boldsymbol{c}\left\{\int_0^T \mathrm{e}^{\mathbf{A}_c\tau}\,\mathrm{d}\tau\mathbf{B}_{nc}(1+\Delta_b) - \int_0^T \mathrm{e}^{\mathbf{A}_{nc}\tau}\,\mathrm{d}\tau\mathbf{B}_{nc}\left(1+\Delta_a\mathbf{B}_{nc}\frac{T}{2}\right)(1+\Delta_b)\right\}\right\|U_M \\
&\leq \left\|\boldsymbol{c}\left\{\mathbf{B}_{nc}\Delta_a\frac{T^2}{2} + \left(\mathbf{A}_c^2\frac{T^3}{3!} + \mathbf{A}_c^3\frac{T^4}{4!} + \mathbf{A}_c^4\frac{T^5}{5!} + \cdots\right)\right.\right. \\
&\quad - \left(\mathbf{A}_{nc}^2\frac{T^3}{3!} + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right) - \left(TI + \mathbf{A}_{nc}\frac{T^2}{2!} + \mathbf{A}_{nc}^2\frac{T^3}{3!}\right. \\
&\quad \left.\left.\left. + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right)\Delta_a\mathbf{B}_{nc}\frac{T}{2}\right\}\mathbf{B}_{nc}(1+\Delta_b)\right\|U_M \\
&\leq \left\|\boldsymbol{c}\left\{\left(\mathbf{A}_c^2\frac{T^3}{3!} + \mathbf{A}_c^3\frac{T^4}{4!} + \mathbf{A}_c^4\frac{T^5}{5!} + \cdots\right)\mathbf{B}_{nc}(1+\Delta_b)\right.\right. \\
&\quad - \left(\mathbf{A}_{nc}^2\frac{T^3}{3!} + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right)\mathbf{B}_{nc}(1+\Delta_b) - \left(\mathbf{A}_{nc}\frac{T^2}{2!} + \mathbf{A}_{nc}^2\frac{T^3}{3!}\right. \\
&\quad \left.\left.\left. + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right)\mathbf{B}_{nc}\Delta_a\mathbf{B}_{nc}(1+\Delta_b)\frac{T}{2}\right\}\right\|U_M \\
&= \left\|\boldsymbol{c}\left\{\left(\mathbf{A}_c^2\frac{T^3}{3!} + \mathbf{A}_c^3\frac{T^4}{4!} + \mathbf{A}_c^4\frac{T^5}{5!} + \cdots\right) - \left(\mathbf{A}_{nc}^2\frac{T^3}{3!} + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right)\right.\right. \\
&\quad \left.\left. - \left(\mathbf{A}_{nc}\frac{T^2}{2!} + \mathbf{A}_{nc}^2\frac{T^3}{3!} + \mathbf{A}_{nc}^3\frac{T^4}{4!} + \cdots\right)\Delta_a\mathbf{B}_{nc}\frac{T}{2}\right\}\mathbf{B}_{nc}(1+\Delta_b)\right\|U_M \\
&\leq \|\boldsymbol{c}\|\left\{\int_0^T \|\mathrm{e}^{\mathbf{A}_c\tau} - (I + \mathbf{A}_c\tau)\|\,\mathrm{d}\tau + \int_0^T \|\mathrm{e}^{\mathbf{A}_{nc}\tau} - (I + \mathbf{A}_{nc}\tau)\|\,\mathrm{d}\tau\right. \\
&\quad \left. + \int_0^T \|\mathrm{e}^{\mathbf{A}_{nc}T} - I\|\,\mathrm{d}\tau\cdot\|\Delta_a\mathbf{B}_{nc}\|\frac{T}{2}\right\}\|\mathbf{B}_c\|U_M \qquad \text{(B 3)}
\end{aligned}$$

where $\|\cdot\|$ denotes the Frobenius norm. Consider a truncation of a power series (Middleton and Goodwin 1990),

$$E(\mathbf{A}T) = \sum_{k=0}^{N} \frac{(\mathbf{A}T)^k}{k!}$$

If $\|\mathbf{A}T\| < N + 1$, then

$$\|e^{\mathbf{A}T} - E(\mathbf{A}T)\| \leqslant \frac{\|\mathbf{A}T\|^{N+1}}{(N+1)!}\left\{\frac{1}{1 - \dfrac{\|\mathbf{A}T\|}{(N+1)}}\right\}$$

Applying this result to (B 3),

$$\|\bar{y}(k, T) - y(k, T)\| \leqslant \|c\|\left\{\int_0^T \frac{\|\mathbf{A}_c\|^2\tau^2}{2!}\left\{\frac{1}{1 - \dfrac{\|\mathbf{A}_c\|\tau}{2}}\right\} d\tau\right.$$

$$+ \int_0^T \frac{\|\mathbf{A}_{nc}\|^2\tau^2}{2!}\left\{\frac{1}{1 - \dfrac{\|\mathbf{A}_{nc}\|\tau}{2}}\right\} d\tau$$

$$\left. + \int_0^T \|\mathbf{A}_{nc}\|\tau\left\{\frac{1}{1 - \|\mathbf{A}_{nc}\|\tau}\right\} d\tau \cdot \|\Delta_a\mathbf{B}_{nc}\frac{T}{2}\right\}\|\mathbf{B}_c\|U_M$$

Integrating and making the approximation

$$\ln\left(1 - \frac{\|\mathbf{A}\|T}{2}\right) \approx \left(-\frac{\|\mathbf{A}\|T}{2} - \frac{\|\mathbf{A}\|^2T^2}{4}\right)$$

$$\ln(1 - \|\mathbf{A}\|T) \approx \left(-\|\mathbf{A}\|T - \frac{\|\mathbf{A}\|^2T^2}{2}\right)$$

gives

$$\|\bar{y}(k, T) - y(k, T)\| \leqslant \frac{T^2}{2}\|c\|\left\{\|\mathbf{A}_c\| + \|\mathbf{A}_{nc}\| + \|\mathbf{A}_{nc}\|\|\Delta_a\mathbf{B}_{nc}\|\frac{T}{2}\right\}\|\mathbf{B}_c\|U_M$$

From this equation it can be concluded that the maximum error between the true output, $\bar{y}(k, T)$, and the approximate output, $y(k, T)$, over one sampling interval is dependent on the square of the sampling interval as well as on the magnitude of the uncertain terms. The error can be made small by reducing the sampling time. In this case reducing the sampling interval by half reduces the output error by a quarter.

References

Furuta, K., 1990, Sliding mode control of a discrete system. *System and Control Letters*, **14**, 145–152.

Furuta, K., and Morisada, M., 1989, Sliding mode control of discrete systems. *Transactions of the Society of Instrument and Control Engineers*, **25**, 574–578.

Goodwin, G. C., Lozano-Leal, R., Mayne, D. Q., and Middleton, R. H., 1986, Rapprochement between continuous and discrete model reference adaptive control. *Automatica*, **22**, 199–207.

Jabbari, A., and Tomizuka, M., 1992, Sliding mode control, discrete time approach. *Mechatronics and Manufacturing Systems*, Kobe, Japan, pp. 453–458.

Manela, J., 1985, Deterministic control of uncertain linear discrete and sampled-data systems. Ph.D. dissertation, University of California at Berkeley, California, U.S.A.

MIDDLETON, R. H., and GOODWIN, G. C., 1986, Improvided finite word length characteristics in digital control using delta operators. *IEEE Transactions on Automatic Control*, **31**, 1015–1021; 1990, *Digital Control and Estimation—A Unified Approach* (Englewood Cliffs, NJ: Prentice-Hall).

RYAN, E. P., 1990, A short graduate course on aspects of robust control system design, U.C. Berkeley, California, U.S.A.

SARPTURK, S. Z., ISTEFANOPULOS, Y., and KAYNAK, O., 1987, On the stability of discrete-time sliding mode control systems. *IEEE Transactions on Automatic Control*, **32**, 930–932.

SLOTINE, J.-J.E., and SASTRY, S. S., 1983, Tracking control of nonlinear systems using sliding surfaces with applications to robot manipulators. *International Journal of Control*, **38**, 465–492.

TESFAYE, A., and TOMIZUKA, M., 1993, Robust digital tracking with perturbation estimation via the euler operator. *Proceedings of the Winter Annual Meeting of ASME Dynamic Systems and Control*, Vol. 50, New Orleans, Louisiana, U.S.A., pp. 49–54.

UTKIN, V. I., 1977, Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, **22**, 212–222.

YOUCEF-TOUMI, K., and ITO, O., 1990, A time delay controller for systems with unknown dynamics. *Journal of Dynamic Systems Measurement and Control*, **112**, 133–142.