The central, Gaussian or normal distribution

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Exercise 7.1

TO DO

Exercise 7.2

Consider the family of distributions $p_{\mu,\sigma}(v)$.

We want to express fact that convolution of $p_{\mu,\sigma}(v)$ with $q(v)$ still belongs to the same family. I will prefer to work with parameter $\nu = \sigma^2$ instead. To avoid confusion the variable v will be replaced by x. So we work with the family $p_{\mu,\nu}(x)$.

Let $\mu_q = \langle\epsilon\rangle_q$ be the mean and $\nu_q = \langle\epsilon^2\rangle_q - \langle\epsilon\rangle_q^2$ the variance of q. Then the new distribution must be $p_{\mu+\mu_q,\nu+\nu_q}(x)$. At the same time the expansion 7.20 becomes

$$p_{\mu+\mu_q,\nu+\nu_q}(x) = p_{\mu,\nu}(x) - \mu_q \frac{\partial}{\partial x} p_{\mu,\nu}(x) + \frac{1}{2}(\nu_q + \mu_q^2)\frac{\partial^2}{\partial^2 x} p_{\mu,\nu}(x) + \dots$$

Taylor expanding around μ, ν

$$p_{\mu+\mu_q,\nu+\nu_q}(x) = p_{\mu,\nu}(x) + \mu_q \frac{\partial}{\partial \mu} p_{\mu,\nu}(x) + \nu_q \frac{\partial}{\partial \nu} p_{\mu,\nu}(x) +$$

$$\frac{1}{2}\mu_q^2 \frac{\partial^2}{\partial^2 \mu} p_{\mu,\nu}(x) + \frac{1}{2}\nu_q^2 \frac{\partial^2}{\partial^2 \nu} p_{\mu,\nu}(x) + \mu_q \nu_q \frac{\partial^2}{\partial \mu \partial \nu} p_{\mu,\nu}(x)$$

Now **if** we wanted this to be true for arbitrary (small) μ_q, ν_q we would have equality of Taylor coefficients:

$$-\frac{\partial}{\partial x} p_{\mu,\nu}(x) = \frac{\partial}{\partial \mu} p_{\mu,\nu}(x)$$

$$\frac{1}{2}\frac{\partial^2}{\partial^2 x} p_{\mu,\nu}(x) = \frac{\partial}{\partial \nu} p_{\mu,\nu}(x) = \frac{1}{2}\frac{\partial^2}{\partial^2 \mu} p_{\mu,\nu}(x)$$

where the first equation says that $p_{\mu,\nu}(x)$ is a function of $x-\mu$ and not of μ and x separately, $p_{\mu,\nu}(x) = f_\nu(x-\mu)$. From this $\frac{\partial^2}{\partial^2 x} p_{\mu,\nu}(x) = \frac{\partial^2}{\partial^2 \mu} p_{\mu,\nu}(x)$ follows, and we simply recover the more general Gaussina family $p_{\mu,\nu}(x) = \frac{1}{\sqrt{2\pi\nu}} \exp\{-\frac{(x-\mu)^2}{2\nu}\}$ as in 7.23.

However, **if** we instead think of μ and ν as $\mu(t)$ and $\nu(t)$ so that the family $p_t(x) = p_{\mu(t),\nu(t)}(x)$ is a single-parameter family, then the expansions become expansions in terms of t: with $\mu_q(t) = \mu_q'(0)t + o(t^2)$, $\nu_q(t) = \nu_q'(0)t + o(t^2)$

$$p_{\mu+\mu_q(t),\nu+\nu_q(t)}(x) = p_{\mu,\nu}(x) + [-\mu_q'(0)\frac{\partial}{\partial x} p_{\mu,\nu}(x) + \frac{1}{2}\nu_q'(0)\frac{\partial^2}{\partial^2 x} p_{\mu,\nu}(x)]t +$$

$$p_{\mu+\mu_q(t),\nu+\nu_q(t)}(x) = p_{\mu,\nu}(x) + \frac{\partial}{\partial t} p_{\mu,\nu}(x)t + o(t^2)$$

Equating Taylor coefficients:

$$\frac{\partial}{\partial t} p_t(x) = -\mu_q' \frac{\partial}{\partial x} p_t(x) + \frac{1}{2}\nu_q' \frac{\partial^2}{\partial^2 x} p_t(x)$$

This is a Fokker-Plank equation, albeit a very special one, with $\mu(x,t) = \mu'(0)$, $\sigma^2(x,t) = \nu'(0)$, corresponding to the stochastic process where the drift μ and diffusion coefficient $\nu/2$ are both constant. Denote $\mu'(0) = m$ and $\nu'(0) = v$.

Changing coordinates to $y(x,t) = x - mt$ aka $x(y,t) = y + mt$, we have

$$p_t(x(y,t)) = p_t(y+mt) =: q_t(y)$$

and compute by chin rule

$$\frac{\partial}{\partial t} q_t(y) = \frac{\partial}{\partial t} p_t(x(y,t)) = \frac{\partial}{\partial t} p_t(y+mt) + m\frac{\partial}{\partial x} p_t(y+mt)$$

while

$$\frac{1}{2} v \frac{\partial^2}{\partial^2 y} q_t(y) = \frac{1}{2} v \frac{\partial^2}{\partial^2 y} p_t(x(y,t)) = \frac{1}{2} v \frac{\partial^2}{\partial^2 x} p_t(y+mt)$$

So the substitution we made reduces the Fokker-Plank equation we have (with drift) to the diffusion equation (without drift) i.e. 7.22 (with $\sigma^2 = t$), which by 7.23 has solution $q_t(y) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{y^2}{2t}\}$, or, after substitution

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(x-mt)^2}{2t}\}$$

This has variance $\sigma^2 = t$ so we can rewrite it as $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-m\sigma^2)^2}{2\sigma^2}\}$, as in the formulation of the exercise.