Elementary Sampling theory

3.26, 3.27 The most probable value of r

The sequence $h(r|N, M, n)$ for fixed N, M, n is unimodal, meaning it first increases, then decreases. To see this we argue as follows.

We want to see wether $h(r+1|N, M, n)$ is bigger than $h(r|N, M, n)$, so we need to compare their fraction to 1. We compute using 3.22

$$h(r+1|N, M, n)/h(r|N, M, n) =$$

$$\frac{(M-r)/(r+1)}{(N-M-n+r+1)/(n-r)} =$$

$$\frac{(r-M)(r-n)}{(r+1)(r+N-M-n+1)} \overset{?}{\gtrless} 1$$

$$(r-M)(r-n) \overset{?}{\gtrless} (r+1)(r+N-M-n+1)$$

$$Mn-(M+n)r \overset{?}{\gtrless} (N-M-n+1)+r(N-M-n+2)$$

$$Mn-(N-M-n+1) \overset{?}{\gtrless} r(N+2)$$

$$\frac{Mn-(N-M-n+1)}{N+2} \overset{?}{\gtrless} r$$

$$\frac{Mn+M+n+1}{N+2} \overset{?}{\gtrless} r+1$$

$$\frac{(M+1)(n+1)}{N+2} \overset{?}{\gtrless} r+1$$

The sequence $h(r)$ increases while the left hand side is bigger.

Thus denoting by r' the number $\frac{(M+1)(n+1)}{N+2}$ we see that if r' is an integer, then $h(r)$ increase until $r = r'-1$, then $h(r'-1) = h(r')$, then the $h(r)$ decrease. If r' is not an integer, then $h(r)$ increase until $h(INT(r'))$, then decrease.

Remak 1: Note that the expected number of red balls is just the "naive" $n\frac{M}{N}$ (this is not hard to show using linearity of expectation, see Example 4.2.3 in Blitzstein-Hwang "Introduction to Probability").

Remark 2: The above result can be restated in the following way: add one red and one white ball to the urn (for a total of $N+2$) and draw $n+1$ balls from it. Compute the "naive" most likely fraction of red balls $\frac{M+1}{N+2}$ and the "naive" most likely number of red balls $\frac{(n+1)(M+1)}{N+2}$. Now subtract 1. This is (up to rounding) the most likely number of red balls drawn in the original procedure. This seems somewhat reminiscent of the correction that putting a beta prior on Bernoulli makes to the posterior expectation, but I have no idea if there is more to this connection than that.

3.29 Symmetry of $h(r|N, M, n)$

Combinatorial proof that

$$h(r|N, M, n) = h(r|N, n, M).$$

Remark: This is Theorem 3.4.5 in Blitzstein - Hwang "Introduction to Probability". See also Theorem 3.9.2.

By definition, $h(r|N, M, n)$ is computed as follows. Lay down N balls, labeled $1, ..., N$. Pick the subset $R_0 = \{1, ..., M\}$ of them and paint it red. Then pick a subset D of size n of all the ball, and compute $r = |D \cap R_0|$. The fraction of Ds that give specific answer r is by definition $h(r|N, M, n)$.

Now suppose instead we pick a different subset R_1 of size M to be red, and repeat the procedure above: pick D, and comute $r = |D \cap R_1|$. We claim that the fraction of Ds that give specific answer r is still $h(r|N, M, n)$. In deed, there exists a permutation of $\{1, ..., N\}$ taking R_1 to R_0 ("sort the reds to be first"); the same permutation takes Ds that give $D \cap R_1 = r$ to those that give $D \cap R_0 = r$. Hence there are the same number of Ds in both circumstances.

The above argument means that $h(r|N, M, n)$ can be also computed as follows. Lay down N balls, labeled $1, ..., N$. Pick **any** subset R of them of size M and paint it red. Then pick a subset D of size n of all the ball, and compute $r = |D \cap R|$. The fraction of **Rs and Ds** that give specific answer r is then $h(r|N, M, n)$.

But the above procedure remains the same if we exchange M and n and rename "paint red" into "pick" and "pick" into "paint red". Then it computes $h(r|N, n, M)$. So the two numbers are equal.

Remark: This view of hypergeometric distribution as giving probabilities of overlap of two subsets ("the red" and "the picked") removes all time dependence and, in my opinion, sheds a lot of light on the discussion at the end of Section 3.2.

Exercise 3.4

Denote by F_i be the event "i is fixed", and, for any $I \subset \{1, ..., n\}$, denote by F_I the event "all i in I are fixed", i.e. $F_I = \prod_{i \in I} F_i$.

We are looking for $P(\sum F_i)$. By inclusion exclusion this is

$$P(\sum F_i) = \sum_{I \subset \{1,...,n\}, I \neq \emptyset} (-1)^{|I|+1} P(F_I).$$

For a given subset of size k probability that it is fixed is $\frac{(n-k)!}{n!}$, and there are $\binom{n}{k}$ such subsets, so the sum over those I with size k gives $(-1)^{k+1}\frac{1}{k!}$. Plugging this in we obtain

$$h = P(\sum F_i) = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!},$$

as wanted.

Observe that $1 - h$ is the value of k-th order Taylor series for e^x evaluated at $x = -1$, which, as $k \to \infty$, converges to $e^{-1} = 1/e$.

Exercise 3.5

Similarly to 3.4, consider the event E_I=the bins with labels $i \in I$ are left empty; then $P(E_I) = (M - |I|)^N / M^N$ and by inclusion-exclusion $P(\overline{\sum E_i})$ is

$$\frac{1}{M^N} \sum_{k=0}^{M} (-1)^k \binom{M}{k} (M-k)^N.$$

Remark: We are computing probability that a function from a set of size N to a set of size M is onto. There are M^N total functions, and the number of surjective ones is $M!$ times a Stirling number of second kind.

Some of Exercise 3.6

Remark: If the initial distribution (for R_0, i.e. $P(red) = P(R_0) = p$, $P(white) = P(\bar{R}_0) = q$) were the same as the limit distribution π (formula 3.125, $\pi(red) = \lim P(R_k) = \frac{p-\delta}{1-\epsilon-\delta}$, $\pi(white) = \lim P(\bar{R}_k) = \frac{q-\epsilon}{1-\epsilon-\delta}$), this would be a steady state Markov chain, whose time-reverse process is also a Markov chain with transition probabilities $M^r_{ij} = \frac{\pi_j}{\pi_i} M_{ji}$ (note that for 2 state chains one always has $M^r_{ij} = M_{ij}$). This is precisely condition 3.131. Under this condition it is easy to compute $P(R_j|R_k)$ with $j < k$, and in the 2-state case that we are considering, they would be the same as $P(R_k|R_j)$ (as in 3.134). However in this exercise the Markov chain starts from the initial distribution that, in general, is not the steady state distribution, so reversing the time produces a process (indexed by negative integers) which is not a Markov chain. Maybe there is still a way to apply general theory of Markov chains to the problem of "backward inference" in this setting; absent that, we proceed by a direct computation (but observe that the reversed process is connected to the limiting behavior of the result, see below).

As usual, all probabilities are conditioned on C. Equation 3.129 is

$$P(R_k|R_j)P(R_j) = P(R_j|R_k)P(R_k)$$

Equation 3.118 is

$$P(R_k) = \frac{(p-\delta) + (\epsilon+\delta)^{k-1}(p\epsilon - q\delta)}{1-\epsilon-\delta}$$

Finally equation 3.128 is

$$P(R_k|R_j) = \frac{(p-\delta) + (\epsilon+\delta)^{k-j}(q-\epsilon)}{1-\epsilon-\delta}$$

Plugging in

$$P(R_j|R_k) = \frac{(p-\delta) + (\epsilon+\delta)^{j-1}(p\epsilon - q\delta)}{(p-\delta) + (\epsilon+\delta)^{k-1}(p\epsilon - q\delta)} \frac{(p-\delta) + (\epsilon+\delta)^{k-j}(q-\epsilon)}{1-\epsilon-\delta}$$

If both j and k go to infinity but $k-j$ is kept constant, this converges to

$$P(R_{\infty+d}|R_\infty) = \frac{(p-\delta) + (\epsilon+\delta)^{d}(q-\epsilon)}{1-\epsilon-\delta},$$

which is precisely the "reversed process" result (for large j and k the influence of the initial distribution not being the stationary one has dissipated; in the 2 state case the reversed process is the same as the forward one, but that's a special feature; when the number of states (i.e. colors) is higher limit behavior of "backward inference" is given by the reversed process).