Decision theory, historical background.

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Exercise 13.1

We interpret this problem as follows: Starting from current fortune M one has to bet unknown amount N which with probability $p > 0.5$ will yield success (and new fortune of $M + N$) and $1 - p$ will yield failure (and new fortune of $M - N$). The question is to find y which maximizes expected log of the new fortune.

Thus the expected log fortune is

$$p \log(M + N) + (1 - p) \log(M - N)$$

Maximizing this in N yields

$$\frac{p}{M + N} = \frac{1 - p}{M - N}.$$

Denoting $r = \frac{N}{M}$ we get $p(1 - r) = (1 - p)(1 + r)$, or $p - rp = 1 - p + r - rp$, and $r = 2p - 1$.

The expected log fortune is then

$$\log M + p \log 2p + (1 - p) \log 2(1 - p) = \log M + H_{0.5} - H_p,$$

where H_p is the entropy of the entropy of the Bernoulli distribution with parameter p.

(The fact that this is higher than $\log M$ which is what one would obtain at $N = 0$ (and of course higher than $-\infty$ that one would obtain at $N = 1$) also shows that this is a global maximum (as opposed to just local stationary point), which we could've verified by second derivative test, but were too lazy to do.)

After n iterations of this the expected log fortune is $\log M + n(H_{0.5} - H_p)$, and the exponent of that is $M \exp\{n\alpha\}$ with $\alpha = H_{0.5} - H_p$, as per the exercise.

Demonstration that this is actually the best (with respect to expected log fortune) strategy not only for single step, but over any number n of steps is via dynamic programming aka Bellman optimality principle, and can be found in Bellman-Kalaba, "On the Role of Dynamic Programming in Statistical Communication Theory", Sections VII-IX.