Elementary hypothesis testing

← Back to Chapters

Exercise 4.1

Given

$$P(D_1...D_m|H_iX) = \prod_j P(D_j|H_iX)$$

and

$$P(D_1...D_m|\overline{H}_iX) = \prod_j P(D_j|\overline{H}_iX)$$

for $1 \leq i \leq n$ and $n > 2$ show that for any fixed i at most one of

$$\frac{P(D_j|H_iX)}{P(D_j|\overline{H}_iX)}$$

is not equal to 1.

Proof:

Firstly, we claim that the case of 2 pieces of data implies the general result.

Indeed, independence assumptions of all D_j together imply analogous pairwise independence for any pair D_k and D_l. So, assuming the case with two data pieces is solved, we have, for a fixed i and for any pair of k, l, either $\frac{P(D_k|H_iX)}{P(D_k|\overline{H}_iX)} = 1$, or $\frac{P(D_l|H_iX)}{P(D_l|\overline{H}_iX)} = 1$ (or both), so of the whole set of $\frac{P(D_j|H_iX)}{P(D_j|\overline{H}_iX)}$ at most one is not equal to 1, as wanted.

So it is enough to sole the case of only two data sets, D_1 and D_2.

We will denote probability density/mass function of D_1 under hypothesis H_iX by V_i and that of D_2 by U_i. We will also denote probability density/mass function of D_1 under hypothesis $\overline{H}_iX$ by V_i^c and that of D_2 by U_i^c, though we will not use these until the very end.

Remark 1: The proof actually works for arbitrary (not necessarily discrete or continuous) real-valued random variables, one just has to say that Vs and Us are CDFs instead of PMFs/PDFs. The reason for that is, firstly, that independence of two random variables can be equivalently written as joint CDF being a product or as joint PMF/PDF being a product, and, secodnly, that equality of CDFs is equivalent to equality of PMFs/PDFs. We work with PDFs/PMFs out of a strange esthetic choice, rather than for any other reason.

We will denote value of V_i at some $D_1 = x_1$ by v_{i1} (and at $D_1 = x_2$ by v_{i2}). Similarly the values of U at $D_2 = y_1$ will be denoted by u_{i1}.

We will use independence of D_1 and D_2 conditional on various hypotheses to prove independence under other hypotheses. Note that independence always means $P(D_1 = x, D_2 = y|H) = P(D_1 = x|H)P(D_2 = y|H)$ and can be checked by checking this for (arbitrary) specific values.

With this in mind, we pick any pair of possible value pairs $D_1 = x_1, D_2 = y_1$ and $D_1 = x_2, D_2 = y_2$, fixed from now on, and form vectors $v_i = \begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix}$ and $u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix}$

Then independence of D_1 and D_2 (conditional on H_iX) says that the joint distribution of $P(D_1 = x, D_2 = y|H_iX)$ is a product of distributions of D_1 and D_2 and is given (at $x = x_1, x_2$ and $y = y_1, y_2$) by the matrix

$$M_i = v_i u_i^T = \begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix} \begin{bmatrix} u_{i1} & u_{i2} \end{bmatrix} =$$

$$= \begin{bmatrix} v_{i1}u_{i1} & v_{i1}u_{i2} \\ v_{i2}u_{i1} & v_{i2}u_{i2} \end{bmatrix}$$

It follows from this that the (joint) probability matrix of $D_1 D_2$ (again, at $x = x_1, x_2$ and $y = y_1, y_2$) conditional on $\overline{H_i}X$ is obtained by taking all the matrices of H_j with $j \neq i$ weighing them by (prior) probabilities h_j of H_j and adding them (and then dividing by the sum of the weights, but this is an overall normalizing factor which will not be important for us). That is, the matrix is proportional to

$$\sum_{j \neq i} h_j M_i = \sum_{j \neq i} h_j v_j u_j^T =$$

$$\begin{bmatrix} \sum_{j \neq i} h_j v_{i1}u_{i1} & \sum_{j \neq i} h_j v_{i1}u_{i2} \\ \sum_{j \neq i} h_j v_{i2}u_{i1} & \sum_{j \neq i} h_j v_{i2}u_{i2} \end{bmatrix}$$

Now the assumption that D_1 and D_2 are independent conditional on $\overline{H}_i$ means this matrix is also a product of "marginal" matrices of $D_1|\overline{H}_iX$ and $D_2|\overline{H}_iX$, i.e. is of rank 1. This means that it has determiant 0.

Three hypothesis.

Let's start with the case of only 3 hypothesis H_1, H_2, H_3. Start with $i = 3$.

#####Lemma: A sum M of two rank 1 matrices $M = h_1 v_1 u_1^T + h_2 v_2 u_2^T$ can only be rank 1 if either v_1 and v_2 are linearly dependent or u_1 and u_2 are linearly dependent.

A "conceptual" proof is as follows: Consider the image of M. Vectors $M(u_1)$ and $M(u_2)$ are both linear combinations of $h_1 v_1$ and $h_2 v_2$, so, to get that the image of M is the span of v_1 and v_2, it is enough that the matrix of coefficients $G = \begin{pmatrix} u_1^T u_1 & u_1^T u_2 \\ u_2^T u_1 & u_2^T u_2 \end{pmatrix}$ is invertible. But G is the Grammian of u_1, u_2 and is invertible precisely when u_1 and u_2 are linearly independent (its determinant is the square of the area of the parallelogram spanned by u_1 and u_2, as you can easily verify). In that case (of independent us), the rank of M is the dimension of the span of v_1, v_2 and if v_1, v_2 were independent, it would be 2. So if rank of M is below 2, then either us or vs are dependent, as wanted.

Remark 2: Those familiar with tensors may realize that we use metric in which u_1 and u_2 are orthonormal (that's the inverse of G) to "raise and index" and go from a bilinear form encoded by M to a linear map, whose range is then the span of vs.

Remark 3: Alternatively, for those who don't like linear algebra, a computational proof of Lemma 1 is as follows: A (non-zero) 2 by 2 matrix has rank one when its determinant is zero. Writing this out in our case we get:

$$[h_1 v_{11} u_{11} + h_2 v_{21} u_{21}][h_1 v_{12} u_{12} + h_2 v_{22} u_{22}] =$$

$$[h_1 v_{11} u_{12} + h_2 v_{21} u_{22}][h_1 v_{12} u_{11} + h_2 v_{22} u_{21}]$$

Additively canceling $h_1^2 v_{11} v_{12} u_{11} u_{12}$ and $h_2^2 v_{21} v_{22} u_{21} u_{22}$ and then dividing by $h_1 h_2$ we have

$$v_{11} u_{11} v_{22} u_{22} + v_{21} u_{21} v_{12} u_{12} =$$

$$v_{11} u_{12} v_{22} u_{21} + v_{21} u_{22} v_{12} u_{11}$$

or
$$(v_{11} v_{22} - v_{12} v_{21})(u_{11} u_{22} - u_{12} u_{21}) = 0,$$

so either v_1 and v_2 are linearly dependent, or u_1 and u_2 are.

Continuing with the case of three hypothesis, recall v_i and u_i were likeliehood/probability vectors of D_1 taking values x_1, x_2 and D_2 taking values y_1, y_2 (under hypothesis $H_i X$).

Observe that a pair of non-zero functions V_1 and V_2 such that $(V_1(x_1), V_1(x_2))$ is always proportional to $(V_2(x_1), V_2(x_2))$ are "globally" proportional meaning $V_1 = kV_2$ (take any x with $V_2(x) \neq 0$ and make $k = V_1(x)/V_2(x)$).

If distributions of D_1 under H_1X and H_2X are different, then they are also not proportional. By the previous paragraph, this implies that there will be two values x_1 and x_2 giving unproportional v_1 and v_2. Then for arbitrary pair of values y_1, y_2 of D_2 the corresponding vectors u_1 and u_2 are proportional, so, again, by the previous paragraph, the whole probability mass/density functions U_1 and U_2 of D_2 under H_1X and H_2X are proportional, ergo equal.

So either $V_1 = V_2$ or $U_1 = U_2$.

Now, in the same way as we just did for $i = 3$, from $i = 1$ and $i = 2$ we get that (either $V_2 = V_3$ or $U_2 = U_3$) and (either $V_1 = V_3$ or $U_1 = U_3$). Since we have 3 equalities and only two types of deistributions (U and V), either the Vs are equal twice, and $V_1 = V_2 = V_3$, or the Us are (and $U_1 = U_2 = U_3$). Correspondingly either D_1 has same distribution under all 3 hypothesis, and then $\frac{P(D_1|H_iX)}{P(D_1|\overline{H}_iX)}$ are all equal to 1, or D_2 does (and then all $\frac{P(D_2|H_iX)}{P(D_2|\overline{H}_iX)}$ are equal to 1). In either case, we get what we want.

This completes the case of 3 hypothesis.

The general case.

To get the extension to more than 3 hypothesis we use the following approach. As we mentioned before, a 2 by 2 matrix is or rank at most 1 if its determinant is zero. So we need some an efficient way of telling when the deterimnant of 2 by 2 matrix is zero.

Remark 4: More generally, a matrix is of rank at most 1 if all 2 by 2 minors have determinannt zero i.e. all $M^{\wedge 2}_{(i,j),(k,l)} = M_{ik}M_{jl} - M_{il}M_{jk}$ are zero. In tensor analysis, these are the entries of the second exterior power $M^{\wedge 2}$ of M. When dimension is 2 there is only one minor, and the $M^{\wedge 2}$ is a scalar, equal to $\det M$. So in dimensions above 2, we can formulate everything that follows in terms of determinants.

We will use the following property of 2D determinants. If M and N are 2 by 2 matrices then

$$D(M, N) := \frac{1}{2}(\det(M + N) - \det M - \det N)$$

is symmetric and bilinear in M, N. This means

$$D(M, N) = D(N, M)$$

and

4

$$D(M_1 + M_2, N) = D(M_1, N) + D(M_1, N)$$

(and hence the same for second variable). Indeed, one computes

$D(M, N) = \frac{1}{2}(M_{11}N_{22} + M_{22}N_{11} - M_{12}N_{21} - M_{21}N_{12})$

and the resulting formula is linear in M and in N, i.e. bilinear.

Observe that $D(M, M) = \det M$. We then have, by induction on the number of summands,

$\det(\sum M_i) = D(\sum M_i, \sum M_j) = \sum_{i,j} D(M_i, M_j)$

Remark 5: We also have $D(\lambda M, N) = \lambda D(M, N)$, as usual in bilinearity, but we don't need this.

Remark 6: In higher (possibly) dimensions, and using tensor language, we are saying that taking second exterior power, which is quadratic in the matrix input, is a restriction of a symmetric bilinear operation (on two inputs), $(M \wedge N)(\vec{a} \wedge \vec{b}) = \frac{1}{2}[(M\vec{a}) \wedge (N\vec{b}) + (N\vec{a}) \wedge (M\vec{b})]$

Now we can apply this to our problem. Let $M_i = h_i v_i u_i^T$ and $N_i = \sum_{i \neq j} M_i$, and $M = M_i + N_i = \sum_j M_j$.

Our assumptions are that all M_i and N_j are rank 1 (i.e. have zero determinant). We now show that M has rank 1 (i.e. has zero determinant).

To that end we write

$$\det M = \sum_{j,k} D(M_j, M_k)$$

We want to see that this is zero. We know

$$0 = \det(N_i) = \sum_{j \neq i, k \neq i} D(M_j, M_k)$$

Summing over i we get (taking note that each $D(M_l, M_l)$ will apear $n - 1$ times, while those $D(M_j, M_k)$ with $j \neq k$ will appear only $n - 2$ times):

$$\sum_l D(M_l, M_l) + (n - 2) \sum_{j,k} D(M_j, M_k) = 0$$

So, since $D(M_l, M_l) = 0$, as long as $n \neq 2$ we have what we want.

This gives $M = vu^T$. Going back to $M = M_i + N_i$ we again see two rank one matrices add up to a rank one matrix. We conclude, just as in the case of 3 hypothesis, that for each specific i, either $V_i^c = V_i$ and hence $\frac{P(D_1|H_i X)}{P(D_1|\overline{H}_i X)} = 1$

OR

$U_i^c = U_i$ and hence $\frac{P(D_2|H_iX)}{P(D_2|\overline{H}_iX)} = 1$. This is exactly what we wanted to prove.

Some remarks on sequential vs. batch updates.

(See formula 4.11; the discussion is in the context of section 4.4 and formula 4.44.)

We are considering a batch of widgets from a single machine. We have three hypothesis, A – the machine has failure rate $\frac{1}{3}$ (prior probability $\frac{1}{11}$), B – the machine has failure rate $\frac{1}{6}$ (prior probability $\frac{10}{11}$), and C – the machine has failure rate $\frac{99}{100}$ (prior probability $\frac{1}{10^6}$).

Prior odds ratio is then (to a very good approximation) $\frac{10^6}{11} : \frac{10^7}{11} : 1$.

The update rule for arbitrary data tells us how to compute the posterior odds A vs B vs C: one takes the prior odds and multiplies each number by the likeliehood of data under corresponding hypothesis. For data of m bad widgets the likeliehhods are $\frac{1}{3^m}, \frac{1}{6^m}, \frac{99^m}{100^m}$. This gives posterior odds:

$$\frac{10^6}{11}\frac{1}{3^m} : \frac{10^7}{11}\frac{1}{6^m} : \frac{1}{10^6}\frac{99^m}{100^m}.$$

Note that since the individual draws are independent **conditional on one specific hypothesis**, the likeliehoods for m widgets are each a product of individual likeliehoods of draws, and we see very clearly that the same posterior odds will be obtained by updating on all the data of m bad widgets, or by splitting $m = m_1 + m_2$ and updating first to odds ratio

$$\frac{10^6}{11}\frac{1}{3^{m_1}} : \frac{10^7}{11}\frac{1}{6^{m_1}} : \frac{1}{10^6}\frac{99^{m_1}}{100^{m_1}},$$

using that as new prior odds and then updating to the final posterior odds

$$\frac{10^6}{11}\frac{1}{3^{m_1}}\frac{1}{3^{m_2}} : \frac{10^7}{11}\frac{1}{6^{m_1}}\frac{1}{6^{m_2}} : \frac{1}{10^6}\frac{99^{m_1}}{100^{m_1}}\frac{99^{m_2}}{100^{m_2}}$$

getting the same result.

It is only when merging A and B into a single hypothesis $\overline{C} = A + B$ that we have trouble: individual draws are not independent under $\overline{C}$. To illustrate, consider $m = 2$. Firstly, the prior odds of $C : \overline{C}$ are

$$C_0 : \overline{C} = C_0 : A_0 + B_0 = 1 : \frac{10^6}{11} + \frac{10^7}{11} = 1 : 10^6$$

The likeliehood of the first widget being bad under C is as before $\frac{99}{100}$. The likeliehood of it being bad under $\overline{C} = A + B$ is trickier: B is 10 times more likely

6

than A, so the probability of a bad widget under $A + B$ is weighted average of probabilities under A and B with the one under B getting ten times more weight. This is simply the "total probability formula"

$$P(D_1|A + B) = P(D_1|A)P(A|A + B) + P(D_1|B)P(B|A + B) =$$

$$= \frac{1}{3}\frac{1}{11} + \frac{1}{6}\frac{10}{11} = \frac{2}{11}$$

After the first widget is drawn, the posterior odds are prior odds times likeliehood:

$$C_1 : \overline{C}_1 = 1\frac{99}{100} : 10^6\frac{2}{11}$$

We can check it's the same result as one obtained by first updating the odds of $A : B : C$ and then summing the ones for A and B:

$$C_1 : A_1 + B_1 = \frac{99^1}{100^1} : \frac{10^6}{11}\frac{1}{3^1} + \frac{10^7}{11}\frac{1}{6^1} = \frac{99}{100} : \frac{2 \cdot 10^6}{11}$$

Now **if** the likeliehood of second bad widget under $\overline{C}$ was again $P(D_2|A+B) = \frac{2}{11}$ we would get the **WRONG** $C_2 : \overline{C}_2 = \frac{99}{100}\frac{99}{100} : \frac{2\cdot10^6}{11}\frac{2}{11}$.

However, it is not - D_2 is not independent of D_1 under $\overline{C}$ – having learned that the first draw was defective we now think it is relatively more likely that the batch came from A rather than B, as follows. The odds ratios after D_1 of $A_1 : B_1 = 1\frac{1}{3} : \frac{1}{6}10 = 1 : 5$, so $P(A|(A + B)D_1) = \frac{1}{6}$ and $P(B|(A + B)D_1) = \frac{5}{6}$. This means that the second widget is more likely to come from A rather than B and is more likely to be bad:

$$P(D_2|(A + B)D_1) =$$

$$P(D_2|AD_1)P(A|(A + B)D_1) + P(D_2|BD_1)P(B|(A + B)D_1) =$$

$$= \frac{1}{3}\frac{1}{6} + \frac{1}{6}\frac{5}{6} = \frac{7}{36} > \frac{2}{11}$$

This means that seeing the second bad widget is giving somewhat less evidence for C over $\overline{C}$ than seeing the first one did (some evidence in D_2 is "already" in D_1).

We can now finish the computation for the second update:

$$C_2 : \overline{C}_2 = \frac{99}{100}\frac{99}{100} : \frac{2 \cdot 10^6}{11}\frac{7}{36} = \frac{99^2}{100^2} : \frac{14 \cdot 10^6}{11 \cdot 6^2}$$

and compare it to the "batch" calculation:

$$C_2 : A_2 + B_2 = \frac{99^2}{100^2} : \frac{10^6}{11}\frac{1}{3^2} + \frac{10^7}{11}\frac{1}{6^2} = \frac{99^2}{100^2} : \frac{14 \cdot 10^6}{11 \cdot 6^2}.$$

Most of Exercise 4.2

Suppose prior odds of hypothesis H_i are s_i, and the probability of good widget under H_i is p_i and of a bad one is $q_i = 1 - p_i$.

For n_0 bad widgets and n_1 good ones the posterior odds are

$$\ldots : L_i : \ldots = \ldots : s_i p_i^{n_0} q_i^{n_1} : \ldots$$

If the fraction of bad widgets is f then we have $n_0 = fn$, $n_1 = (1 - f)n$ and posterior odds

$$\ldots : L_i : \ldots = \ldots : s_i (p_i^{1-f})^n (q_i^f)^n : \ldots$$

For two different hypotheses the ratio $L_i : L_j$ does not go to zero or infinity precisely when $p_i^{1-f} q_i^f = p_j^{1-f} q_j^f$. Solving for f we get:

$$\frac{p_i}{p_j} = \left(\frac{q_j p_i}{q_i p_j}\right)^f$$

$$\ln \frac{p_i}{p_j} = f \ln \frac{q_j p_i}{q_i p_j}$$

$$f = \frac{\ln \frac{p_i}{p_j}}{\ln \frac{q_j}{q_i} + \ln \frac{p_i}{p_j}}$$

$$f = \frac{1}{\frac{\ln \frac{q_j}{q_i}}{\ln \frac{p_j}{p_i}} + 1}$$

For the case in section 4.4 we have $p_C = 1/100$, $p_A = 2/3$, $q_C = 99/100$, $q_A = 1/3$,

$$f = \frac{1}{\frac{\ln \frac{q_j}{q_i}}{\ln \frac{p_j}{p_i}} + 1} = \frac{1}{\frac{\ln \frac{297}{100}}{\ln \frac{200}{3}} + 1} = 0.79415525984291269327755075440066871236492523334748242$$

Additional notes on 4.2 proof

The likelihood of hypothesis B is irrelevant to calculating the threshold f_t because the the point where the evidence is balanced (in the limit) between hypotheses A and C only depends on the relationship between the two likelihoods. The prior probability of C is also irrelevant to the calculation.

Note: the value of f_t given in the book is incorrect, the correct answer is above. This is noted in the errata.

Some of Exercise 4.3

We now consider the case of 4 hypothesis A, B, C as before and F. We remark that since we will study the evolution of our beliefs in the scenario where about $1/4$ of widgets are defective, neither C nor the hypothetical E from the first part of this exercise under which most widgets are good are of any importance, so we can, in fact just study how F fairs against A and B.

The likeliehoods of data of $n/4$ bad widgets and $3n/4$ good ones under $A : B : F$ are

$$\left[\left(\frac{1}{3}\right)^{\frac{1}{4}}\left(\frac{2}{3}\right)^{\frac{3}{4}}\right]^n : \left[\left(\frac{1}{6}\right)^{\frac{1}{4}}\left(\frac{5}{6}\right)^{\frac{3}{4}}\right]^n : \left[\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{3}{4}\right)^{\frac{3}{4}}\right]^n$$

$$0.5606^n : 0.5573^n : 0.5699^n$$

So the evidence for F is (in Jaynesean db) $10(\log_{10} 0.5699^n - \log_{10}(0.5606^n + 0.5573^n))$ which for large n is $n \cdot 10 \log_{10} \frac{0.5699}{0.5606} \approx n \cdot 0.0714556 \approx \frac{n}{14}$. So we would need about 14 samples per decibel of evidence in favor of F, i.e. multiple hundreds before something like 20db of evidence for F over A and B can be collected.

Exercise 4.4

This problem is simplified if we assume that A and B are the only hypothesis under consideration. Then evidence for A is evidence against B and vice versa. In addition, the individual widget tests are independent both under $A = \overline{B}$ and $B = \overline{A}$.

So each negative test gives $10 \log_{10} \frac{1/3}{1/6} = 3$ db evidence for A. If B is true, the probability of this is $\frac{1}{6}$.

And a positive gives $10 \log_{10} \frac{2/3}{5/6} = -0.97$ db evidence against A. If B is true, the probability of this is $\frac{5}{6}$.

We now indeed have a random walk. We start at 0 db of evidence, and then at each step we either, with probability $\frac{1}{6}$ take a step of size 3, or, with probability $\frac{5}{6}$ take a step of size -0.97.

We now ask: 1) what is the probability of ever getting to 20 db? 2) how long will it take to get -20 db?

This is an exercise in "Wald theory" a la A. Wald, "Sequential analysis" or W. Feller, "An introduction to probability theory and its applications", Vol 1 Ch. XIV., Vol 2 Ch.XII (Abraham Wald is the statistician of the "holes you see in the returning bombers are in the areas where a bomber can take a hit and still come back, it's all the other areas you should worry about, you dolts!" fame; someone really should make a movie about his life.)

One gets a simplification by rounding -0.97 to -1, for then the possible amounts of evidence are all integers, and the random walk is on $\mathbb{Z}$. Moreover, the resulting random walk is "skip-free to the left" - one can not move left by more than one unit at a time. There are a more or less standard approaches for analyzing such a walk.

(Without simplifying exact solution becomes intractable, but one can get various approximations. We derive only very basic such approximation at the end this write-up, but it is discussed by Wald and in Vol II of Feller.)

Probability of getting 20 db for A**.**   Let p_i be the probability that the walk will ever reach the state i decibel.

Firstly, the expected evidence is $3\frac{1}{6} + (-1)\frac{5}{6} = -\frac{1}{3}$ is negative, so by the strong law of large numbers the walk goes off to $-\infty$ with probability 1. Since the only way to reach it starting from 0 is to go through all negative values, $p(i) = 1$ for all $i \leq 0$.

In addition, this means that, if we denote by X_j the set of all walks that reach state i but don't reach state $i + 1$ then $X = \cup_{j \geq 0} X_j$ has probability 1 (each walk that converges to $-\infty$ will belong to one of the X_i). This means that $p_i = P(\cup_{j \geq i-1} X_i)$ converges to zero as $i \to \infty$.

The above two paragraphs give boundary conditions for p_i. Now let's find a relation between different p_is (for positive i). Consider the first step of the walk. Either, with probability $\frac{1}{6}$ it is 3 to the right, and so what remains is to reach $i - 3$ steps to the right, probability of which is p_{i-3}. Or, with probability $\frac{5}{6}$ it is 1 to the left, and so now the walk needs to reach $i + 1$ steps to the right, probability of which is p_{i+1}. So

$$p_i = \frac{1}{6}p_{i-3} + \frac{5}{6}p_{i+1}$$

The first few are $p_1 = \frac{1}{6} + \frac{5}{6}p_2$, $p_2 = \frac{1}{6} + \frac{5}{6}p_3$, $p_3 = \frac{1}{6} + \frac{5}{6}p_4$, $p_4 = \frac{1}{6}p_1 + \frac{5}{6}p_5$ etc.

Remark: $\frac{1}{6}p_{-3} + \frac{5}{6}p_1 = \frac{1}{6} + \frac{5}{6}p_1$ is the probability of returning to 0 **after** the firs step, and so in **not** equal to $p_0 = 1$.

There is a standard way to solve such recurrence relations, and even several variations on it.

In a more direct way, one starts with $p_i = r^i$ to get characteristic equation

$$r^3 = \frac{1}{6} + \frac{5}{6}r^4.$$

This has roots $r_1 = 1$, $r_2 \approx 0.8$, $r_{3,4} \approx -0.3 \pm 0.4i$ (so $|r_{3,4}| \approx 0.5$). A general solution is a linear combination of these, $p_i = c_1 r_1^i + c_2 r_2^i + c_3 r_3^i + c_4 r_4^i$. The boundary condition $\lim_{i \to \infty} p_i = 0$ means $c_1 = 0$, and the other conditions $p_0 = p_{-1} = p_{-2} = 1$ give

$$\begin{pmatrix} 1 & 1 & 1 \\ r_2^{-1} & r_3^{-1} & r_4^{-1} \\ r_2^{-2} & r_3^{-2} & r_4^{-2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ r_2^{-1} & r_3^{-1} & r_4^{-1} \\ r_2^{-2} & r_3^{-2} & r_4^{-2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

I seem to get $c_4 = 0.07492831 - 0.0259995i$, $c_3 = 0.07492831 + 0.0259995i$, $c_2 = 0.85014338$. This means that we can ignore the smaller modulus complex roots, and write $p_i \approx 0.85 \cdot (0.7823728)^i$. For $i = 20$ we have

$$p_{20} \approx 0.006276157.$$

This can probably be described as "negligibly small".

Alternatively, another way (Feller, Chapter XI.1 and XI.4, particularly Example b) is to define $Q(s) = \sum_{i=-2}^{\infty} p_i s^i$ and observe that starting from

$$p_i = \frac{1}{6}p_{i-3} + \frac{5}{6}p_{i+1}$$

multiplying by s^i, and summing over $i \geq 1$ and summing, using $p_{-2} = p_{-1} = p_0 = 1$ and $p_1 = p$ gives:

$$Q(s) - 1 - s^{-1} - s^{-2} = \frac{1}{6}s^3 Q(s) + \frac{5}{6}\left(\frac{1}{s}(Q(s) - sp - 1 - s^{-1} - s^{-2})\right)$$

$$\frac{5}{6}(ps^3 + s^2 + s + 1) - (s^3 + s^2 + s) = Q(s)s^2\left(\frac{1}{6}s^4 - s + \frac{5}{6}\right)$$

11

$$Q(s)s^2 = \frac{\frac{5}{6}(ps^3 + s^2 + s + 1) - (s^3 + s^2 + s)}{\frac{1}{6}s^4 - s + \frac{5}{6}}$$

This means that $Q(s)s^2 = \frac{U(s)}{V(s)}$ is a rational function and has partial fraction expansion $\sum_{j=1}^{4} \frac{a_k}{s_k - s}$ with s_k the roots of $V(s)$ (luckily in our case the roots of V are distinct) and

$$a_k = \frac{-U(s_k)}{V'(s_k)}$$

and

$$p_{i-2} = \frac{a_1}{s_1^{i+1}} + \frac{a_2}{s_2^{i+1}} + \frac{a_3}{s_3^{i+1}} + \frac{a_4}{s_4^{i+1}}.$$

We note that the roosts s_k of $V(s)$ are the inverses of the roots r_k in our previous "direct approach", and the formula above is thus identical with the one we got before.

One of the roots of $V(s)$ is 1, but the corresponding coefficient c_1 must be 0 otherwise the limit p_i will not be zero. This gives $U(1) = 0$ meaning $\frac{5}{6}(p+3) - 3 = 0$, or $p = \frac{3}{5}$.

The next smallest in absolute value root is $s_2 \approx 1.278$. The corresponding coefficient

$$a_2 = \frac{-U(s_2)}{V'(s_2)} \approx \frac{0.696}{1.225424} \approx 1.775215$$

giving

$$p_{20} \approx \frac{1.775215}{(1.2781)^{23}} \approx 0.0062772128$$

as before (within precision of calculations). We could compute a_3 and a_4 to make sure they are not huge and ignoring the other terms is justified, but we have already seen this in the previous method, so we omit this calculation.

Remark: The fact that one can use these two methods to compute the coefficients c_k in the expansion of p_i means that the inverse of the Vandermonde matrix is computable via partial fractions expansion. This has been noted in the literature.

Time to get 20 db for B. For a shortcut skip to the bottom "Wald identities".

Since the the walk is skip-free on the left, to get 20 db for B one has to first get 19 db, and for that one needs to first get 18 db etc. So we start by getting a distribution of times to get 1 db for B. Let T be the (random) time until the walk hits -1. We define the formal power series

$$f(x) = E(x^T) = xP(T = 1) + x^2 P(T = 2) + ...$$

encoding the distribution of T. We have $f(1) = 1$ and $f(0) = 0$. We also have $f'(1) = E(T)$ and $f''(1) = E(T(T - 1))$ and so on.

Remark: This annoying $T^2 - T$ as opposed to just T can be avoided by changing variables $x = e^u$ (giving moment generating function of T), or, even better, $x = e^{iu}$ (giving characteristic function of T). This is in some ways nicer, but we stick to generating function version and its polynomial look and feel.

Then the distribution of (random) time to hit $-k$ is given by writing $T_k = Z_1 + Z_2 + \ldots + Z_k$ with Z_i i.i.d. with same distribution as T, so that the formal power series encoding the distribution of T_k is

$$f_k(x) = E(x^{T_k}) = E(x^{\sum Z_i}) = E(\prod x^{Z_i}) = \prod E(x^{Z_i}) = f(x)^k$$

Now consider T again, and condition on the first step of the walk. With probability $\frac{5}{6}$ it is to the left and $T = 1$. With probability $\frac{1}{6}$ it is to the right and then T is distributed as encoded by $f_4(x) = xf(x)^4$ (the extra x is for the one extra time beat that the first step used). So overall T, being a mixture of these two possibilities, is distributed as $\frac{5}{6}x + \frac{1}{6}f(x)^4$. Hence

$$f(x) = \frac{5}{6}x + \frac{1}{6}xf(x)^4$$

One could write $f(x) = c_1 x + c_2 x^2 + ...$ and plug in to get

$$c_1 x + c_2 x^2 + ... = \frac{5}{6}x + \frac{1}{6}x(c_1 x + c_2 x^2 + ...)^4$$

which can be rewritten as a "upper triangular" system of equations to be solved one after another, getting $c_1 = \frac{5}{6}, c_2 = c_3 = c_4 = 0, c_5 = \frac{1}{6}c_1^4 = \frac{5^4}{6^5}, c_6 = \frac{1}{6}4c_1^3 c_2 = 0$ etc. Then, of course it is $f(x)^{20}$ that gives the distribution of times to hit 20 db that we are after.

Abandoning the pursuit of full distribution of T_{20} we instead compute its first and second moments.

First we deal with the moments of T. Differentiating the equationfor f we have

$$f' = \frac{5}{6} + \frac{1}{6}f^4 + \frac{1}{6}x4f'f^3$$

Since $f(1) = 1$ we get $f'(1) = 1 + \frac{4}{6}f'(1)$, $f'(1) = 3$.

We have $E(T_{20}) = (f^{20})'(1) = 20f'(1)f(1)^{19} = 20 \times 3 = 60$. This tells us the expected time to get 20 db for B is 60 samples. (For general k we get $E(T_k) = 3k$.)

We will now find the variance of this estimate.

Differentiating again

$$f'' = \frac{1}{6}4f'f^3 + \frac{1}{6}4f'f^3 + \frac{1}{6}x4f''f^3 + \frac{1}{6}x4f'3f'f^2$$

from which $f''(1) = 2 + 2 + \frac{2}{3}f''(1) + 18$, $f''(1) = 66$. Thus $E(T^2) = 69$ and $Var(T) = E(T^2) - E(T)^2 = 60$, and standard deviation is $\sqrt{60} \approx 7.75$

Remark 1: This low expectation and high variance are due to the high probability of $T = 1$; conditional on $T \neq 1$ the expectation of T is $(3 - \frac{5}{6})/\frac{1}{6} = 13$ (which coincides with the (unconditional) expectation of T_4 plus 1, i.e. $3 \cdot 4 + 1 = 13$). The conditional variance is the variance of T_4 and is 240, so the standard deviation is ≈ 15.5 (see below). This is still indicating high probability of low values and a "fat tail", but a bit less so than the unconditional case.

On the other hand

$$(f^{20})'' = (20f^{19}f')' = 20(19(f')^2f^{18} + f^{19}f'') = 20(19 \cdot 9 + 66) = 4740$$

so $E(T_{20}^2) = 4740 + 60 = 4800$ and $Var(T) = 4800 - 60^2 = 1200$, and standard deviation $\sqrt{1200} \approx 34.64$, which is more reasonable. It indicates that we may get lucky and get 20 db evidence much sooner than 60 tests, but may get unlucky and have to wait substantially longer, maybe as long as 130 trials or even more. One could compute further moments using the same method (with a longer calculation) to get more of an idea.

For general k we get $E(T_k^2) = k((k-1)9 + 66) + 3k$, $Var(T_k) = 60k$

Alternative solution using Wald identities.

We have in general $E(T_k) = 3k$. Can be deduced much more quickly from the general First Wald Identity: if T is a stopping time with respect to i.i.d. Z_is with $E(T)$ finite, and $X = \sum_{i=1}^{T} Z_i$ then

$$E(X) = E(T)E(Z_1).$$

In our case, assuming $E(T_k)$ is finite, we have $E_{Z_1} = 3\frac{1}{6} - \frac{5}{6} = -\frac{1}{3}$, $X = -k$ (always), so $E(T_k) = 3k$.

We still need to check $E(T_k) < \infty$. This follows from (a generalization) of our analysis of the random walk on $\mathbb{Z}$, but in fact Hoeffding's inequality implies exponentially small bond on probability $T_k > n$, so T_k actually has all moments finite.

Wald's second identity (under same assumption $E(T)$ finite) is

$$E[(X - E(Z_i)T)^2] = Var(Z_i)E(T).$$

For us $X = -k$, $E(Z_i) = -\frac{1}{3}$, $E(T_k) = 3k$ and

$$VarZ_i = \frac{1}{6}(3 + \frac{1}{3})^2 + \frac{5}{6}(-1 + \frac{1}{3})^2 = 100/54 + 20/54 = 120/54$$

Plugging in, this gives:

$$k^2 - 2(-k)(-1/3)3k + 1/9E(T_k^2) = 120/54 \cdot 3k$$

$$-9k^2 + E(T_k^2) = 60k$$

$$E(T_k^2) = 9k^2 + 60k$$

$$Var(T_k) = 9k^2 + 60k - 9k^2 = 60k$$

as before.

One advantge of using this method is that it can handle the unapproximated problem as well.

Indeed, if we keep the leftward step 0.97 instead of making it 1, $E(Z_1)$ changes a bit to $-\frac{1}{3} + \frac{5}{6}0.03 = \frac{-200+15}{600} = -\frac{37}{120}$, and the $E(X)$ is no longer $-k$, but is between $-k$ and $-k - 0.97$, so we get an estimate $\frac{120}{54}k \leq E(T_k) < \frac{120}{54}(k+0.97)$. Similar estimate can be found for the variance of T_k as well.

Exercise 4.5

It seems that from a Bayesian perspective the conjecture is tautological. If "reliability of conclusion" means some precise minimum of posterior odds ratio in favor of the ith hypothesis, then one has to keep collecting data until that odds ratio is achieved. So, tautologically, the minimum ASN is obtained by the procedure in which one stops collecting further data after the requisite odds ratio (for any of the hypothesis under consideration) is attained for the first time.

From the frequentist point of view the situation is more complicated. Instead of posterior probability/odds for a specific hypothesis, the frequentist guarantee on a procedure is in a form of a bound on probability of Type I (falsely rejecting a true hypothesis, low p value) and Type II (accepting a false hypothesis, high power) errors is posited, and a procedure that confirms to these bounds is evaluated based on its ASN. It is relatively easy to see tha for testing binary hypothesis, if Bayesian/sequential procedure always hits the required posterior odds exactly, without overshooting (for example, in the setup of Exercise 4.4 the evidence is always either plus or minus 1 and the bounds are integers), then sequential test will have the smallest ASN, and, thus, in general, it will have closed to the smallest ASN (this is Section A.7 of Wald's "Sequential Analysis" book). However without such an assumption such optimality of sequential tests is harder to establish. Nonetheless it was done by Wald and Wolfowitz in 1948 ("Optimum character of the sequential ratio test").