

UNIVERSITÄT  
BAYREUTH

# Feuerzangenbowle

Physics of Imbibition and Porous Media Flow

Manuel Lippert

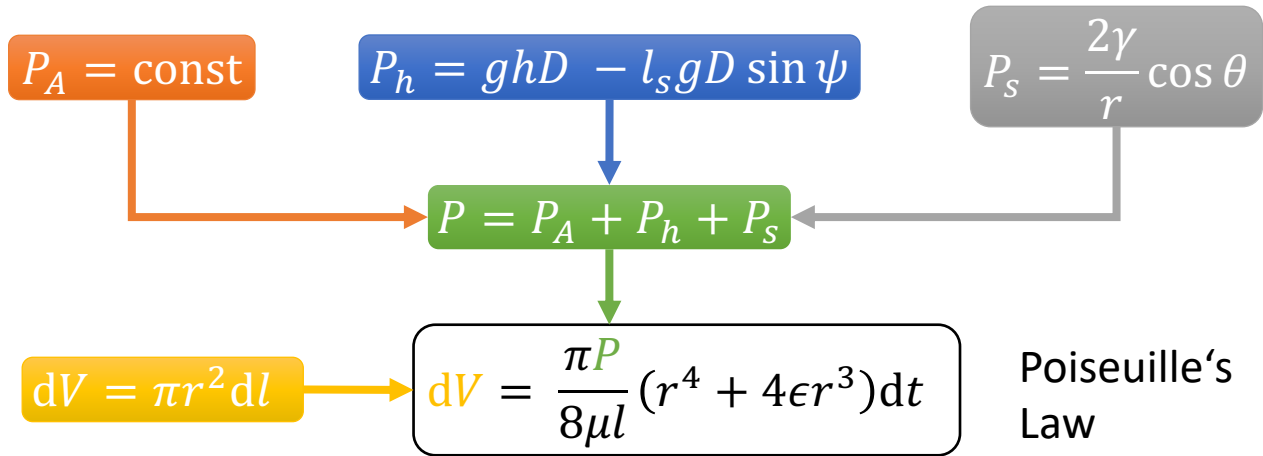
Physik (Master of Science)



# WHAT IS IMBIBITION?

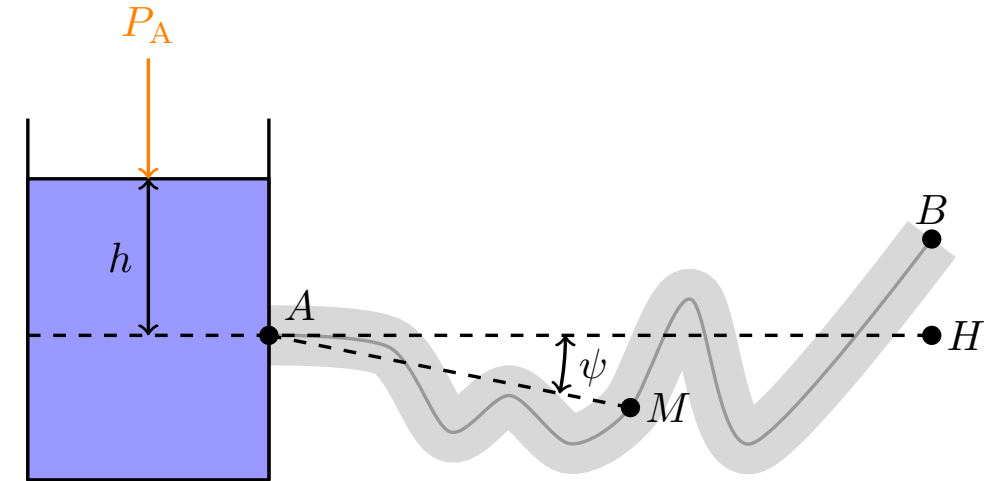


# IMBIBITION AND DYNAMICS OF CAPILLARY FLOW



$$\Rightarrow \frac{dl}{dt} = \frac{\left( P_A + gD(h - l_s \sin \psi) + \frac{2\gamma}{r} \cos \theta \right)}{8\mu l} (r^2 + 4\epsilon r)$$

$$\Rightarrow \frac{dl}{dt} = \frac{(r^2 + 4\epsilon r)}{8\mu} \frac{\left( P_A + gDh + \frac{2\gamma}{r} \cos \theta - gDl \sin \psi \right)}{l}$$

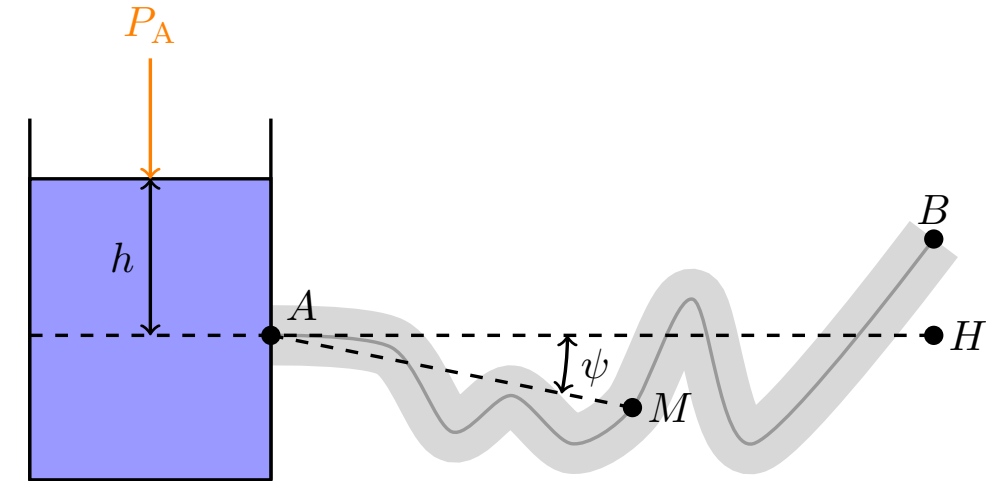


- $\theta = \text{const}$
- $\epsilon = \text{const}$
- $\psi = 0, \pi/2$
- $l_s = l$  for  $\psi = \pi/2$

# IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

$$\begin{aligned}\frac{dl}{dt} &= \frac{(r^2 + 4\epsilon r)}{8\mu} \frac{\left(P_A + gDh + \frac{2\gamma}{r} \cos \theta - gDl \sin \psi\right)}{l} \\ &= \frac{(r^2 + 4\epsilon r)}{8\mu} \frac{\left(P_A + gDh + \frac{2\gamma}{r} \cos \theta - gD \sin \psi l\right)}{l} \\ &= R \frac{(P_R - C_h l)}{l} = \frac{RC_h(1 - (C_h/P_R)l)}{(C_h/P_R)l}\end{aligned}$$

$$\int_0^l dl' \frac{dt}{dl'} \Rightarrow RC_h t = -l - \frac{1}{(C_h/P_R)} \ln(1 - (C_h/P_R)l)$$



- $\theta = \text{const}$
- $\epsilon = \text{const}$
- $\psi = 0, \pi/2$
- $l_s = l$  for  $\psi = \pi/2$
- $R = \frac{(r^2 + 4\epsilon r)}{8\mu}$
- $P_R = P_A + gDh + \frac{2\gamma}{r} \cos \theta$
- $C_h = gD \sin \psi$

# IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

$$RC_h t = -l - \frac{1}{(C_h/P_R)} \ln(1 - (C_h/P_R)l)$$

$$\ln(1 - ax) \approx -ax - \frac{a^2}{2}x^2 + \text{h.o.t}$$

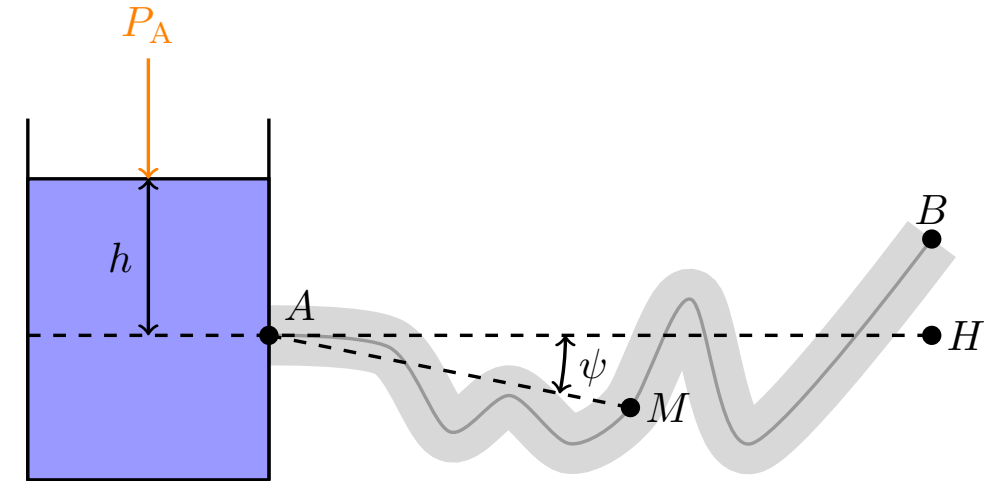
$$\Rightarrow RC_h t = -l + \frac{1}{(C_h/P_R)} \left( (C_h/P_R)l + \frac{(C_h/P_R)^2}{2} l^2 \right)$$

$$= \frac{(C_h/P_R)}{2} l^2$$

$$\Rightarrow l^2 = \left( \frac{\gamma \cos \theta}{\mu} \right) rt$$

„Lucas-Washburn Law“

Same result  
for  $\psi = 0$



- $\theta = \text{const}$
- $\epsilon = 0$
- $\psi = \pi/2$
- $l_s = l$  for  $\psi = \pi/2$
- $R = \frac{(r^2 + 4\epsilon r)}{8\mu} = \frac{r^2}{8\mu}$
- $P_R = P_A + gDh + \frac{2\gamma}{r} \cos \theta \approx \frac{2\gamma}{r} \cos \theta$
- $C_h = gD \sin \psi = gD$
- $C_h l \ll P_R$

# IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

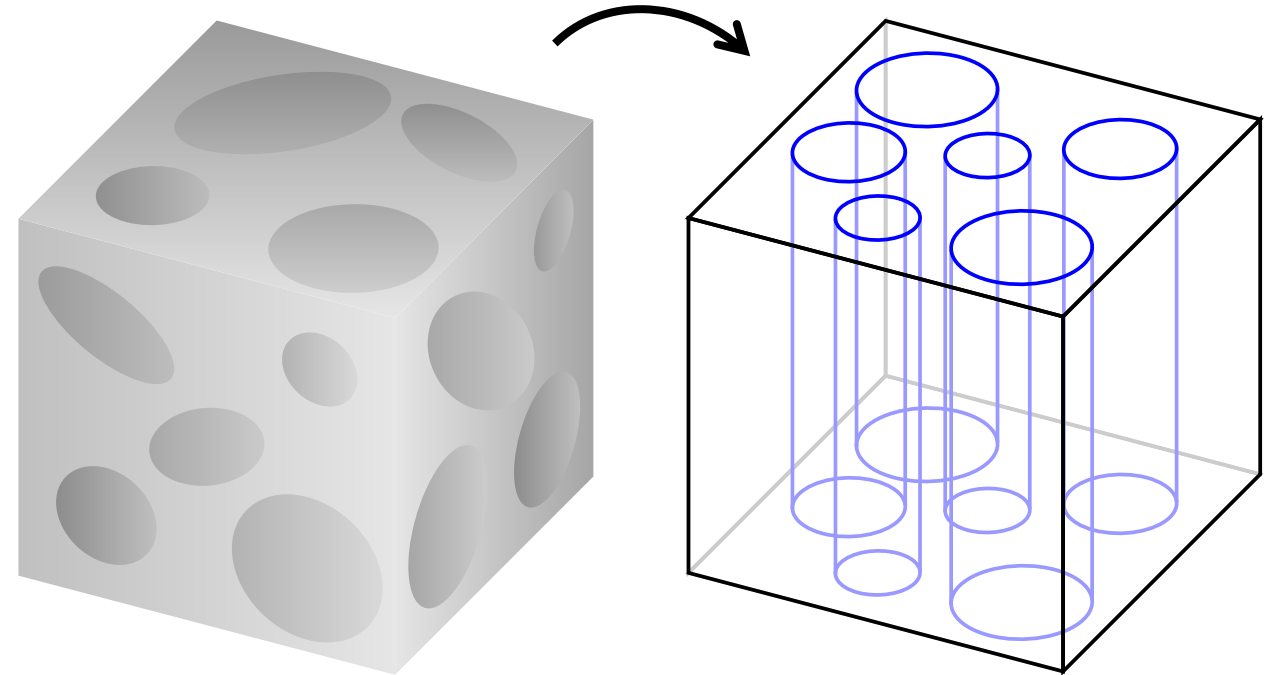
Penetrativity

$$\Rightarrow l^2 = \left( \frac{\gamma \cos \theta}{\mu} \right) r t$$

„Lucas-Washburn Law“

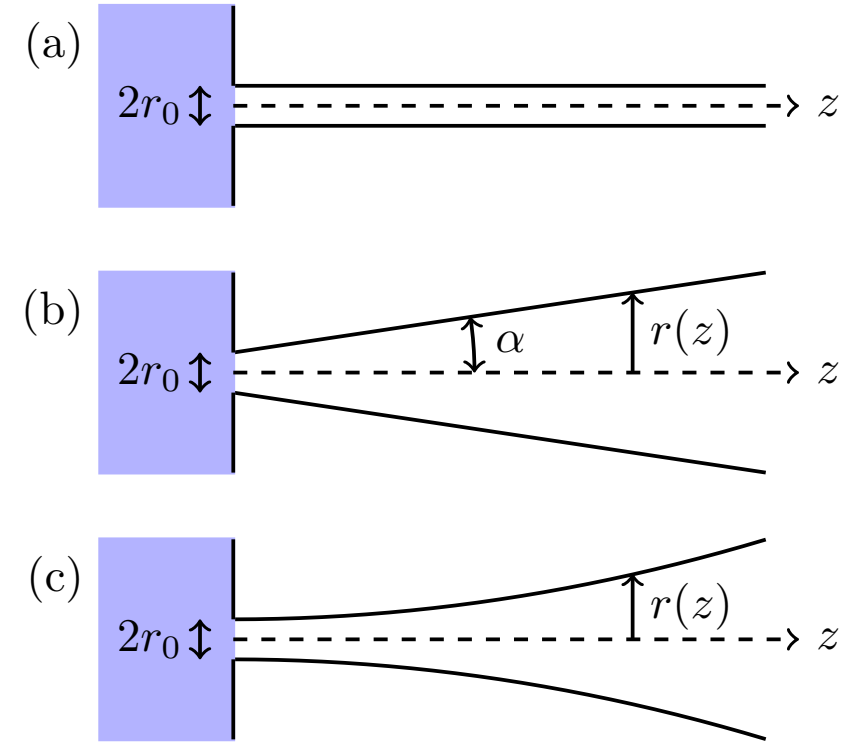
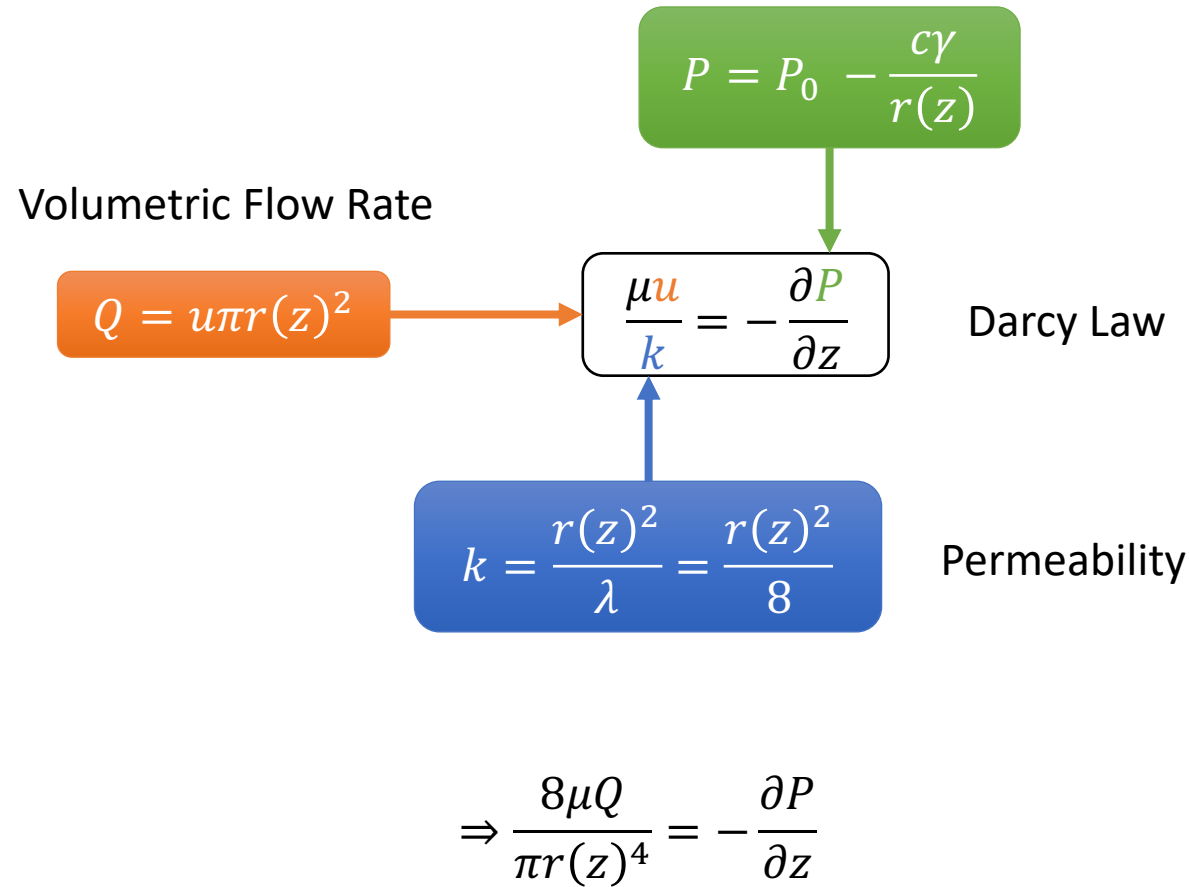
$$\Rightarrow \frac{dl}{dt} = \frac{r^2}{8\mu l} \frac{2\gamma}{r} \cos \theta$$

Rate of Penetration



$$\Rightarrow V = \sum_{i=1}^N \pi r_i^2 l = \frac{\pi}{2} \left( \frac{t}{\mu} \right)^{1/2} \sum_{i=1}^N \left( \frac{2\gamma}{r_i} \cos \theta \right)^{1/2} r_i^3 = S \left( \frac{\gamma}{\mu} \right)^{1/2} t^{1/2}$$

# IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D



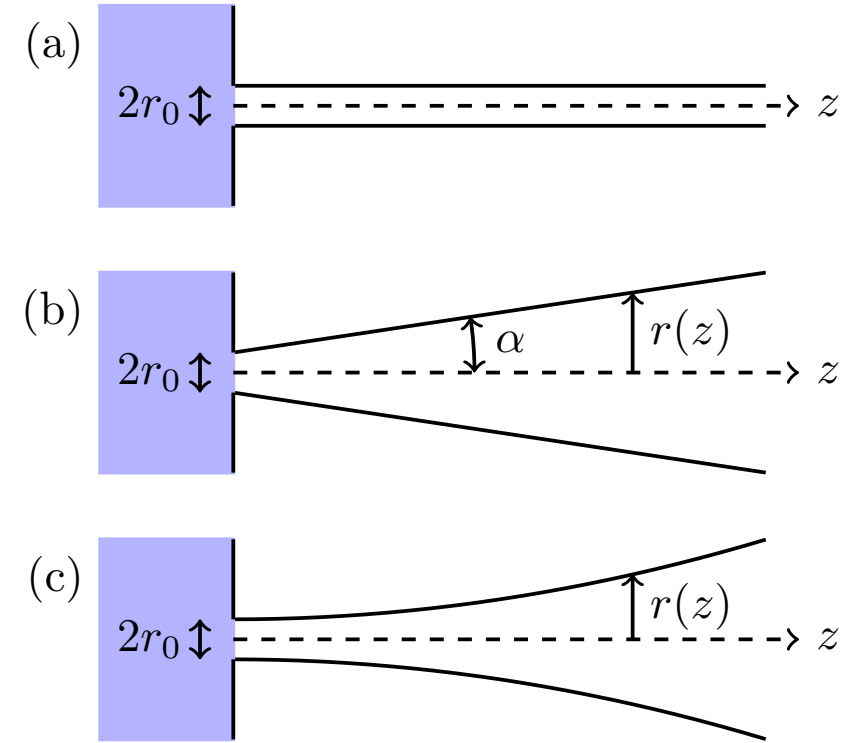
# IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

$$\frac{8\mu Q}{\pi r(z)^4} = -\frac{\partial P}{\partial z} \quad \Rightarrow \quad -\int_0^l dz \frac{\partial P}{\partial z} = P_0 - \left(P_0 - \frac{c\gamma}{r(l)}\right) = \frac{c\gamma}{r(l)}$$

$$\boxed{\frac{dl}{dt} = \frac{Q}{\pi r(l)^2}} \Rightarrow Q = \frac{c\pi\gamma}{8\mu} \left[ r(l) \int_0^l dz r(z)^{-4} \right]^{-1}$$

$$\boxed{\Rightarrow \frac{dl}{dt} = \frac{c\gamma}{8\mu} \left[ r(l)^3 \int_0^l dz r(z)^{-4} \right]^{-1}}$$

General Imbibition Equation for Invasion into structured Solid in 3D





# IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

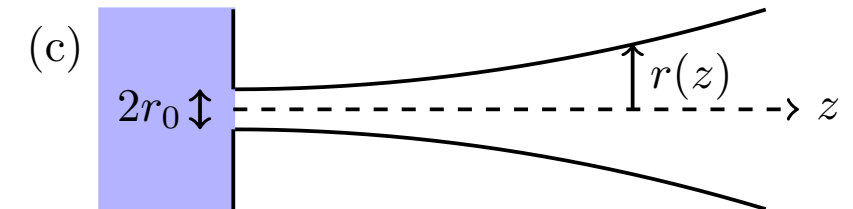
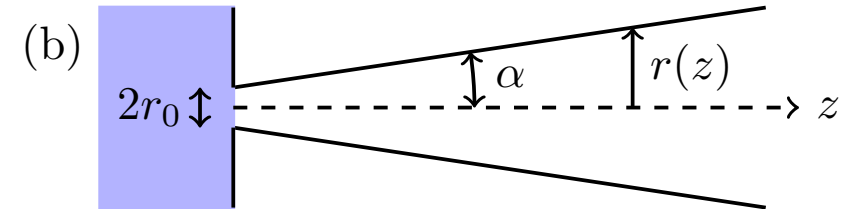
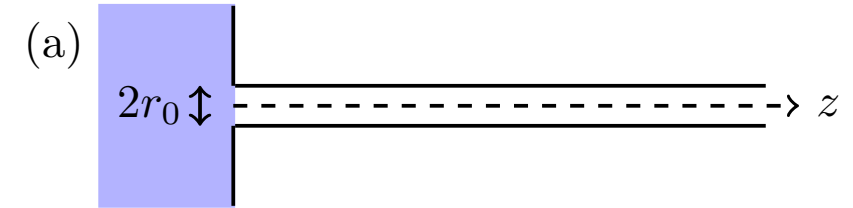
Power-Law shaped Structure

$$\frac{dl}{dt} = \frac{c\gamma}{8\mu} \left[ r(l)^3 \int_0^l dz r(z)^{-4} \right]^{-1}$$

$$r(z) = r_0 + \alpha z^n$$

$$\Rightarrow \frac{dl}{dt} = \frac{c\gamma r_0}{8\mu} \left[ \left( 1 + \frac{\alpha}{r_0} l^n \right)^3 \int_0^l dz \left( 1 + \frac{\alpha}{r_0} z^n \right)^{-4} \right]^{-1}$$

$$\Rightarrow \frac{dL}{dT} = \left[ (1 + L^n)^3 \int_0^L dZ (1 + Z^n)^{-4} \right]^{-1}$$



- $c(l) = 2 \cos \left( \theta + \arctan \left( \frac{dr}{dz}(l) \right) \right) \approx \text{const}$
- $L = (\alpha/r_0)^{1/n} l$  and  $T = \frac{c\gamma r_0}{8\mu} (\alpha/r_0)^{2/n} t$

# IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

$$\frac{dL}{dT} = \left[ (1 + L^n)^3 \int_0^L dZ (1 + Z^n)^{-4} \right]^{-1}$$

(I)  $n = 0$  ( $r(z) = r_0$ ):  $\Rightarrow L \frac{dL}{dT} = \text{const} \Rightarrow L \sim T^{1/2}$

(II)  $n \geq 1$ : „Lucas-Washburn Law“

(i) Short Times:

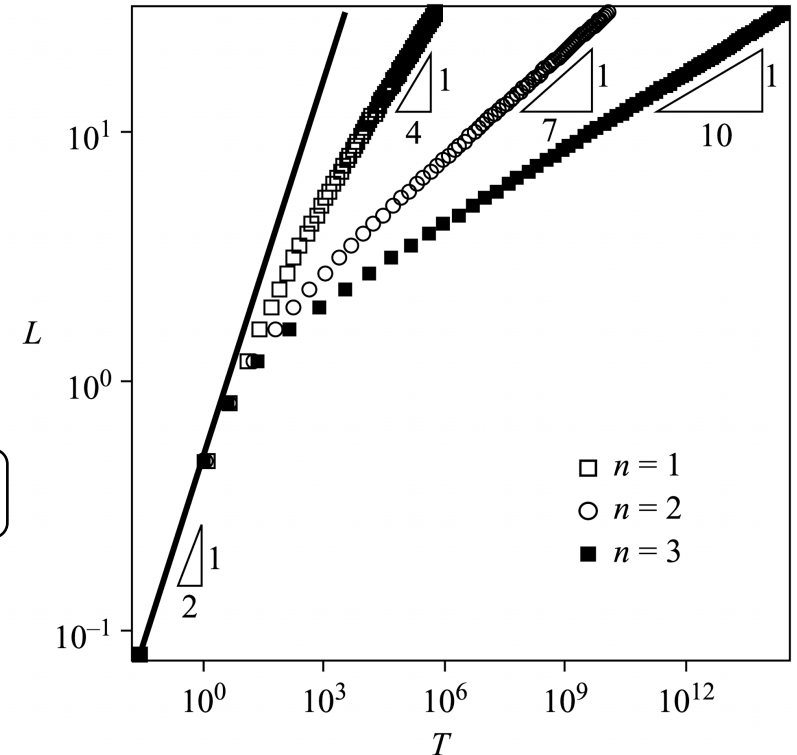
$$L \ll 1 \text{ and } \int_0^L dZ (1 + Z^n)^{-4} \simeq L \Rightarrow L \frac{dL}{dT} \simeq \text{const} \Rightarrow L \sim T^{1/2}$$

(ii) Long Times:

$$L \gg 1, (1 + L^n) \simeq L^n \text{ and}$$

$$\int_0^L dZ (1 + Z^n)^{-4} \approx \int_0^\infty dZ (1 + Z^n)^{-4} = \text{const for } n > 1/4$$

$$\Rightarrow L^{3n} \frac{dL}{dT} \simeq \text{const} \Rightarrow L \sim T^{1/(3n+1)}$$



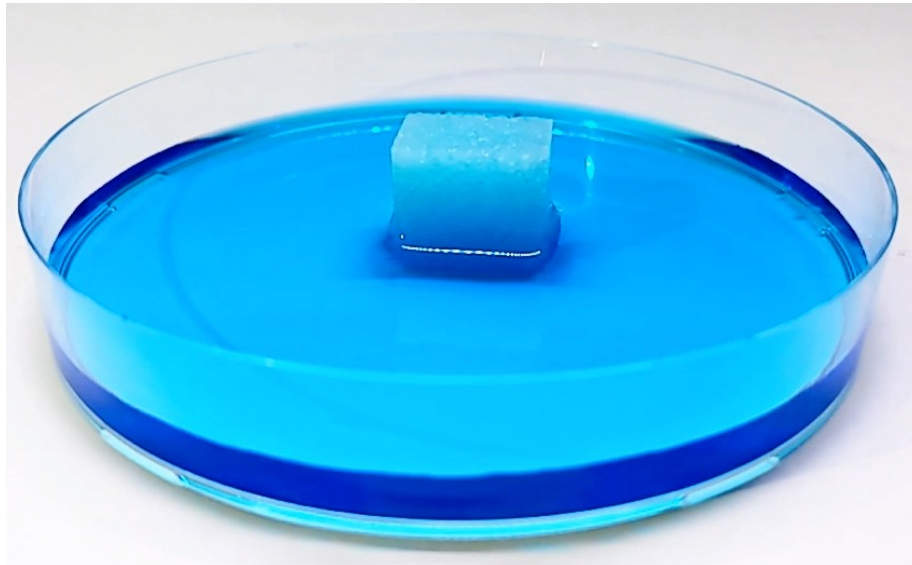
- $c(l) = \text{const}$

- $L = (\alpha/r_0)^{1/n} l$  and  $T = \frac{c\gamma r_0}{8\mu} (\alpha/r_0)^{2/n} t$

# IMBIBITION WITH SUGAR IN WATER AND GLYCERIN

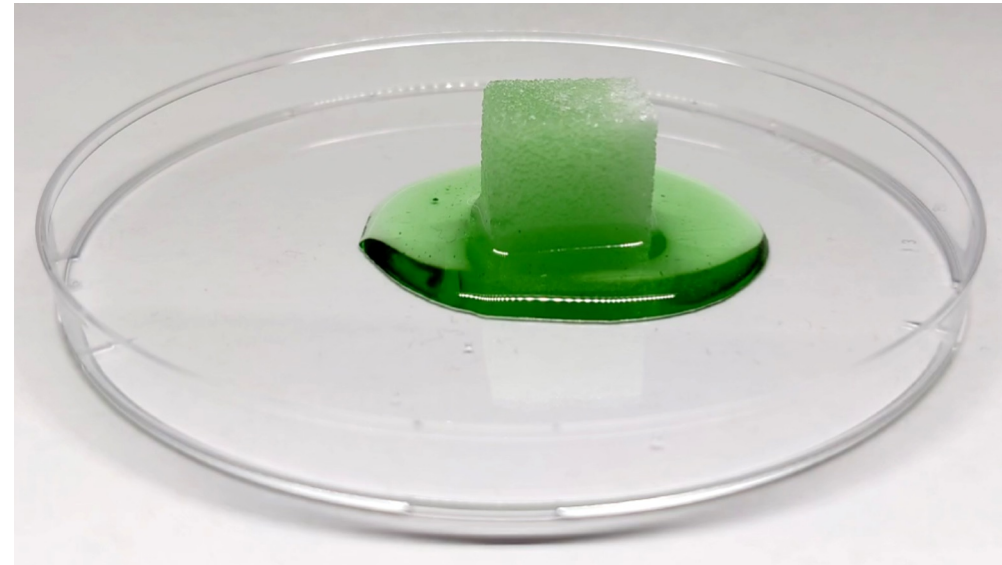
$$\Rightarrow t \sim \mu$$

Water



Slowed down with 1s = 0.125s

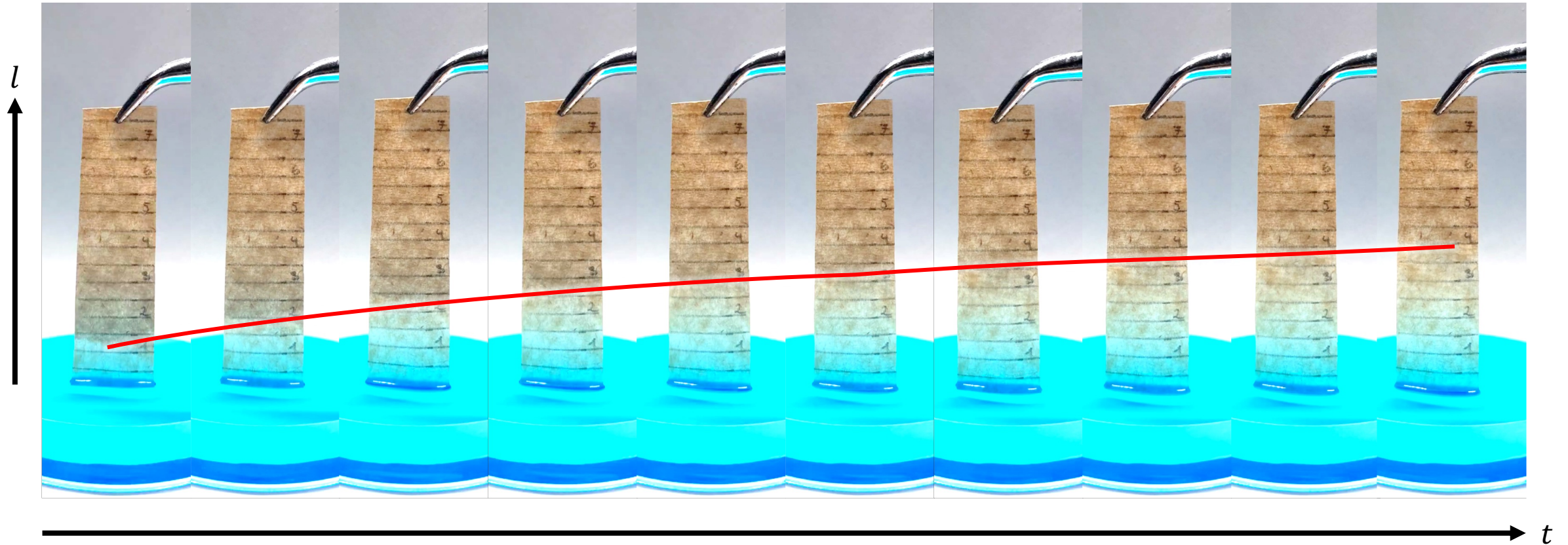
Glycerin



Sped up with 1s = 30s

# IMBIBITION WITH FILTER PAPER IN WATER

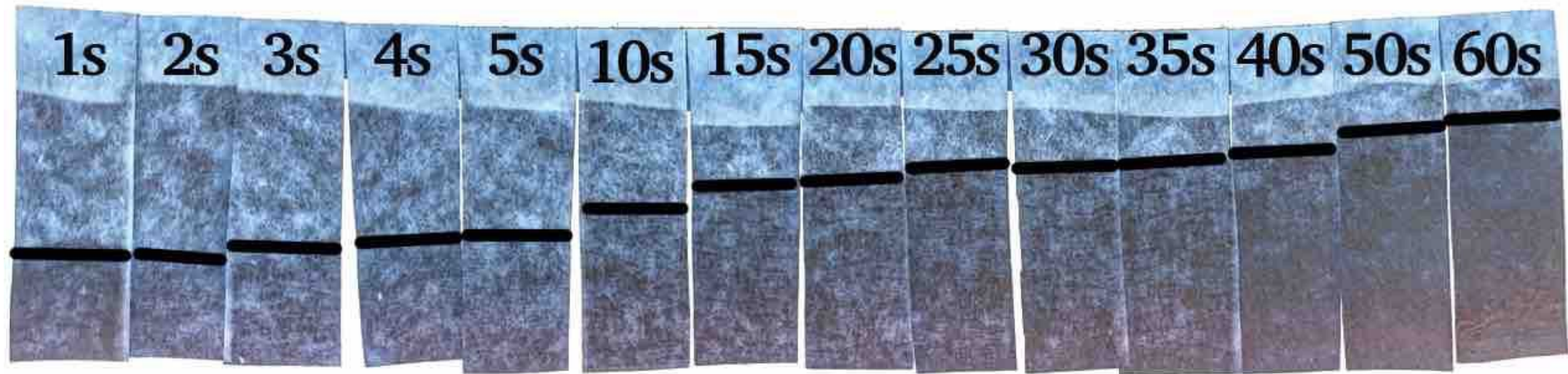
Sped up with 1s = 10s



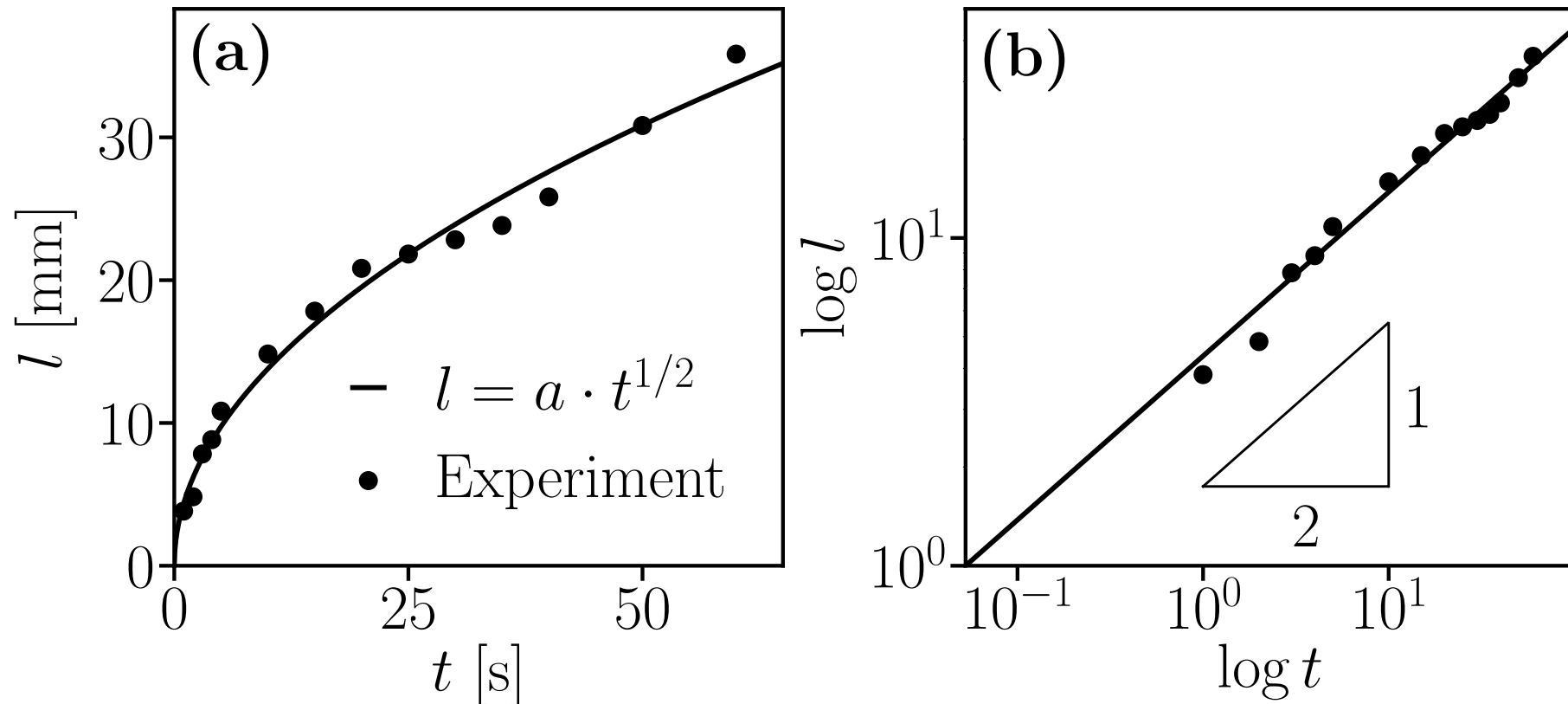
$$\Rightarrow l \sim t^{1/2}$$



# IMBIBITION WITH FILTER PAPER IN WATER



# IMBIBITION WITH FILTER PAPER IN WATER



$$\Rightarrow l \sim t^{1/2}$$

# IMBIBITION: TAKE HOME MESSAGES

- Imbibition is an important process in nature and industry
- Capillarity-driven imbibition can be described by „Lucas-Washburn Law“

$$l^2 = \left( \frac{\gamma \cos \theta}{\mu} \right) r t$$

- Varying the geometry of the capillary (3D) results in two time regimes:

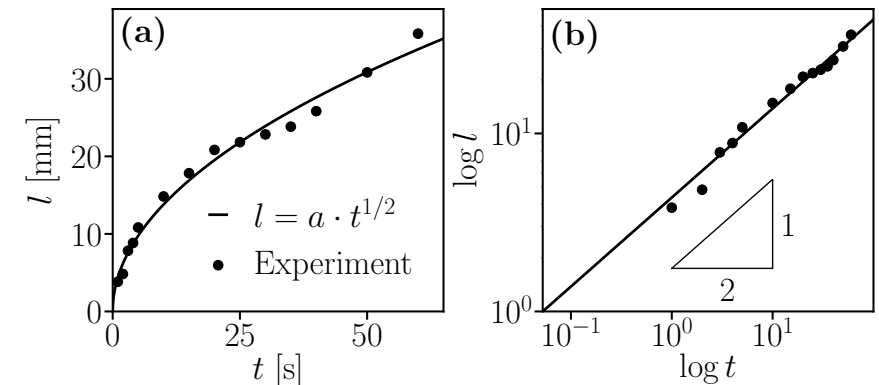
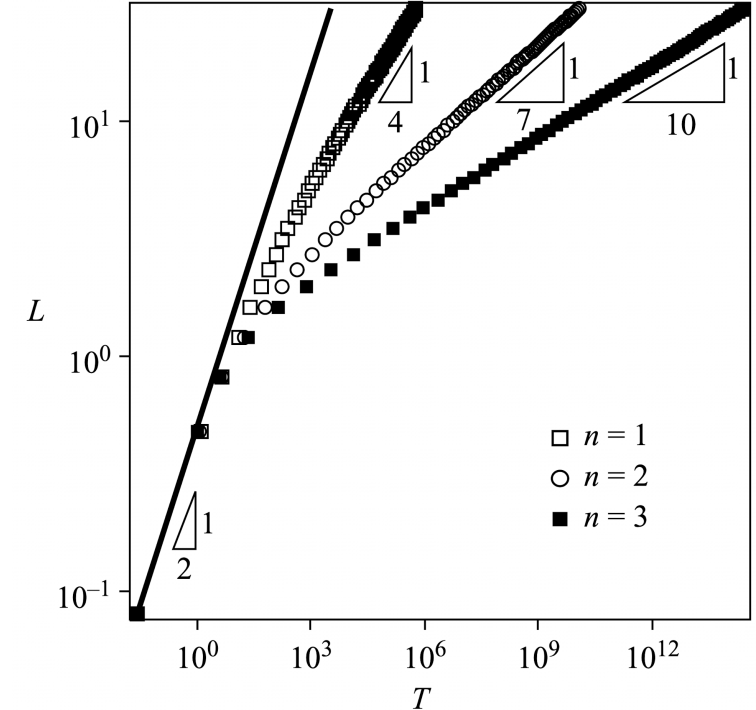
$$L \sim T^{1/2}$$

Short times

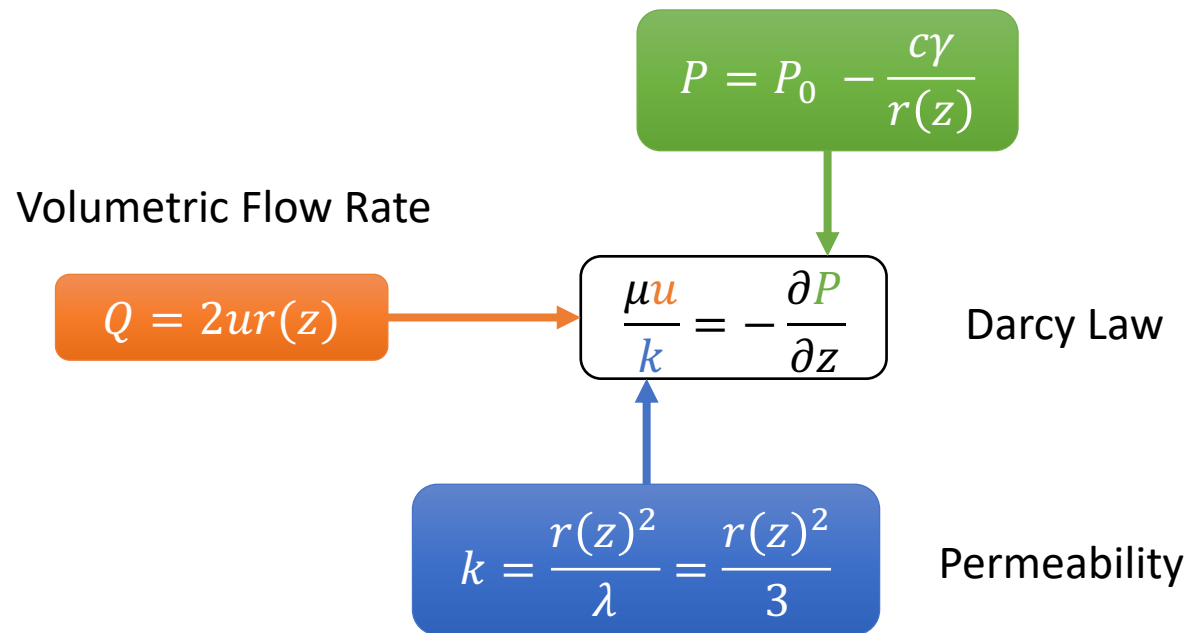
$$L \sim T^{1/(3n+1)}$$

Long times

- The more diverging the geometry, the slower the dynamics become at long time
- Simple experiment with coffee filter paper proves the „Lucas-Washburn Law“



# IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 2D



$$\Rightarrow \frac{3\mu Q}{\pi r(z)^3} = -\frac{\partial P}{\partial z}$$

$$\Rightarrow \frac{dl}{dt} = \frac{c\gamma}{3\mu} \left[ r(l)^2 \int_0^l dz r(z)^{-3} \right]^{-1}$$

$$\Rightarrow \frac{dL}{dT} = \left[ (1 + L^n)^2 \int_0^L dZ (1 + Z^n)^{-3} \right]^{-1}$$

$$L \sim T^{1/2}$$

Short times

$$L \sim T^{1/(2n+1)}$$

Long times

- $c(l) = \text{const}$

- $L = (\alpha/r_0)^{1/n} l$  and  $T = \frac{c\gamma r_0}{8\mu} (\alpha/r_0)^{2/n} t$



## WHY $c(\ell)$ IS CONSTANT?

$$c(\ell) \propto \cos \left( \theta_e + \arctan \left( \frac{dh}{dz}(\ell) \right) \right), \quad (\text{A } 1)$$

Let us consider the profile  $h(z) = h_0 + \alpha z^n$ , with  $n > 1$ . The lubrication theory requirement ( $dh/dz \ll 1$ ) leads to the simplification  $\arctan(dh/dz(\ell)) \simeq dh/dz(\ell) = n\alpha\ell^{n-1}$ . If we now consider a completely wetting case with  $\theta_e \ll 1$ ,

$$c \propto 1 - \frac{n^2}{2} \alpha^2 \ell^{2n-2} + \dots \quad (\text{A } 2)$$

The approximation  $c \simeq \text{constant}$  can then hold if  $n^2 \alpha^2 \ell^{2n-2} \ll 1$ , i.e.

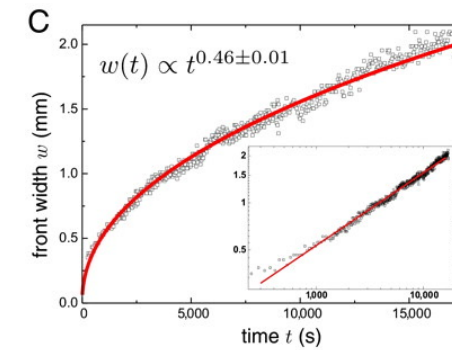
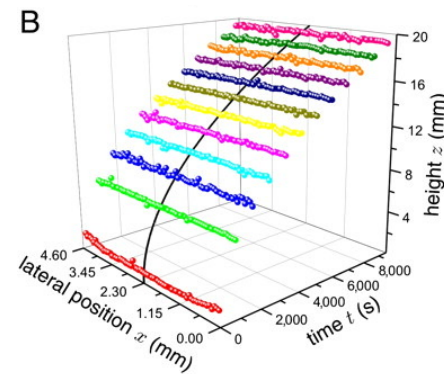
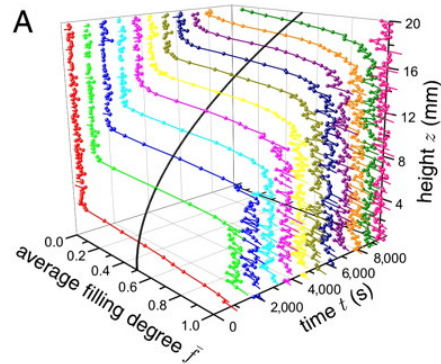
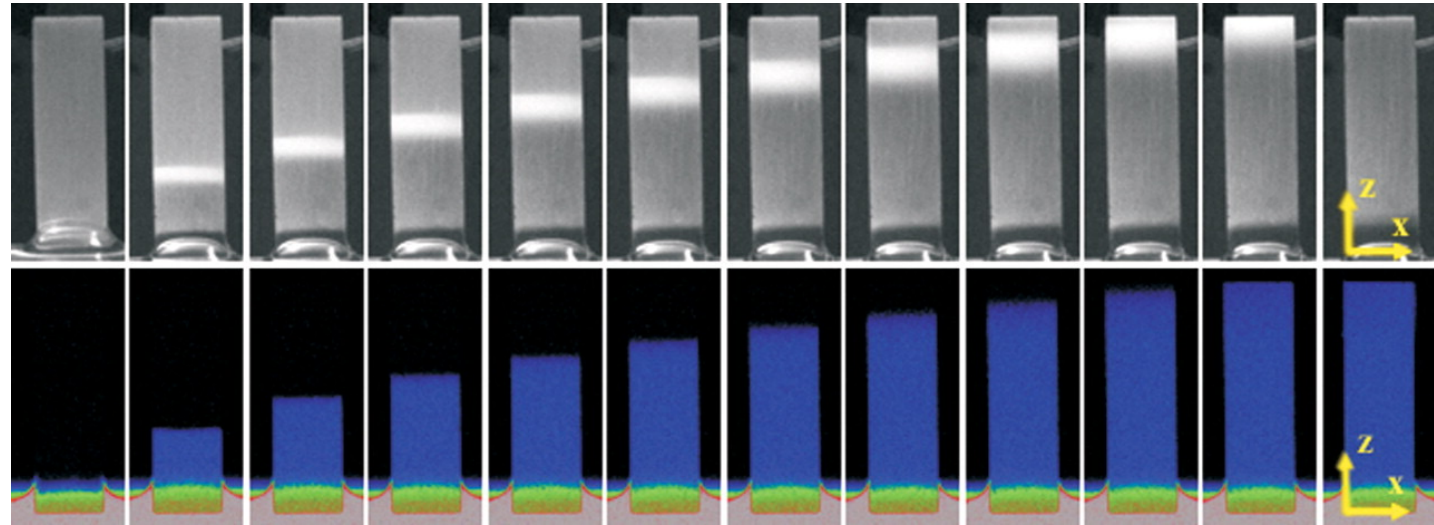
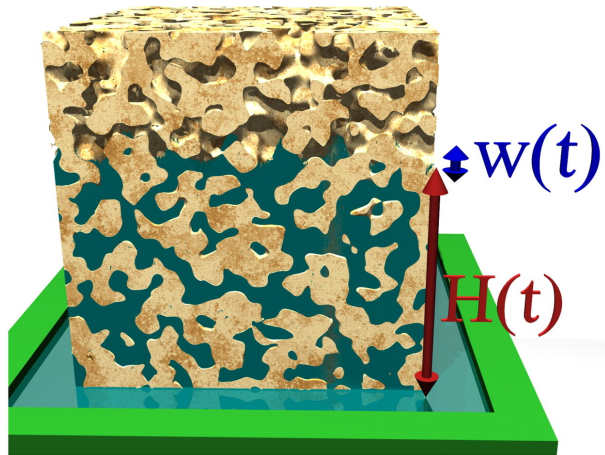
$$\ell \ll \left( \frac{1}{\alpha n} \right)^{1/(n-1)}. \quad (\text{A } 3)$$

On the other hand, we know that the cross-over time between the two asymptotic limits occurs on a length scale  $L \simeq 1$  or  $\ell \simeq (h_0/\alpha)^{1/n}$ . Thus, the approximation  $c \simeq \text{constant}$  remains valid if

$$\left( \frac{h_0}{\alpha} \right)^{1/n} \ll \left( \frac{1}{\alpha n} \right)^{1/(n-1)} \quad \text{or} \quad h_0 \ll (\alpha n^n)^{1/(1-n)}. \quad (\text{A } 4)$$

We assume the cases considered here to be under this condition.

# IMBIBITION EXPERIMENTS WITH VYCOR GLASS



# OUTTAKES

