UNIVERSITÄT
BAYREUTH

Feuerzangenbowle

Physics of Imbibition and Porous Media Flow

Manuel Lippert

Physik (Master of Science)

WHAT IS IMBIBITION?

IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

$P_A = \text{const}$

$P_h = ghD \; - l_s gD \sin\psi$

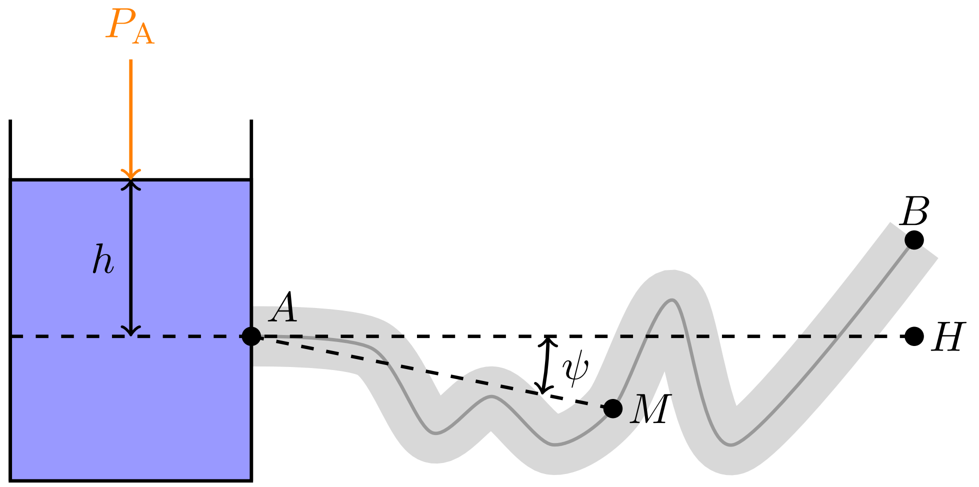
$P_s = \dfrac{2\gamma}{r}\cos\theta$

$P = P_A + P_h + P_s$

$\mathrm{d}V = \pi r^2 \mathrm{d}l$

$$\mathrm{d}V = \frac{\pi P}{8\mu l}(r^4 + 4\epsilon r^3)\mathrm{d}t$$

Poiseuille‘s Law



- $\psi = \theta = \text{const}$
- $l_s = l$ for $\psi = 0, \pi/2$
- $\epsilon = \;\text{const}$

$$\Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\left(P_A + gD(h - l_s \sin\psi) + \frac{2\gamma}{r}\cos\theta\right)}{8\mu l}(r^2 + 4\epsilon r)$$

$$\boxed{\psi = 0, \pi/2} \quad \Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{(r^2 + 4\epsilon r)}{8\mu}\frac{\left(P_A + gDh + \frac{2\gamma}{r}\cos\theta \; - gDl\sin\psi\right)}{l}$$

E. W. Washburn (1921) “The dynamics of capillary flow“

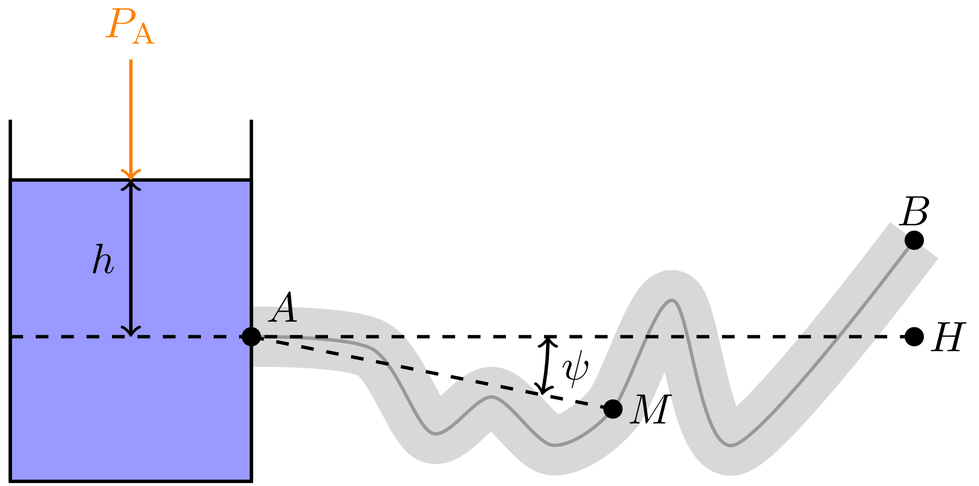
IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{(r^2+4\epsilon r)}{8\mu}\frac{\left(P_A + gDh + \frac{2\gamma}{r}\cos\theta \ - gDl\sin\psi\right)}{l}$$

$$= \frac{(r^2+4\epsilon r)}{8\mu}\frac{\left(P_A + gDh + \frac{2\gamma}{r}\cos\theta \ - gD\sin\psi \ l\right)}{l}$$

$$= R\frac{(P_R - C_h l)}{l} = \frac{RC_h(1-(C_h/P_R)l)}{(C_h/P_R)l}$$

$$\int_0^l \mathrm{d}l' \frac{\mathrm{d}t}{\mathrm{d}l'} \ \Rightarrow RC_h t = -l - \frac{1}{(C_h/P_R)}\ln(1-(C_h/P_R)l)$$



- $\psi = \theta = \text{const}$
- $l_s = l$ for $\psi = 0, \pi/2$
- $\epsilon = \text{const}$

- $R = \frac{(r^2+4\epsilon r)}{8\mu}$
- $P_R = P_A + gDh + \frac{2\gamma}{r}\cos\theta$
- $C_h = gD\sin\psi$

IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

$$RC_h t = -l - \frac{1}{(C_h/P_R)} \ln(1 - (C_h/P_R)l)$$

$$\ln(1 - ax) \approx -ax - \frac{a^2}{2}x^2 + \text{h.o.t}$$

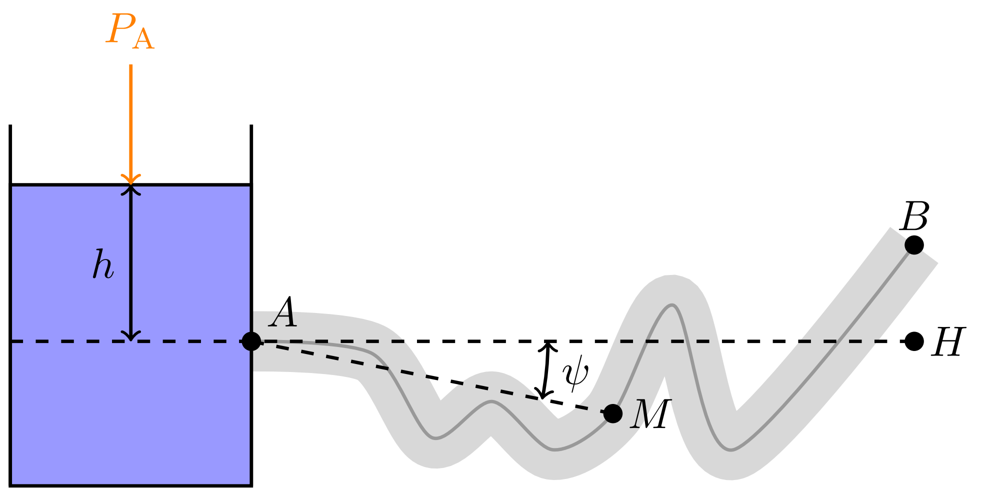
$$\Rightarrow RC_h t = -l + \frac{1}{(C_h/P_R)}\left((C_h/P_R)l + \frac{(C_h/P_R)^2}{2} l^2\right)$$

$$= \frac{(C_h/P_R)}{2} l^2$$

$$\boxed{\Rightarrow l^2 = \left(\frac{\gamma}{\mu}\frac{\cos\theta}{2}\right) rt}$$

„Lucas-Washburn Law“

Same result for $\psi = 0$



- $\psi = \theta = \text{const}$
- $l_s = l$ for $\psi = 0, \pi/2$
- $\epsilon = 0$
- $R = \frac{(r^2 + 4\epsilon r)}{8\mu} = \frac{r^2}{8\mu}$
- $P_R = P_A + gDh + \frac{2\gamma}{r}\cos\theta \approx \frac{2\gamma}{r}\cos\theta$
- $C_h = gD\sin\psi = gD$ for $\psi = \pi/2$

E. W. Washburn (1921) “The dynamics of capillary flow“

IMBIBITION AND DYNAMICS OF CAPILLARY FLOW

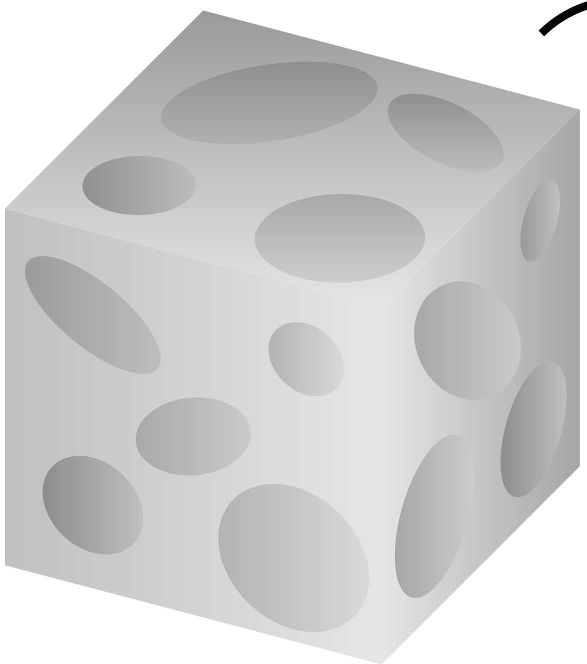
Penetrativity

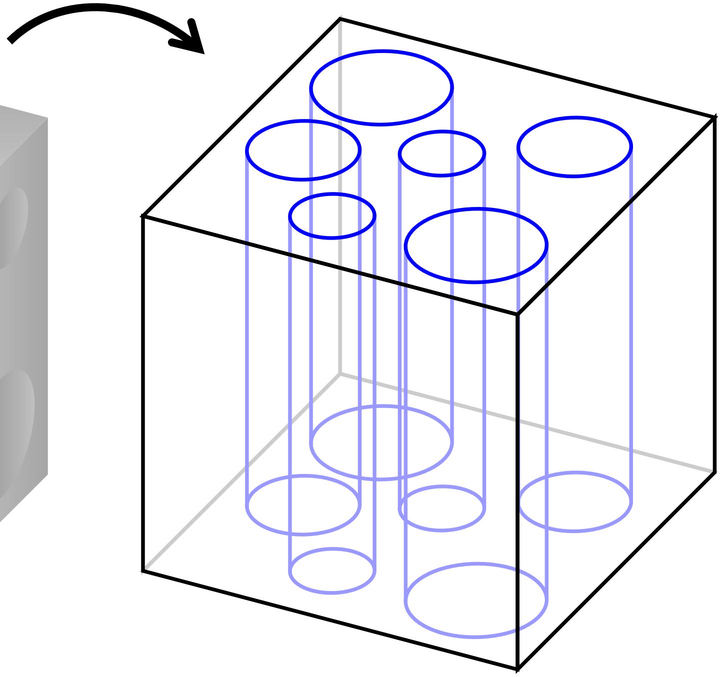
$$\Rightarrow l^2 = \left(\frac{\gamma}{\mu}\frac{\cos\theta}{2}\right) r t$$

„Lucas-Washburn Law“

$$\Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{r^2}{8\mu l}\frac{2\gamma}{r}\cos\theta$$

Rate of Penetration





$$\Rightarrow V = \sum_{i=1}^{N} \pi r_i^2 l = \frac{\pi}{2}\left(\frac{t}{\mu}\right)^{1/2} \sum_{i=1}^{N}\left(\frac{2\gamma}{r_i}\cos\theta\right)^{1/2} r_i^3 = S\left(\frac{\gamma}{\mu}\right)^{1/2} t^{1/2}$$

IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

$$P = P_0 - \frac{c\gamma}{r(z)}$$

Volumetric Flow Rate

$$Q = u\pi r(z)^2$$

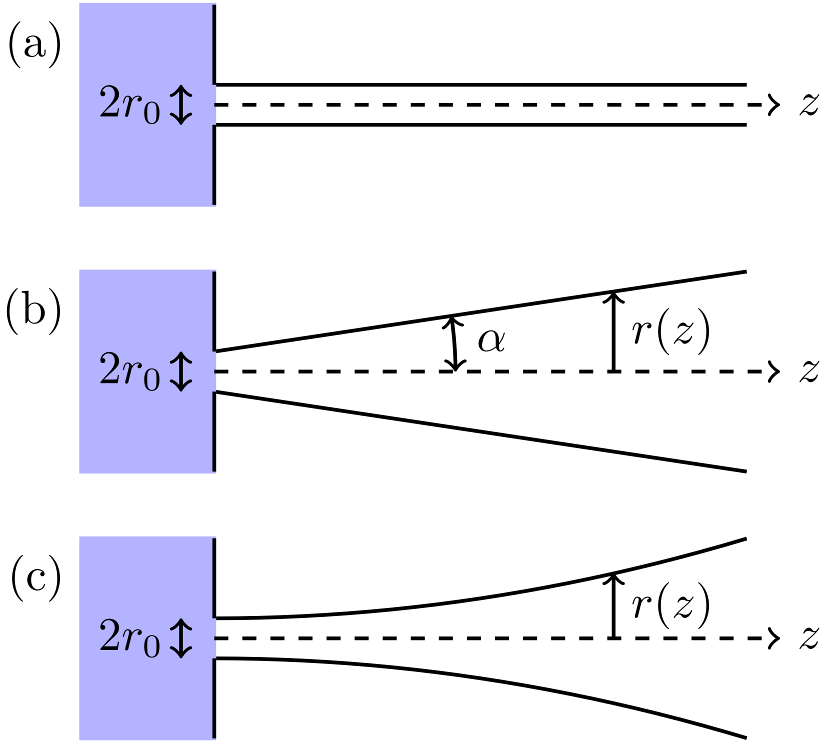
$$\frac{\mu u}{k} = -\frac{\partial P}{\partial z}$$

Darcy Law

$$k = \frac{r(z)^2}{\lambda} = \frac{r(z)^2}{8}$$

Permeability

$$\Rightarrow \frac{8\mu Q}{\pi r(z)^4} = -\frac{\partial P}{\partial z}$$



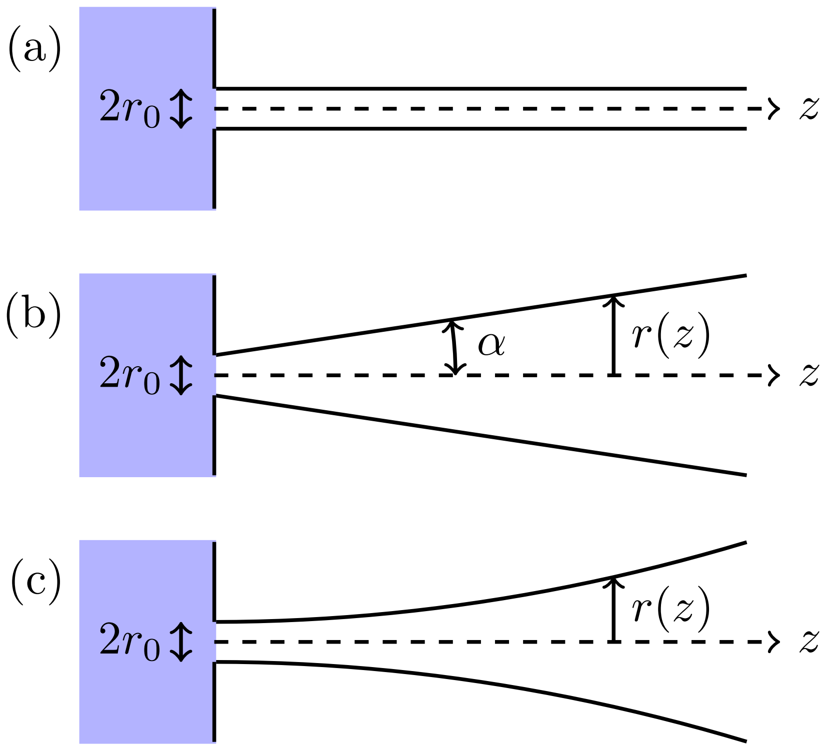
IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

$$\frac{8\mu Q}{\pi r(z)^4} = -\frac{\partial P}{\partial z} \qquad \Rightarrow -\int_0^l \mathrm{d}z \frac{\partial P}{\partial z} = P_0 - \left(P_0 - \frac{c\gamma}{r(l)}\right) = \frac{c\gamma}{r(l)}$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{Q}{\pi r(l)^2} \longrightarrow \Rightarrow Q = \frac{c\pi\gamma}{8\mu}\left[r(l)\int_0^l \mathrm{d}z\, r(z)^{-4}\right]^{-1}$$

$$\boxed{\Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{c\gamma}{8\mu}\left[r(l)^3\int_0^l \mathrm{d}z\, r(z)^{-4}\right]^{-1}}$$

General Imbibition Equation for Invasion into structured Solid in 3D



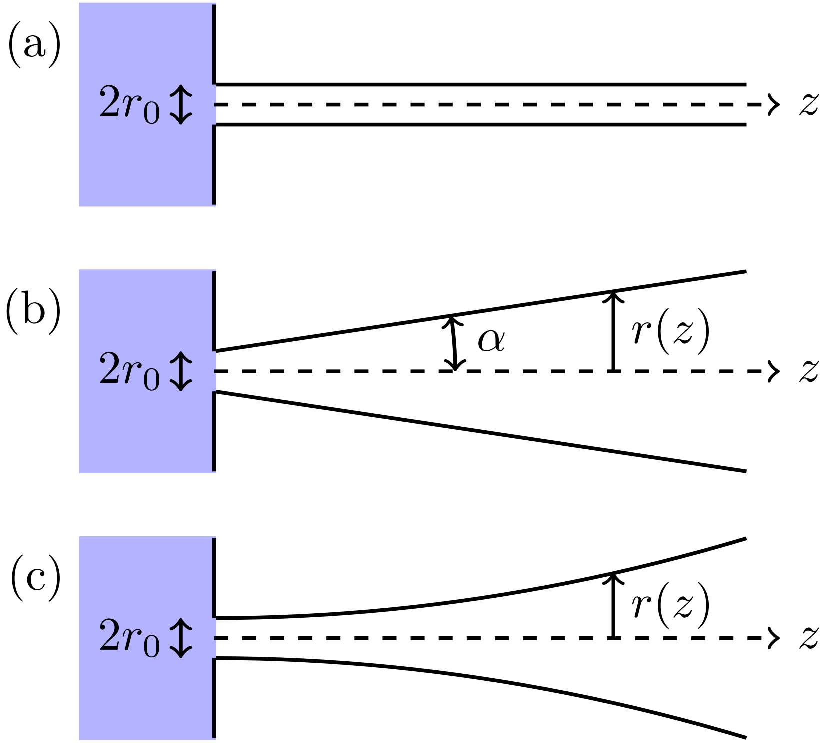
IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

Power-Law shaped Structure

$$\boxed{\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{c\gamma}{8\mu}\left[r(l)^3\int_0^l \mathrm{d}z\, r(z)^{-4}\right]^{-1}} \longleftarrow r(z) = r_0 + \alpha z^n$$

$$\Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{c\gamma r_0}{8\mu}\left[\left(1+\frac{\alpha}{r_o}l^n\right)^3\int_0^l \mathrm{d}z\left(1+\frac{\alpha}{r_o}z^n\right)^{-4}\right]^{-1}$$

$$\boxed{\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}T} = \left[(1+L^n)^3\int_0^L \mathrm{d}Z\,(1+Z^n)^{-4}\right]^{-1}}$$



- $c(l) = 2\cos\left(\theta + \arctan\left(\frac{\mathrm{d}r}{\mathrm{d}z}(l)\right)\right) \approx \text{const}$
- $L = (\alpha/r_0)^{1/n}l$ and $T = \frac{c\gamma r_0}{8\mu}(\alpha/r_0)^{2/n}t$

IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 3D

$$\boxed{\frac{\mathrm{d}L}{\mathrm{d}T} = \left[(1+L^n)^3 \int_0^L \mathrm{d}Z\,(1+Z^n)^{-4}\right]^{-1}}$$

(I) $\boldsymbol{n = 0\ (r(z) = r_0)}$: $\Rightarrow L \sim T^{1/2}$ „Lucas-Washburn Law“

(II) $\boldsymbol{n \geq 1}$:

(i) Short Times:

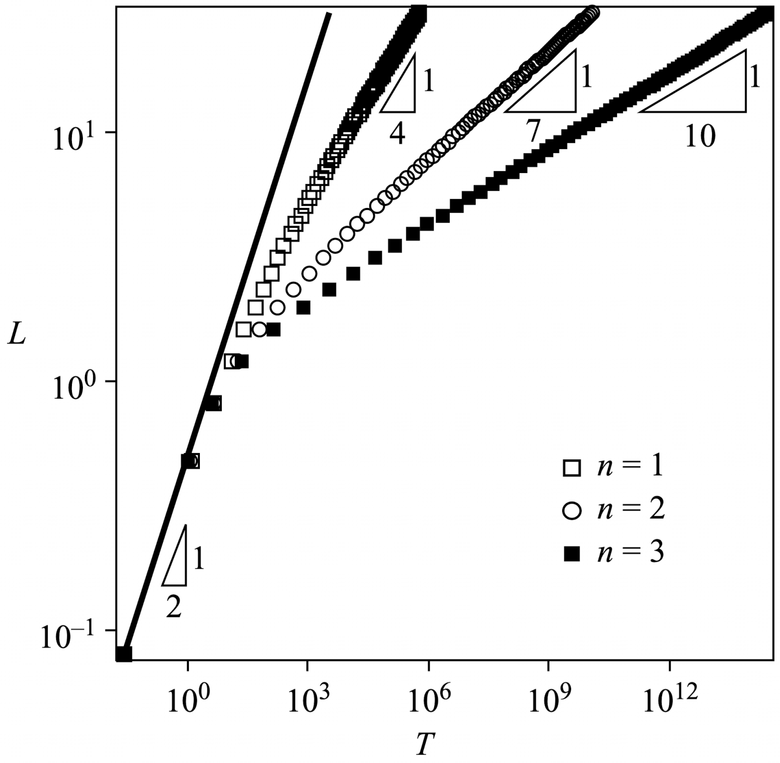
$$L \ll 1 \text{ and } \int_0^L \mathrm{d}Z\,(1+Z^n)^{-4} \simeq L \;\Rightarrow L\frac{\mathrm{d}L}{\mathrm{d}T} \simeq \text{const} \qquad \boxed{\Rightarrow L \sim T^{1/2}}$$

(ii) Long Times:

$$L \gg 1, (1+L^n) \simeq L^n \text{ and}$$

$$\int_0^L \mathrm{d}Z\,(1+Z^n)^{-4} \approx \int_0^\infty \mathrm{d}Z\,(1+Z^n)^{-4} = \text{const for } n > 1/4$$

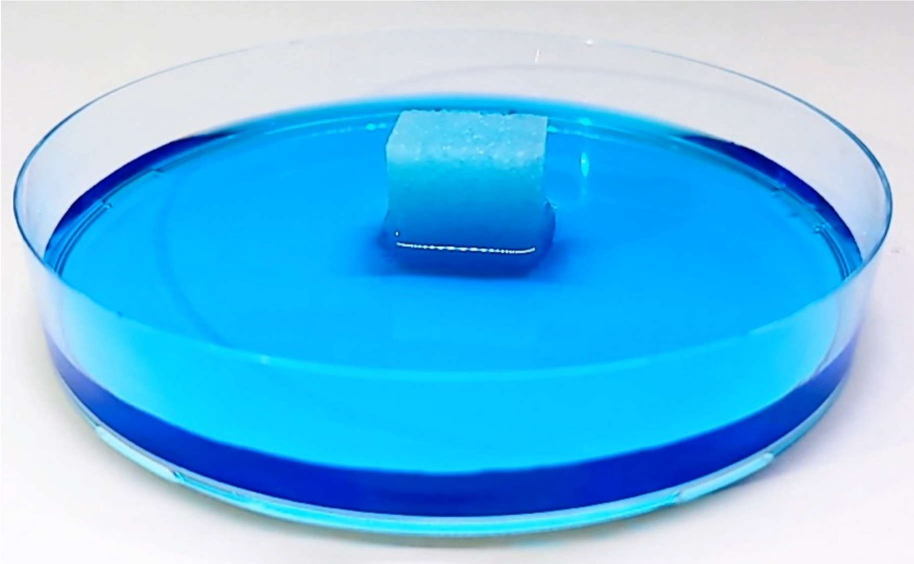
$$\Rightarrow L^{3n}\frac{\mathrm{d}L}{\mathrm{d}T} \simeq \text{const} \qquad \boxed{\Rightarrow L \sim T^{1/(3n+1)}}$$



- $c(l) = \text{const}$
- $L = (\alpha/r_0)^{1/n} l$ and $T = \frac{c\gamma r_0}{8\mu}(\alpha/r_0)^{2/n} t$

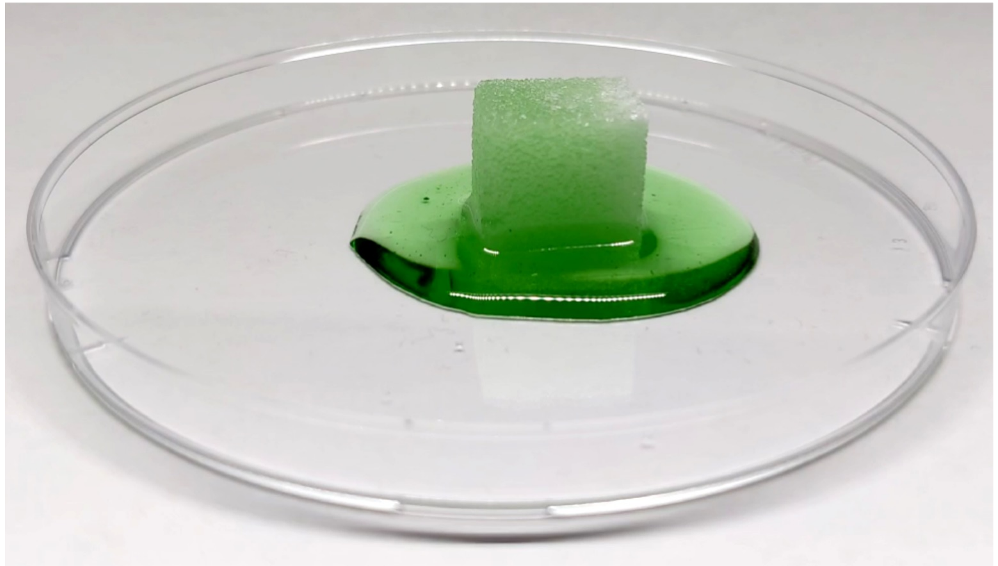
IMBIBITION WITH SUGAR IN WATER AND GLYCERIN

Water



Slowed down with 1s = 0.125s

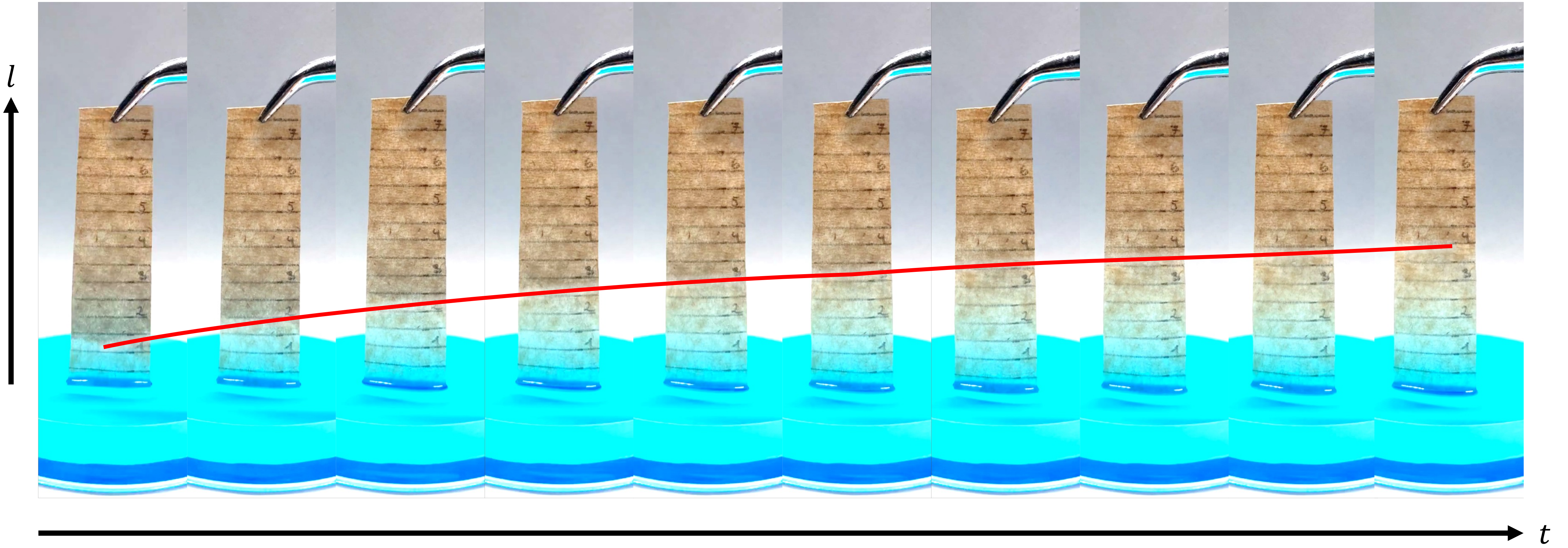
Glycerin



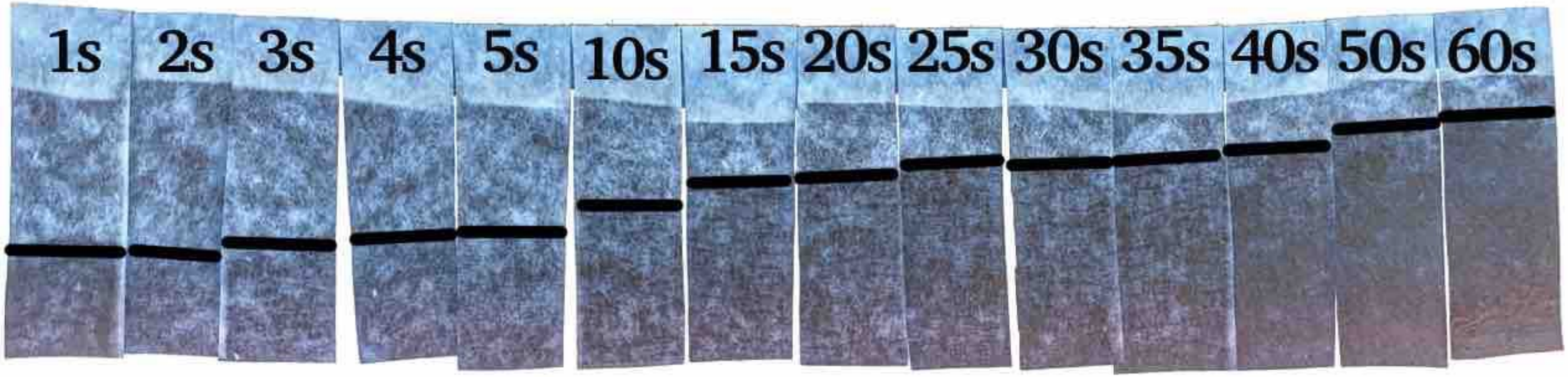
Sped up with 1s = 30s

IMBIBITION WITH FILTER PAPER IN WATER

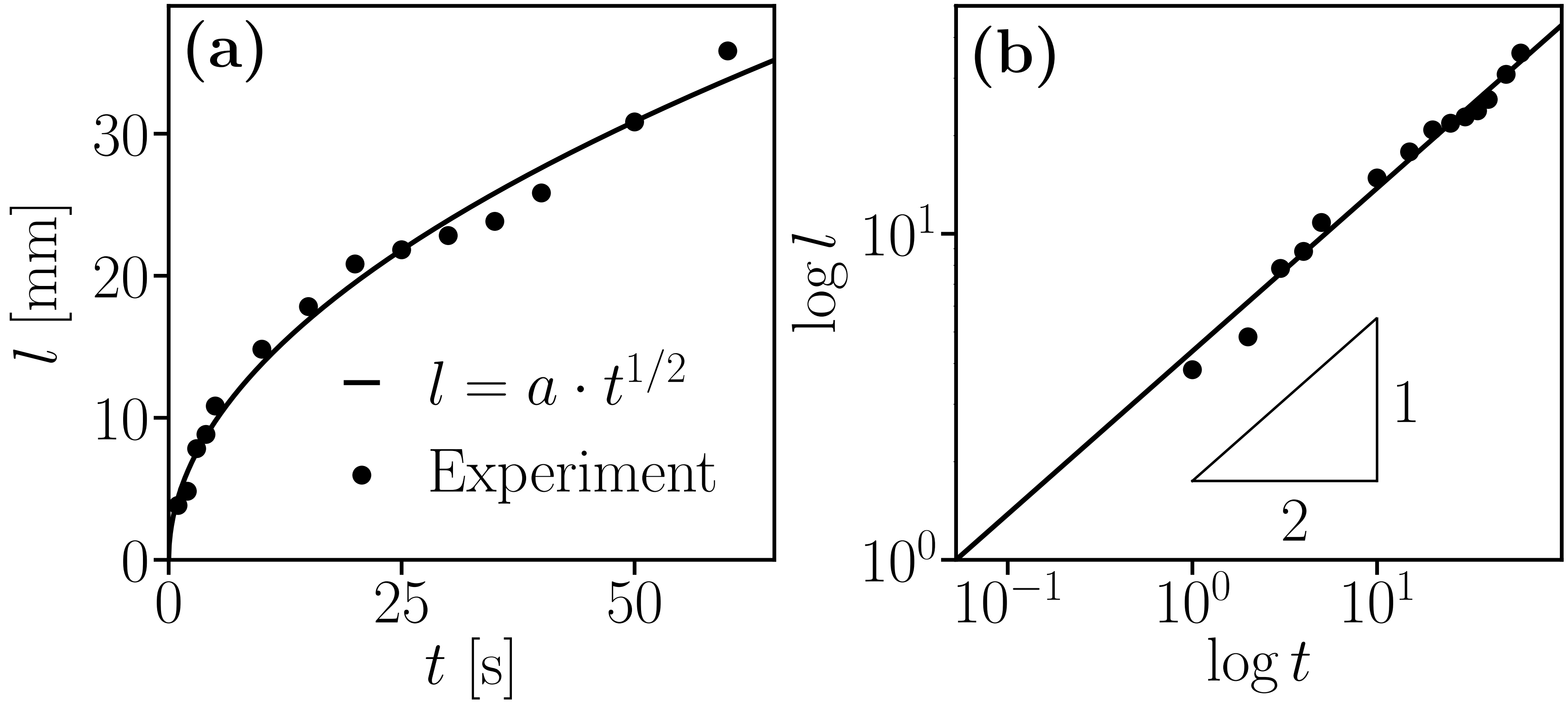
Sped up with 1s = 10s



IMBIBITION WITH FILTER PAPER IN WATER



IMBIBITION WITH FILTER PAPER IN WATER



IMBIBITION: WHAT WE LEARNED TODAY?

- Imbibition is an important process in nature and industry
- Capillarity-driven imbibition can be described by „Lucas-Washburn Law“

$$\boxed{l^2 = \left(\frac{\gamma}{\mu}\frac{\cos\theta}{2}\right) rt}$$

- Varying the geometry of the capillary (3D) results in two time regimes:

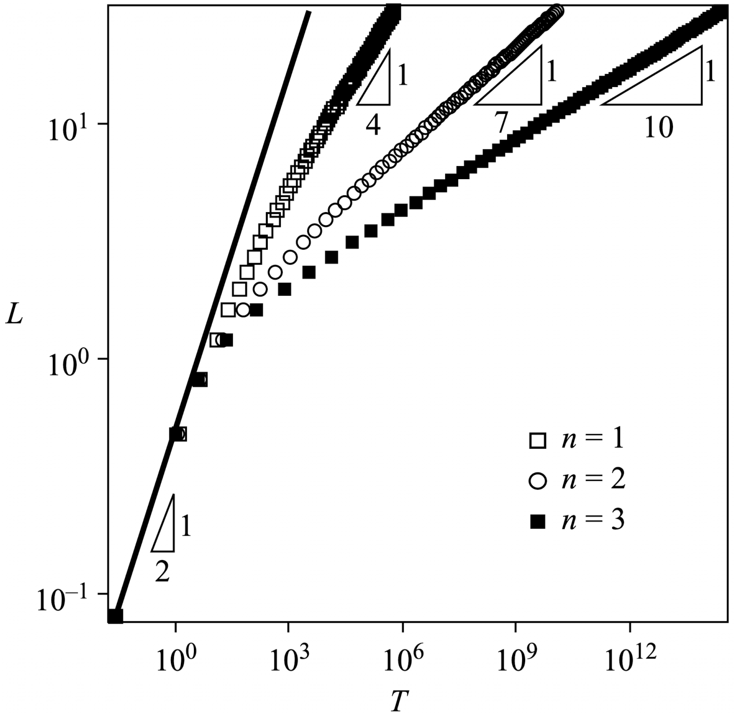
$$\boxed{L \sim T^{1/2}}$$

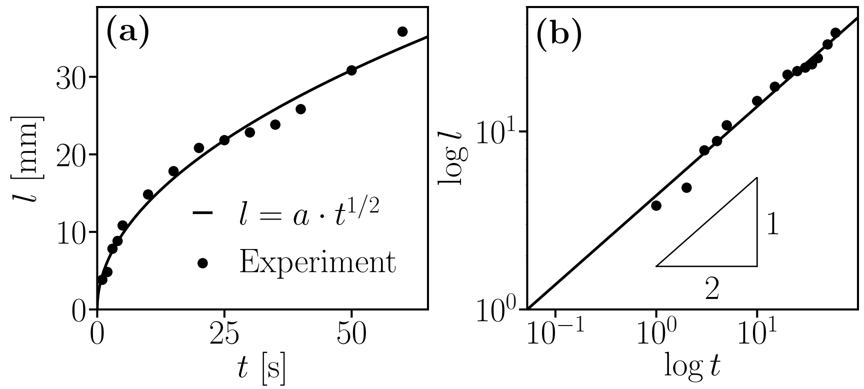
Short times

$$\boxed{L \sim T^{1/(3n+1)}}$$

Long times

- The more diverging the geometry, the slower the dynamics become at long time
- Simple experiment with coffee filter paper proofs the „Lucas-Washburn Law“





IMBIBITION IN GEOMETRIES WITH AXIAL VARIATIONS IN 2D

$$P = P_0 - \frac{c\gamma}{r(z)}$$

Volumetric Flow Rate

$$Q = 2ur(z)$$

$$\frac{\mu u}{k} = -\frac{\partial P}{\partial z}$$

Darcy Law

$$k = \frac{r(z)^2}{\lambda} = \frac{r(z)^2}{3}$$

Permeability

$$\Rightarrow \frac{3\mu Q}{\pi r(z)^3} = -\frac{\partial P}{\partial z}$$

$$\Rightarrow \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{c\gamma}{3\mu}\left[r(l)^2 \int_0^l \mathrm{d}z\, r(z)^{-3}\right]^{-1}$$

$$\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}T} = \left[(1+L^n)^2 \int_0^L \mathrm{d}Z\,(1+Z^n)^{-3}\right]^{-1}$$

$$\boxed{L \sim T^{1/2}}$$

Short times

$$\boxed{L \sim T^{1/(2n+1)}}$$

Long times

- $c(l) = \text{const}$
- $L = (\alpha/r_0)^{1/n} l$ and $T = \frac{c\gamma r_0}{8\mu}(\alpha/r_0)^{2/n} t$

WHY $c(l)$ IS CONSTANT?

$$c(\ell) \propto \cos\left(\theta_e + \arctan\left(\frac{\mathrm{d}h}{\mathrm{d}z}(\ell)\right)\right), \tag{A 1}$$

Let us consider the profile $h(z) = h_0 + \alpha z^n$, with $n > 1$. The lubrication theory requirement ($\mathrm{d}h/\mathrm{d}z \ll 1$) leads to the simplification $\arctan(\mathrm{d}h/\mathrm{d}z(\ell)) \simeq \mathrm{d}h/\mathrm{d}z(\ell) = n\alpha\ell^{n-1}$. If we now consider a completely wetting case with $\theta_e \ll 1$,

$$c \propto 1 - \frac{n^2}{2}\alpha^2\ell^{2n-2} + \cdots \tag{A 2}$$

The approximation $c \simeq$ constant can then hold if $n^2\alpha^2\ell^{2n-2} \ll 1$, i.e.

$$\ell \ll \left(\frac{1}{\alpha n}\right)^{1/(n-1)}. \tag{A 3}$$

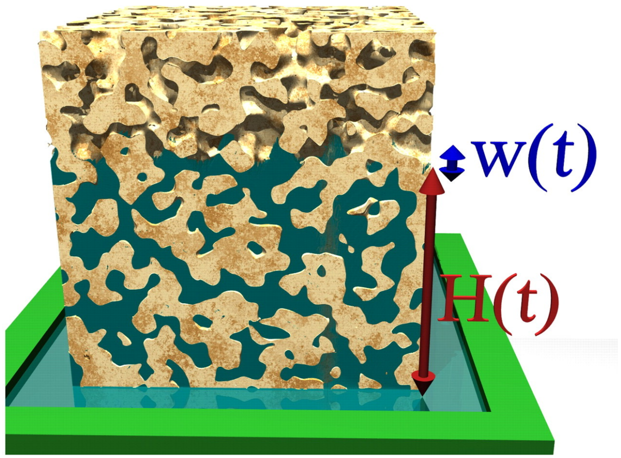
On the other hand, we know that the cross-over time between the two asymptotic limits occurs on a length scale $L \simeq 1$ or $\ell \simeq (h_0/\alpha)^{1/n}$. Thus, the approximation $c \simeq$ constant remains valid if

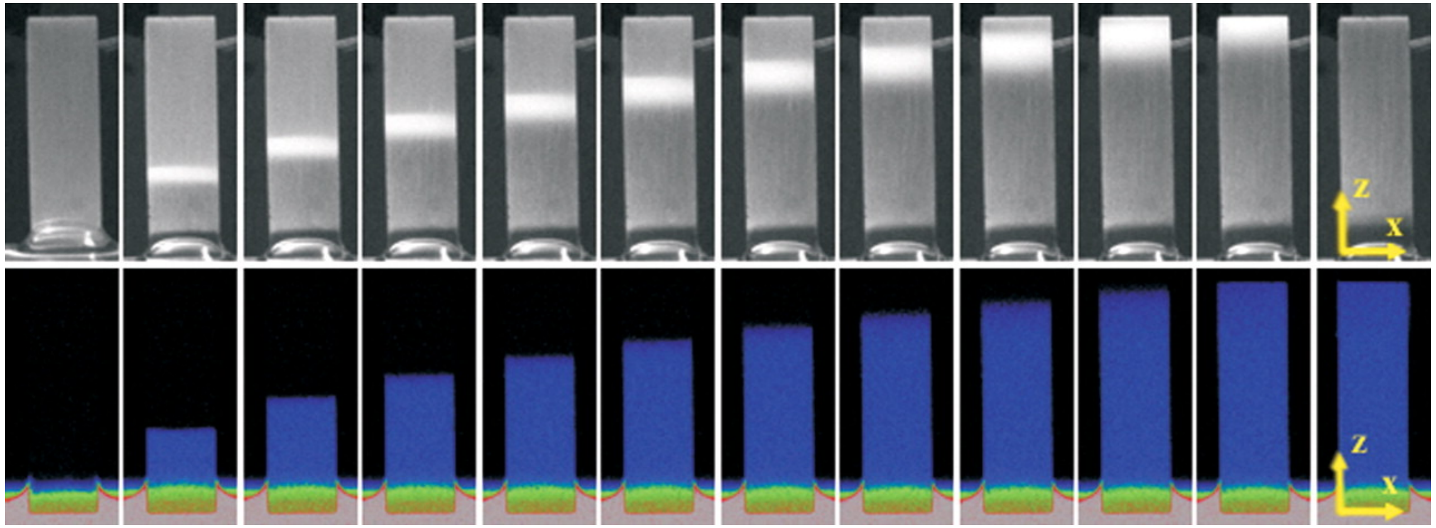
$$\left(\frac{h_0}{\alpha}\right)^{1/n} \ll \left(\frac{1}{\alpha n}\right)^{1/(n-1)} \quad \text{or} \quad h_0 \ll (\alpha n^n)^{1/(1-n)}. \tag{A 4}$$

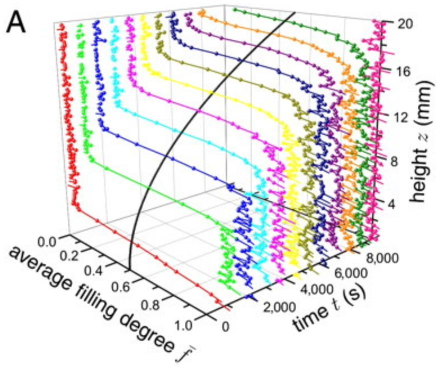
We assume the cases considered here to be under this condition.

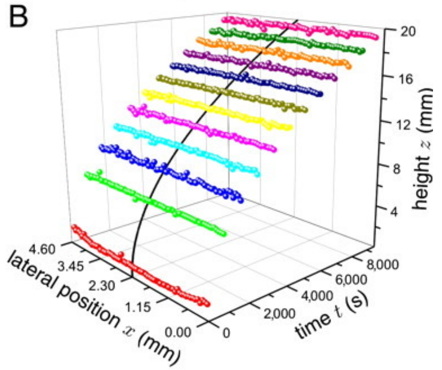
S. Gruener et al. (2012) “Anomalous front broadening during spontaneous imbibition in a matrix with elongated pores”

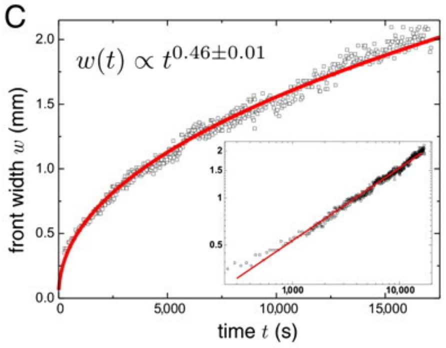
IMBIBITION EXPERIMENTS WITH VYCOR GLASS











OUTTAKES