Computational Fluid Dynamics: SPC 506

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# Two-Dimensional Navier Stokes Equations Solution in Vorticity Stream Formulation

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## Table of Contents

## Problem I

The governing equation of the incompressible, adiabatic viscous flow can be written as:

$$\vec{\nabla} \cdot \vec{V} = 0, \qquad \rho \frac{DV}{Dt} = -\nabla P + \mu \nabla^2 V$$

Where $V, \rho$ and $P$ are the velocity, density, and pressure of the flow, respectively.

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho \left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Dividing by $\rho$ and rearranging:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

The vorticity in two-dimensional flow is defined as

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The stream function is defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

Eliminating pressure from the momentum equations by cross differentiation:

$$\frac{\partial^2 u}{\partial t \partial y} + \frac{\partial^2 u}{\partial x\, \partial y} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 y}{\partial y^2} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} = \nu\left(\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^3}\right)$$

$$\frac{\partial^2 v}{\partial t \partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + v\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} = \nu\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial y^2 \partial x}\right)$$

Subtract the two equations from each other yields:

$$\frac{\partial^2 u}{\partial t \partial y} + \frac{\partial^2 u}{\partial x\, \partial y} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 y}{\partial y^2} - \frac{\partial^2 v}{\partial t \partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y \partial x} - v\frac{\partial^2 v}{\partial x \partial y}$$
$$= \nu\left(\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^3} - \frac{\partial^3 v}{\partial x^3} - \frac{\partial^3 v}{\partial y^2 \partial x}\right)$$

Rearranging those terms:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$
$$=\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad from\ continuity$$

Substituting by the vorticity:

$$\frac{\partial \Omega}{\partial t}+u\frac{\partial \Omega}{\partial x}+v\frac{\partial \Omega}{\partial y}=\nu\left(\frac{\partial^2 \Omega}{\partial x^2}+\frac{\partial^2 \Omega}{\partial y^2}\right)$$

Substituting back by stream function

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\Omega_z$$

Which are both the dimensional governing equations for incompressible, viscid flow.

To nondimensionalize the equation:

$$\Omega^* = \frac{\Omega L}{u_\infty}, \qquad \psi^* = \frac{\psi}{u_\infty L}$$
$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{L}$$
$$u^* = \frac{u}{u_\infty}, \qquad v^* = \frac{v}{u_\infty}$$
$$t^* = \frac{t u_\infty}{L}, \qquad Re = \frac{u_\infty L}{\nu_\infty}$$

$$\frac{\partial\left(\frac{\Omega^* u_\infty}{L}\right)}{\partial\left(\frac{t^* L}{u_\infty}\right)}+u^* u_\infty\frac{\partial\left(\frac{\Omega^* u_\infty}{L}\right)}{\partial(x^* L)}+v^* u_\infty\frac{\partial\left(\frac{\Omega^* u_\infty}{L}\right)}{\partial(y^* L)}=\frac{u_\infty L}{Re}\left(\frac{\partial^2\left(\frac{\Omega^* u_\infty}{L}\right)}{\partial(x^* L)^2}+\frac{\partial^2\left(\frac{\Omega^* u_\infty}{L}\right)}{\partial(y^* L)^2}\right)$$

Yielding,

$$\frac{\partial \Omega^*}{\partial t^*}+u^*\frac{\partial \Omega^*}{\partial x^*}+v^*\frac{\partial \Omega^*}{\partial y^*}=\frac{1}{Re}\left(\frac{\partial^2 \Omega^*}{\partial x^{*2}}+\frac{\partial^2 \Omega^*}{\partial y^{*2}}\right)$$

And

$$\frac{\partial^2(\psi^* u_\infty L)}{\partial(x^* L)^2}+\frac{\partial^2(\psi^* u_\infty L)}{\partial(y^* L)^2}=-\frac{\Omega^* u_\infty}{L}$$
$$\frac{u_\infty L}{L^2}\frac{\partial^2 \psi^*}{\partial x^{*2}}+\frac{u_\infty L}{L^2}\frac{\partial^2 \psi^*}{\partial y^{*2}}=-\frac{u_\infty}{L}\Omega^*$$

Then

$$\frac{\partial^2 \psi^*}{\partial x^{*2}} + \frac{\partial^2 \psi^*}{\partial y^{*2}} = -\Omega^*$$

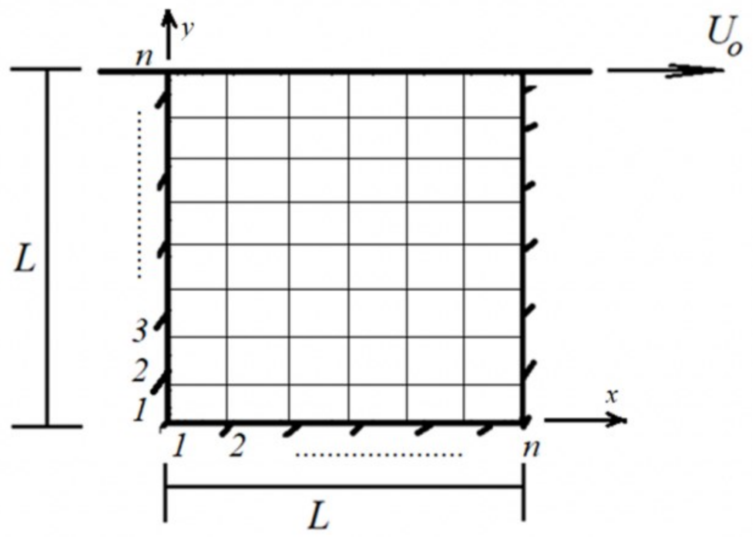
Which represents nondimensional governing equations for incompressible, viscid flow.

Dropping the dot notation, in another format, they can be expressed as:

$$\Omega_t + (\vec{\nabla} \cdot \vec{V})\Omega = \frac{1}{Re}\nabla^2 \Omega$$
$$\nabla^2 \psi = -\Omega$$

# Problem 2

A square cavity within which an unsteady fluid motion is generating by sliding an infinitely long plate lying on the top of the cavity. If there is no flow squeezed out of the cavity below the moving plate, the fluid motion forms closed paths within the cavity. Suppose that the size of the cavity is L×L and the sliding velocity is U0 in the positive x direction.




## Boundary Conditions

### Left boundary: wall

$$u_{1,j} = 0$$
$$v_{1,j} = 0$$
$$\psi_{1,j} = \psi_0$$

### Right boundary: wall

$$u_{IM,j} = 0$$
$$v_{IM,j} = 0$$
$$\psi_{IM,j} = \psi_0$$

### Top moving lid: moving wall

$$u_{i,JM} = U_0$$
$$v_{i,JM} = 0$$
$$\psi_{i,JM} = \psi_0$$

## Bottom boundary: wall

$$u_{i,1} = 0$$
$$v_{i,1} = 0$$
$$\psi_{i,1} = \psi_0$$

For vorticity

$$\psi_{i,j+1} = \psi_{i,j} + \Delta y \frac{\partial \psi_{i,j}}{\partial y} + \frac{\Delta y^2}{2}\frac{\partial^2 \psi_{i,j}}{\partial y^2} + \frac{\Delta y^3}{6}\frac{\partial^3 \psi_{i,j}}{\partial y^3} + O(\Delta y^4)$$

$$\psi_{i,j+2} = \psi_{i,j} + 2\Delta y \frac{\partial \psi_{i,j}}{\partial y} + 2\Delta y^2 \frac{\partial^2 \psi_{i,j}}{\partial y^2} + \frac{8\,\Delta y^3}{6}\frac{\partial^3 \psi_{i,j}}{\partial y^3} + O(\Delta y^4)$$

Multiplying the first equation by 8 and subtracting the second equation from it,

$$8\,\psi_{i,j+1} - \psi_{i,j+2} = 7\psi_{i,j} + 6\,\Delta y \frac{\partial \psi_{i,j}}{\partial y} + 2\Delta y^2 \frac{\partial^2 \psi_{i,j}}{\partial y^2} + O(\Delta y^4)$$

$$\frac{\partial \psi_{i,j}}{\partial y} = 0, \qquad at\ any\ stationary\ wall$$

Then

$$8\,\psi_{i,j+1} - \psi_{i,j+2} = 7\psi_{i,j} + 2\Delta y^2 \Omega_{i,j} + O(\Delta y^4)$$

$$\Omega_{i,j} = \frac{8\,\psi_{i,j+1} - \psi_{i,j+2} - 7\psi_{i,j}}{2\Delta y^2} + O(\Delta y^2), \qquad at\ any\ stationary\ wall$$

$$\Omega_{i,j} = \frac{8\,\psi_{i,j+1} - \psi_{i,j+2} - 7\psi_{i,j}}{2\Delta y^2} - 3\frac{U0}{\Delta y} + O(\Delta y^2), \qquad for\ moving\ walls$$

# Scheme

Using explicit forward differencing for time, second order upwind scheme for convective terms, and central differencing for diffusive terms to drive the finite difference equation for the vorticity.

## Vorticity

$$
\begin{aligned}
\frac{\boldsymbol{\Omega^{k+1}}_{i,j} - \boldsymbol{\Omega^{k}}_{i,j}}{\Delta t} &+ \frac{1}{2}(1-\epsilon_x)\left(\frac{-3\Omega^{k}_{i,j} + 4\Omega^{k}_{i+1,j} - \Omega^{k}_{i+2,j}}{2\Delta x}\right) \\
&+ \frac{1}{2}(1+\epsilon_x)\left(\frac{3\Omega^{k}_{i,j} - 4\Omega^{k}_{i-1,j} + \Omega^{k}_{i-2,j}}{2\Delta x}\right) \\
&+ \frac{1}{2}(1-\epsilon_y)\left(\frac{-3\Omega^{k}_{i,j} + 4\Omega^{k}_{i,j+1} - \Omega^{k}_{i,j+2}}{2\Delta y}\right) \\
&+ \frac{1}{2}(1+\epsilon_y)\left(\frac{3\Omega^{k}_{i,j} - 4\Omega^{k}_{i,j-1} + \Omega^{k}_{i,j-2}}{2\Delta y}\right) \\
&= \frac{1}{Re}\left(\frac{\Omega^{k}_{i+1,j} - 2\Omega^{k}_{i,j} + \Omega^{k}_{i-1,j}}{\Delta x^2} + \frac{\Omega^{k}_{i,j+1} - 2\Omega^{k}_{i,j} + \Omega^{k}_{i,j-1}}{\Delta y^2}\right)
\end{aligned}
$$

If u is positive, $\epsilon_x = 1$. If it is negative $\epsilon_x = -1$. Else $\epsilon_x = 0$. Similarly for $\epsilon_y$.

## Stream Function

Using Gauss-Seidel scheme to drive the stream function finite difference equation.

$$
\frac{\psi^{k}_{i+1,j} - 2\psi^{k}_{i,j} + \psi^{k}_{i-1,j}}{\Delta x^2} + \frac{\psi^{k}_{i,j+1} - 2\psi^{k}_{i,j} + \psi^{k}_{i,j-1}}{\Delta y^2} = -\Omega^{k}_{i,j}
$$

$$
\psi^{k}_{i+1,j} - 2\psi^{k}_{i,j} + \psi^{k}_{i-1,j} + \Delta\beta^2\left(\psi^{k}_{i,j+1} - 2\psi^{k}_{i,j} + \psi^{k}_{i,j-1}\right) = -\Delta x^2 \Omega^{k}_{i,j}
$$

Where $\beta^2 = \frac{\Delta x^2}{\Delta y^2}$

$$
2\psi^{k}_{i,j}\left(1+\beta^2\right) = \psi^{k}_{i+1,j} + \psi^{k}_{i-1,j} + \beta^2\left(\psi^{k}_{i,j+1} + \psi^{k}_{i,j-1}\right) + \Delta x^2 \Omega^{k}_{i,j}
$$

$$
\psi^{k}_{i,j} = \frac{1}{2\left(1+\beta^2\right)}\left[\psi^{k}_{i+1,j} + \psi^{k}_{i-1,j} + \beta^2\left(\psi^{k}_{i,j+1} + \psi^{k}_{i,j-1}\right) + \Delta x^2 \Omega^{k}_{i,j}\right]
$$

For equal grid size, $\beta^2 = 1$

$$
\boldsymbol{\psi^{k+1}_{i,j} = \frac{1}{4}\left(\psi^{k}_{i+1,j} + \psi^{k+1}_{i-1,j} + \psi^{k}_{i,j+1} + \psi^{k+1}_{i,j-1} + \Delta x^2 \Omega^{k+1}_{i,j}\right)}
$$

# Stability

## Vorticity

$$\begin{aligned}\Omega^{k+1}{}_{i,j} = \Omega^{k}{}_{i,j} - \frac{\Delta t}{4\Delta x}\Big[&(1-\epsilon_x)\left(-3u^k_{i,j}\,\Omega^k{}_{i,j} + 4u^k_{i+1,j}\Omega^k{}_{i+1,j} - u^k_{i+2,j}\Omega^k{}_{i+2,j}\right)\\ &+ (1+\epsilon_x)\left(3u^k_{i,j}\Omega^k{}_{i,j} - 4u^k_{i-1,j}\Omega^k{}_{i-1,j} + u^k_{i-2,j}\Omega^k{}_{i-2,j}\right)\\ &+ (1-\epsilon_y)\left(-3v^k_{i,j}\Omega^k{}_{i,j} + 4v^k_{i,j+1}\Omega^k{}_{i,j+1} - v^k_{i,j+2}\Omega^k{}_{i,j+2}\right)\\ &+ (1+\epsilon_y)\left(3v^k_{i,j}\Omega^k{}_{i,j} - 4v^k_{i,j-1}\Omega^k{}_{i,j-1} + v^k_{i,j-2}\Omega^k{}_{i,j-2}\right)\Big]\\ &+ \frac{\Delta t}{Re}\left(\frac{\Omega^k{}_{i+1,j} - 4\Omega^k{}_{i,j} + \Omega^k{}_{i-1,j} + \Omega^k{}_{i,j+1} + \Omega^k{}_{i,j-1}}{\Delta x^2}\right)\end{aligned}$$

$$\Omega^k{}_{i,j} = O^k e^{I(i\theta + j\varphi)}$$

$$\begin{aligned}O^k O^1 e^{I(i\theta+j\varphi)} = O^k e^{I(i\theta+j\varphi)}\\ - \frac{\Delta \boldsymbol{t}\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}\Big[&(\boldsymbol{1}-\boldsymbol{\epsilon_x})\left(-3O^k e^{I(i\theta+j\varphi)} + \boldsymbol{4}O^k e^{I(i\theta+j\varphi)} e^{I\theta} - O^k e^{I(i\theta+j\varphi)} e^{2I\theta}\right)\\ &+ (\boldsymbol{1}+\boldsymbol{\epsilon_x})\left(\boldsymbol{3}O^k e^{I(i\theta+j\varphi)} - \boldsymbol{4}O^k e^{I(i\theta+j\varphi)} e^{-I\theta} + O^k e^{I(i\theta+j\varphi)} e^{-2I\theta}\right)\\ &+ (\boldsymbol{1}-\boldsymbol{\epsilon_y})\left(-3O^k e^{I(i\theta+j\varphi)} + \boldsymbol{4}O^k e^{I(i\theta+j\varphi)} e^{I\varphi} - O^k e^{I(i\theta+j\varphi)} e^{2I\varphi}\right)\\ &+ (\boldsymbol{1}+\boldsymbol{\epsilon_y})\left(\boldsymbol{3}O^k e^{I(i\theta+j\varphi)} - \boldsymbol{4}O^k e^{I(i\theta+j\varphi)} e^{-I\varphi} + O^k e^{I(i\theta+j\varphi)} e^{-2I\varphi}\right)\Big]\\ &+ \frac{\Delta \boldsymbol{t}}{\boldsymbol{Re}\Delta \boldsymbol{x}^2}\Big(O^k e^{I(i\theta+j\varphi)} e^{I\theta} + O^k e^{I(i\theta+j\varphi)} e^{-I\theta} + O^k e^{I(i\theta+j\varphi)} e^{I\varphi}\\ &+ +O^k e^{I(i\theta+j\varphi)} e^{-I\varphi} - \boldsymbol{4}O^k e^{I(i\theta+j\varphi)}\Big)\end{aligned}$$

Divide by $O^k e^{I(i\theta+j\varphi)}$

$$\begin{aligned}G = 1 - \frac{\Delta \boldsymbol{t}\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}\Big[&(\boldsymbol{1}-\boldsymbol{\epsilon_x})\left(-3 + \boldsymbol{4}e^{I\theta} - e^{2I\theta}\right) + (\boldsymbol{1}+\boldsymbol{\epsilon_x})\left(\boldsymbol{3} - \boldsymbol{4}e^{-I\theta} + e^{-2I\theta}\right)\\ &+ (\boldsymbol{1}-\boldsymbol{\epsilon_y})(-3 + \boldsymbol{4}e^{I\varphi} - e^{2I\varphi}) + (\boldsymbol{1}+\boldsymbol{\epsilon_y})(\boldsymbol{3} - \boldsymbol{4}e^{-I\varphi} + e^{-2I\varphi})\Big]\\ &+ \frac{\Delta \boldsymbol{t}}{\boldsymbol{Re}\Delta \boldsymbol{x}^2}\left(e^{I\theta} + e^{-I\theta} + e^{I\varphi} + +e^{-I\varphi} - \boldsymbol{4}\right)\end{aligned}$$

$$\begin{aligned}G = 1 - \frac{\Delta \boldsymbol{t}\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}\Big[&(\boldsymbol{1}-\boldsymbol{\epsilon_x})(-3 + 4\cos(\theta) - \cos(2\theta) + I(4\sin(\theta) - \sin(2\theta)))\\ &+ (\boldsymbol{1}+\boldsymbol{\epsilon_x})(3 - 4\cos(\theta) + \cos(2\theta) + I(4\sin(\theta) - \sin(2\theta)))\\ &+ (\boldsymbol{1}-\boldsymbol{\epsilon_y})(-3 + 4\cos(\varphi) - \cos(2\varphi) + I(4\sin(\varphi) - \sin(2\varphi)))\\ &+ (\boldsymbol{1}+\boldsymbol{\epsilon_y})(3 - 4\cos(\varphi) + \cos(2\varphi) + I(4\sin(\varphi) - \sin(2\varphi)))\Big]\\ &+ \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re}\Delta \boldsymbol{x}^2}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})\end{aligned}$$

*For* $\boldsymbol{\epsilon_x} = 1$

$\boldsymbol{\epsilon_y} = 1$:

$$G = 1 - \frac{\boldsymbol{\Delta t\, u_{max}}}{\boldsymbol{4\Delta x}}[2(3 - 4\cos(\theta) + \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))) + 2(3 - 4\cos(\varphi) + \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})$$

$\boldsymbol{\epsilon_y} = -1$:

$$G = 1 - \frac{\boldsymbol{\Delta t\, u_{max}}}{\boldsymbol{4\Delta x}}[2(3 - 4\cos(\theta) + \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))) + 2(-3 + 4\cos(\varphi) - \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})$$

$\boldsymbol{\epsilon_y} = 0$:

$$G = 1 - \frac{\boldsymbol{\Delta t\, u_{max}}}{\boldsymbol{4\Delta x}}\left[2\left(3 - 4\cos(\theta) + \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))\right) + 2\mathrm{I}(4\sin(\varphi) - \sin(2\varphi))\right] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})$$

*For* $\boldsymbol{\epsilon_x} = -1$

$\boldsymbol{\epsilon_y} = 1$:

$$G = 1 - \frac{\boldsymbol{\Delta t\, u_{max}}}{\boldsymbol{4\Delta x}}[2(-3 + 4\cos(\theta) - \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))) + 2(3 - 4\cos(\varphi) + \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})$$

$\boldsymbol{\epsilon_y} = -1$:

$$G = 1 - \frac{\boldsymbol{\Delta t\, u_{max}}}{\boldsymbol{4\Delta x}}\left[2\left(-3 + 4\cos(\theta) - \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))\right) + 2(-3 + 4\cos(\varphi) - \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))\right] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \boldsymbol{2})$$

$\boldsymbol{\epsilon_y} = 0$:

$$G = 1 - \frac{\Delta t\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}\big[2\big(-3 + 4\cos(\theta) - \cos(2\theta) + \mathrm{I}(4\sin(\theta) - \sin(2\theta))\big) + 2\mathrm{I}(4\sin(\varphi) - \sin(2\varphi))\big] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \mathbf{2})$$

*For* $\boldsymbol{\epsilon_x} = 0$

$\boldsymbol{\epsilon_y} = 1$:

$$G = 1 - \frac{\Delta t\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}[2\mathrm{I}(4\sin(\theta) - \sin(2\theta)) + 2(3 - 4\cos(\varphi) + \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \mathbf{2})$$

$\boldsymbol{\epsilon_y} = -1$:

$$G = 1 - \frac{\Delta t\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}\big[2\mathrm{I}\big(4\sin(\theta) - \sin(2\theta)\big) + 2(-3 + 4\cos(\varphi) - \cos(2\varphi) + \mathrm{I}(4\sin(\varphi) - \sin(2\varphi)))\big] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \mathbf{2})$$

$\boldsymbol{\epsilon_y} = 0$:

$$G = 1 - \frac{\Delta t\, \boldsymbol{u_{max}}}{\boldsymbol{4\Delta x}}[2\mathrm{I}(4\sin(\theta) - \sin(2\theta)) + 2\mathrm{I}(4\sin(\varphi) - \sin(2\varphi))] + \frac{2\Delta \boldsymbol{t}}{\boldsymbol{Re\Delta x^2}}(\cos(\theta) + \cos(\varphi) - \mathbf{2})$$

## Stream

To analyse the stability of the Gauss-Seidel scheme, a perturbation in the form of

$$u_{i,j}^{k} = U^{k} e^{I(i\theta + j\varphi)}$$

Given the scheme:

$$\psi^{k+1}{}_{i,j} = \frac{1}{4}\left(\psi^{k}{}_{i+1,j} + \psi^{k+1}{}_{i-1,j} + \psi^{k}{}_{i,j+1} + \psi^{k+1}{}_{i,j-1} + \Delta x^2 \Omega^{k+1}{}_{i,j}\right)$$

Substituting the perturbation into the iteration equation, we have:

$$\begin{aligned} U^{n+1} e^{I(i\theta\,+\,j\,\varphi)} \\ &= \frac{1}{4}\left(U^{n} e^{I(i\theta\,+\,j\varphi)} e^{I\theta} + U^{n-1} e^{I(i\theta +\,j\,\varphi)} e^{-I\theta} + U^{n} e^{I(i\theta\,+\,j\varphi)} e^{I\varphi} \right. \\ &\left. + \; U^{n-1} e^{I(i\theta\,+\,j\varphi)} e^{-I\varphi} + \Delta x^2 \Omega^{k+1}{}_{i,j}\right) \end{aligned}$$

Dividing both sides by $U^{n} e^{I(i\,\theta +\,j\varphi)}$,

$$G = \frac{1}{4}\left(e^{I\theta} + \frac{1}{G} e^{-I\theta} + e^{I\varphi} + \frac{1}{G} e^{-I\varphi} + U^{-n} e^{-I(i\,\theta +\,j\varphi)} \Delta x^2 \Omega^{k+1}{}_{i,j}\right)$$

$$\begin{aligned} G^2 &= \frac{1}{4}\left(G\left(e^{I\theta} + e^{I\varphi}\right) + e^{-I\theta} + e^{-I\varphi} + U^{-n+1} e^{-I(i\,\theta +\,j\varphi)} \Delta x^2 \Omega^{k+1}{}_{i,j}\right) \\ &= \frac{1}{4}\Big[G\big(cos(\theta) + cos(\varphi) + I(sin(\theta) + sin(\varphi))\big) + cos(\theta) + cos(\varphi) - I\big(sin(\theta) + sin(\varphi)\big) \\ &\qquad + U^{-n+1} e^{-I(i\,\theta +\,j\varphi)} \Delta x^2 \Omega^{k+1}{}_{i,j}\Big] \end{aligned}$$

Then

$$\begin{aligned} &G^2 - \frac{1}{4} BG - \frac{1}{4} C = 0 \\ &B = 2\left[cos\left(\frac{\theta+\varphi}{2}\right) cos\left(\frac{\theta-\varphi}{2}\right) + I\left(\sin\left(\frac{\theta+\varphi}{2}\right) \cos\left(\frac{\theta-\varphi}{2}\right)\right)\right] \\ &C = 2\left[cos\left(\frac{\theta+\varphi}{2}\right) cos\left(\frac{\theta-\varphi}{2}\right) - I\left(\sin\left(\frac{\theta+\varphi}{2}\right) \cos\left(\frac{\theta-\varphi}{2}\right)\right)\right] \\ &\qquad + U^{-n+1}(\cos(i\theta + j\varphi) - I \sin(i\theta + j\varphi)) \Delta x^2 \Omega^{k+1}{}_{i,j} \end{aligned}$$

## Algorithm

1- While vorticity $\Omega$has not converged
   1. Calculate $\Omega$ on boundaries using **second order forward or backward differencing**.
   2. Calculate $\Omega$ near boundaries using **first order upwind** scheme.
      - For each node, calculate $\boldsymbol{\epsilon_x}$ and $\boldsymbol{\epsilon_y}$
   3. Calculate $\Omega$ in the interior domain using **second-order upwind** scheme.
   4. While stream function $\psi$ has not converged
      - Solve for it using gauss seidel scheme.
   5. Calculate velocity components.

The convergence criteria for $\psi$ is as follows.

$$Error_{\psi} = 100 * \frac{\sum_{\substack{i=1 \\ j=1}}^{\substack{i=IM \\ j=JM}} \left| \psi^{k2+1}{}_{i,j} - \psi_{i,j}^{k2} \right|}{\sum_{\substack{i=1 \\ j=1}}^{\substack{i=IM \\ j=JM}} \left| \psi^{k2+1}{}_{i,j} \right|} \leq 0.1$$

The convergence criteria for $\Omega$ is as follows.

$$Error_{\Omega} = \frac{\sum_{\substack{i=1 \\ j=1}}^{\substack{i=IM \\ j=JM}} \left| \Omega^{k+1}{}_{i,j} - \Omega_{i,j}^{k} \right|}{\sum_{\substack{i=1 \\ j=1}}^{\substack{i=IM \\ j=JM}} \left| \psi^{k2+1}{}_{i,j} \right|} \leq 2$$

## Results

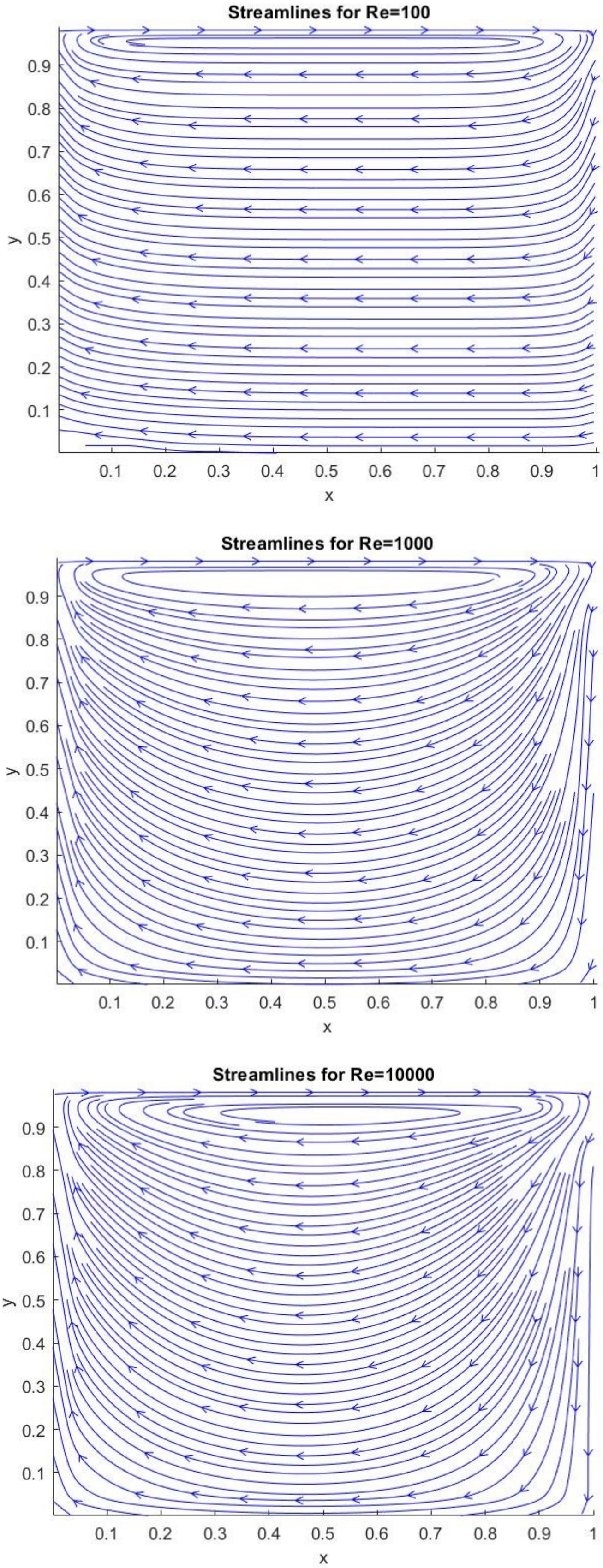
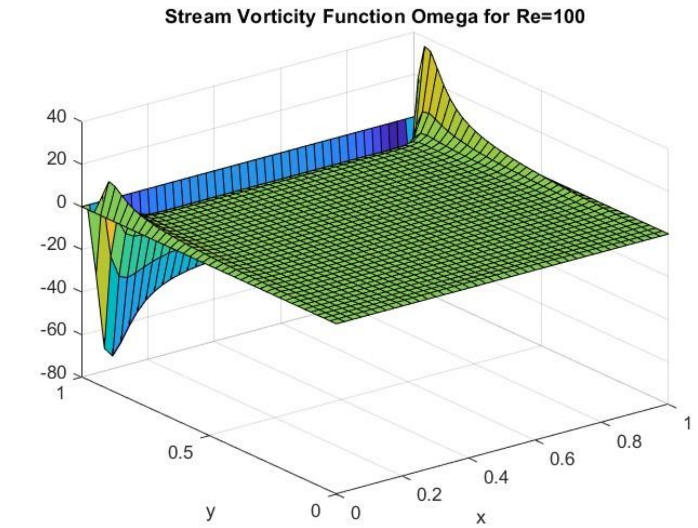
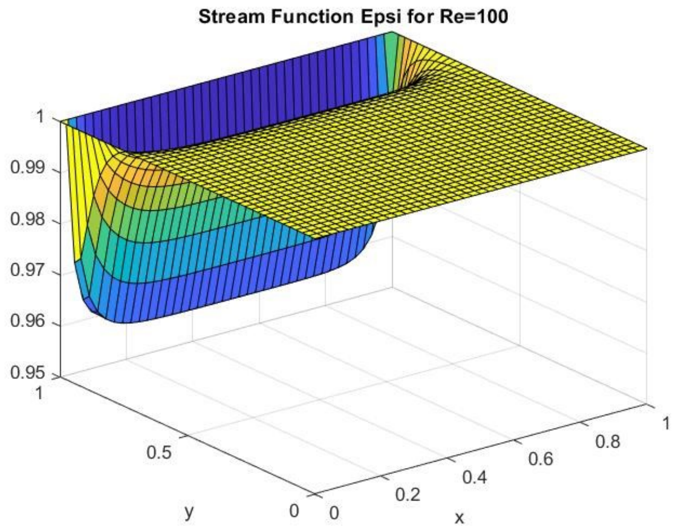
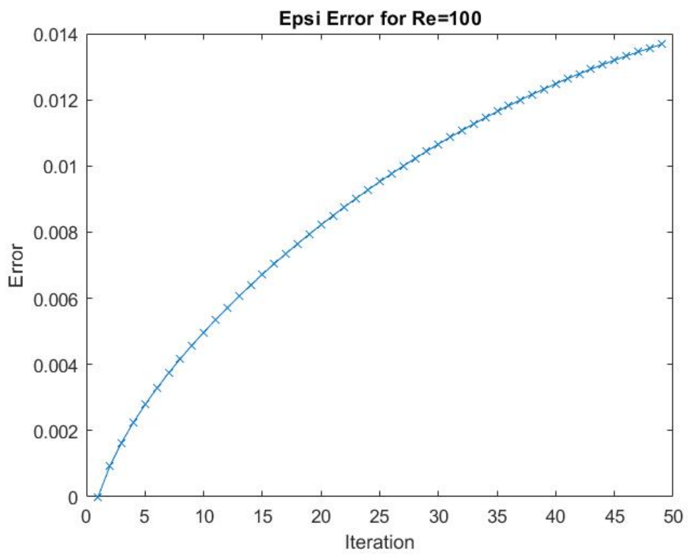


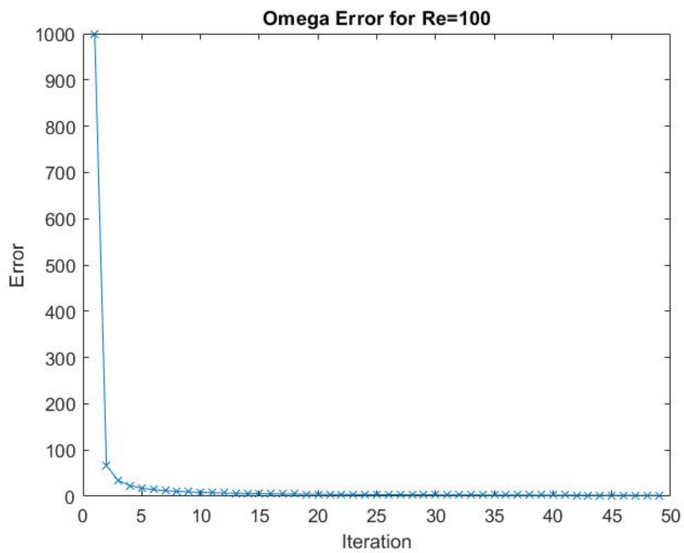

*Figure 1 The streamlines for each Reynolds number. Increasing Re increases vorticity.*

## Re=10^2

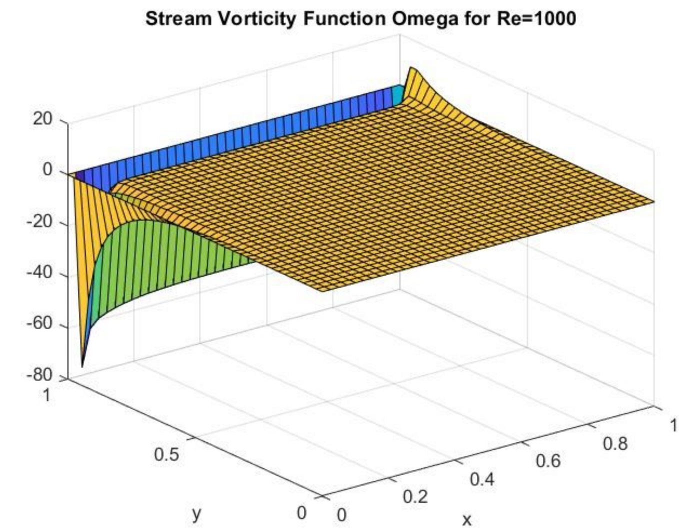

Stream Vorticity Function Omega for Re=100
y
x

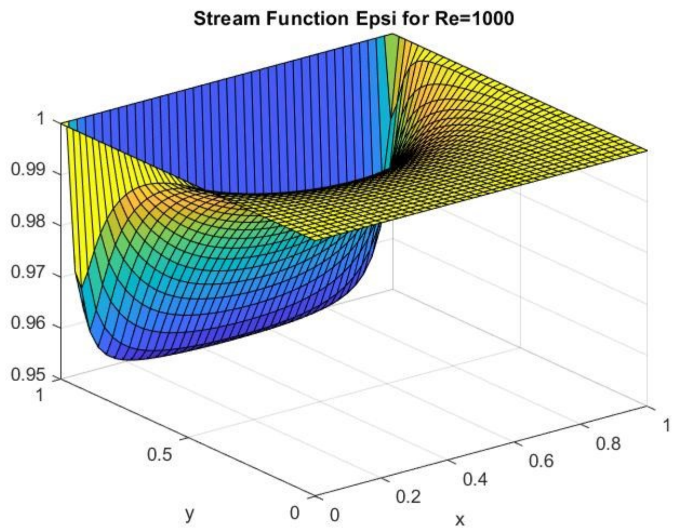

Stream Function Epsi for Re=100
y
x

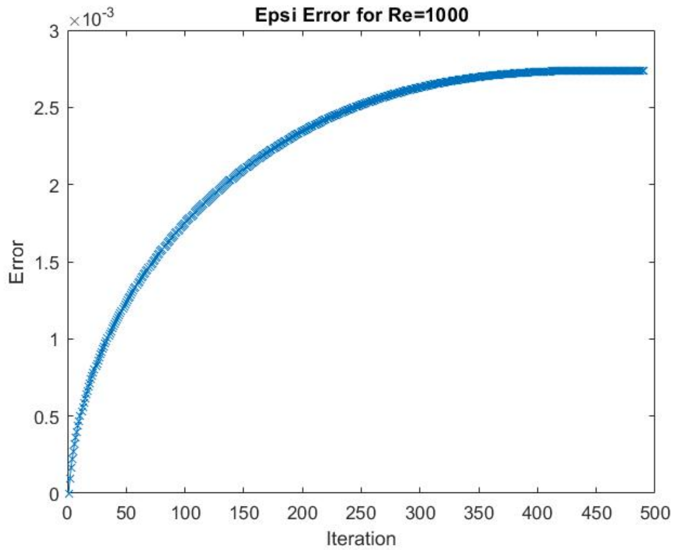

Epsi Error for Re=100
Error
Iteration

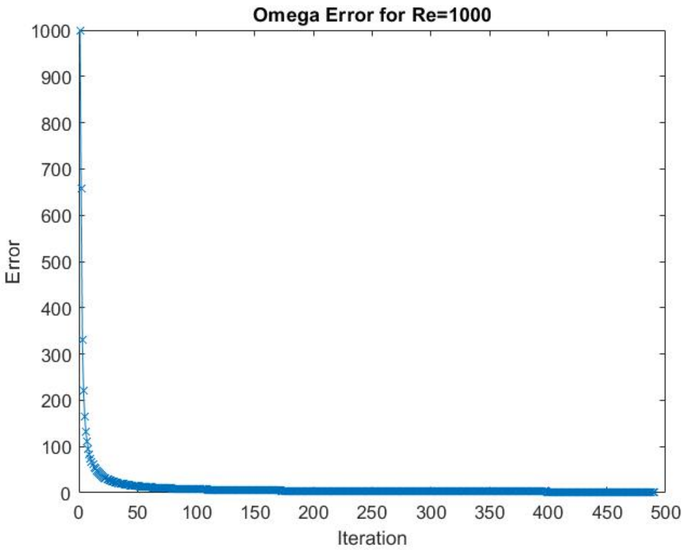

Omega Error for Re=100
Error
Iteration

## Re=10^3

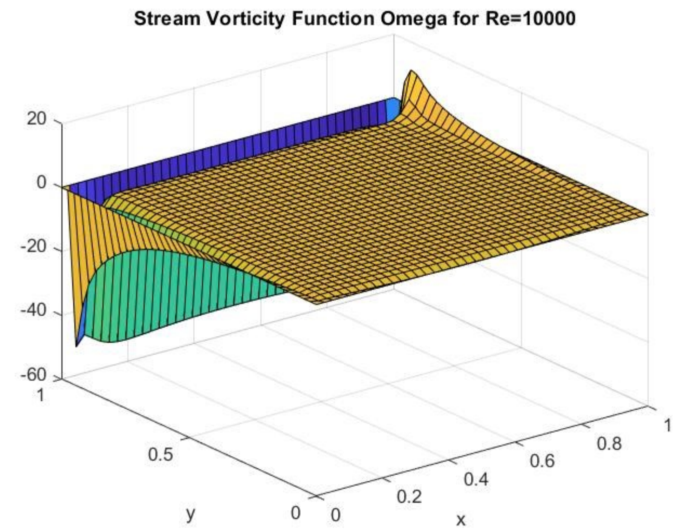

Stream Vorticity Function Omega for Re=1000
20
0
-20
-40
-60
-80
1
0.5
y
0
0
0.2
0.4
0.6
0.8
1
x

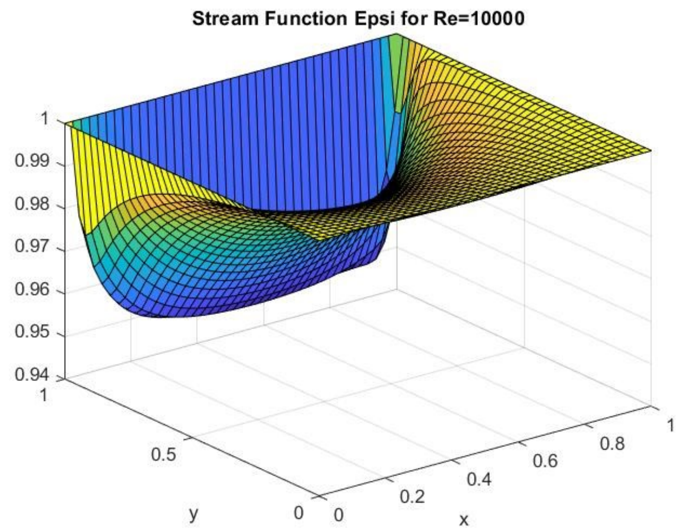

Stream Function Epsi for Re=1000
1
0.99
0.98
0.97
0.96
0.95
1
0.5
y
0
0
0.2
0.4
0.6
0.8
1
x

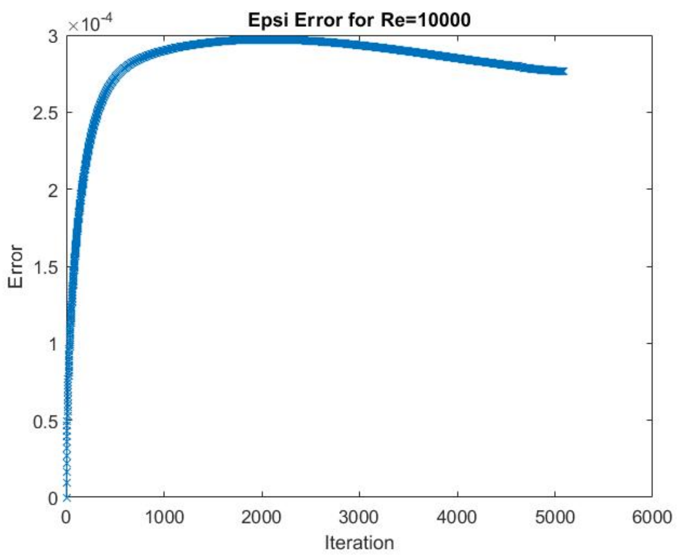

Epsi Error for Re=1000
$\times 10^{-3}$
3
2.5
2
1.5
1
0.5
0
Error
0
50
100
150
200
250
300
350
400
450
500
Iteration

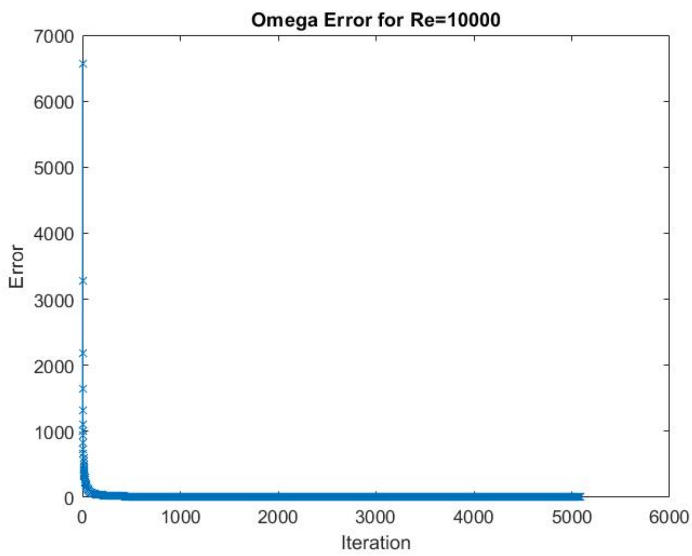

Omega Error for Re=1000
1000
900
800
700
600
500
400
300
200
100
0
Error
0
50
100
150
200
250
300
350
400
450
500
Iteration

## Re=10^4


Stream Vorticity Function Omega for Re=10000
20
0
-20
-40
-60
1
0.5
y
0
0
0.2
0.4
0.6
0.8
1
x


Stream Function Epsi for Re=10000
1
0.99
0.98
0.97
0.96
0.95
0.94
1
0.5
y
0
0
0.2
0.4
0.6
0.8
1
x


Epsi Error for Re=10000
×10^-4
3
2.5
2
1.5
1
0.5
0
Error
0
1000
2000
3000
4000
5000
6000
Iteration


Omega Error for Re=10000
7000
6000
5000
4000
3000
2000
1000
0
Error
0
1000
2000
3000
4000
5000
6000
Iteration

## Comparison Between Upwind And Central Schemes For Re=100,

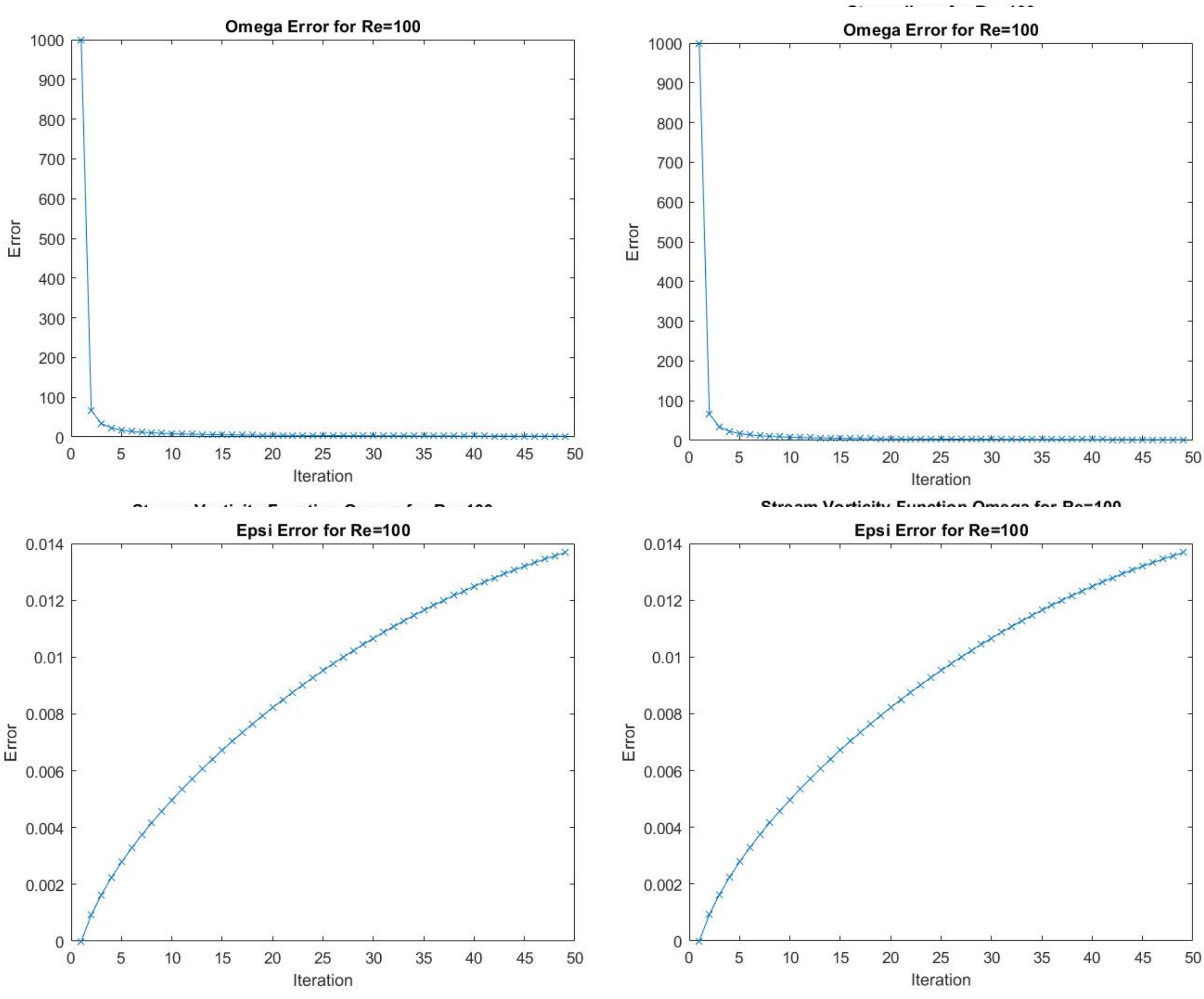
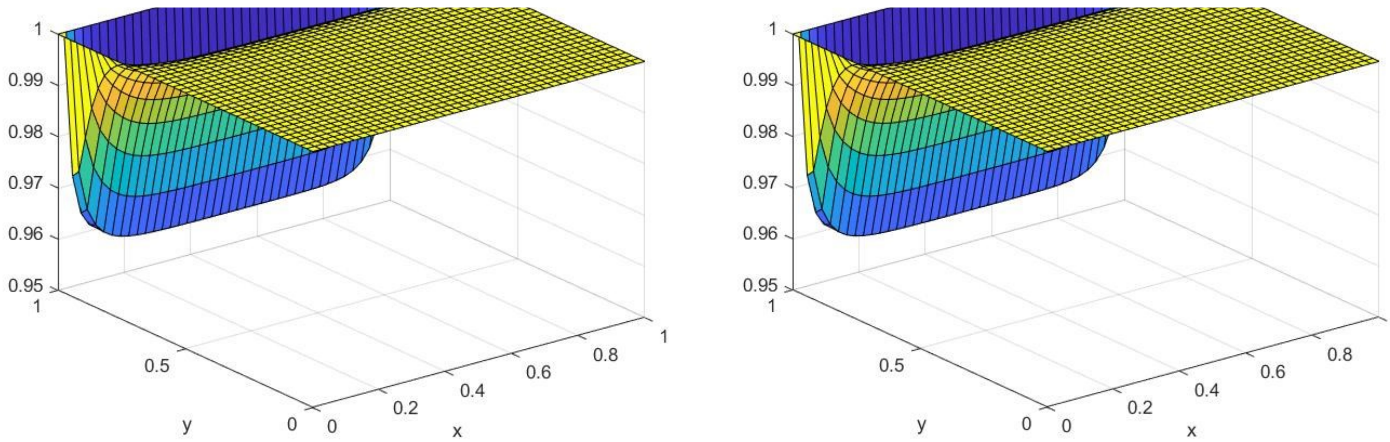
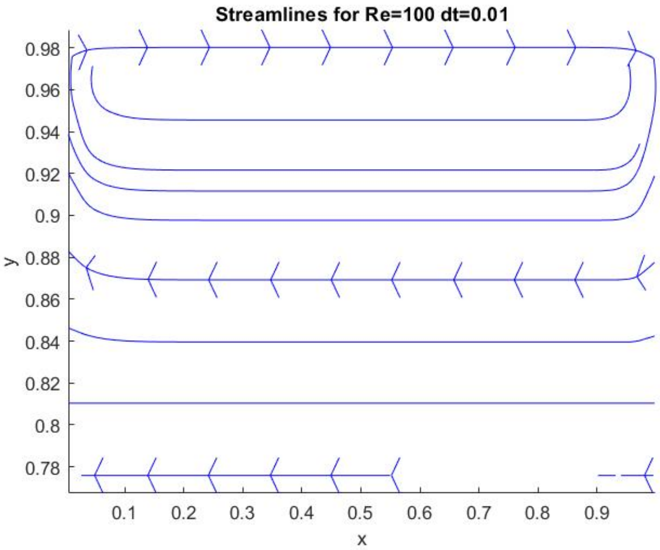
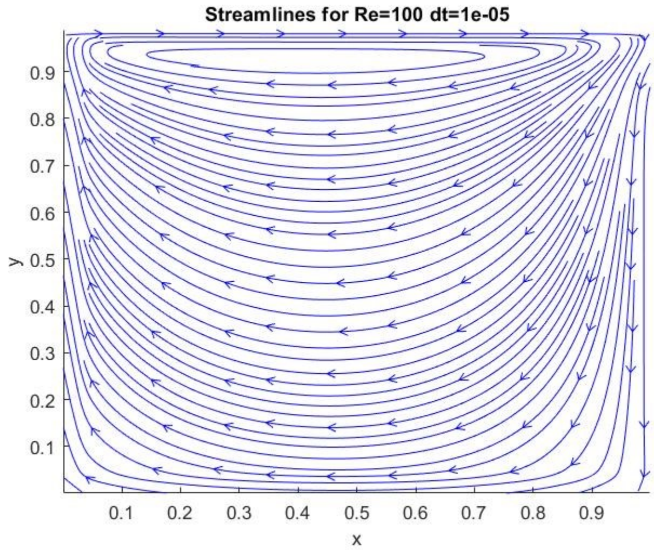


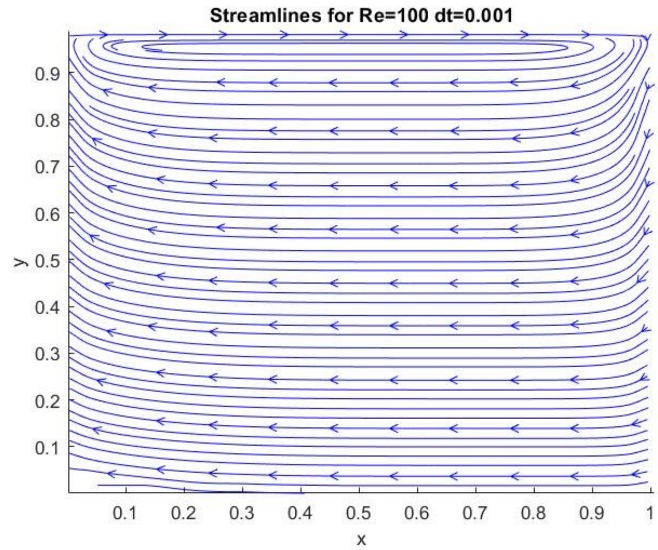

*Figure 2 On the left is the central space solution, and on the right is the second-order upwind scheme solution. Both for Re=100, and dt=0.001. Both took the same number of iterations (49), with a slight difference in runtime.*

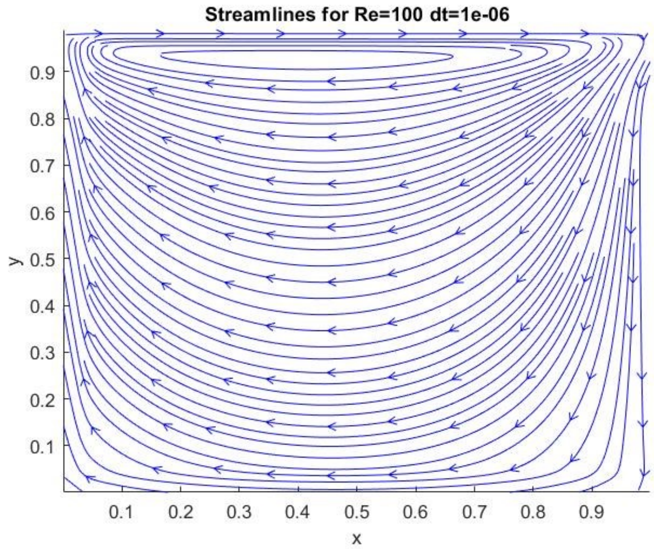


## Studying The Effect On Increasing Timestep



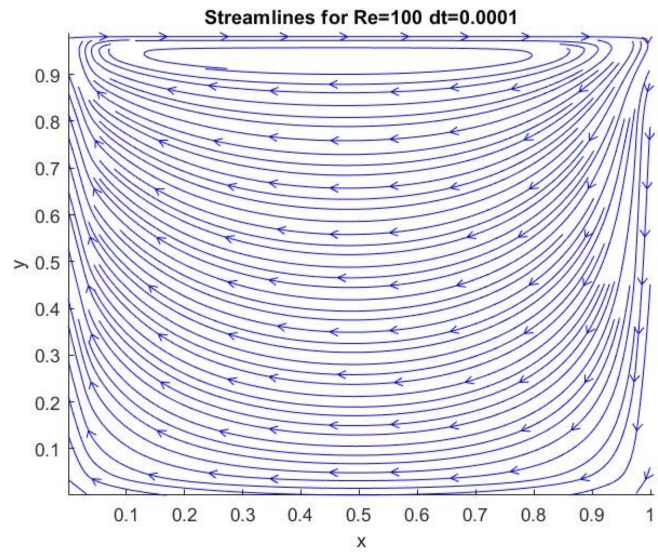
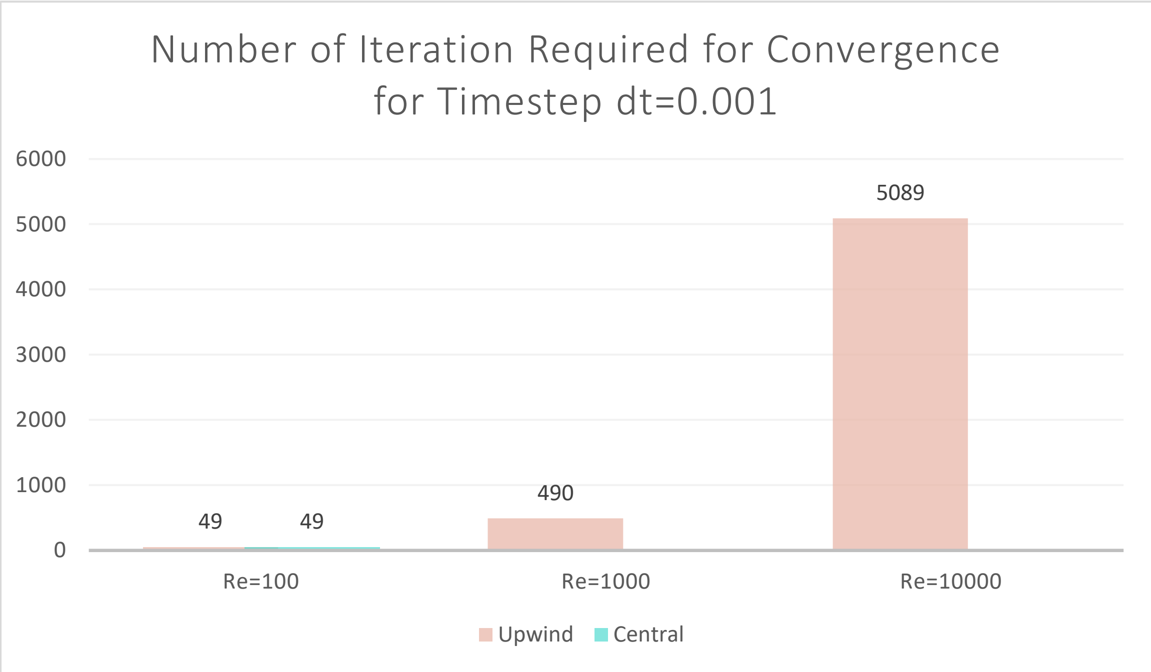
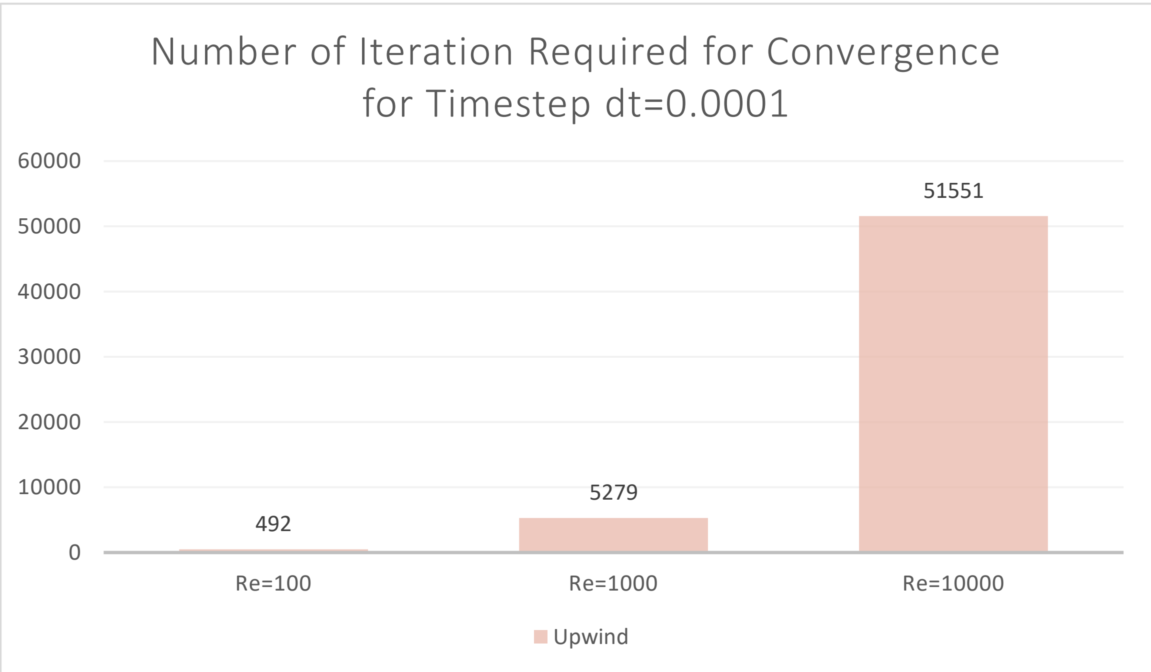
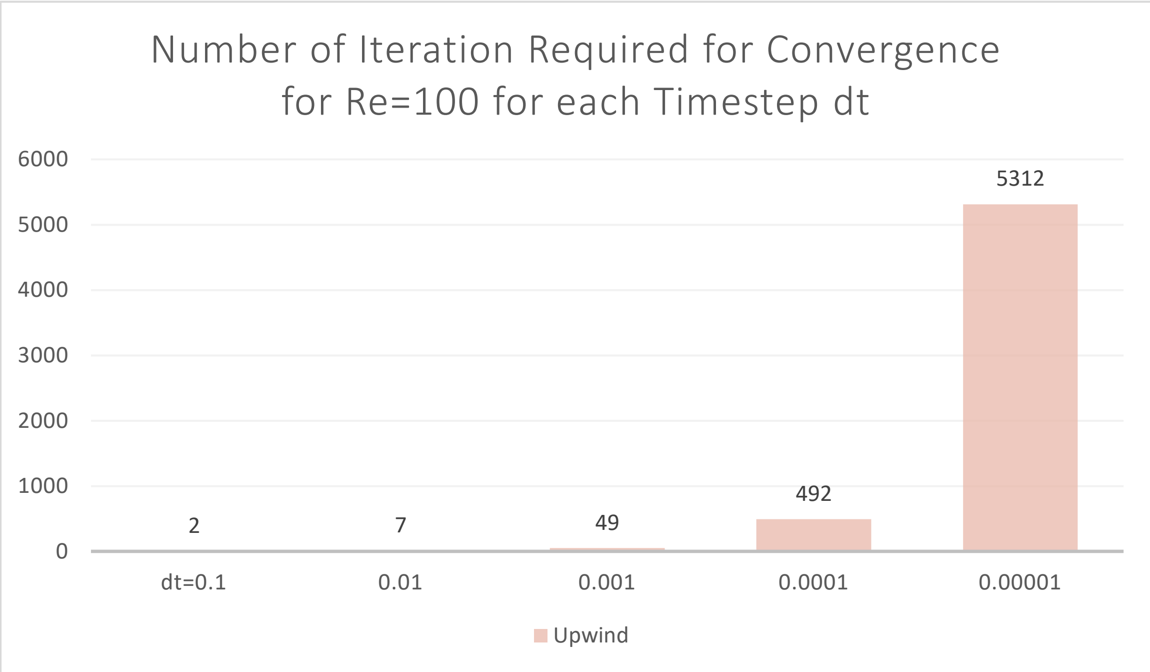




*Figure 3 For Re=100 and the same convergence criteria, the effect of increasing the timestep is evident. The maximum timestep is in the order of 10^-3 seconds. Decreasing the timestep allows for more precise results.*

## Discussion


Number of Iteration Required for Convergence for Timestep dt=0.001
6000
5000
4000
3000
2000
1000
0
49
49
490
5089
Re=100
Re=1000
Re=10000
Upwind
Central


Number of Iteration Required for Convergence for Timestep dt=0.0001
60000
50000
40000
30000
20000
10000
0
492
5279
51551
Re=100
Re=1000
Re=10000
Upwind




Decreasing the timestep by a power of 10 increases the number of iterations by almost the same power, and roughly doubles the runtime.

Increasing the Reynolds number by a power of 10, increases the number of iterations by the same power.

Decreasing time step allows for more precise results.

When increasing the timestep above a certain threshold (dt =0.001) the solution converges but gives nonsense results.