Rhythms

Analyzing Rhythms (Part 2)

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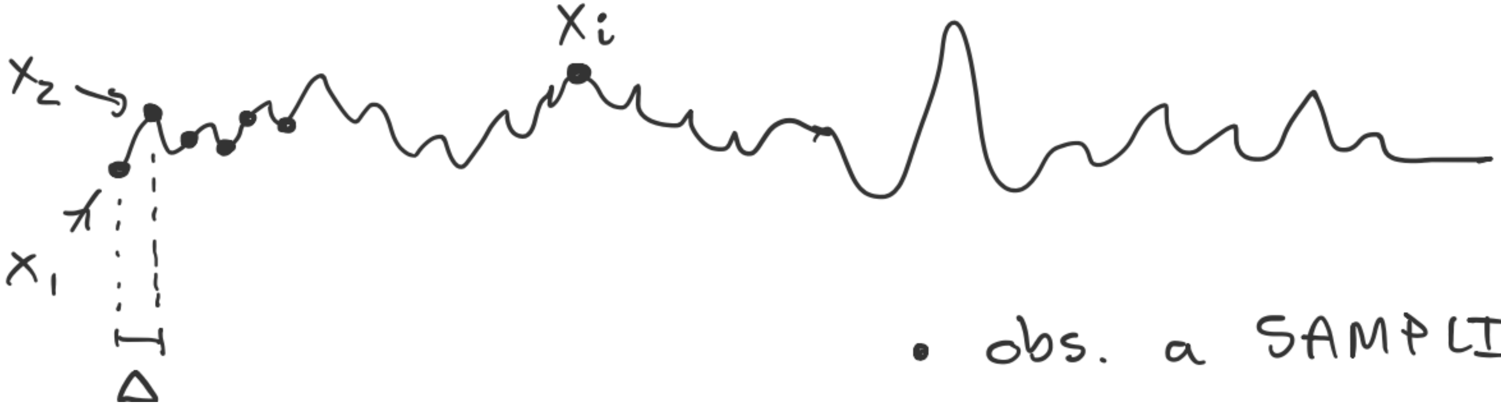
Today

Autocovariance

Notation



- obs. a SAMPLING.

- Δ = sampling interval (Ex. 1ms)
- $\frac{1}{\Delta}$ = sampling freq. (Ex. 1000 Hz)

N = length of data (# of data points

or $T = N\Delta$ = total duration of recording

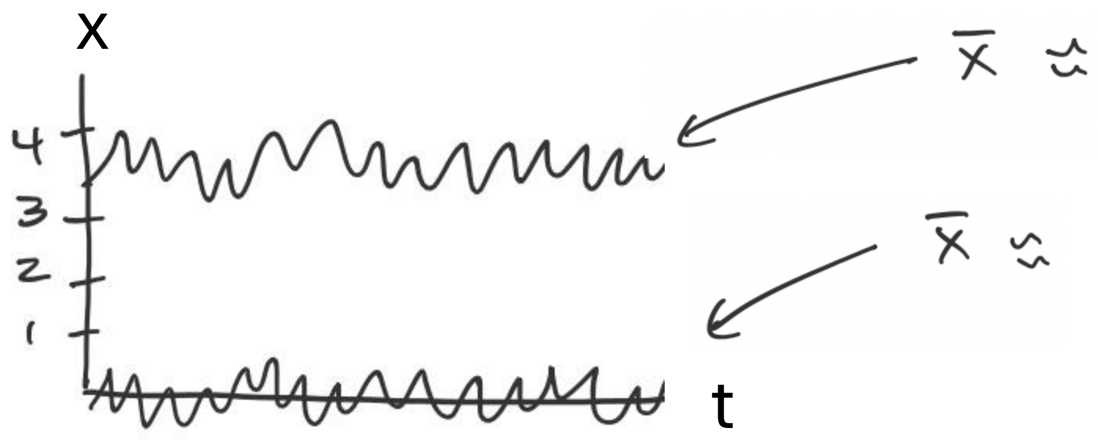
Mean & variance

Define x_n data at index n

Define $\bar{x}$ = mean of x

$$\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

Ex.



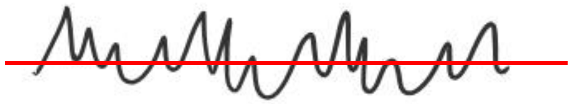
Mean & variance

Define x_n data at index n

Define σ^2 = variance of x

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})^2$$

<u>Ex</u>.



σ^2 small

σ^2 big

Autocovariance (equation)

Define:

$$r_{xx}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L} - \bar{x})(x_n - \bar{x})$$

The autocovariance of *x* at lag L

sum over data *x*

the data with mean subtracted at index $n + L$

... multiplied by itself at index n

Autocovariance (intuition)

$$r_{xx}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L}-\bar{x})(x_n-\bar{x})$$

Fix **<u>L=0</u>**

x

assume mean = 0 (i.e., $\bar{x} = 0$)

x_n

multiply

x_{n+0}

- at each index, multiply the two signals → get a number
- sum these numbers to get $r_{xx}[0]$

<u>Note</u>: compare to variance

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n-\bar{x})^2$$

$$\boxed{r_{xx}[0] = \sigma^2}$$

Q. Will this number be big or small?

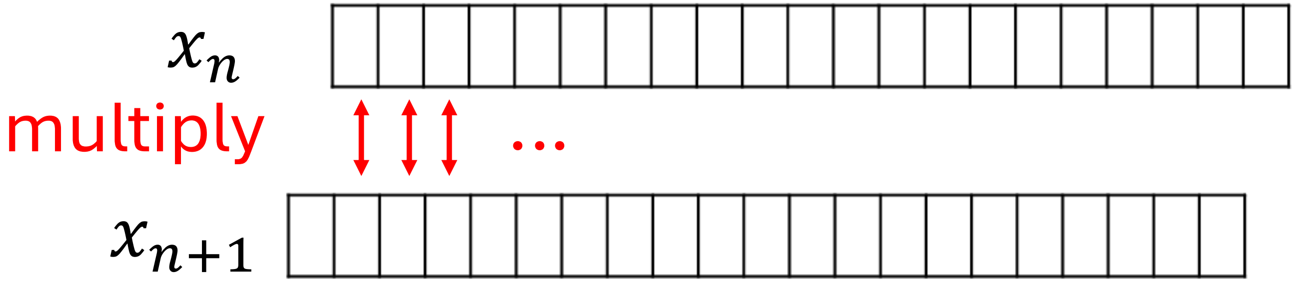
Autocovariance (intuition)

$$r_{xx}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L} - \bar{x})(x_n - \bar{x})$$

Fix **L=1**

x 

assume mean = 0 (i.e., $\bar{x} = 0$)



- at each index, multiply the two signals → get a number
- sum these numbers to get $r_{xx}[1]$

$x_0\ x_1$

$x_1\ x_2$

$x_2\ x_3$

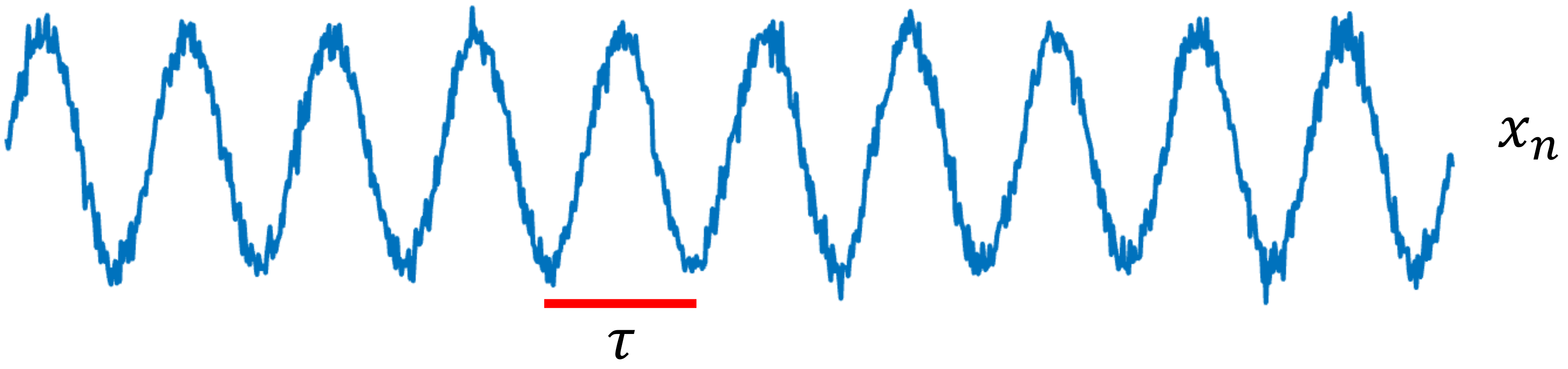
...

Q. Will this number be big or small?

Repeat for L = 2,3, …, L = -1, -2, … to get $r_{xx}[L]$

Autocovariance (example)

Ex. Consider these nearly sinusoidal data

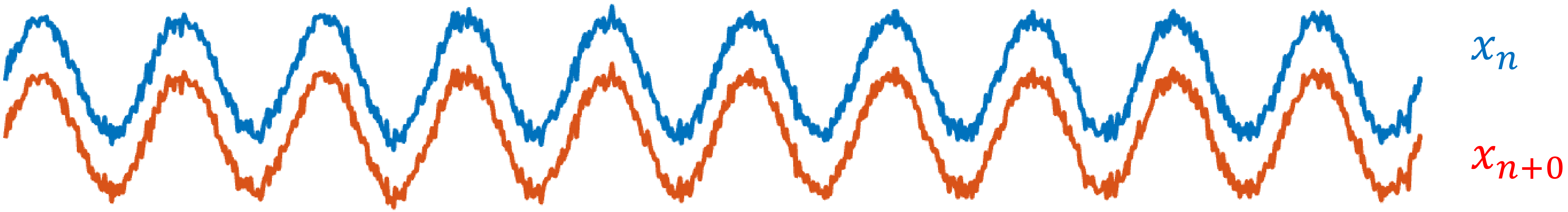


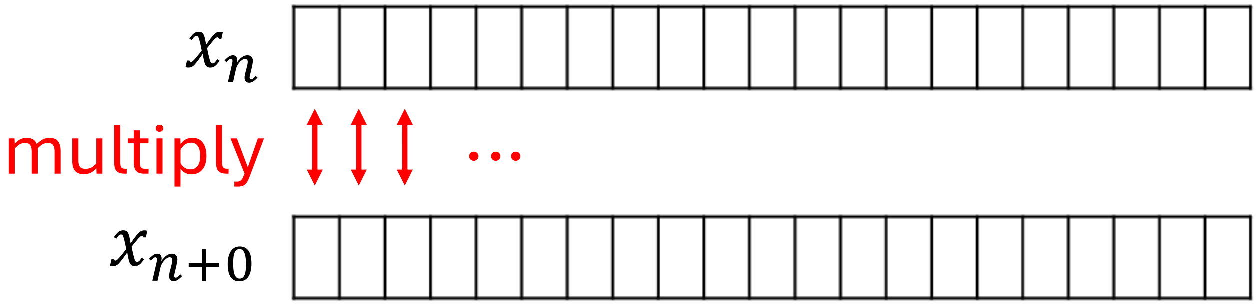
Here, x_n approximately sinusoid with period τ

assume mean = 0 (i.e., $\bar{x} = 0$)

Autocovariance (example)

Q. What is $r_{xx}[0]$?





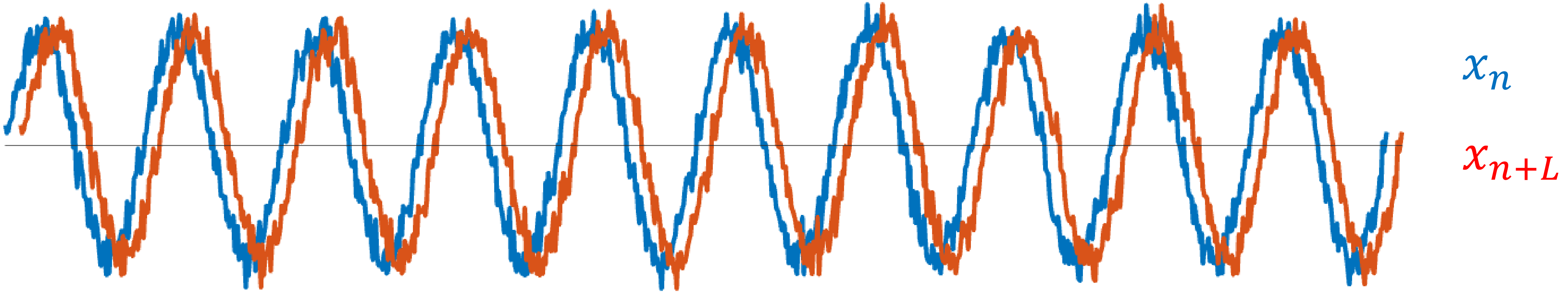
$x_n * x_n \geq 0$

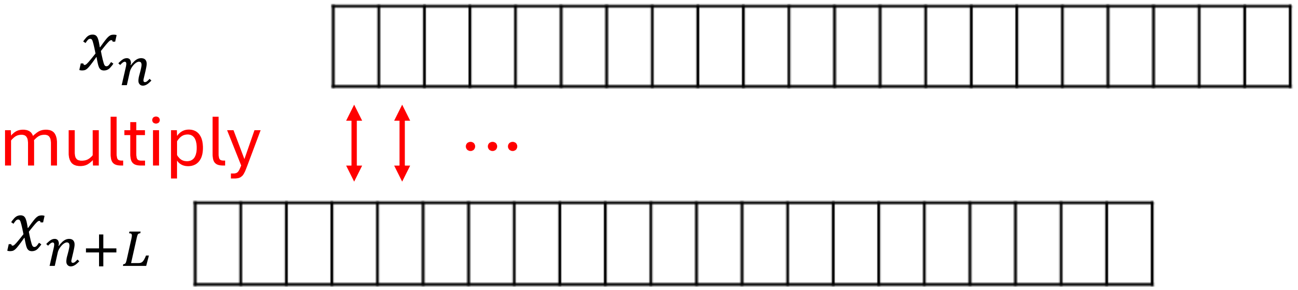
$r_{xx}[0]$~ big positive number

good match

Autocovariance (example)

Q. What is $r_{xx}[L]$ for L>0 but small ?





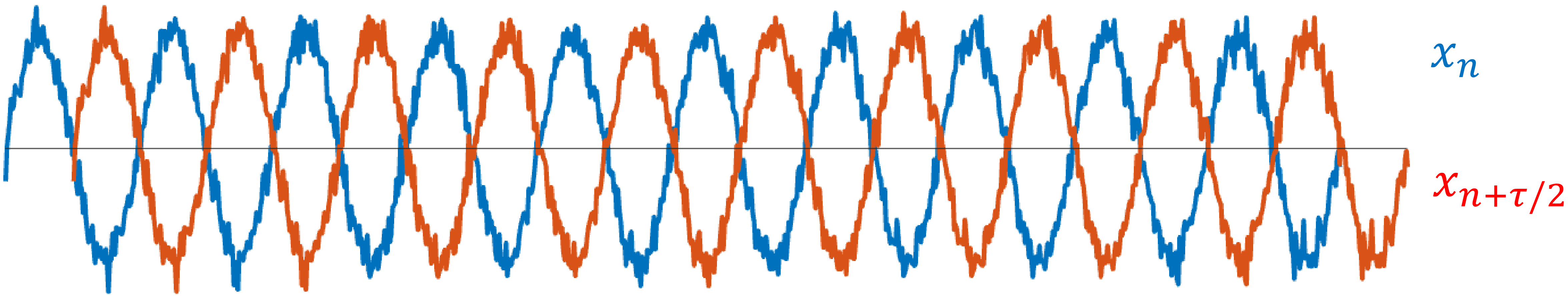
$x_n * x_{n+L} > 0$ and $x_n * x_{n+L} < 0$

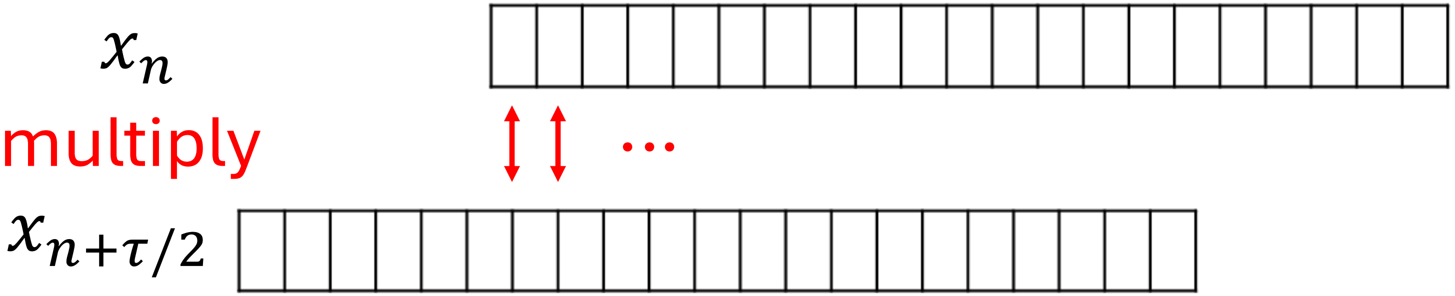
$r_{xx}[0]$~ less big positive number

not as good a match

Autocovariance (example)

Q. What is $r_{xx}[L]$ for $\mathrm{L} = \tau/2$?





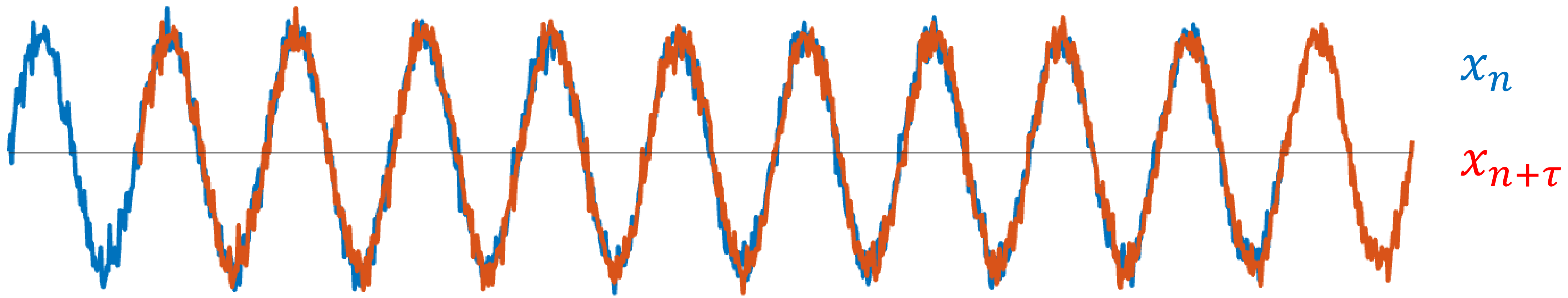
$x_n * x_{n+\tau/2} \leq 0$

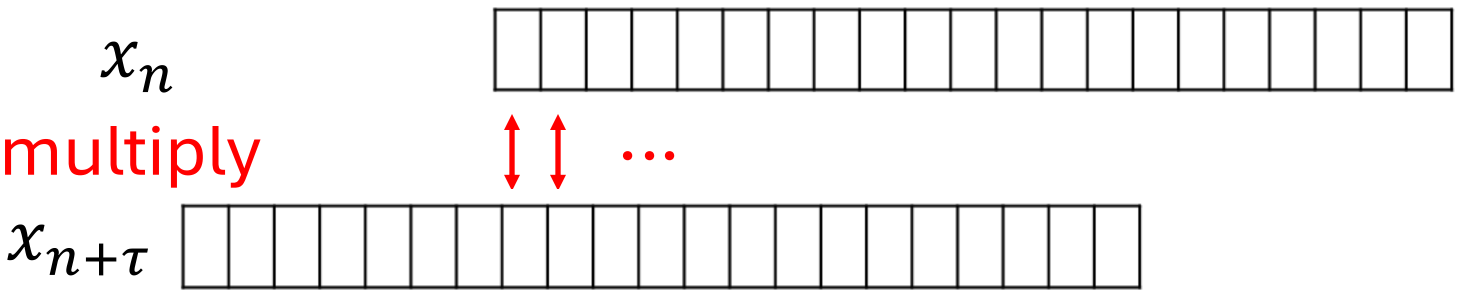
$r_{xx}[\tau/2]\sim$ big negative number

good anti-match

Autocovariance (example)

Q. What is $r_{xx}[L]$ for $\mathrm{L} = \tau$?





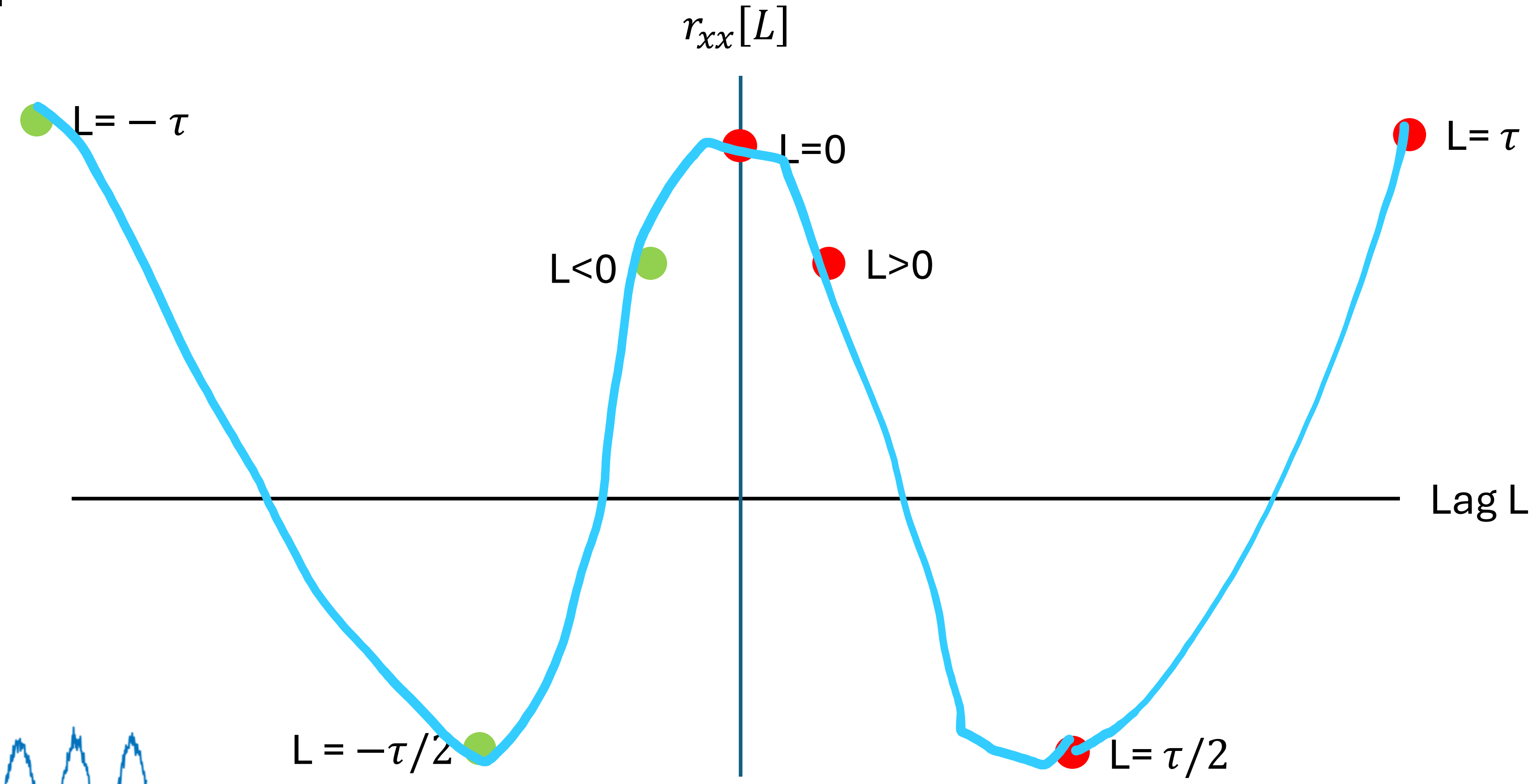
$x_n * x_{n+\tau} \geq 0$ mostly

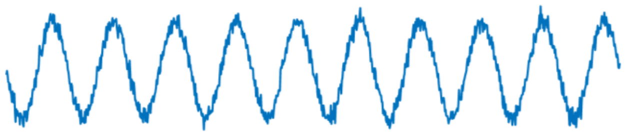
$r_{xx}[\tau]$~ big positive number

good match (again)

Autocovariance (example)

Q. Plot $r_{xx}[L]$ versus L?

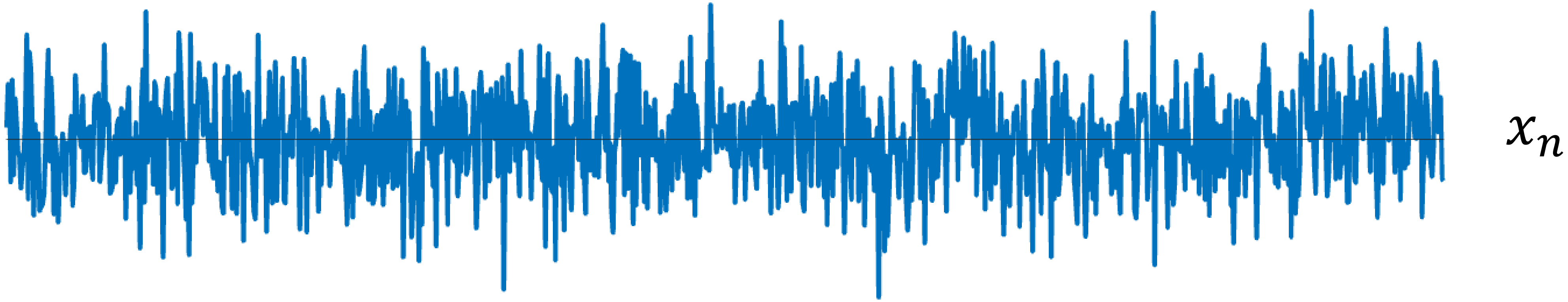




Periodic x → periodic $r_{xx}[L]$

Autocovariance (example 2)

Consider these noisy data



Here, x_n (Gaussian) random noise

assume mean = 0 (i.e., $\bar{x} = 0$)

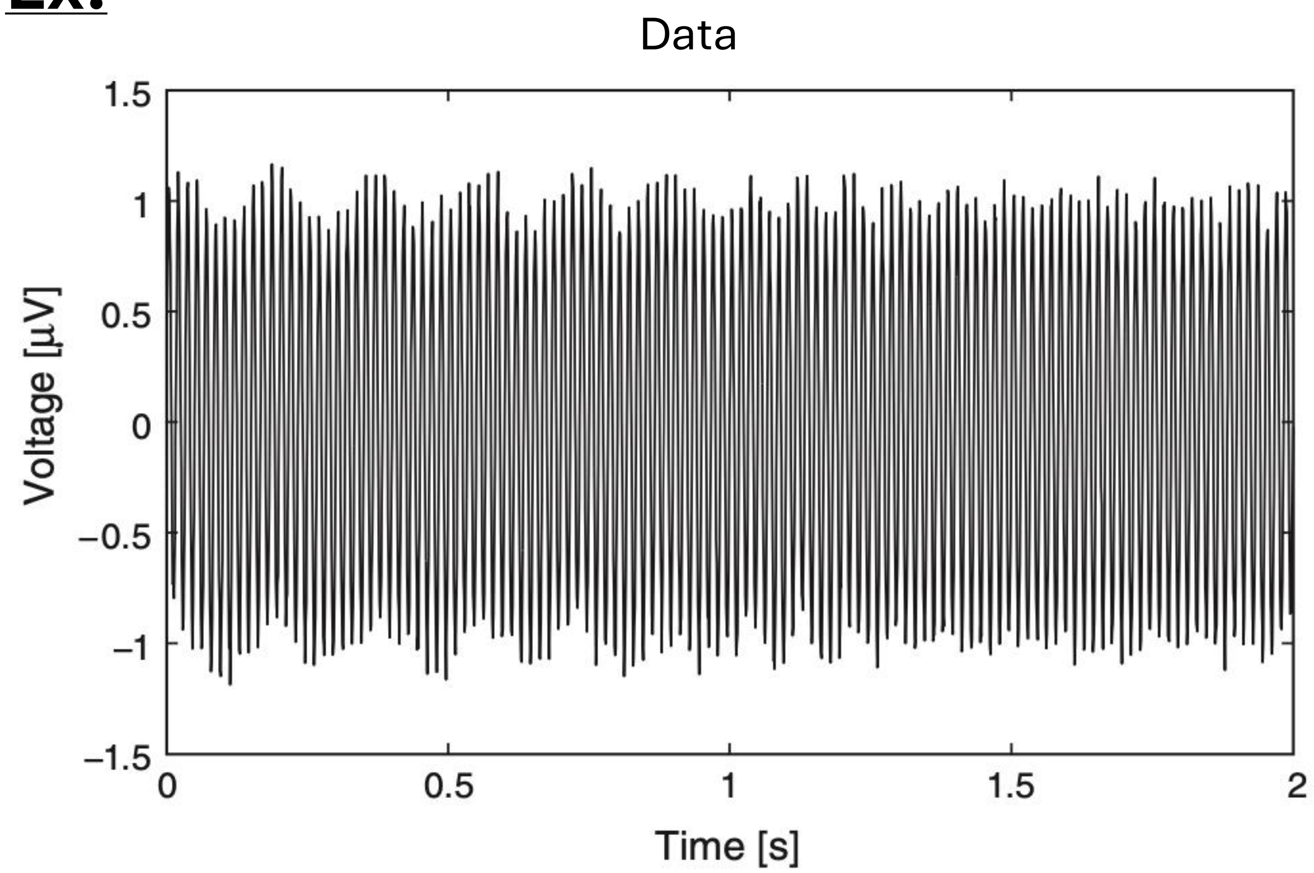
Q. What is $r_{xx}[0]$?

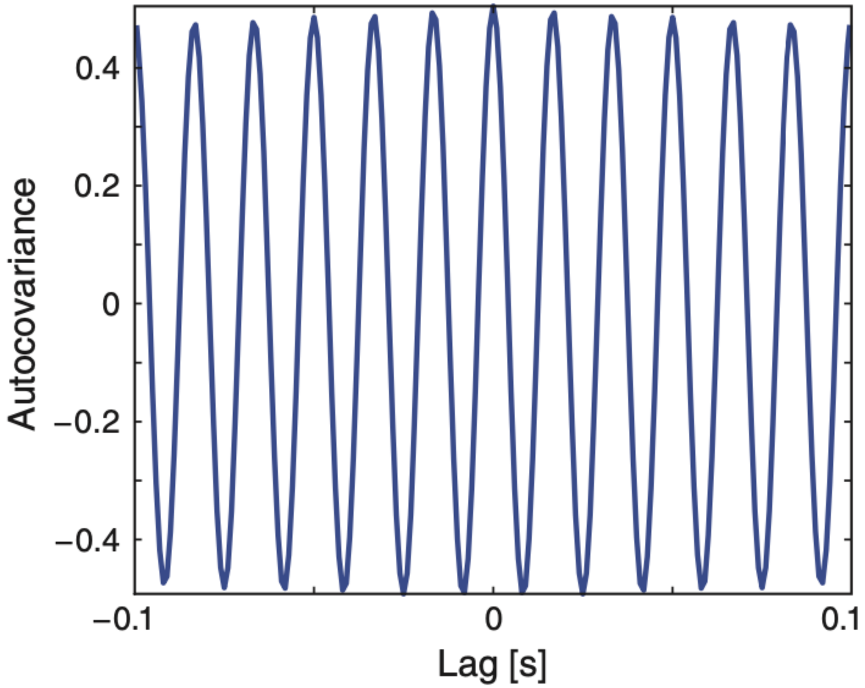
Q. What is $r_{xx}[L]$ versus L?

Python

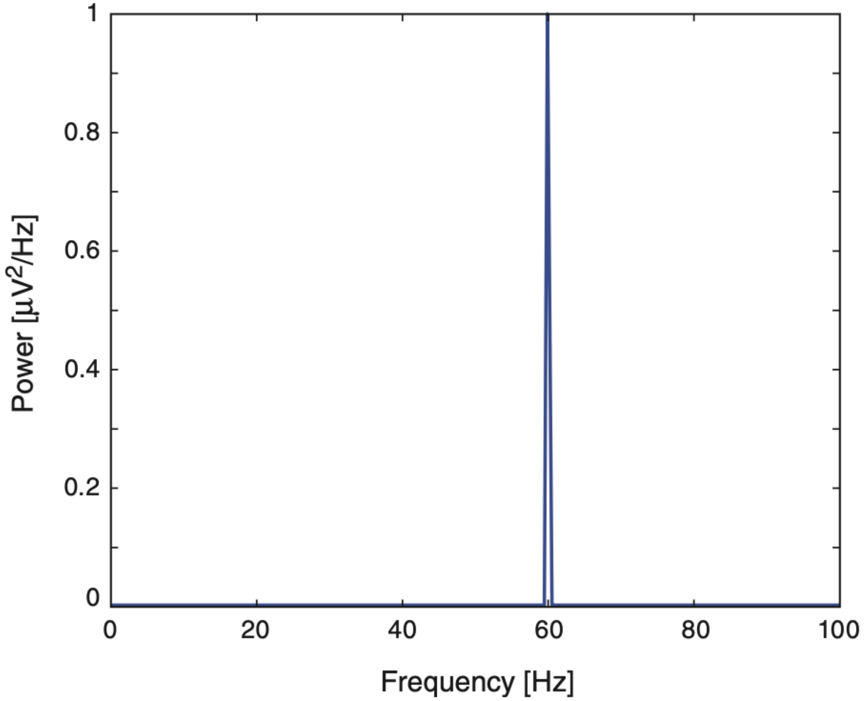
Autocovariance vs spectrum

Ex.





r_{xx}



S_{xx}

Q. Are r_{xx} and S_{xx} related?

Autocovariance vs spectrum

Remember the spectrum:

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

Fourier transform of x

where $$X_j = \sum_{n=1}^{N} x_n \exp(-2\pi i f_j t_n)$$

substitute in ...

$$S_{xx,j} = \frac{2\Delta^2}{T} \Big(\sum_n x_n \exp(-2\pi i f_j t_n) \Big) \Big(\sum_m x_m^* \exp(2\pi i f_j t_m) \Big)$$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \underbrace{\boxed{\left(\sum_n x_n \exp(-2\pi i f_j t_n)\right)}}_{X_j} \underbrace{\boxed{\left(\sum_m x_m^* \exp(2\pi i f_j t_m)\right)}}_{X_j^*}$$

New dummy time index

<u>Note</u>: replace i with $-i$

<u>Note</u>: replace x_m with x_m^*

But x_m^* is real, so $\boxed{x_m^* = x_m}$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \boxed{\left(\sum_n x_n \exp(-2\pi i f_j t_n)\right)} \boxed{\left(\sum_m x_m \exp(2\pi i f_j t_m)\right)}$$

X_j $\quad$ X_j^*

Replace

$$f_j = j * df \qquad df = \frac{1}{T} \qquad f_j = \frac{j}{T} = \frac{j}{N\Delta}$$

$$t_n = n\ \Delta \qquad t_m = m\ \Delta$$

$$S_{xx,j} = \frac{2\Delta^2}{T} \sum_n \sum_m x_n x_m \exp(-\frac{2\pi i}{N} j(n-m))$$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \sum_n \sum_m x_n x_m \exp(-\frac{2\pi i}{N} j \boxed{(n-m)})$$

l

$T = N\Delta$

$n = m + l$

sum over all n → sum over all l

Define a new variable $l = n - m$

$$S_{xx,j} = \frac{2\Delta^2}{\Delta} \sum_l \left(\frac{1}{N} \sum_m x_{m+l} x_m \right) \exp(-\frac{2\pi i}{N} j\, l)$$

Autocovariance vs spectrum

$$\boxed{r_{xx}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L} - \bar{x})(x_n - \bar{x})}$$

$$S_{xx,j} = \frac{2\Delta^2}{\Delta}\sum_{l}\underbrace{\boxed{\left(\frac{1}{N}\sum_{m} x_{m+l}\,x_m\right)}}_{r_{xx}[l]}\exp(-\frac{2\pi i}{N}j\,l)$$

Q. What is this?

$$S_{xx,j} = 2\Delta\sum_{l} r_{xx}[l]\exp(-\frac{2\pi i}{N}j\,l)$$

spectrum

autocovariance

Autocovariance vs spectrum

$$S_{xx,j} = 2\Delta \sum_l r_{xx}[l] \exp(-\frac{2\pi i}{N} j\, l)$$

a few more steps …

$$= 2\Delta \sum_l r_{xx}[l] \exp(-\frac{2\pi i}{N\boxed{\Delta}} j \boxed{\Delta} l)$$

$$= 2\Delta \sum_l r_{xx}[l] \exp(-2\pi i \frac{j}{\boxed{T}} \Delta l)$$

$T = N\Delta$

$$= 2\Delta \sum_l r_{xx}[l] \exp(-2\pi i \boxed{f_j} \boxed{t_l})$$

$f_j = \dfrac{j}{T}$ $\quad$ $t_l = l\ \Delta$

Autocovariance vs spectrum

$$S_{xx,j} = 2\Delta \underbrace{\sum_l}_{\text{sum over time}} r_{xx}[l] \underbrace{\exp(-2\pi i f_j t_l)}_{\text{complex exponentials at frequency } f_j}$$

Remember the Fourier transform of x_n

$$X_j = \underbrace{\sum_{n=1}^{N}}_{\text{sum over time}} x_n \underbrace{\exp(-2\pi i f_j t_n)}_{\text{complex exponentials at frequency } f_j} \qquad \text{or} \qquad X_j = FT\{x_n\}$$

So

$$\boxed{S_{xx,j} = 2\,\Delta\, FT\{\, r_{xx}\}}$$

The spectrum is the Fourier transform of the autocovariance.

Autocovariance vs spectrum

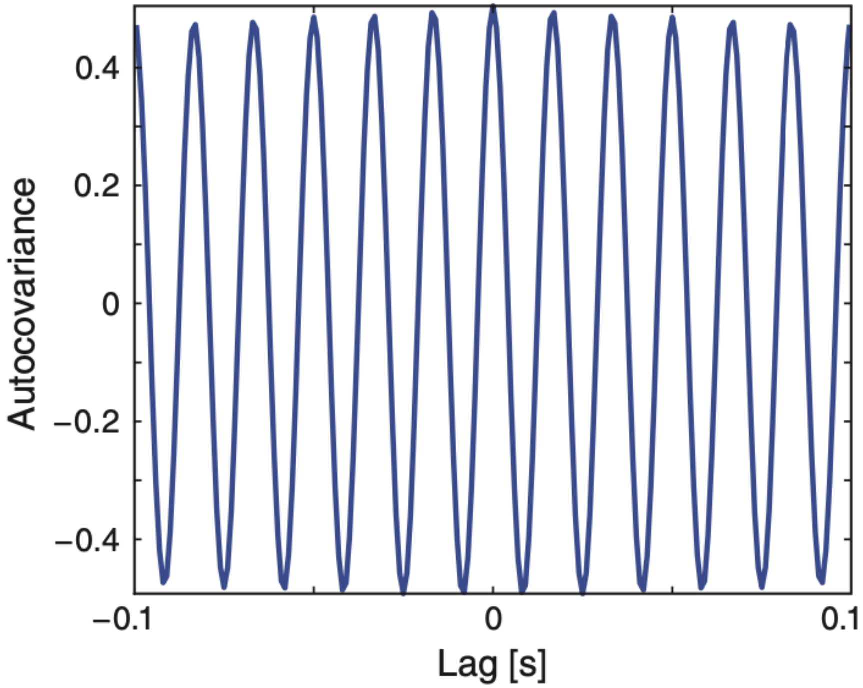
The spectrum is the Fourier transform of the autocovariance.

Autocovaraince: time-domain, lag L

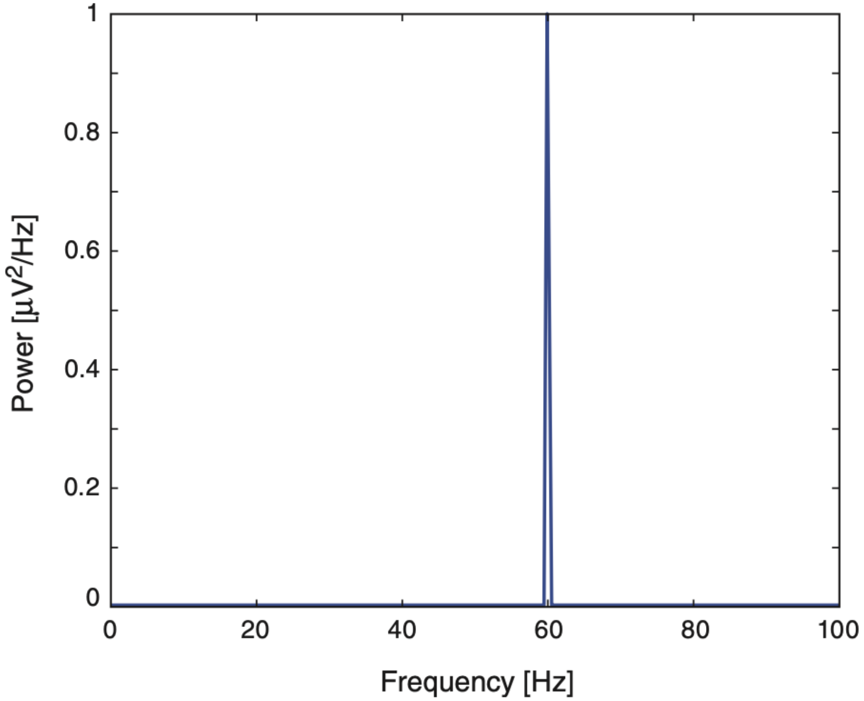
Spectrum: freq-domain measure, f_j

Different perspective on the dependent structure in the data.

In practice, consider both (sometimes)



r_{xx}



S_{xx}

Autocovariance vs spectrum

Python