

Rhythms

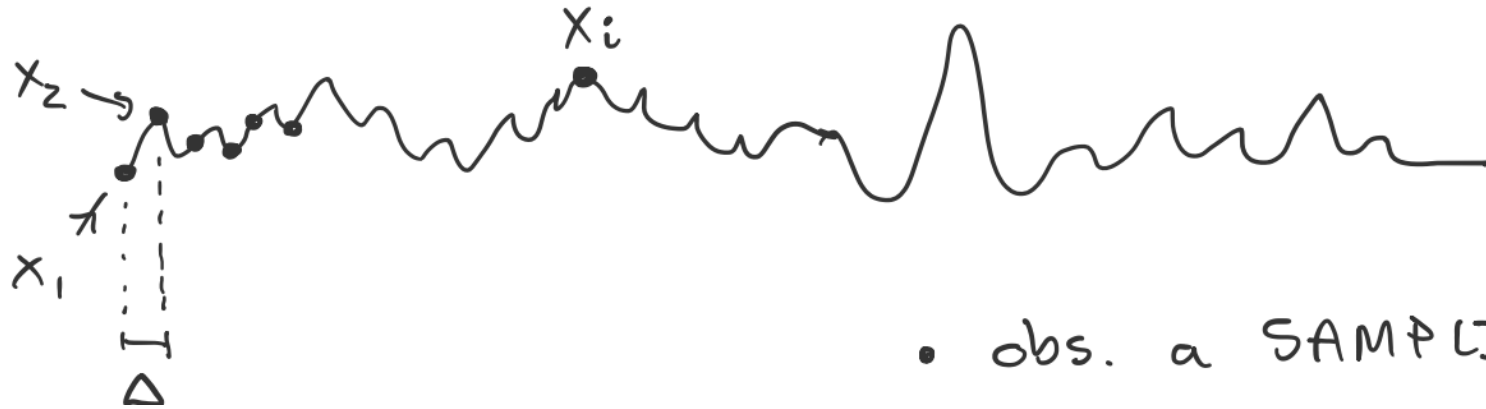
Analyzing Rhythms (Part 2)

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Today

Autocovariance

Notation



- obs. a SAMPLING.

- Δ = sampling interval (Ex. 1ms)
- $\frac{1}{\Delta}$ = sampling freq. (Ex. 1000 Hz)



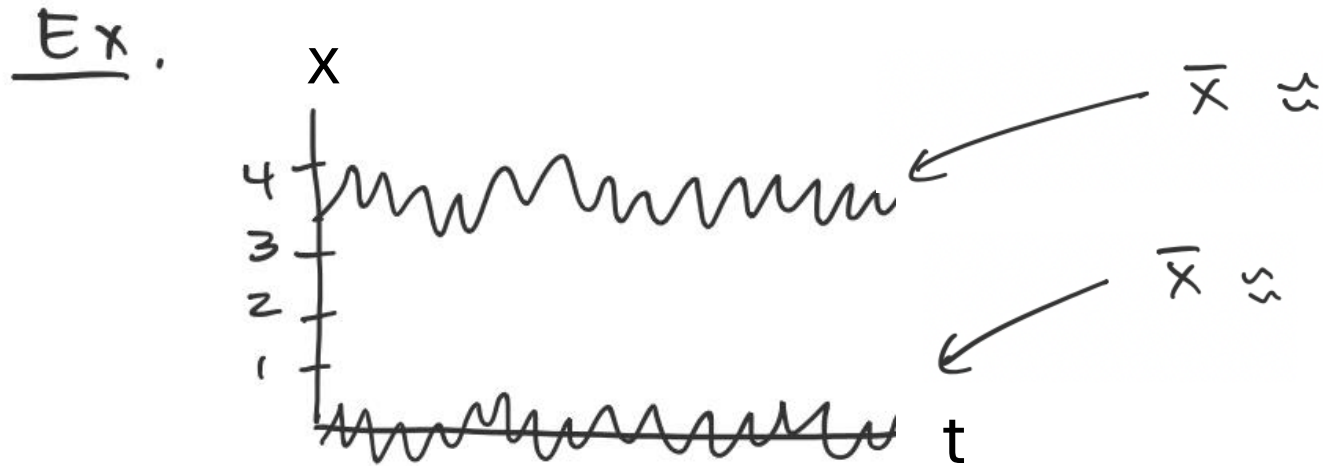
N = length of data (# of data points)

or $T = N\Delta$ = total duration of recording

Mean & variance

Define x_n data at index n

Define \bar{x} = mean of x $\bar{x} = \frac{1}{N} \sum_{n=1}^N x_n$



Mean & variance

Define x_n data at index n

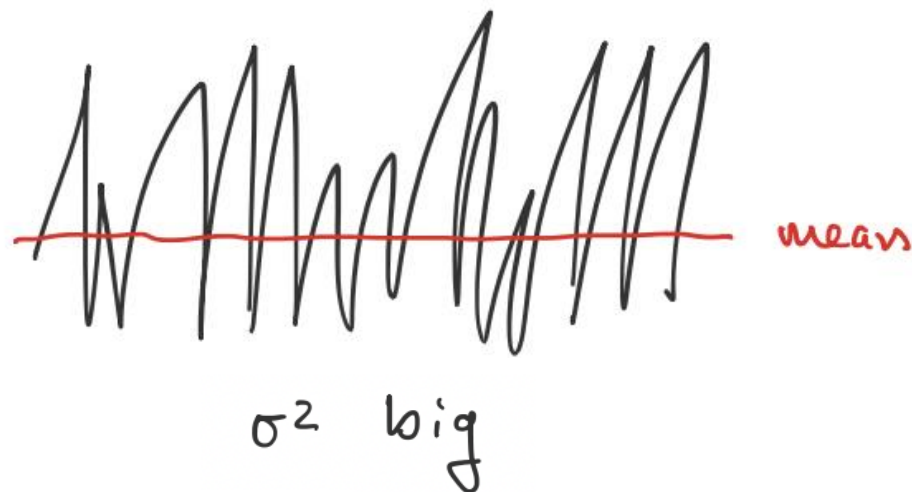
Define σ^2 = variance of x

$$\sigma^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \bar{x})^2$$

Ex.



σ^2 small



σ^2 big

Autocovariance (equation)

Define:

$$r_{xx}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(x_n - \bar{x})$$

The autocovariance
of x at lag L

number
of data
points

sum over
all data
points

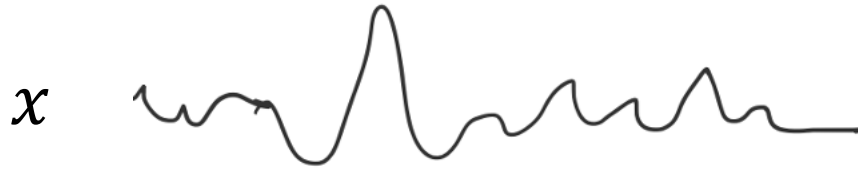
the data
with mean
subtracted
at index $n + L$

... multiplied
by itself
at index n

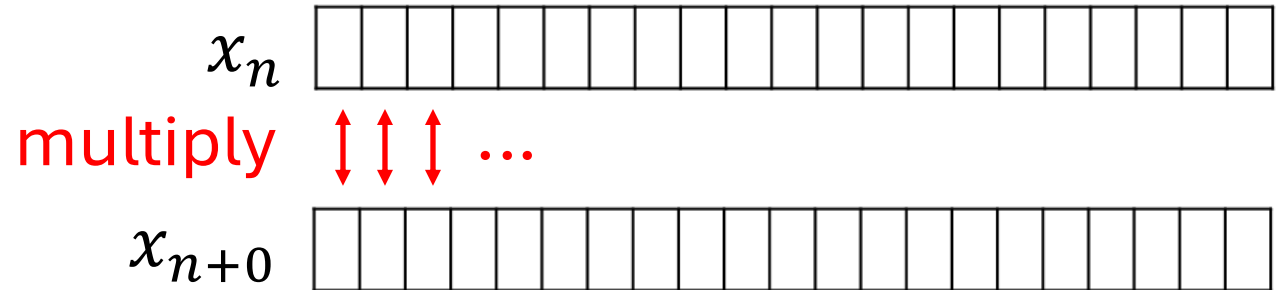
Autocovariance (intuition)

$$r_{xx}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(x_n - \bar{x})$$

Fix **L=0**



assume mean = 0 (i.e., $\bar{x} = 0$)



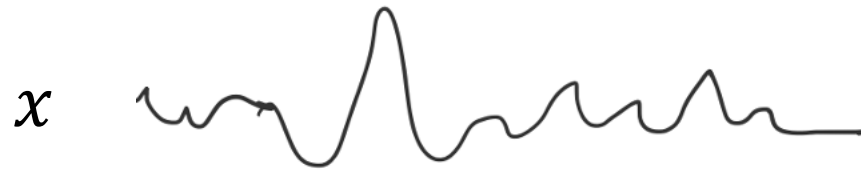
- at each index, multiply the two signals \rightarrow get a number
- sum these numbers to get $r_{xx}[0]$

Q. Will this number be big or small?

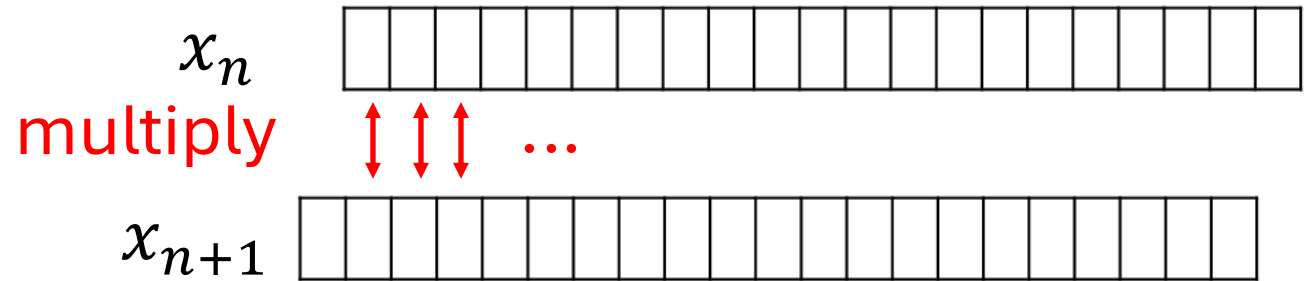
Autocovariance (intuition)

$$r_{xx}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(x_n - \bar{x})$$

Fix **L=1**



assume mean = 0 (i.e., $\bar{x} = 0$)



- at each index, multiply the two signals \rightarrow get a number
- sum these numbers to get $r_{xx}[1]$

$x_0 \ x_1$

$x_1 \ x_2$

$x_2 \ x_3$

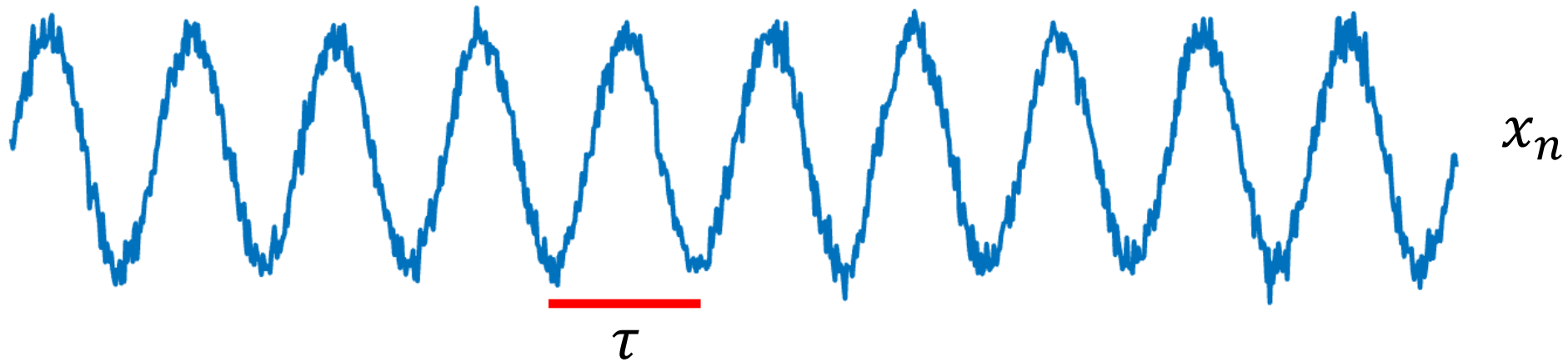
...

Q. Will this number be big or small?

Repeat for $L = 2, 3, \dots, L = -1, -2, \dots$ to get $r_{xx}[L]$

Autocovariance (example)

Ex. Consider these nearly sinusoidal data

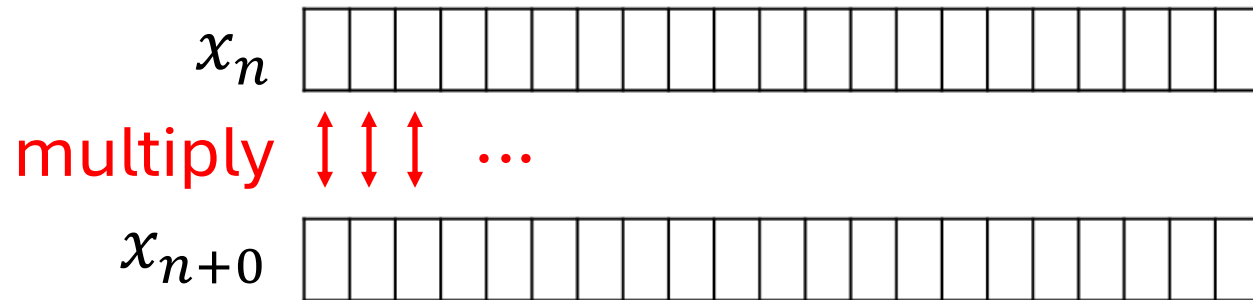
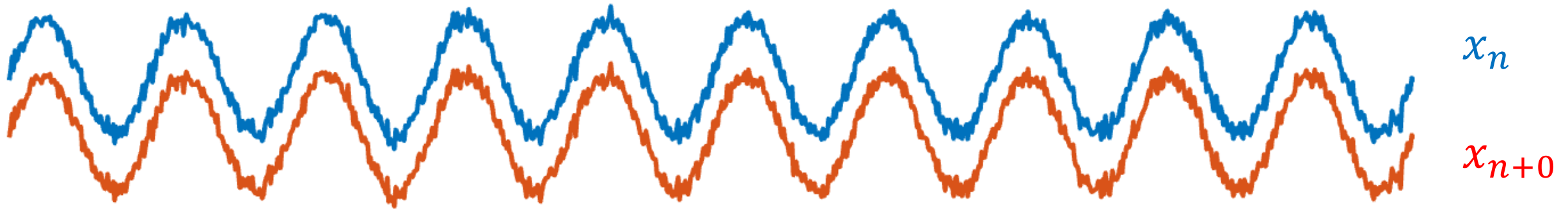


Here, x_i approximately sinusoid with period τ

assume mean = 0 (i.e., $\bar{x} = 0$)

Autocovariance (example)

Q. What is $r_{xx}[0]$?

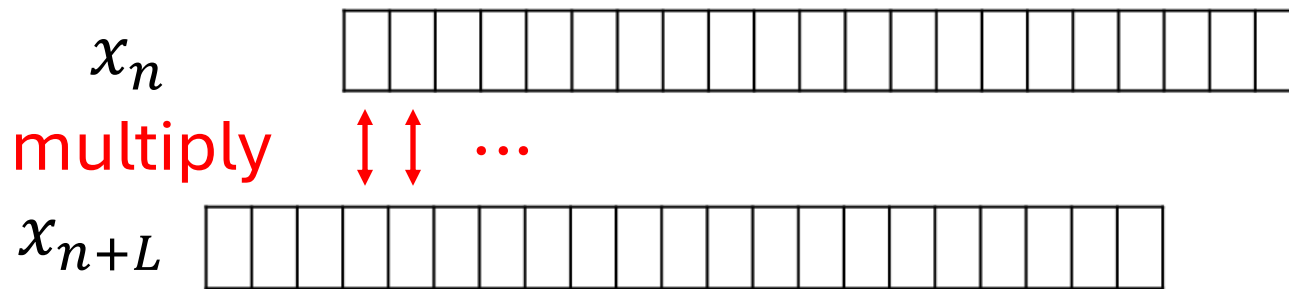
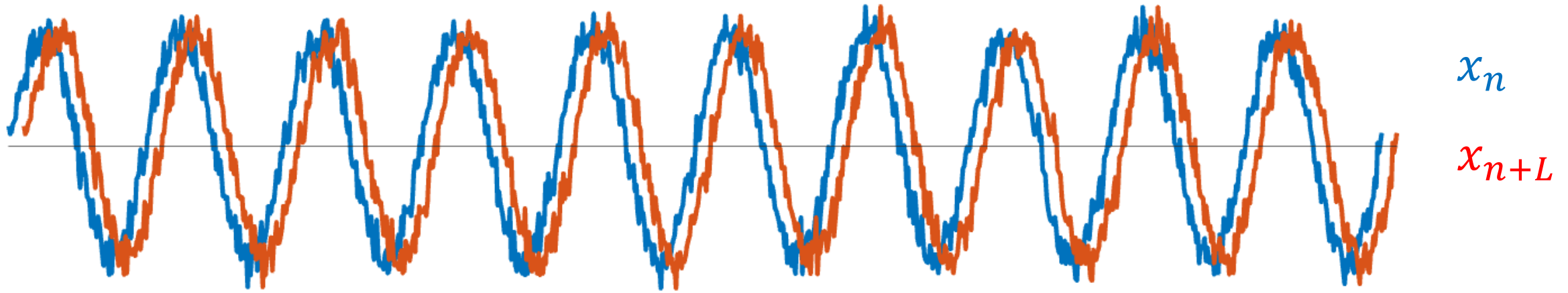


$$x_n * x_n \geq 0$$

$r_{xx}[0] \sim$ big positive number
good match

Autocovariance (example)

Q. What is $r_{xx}[L]$ for $L > 0$ but small ?



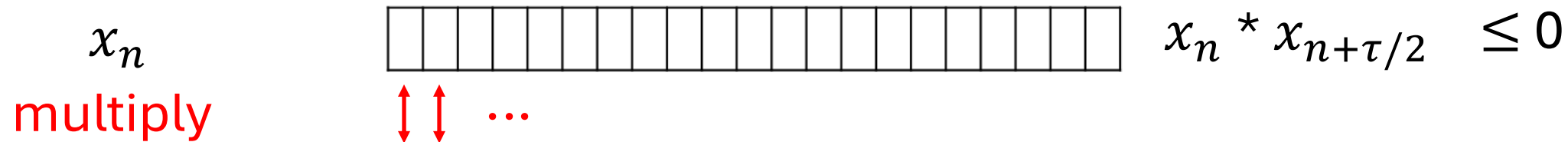
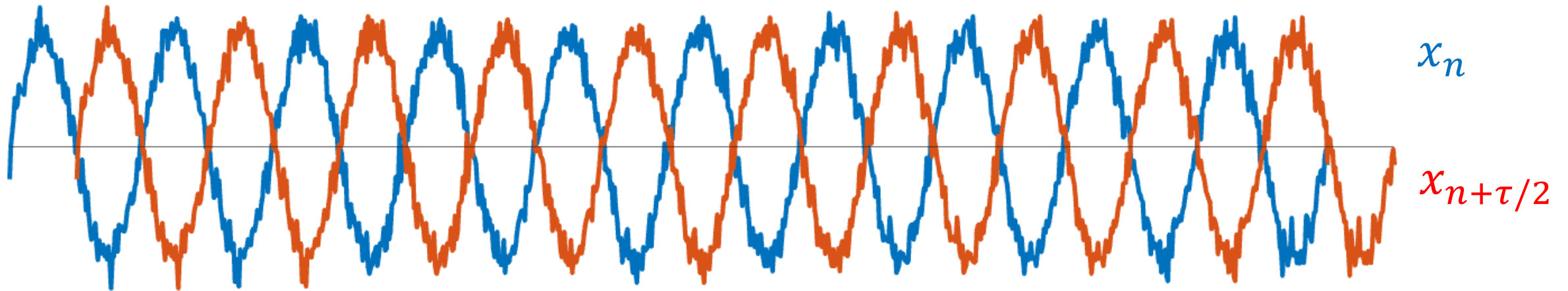
$$x_n * x_{n+L} > 0 \text{ and } x_n * x_{n+L} < 0$$

$r_{xx}[0] \sim$ less big positive number

not as good a match

Autocovariance (example)

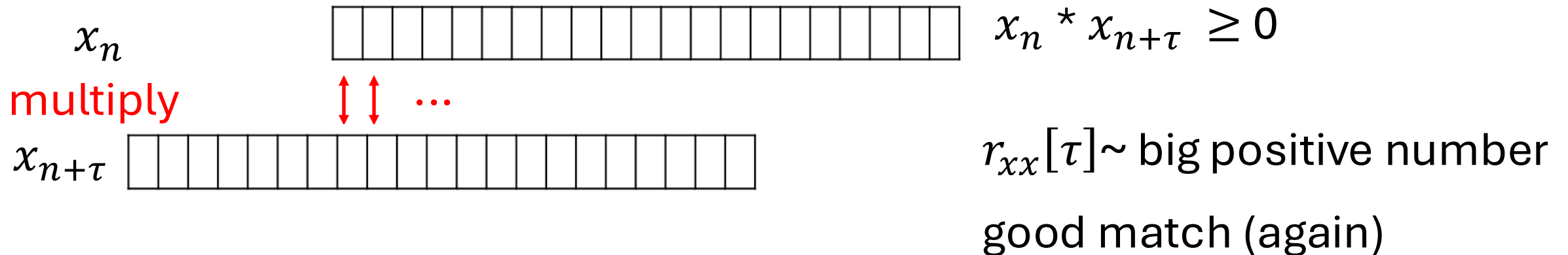
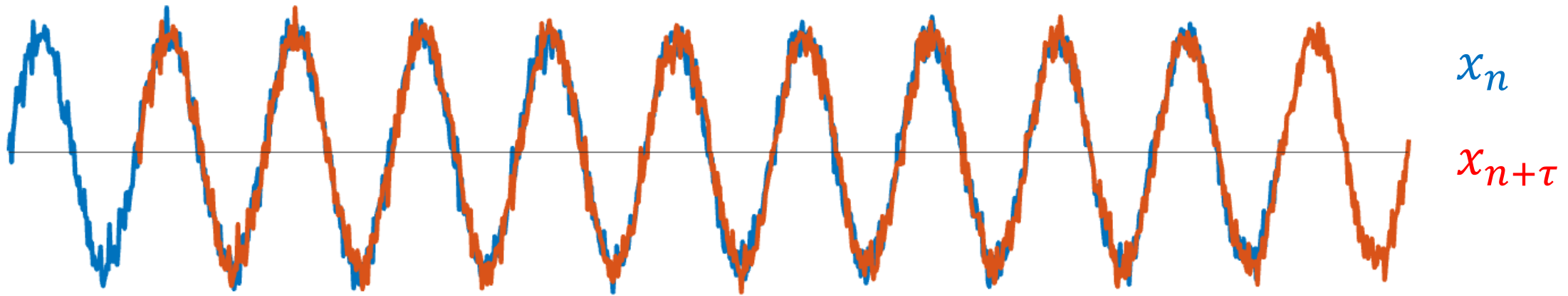
Q. What is $r_{xx}[L]$ for $L = \tau/2$?



$r_{xx}[\tau/2] \sim$ big negative number
good anti-match

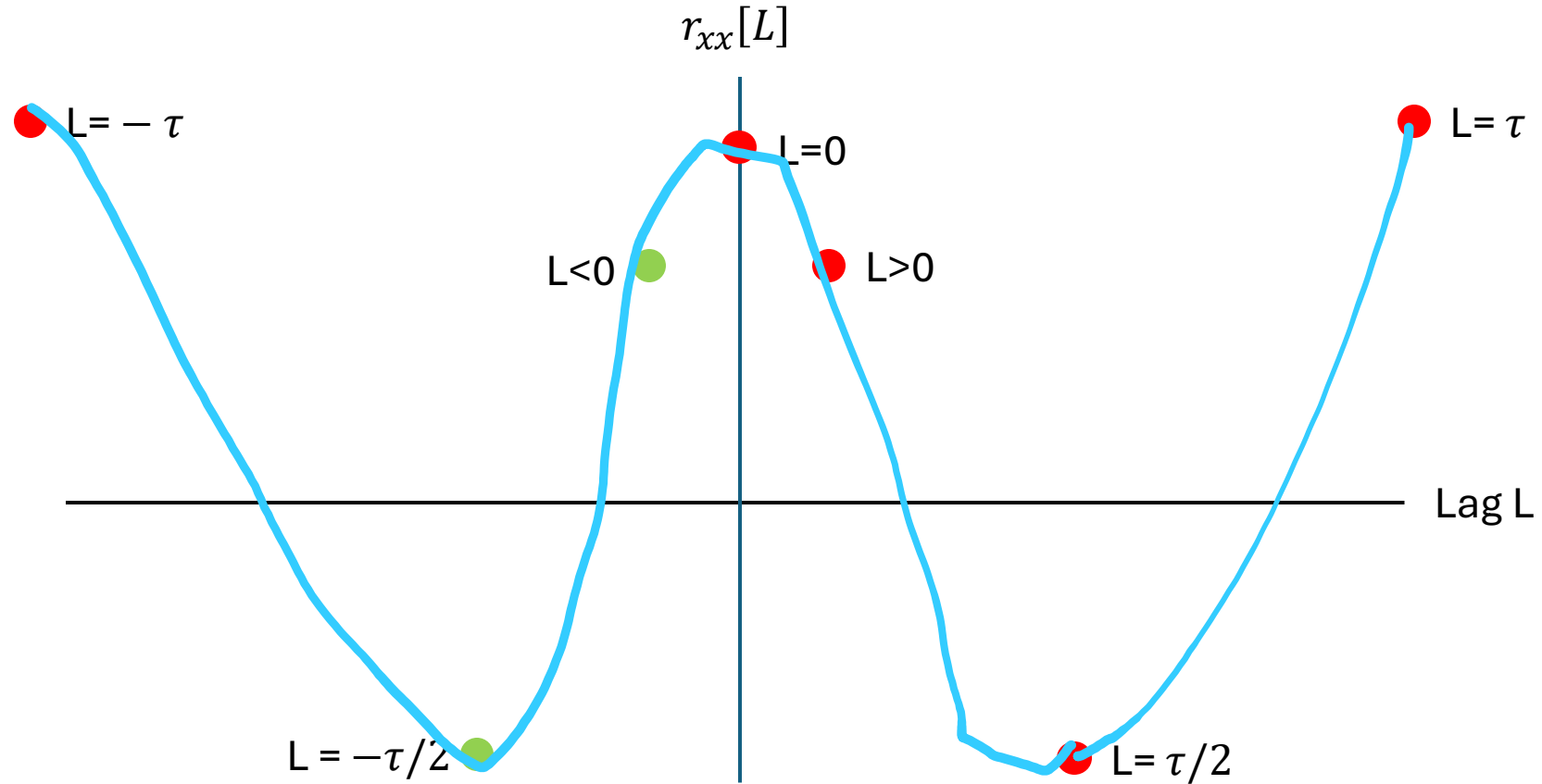
Autocovariance (example)

Q. What is $r_{xx}[L]$ for $L = \tau$?



Autocovariance (example)

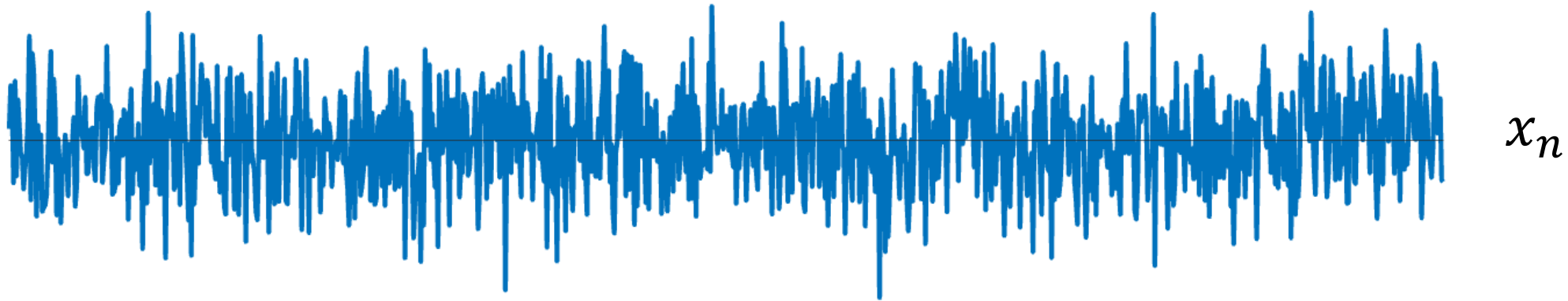
Q. Plot $r_{xx}[L]$ versus L ?



Periodic $x \rightarrow$ periodic $r_{xx}[L]$

Autocovariance (example 2)

Consider these noisy data



Here, x_n (Gaussian) random noise

assume mean = 0 (i.e., $\bar{x} = 0$)

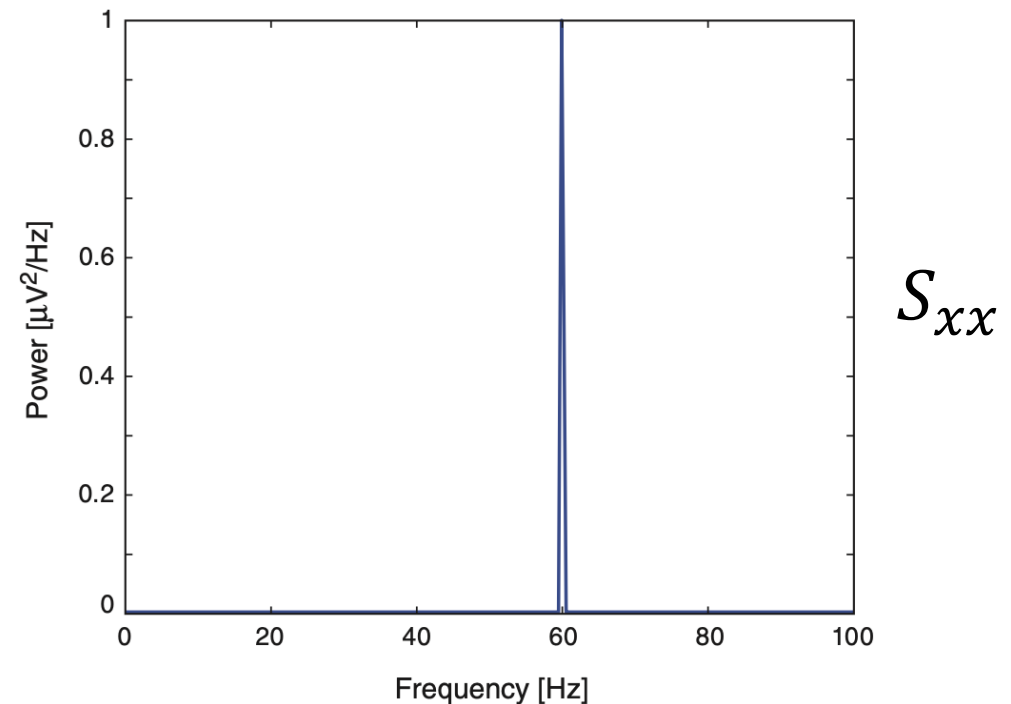
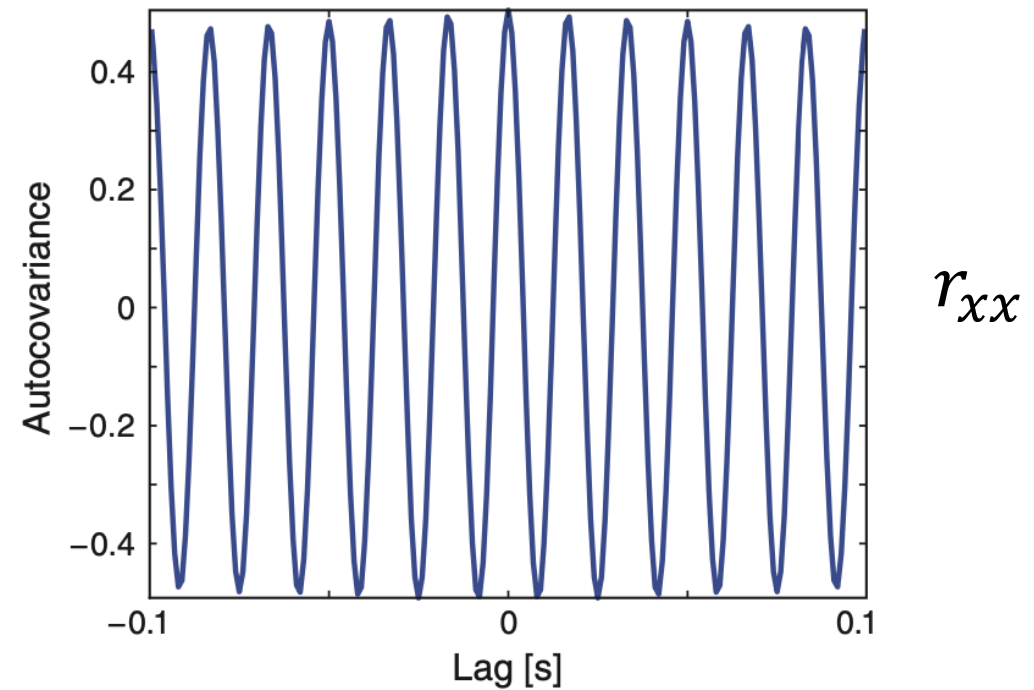
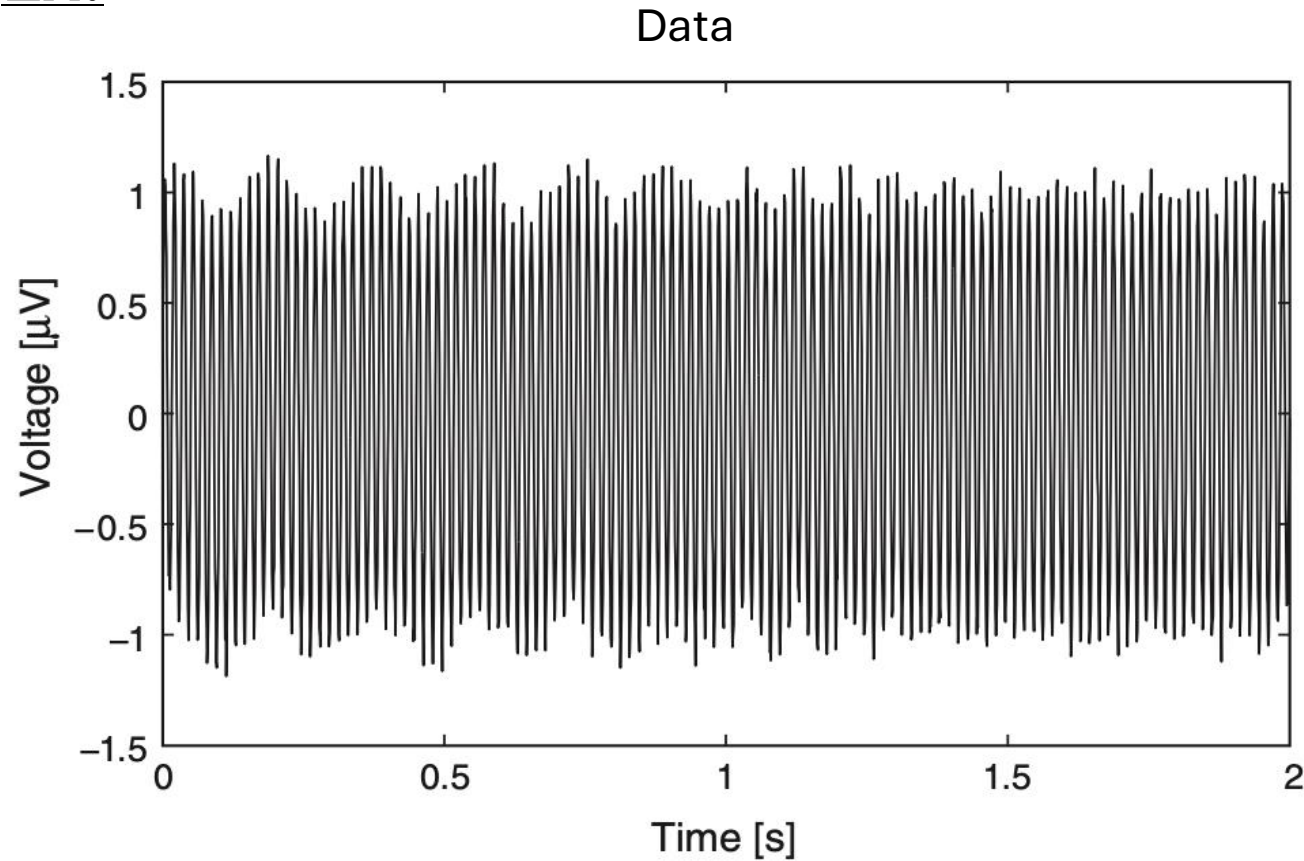
Q. What is $r_{xx}[0]$?

Q. What is $r_{xx}[L]$ versus L ?

Python

Autocovariance vs spectrum

Ex.



Q. Are r_{xx} and S_{xx} related?

Autocovariance vs spectrum

Remember the spectrum:

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$



where $X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n)$ Fourier transform of x

substitute in ...

$$S_{xx,j} = \frac{2\Delta^2}{T} \left(\sum_n x_n \exp(-2\pi i f_j t_n) \right) \left(\sum_m x_m^* \exp(2\pi i f_j t_m) \right)$$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \left(\sum_n x_n \exp(-2\pi i f_j t_n) \right) \left(\sum_m x_m^* \exp(2\pi i f_j t_m) \right)$$

X_j



X_j^*

New dummy time index

Note: replace i with $-i$

Note: replace x_m with x_m^*

But x_m^* is real, so $x_m^* = x_m$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \left(\sum_n x_n \exp(-2\pi i f_j t_n) \right) \left(\sum_m x_m \exp(2\pi i f_j t_m) \right)$$

Replace

X_j

X_j^*


$$f_j = j * df \quad df = \frac{1}{T} \quad f_j = \frac{j}{T}$$



$$t_n = n \Delta \quad t_m = m \Delta$$

$$S_{xx,j} = \frac{2\Delta^2}{T} \sum_n \sum_m x_n x_m \exp\left(-\frac{2\pi i}{N} j(n - m)\right)$$

Autocovariance vs spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} \sum_n \sum_m x_n x_m \exp\left(-\frac{2\pi i}{N} j \boxed{n - m}\right)$$

$T = N\Delta$ 

  $n = m + l$

sum over all $n \rightarrow$ sum over all l

Define a new variable $l = n - m$

$$S_{xx,j} = \frac{2\Delta^2}{\Delta} \sum_l \left(\frac{1}{N} \sum_m x_{m+l} x_m \right) \exp\left(-\frac{2\pi i}{N} j l\right)$$

Autocovariance vs spectrum

$$r_{xx}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(x_n - \bar{x})$$

$$S_{xx,j} = \frac{2\Delta^2}{\Delta} \sum_l \left(\frac{1}{N} \sum_m x_{m+l} x_m \right) \exp\left(-\frac{2\pi i}{N} j l\right)$$

Q. What is this? $r_{xx}[l]$

$$S_{xx,j} = 2\Delta \sum_l r_{xx}[l] \exp\left(-\frac{2\pi i}{N} j l\right)$$

spectrum autocovariance

Autocovariance vs spectrum

$$\begin{aligned} S_{xx,j} &= 2\Delta \sum_l r_{xx}[l] \exp\left(-\frac{2\pi i}{N} j l\right) \\ &= 2\Delta \sum_l r_{xx}[l] \exp\left(-\frac{2\pi i}{N\Delta} j \Delta l\right) \\ &= 2\Delta \sum_l r_{xx}[l] \exp\left(-2\pi i \frac{j}{T} \Delta l\right) \\ &= 2\Delta \sum_l r_{xx}[l] \exp\left(-2\pi i f_j t_l\right) \end{aligned}$$

a few more steps ...

$$T = N\Delta$$

$$f_j = \frac{j}{T}$$

Autocovariance vs spectrum

$$S_{xx,j} = 2\Delta \sum_l r_{xx}[l] \exp(-2\pi i f_j t_l)$$

sum over time complex exponentials at frequency f_j

Remember the Fourier transform of x_n

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n) \quad \text{or} \quad X_j = FT\{x_n\}$$

sum over time complex exponentials at frequency f_j

So

$$S_{xx,j} = 2 \Delta FT\{r_{xx}\}$$

The spectrum is the Fourier transform of the autocovariance.

Autocovariance vs spectrum

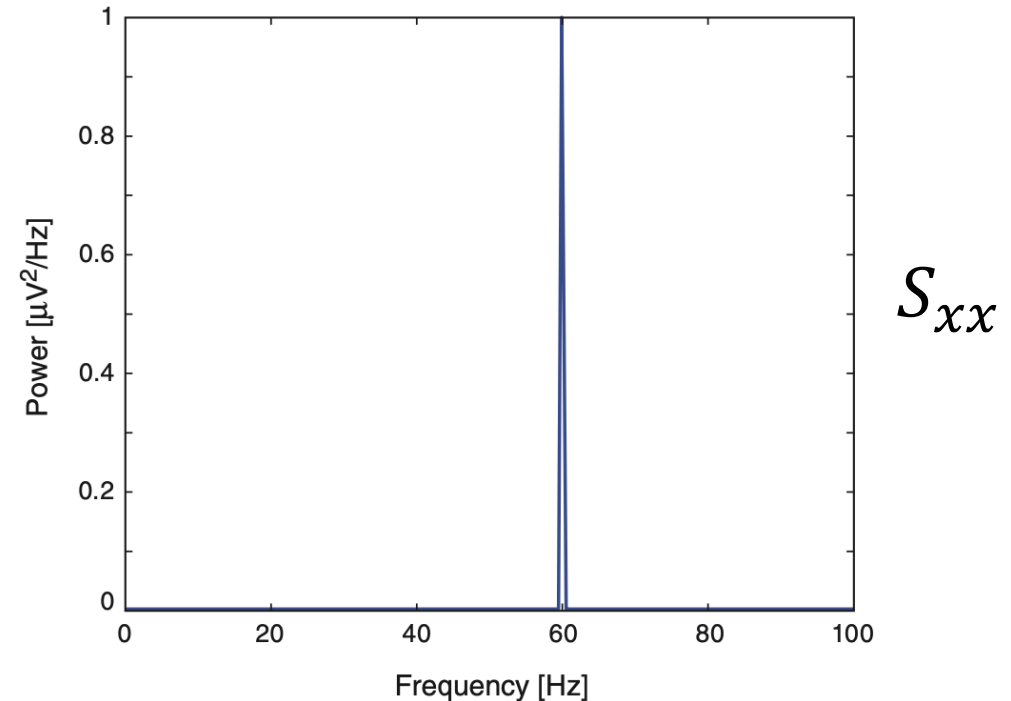
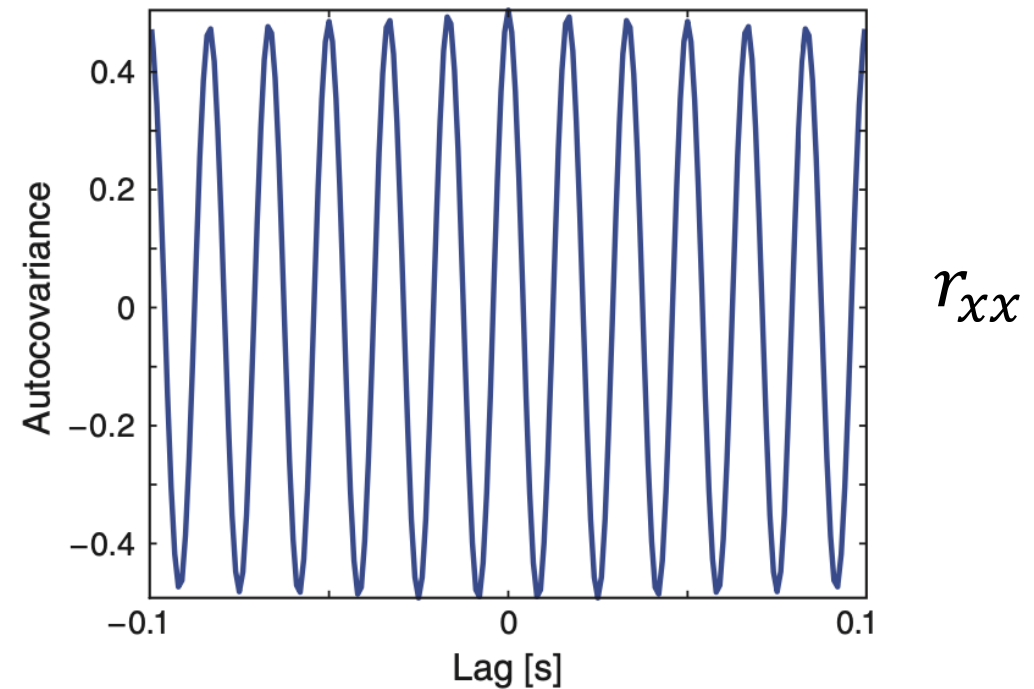
The spectrum is the Fourier transform of the autocovariance.

Autocovariance: time-domain, lag L

Spectrum: freq-domain measure, f_j

Different perspective on the dependent structure in the data.

In practice, consider both (sometimes)



Autocovariance vs spectrum

Python