Coherence
(and cross-covariance)

Computing the coherence (Part 1)

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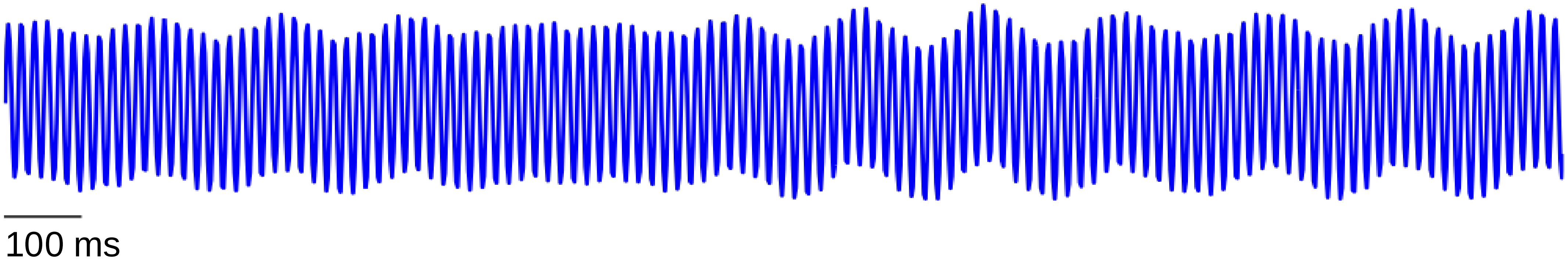
Today

Review/reminder

Coherence

Rhythms

Consider these (experimental data)

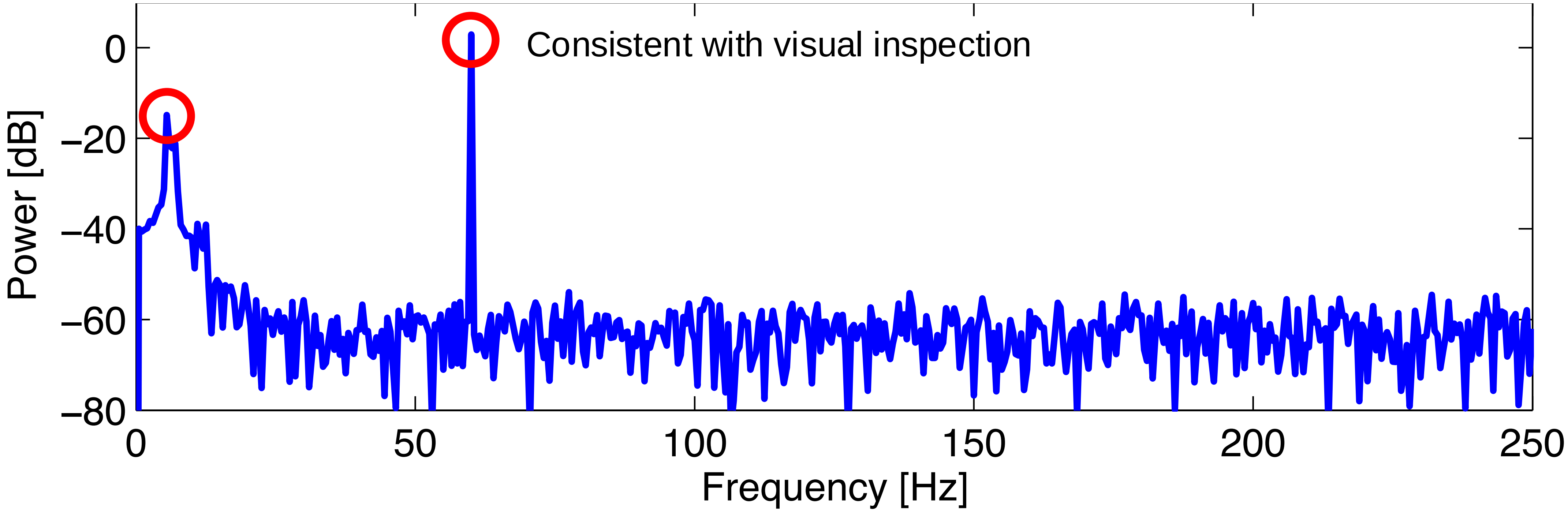


Visual inspection

- Rhythmic (dominant fast rhythm)
- It’s complicated
- Beyond visual inspection . . . quantitative characterization?

Spectrum

Beyond visual inspection . . .



<u>Axes</u>: Power [dB] vs Frequency [Hz]

A simpler representation in frequency domain. Two peaks at ~5-8 Hz, 60 Hz

An improved understanding of rhythmic activity.

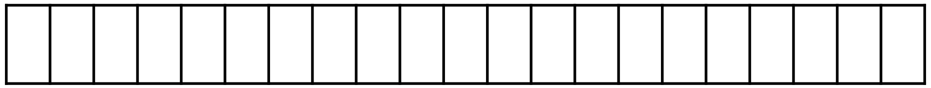
How?

Spectrum

Here, x is activity recorded from a single trial:

$$S_{xx,j} = \frac{2\Delta^2}{T} \boxed{X_j} X_j^*$$

x (LFP, EEG, ...)



Note: Time is discrete

x_n = Data at index n

(Auto-)spectrum of signal *x*

complex conjugate

Δ = sampling interval

T = total time of observation

X_j = Fourier transform of the data (*x*) at frequency *j*

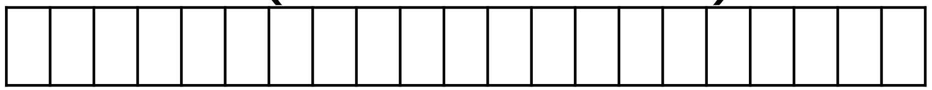
Spectrum

$$i = \sqrt{-1}$$

$$X_j = \sum_{n=1}^{N} x_n \exp(-2\pi i f_j t_n)\,.$$

Fourier transform of the data *x*.

x (LFP, EEG, ...)



x_n = Data at index n

t_n = Time at index n $\quad t_n = \Delta n \quad$ where Δ = sampling interval

f_j = Frequency at index j $\quad f_j = j/T \quad$ where T = total time of observation

Spectrum

Two important quantities

Frequency resolution $$df = \frac{1}{T}$$

Nyquist frequency $$f_{\mathrm{NQ}} = \frac{f_0}{2}$$

Q. We record data at 250 Hz. What is the highest frequency we can observe? What is the frequency resolution?

Coherence: words

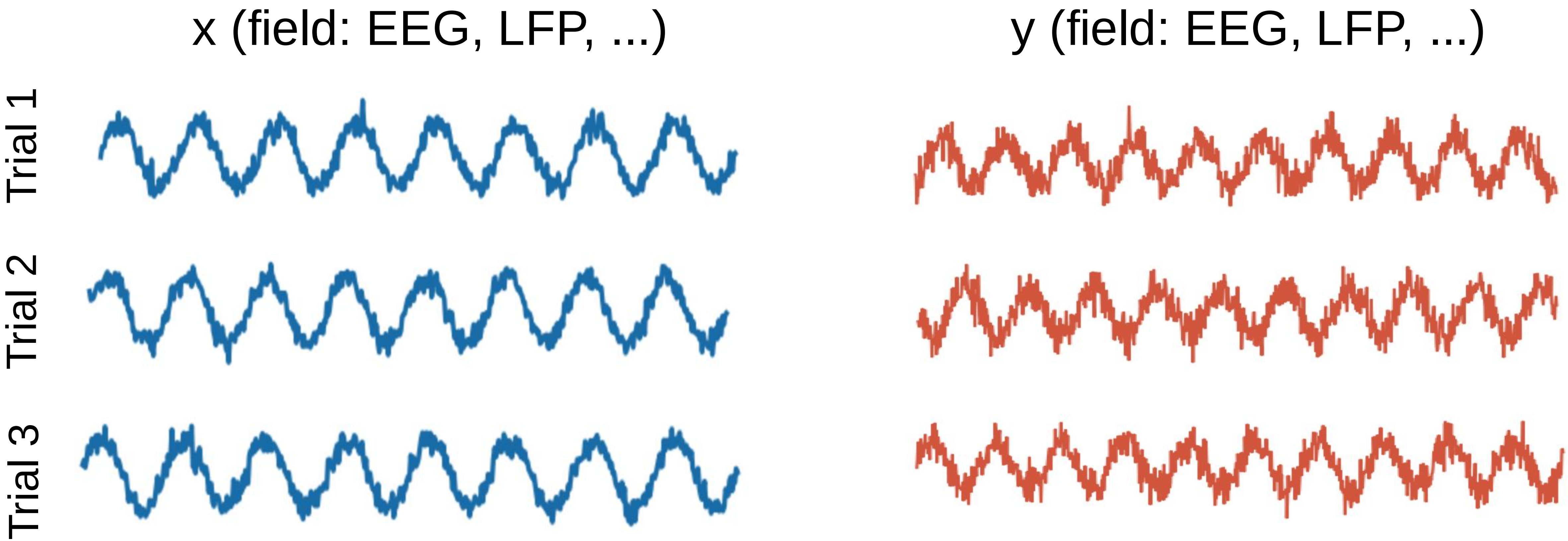
A constant phase relationship between two signals, at the same frequency, across trials.

Note

- “*same frequency*”
- “*across trials*”

Coherence: idea

Ex: Record data simultaneously from two sensors, across multiple trials



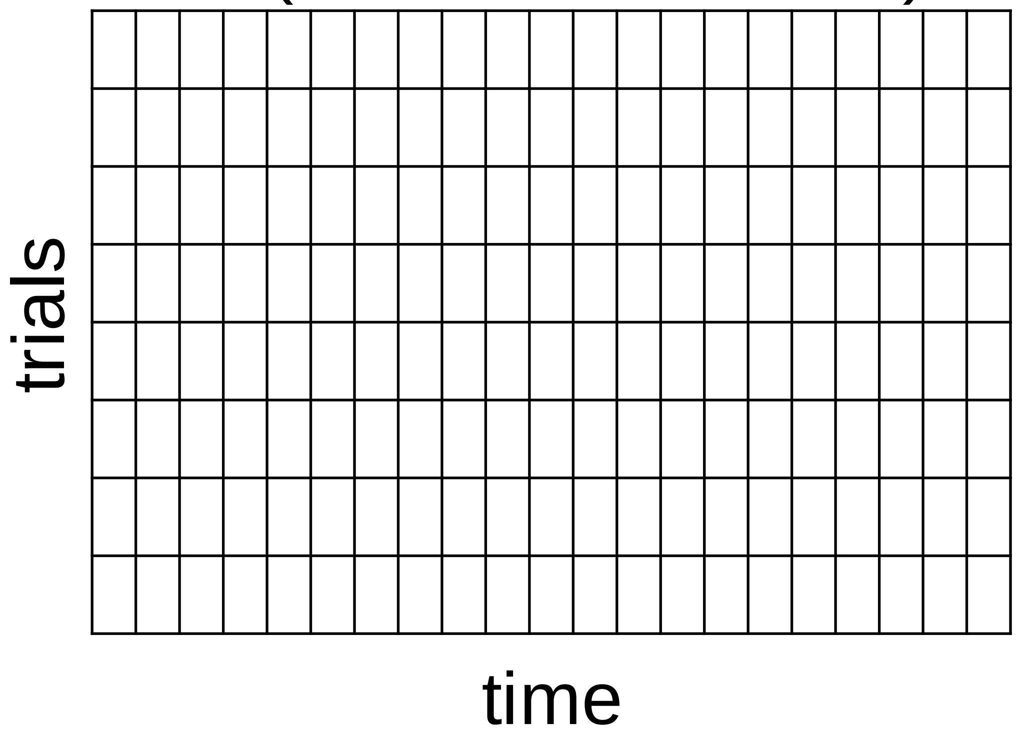
Is there a *constant phase relationship between x & y, at the same f, across trials?*

Coherence: idea

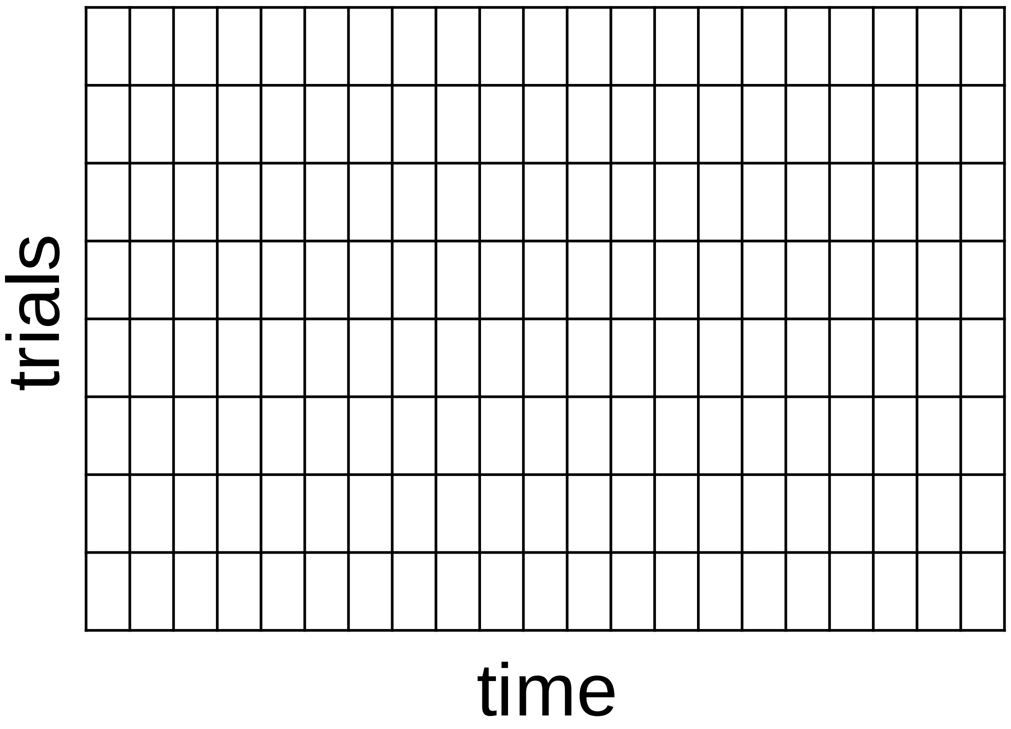
Ex: Record data simultaneously from two sensors, across multiple trials

Organize the data ...

x (field: EEG, LFP, ...)



y (field: EEG, LFP, ...)



Each row is a trial, each column is a time point, organize data in matrices.

Coherence: equations

This is what we'll compute:

$$\kappa_{xy,\,j} = \frac{|< S_{xy,\,j} >|}{\sqrt{< \boxed{S_{xx,\,j}} >}\sqrt{< S_{yy,\,j} >}}$$

$S_{xy,j}$ = Cross-spectrum at frequency index j

$S_{xx,j}, S_{yy,j}$ = Auto-spectra at frequency index j

$< S >$ = Average of S over trials

Define each piece ...

Spectrum: intuition

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

More spectrum intuition ...

Fourier transform of the data *x*.

Data as a function of time index n

$$X_j = \sum_{n=1}^{N} \boxed{x_n} \boxed{\exp(-2\pi i f_j t_n)} .$$

Replace with Euler's formula

Sinusoids at frequency f_j

$$X_j = \left(\sum_{n=1}^{N} x_n \cos(-2\pi f_j t_n) \right) + i \left(\sum_{n=1}^{N} x_n \sin(-2\pi f_j t_n) \right)$$

Real

Imaginary

Spectrum: intuition

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

More spectrum intuition ...

Fourier transform of the data *x*.

$$X_j = \left(\sum_{n=1}^{N} x_n \cos(-2\pi f_j t_n)\right) + i\left(\sum_{n=1}^{N} x_n \sin(-2\pi f_j t_n)\right)$$

X_j can be **complex**

- the Fourier transform of x_n can have both real and imaginary parts.

So, X_j lives in the complex-plane ...

Spectrum: intuition

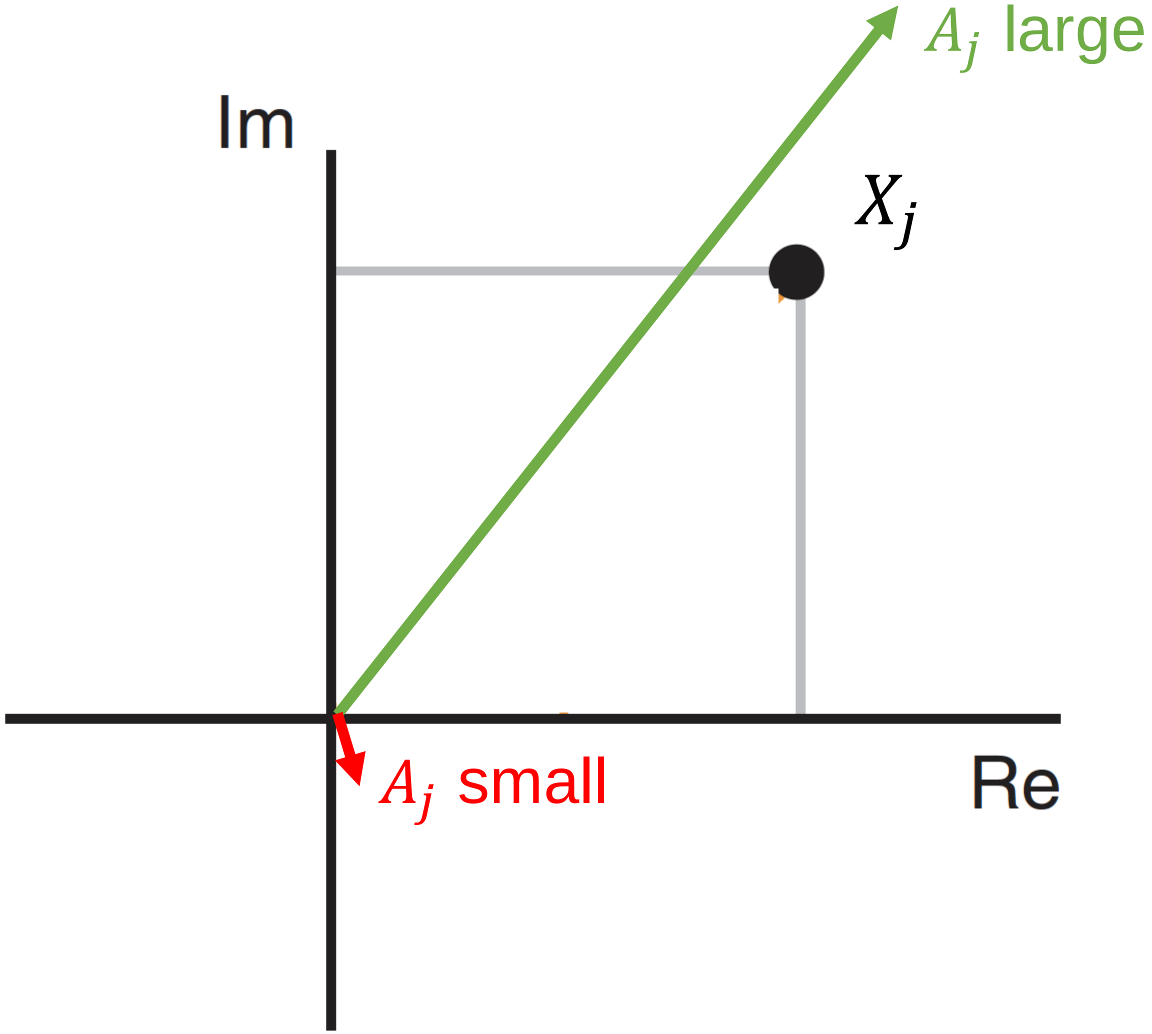
X_j lives in the complex-plane:

Express X_j in polar coordinates:

$$X_j = A_j \exp(i\phi_j)$$

A_j = Amplitude at frequency index j

ϕ_j = Phase at frequency index j



Match: A_j at frequency f_j is <u>large</u>

Mismatch: A_j at frequency f_j is <u>small</u>

Spectrum: intuition

Consider the spectrum of x_n:

Express X_j in polar coordinates:

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^* = \frac{2\Delta^2}{T} \left(A_j \exp(i\phi_j)\right)\left(A_j \exp(-i\phi_j)\right)$$

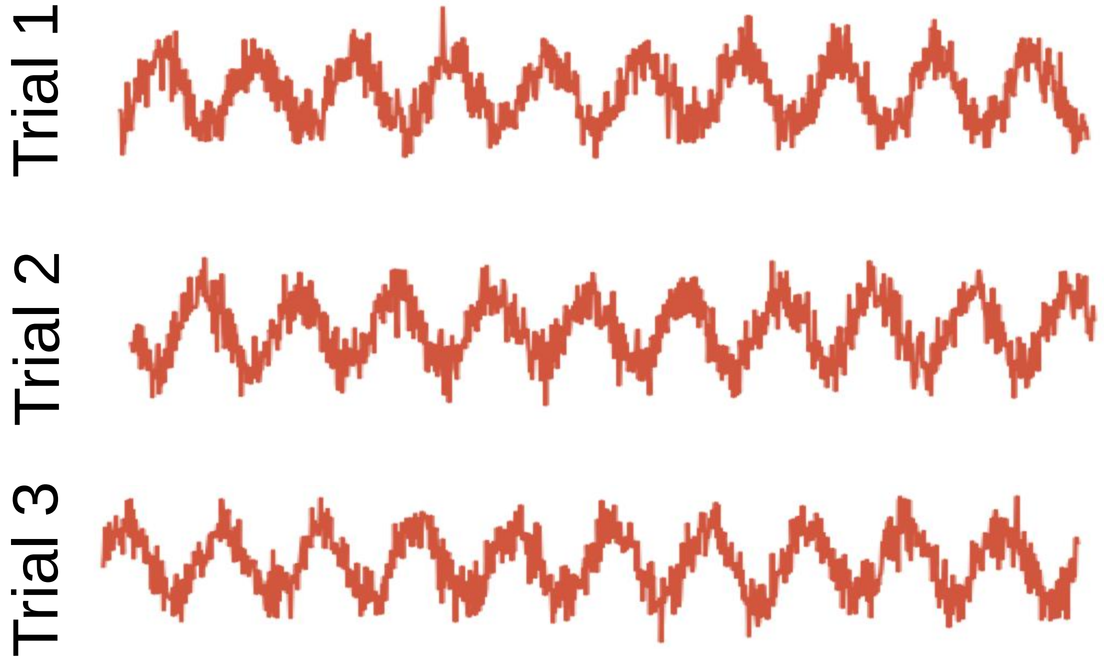
$$= \frac{2\Delta^2}{T} A_j^2 \exp(i\phi_j - i\phi_j) \boxed{= \frac{2\Delta^2}{T} A_j^2 \,.}$$

More direct interpretation of the spectrum at frequency f_j :

better match → larger amplitude of X_j in the complex plane → more power

Spectrum from <u>multiple trials</u>

Ex: Record data across multiple trials



K total trials

Q. Spectrum?

Spectrum from multiple trials

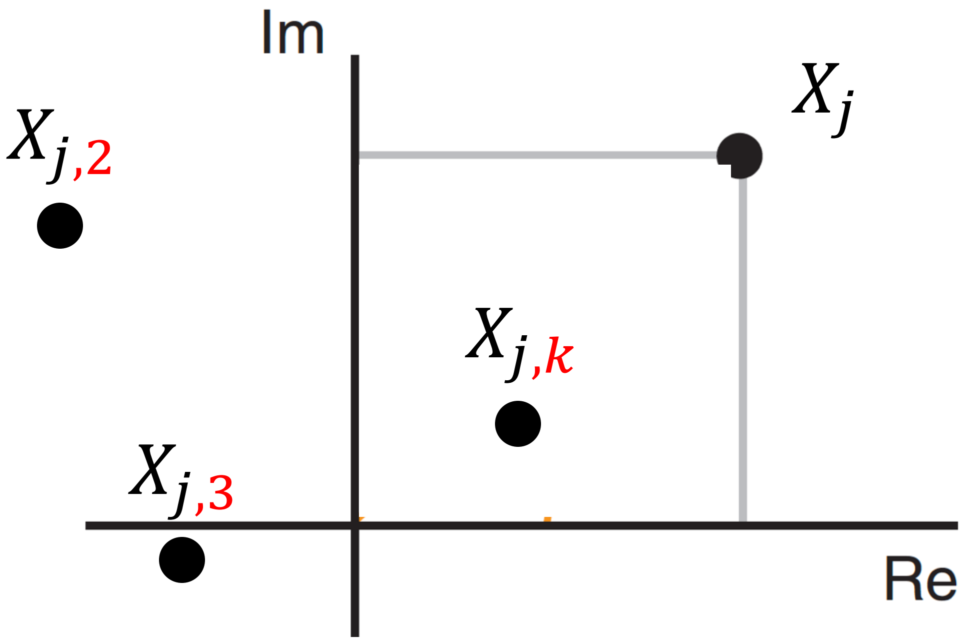
X_j lives in the complex-plane:

Fourier transform for each trial lives in the complex-plane:

In polar coordinates:

$A_{j,k}$ = Amplitude at frequency index j and trial index k

$\phi_{j,k}$ = Phase at frequency index j and trial index k



Spectrum from multiple trials

To compute coherence, we need the trial-averaged spectrum:

Single trial spectrum

Average squared amplitude over trials

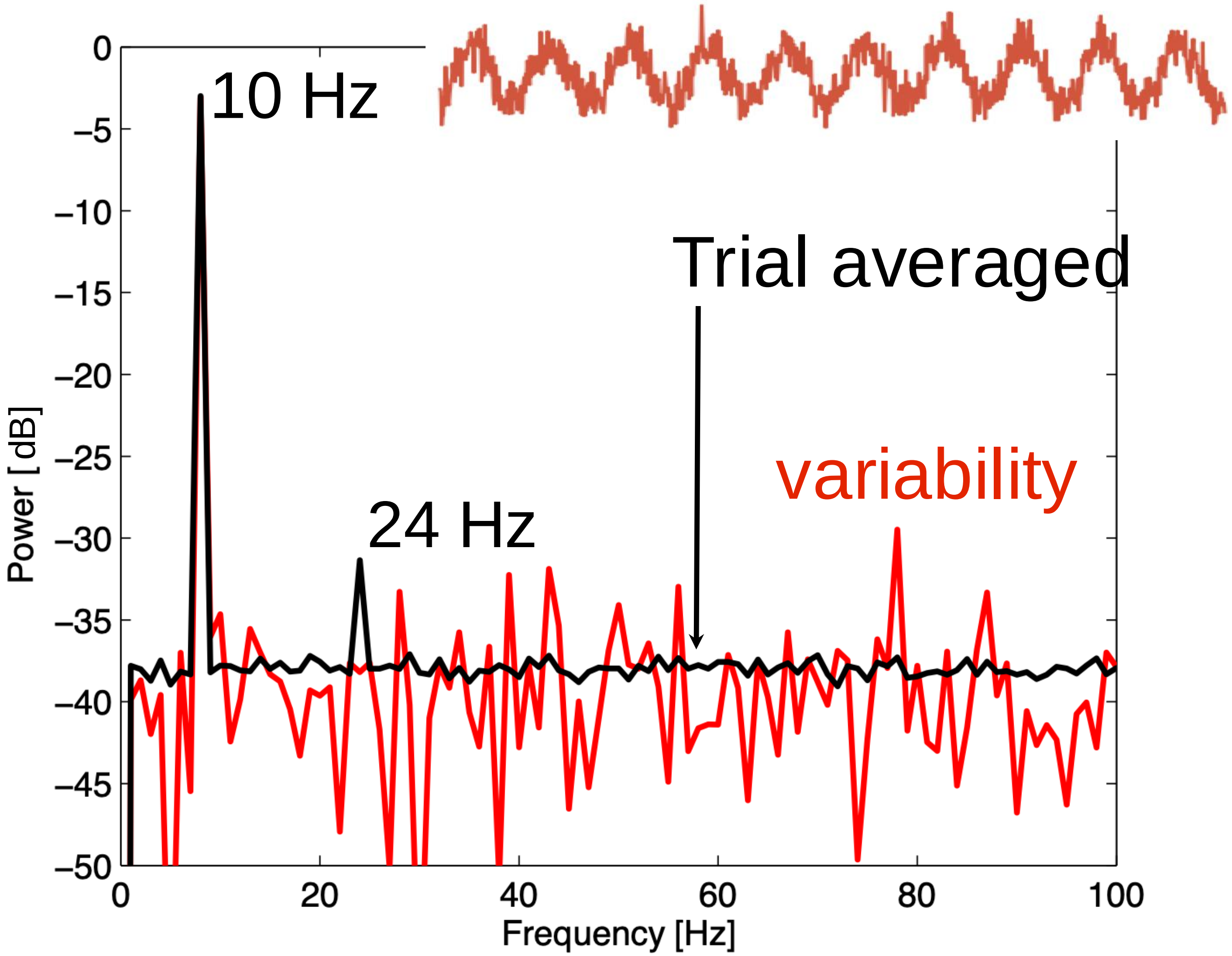
$$< S_{xx,j} = \frac{2\Delta^2}{T} A_j^2 \cdot \sum_{k=1}^{K} A_{j,k}^2$$

$A_{j,k}$ = the amplitude of the signal x, at frequency index j, and trial index k .

K = total number of trials

Spectrum from multiple trials

Single trial:



Trial averaged spectrum:

reduced variability.
reveals another peak . . .

Coherence: equations

Similarly, for signal y_n. <u>Fourier transform</u> of y at frequency j, and trial k:

$$Y_{j,k} = B_{j,k} \exp(i\,\theta_{j,k})$$

$B_{j,k}$ = the amplitude of the signal y at frequency index j and trial index k .

$\theta_{j,k}$ = the phase of the signal y at frequency index j and trial index k .

The <u>trial-averaged spectrum</u> of y at frequency index j

$$< S_{yy,j} > = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^{K} B_{j,k}^2$$

Coherence: equations

$$\kappa_{xy,\,j} = \frac{|\boxed{< S_{xy,\,j} >}|}{\sqrt{< S_{xx,\,j} >}\sqrt{< S_{yy,\,j} >}}$$

✓ ✓

$$< S_{xx,j} >= \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} A_{j,k}^2 \qquad < S_{yy,j} >= \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} B_{j,k}^2$$

Consider the trial averaged cross-spectrum ...

Coherence: equations

The trial averaged <u>cross-spectrum</u> at frequency index j:

Like the auto-spectrum, but use X and Y.

$$< S_{xy,j} > = \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} X_{j,k} Y_{j,k}^{*}$$

In polar coordinates:

$$< S_{xy,j} > = \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} A_{j,k} B_{j,k} \exp\left(i\Phi_{j,k}\right)$$

Phase of x Phase of y

where $\Phi_{j,k} = \phi_{j,k} - \theta_{j,k}$ is the <u>phase difference</u> between the two signals, at frequency index j and trial k.

Coherence: equations

Put it all together ...

$$\kappa_{xy,\,j} = \frac{|< S_{xy,\,j} >|}{\sqrt{< S_{xx,\,j} >}\sqrt{< S_{yy,\,j} >}}$$

In polar coordinates

cross-spectrum of x & y,
depends on trial averaged
amplitudes, phase differences.

$$= \frac{\left| \sum_{k=1}^{K} A_{j,k} B_{j,k} \exp\left(i \Phi_{j,k} \right) \right|}{\sqrt{\sum_{k=1}^{K} A_{j,k}^2} \sqrt{\sum_{m=1}^{K} B_{j,m}^2}}$$

x trial averaged spectrum,
at frequency index j

y trial averaged spectrum,
at frequency index j

Coherence: intuition

To build intuition, assume: the amplitude is <u>identical</u> for both signals and all trials.

$$A_{j,k} = B_{j,k} = C_j$$

<u>Note</u>: no trial dependence

then

$$\kappa_{xy,j} = \frac{\left| \sum_{k=1}^{K} A_{j,k} B_{j,k} \exp\left(i\Phi_{j,k}\right) \right|}{\sqrt{\sum_{k=1}^{K} A_{j,k}^2} \sqrt{\sum_{m=1}^{K} B_{j,m}^2}}$$

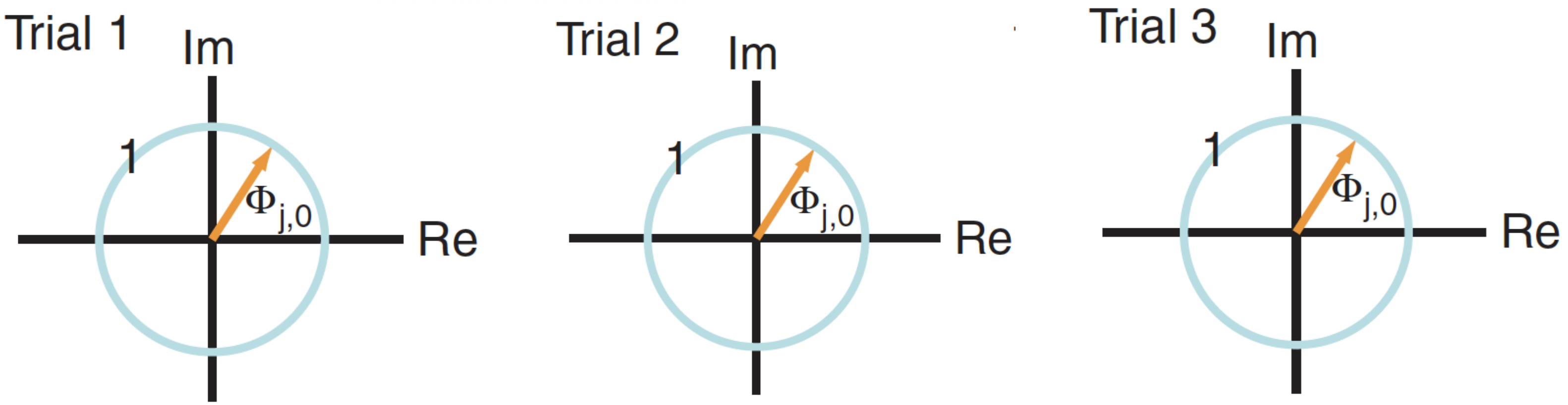
only involves the <u>phase difference</u> between the two signals averaged across trials.

Coherence: intuition

$$\kappa_{xy,j} = \frac{1}{K}\left|\sum_{k=1}^{K}\exp\left(i\Phi_{j,k}\right)\right|$$

Case 1: Phases align across trials. $\boxed{\Phi_{j,k} = \Phi_{j,0}}$

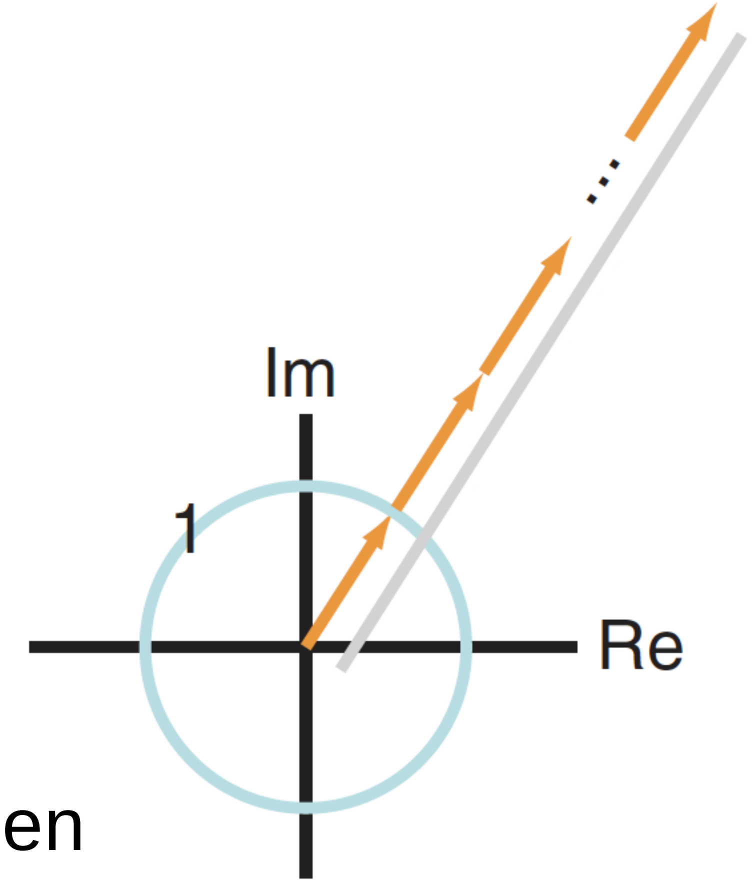
Plot $\exp\left(i\Phi_{j,k}\right)$ in the complex plane.



sum these vectors end to end across trials

divide by K

$$\kappa_{xy,j} \approx 1$$



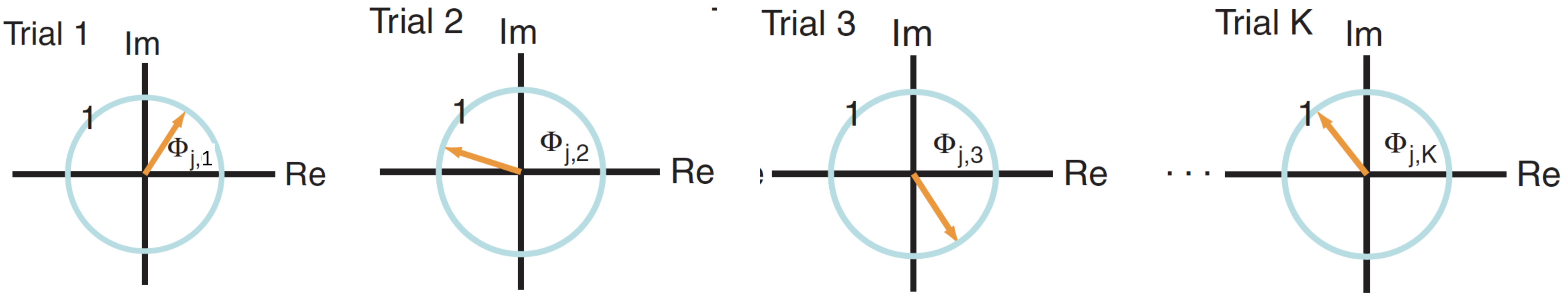
strong coherence - constant phase relation between the two signals across trials at frequency index j.

Coherence: intuition

$$\kappa_{xy,j} = \frac{1}{K}\left|\sum_{k=1}^{K} \exp\left(i\Phi_{j,k}\right)\right|$$

Case 2: Random phase differences across trials.

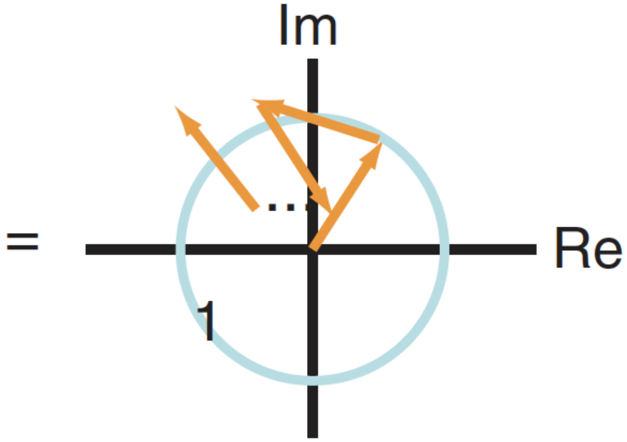
Plot $\exp\left(i\Phi_{j,k}\right)$ in the complex plane.



sum these vectors end to end across trials

divide by K

$$\kappa_{xy,j} \approx 0$$



weak coherence - random phase relation between the two signals across trials at frequency index j.

Coherence: summary

$$0 \leq \kappa_{xy,j} \leq 1$$

0: no coherence between signals x and y at frequency index j

1: strong coherence between signals x and y at frequency index j .

The coherence is a measure of the phase consistency between two signals at frequency index j across trials.

Cross-covariance

Remember autocovariance

$$r_{xx}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L} - \bar{x})(x_n - \bar{x})$$

Now **cross**-covariance

$$r_{xy}[L] = \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L} - \bar{x})\boxed{(y_n - \bar{y})}$$

Q. What's different?

Cross-covariance

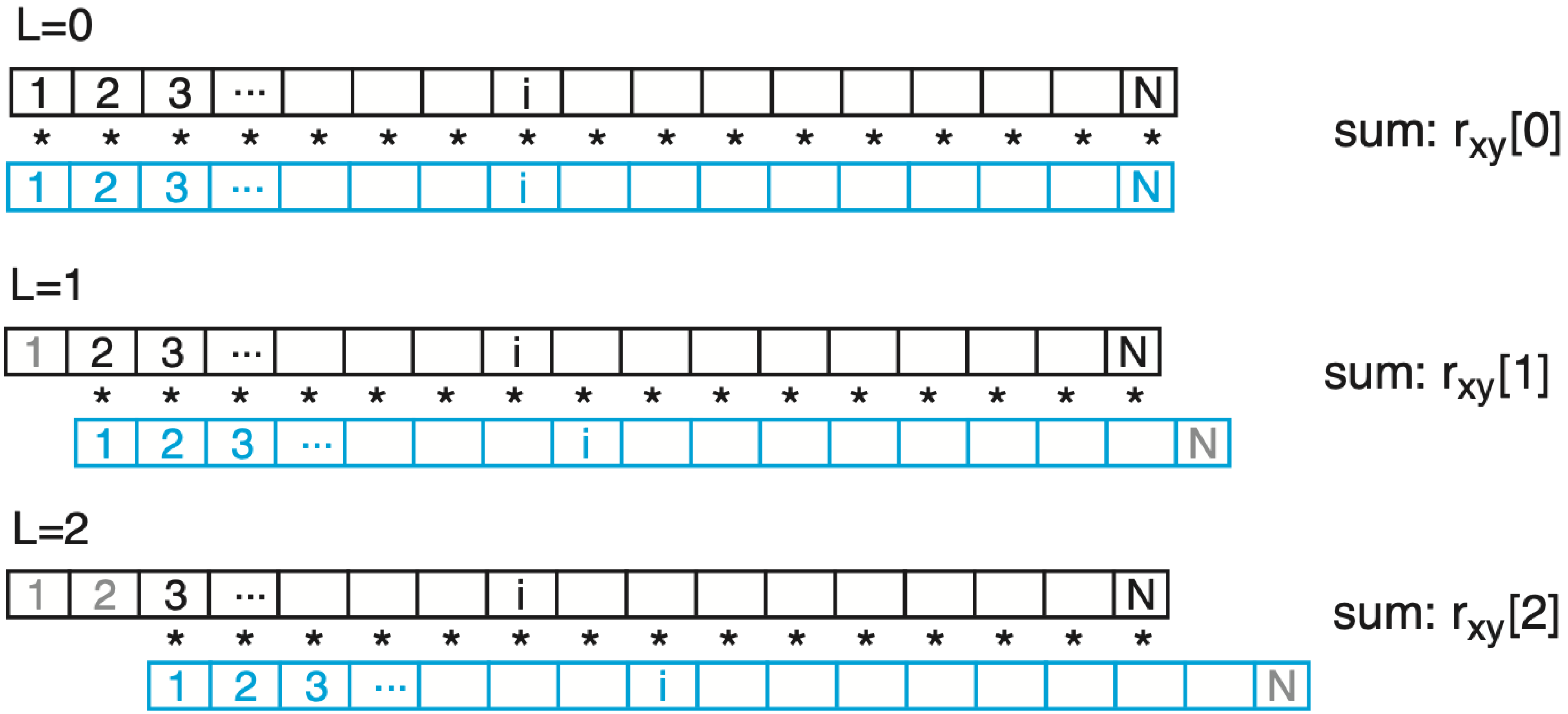
$$r_{xy}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(y_n - \bar{y})$$

Idea: compare two time series (x and y)

Do they match?

Cross-covariance

Compute cross-covariance at different lags L



Cross-covariance

Trial-averaged cross-covariance

mean of signal y,
trial k

$$r_{xy}[L] = \boxed{\frac{1}{K}\sum_{k=1}^{K}} \frac{1}{N}\sum_{n=1}^{N-L}(x_{n+L,k} - \bar{x}_k)(y_{n,k} - \bar{y}_k)$$

average over trials

signal x,
time n+L,
trial k.

signal y,
time n,
trial k.

mean of signal x,
trial k

Cross-covariance vs coherence

Remember:

$$\underbrace{S_{xx,j}}_{\text{auto-spectrum}} = 2\Delta \sum_l \underbrace{r_{xx}}_{\text{auto-covariance}}[l] \exp\left(-\frac{2\pi i}{N} j\, l\right)$$

In the same way:

$$\underbrace{S_{xy,j}}_{\textbf{cross-}\text{spectrum}} = 2\Delta \sum_l \underbrace{r_{xy}}_{\textbf{cross-}\text{covariance}}[l] \exp\left(-\frac{2\pi i}{N} j\, l\right)$$

$\boxed{\text{The cross-spectrum is the Fourier transform of the cross-covariance.}}$

(frequency domain) *(time domain)*

Coherence: examples

Python