Rhythms

Analyzing Rhythms (Part 1)

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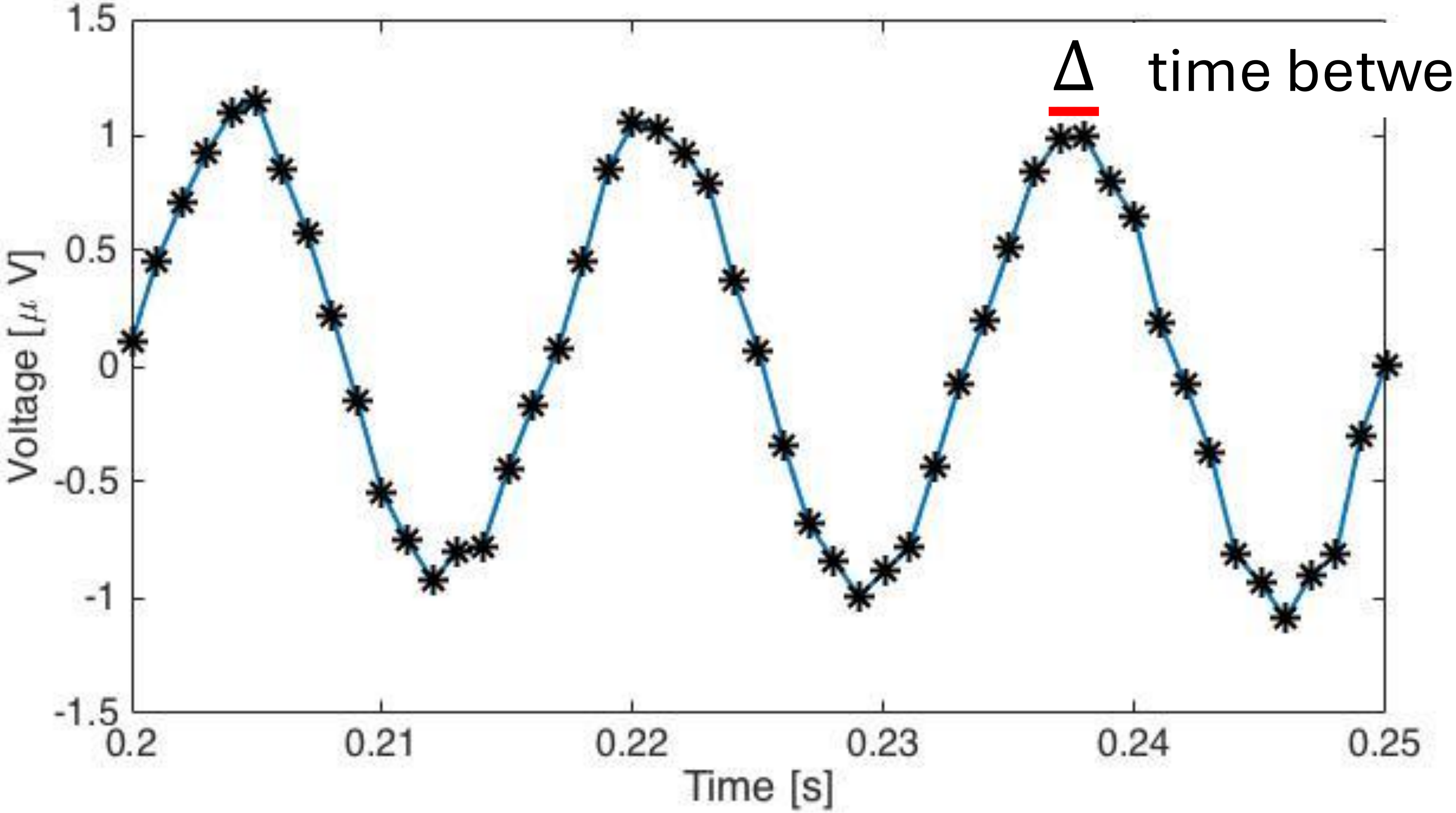
Today

- Practical notions

  Sampling frequency, Nyquist frequency, tapering

Data is not continuous

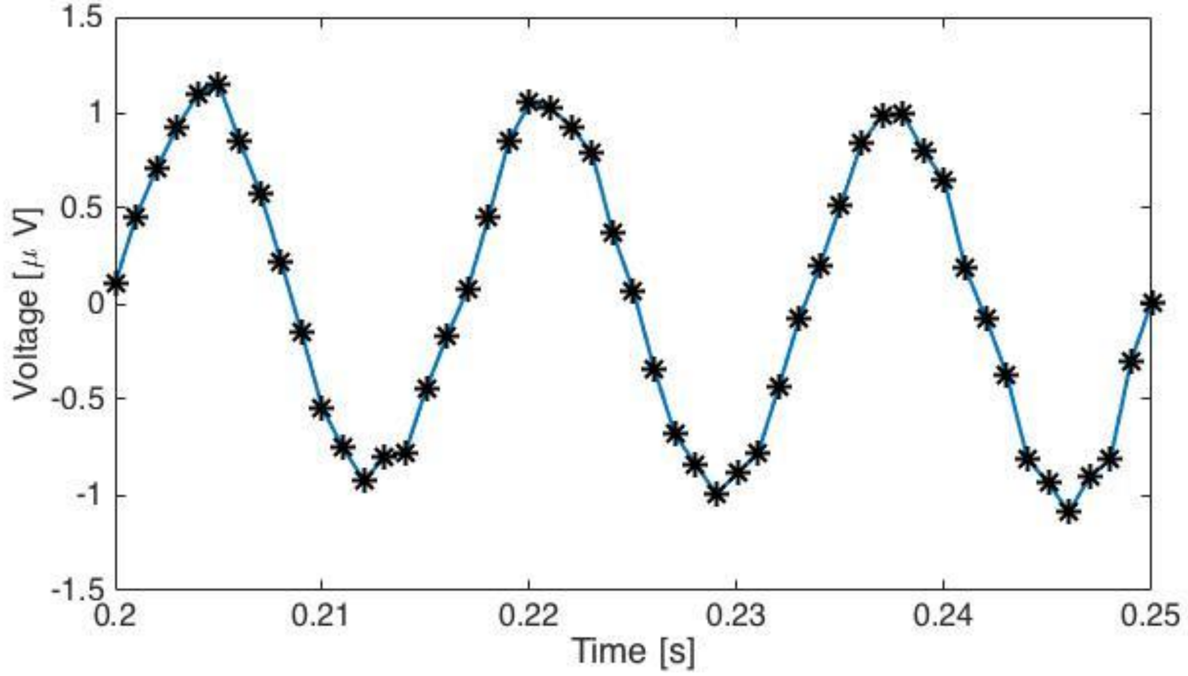
Consider a small snippet of data



Δ time between samples

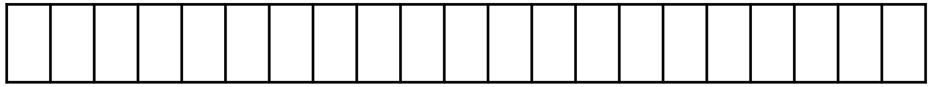
Notation: x_n = Data at index n

Data is not continuous



Notation

x (EEG, LFP, MEG, ...)



x_n = Data at index n

t_n = Time at index n, $t_n = \Delta n$ where Δ = sampling interval

f_j = Frequency at index j, $f_j = j/T$ where T = total time of observation

Discrete time Fourier transform

Previously $$V[t] = \sum_j A_j \cos(2\pi f_j t) + B_j \sin(2\pi f_j t)$$

$$A_k = \frac{2}{T}\int_0^T V[t]\cos(2\pi f_k t)dt$$ and $$B_k = \frac{2}{T}\int_0^T V[t]\sin(2\pi f_k t)dt$$

Compare data V[t] to cosine at frequency f_k , does it match?

Now replace A_k, B_k: $$\boxed{X_j = \sum_{n=1}^{N} x_n \exp(-2\pi i f_j t_n)}$$????

<u>Fourier transform</u> of the data *x*.

Discrete time Fourier transform

Define the parts

$i^2 = -1$

$$X_j = \sum_{n=1}^{N} x_n \boxed{\exp(-2\pi i f_j t_n)}$$????

FT of x
index j

sum over
all time n

data x
index n

frequency
index j

$f_j = j/T$

time
index n

$t_n = \Delta n$

Discrete time Fourier transform

Euler's formula
$e^{i\theta} = \cos(\theta) + i\,\sin(\theta)$

Feynman: "*the most remarkable formula in mathematics*"

Fourier transform intuition:

Data as a function of time index n

Data as a function of frequency index j

$$X_j = \sum_{n=1}^{N} \boxed{x_n} \boxed{\exp(-2\pi\, i\, f_j\, t_n)}$$

Sinusoids at frequency f_j

Euler's formula:

$$\exp(-2\pi\; i f_j\, t_n) = \;\cos(-2\pi\, f_j\, t_n) + i\sin(-2\pi\, f_j\, t_n).$$

So, at each time (index n) multiply data x_n by sinusoids at frequency f_j
Then sum up over all time.

Discrete time Fourier transform

Fourier transform intuition:

Data as a function of time index n

Data as a function of frequency index j

$$X_j = \sum_{n=1}^{N} \boxed{x_n} \boxed{\exp(-2\pi i f_j t_n)}\,.$$

Sinusoids at frequency f_j

Idea: compare our data x_n to sinusoids at frequency f_j and see how well they "match".

Good match: X_j = big

Bad match: X_j = small

X_j reveals the frequencies f_j that match our data.

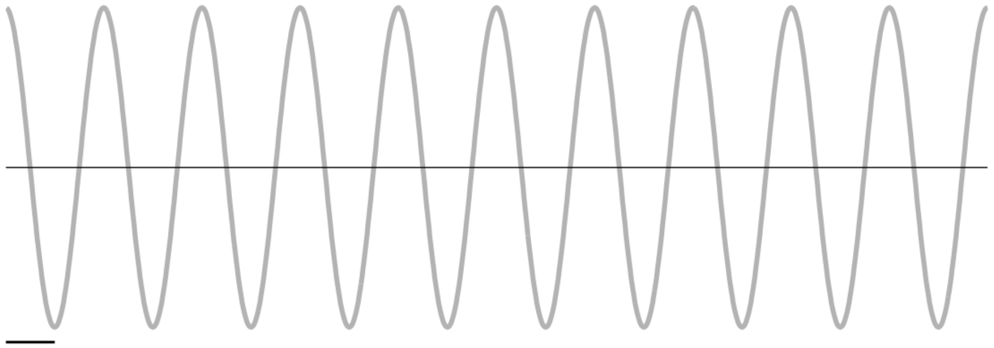
Spectrum: idea

Fourier transform intuition: **"Compare" data to sinusoids at different frequencies**

Match: X_j at frequency f_j is large

Mismatch: X_j at frequency f_j is small

Example:



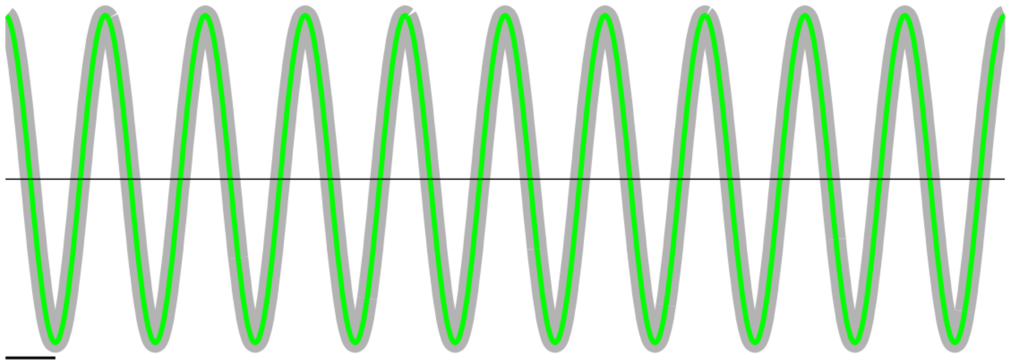
"Data"
10 Hz cosine

4 Hz

Multiply (+,-,+,-,...) & add

... small value

4 Hz does not match data



10 Hz

Multiple (+,+,+,+,...) & add

... large value

10 Hz matches data

Discrete time Fourier transform

Sound familiar? $$X_j = \sum_{n=1}^{N} x_n \exp(-2\pi i f_j t_n)$$ Fourier transform of the data *x*.

replace with Euler's formula

$$\cos(-2\pi f_j t_n) + i \sin(-2\pi f_j t_n).$$

$$X_j = \boxed{\left(\sum_{n=1}^{N} x_n \cos(-2\pi f_j t_n)\right)} + i \boxed{\left(\sum_{n=1}^{N} x_n \sin(-2\pi f_j t_n)\right)}$$

Looks like $$\boxed{A_k} = \frac{2}{T}\int_0^T V[t]\cos(2\pi f_k t)dt \qquad \boxed{B_k} = \frac{2}{T}\int_0^T V[t]\sin(2\pi f_k t)dt$$

Same idea: compare data to sinusoids and see how well they match

Spectrum: definition

The power of data x
at frequency index j

(change sign of i everywhere)

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

Previously $\dfrac{A_j^2 + B_j^2}{2}$

Same idea!

Constant that depends on sampling interval, duration of recording

FT of data

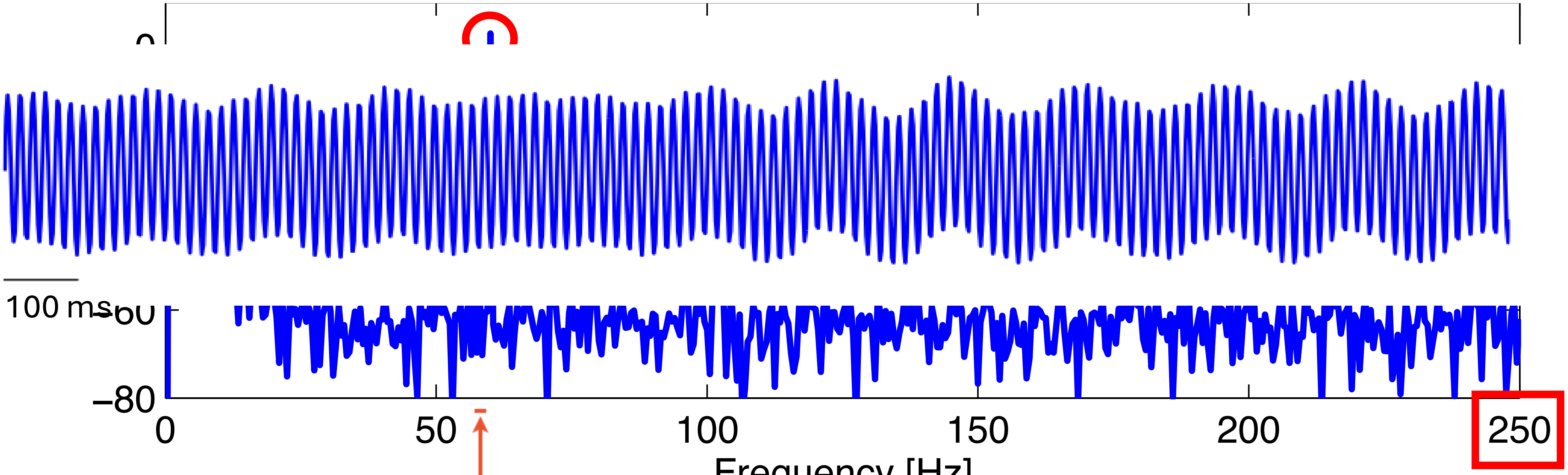
Complex conjugate of FT of data

Power at frequency f_j indicates how well sinusoids at f_j "match" our data.

Good match → High power at frequency f_j

Spectrum

... reveals the dominant frequencies that “match” the data.



Important quantities

Frequency resolution: df

Nyquist frequency: F_{NQ}

Define these two quantities.

Frequencies

f_j = Frequency at index j, $f_j = j/T$ where T = total time of observation

$$f_j = \left\{0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{1}{2\Delta} - \frac{1}{T}, \boxed{\frac{1}{2\Delta}}, -\left(\frac{1}{2\Delta} - \frac{1}{T}\right), -\left(\frac{1}{2\Delta} - \frac{2}{T}\right), \ldots, \frac{-2}{T}, \frac{-1}{T}\right\}$$

<u>Two important quantities</u>

Largest frequency: **Nyquist frequency** $f_{NQ} = \dfrac{1}{2\Delta} = \dfrac{f_0}{2}$ half the sampling frequency

$f_0 = \dfrac{1}{\Delta}$ sampling frequency

Frequency resolution $df = \dfrac{1}{T}$ reciprocal of total recording duration

Frequencies

Visualize f_j as a **vector**

index	0	1	2	...	$N/2-1$	$\frac{N}{2}$	$\frac{N}{2}+1$		N-2	N-1
freq	0	df	$2df$	...	$f_{NQ}-df$	f_{NQ}	$-(f_{NQ}-df)$	...	$-2df$	$-df$

Step forward in intervals of frequency resolution *df*

... until reaching maximum frequency f_{NQ}

then, frequencies are <u>negative</u>, starting with the largest magnitude.

decrease in magnitude until -*df*

Frequencies

Note: length(t_n) = N

length(f_j) = N

time and frequency vectors have the same length N

- If we record N data points, then we have N frequencies to examine.

Note: Frequencies f_j include negative values.

Important fact: when data x_n is real (no imaginary component), then negative frequencies are redundant.

$S_{xx,j}$ at f_j = $S_{xx,j}$ at $-f_j$

Frequencies

Q: Is x_n real?

A: Yes (in neuroscience)

Only inspect $f_j > 0$

$$f_j = \left\{ 0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{1}{2\Delta} - \frac{1}{T}, \frac{1}{2\Delta}, \boxed{-\left(\frac{1}{2\Delta} - \frac{1}{T}\right), -\left(\frac{1}{2\Delta} - \frac{2}{T}\right), \ldots, \frac{-2}{T}, \frac{-1}{T}} \right\}$$

Ignore negative frequencies (redundant)

Spectrum: df

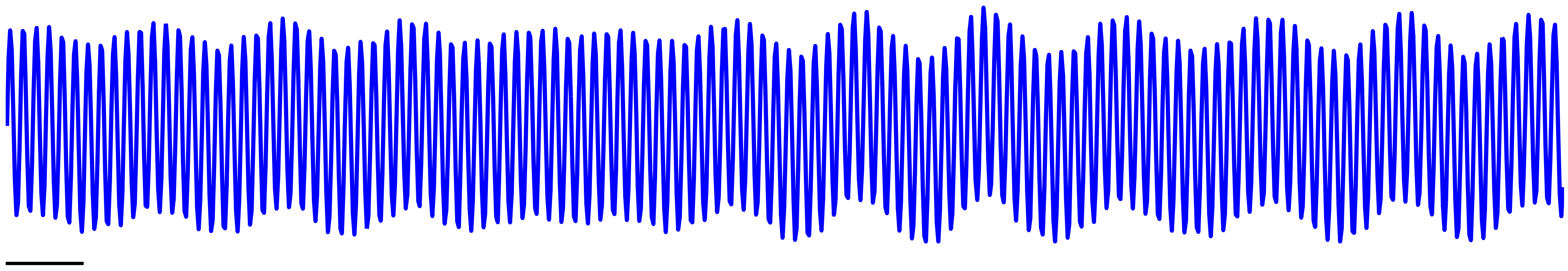
- What is df?

$$\boxed{df = \frac{1}{T}}$$

frequency resolution

where T = Total duration of recordings.

<u>Ex.</u>



T = 2 s so df = 0.5 Hz

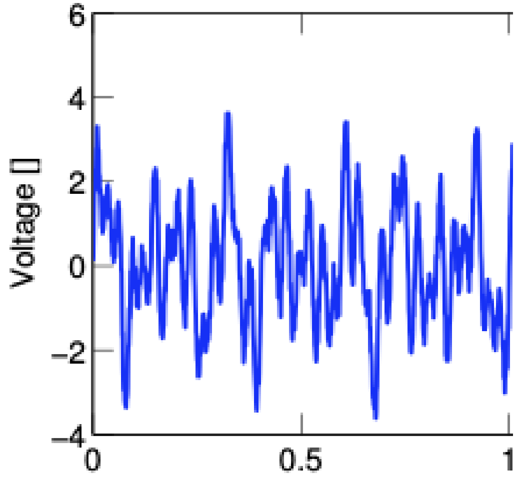
Q: How do we improve frequency resolution?

A: Increase T or record for longer time.

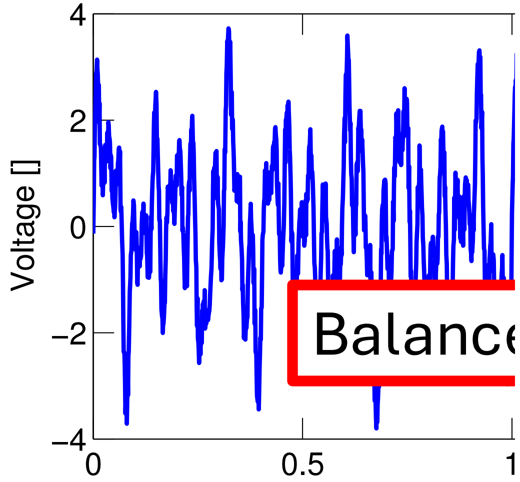
Spectrum: df

- Demand 0.2 Hz frequency resolution

df = 0.2 Hz = 1/T , so T = 5 s



But, data may change during longer recordings . . .



Spectrum: F_{NQ}

- What is F_{NQ}?

$$\boxed{F_{NQ} = f_0/2}$$

Nyquist frequency

where f_0 = sampling frequency.

The **highest** frequency we can observe.

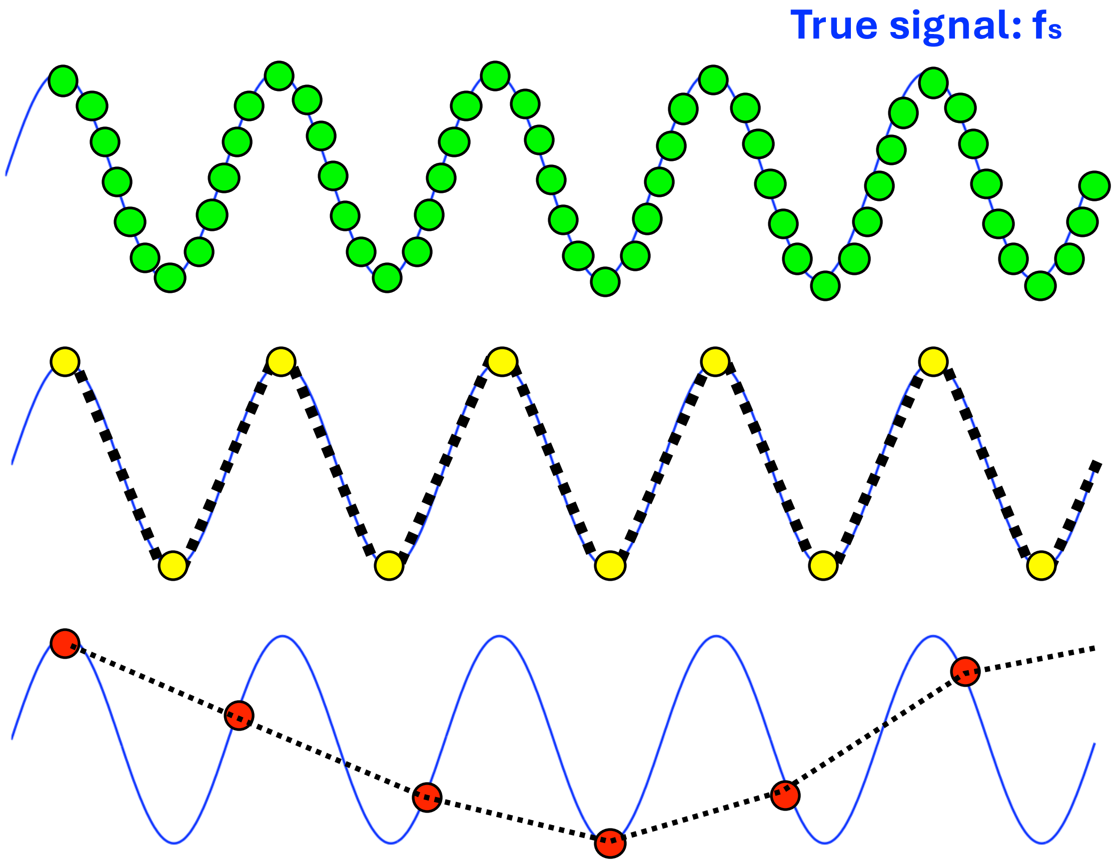
Sample:

$f_0 >> 2\ f_s$

$f_0 = 2\ \boxed{f_s}$

Max freq we can observe at this sample rate!

$f_0 < 2\ f_s$



Accurate reconstruction

2 samples/cycle

Enough to reconstruct signal, but just barely.

High frequency (in data) mapped to low frequency (**aliased**).

All hope lost! Indistinguishable from true low frequency signals.

Spectrum: $df,\ F_{NQ}$

Summary

Frequency resolution $$df = \frac{1}{T}$$ ← Duration of recording

Nyquist frequency $$f_{\mathrm{NQ}} = \frac{f_0}{2}$$ ← Sampling frequency

For finer frequency resolution: record more data.

To observe higher frequencies: increase sampling rate.

Spectrum: four (important) asides

- Units
- Scale
- Tapers
- Spectrogram

Spectrum: units

Q. What are the units of the spectrum?

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

$$\frac{[s]^2}{[s]} \quad [V][V]$$

$$[s] \qquad [V]^2$$

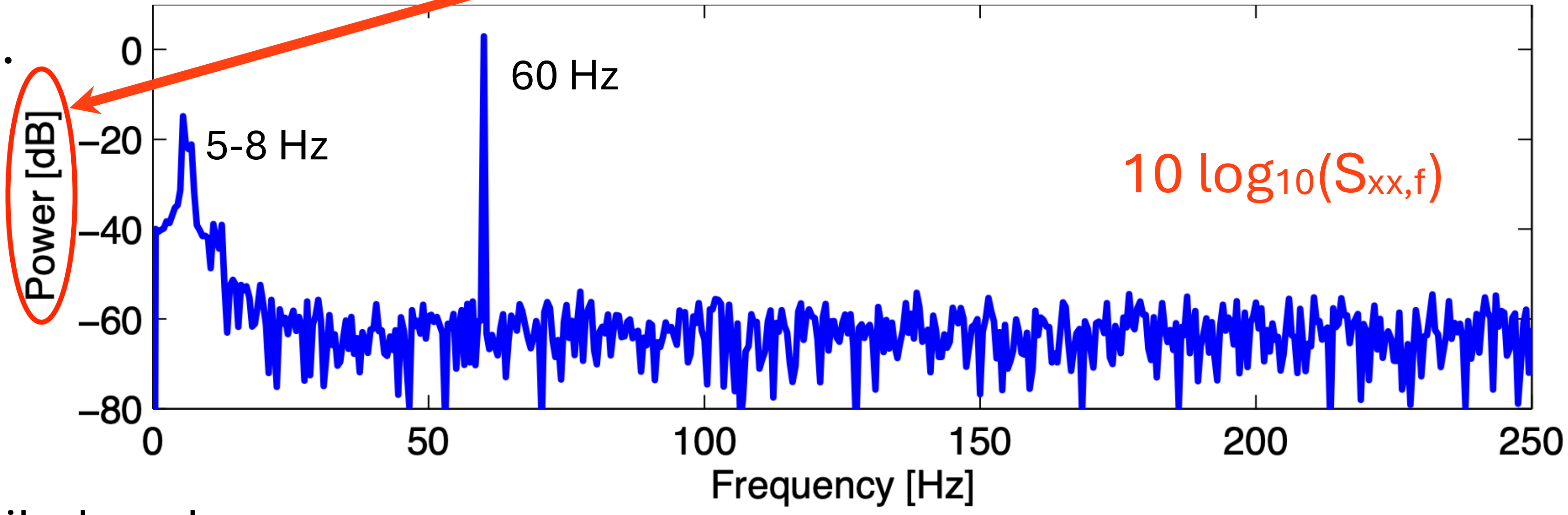
$$\boxed{\frac{[V]^2}{[Hz]}}$$

"volts squared per Hertz"

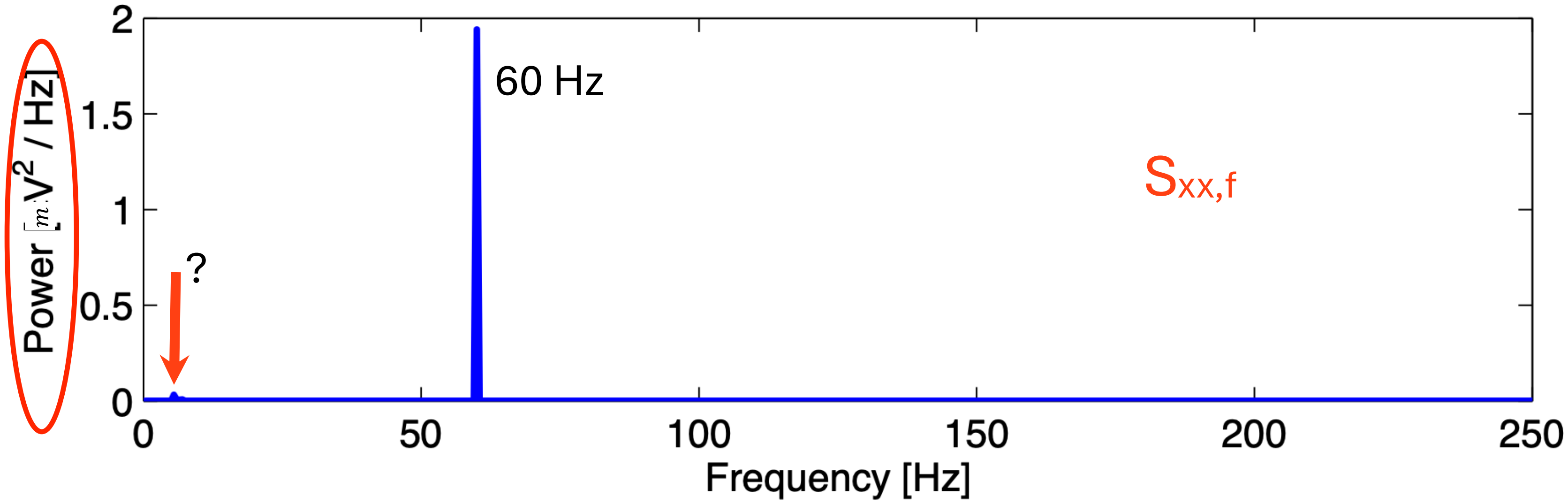
Spectrum: scale

Emphasize lower amplitude signals

A note on scale .



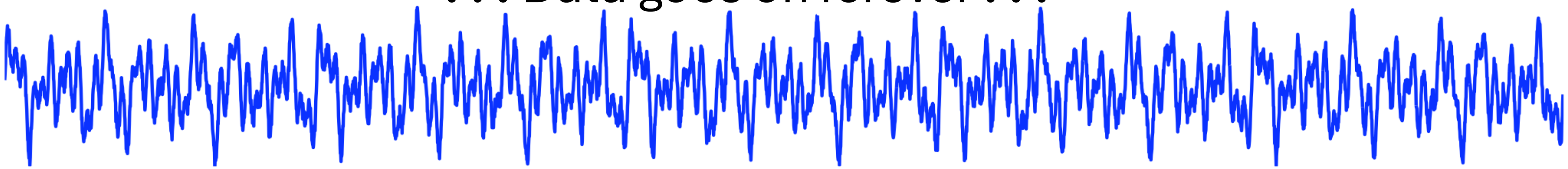
Without the decibel scale . . .



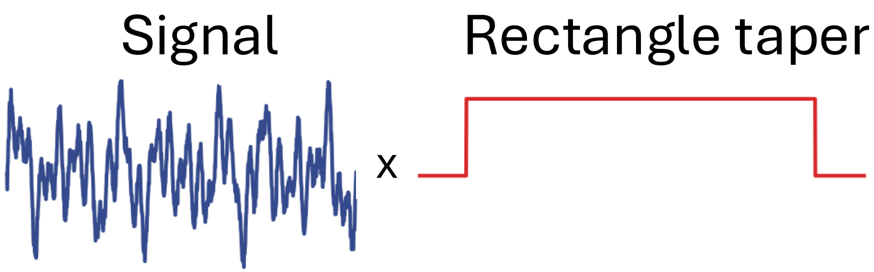
Spectrum: tapers

Doing nothing, we make an implicit taper choice . . .

. . . Data goes on forever . . .



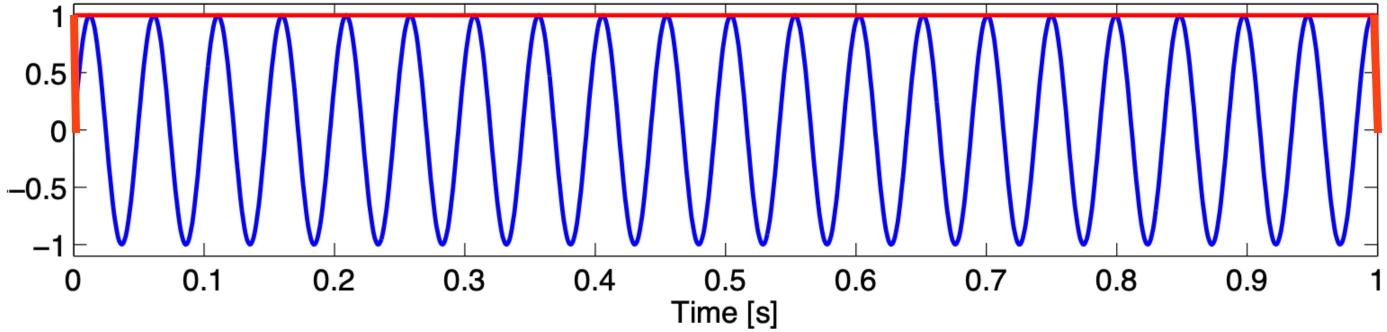
What we're observing:

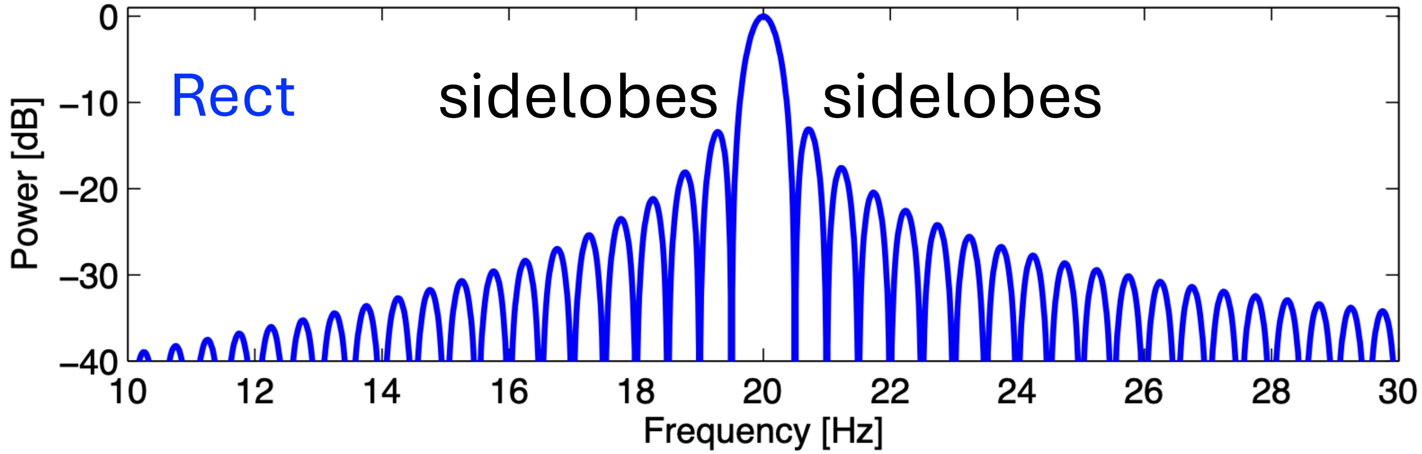


Spectrum: tapers

The rectangle taper impacts the power spectrum.

Pure sinusoid at 20 Hz

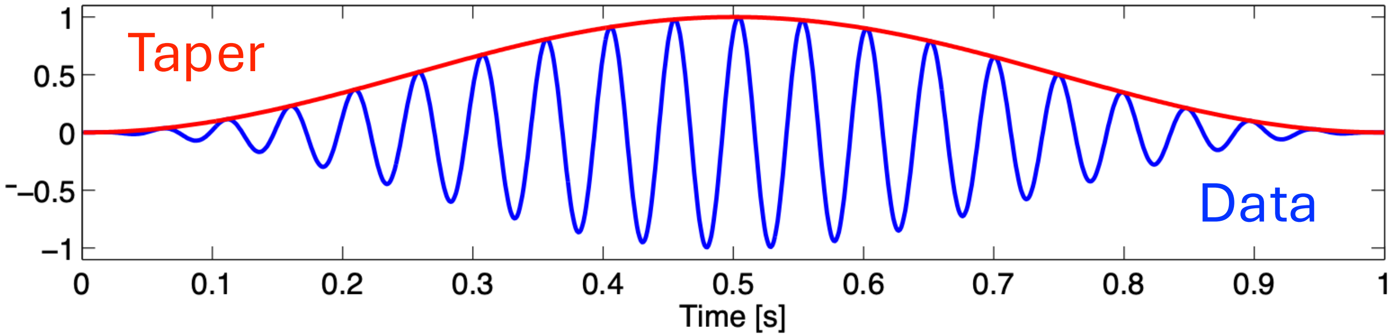




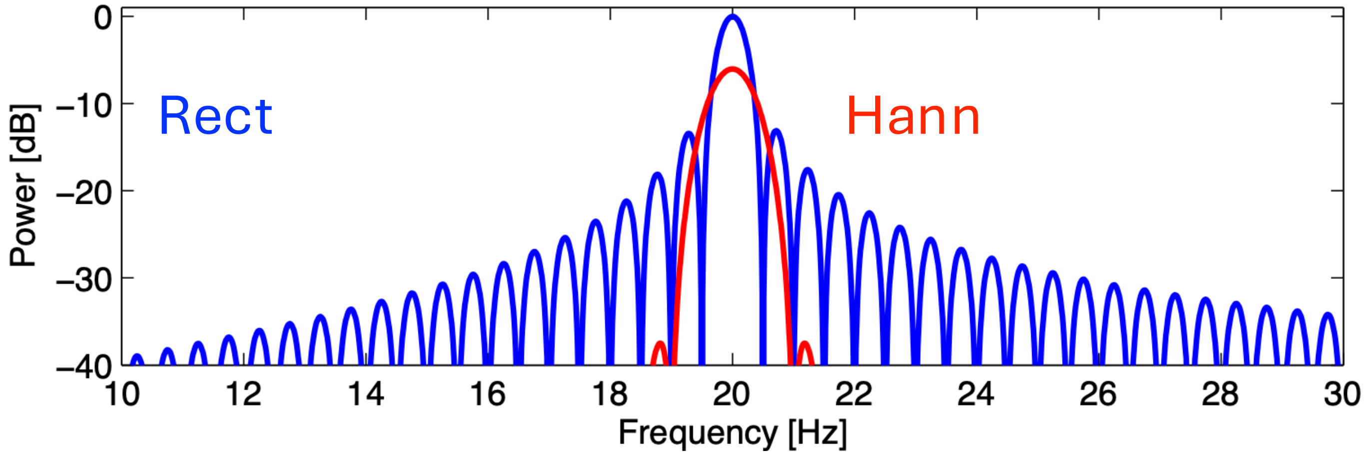
Sharp peak is "smeared out" . . .

Spectrum: tapers

Idea: smooth the sharp edges of rectangle taper.



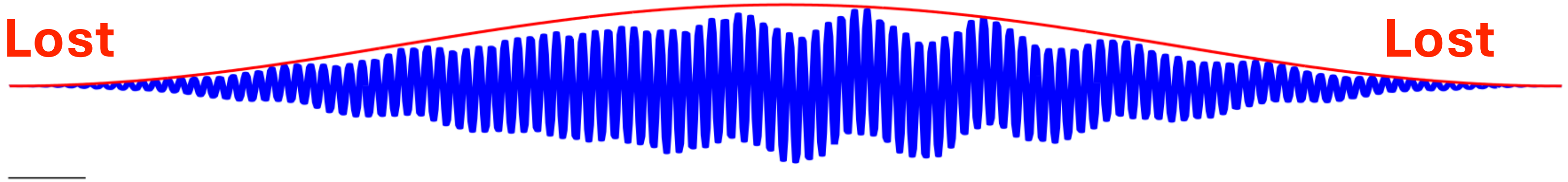
Compute spectrum of tapered data.

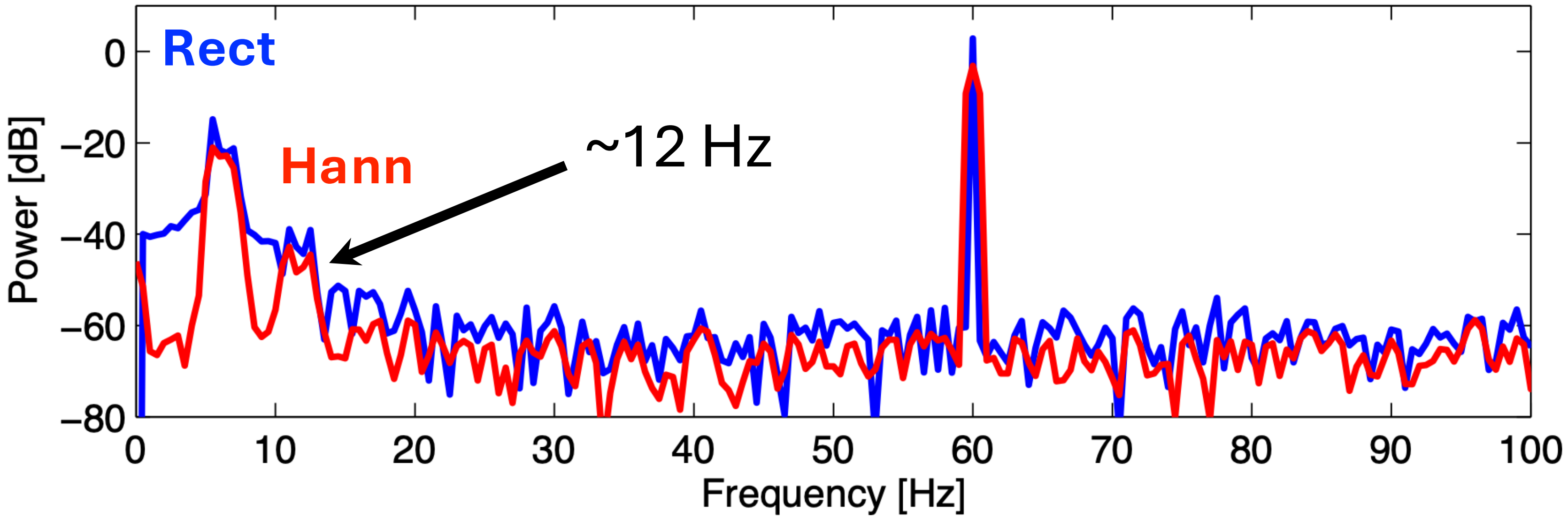


Taper reduces the sidelobes.

Spectrum: tapers

Ex:





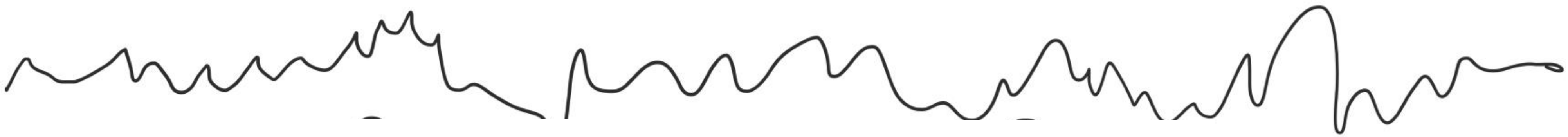
Good: Reduced sidelobes reveals a new peak.

Bad: Broader peaks & lose data at edges.

"More lives have been lost looking at the [rectangular tapered spectrum] than by any other action involving time series." [Tukey 1980]

Spectrum: spectrogram

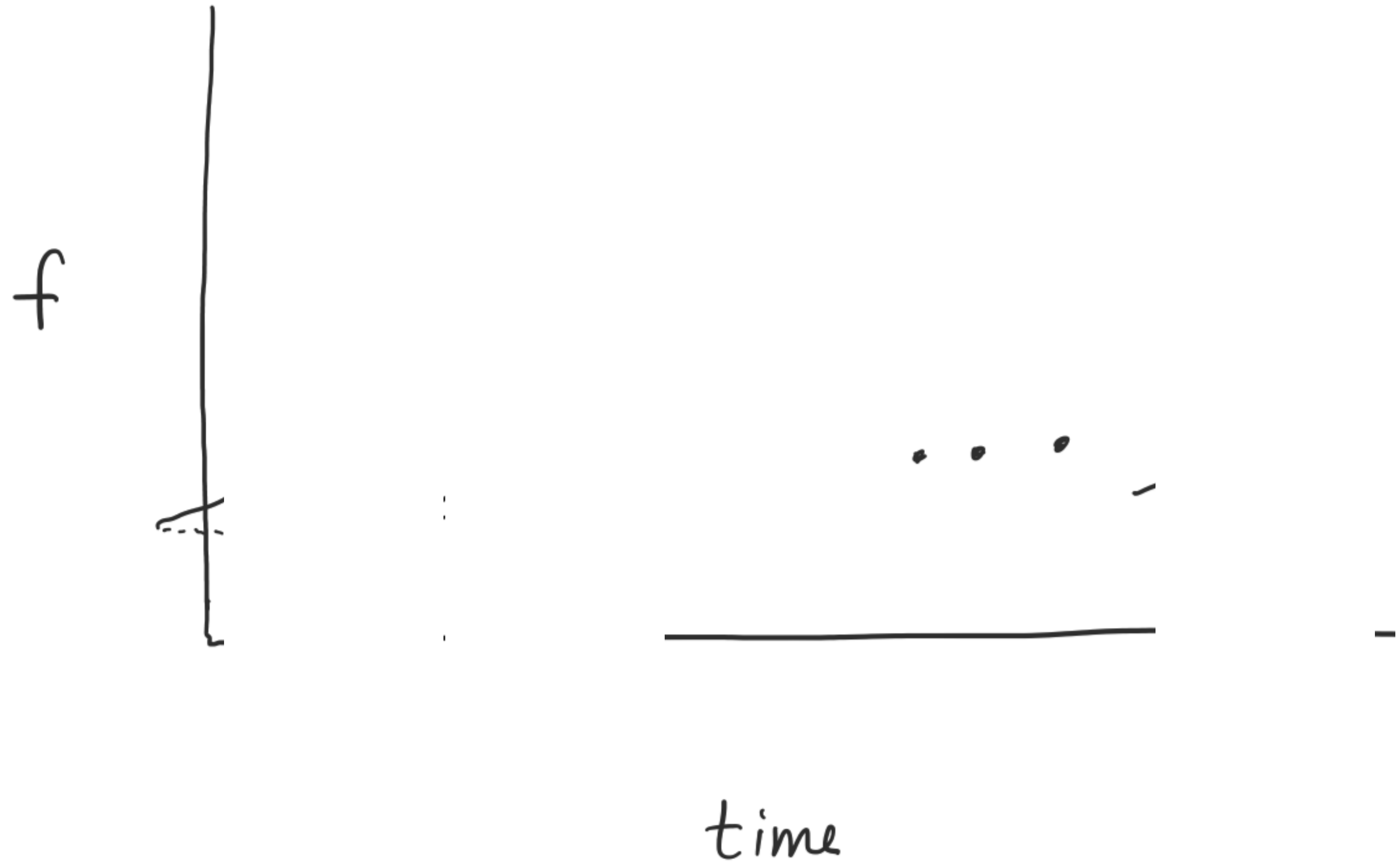
Idea: Divide data into smaller intervals, then compute the spectrum in each interval



Spectrum: spectrogram

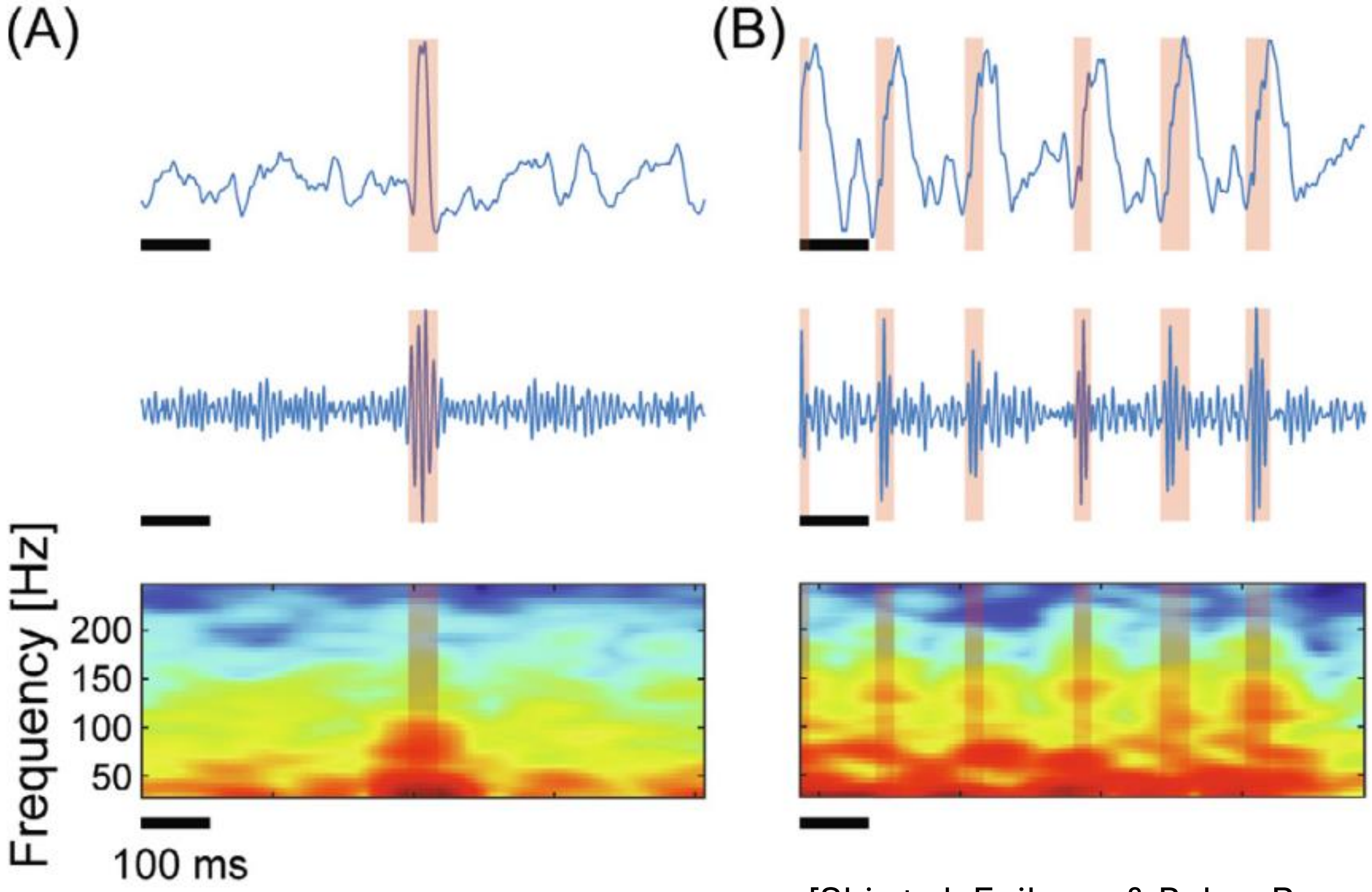
spectrogram

Display as



Usually, color indicates power

Spectrum: spectrogram



[Shi et al, Epilepsy & Behav Reports, 2022]

Spectrum: spectrogram

Q. What happens to df ?

Original data 10 s , df =

Intervals 1 s , df =

Q. What happens to f_{NQ} ?

Q. When is this a good idea?

Spectrum: summary

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

Data as a function of time index n

$$X_j = \sum_{n=1}^{N} \boxed{x_n} \boxed{\exp(-2\pi i f_j t_n)} .$$

Sinusoids at frequency f_j

Frequency resolution

$$df = \frac{1}{T}$$

Nyquist frequency

$$f_{\mathrm{NQ}} = \frac{f_0}{2}$$

Units, decibel scale, tapers, spectrograms, ...

Spectrum practicals

Python