

Coherence

(and cross-covariance)

Computing the coherence (Part 1)

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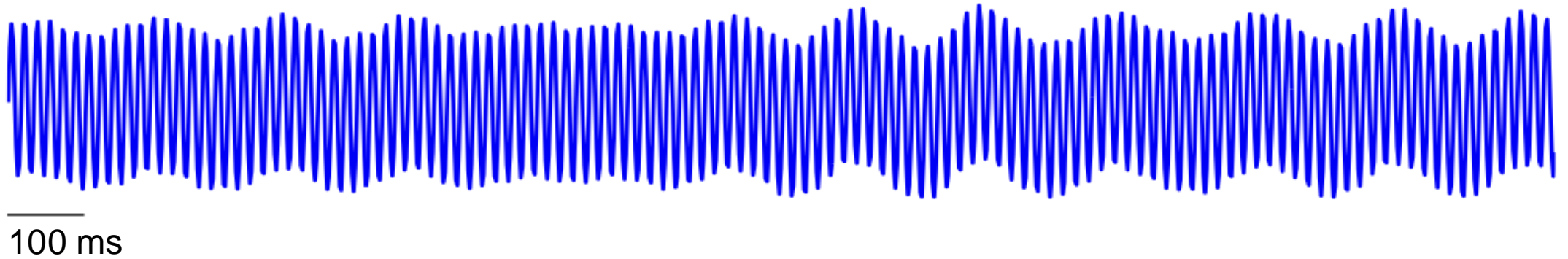
Today

Review/reminder

Coherence

Rhythms

Consider these (experimental data)

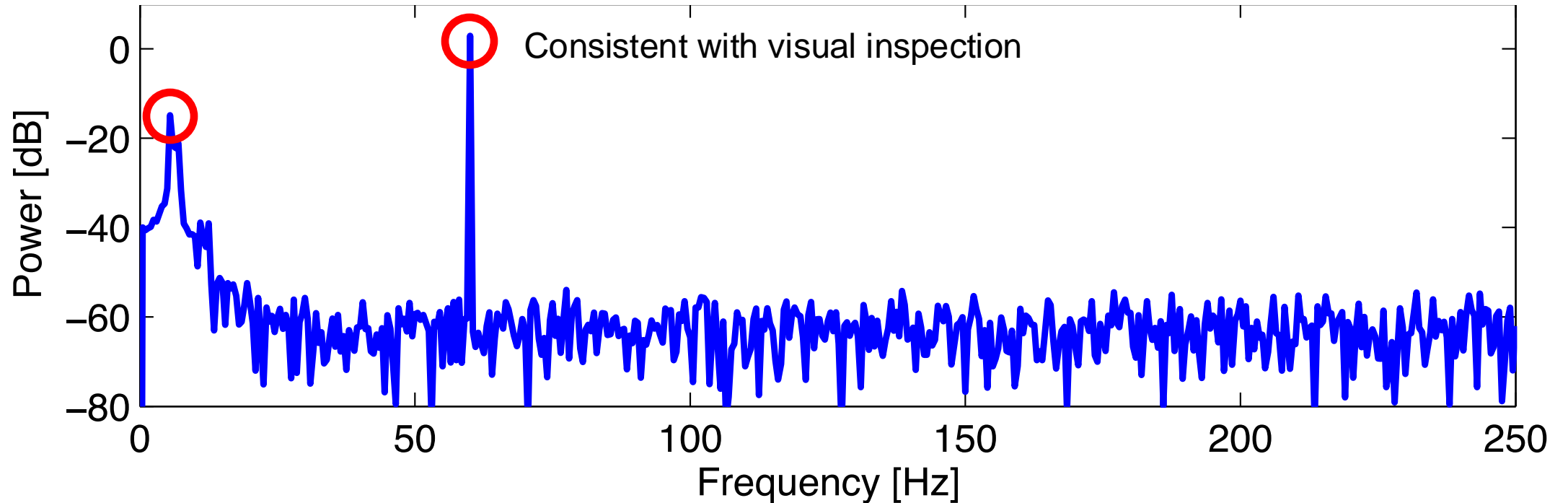


Visual inspection

- Rhythmic (dominant fast rhythm)
- It's complicated
- Beyond visual inspection . . . quantitative characterization?

Spectrum

Beyond visual inspection . . .



Axes: Power [dB] vs Frequency [Hz]

A simpler representation in frequency domain. ^[SEP] Two peaks at ~5-8 Hz, 60 Hz

An improved understanding of rhythmic activity.

How?

Spectrum

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n).$$

Data as a function of time index n

Sinusoids at frequency f_j

**Frequency
resolution**

$$df = \frac{1}{T}$$

**Nyquist
frequency**

$$f_{\text{NQ}} = \frac{f_0}{2}$$

Spectrum

So, x is activity recorded from a single trial:

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

(Auto-)spectrum of signal x

Δ = sampling interval

T = total time of observation

X_j = Fourier transform of the data (x) at frequency j

x (LFP, EEG, ...)



Note: Time is discrete

x_n = Data at index n

complex conjugate

Spectrum

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n) . \quad \text{Fourier transform of the data } x.$$

$i = \sqrt{-1}$
↓

x_n = Data at index n

x (LFP, EEG, ...)

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t_n = Time at index n $t_n = \Delta n$ where Δ = sampling interval

f_j = Frequency at index j $f_j = j/T$ where T = total time of observation

Coherence: words

A constant phase relationship between two signals, at the same frequency, across trials.

Note

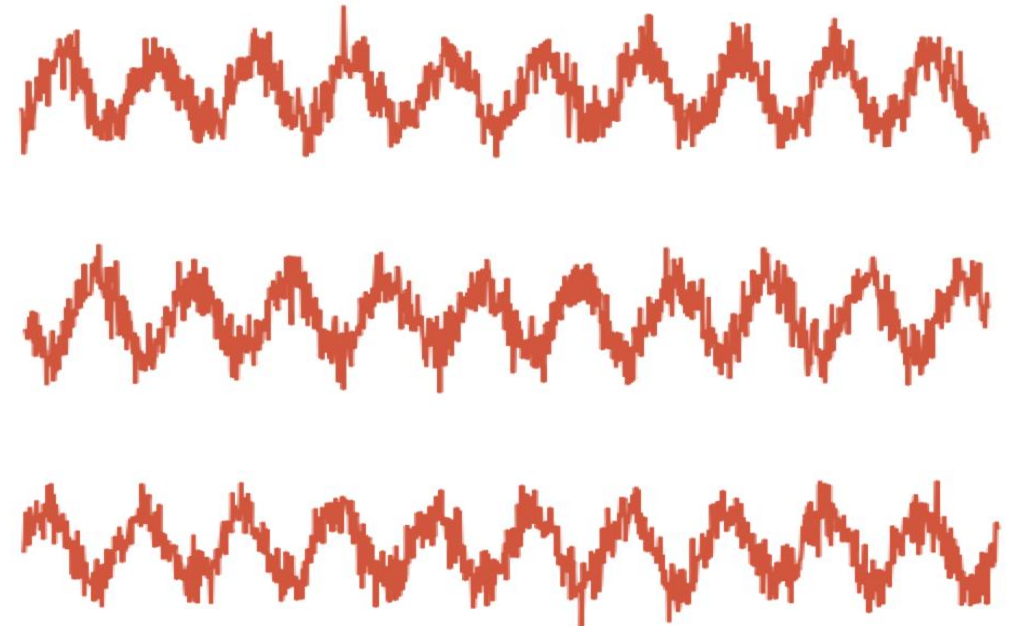
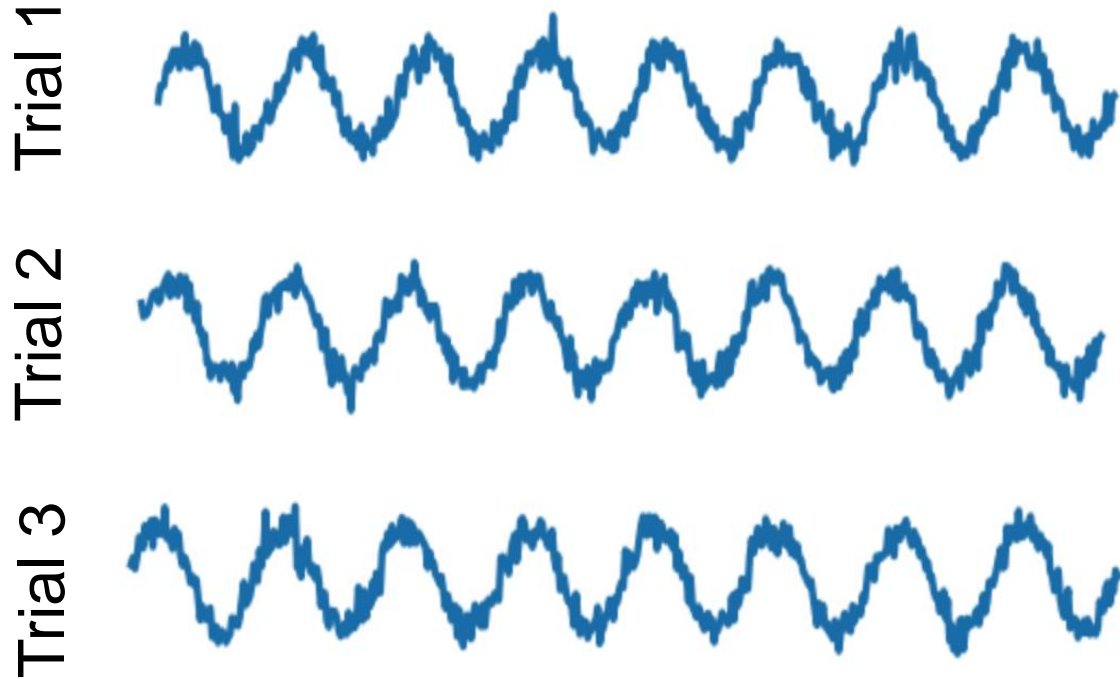
- “*same frequency*”
- “*across trials*”

Coherence: idea

Ex: Record data simultaneously from two sensors, across multiple trials

x (field: EEG, LFP, ...)

y (field: EEG, LFP, ...)

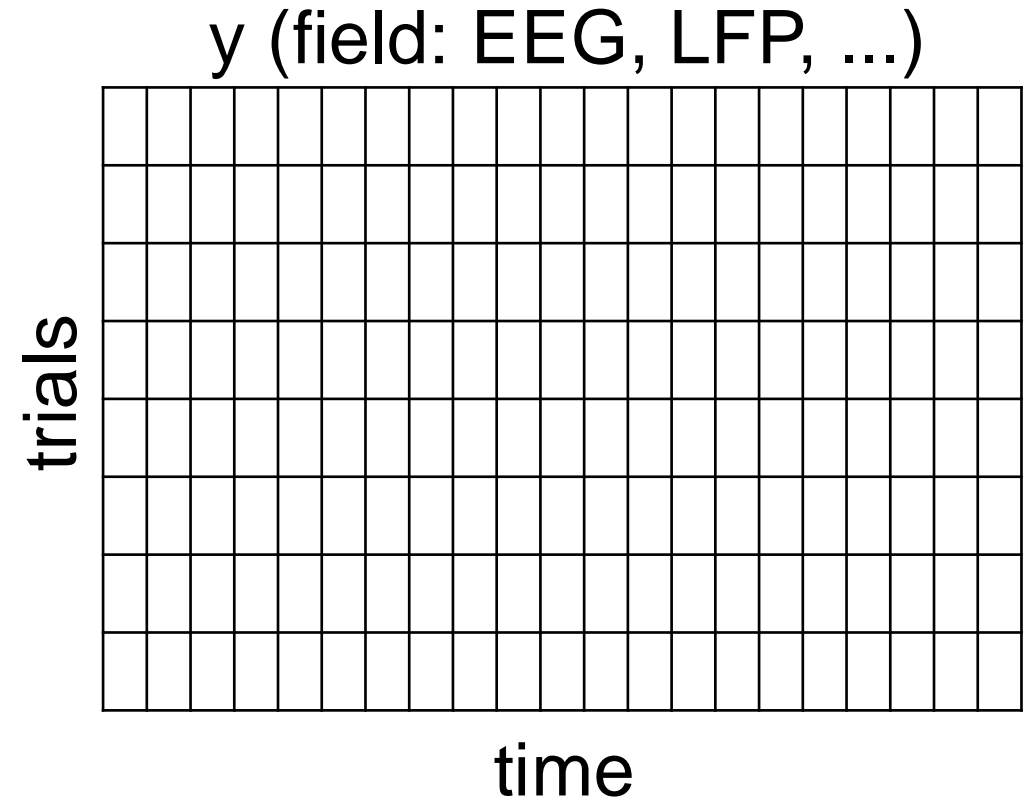
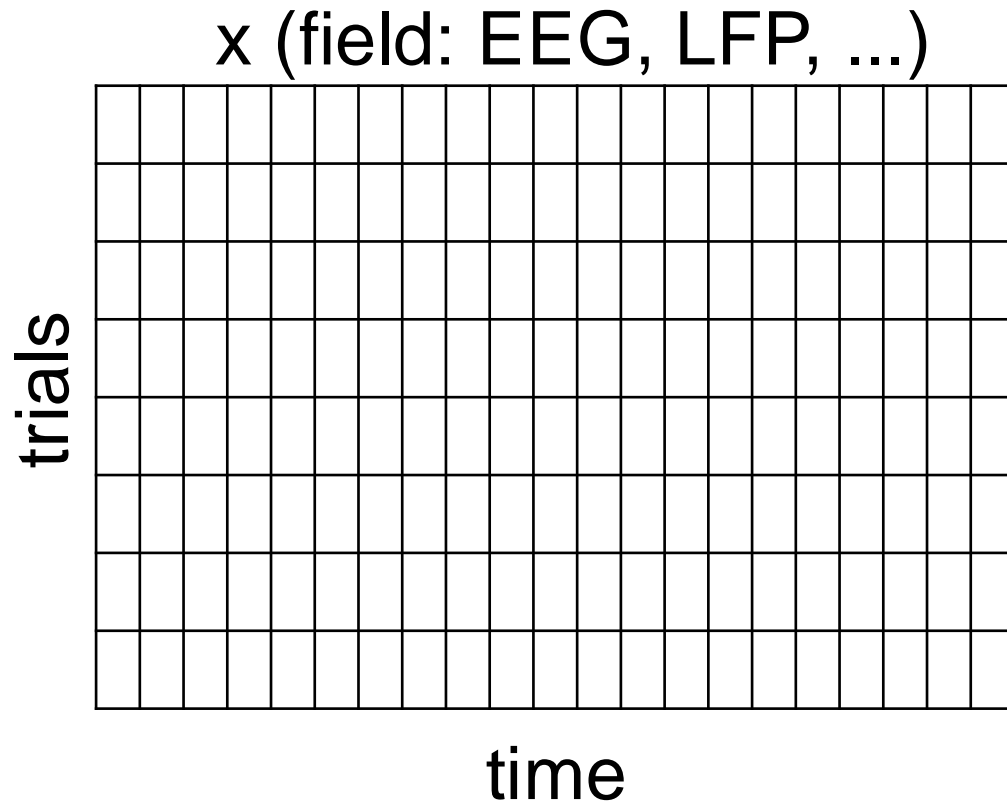


Is there a *constant phase relationship* between x & y , at the same f , across trials?

Coherence: idea

Ex: Record data simultaneously from two sensors, across multiple trials

Organize the data ...



Each row is a trial, each column is a time point, organize data in matrices.

Coherence: equations

This is what we'll compute:

$$K_{xy, j} = \frac{|\langle S_{xy, j} \rangle|}{\sqrt{\langle S_{xx, j} \rangle} \sqrt{\langle S_{yy, j} \rangle}}$$

$S_{xy, j}$ = Cross-spectrum at frequency index j

$S_{xx, j}, S_{yy, j}$ = Auto-spectra at frequency index j 

$\langle S \rangle$ = Average of S over trials

Define each piece ...

Spectrum: intuition

More spectrum intuition ...

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

Fourier transform of the data x .

$$X_j = \sum_{n=1}^N \boxed{x_n} \boxed{\exp(-2\pi i f_j t_n)}.$$

Data as a function of time index n

Sinusoids at frequency f_j

Replace with Euler's formula

$$X_j = \underbrace{\left(\sum_{n=1}^N x_n \cos(-2\pi f_j t_n) \right)}_{\text{Real}} + \underbrace{i}_{\uparrow} \underbrace{\left(\sum_{n=1}^N x_n \sin(-2\pi f_j t_n) \right)}_{\text{Imaginary}}$$

Spectrum: intuition

More spectrum intuition ...

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

Fourier transform of the data x .

$$X_j = \left(\sum_{n=1}^N x_n \cos(-2\pi f_j t_n) \right) + i \left(\sum_{n=1}^N x_n \sin(-2\pi f_j t_n) \right)$$

X_j can be **complex**

- the Fourier transform of x_n can have both real and imaginary parts.

So, X_j lives in the complex-plane ...

Spectrum: intuition

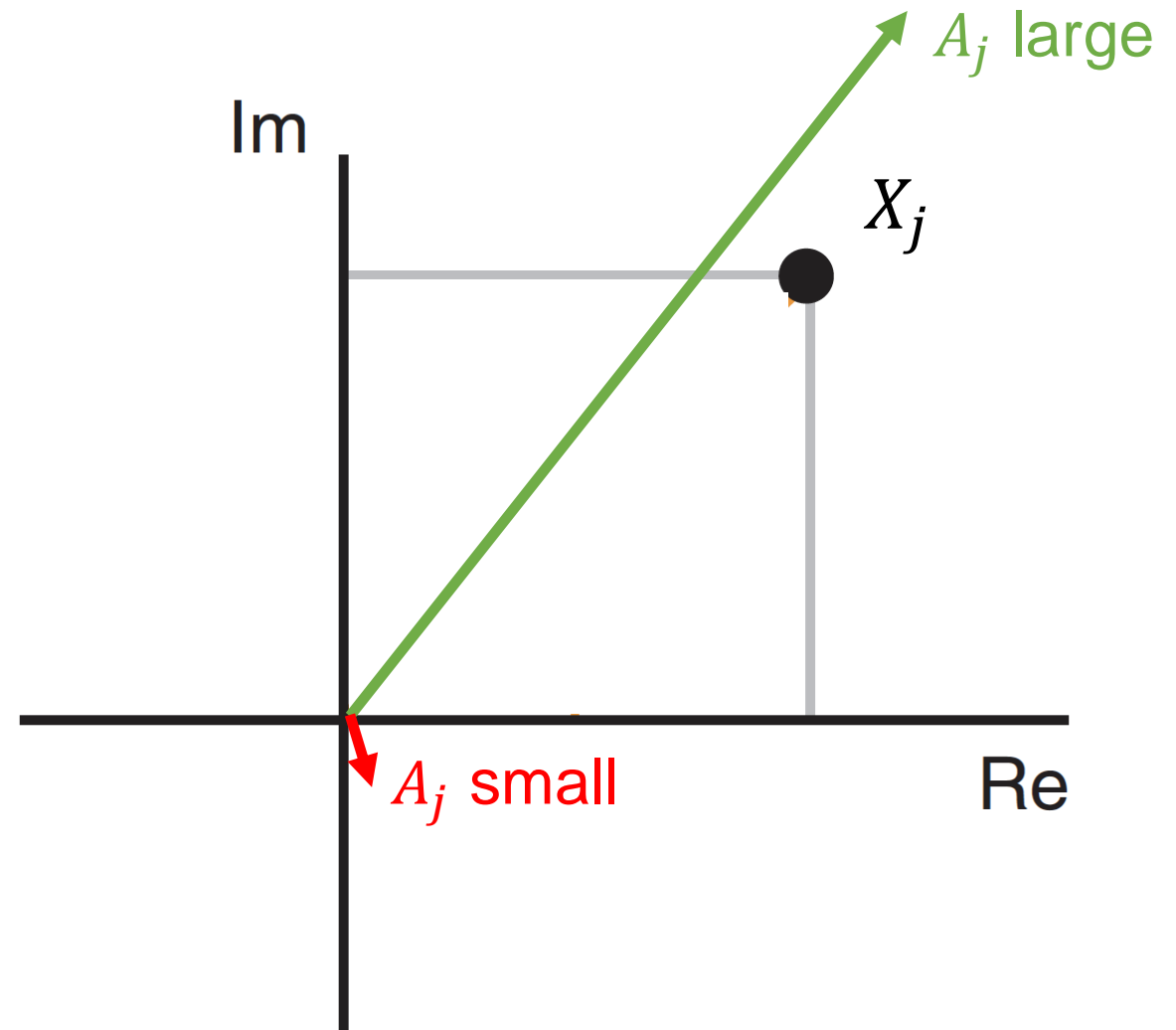
X_j lives in the complex-plane:

Express X_j in polar coordinates:

$$X_j = A_j \exp(i\phi_j)$$

A_j = Amplitude at frequency index j

ϕ_j = Phase at frequency index j



Match: A_j at frequency f_j is large

Mismatch: A_j at frequency f_j is small

Spectrum: intuition

Consider the spectrum of x_n :

Express X_j in polar coordinates:

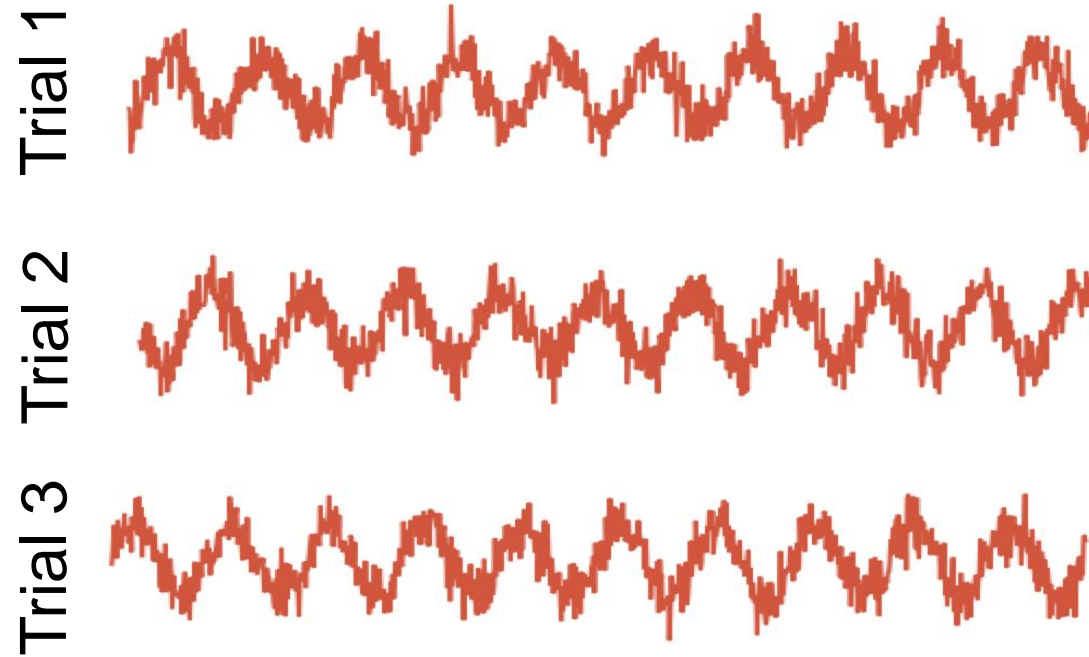
$$\begin{aligned} S_{xx,j} &= \frac{2\Delta^2}{T} X_j X_j^* = \frac{2\Delta^2}{T} (A_j \exp(i\phi_j)) (A_j \exp(-i\phi_j)) \\ &= \frac{2\Delta^2}{T} A_j^2 \exp(i\phi_j - i\phi_j) = \frac{2\Delta^2}{T} A_j^2. \end{aligned}$$

More direct interpretation of the spectrum at frequency f_j :

better match \rightarrow larger amplitude of X_j in the complex plane \rightarrow more power

Spectrum from multiple trials

Ex: Record data across multiple trials



K total trials

Q. Spectrum?

Spectrum from multiple trials

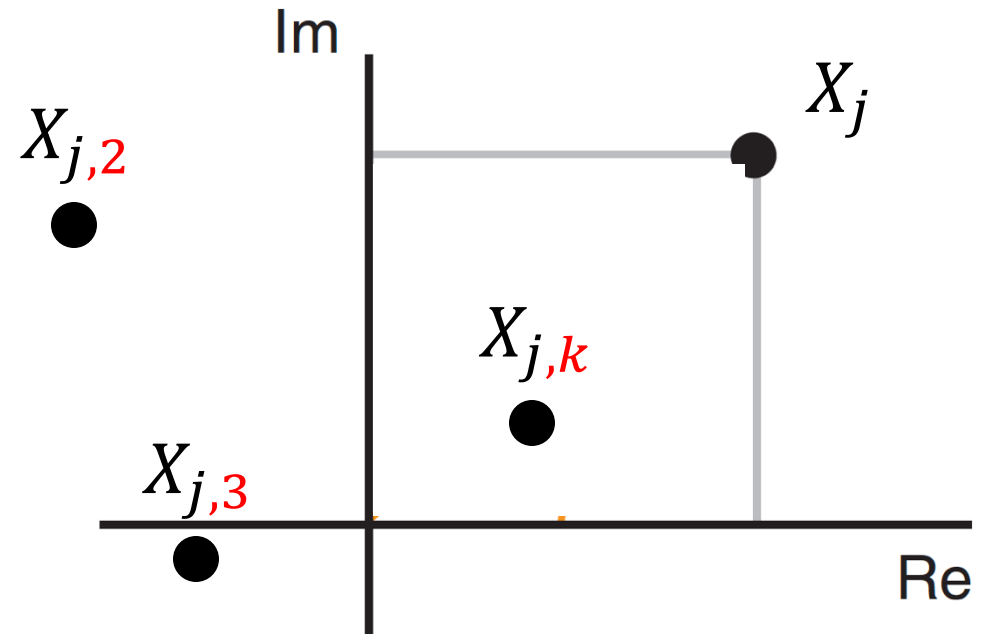
X_j lives in the complex-plane:

Fourier transform for each trial lives in the complex-plane:

In polar coordinates:

$A_{j,k}$ = Amplitude at frequency index j
and trial index k

$\phi_{j,k}$ = Phase at frequency index j
and trial index k



Spectrum from multiple trials

To compute coherence, we need the trial-averaged spectrum:

Single trial spectrum

$$\langle S_{xx,j} \rangle = \frac{2\Delta^2}{T} A_j^2 \cdot \left[\sum_{k=1}^K A_{j,k}^2 \right]$$

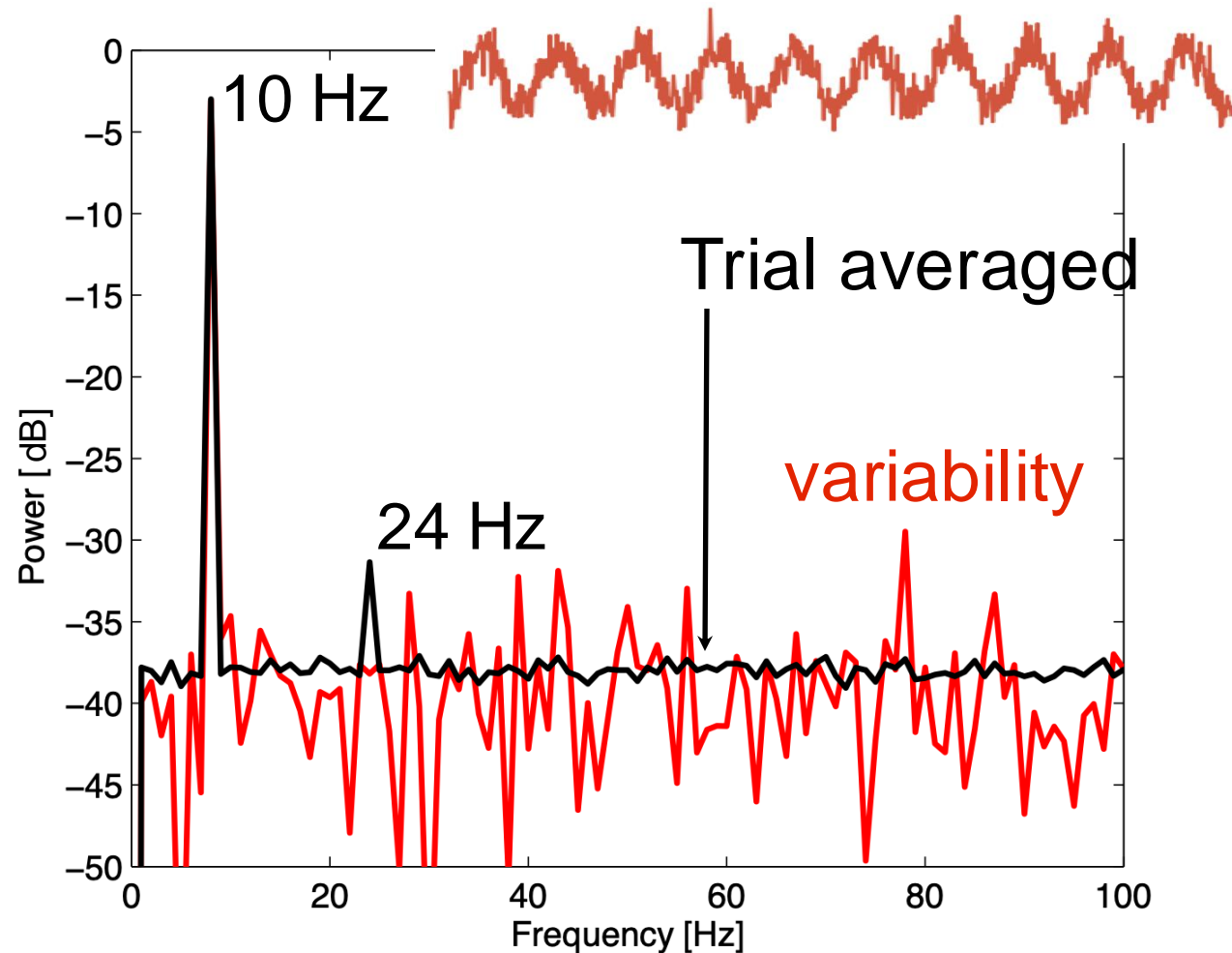
Average squared amplitude over trials

$A_{j,k}$ = the amplitude of the signal x , at frequency index j , and trial index k .

K = total number of trials

Spectrum from multiple trials

Single trial:



Trial averaged spectrum:

reduced variability.
reveals another peak . . .

Coherence: equations

Similarly, for signal y_n . Fourier transform of y at frequency j , and trial k :

$$Y_{j,k} = B_{j,k} \exp(i \theta_{j,k})$$

$B_{j,k}$ = the amplitude of the signal y at frequency index j and trial index k .

$\theta_{j,k}$ = the phase of the signal y at frequency index j and trial index k .

The trial-averaged spectrum of y at frequency index j

$$< S_{yy,j} > = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^K B_{j,k}^2$$

Coherence: equations

$$K_{xy, j} = \frac{|\langle S_{xy, j} \rangle|}{\sqrt{\langle S_{xx, j} \rangle} \sqrt{\langle S_{yy, j} \rangle}}$$

$$\langle S_{xx, j} \rangle = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^K A_{j,k}^2 \quad \langle S_{yy, j} \rangle = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^K B_{j,k}^2$$

Consider the trial averaged cross-spectrum ...

Coherence: equations

The trial averaged cross-spectrum at frequency index j:

$$\langle S_{xy,j} \rangle = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^K X_{j,k} Y_{j,k}^*$$

Like the auto-spectrum, but use X and Y.

In polar coordinates:

$$\langle S_{xy,j} \rangle = \frac{2\Delta^2}{T} \frac{1}{K} \sum_{k=1}^K A_{j,k} B_{j,k} \exp(i\Phi_{j,k})$$

Phase of x Phase of y

where $\Phi_{j,k} = \phi_{j,k} - \theta_{j,k}$ is the phase difference between the two signals, at frequency index j and trial k.

Coherence: equations

Put it all together ...

$$\kappa_{xy, j} = \frac{|\langle S_{xy, j} \rangle|}{\sqrt{\langle S_{xx, j} \rangle} \sqrt{\langle S_{yy, j} \rangle}}$$

In polar coordinates

cross-spectrum of x & y,
depends on trial averaged
amplitudes, phase differences.

$$= \frac{\left| \sum_{k=1}^K A_{j,k} B_{j,k} \exp(i\Phi_{j,k}) \right|}{\sqrt{\sum_{k=1}^K A_{j,k}^2} \sqrt{\sum_{m=1}^K B_{j,m}^2}}$$

x trial averaged spectrum,
at frequency index j

$$\sqrt{\sum_{k=1}^K A_{j,k}^2} \sqrt{\sum_{m=1}^K B_{j,m}^2}$$

y trial averaged spectrum,
at frequency index j

Coherence: intuition

To build intuition, assume: the amplitude is identical for both signals and all trials.

$$A_{j,k} = B_{j,k} = C_j \quad \text{Note: no trial dependence}$$

then

$$\kappa_{xy,j} = \frac{\left| \sum_{k=1}^K A_{j,k} B_{j,k} \exp(i\Phi_{j,k}) \right|}{\sqrt{\sum_{k=1}^K A_{j,k}^2} \sqrt{\sum_{m=1}^K B_{j,m}^2}}$$

only involves the phase difference between the two signals averaged across trials.

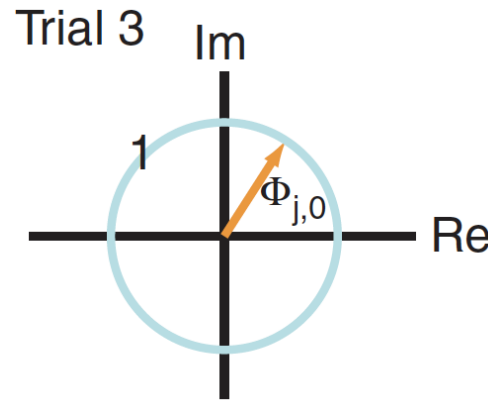
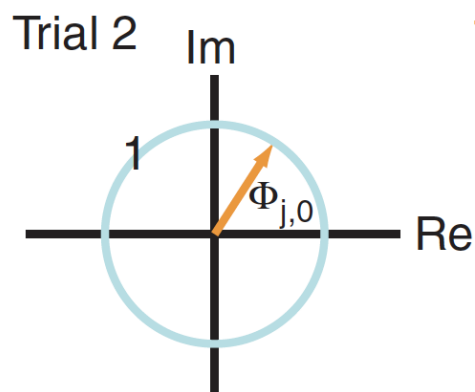
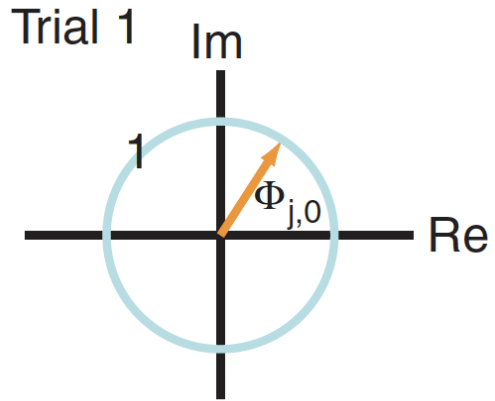
Coherence: intuition

Case 1: Phases align across trials.

$$\Phi_{j,k} = \Phi_{j,0}$$

$$K_{xy,j} = \frac{1}{K} \left| \sum_{k=1}^K \exp(i\Phi_{j,k}) \right|$$

Plot $\exp(i\Phi_{j,k})$ in the complex plane.

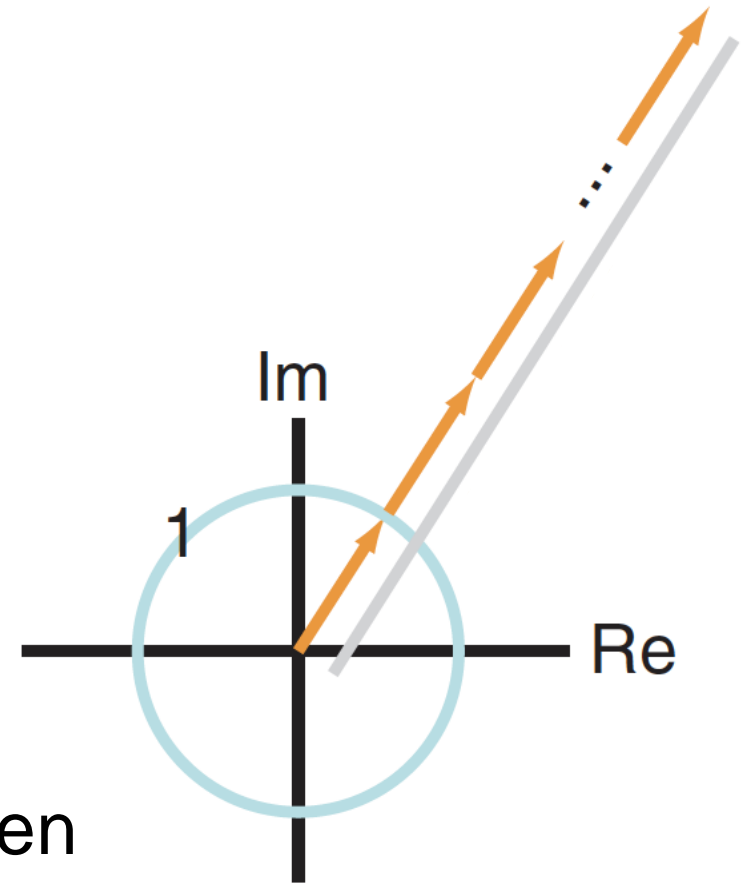


sum these vectors end to end across trials

divide by K

$$K_{xy,j} \approx 1$$

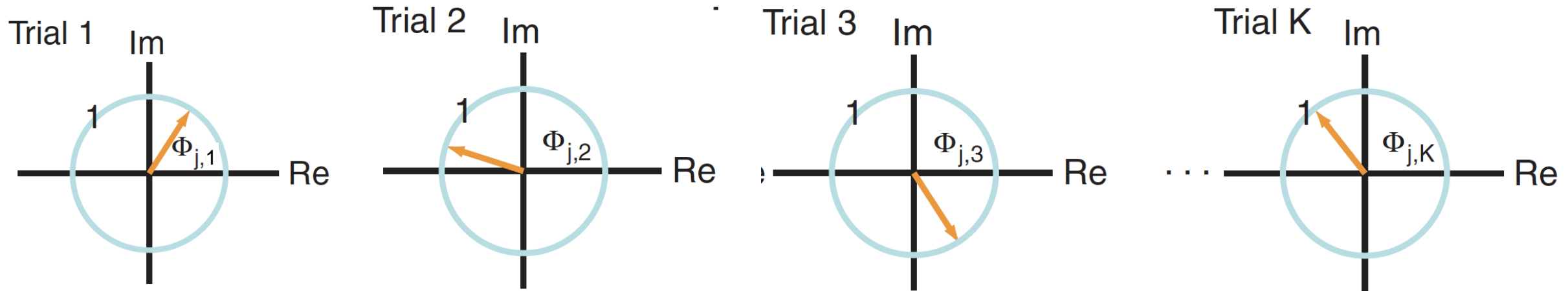
strong coherence - constant phase relation between the two signals across trials at frequency index j .



Coherence: intuition

Case 2: Random phase differences across trials.

Plot $\exp(i\Phi_{j,k})$ in the complex plane.

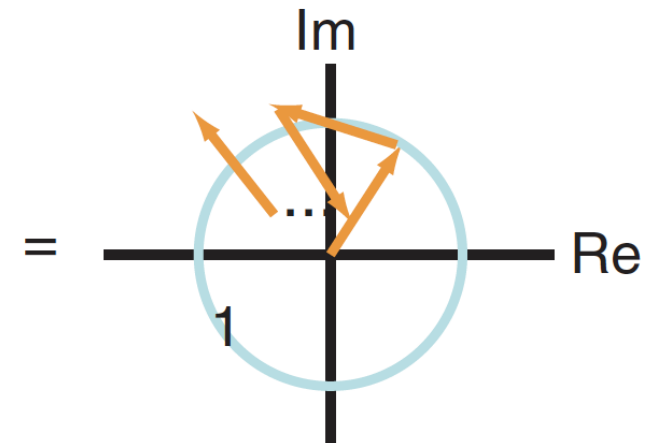


sum these vectors end to end across trials

divide by K

$$K_{xy,j} \approx 0$$

weak coherence - random phase relation between the two signals across trials at frequency index j .



$$K_{xy,j} = \frac{1}{K} \left| \sum_{k=1}^K \exp(i\Phi_{j,k}) \right|$$

Coherence: summary

$$0 \leq \kappa_{xy,j} \leq 1$$

0: no coherence between signals x and y at frequency index j

1: strong coherence between signals x and y at frequency index j .

The coherence is a measure of the phase consistency between two signals at frequency index j across trials.

Cross-covariance

Remember autocovariance

$$r_{xx}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(x_n - \bar{x})$$

Now **cross**-covariance

$$r_{xy}[L] = \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L} - \bar{x})(y_n - \bar{y})$$

Q. What's different?

Cross-covariance

Compute cross-covariance at different lags L

$L=0$



sum: $r_{xy}[0]$

$L=1$



sum: $r_{xy}[1]$

$L=2$



sum: $r_{xy}[2]$

Cross-covariance

Trial-averaged cross-covariance

$$r_{xy}[L] = \boxed{\frac{1}{K} \sum_{k=1}^K} \frac{1}{N} \sum_{n=1}^{N-L} (x_{n+L,k} - \bar{x}_k)(y_{n,k} - \bar{y}_k)$$

average over trials

signal x,
time $n+L$,
trial k.

mean of signal x,
trial k

signal y,
time n,
trial k.

mean of signal y,
trial k

Cross-covariance vs coherence

Remember:

$$S_{xx,j} = 2\Delta \sum_l r_{xx}[l] \exp\left(-\frac{2\pi i}{N} j l\right)$$

↑ ↑
auto-spectrum auto-covariance

In the same way:

$$S_{xy,j} = 2\Delta \sum_l r_{xy}[l] \exp\left(-\frac{2\pi i}{N} j l\right)$$

↑ ↑
cross-spectrum **cross-covariance**

The cross-spectrum is the Fourier transform of the cross-covariance.

(frequency domain)

(time domain)

Coherence: examples



Python