

MA666: Neural Networks and Learning

Part 1 Backpropagation

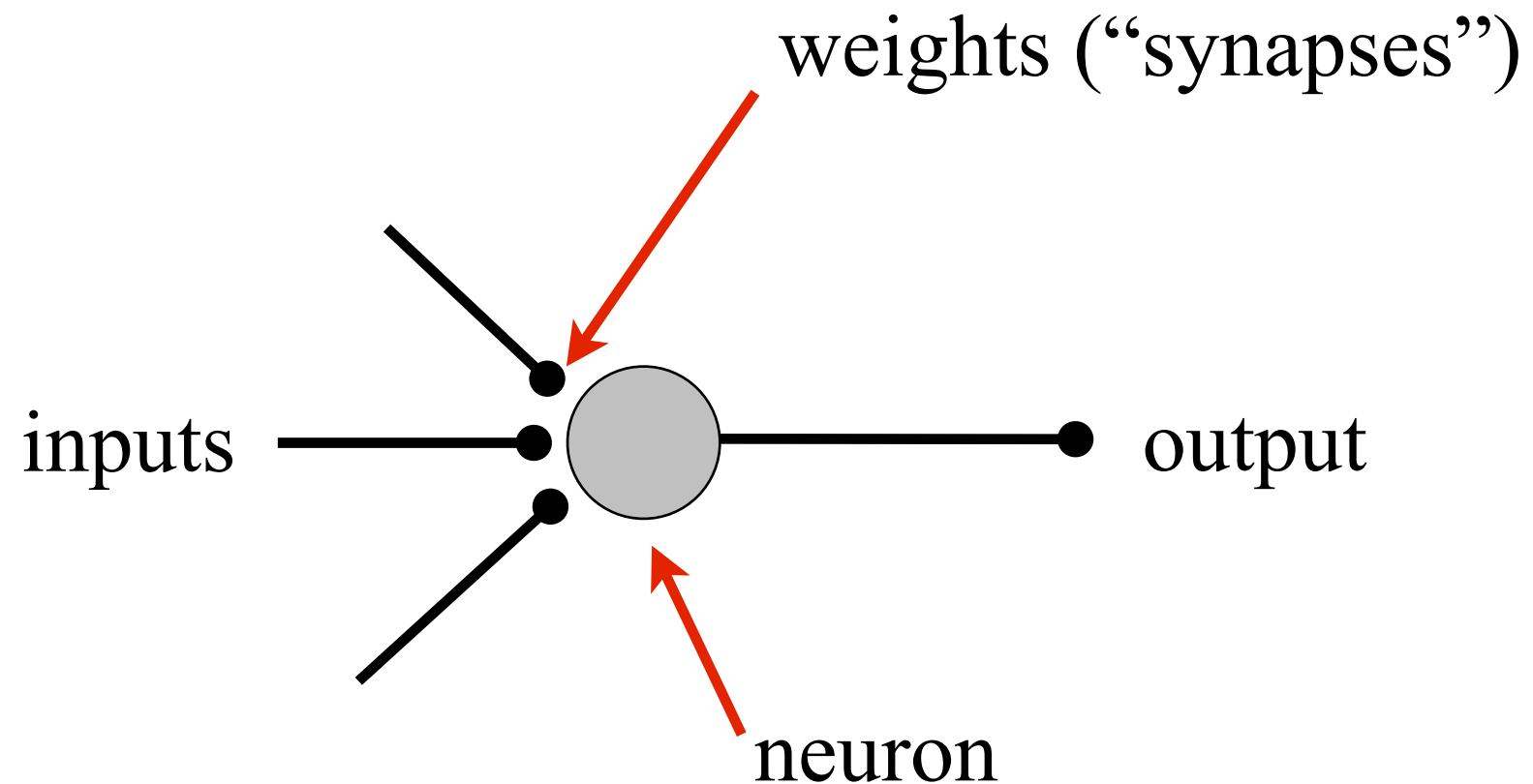
Today

We'll study learning in a “simple” neural network:

- Backpropagation

Remember, the Perceptron

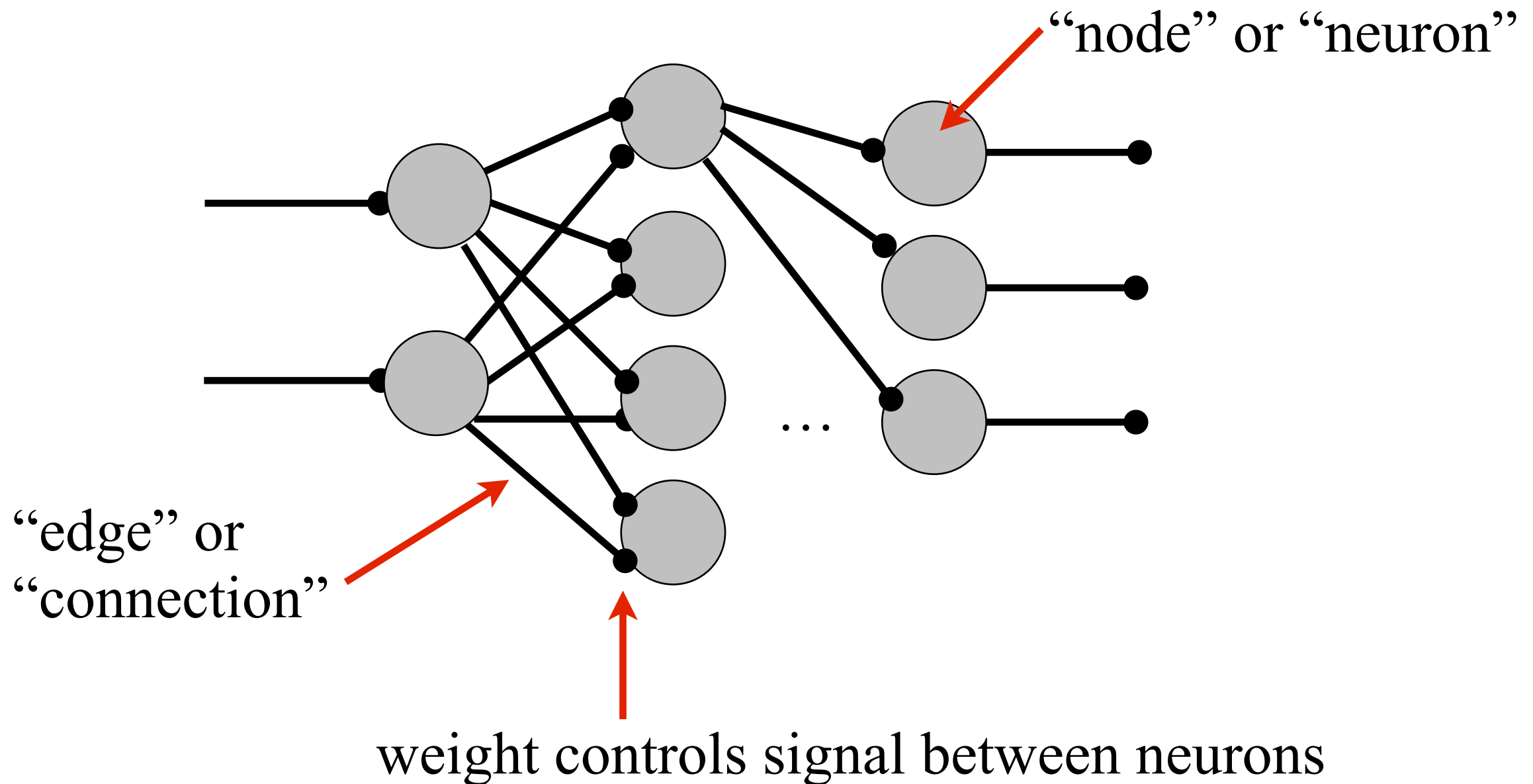
Cartoon & Cast of Characters



Note: there's only one neuron.

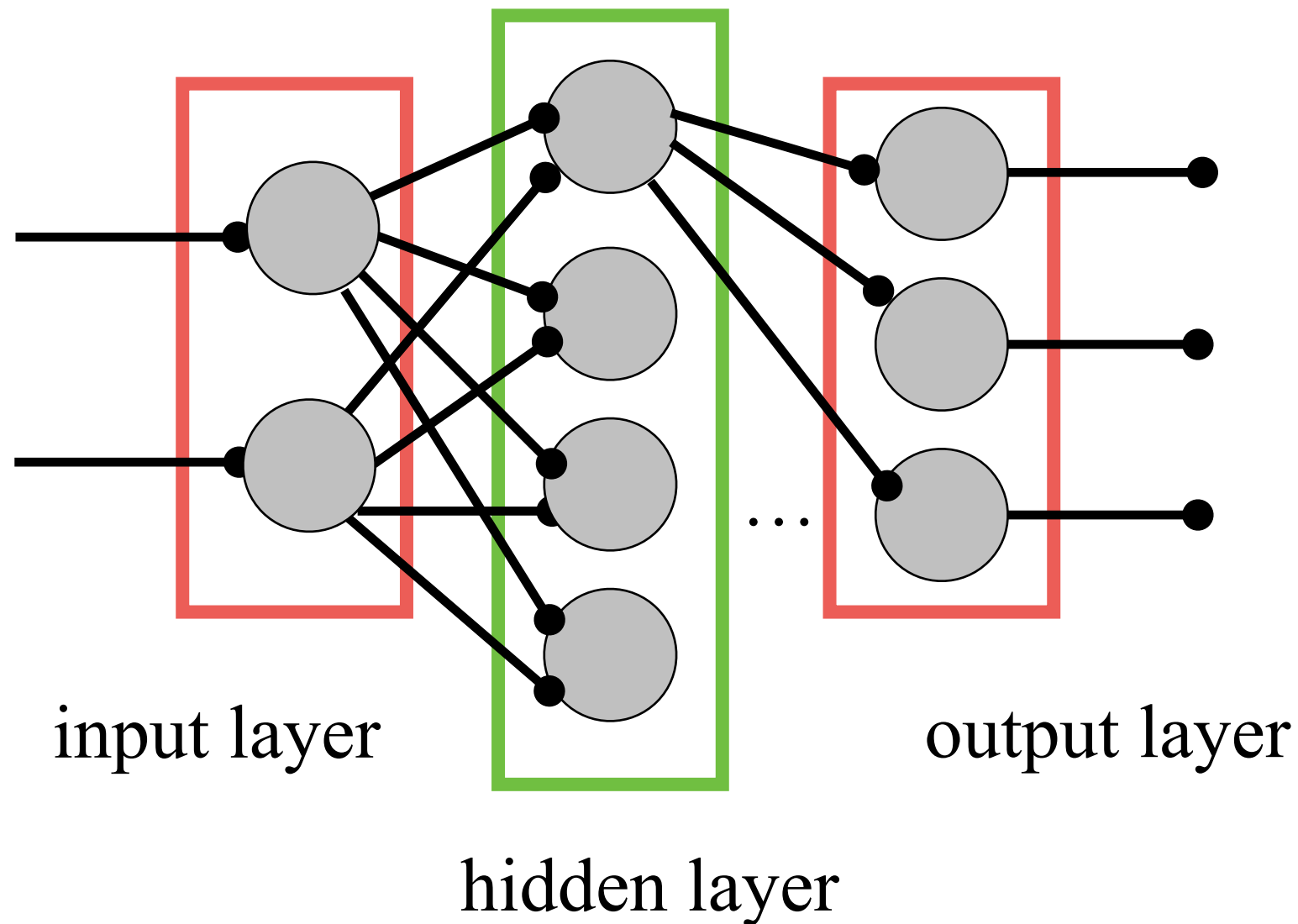
Today, a neural network

Today, **neural networks**: collections of abstracted neurons connected to each other through weighted connections (simplified “synapses”).



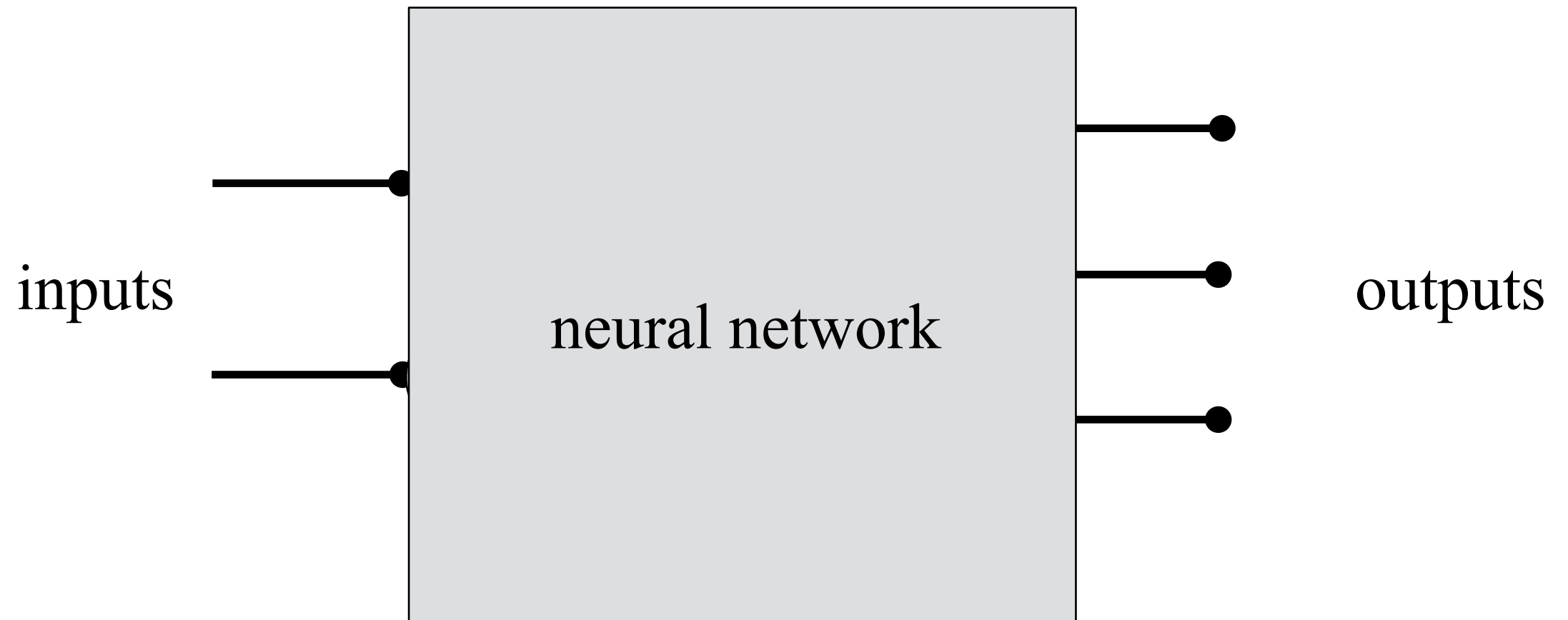
Today, a neural network

Today, **neural networks**: collections of abstracted neurons connected to each other through weighted connections (simplified “synapses”).



Information processed through the network

Abstractly:



Neural networks can exhibit rich behavior.

Cool example: playground.tensorflow.org

Neural networks can learn

Neural networks are:

- **adaptive**

- internal structure changes based on information flowing through the network.

- To do so, **adjust weights**.

- Idea:

- When network outputs are “good”, preserve the weights.
- When network output are “bad”, changes the weights.
 - When the network makes errors, adapt.

We trained a perceptron ...

Now, we'll train a neural network to do what we want ...

Neural networks can learn

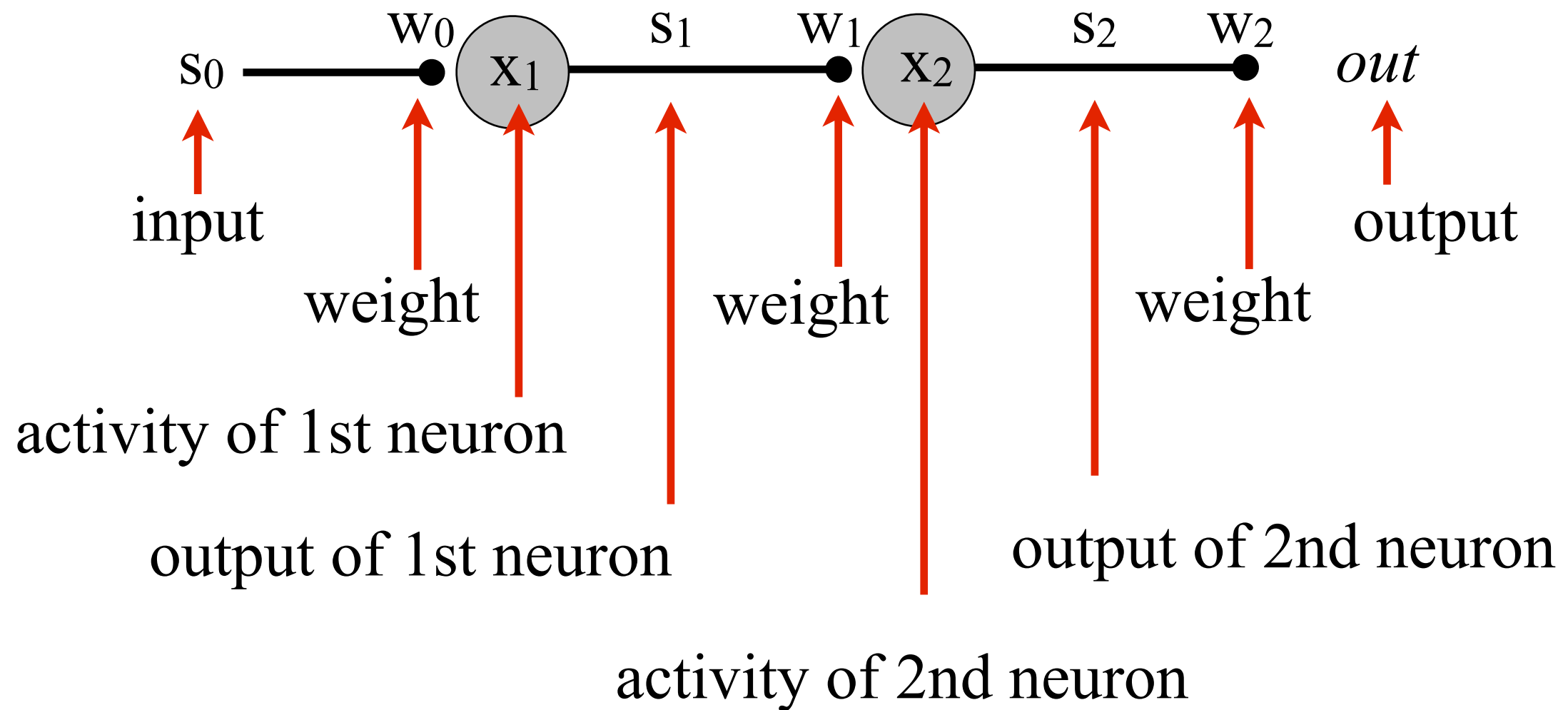
Some terminology:

- “*training a neural network*”
calibrate weights to get output we want.
- **Forward propagation**
For a set of weights & input, calculate output.
- **Backpropagation**
Determine error in output, and adjust weights to decrease error.

Let’s train a “simple” neural network to do our bidding ...

A “simple” neural network

Start with perceptron ... add a node ... and label everything.



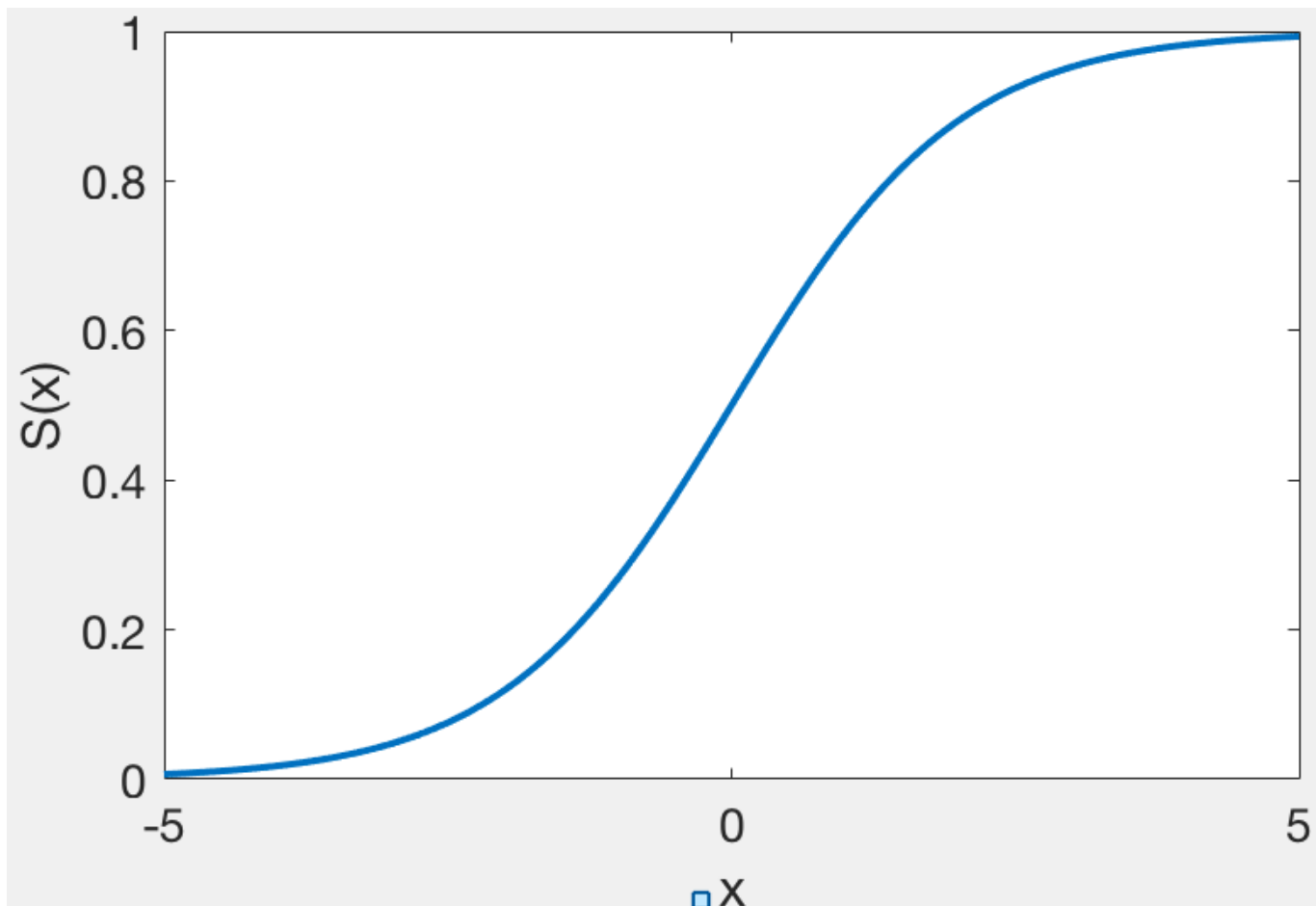
Activation function

Remember, the activation function:

x (activity) $\xrightarrow{\text{activation function}}$ output

Here we'll use a **sigmoid** activation function:

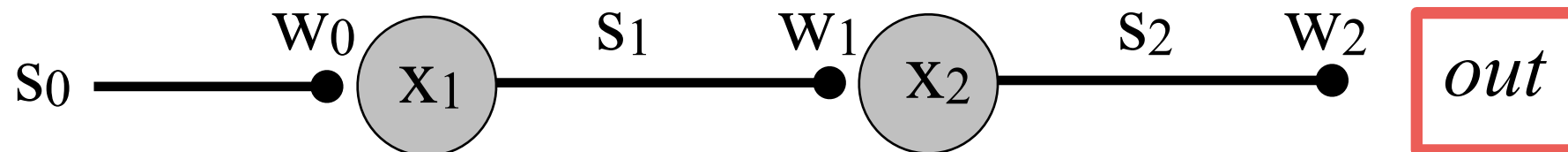
$$S(x) = 1 / (1 + e^{-x})$$



NOTE: It's like a
“smoothed” binary
threshold.

A “simple” neural network

We want our network to **learn** ...



so that when then input $s_0=2$,

the input $out=0.7$

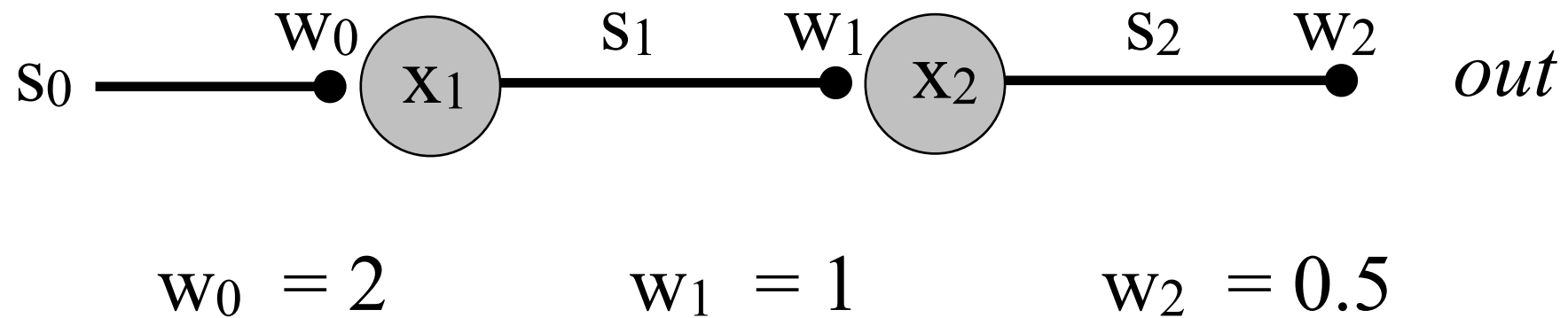
Q: How do we do it?

A: We need to choose the right weights: w_0 w_1 w_2

So, how do we find the right weights?

What are the right model weights?

Let's guess:

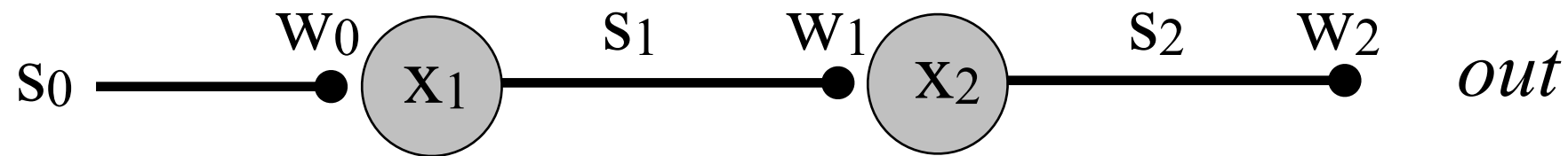


Q: How did I choose these?

Q: Do they work?

A: Let's check ... **forward propagation**

Forward propagation



$$s_0 = 2 \qquad w_0 = 2 \qquad w_1 = 1 \qquad w_2 = 0.5$$

target:
out = 0.7

Let's do it.

$$x_1 = w_0 s_0 = 2 * 2 = 4$$

$$s_1 = S(x_1) = S(4) = 0.982$$

$$x_2 = w_1 s_1 = 1 * 0.982 = 0.982$$

$$s_2 = S(x_2) = S(0.982) = 0.7275$$

$$\text{out} = w_2 s_2 = 0.5 * 0.7275 = 0.3638$$

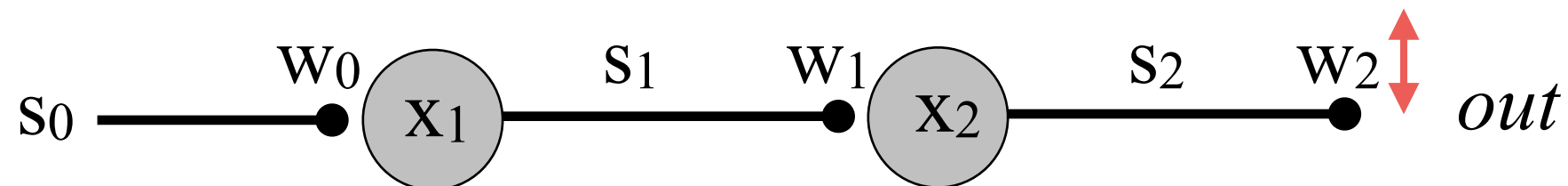
Match?

NO

How does a change in weight w_2 impact output?

Q: So now what?

(intermediate) Goal: get output (*out*) closer to target (0.7)



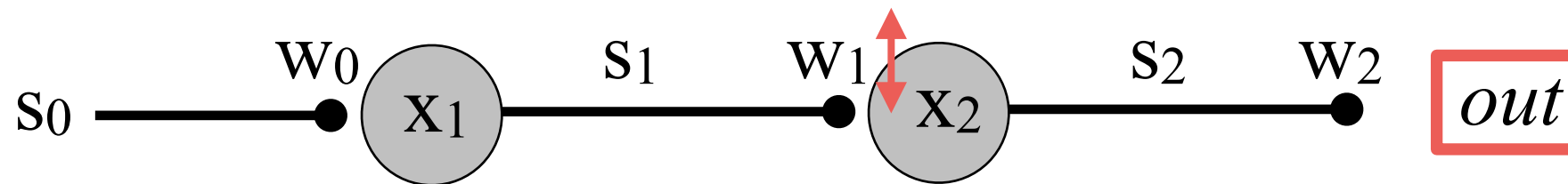
Q: How does a change in weight w_2 impact the output?

Idea: wiggle w_2 how does *out* change? $out = s_2 w_2$

Mathematically,
$$\frac{d \text{ out}}{d w_2} = \frac{d (s_2 w_2)}{d w_2} = s_2$$

How does a change in weight w_1 impact output?

Let's keep going, working backwards.



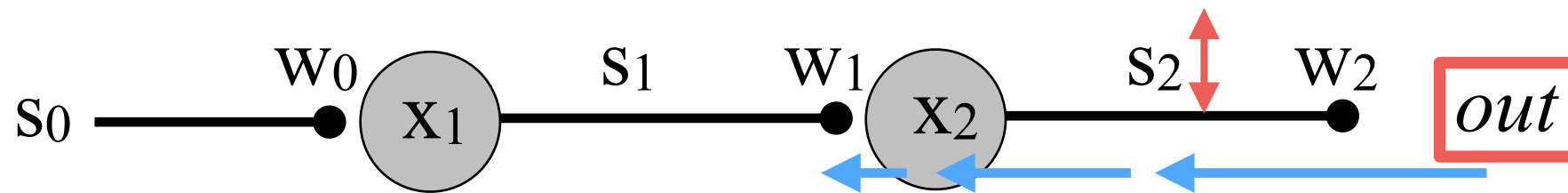
Q: How does a change in weight w_1 impact the output?

Idea: wiggle w_1 how does *out* change?

Mathematically, $\frac{d \text{out}}{d w_1} =$ Hmm ...

out does not depend directly on w_1

How does a change in weight w_1 impact output?



out does depend on s_2 and s_2 depend on x_2 and x_2 depend on w_1

Mathematically ... the **chain rule**

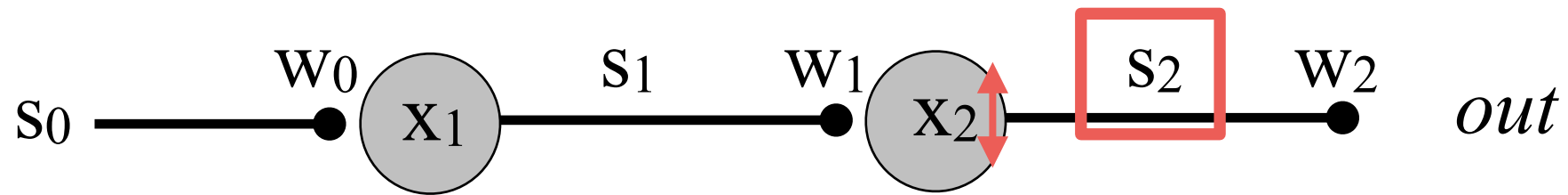
$$\frac{d \text{out}}{d w_1} = \boxed{\frac{d \text{out}}{d s_2}} \frac{d s_2}{d x_2} \frac{d x_2}{d w_1}$$

wiggle s_2 and *out* changes

Remember: $\text{out} = s_2 w_2$

$$\frac{d \text{out}}{d s_2} = \frac{d (s_2 w_2)}{d s_2} \boxed{= w_2}$$

How does a change in weight w_1 impact output?



Continue the **chain rule**

$$\frac{d \text{out}}{d w_1} = \frac{d \text{out}}{d s_2} \boxed{\frac{d s_2}{d x_2}} \frac{d x_2}{d w_1}$$

wiggle x_2 and s_2 changes

$$s_2 = S(x_2) = \left(\frac{1}{1 + e^{-x_2}} \right)$$

$$\text{so } \frac{d s_2}{d x_2} = \frac{d}{d x_2} \left(\frac{1}{1 + e^{-x_2}} \right)$$

A complicated derivative

We need to compute:

$$\frac{d}{dx_2} \left(\frac{1}{1 + e^{-x_2}} \right) = \text{Hmm ... Quotient Rule}$$

$$= \frac{(1 + e^{-x_2}) \frac{d(1)}{dx_2} - (1) \frac{d(1 + e^{-x_2})}{dx_2}}{(1 + e^{-x_2})^2}$$

Q: What is $\frac{d(1 + e^{-x_2})}{dx_2}$?

$$= \frac{d(1)}{dx_2} + \frac{d(e^{-x_2})}{dx_2} = (e^{-x_2}) \frac{d(-x_2)}{dx_2} = -e^{-x_2}$$

A complicated derivative

So,

$$\begin{aligned}\frac{d}{dx_2} \left(\frac{1}{1 + e^{-x_2}} \right) &= \frac{0 - (1)(-e^{-x_2})}{(1 + e^{-x_2})^2} \\ &= \frac{e^{-x_2}}{(1 + e^{-x_2})^2}\end{aligned}$$

Q: Can we simplify this expression?

A: Yes, but requires faith ...

- Split up denominator:
$$\frac{e^{-x_2}}{(1 + e^{-x_2})^2} = \left(\frac{1}{1 + e^{-x_2}} \right) \left(\frac{e^{-x_2}}{1 + e^{-x_2}} \right)$$

A complicated derivative

- Add 0 to the second term:

$$\left(\frac{1}{1+e^{-x_2}}\right)\left(\frac{e^{-x_2}}{1+e^{-x_2}}\right) = \left(\frac{1}{1+e^{-x_2}}\right)\left(\frac{\boxed{1} + e^{-x_2}\boxed{-1}}{1+e^{-x_2}}\right)$$

Sum is 0

Q: Why?

A: Let's organize terms ...

$$= \underbrace{\left(\frac{1}{1+e^{-x_2}}\right)}_{= s_2} \left(\underbrace{\left(\frac{1+e^{-x_2}}{1+e^{-x_2}}\right)}_{= 1} - \underbrace{\left(\frac{1}{1+e^{-x_2}}\right)}_{= s_2} \right)$$

Remember:

$$s_2 = \left(\frac{1}{1+e^{-x_2}}\right)$$

$$= (s_2) (1 - s_2)$$

A complicated derivative

So, for our chain rule calculation:

$$\frac{d \text{ out}}{d w_1} = \frac{d \text{ out}}{d s_2} \boxed{\frac{d s_2}{d x_2}} \frac{d x_2}{d w_1}$$

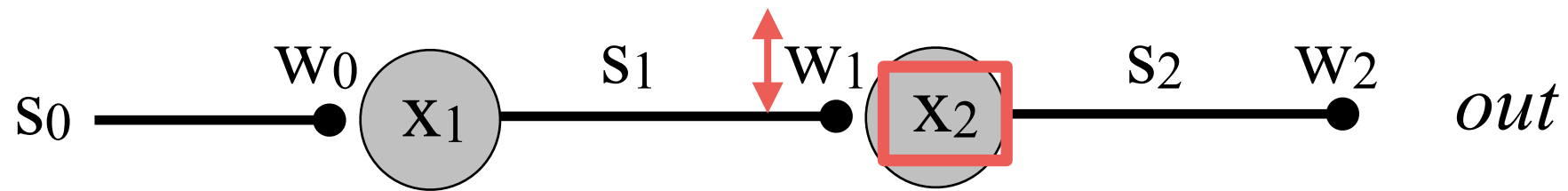
And we found:

$$\frac{d s_2}{d x_2} = \frac{d}{dx_2} \left(\boxed{\frac{1}{1 + e^{-x_2}}} \right) = \dots \text{many steps} \dots \boxed{= s_2 (1 - s_2)}$$

$\boxed{\frac{1}{1 + e^{-x_2}}} = s_2$

To complete the chain rule, one more derivative ...

How does a change in weight w_1 impact output?



Continue the **chain rule**:

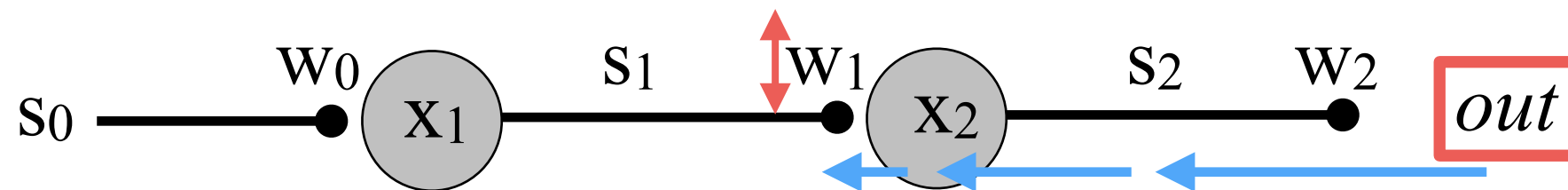
$$\frac{d \text{ out}}{d w_1} = \frac{d \text{ out}}{d s_2} \frac{d s_2}{d x_2} \boxed{\frac{d x_2}{d w_1}}$$

wiggle w_1 and x_2 changes

Remember: $x_2 = s_1 w_1$

$$\frac{d x_2}{d w_1} = \frac{d (s_1 w_1)}{d w_1} = \boxed{s_1}$$

Back to our original question:



Q: How does a change in weight w_1 impact the output?

Mathematically ... the **chain rule**

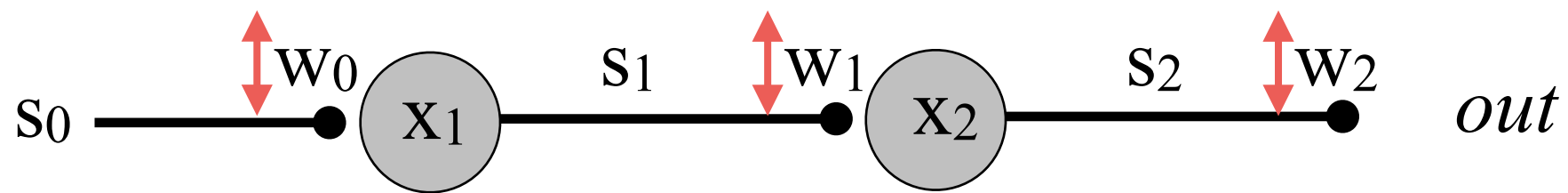
$$\frac{d \text{out}}{d w_1} = \frac{d \text{out}}{d s_2} \quad \frac{d s_2}{d x_2} \quad \frac{d x_2}{d w_1}$$

$$\frac{d \text{out}}{d w_1} = w_2 \quad s_2 (1 - s_2) \quad s_1$$

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How does a change in weight w_0 impact output?

We're almost there ...



$$\frac{d \text{out}}{d w_2} = s_2$$

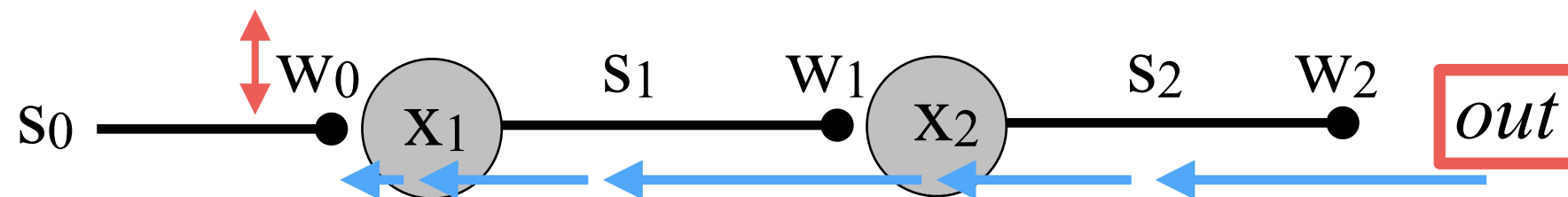
$$\frac{d \text{out}}{d w_1} = w_2 s_2 (1 - s_2) s_1$$

Q: How does a change in weight w_0 impact the output?

A: Chain rule ...

How does a change in weight w_0 impact output?

Q: How does a change in weight w_0 impact the output?



$$\frac{d \text{out}}{d w_0} = \frac{d \text{out}}{d s_2} \frac{d s_2}{d x_2} \boxed{\frac{d x_2}{d s_1} \frac{d s_1}{d x_1} \frac{d x_1}{d w_0}} \quad \text{Ugh ...}$$

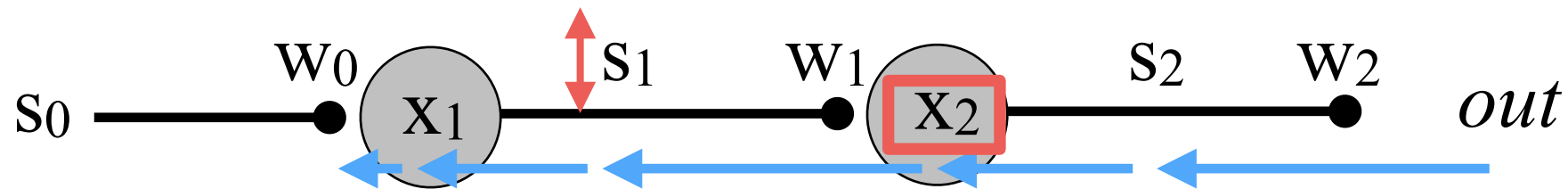
Luckily, we've already calculated two of these.

$$w_2 \quad s_2 (1 - s_2)$$

Let's compute the last 3 terms ...

How does a change in weight w_0 impact output?

3rd term:



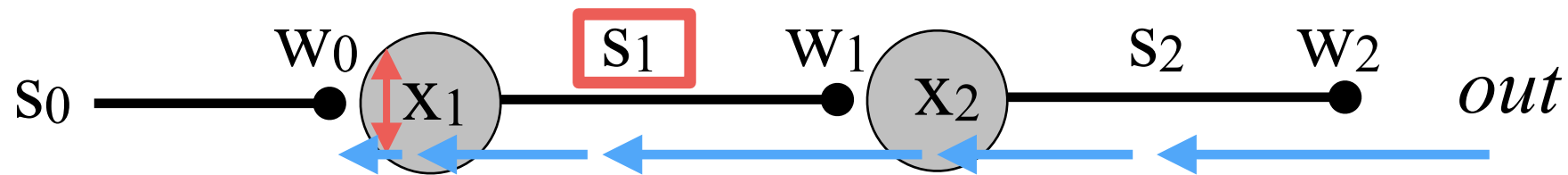
$$\frac{d \text{out}}{d w_0} = \frac{d \text{out}}{d s_2} \frac{d s_2}{d x_2} \boxed{\frac{d x_2}{d s_1}} \frac{d s_1}{d x_1} \frac{d x_1}{d w_0}$$

Remember: $x_2 = s_1 w_1$

$$\frac{d x_2}{d s_1} = \frac{d (s_1 w_1)}{d s_1} = \boxed{w_1}$$

How does a change in weight w_0 impact output?

4th term:



$$\frac{d out}{d w_0} = \frac{d out}{d s_2} \frac{d s_2}{d x_2} \frac{d x_2}{d s_1} \boxed{\frac{d s_1}{d x_1}} \frac{d x_1}{d w_0}$$

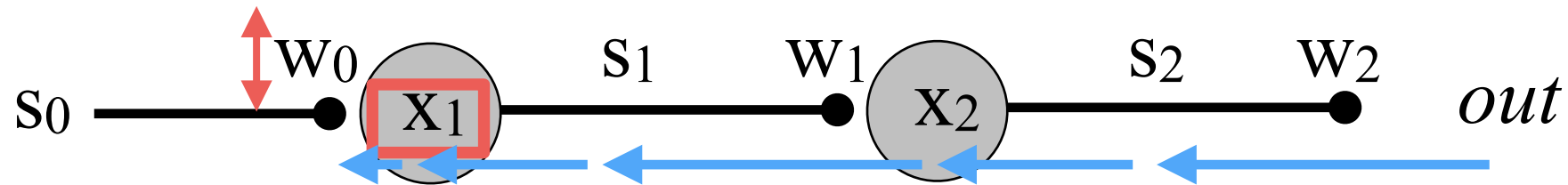
We found earlier that:

$$\frac{d s_2}{d x_2} = s_2 (1 - s_2) \quad \dots \text{ so } \dots \quad \frac{d s_1}{d x_1} = \boxed{s_1 (1 - s_1)}$$

This involved many steps!

How does a change in weight w_0 impact output?

5th term:



$$\frac{d \text{out}}{d w_0} = \frac{d \text{out}}{d s_2} \frac{d s_2}{d x_2} \frac{d x_2}{d s_1} \frac{d s_1}{d x_1} \boxed{\frac{d x_1}{d w_0}}$$

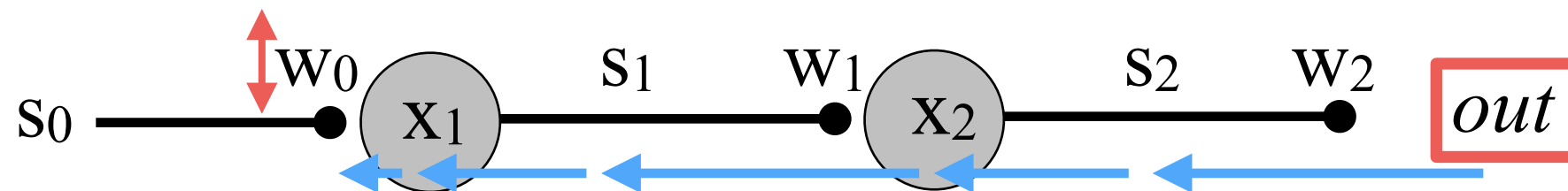
Remember: $x_1 = s_0 w_0$

$$\frac{d x_1}{d w_0} = \frac{d (s_0 w_0)}{d w_0} = \boxed{s_0}$$

How does a change in weight w_0 impact output?

We now have the pieces to answer:

Q: How does a change in weight w_0 impact the output?



$$\frac{d \text{out}}{d w_0} = \frac{d \text{out}}{d s_2} \frac{d s_2}{d x_2} \frac{d x_2}{d s_1} \frac{d s_1}{d x_1} \frac{d x_1}{d w_0}$$

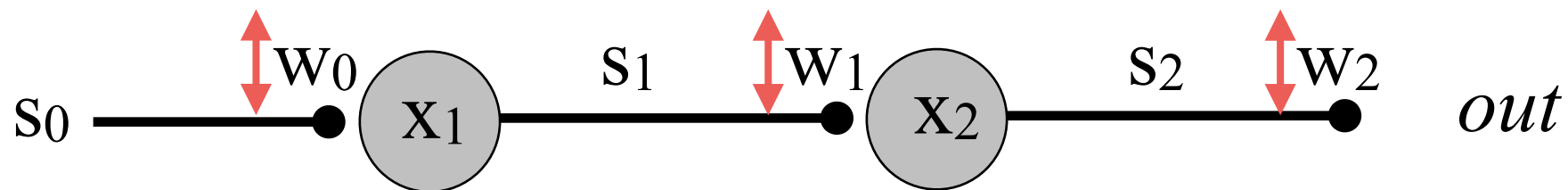
$$= w_2 \quad s_2 (1 - s_2) \quad w_1 \quad s_1 (1 - s_1) \quad s_0$$

Slide 16 *Slide 21* *Slide 26* *Slide 27* *Slide 28*

How does a change in weight impact output?

To summarize:

- We've found how changes in model weights impact output.



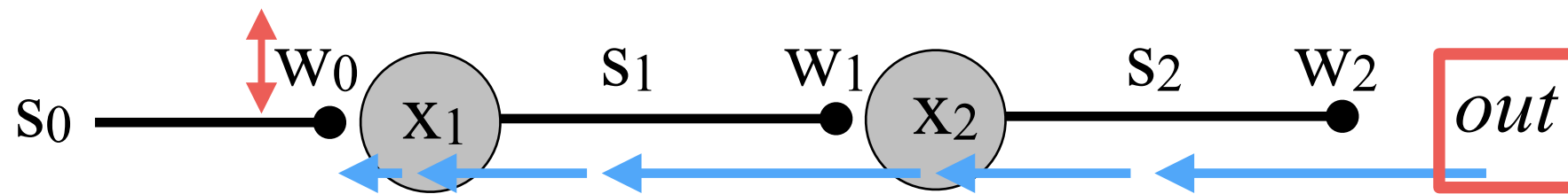
$$\frac{d \text{out}}{d w_2} = s_2$$

$$\frac{d \text{out}}{d w_1} = w_2 s_2 (1 - s_2) s_1$$

$$\frac{d \text{out}}{d w_0} = w_2 s_2 (1 - s_2) w_1 s_1 (1 - s_1) s_0$$

So, how does a change in weight impact output? **backpropagation!**

How does a change in weight impact output?



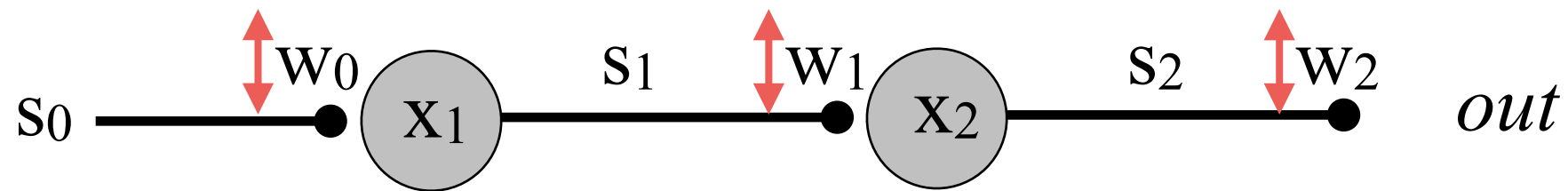
Backpropagation:

Work “**backwards**” from output to weight, computing derivatives along the way

Q: How do these derivatives help us update weights and obtain desired output?

Define our goal

We want:



$$out = target$$

rearrange $out - target = 0$ **GOAL**

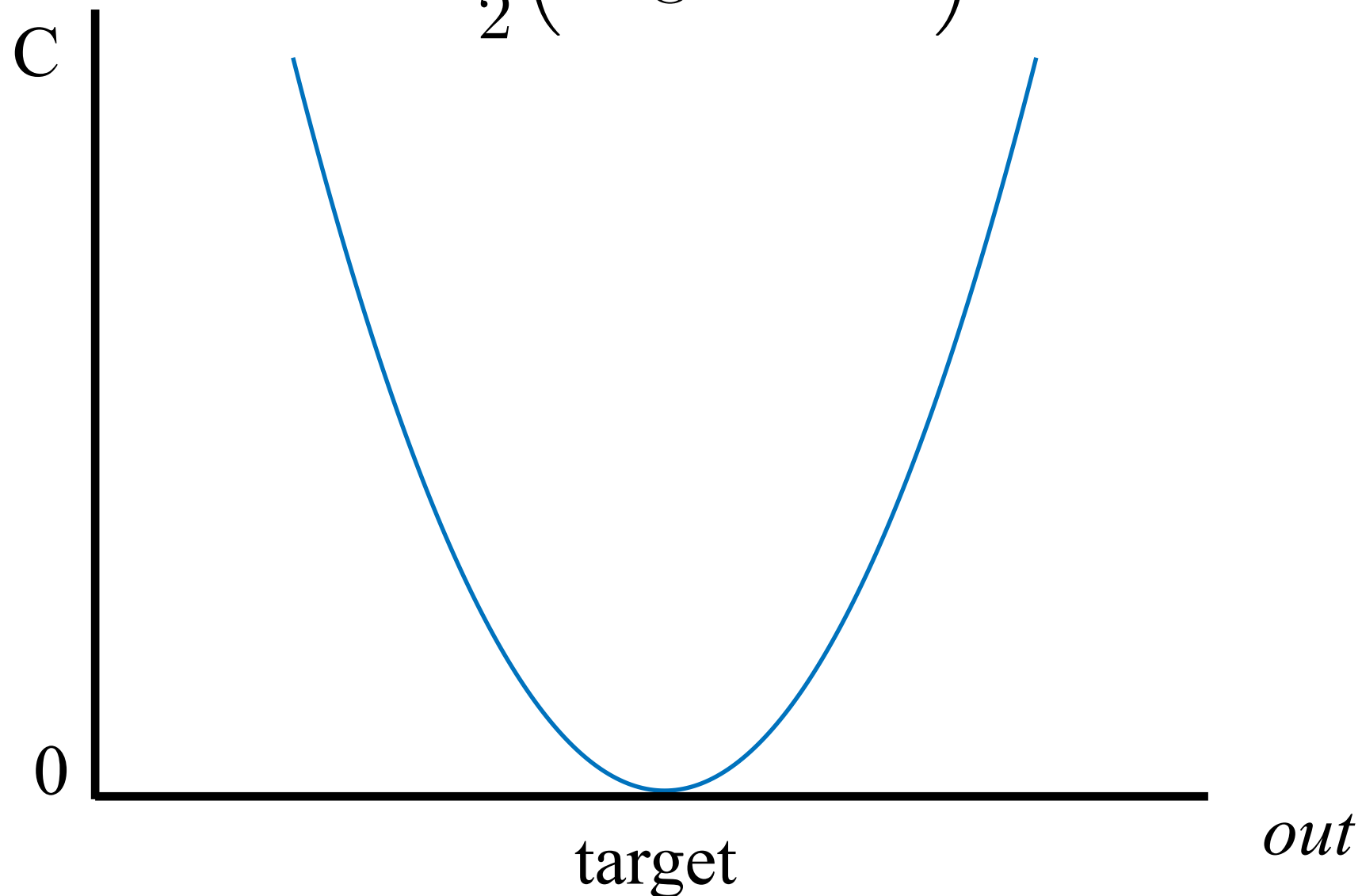
Let's use this to define a **cost function** ...

Create a cost function

Define the cost function:

$$C = \frac{1}{2} (\text{target} - out)^2$$

Plot it:



Q: Where is the cost zero?

A: When $out = \text{target}$.

Create a cost function

Q: Why this cost function?

$$C = \frac{1}{2} \left(\text{target} - out \right)^2$$

- Minimum (the “lowest cost”) when $out = \text{target}$.
- It’s convenient (a quadratic).
- It steadily increases as out deviates from target.
- It’s “easy” to compute derivatives.

Create a cost function

Q: How does the cost function change due to changes in *out*?

A: We need to compute a derivative ...

$$\frac{dC}{dout} = \frac{d}{out} \left[\frac{1}{2} \left(\text{target} - out \right)^2 \right] = ?$$

Chain rule ...

$$\frac{dC}{dout} = 2 \cdot \frac{1}{2} (\text{target} - out) \cdot \left(\frac{d(-out)}{dout} \right)$$

$$\frac{dC}{dout} = -(\text{target} - out)$$

$$\frac{dC}{dout} = out - target$$

Create a cost function

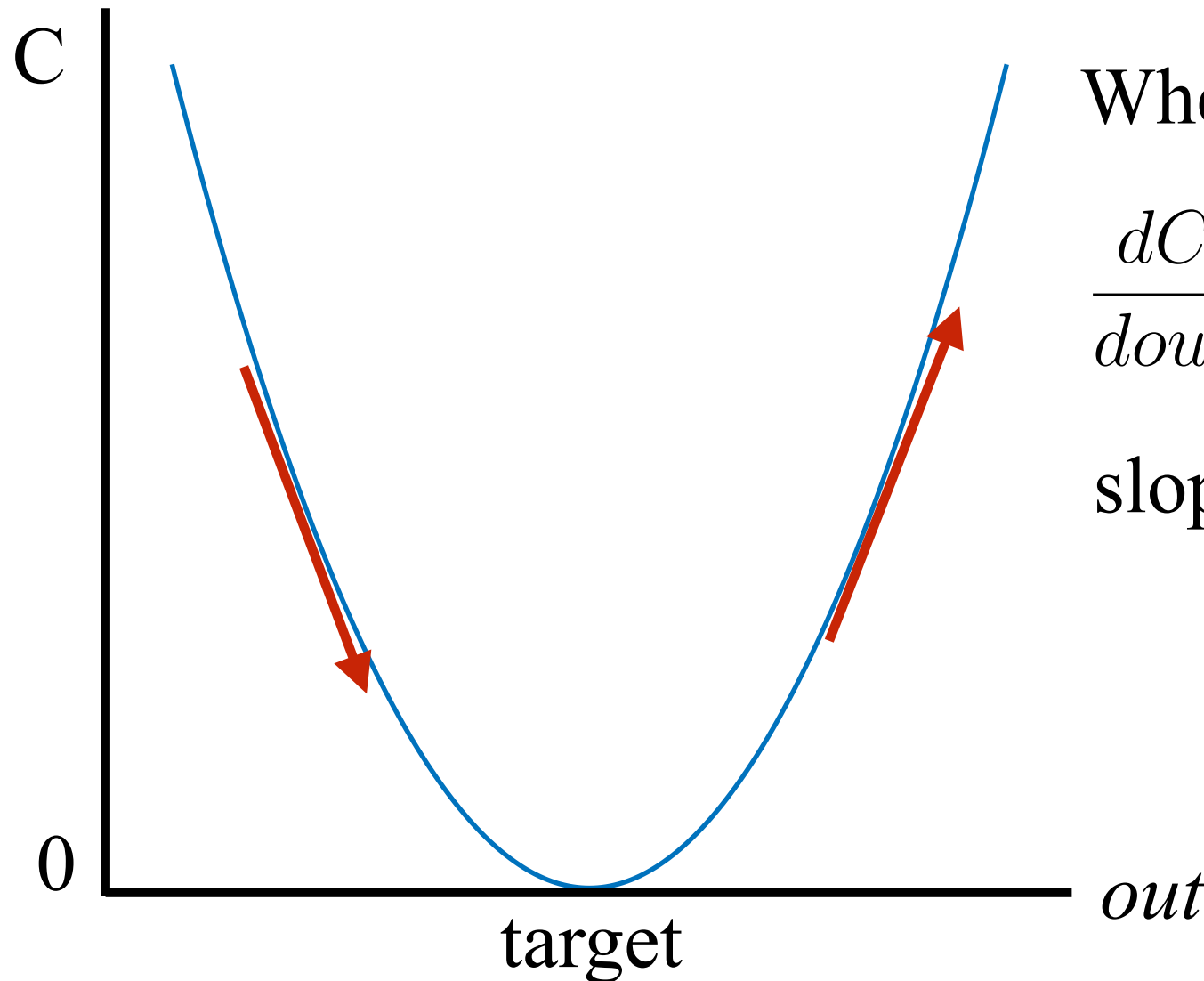
Q: Does this derivative make sense?

$$\frac{dC}{dout} = out - target$$

When $out < target$

$$\frac{dC}{dout} < 0$$

slope < 0



When $out > target$

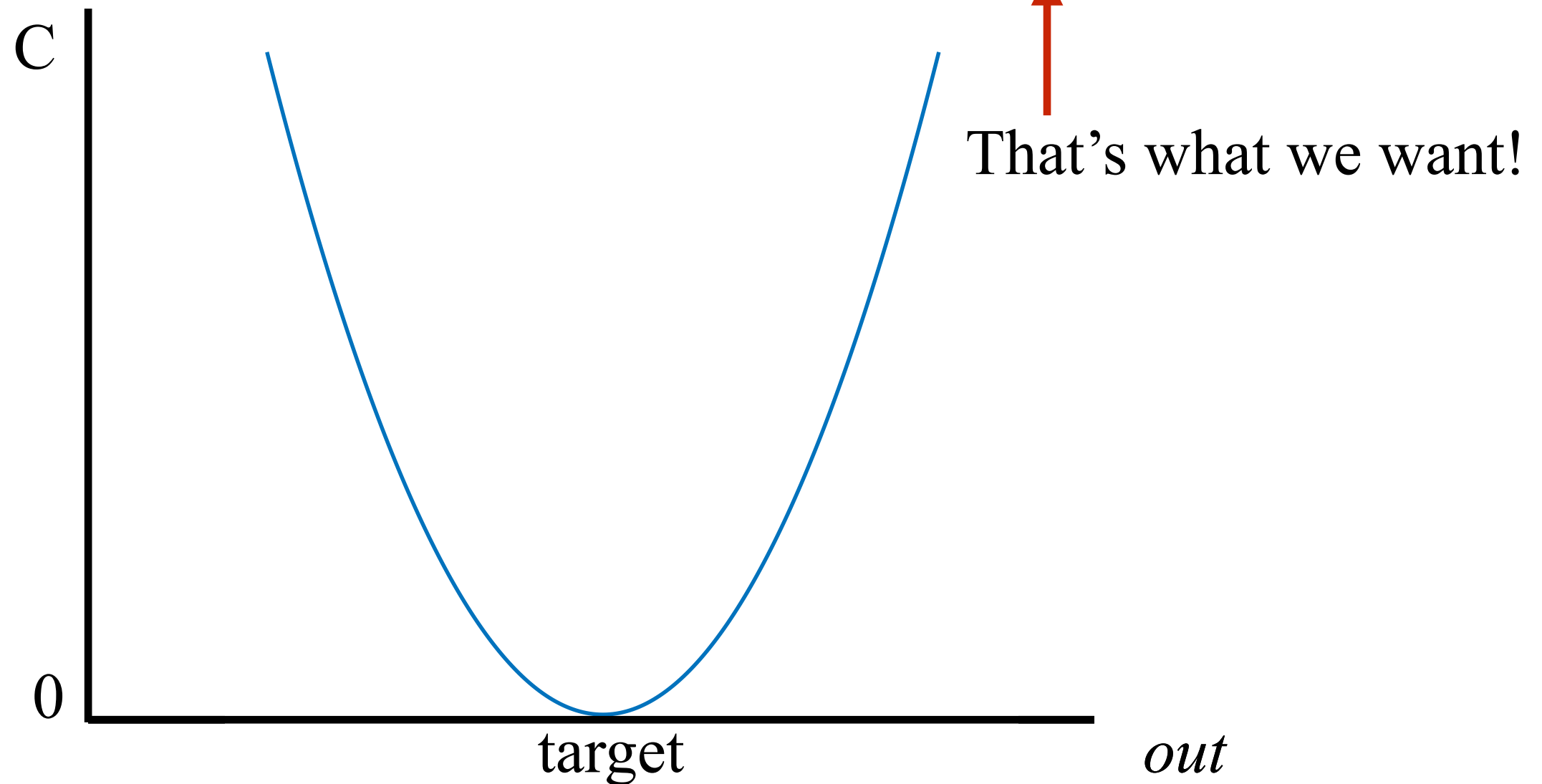
$$\frac{dC}{dout} > 0$$

slope > 0

Create a cost function

Now, our goal: Choose weights to minimize the cost function

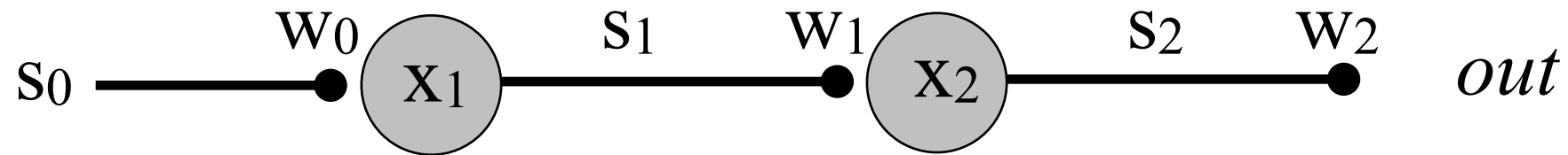
Remember, cost minimized when $out = target$



Here, we plot C versus out . But out depends on weights ...

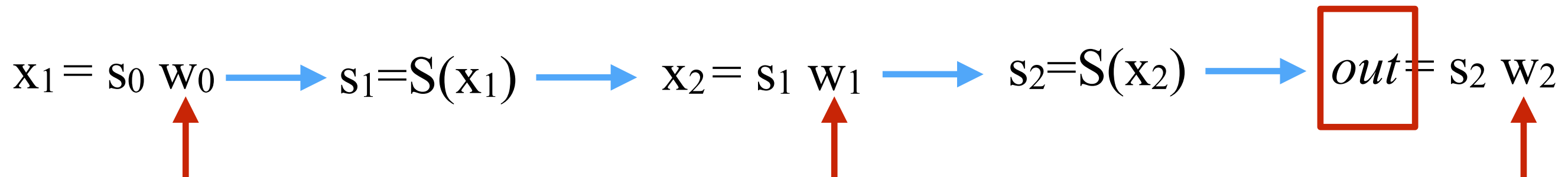
Create a cost function

Q: How does *out* depend on weights?



A: It's complicated.

Consider feedforward solution ...

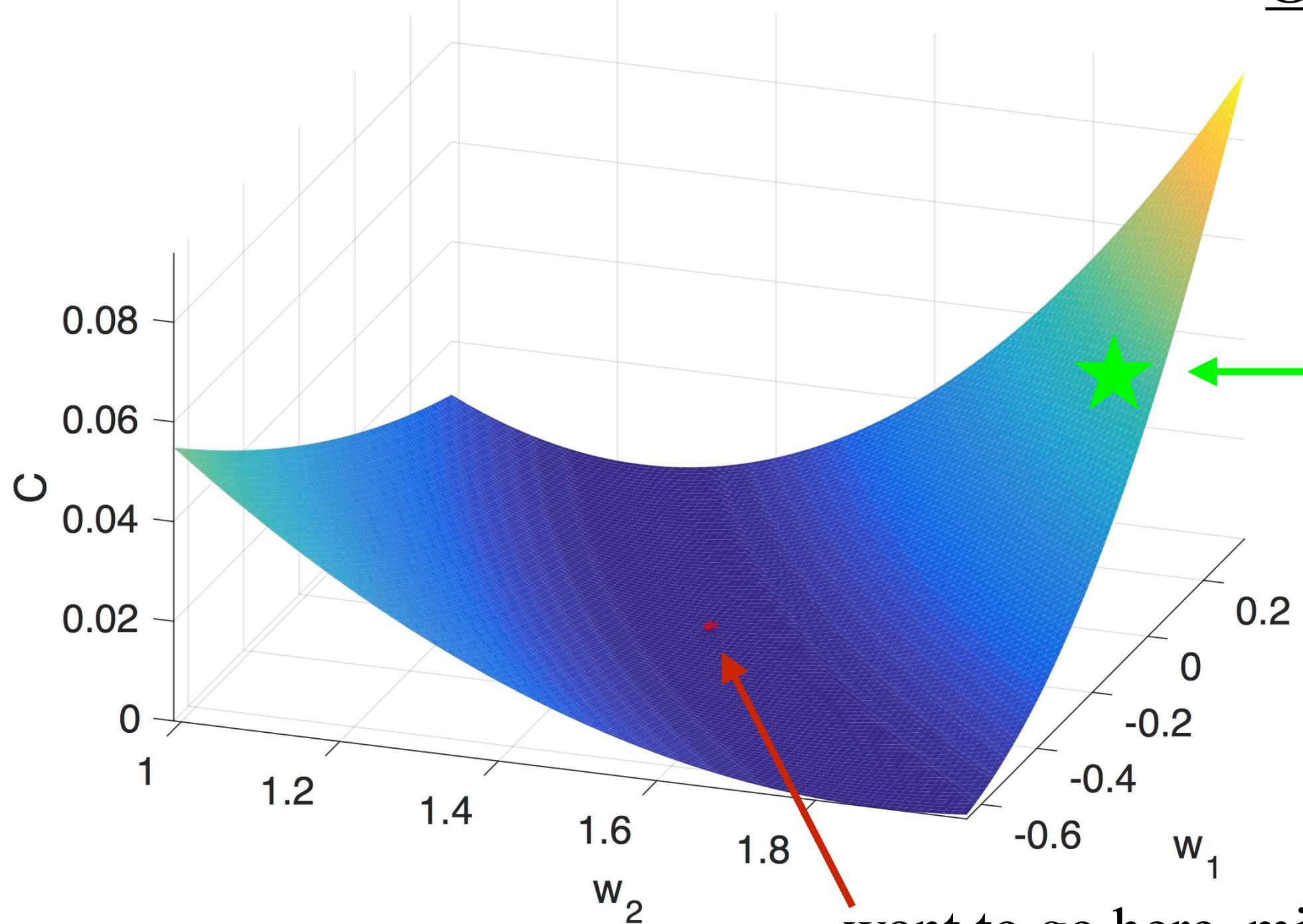


If *out* depends on weights, so does the cost ... $C(w_0, w_1, w_2) = ?$

Create a cost function

Plot cost C versus w_1 and w_2 :

Goal: minimize cost.



But, we have
our initial,
randomly
chosen weights

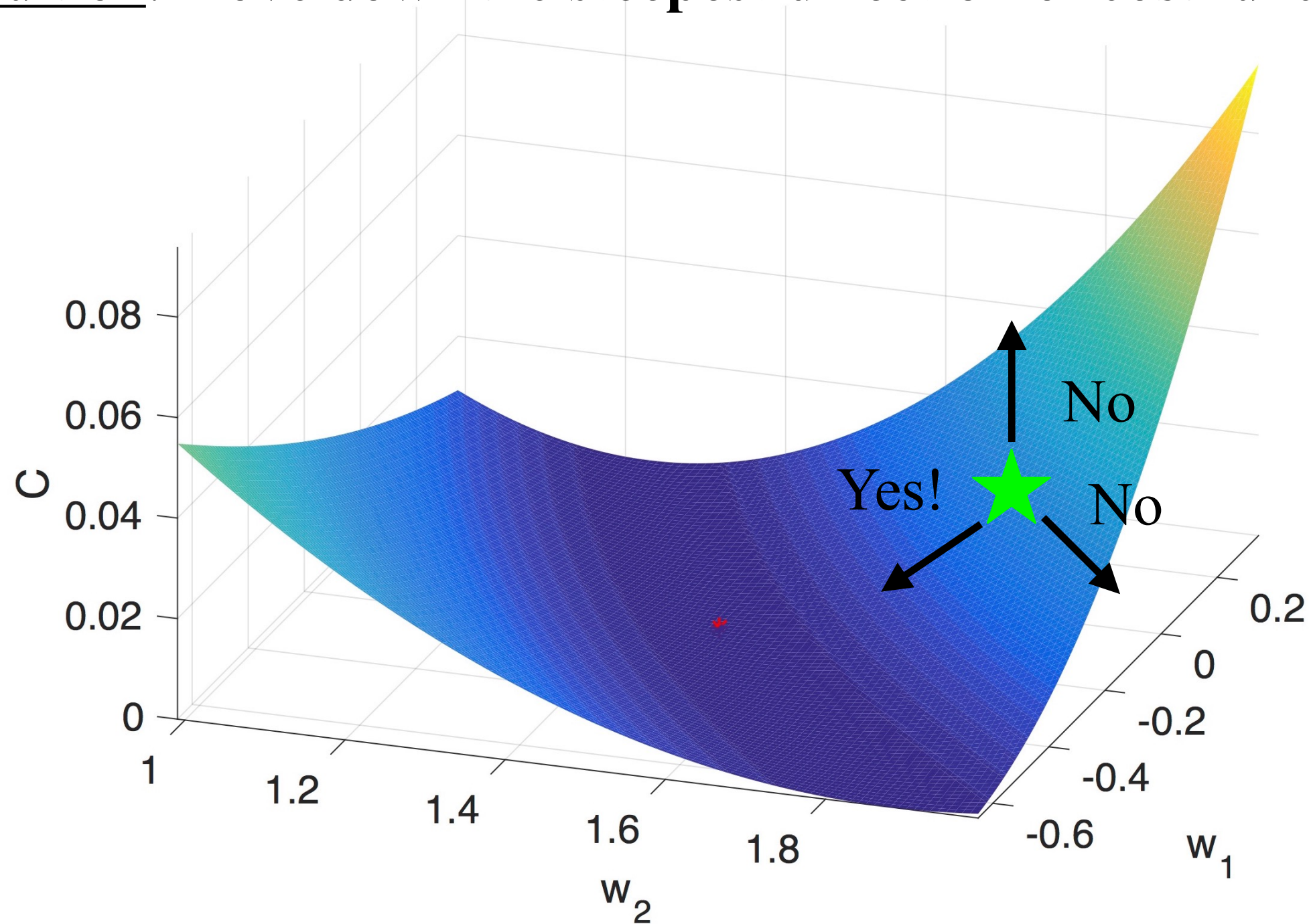
want to go here, minimize cost.

Q: How do we adjust weights to reach the minimum cost?

A: Move “downhill” ...

Follow the cost function

Intuition: move down the **steepest direction** of cost function



Imagine placing a marble ... where does it roll? To the minimum.

Q: How do we find the steepest direction? **A:** Compute the gradient

Follow the cost function

Gradient of the cost function.

How C changes due to small changes in w_0, w_1, w_2 .

We need to compute: $\frac{dC}{dw_0}$ $\frac{dC}{dw_1}$ $\frac{dC}{dw_2}$

Then, update the weights in steps proportional to the negative gradient.

The diagram illustrates the weight update step for w_1 . It shows three update equations: $w_0 \leftarrow w_0 - \alpha \frac{dC}{dw_0}$, $w_1 \leftarrow w_1 - \alpha \frac{dC}{dw_1}$, and $w_2 \leftarrow w_2 - \alpha \frac{dC}{dw_2}$. The term $\alpha \frac{dC}{dw_1}$ in the middle equation is enclosed in a red box and labeled "update" in red text. An upward arrow from the text "new" weight points to the w_1 on the left side of the middle equation. Another upward arrow from the text "original" weight points to the w_1 on the right side of the middle equation. A diagonal arrow from the text "(negative) steepest direction of C" points to the negative sign between the w_1 terms in the middle equation.

$$w_0 \leftarrow w_0 - \alpha \frac{dC}{dw_0} \quad w_1 \leftarrow w_1 - \alpha \frac{dC}{dw_1} \quad w_2 \leftarrow w_2 - \alpha \frac{dC}{dw_2}$$

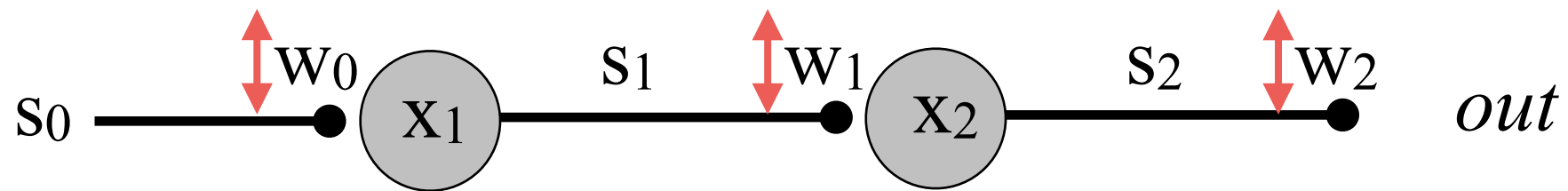
“new” weight “original” weight (negative) steepest direction of C

Procedure: gradient descent

α = learning rate

Follow the cost function

Q: How does the cost function change due to changes in weights?



Consider:

$$\frac{dC}{dw_2} = ???$$

We know how C depends on out : $C = \frac{1}{2} \left(\text{target} - \text{out} \right)^2$

And we know how out depends on w_2 : $out = s_2 w_2$

To compute the derivative, use the **chain rule** ...

Follow the cost function

Our goal:

$$\frac{dC}{dw_2} = ???$$

We know C depends on out , and out depends on w_2 ...

$$\frac{dC}{dw_2} = \frac{dC}{dout} \frac{dout}{dw_2}$$

We've already solved the first derivative!

$$\frac{dC}{dout} = out - target$$

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Let's compute the next derivative ...

Follow the cost function

$$\frac{d \text{out}}{d w_2} = ???$$

We know: $\text{out} = s_2 w_2$

$$\text{So, } \frac{d \text{out}}{d w_2} = \frac{d (s_2 w_2)}{d w_2} = s_2 \frac{d w_2}{d w_2} = \boxed{s_2}$$

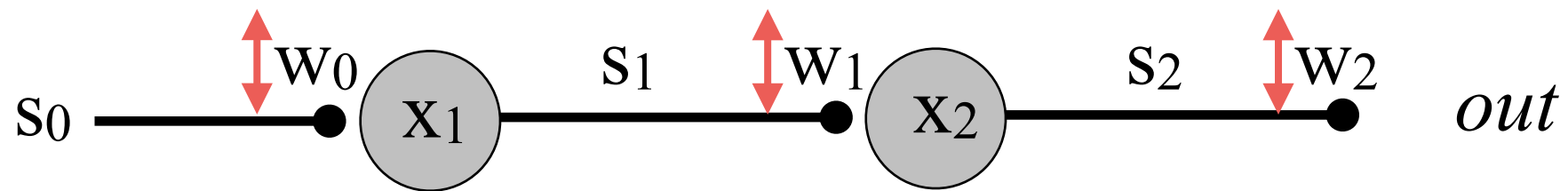
Then,

$$\frac{d C}{d w_2} = \frac{d C}{d \text{out}} \frac{d \text{out}}{d w_2}$$

$$\boxed{\frac{d C}{d w_2} = (\text{out} - \text{target}) s_2}$$

Follow the cost function

Q: How does the cost function change due to changes in weights?



We found, $\frac{dC}{dw_2} = (out - target)$

How bad we're doing.

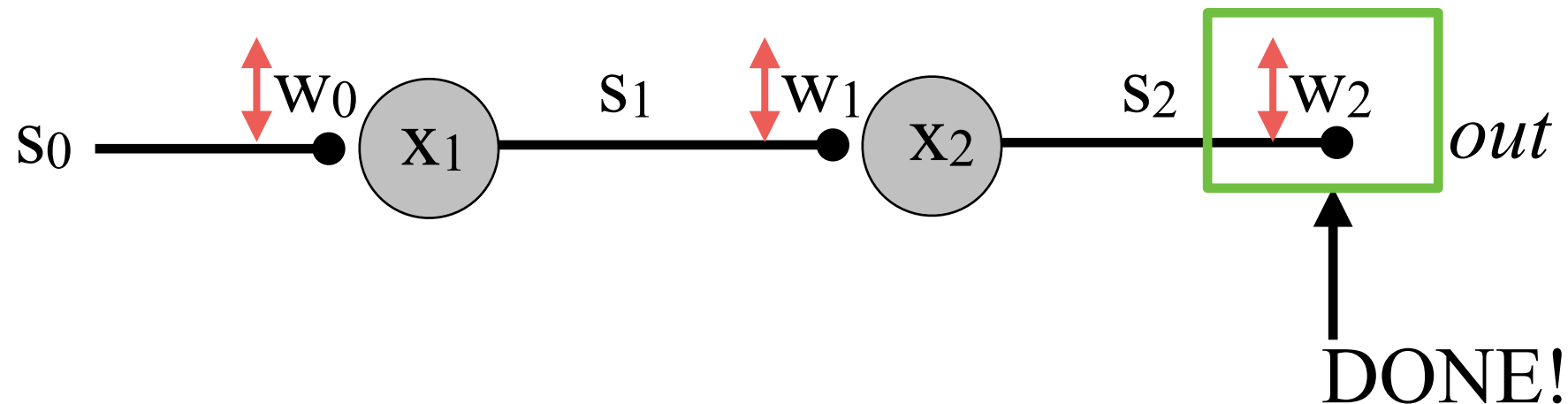
s_2
↑
output from x_2

Update the weight w_2 :

$$w_2 \leftarrow w_2 - \alpha \frac{dC}{dw_2} \quad \text{becomes} \quad w_2 \leftarrow w_2 - \alpha (out - target) s_2$$

Follow the cost function

So, we've now found an equation to update one of the weights w_2 that acts to minimize the cost function.



$$w_2 \text{ update: } w_2 \leftarrow w_2 - \alpha(out - \text{target})s_2$$

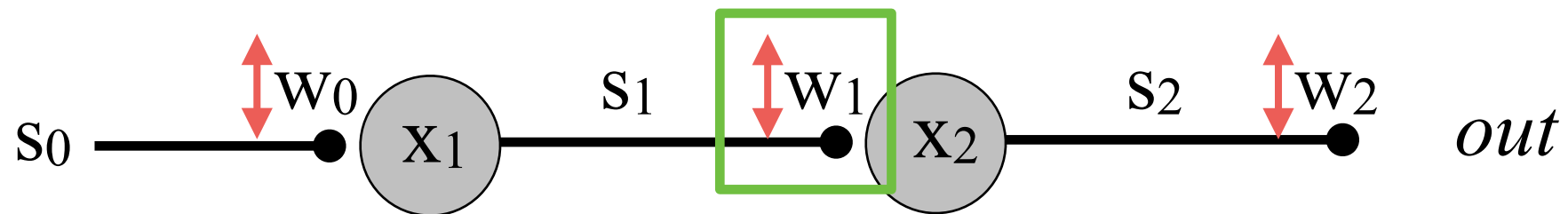
Q: Can we find equations to update w_1 and w_0 ?

A: Yes we can.

It'll seem difficult, but we've already done the hard work ...

Follow the cost function

Q: How does the cost function change due to change in w_1 ?



$$\frac{dC}{dw_1} = ???$$

We don't know this ...
but can write it using
things we do know.

We need the **chain rule**:

$$\frac{dC}{dw_1} = \frac{dC}{dout} \frac{dout}{dw_1}$$

Follow the cost function

Q: How does the cost function change due to change in w_1 ?

$$\frac{dC}{dw_1} = \frac{dC}{dout} \frac{dout}{dw_1}$$

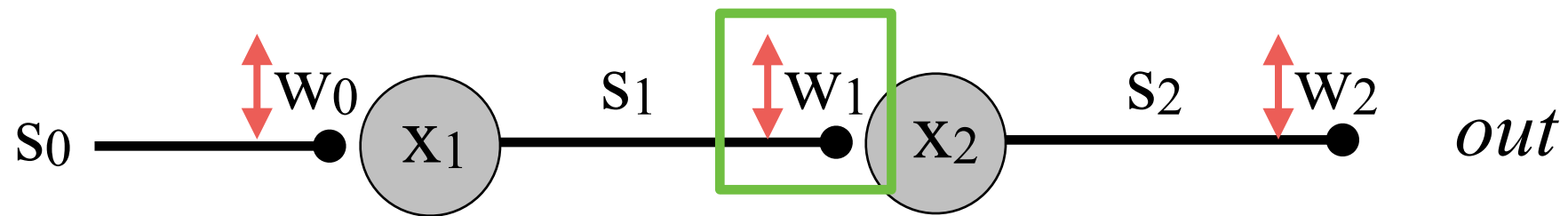
We already found:

$$\frac{dC}{dout} = out - target \quad (Slide\ 35)$$

$$\frac{dout}{dw_1} = w_2 s_2 (1 - s_2) s_1 \quad (Slide\ 23)$$

Follow the cost function

Q: How does the cost function change due to change in w_1 ?



We conclude:

$$\frac{dC}{dw_1} = \frac{dC}{dout} \frac{dout}{dw_1}$$

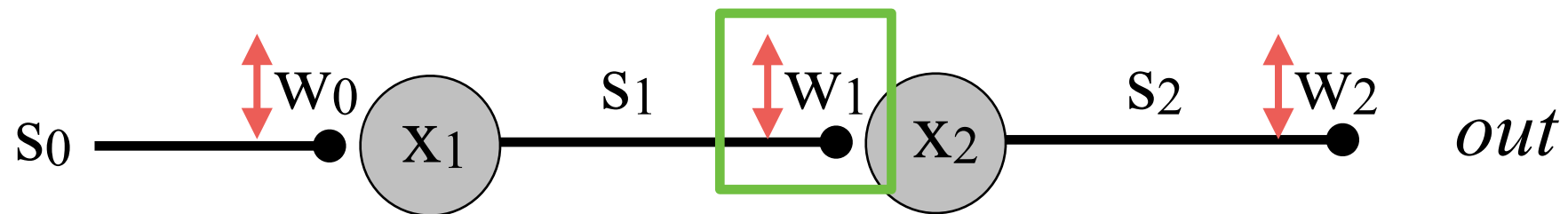
$$\frac{dC}{dw_1} = \boxed{(out - target)} w_2 s_2 (1 - s_2) s_1$$

How bad we're doing.

↑
complicated expression
of outputs and weight

Follow the cost function

Q: How does the cost function change due to change in w_1 ?



Update the weight w_1 :

$$w_1 \leftarrow w_1 - \alpha \frac{dC}{dw_1} \text{ substitute in for this!}$$

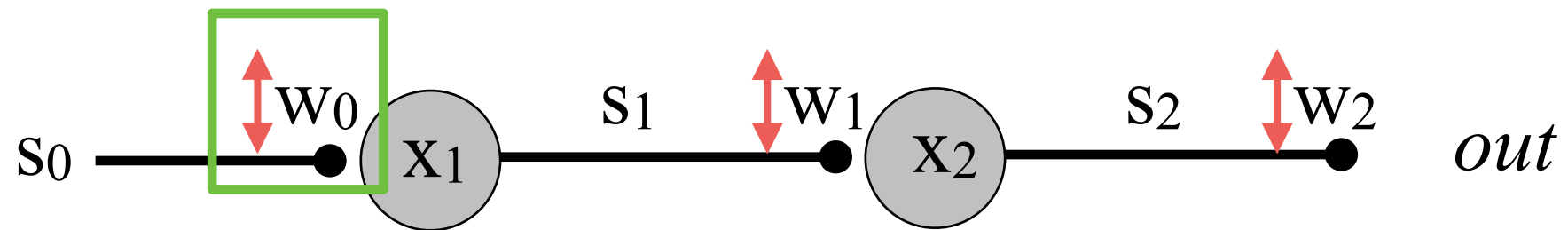
becomes

$$w_1 \leftarrow w_1 - \alpha (out - target) w_2 s_2 (1 - s_2) s_1$$

Q: What happens when $out = target$?

Follow the cost function

Q: How does the cost function change due to change in w_0 ?

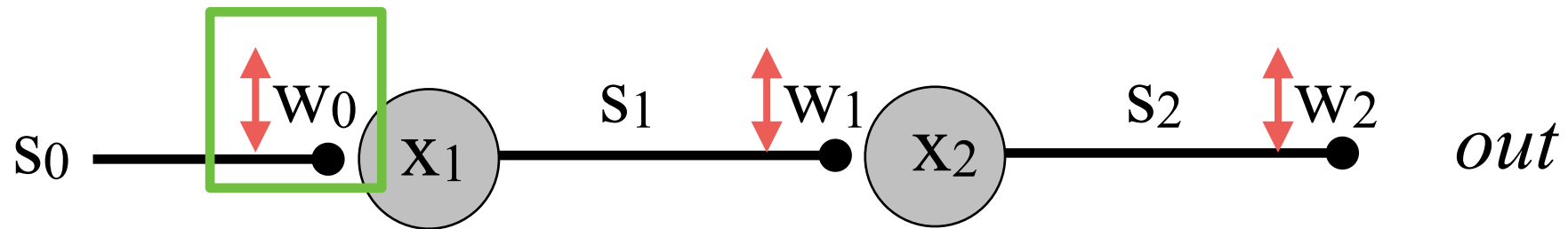


$$\frac{dC}{dw_0} = ???$$

Try it ...

Follow the cost function

Q: How does the cost function change due to change in w_0 ?



We conclude:

$$\frac{dC}{dw_0} = (out - target) w_2 s_2 (1 - s_2) w_1 s_1 (1 - s_1) s_0$$

and

$$w_0 \leftarrow w_0 - \alpha(out - target)w_2s_2(1 - s_2)w_1s_1(1 - s_1)s_0$$

Impressive expression!

Put it all together

Prescription to find the weights that minimize cost function
(so that *out* is near target).

1. Choose random initial weights.

$$w_0 = 2 \qquad w_1 = 1 \qquad w_2 = 0.5$$

2. Fix input at desire value, and calculate *out*.



Put it all together

Prescription (continued)

3. Update the weights

$$w_2 \leftarrow w_2 - \alpha(out - target)s_2$$

$$w_1 \leftarrow w_1 - \alpha(out - target)w_2s_2(1 - s_2)s_1$$

$$w_0 \leftarrow w_0 - \alpha(out - target)w_2s_2(1 - s_2)w_1s_1(1 - s_1)s_0$$

Note: We know all of the values required

α = learning rate, we choose this.

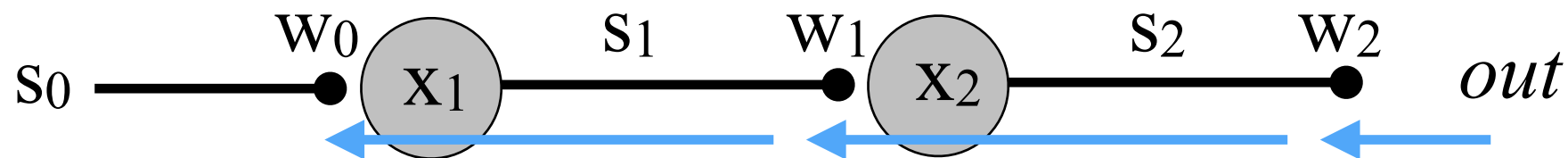
s_0, s_1, s_2 = calculated during forward propagation

Put it all together

Prescription (continued)

4. Repeat Steps 2 & 3 until error is small enough.
or *out* is close enough to target.

This procedure is called **backpropagation**

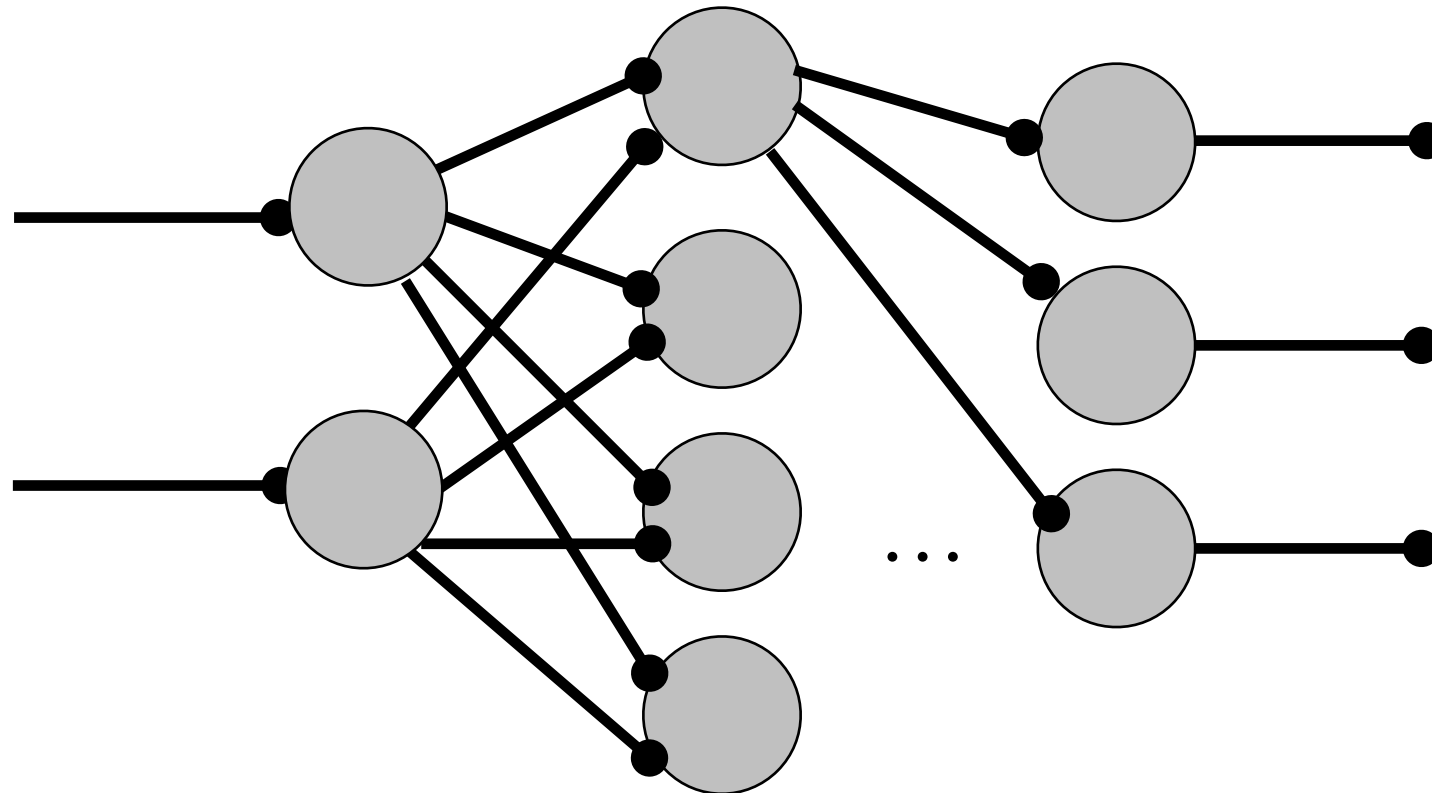


we work “backwards” through our neural network

from *out* to changes in w_2 to changes in w_1 to changes in w_0

Backpropagation

Can evaluate more complicated neural networks



Same ideas apply, but algebra is intense.

Cool example: playground.tensorflow.org

Next ...

Implement backpropagation in Python