

Rhythms

Analyzing Rhythms (Part 1)

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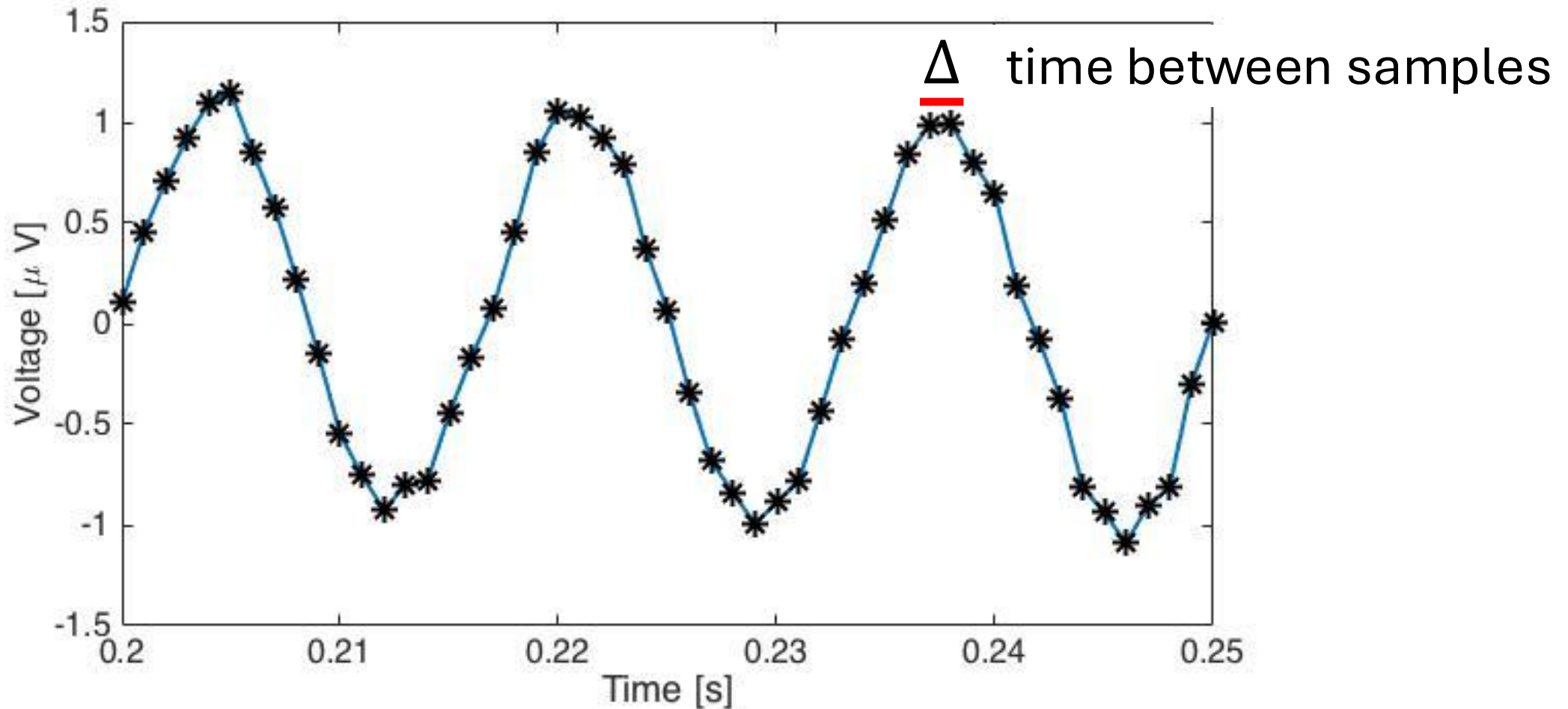
Today

- Practical notions

Sampling frequency, Nyquist frequency, tapering

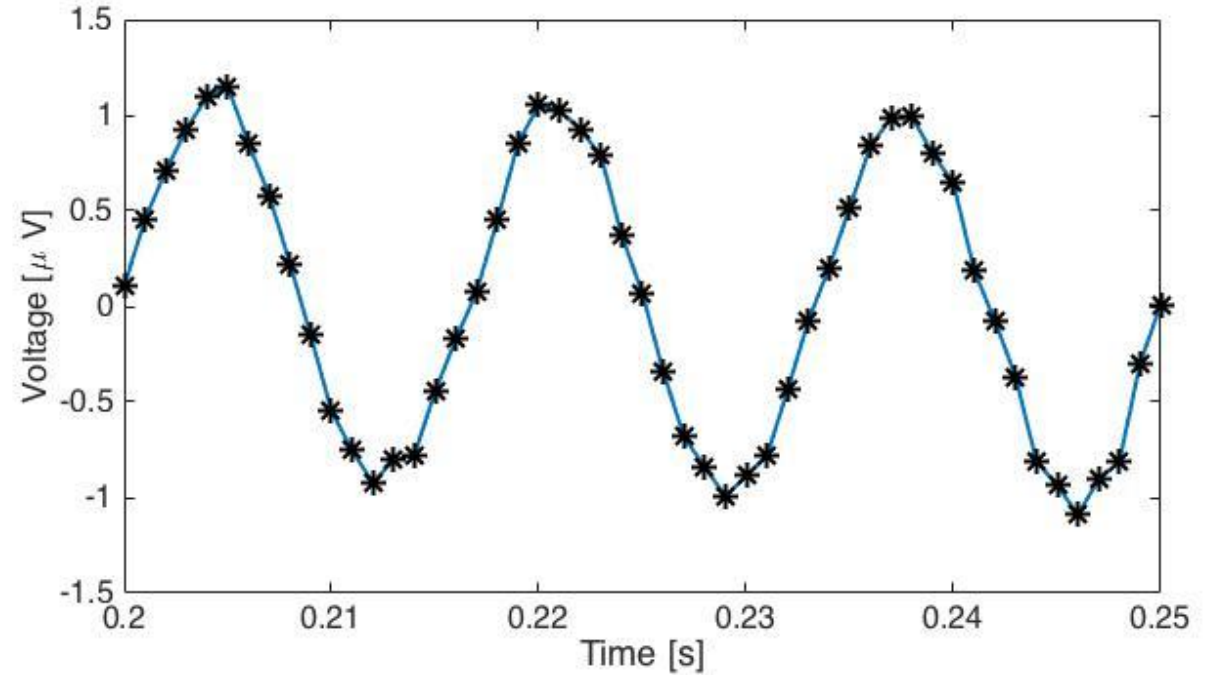
Data is not continuous

Consider a small snippet of data



Notation: x_n = Data at index n

Data is not continuous



Notation

x_n = Data at index n



x (EEG, LFP, MEG, ...)

t_n = Time at index n , $t_n = \Delta n$ where Δ = sampling interval

f_j = Frequency at index j , $f_j = j/T$ where T = total time of observation

Discrete time Fourier transform

Previously $V[t] = \sum_j A_j \cos(2\pi f_j t) + B_j \sin(2\pi f_j t)$

$$A_k = \frac{2}{T} \int_0^T V[t] \cos(2\pi f_k t) dt \quad \text{and} \quad B_k = \frac{2}{T} \int_0^T V[t] \sin(2\pi f_k t) dt$$



Compare data $V[t]$ to cosine at frequency f_k , does it match?

Now replace A_k, B_k :

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n)$$

????

Fourier transform of the data x .

Discrete time Fourier transform

Define the parts

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n) \quad \text{????}$$

$i^2 = -1$

FT of x
index j

data x
index n

frequency
index j
 $f_j = j/T$

time
index n
 $t_n = \Delta n$

sum over
all time n

The diagram illustrates the components of the Discrete Time Fourier Transform equation. The equation is $X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n)$. A red box highlights the exponential term $\exp(-2\pi i f_j t_n)$. Red arrows point from the following labels to their corresponding parts in the equation: 'FT of x index j' points to X_j ; 'data x index n' points to x_n ; 'frequency index j' points to f_j ; 'time index n' points to t_n ; 'sum over all time n' points to the summation symbol \sum ; and ' $i^2 = -1$ ' points to the imaginary unit i in the exponent. The label '????' is placed to the right of the exponential term.

Discrete time Fourier transform

Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Fourier transform intuition:

Feynman: “*the most remarkable formula in mathematics*”

Data as a function of
frequency index j

Data as a function of time index n

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n)$$

Sinusoids at frequency f_j

Euler's formula:

$$\exp(-2\pi i f_j t_n) = \cos(-2\pi f_j t_n) + i \sin(-2\pi f_j t_n).$$

So, at each time (index n) multiply data x_n by sinusoids at frequency f_j
Then sum up over all time.

Discrete time Fourier transform

Fourier transform intuition:

Data as a function of
frequency index j

Data as a function of time index n

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n) .$$

Sinusoids at frequency f_j

Idea: compare our data x_n to sinusoids at frequency f_j and see how well they “match”.

Good match: $X_j =$ big

Bad match: $X_j =$ small

X_j reveals the frequencies f_j that match our data.

Spectrum: idea

Fourier transform intuition:

“Compare” data to sinusoids at different frequencies

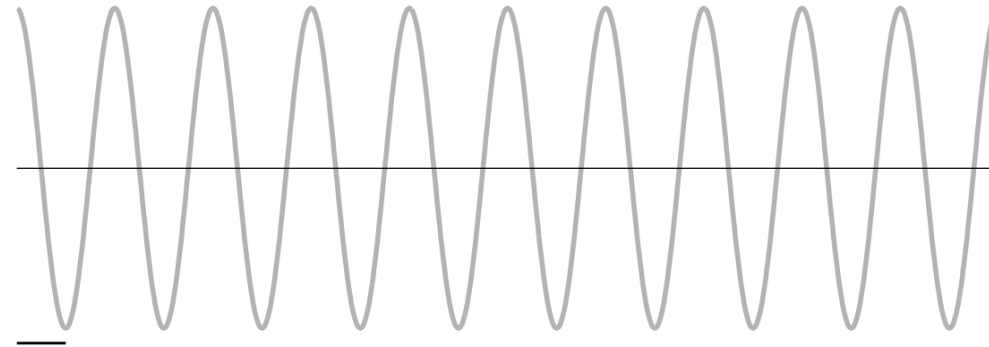
Match:

X_j at frequency f_j is large

Mismatch:

X_j at frequency f_j is small

Example:

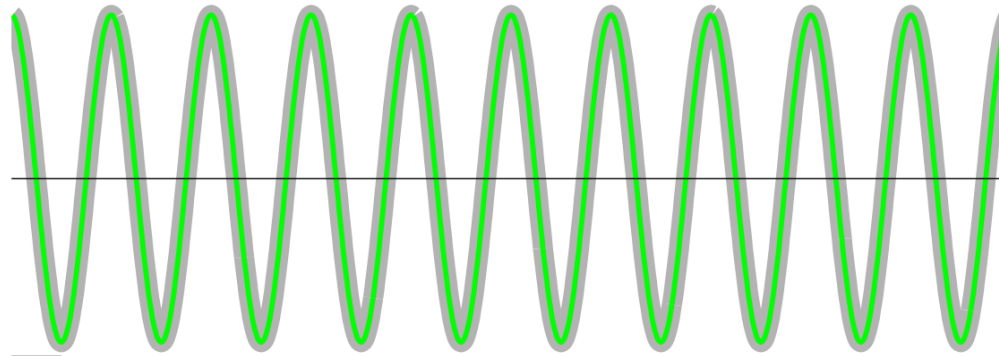


“Data”
10 Hz cosine

4 Hz

Multiply (+,-,+,-,...) & add
... small value

4 Hz does not match data



10 Hz

Multiply (+,+,+,+,...) & add
... large value

10 Hz matches data

Discrete time Fourier transform

Sound familiar? $X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n)$ Fourier transform of the data x .

↑ replace with Euler's formula
 $\cos(-2\pi f_j t_n) + i \sin(-2\pi f_j t_n).$

$$X_j = \left(\sum_{n=1}^N x_n \cos(-2\pi f_j t_n) \right) + i \left(\sum_{n=1}^N x_n \sin(-2\pi f_j t_n) \right)$$

Looks like $A_k = \frac{2}{T} \int_0^T V[t] \cos(2\pi f_k t) dt$ $B_k = \frac{2}{T} \int_0^T V[t] \sin(2\pi f_k t) dt$

Same idea: compare data to sinusoids and see how well they match

Spectrum: definition

The power of data x
at frequency index j

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

(change sign of i everywhere)

Previously $\frac{A_j^2 + B_j^2}{2}$
Same idea!

Constant that depends on
sampling interval,
duration of recording

FT of
data

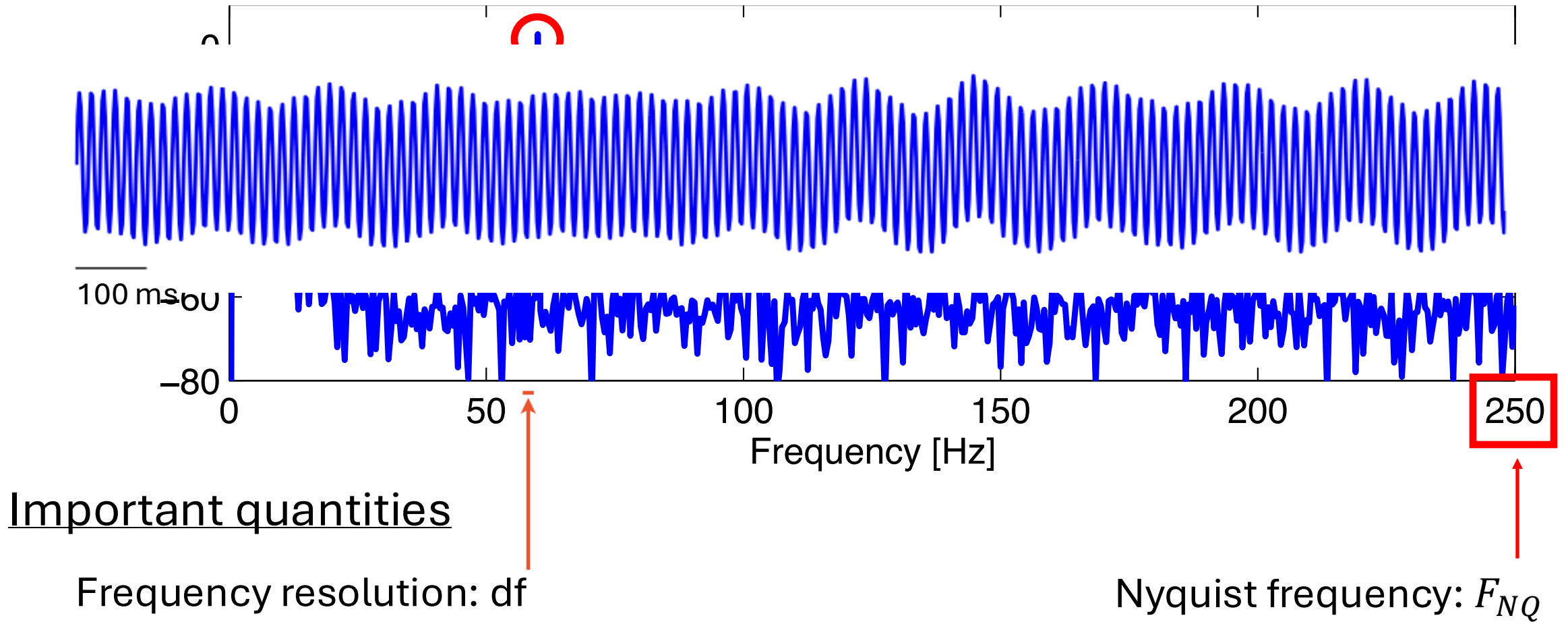
Complex
conjugate
of FT of data

Power at frequency f_j indicates how well sinusoids at f_j “match” our data.

Good match \rightarrow High power at frequency f_j

Spectrum

... reveals the dominant frequencies that “match” the data.



Define these two quantities.

Frequencies

f_j = Frequency at index j , $f_j = j/T$ where T = total time of observation

$$f_j = \left\{ 0, \frac{1}{T}, \frac{2}{T}, \dots, \frac{1}{2\Delta} - \frac{1}{T}, \boxed{\frac{1}{2\Delta}}, -\left(\frac{1}{2\Delta} - \frac{1}{T}\right), -\left(\frac{1}{2\Delta} - \frac{2}{T}\right), \dots, \frac{-2}{T}, \frac{-1}{T} \right\}$$

$\xrightarrow{\quad} \xrightarrow{\quad}$

Two important quantities

Largest frequency: **Nyquist frequency** $f_{\text{NQ}} = \frac{1}{2\Delta} = \frac{f_0}{2}$ half the sampling frequency

$$f_0 = \frac{1}{\Delta} \text{ sampling frequency}$$

Frequency resolution $df = \frac{1}{T}$ reciprocal of total recording duration

Frequencies

Visualize f_j as a **vector**

index	0	1	2	...	$\frac{N}{2}-1$	$\frac{N}{2}$	$\frac{N}{2}+1$		N-2	N-1
freq	0	df	2df	...	$f_{N/2}-df$	$f_{N/2}$	$-(f_{N/2}-df)$...	-2df	-df



Step forward in
intervals of
frequency
resolution df



... until
reaching
maximum
frequency $f_{N/2}$



then,
frequencies
are negative,
starting with
the largest
magnitude.



decrease in
magnitude
until $-df$

Frequencies

Note: $\text{length}(t_n) = N$
 $\text{length}(f_j) = N$ time and frequency vectors have the same length N

- If we record N data points, then we have N frequencies to examine.

Note: Frequencies f_j include negative values.

Consider $\cos(2\pi f t)$ $\xrightarrow{\text{Euler's formula}}$ $\frac{e^{2\pi i f t} + e^{-2\pi i f t}}{2}$

a real signal $f > 0$ $f < 0$

We need positive & negative frequencies to represent a real signal.

Frequencies

Note: Frequencies f_j include negative values.

Important fact: when data x_n is real (no imaginary component), then negative frequency spectrum is redundant.

$$S_{xx,j} \text{ at } f_j = S_{xx,j} \text{ at } -f_j$$

Frequencies

Q: Is x_n real?

A: Yes (in neuroscience)

Only inspect $f_j > 0$

$$f_j = \left\{ 0, \frac{1}{T}, \frac{2}{T}, \dots, \frac{1}{2\Delta} - \frac{1}{T}, \frac{1}{2\Delta}, \underbrace{-\left(\frac{1}{2\Delta} - \frac{1}{T}\right), -\left(\frac{1}{2\Delta} - \frac{2}{T}\right), \dots, \frac{-2}{T}, \frac{-1}{T}} \right\}$$

Ignore negative frequencies (redundant)

Spectrum: df

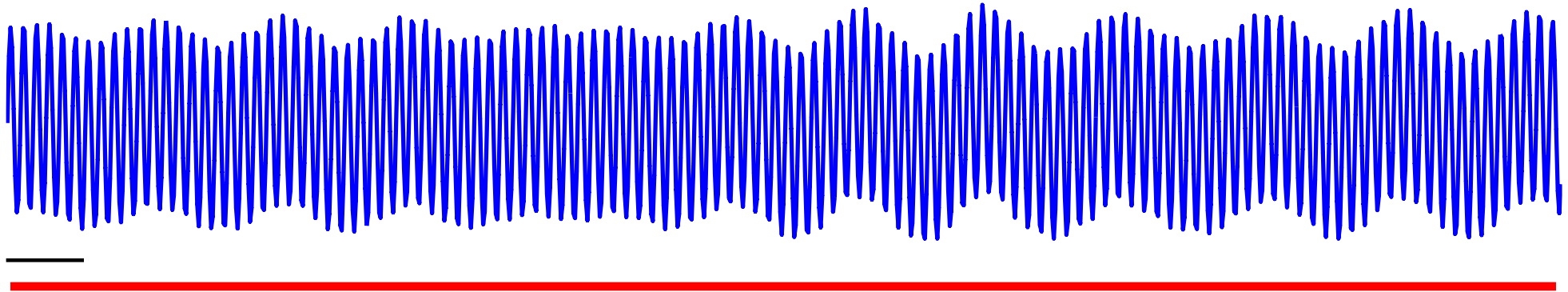
- What is df ?

$$df = \frac{1}{T}$$

frequency resolution

where T = Total duration of recordings.

Ex.



$T = 2 \text{ s}$ so $df = 0.5 \text{ Hz}$

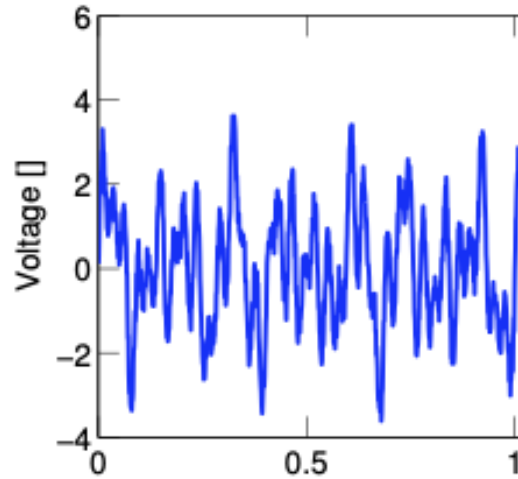
Q: How do we improve frequency resolution?

A: Increase T or record for longer time.

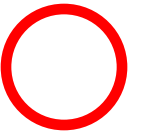
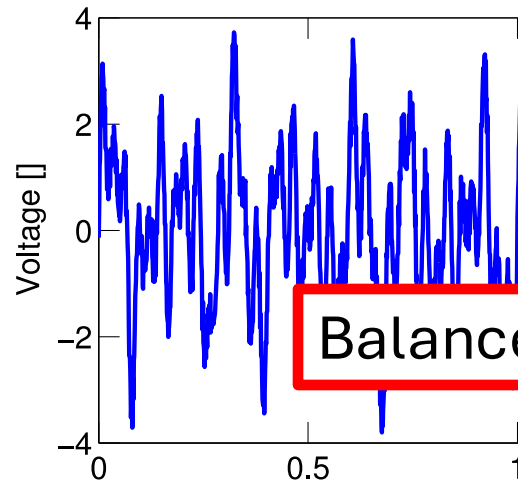
Spectrum: df

- Demand 0.2 Hz frequency resolution

$$df = 0.2 \text{ Hz} = 1/T, \text{ so } T = 5 \text{ s}$$



But, data may change during longer recordings . . .



Spectrum: F_{NQ}

- What is F_{NQ} ?

$$F_{NQ} = f_0/2$$

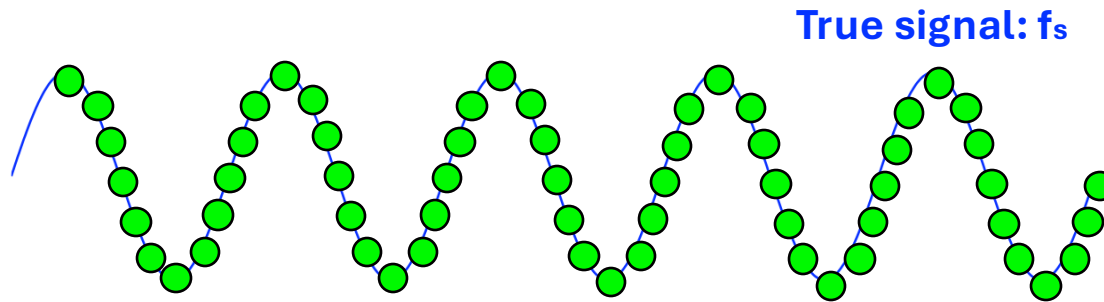
Nyquist frequency

where f_0 = sampling frequency.

The **highest** frequency we can observe.

Sample:

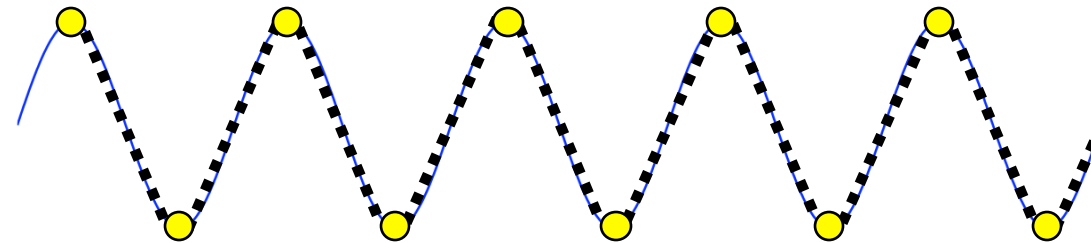
$$f_0 \gg 2 f_s$$



Accurate
reconstruction

$$f_0 = 2 f_s$$

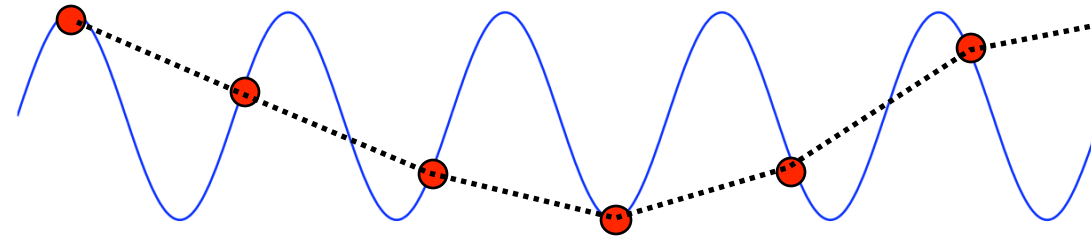
Max freq we can
observe at this
sample rate!



2 samples/cycle

Enough to reconstruct
signal, but just barely.

$$f_0 < 2 f_s$$



High frequency (in data)
mapped to low frequency
(**aliased**).

All hope lost! Indistinguishable from true low frequency signals.

Spectrum: df , F_{NQ}

Summary

**Frequency
resolution**

$$df = \frac{1}{T}$$

← Duration of recording

**Nyquist
frequency**

$$f_{NQ} = \frac{f_0}{2}$$

← Sampling frequency

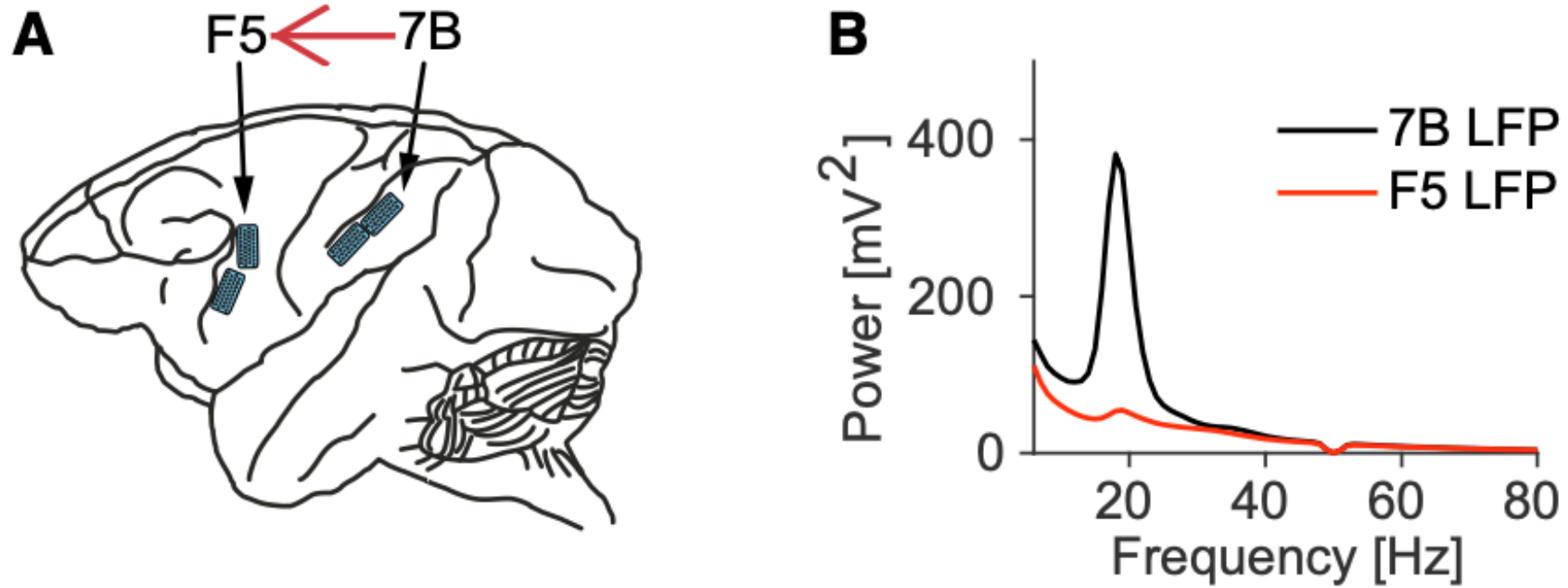
For finer frequency resolution:

record more data.

To observe higher frequencies:

increase sampling rate.

Example



[Schneider et al, Neuron, 2021]

Q. What rhythms dominate?

Quiz

Require a 2 Hz frequency resolution, and to observe rhythms up to 500 Hz. What is the experiment / recording setup?

$$df =$$

$$f_{\text{NQ}} =$$

Spectrum: four (important) asides

- Units
- Scale
- Tapers
- Spectrogram

Spectrum: units

Q. What are the units of the spectrum?

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

$$\frac{[s]^2}{[s]} \quad [V][V]$$

$$[s] \quad [V]^2$$

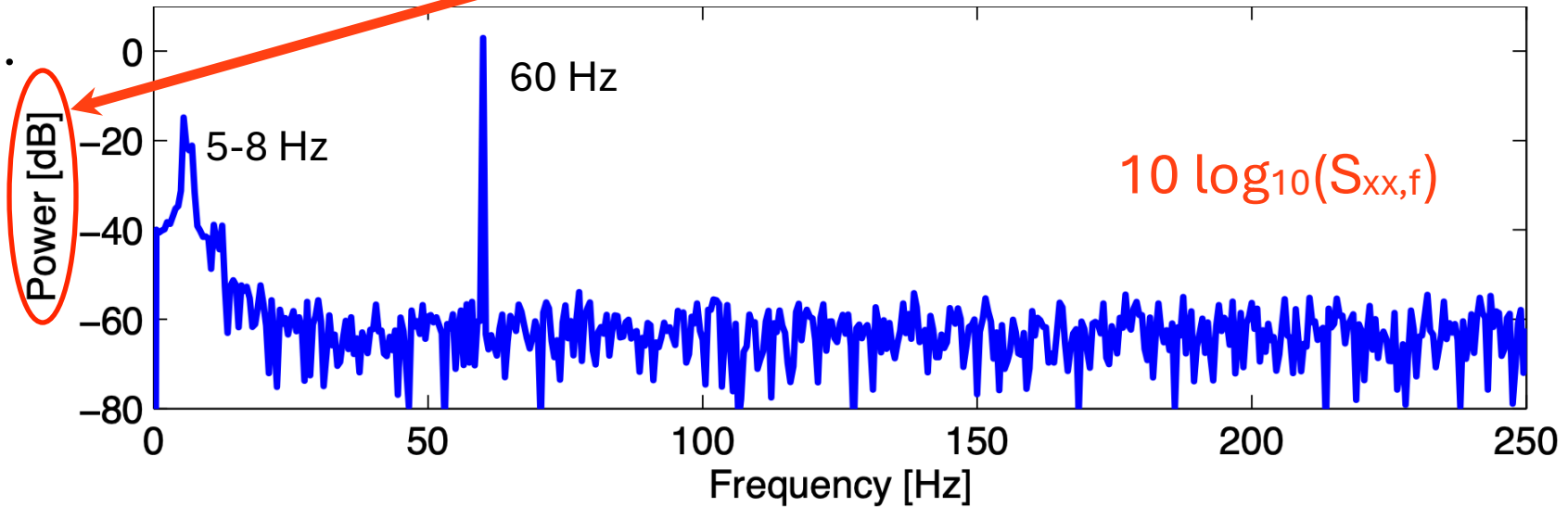
$$\boxed{\frac{[V]^2}{[Hz]}}$$

“volts squared per Hertz”

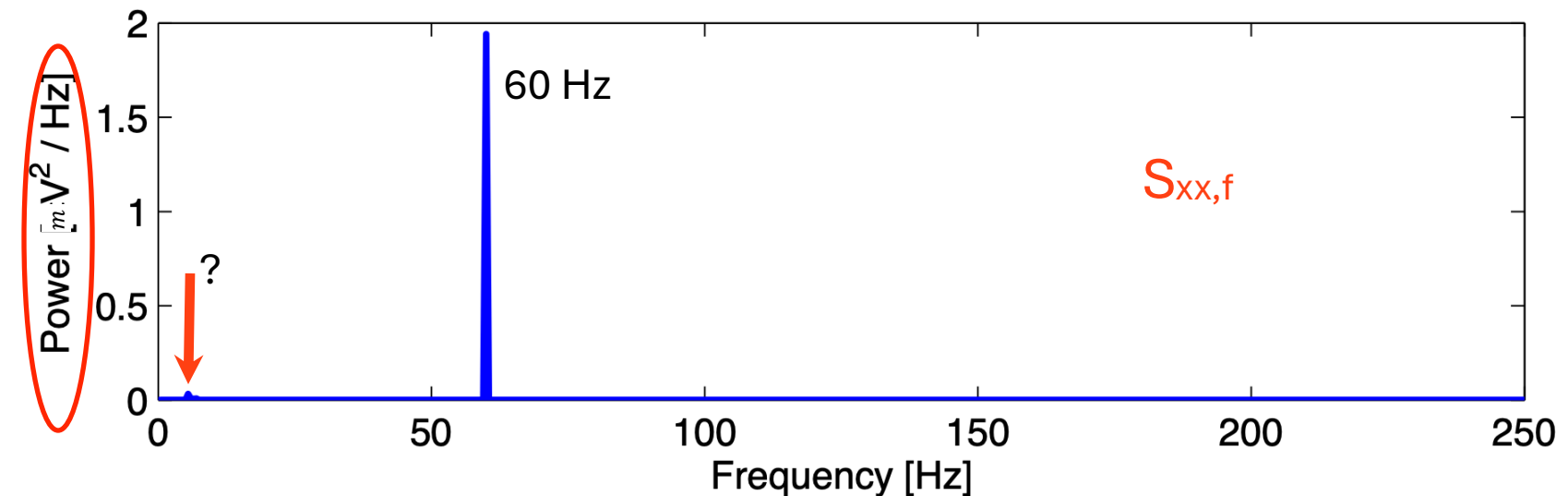
Spectrum: scale

Emphasize lower amplitude signals

A note on scale .



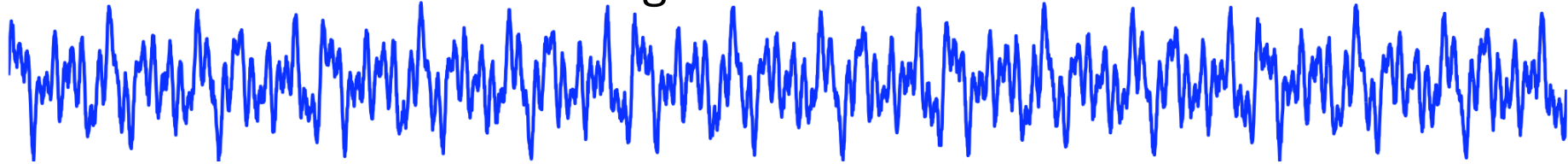
Without the decibel scale . . .



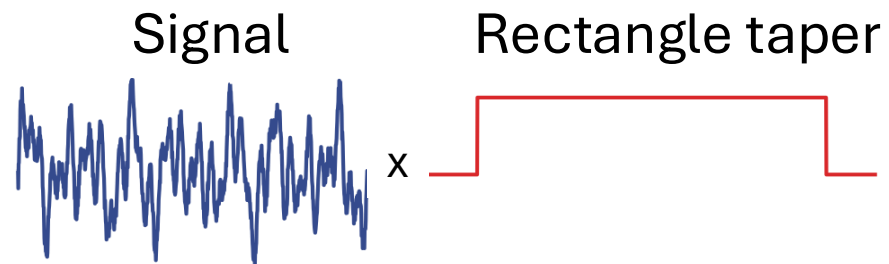
Spectrum: tapers

Doing nothing, we make an implicit taper choice . . .

. . . Data goes on forever . . .



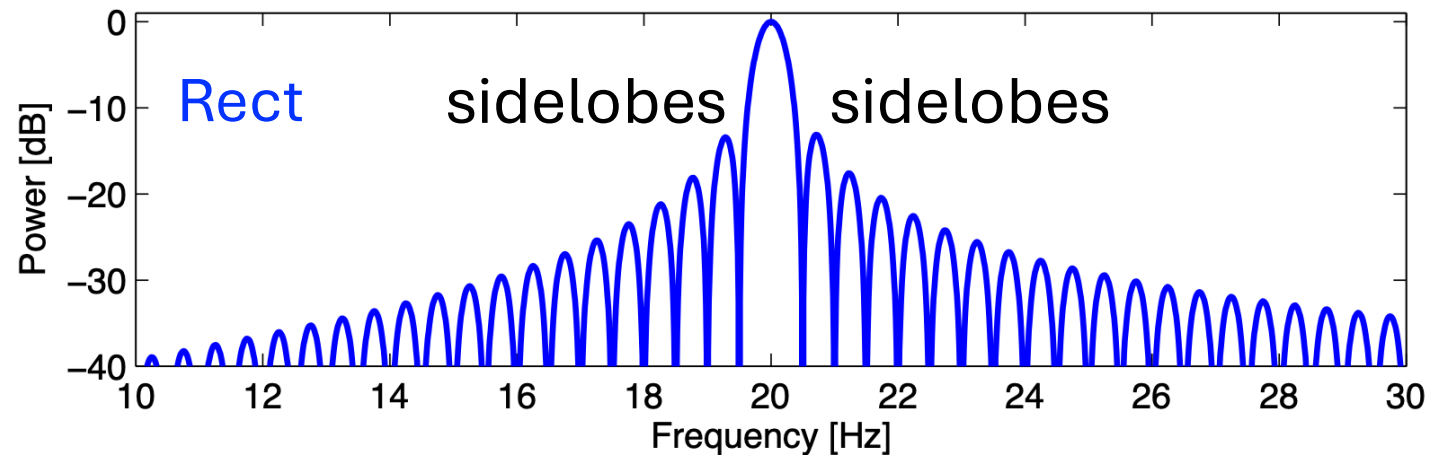
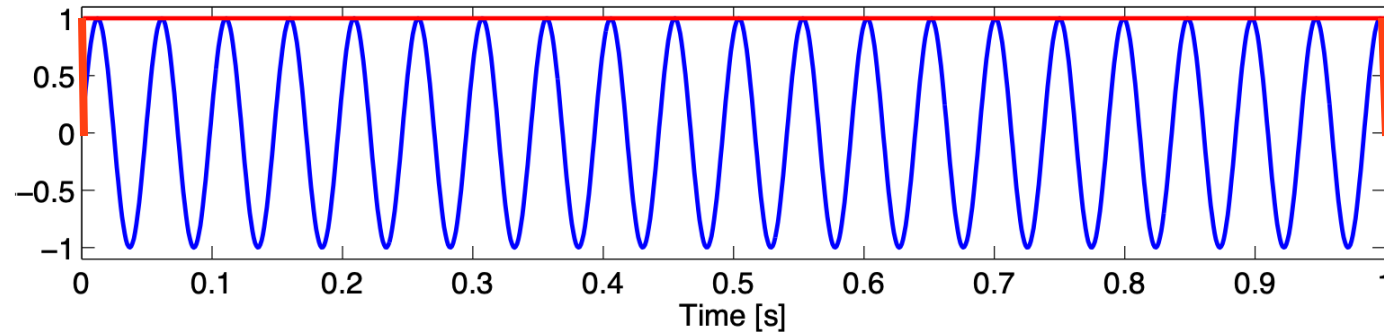
What we're observing:



Spectrum: tapers

The rectangle taper impacts the power spectrum.

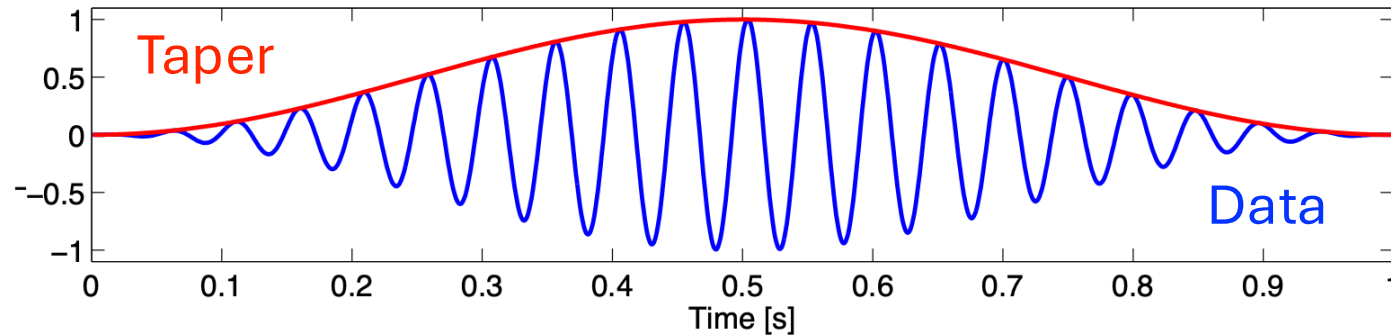
Pure
sinusoid
at 20 Hz



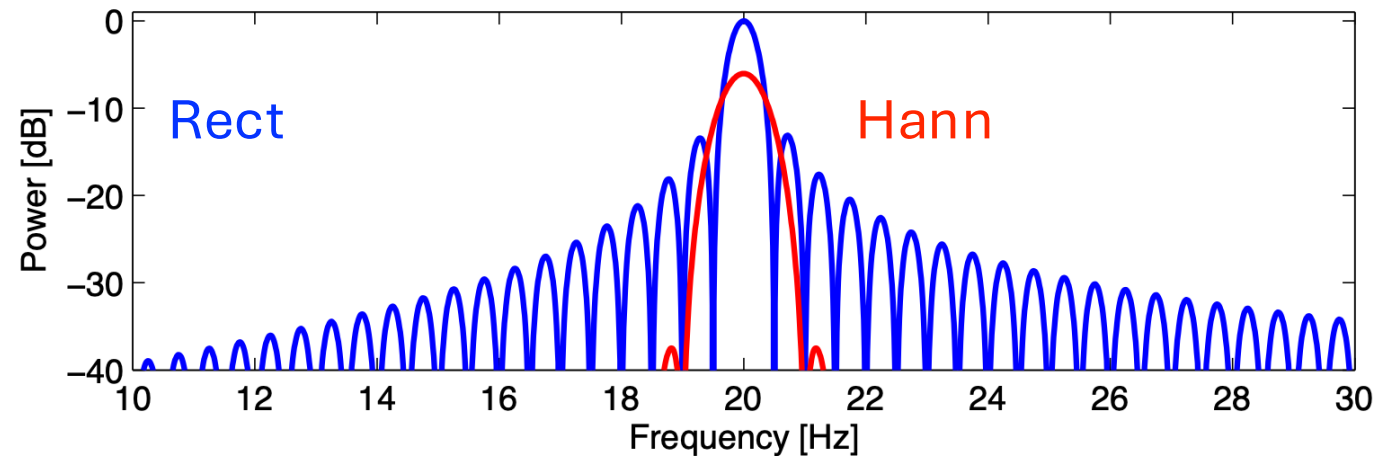
Sharp peak is “smeared out” . . .

Spectrum: tapers

Idea: smooth the sharp edges of rectangle taper.

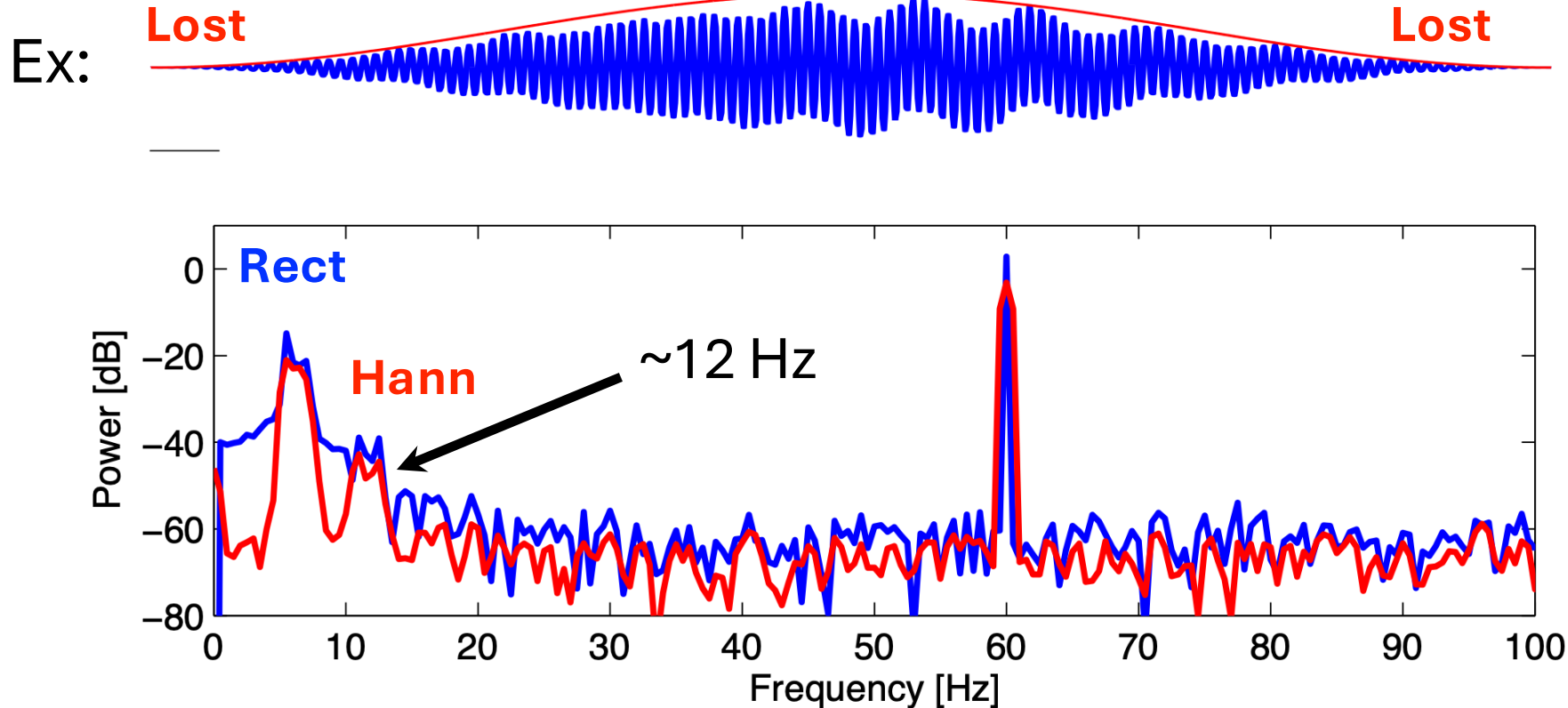


Compute spectrum of tapered data.



Taper reduces the sidelobes.

Spectrum: tapers



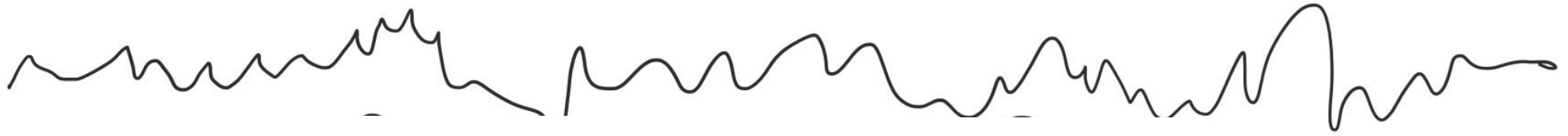
Good: Reduced sidelobes reveals a new peak.

Bad: Broader peaks & lose data at edges.

“More lives have been lost looking at the [rectangular tapered spectrum] than by any other action involving time series.” [Tukey 1980]

Spectrum: spectrogram

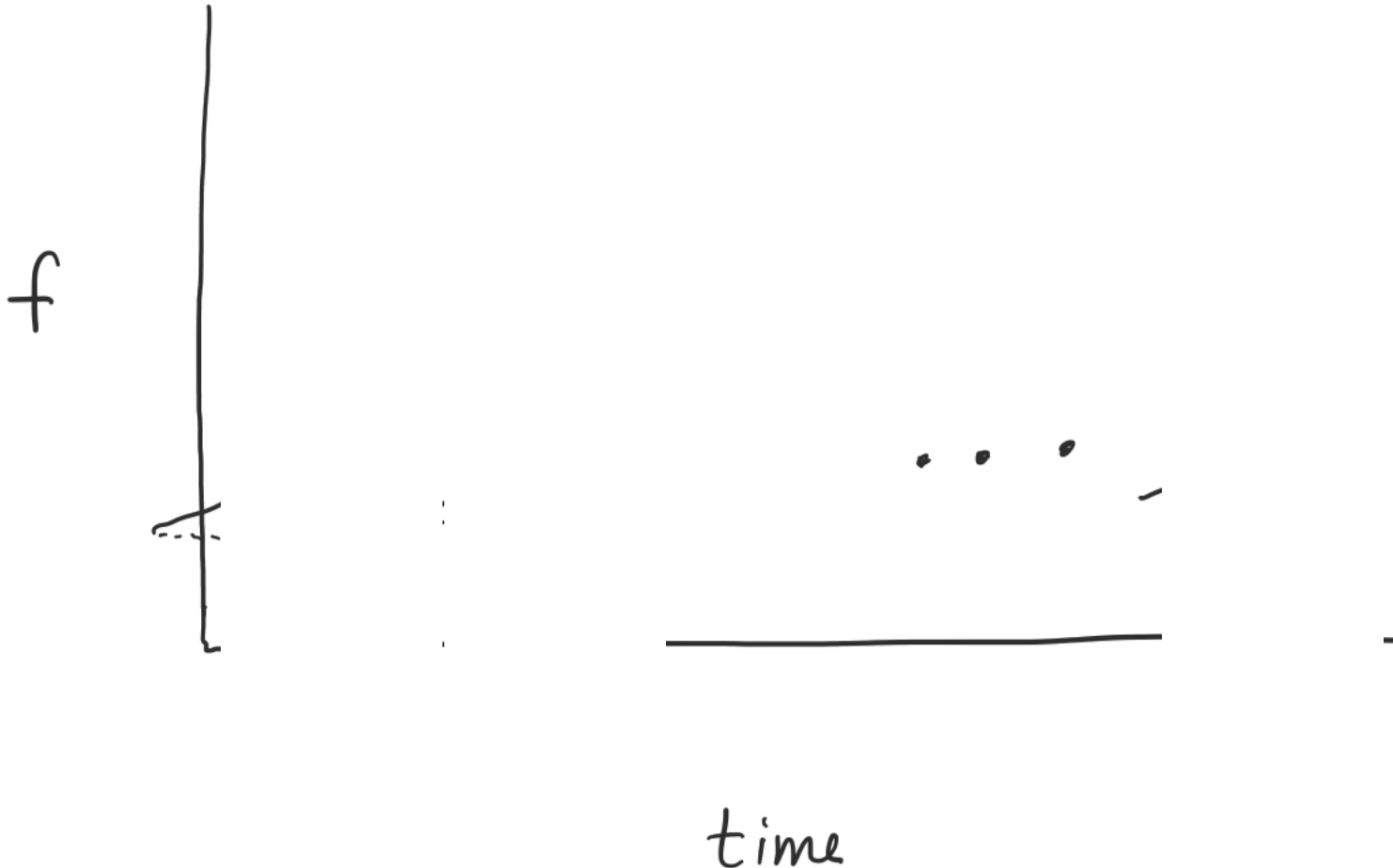
Idea: Divide data into smaller intervals, then compute the spectrum in each interval



Spectrum: spectrogram

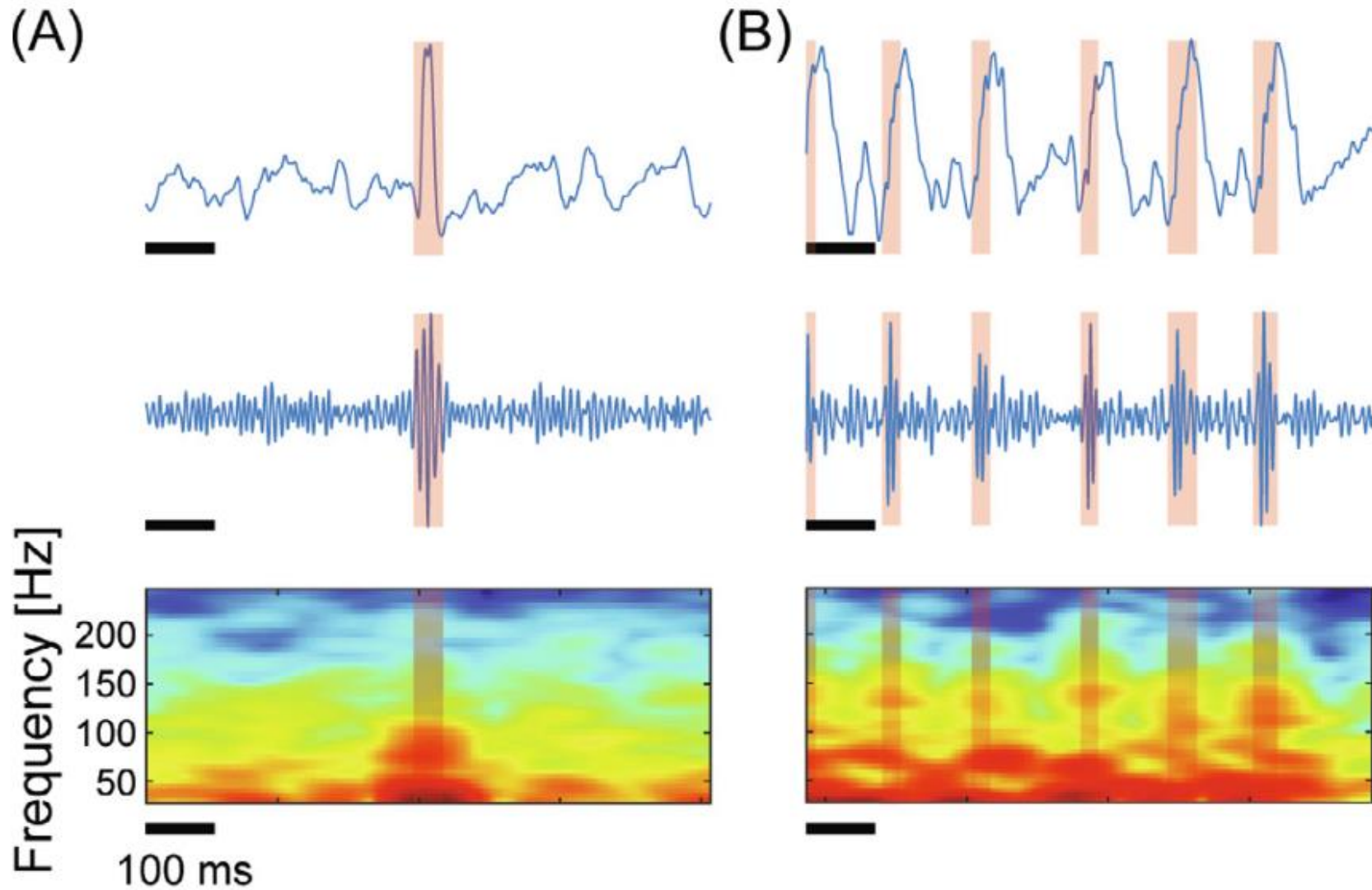
Display as

spectrogram



Usually, color
indicates power

Spectrum: spectrogram



[Shi et al, Epilepsy & Behav Reports, 2022]

Spectrum: spectrogram

Q. What happens to df ?

Original data 10 s , $df =$

Intervals 1 s , $df =$

Q. What happens to f_{NG} ?

Q. When is this a good idea ?

Spectrum: summary

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

$$X_j = \sum_{n=1}^N x_n \exp(-2\pi i f_j t_n).$$

Data as a function of time index n

Sinusoids at frequency f_j

Frequency resolution

$$df = \frac{1}{T}$$

Nyquist frequency

$$f_{\text{NQ}} = \frac{f_0}{2}$$

Units, decibel scale, tapers, spectrograms, ...

Spectrum practicals



Python