⊛ Serie de Fourier

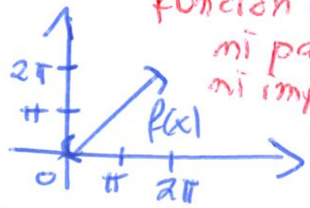
$$\boxed{SF_f(x) = \pi - 2\sum_{n=0}^{\infty} \frac{\operatorname{sen}(nx)}{n}}$$

FDADαH

Apellidos y nombre

AMPLIACIÓN DE MATEMÁTICAS 2019-2020
TRABAJO PRÁCTICO 2: Series de Fourier

función continua
ni par
ni impar



Calcula la serie de Fourier de $f(x) = x$ en el intervalo $(0, 2\pi)$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left(\frac{x^2}{2}\Big|_0^{2\pi} = \pi\right]$$

$$\pi a_n = \int_0^{2\pi} x\cos(nx)\,dx \underset{\text{int. por partes}}{=} \cancel{x\cdot\frac{\operatorname{sen}(nx)}{n}}^{\,0}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\operatorname{sen}(nx)}{n}\,dx = +\frac{1}{n^2}\cos(nx)\Big|_0^{2\pi} =$$

$u = x \to du = dx$

$dv = \cos(nx) \to v = \frac{\operatorname{sen}(nx)}{n}$

$$= \frac{1}{n^2}\left(\cancel{\cos(2\pi n)}^{\,1} - 1\right) = 0\Big] \quad \forall n \neq 0$$

$$\pi b_n = \int_0^{2\pi} x\operatorname{sen}(nx)\,dx \underset{\text{int. por partes}}{=} x\cdot\frac{-\cos(nx)}{n}\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\cos(nx)}{n}\,dx = -\frac{2\pi}{n} + \frac{1}{n^2}\left(\operatorname{sen}(nx)\Big|_0^{2\pi} =\right.$$

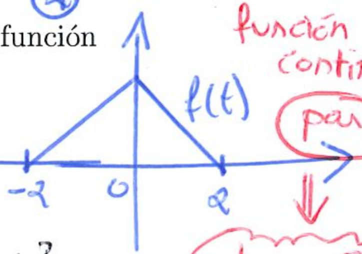
$u = x \to du = dx$

$dv = \operatorname{sen}(nx) \to v = -\frac{1}{n}\cos(nx)$

$$\forall n \in \mathbb{N} \quad = -\frac{2\pi}{n}\Big] \quad ⊛$$

función continua

par $\Rightarrow$ $\boxed{b_n = 0}$



(Examen Septiembre de 2014) Calcula la serie de Fourier de la función

$$f(t) = 2 - |t|$$

en el intervalo $[-2, 2]$. Utiliza el resultado para calcular la suma

$$\sum_{n\in\mathbb{N}} \frac{1}{(2n+1)^2}.$$

$$2t + \frac{t^2}{2}\Big|_{-2}^{0} \qquad 2t - \frac{t^2}{2}\Big|_0^2$$

$$a_0 = \frac{1}{4}\int_{-2}^{2}(2-|t|)\,dt = \frac{1}{4}\left(\int_{-2}^{0}(2+t)\,dt\right) + \frac{1}{4}\left(\int_0^2 (2-t)\,dt\right) = \cancel{1} - \cancel{\tfrac{1}{2}} + 1 - \cancel{\tfrac{1}{2}} = 1\Big]$$

$$a_n = \frac{2}{4}\int_{-2}^{2}(2-|t|)\cos\left(\frac{2\pi n}{4}t\right)dt \overset{\text{paridad}}{=} \int_0^2 (2-t)\cos\left(\frac{\pi n}{2}t\right)dt = \quad \forall n \neq 0$$

$$= \underbrace{\int_0^2 2\cos\left(\frac{\pi n}{2}t\right)dt}_{} - \underbrace{\int_0^2 t\cos\left(\frac{\pi n}{2}t\right)dt}_{\text{lo hemos hecho antes}} = \begin{cases} 0 & \text{si } n \text{ es par} \\ \frac{8}{\pi^2 n^2} & \text{si } n \text{ es impar}\end{cases}$$

$$\cancel{\frac{2\cdot 2}{n\pi}\operatorname{sen}\left(\frac{\pi n}{2}t\right)}^{\,0}\Big|_0^2 \qquad 2t\,\frac{\operatorname{sen}(\pi t)}{\pi n}\Big|_0^2 - \frac{4}{\pi n^2}\cos\left(\frac{\pi n}{2}t\right)\Big|_0^2 = -\frac{4}{\pi^2 n^2}\left(\cos(\pi n t) - 1\right)$$

Serie de Fourier:

$$\boxed{SF_f(t) = 1 + \sum_{k=0}^{\infty}\frac{8}{\pi^2(2k+1)^2}\cos\left(\frac{\pi(2k+1)}{2}t\right)}$$

Como f es continua y $\exists f'(0^-), f'(0^+) \Rightarrow f(0) = 2 = SF_f(0)$

$$\Leftrightarrow \boxed{\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}}$$

Una familia de funciones $\{g_i : A \to \mathbb{C}\}_{i\in I}$, con $A \subset \mathbb{R}$, se dice **ortogonal** sii para todo par de índices $i, j \in I$ es

$$\int_A g_i(t)\overline{g_j(t)}dt = 0.$$

- Considera la familia de funciones $\{f_n(t) = e^{2int}\}_{n\in\mathbb{Z}}$ con $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Demuestra que es una familia ortogonal.

Hay que ver: $\langle f_n, f_m\rangle \equiv \int_{-\pi/2}^{\pi/2} e^{2it(n-m)}dt = 0$, $\forall n \neq m$ enteros. En efecto:

$$\langle f_n, f_m\rangle = \int_{-\pi/2}^{\pi/2} e^{2it(n-m)}dt = \frac{e^{2it(n-m)}}{n-m}\Big|_{-\pi/2}^{\pi/2} = \frac{e^{i\pi(n-m)} - e^{-i\pi(n-m)}}{n-m} =$$

$e^{it} = \cos t + i\operatorname{sen} t$

misma paridad

$e^{it} = \cos t + i\operatorname{sen} t$

$$= \frac{\cos((n-m)\pi) - \cos(-(n-m))}{n-m} = \frac{(-1)^{\boxed{n-m}} - (-1)^{\boxed{m+n}}}{n-m} = 0 \,]$$

- Sabiendo que $\{e^{int}\}_{n\in\mathbb{Z}}$ es ortogonal, calcula el producto de cada función de esta familia por sí misma:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2int}\underbrace{\overline{e^{2int}}}_{e^{-2int}}dt. = \int_{-\pi/2}^{\pi/2} dt = t\Big|_{-\pi/2}^{\pi/2} = \pi \,]$$

$e^{2int} \cdot e^{-2int} = e^0 = 1$

⊛⊛ Serie de Fourier: $SF_f(t) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{i}{n} e^{2int} + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{i}{n} e^{-2int}$

Calcula la serie de Fourier para la función $f(t) = t$ en el intervalo $[-\frac{\pi}{2}, \frac{\pi}{2}]$ con respecto a la familia de funciones $\{e^{2int}\}_{n\in\mathbb{Z}}$ anterior.

$$C_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} t\,dt = \frac{1}{\pi}\left(\frac{t^2}{2}\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{8} - \frac{\pi}{8} = 0 \,\right]$$

$$\pi C_n = \int_{-\pi/2}^{\pi/2} t\, e^{-2int}dt \overset{\text{int. por partes}}{=} -t\cdot\frac{e^{-2int}}{2in}\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \frac{e^{-2int}}{2in}dt =$$

$u = t \to du = dt$

$dv = e^{-2int} \to v = -\frac{e^{-2int}}{2in}$

$\forall n \neq 0$

$$= -\frac{\pi}{2}\cdot\frac{\cancel{e^{-\pi in}}^{(-1)^n}}{2in} - \frac{\pi}{2}\cdot\frac{\cancel{e^{\pi in}}^{(-1)^n}}{2in} + \cancel{\frac{e^{2int}}{4n^2}\Big|_{-\pi/2}^{\pi/2}}^{0} = \frac{-\cancel{2}2\cdot\pi(-1)^n}{\cancel{4}in} = (-1)^n\cdot\frac{\pi i}{n} \,]$$

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