Apellidos y nombre

AMPLIACIÓN DE MATEMÁTICAS
TRABAJO PRÁCTICO 2: Series de Fourier

Calcula la serie de Fourier de $f(x) = |x - \pi|$ en el intervalo $[0, 2\pi]$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} |x-\pi|\,dx = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi}(x-\pi)\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{x^2}{2} - \pi x\right]_{\pi}^{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}|x-\pi|\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx - \frac{1}{\pi}\int_{\pi}^{2\pi}(\pi - x)\cos(nx)\,dx =$$

$u = \pi - x \Rightarrow du = -dx$

$dv = \cos(nx)dx \Rightarrow v = \frac{sen(nx)}{n}$

$$= \frac{1}{\pi}\cancel{\left[\frac{(\pi - x)}{n}sen(nx)\right]_0^{\pi}} + \frac{1}{n\pi}\int_0^{\pi} sen(nx)\,dx - \frac{1}{\pi}\cancel{\left[\frac{(\pi-x)}{n}sen(nx)\right]_{\pi}^{2\pi}} - \frac{1}{n\pi}\int_{\pi}^{2\pi} sen(nx)\,dx =$$

$$= \frac{1}{n\pi}\left(\left[\frac{-\cos(n\pi)}{n}\right]_0^{\pi} - \left[-\frac{\cos(n\pi)}{n}\right]_{\pi}^{2\pi}\right) = \frac{2}{n^2\pi}\left(1 - (-1)^n\right) = \begin{cases} 0 & si\ n = 2k \\ \frac{-4}{n^2\pi} = \frac{-4}{(2k+1)^2\pi} & si\ n = 2k+1 \end{cases}$$

$b_n = 0$ por paridad

$$SF_f(x) = \frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{-4}{(2k+1)^2\pi}\cos((2k+1)x)$$

Calcula la serie de Fourier de $f(x) = \cos(\pi x) - x$ en el intervalo $[-1, 1]$.

Esto es su propia serie de Fourier

$-x$ es impar $\Rightarrow a_n = 0 \ \forall n$

$$b_n = \frac{1}{1}\int_{-1}^{1} -x\, sen(n\pi x)\,dx = \left[\frac{x\cos(n\pi x)}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx =$$

$u = -x \Rightarrow du = -dx$

$dv = sen(n\pi x)dx \Rightarrow v = \frac{-\cos(n\pi x)}{n\pi}$

$$= \left[\frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\pi)}{n\pi}\right] - \cancel{\left[\frac{sen(n\pi x)}{n^2\pi^2}\right]_{-1}^{1}} \to 0 = \frac{2(-1)^n}{n\pi}$$

$$SF_f(x) = -\cos(\pi x) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}sen(n\pi x)$$

Una familia de funciones $\{g_i: A \to \mathbb{C}\}_{i\in I}$, con $A \subset \mathbb{R}$, es **ortogonal** sii $\forall i, j \in I$: $\int_A g(t)\overline{g(t)}dt = 0$. Considera la familia de funciones $f_n(t) = e^{2int}{}_{n\in\mathbb{N}\cup\{0\}}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Demuestra que es una familia ortogonal.

$$n \neq m \qquad \langle f_n, f_m \rangle = \int_{-\pi/2}^{\pi/2} e^{2it(n-m)} dt = \left[\frac{e^{2it(n-m)}}{n-m}\right]_{-\pi/2}^{\pi/2} =$$

$$= \left(\frac{e^{i\pi(n-m)} - e^{-i\pi(n-m)}}{n-m}\right) = \frac{\cos((n-m)\pi) + \cancel{i\,\text{sen}((n-m)\pi)} - \cos(-(n-m)\pi) - \cancel{i\,\text{sen}(-(n-m)\pi)}}{n-m} = \frac{(-1)^{n-m} - (-1)^{m-n}}{n-m} = 0$$

Sabiendo que $\{e^{int}\}_{n\in\mathbb{Z}}$ es ortogonal, calcula el producto de cada función de esta familia por sí misma: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2int}\overline{e^{2int}}dt = \int_{-\pi/2}^{\pi/2} e^0\, dt = [t]_{-\pi/2}^{\pi/2} = \pi$

$$(*) = \frac{1}{\pi}\left(-\frac{e^{-in\pi}}{4n^2} + \frac{1}{4n^2} + \frac{1}{4n^2} - \frac{e^{in\pi}}{4n^2}\right) = \frac{1-(-1)^n}{2n^2\pi} \Rightarrow$$

$$\boxed{SF_f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{2n^2\pi} e^{-2int} + \sum_{n=1}^{\infty} \frac{1-(-1)^{-n}}{2(n^2)\pi} e^{2int}}$$

Calcula la serie de Fourier de $f(x) = |t|$ en el intervalo $[-\frac{\pi}{2}, \frac{\pi}{2}]$ con respecto a la familia de funciones $\{e^{2int}\}_{n\in\mathbb{Z}}$.

$$n = 0 \Rightarrow \quad c_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |t|\,dt = \frac{2}{\pi}\int_0^{\pi/2} t\,dt = \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$n \neq 0 \Rightarrow \quad c_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |t| e^{-2int} dt = \frac{1}{\pi}\int_0^{\pi/2} t e^{-2int} dt - \frac{1}{\pi}\int_{-\pi/2}^{0} t e^{-2int} dt$$

$$= \frac{1}{\pi}\left(\left[\frac{te^{-2int}}{-2in}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{e^{-2int}}{2in} dt + \left[\frac{te^{-2int}}{2in}\right]_{-\frac{\pi}{2}}^{0} - \int_{-\pi/2}^{0} \frac{e^{-2int}}{2in} dt\right) =$$

$u = t \Rightarrow du = dt$

$dv = e^{-2int}dt \Rightarrow v = \dfrac{e^{-2int}}{-2in}$

$$= \frac{1}{\pi}\left(\cancel{\frac{\pi e^{-in\pi}}{4in}} - 0 + \left[\frac{e^{-2int}}{-4n^2}\right]_0^{\pi/2} + 0 + \cancel{\frac{\pi e^{in\pi}}{4in}} - \left[\frac{e^{-2int}}{-4n^2}\right]_{-\pi/2}^{0}\right) (*)$$

se cancelan