Universidad Complutense de Madrid
Facultad de Informática
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Computación Cuántica
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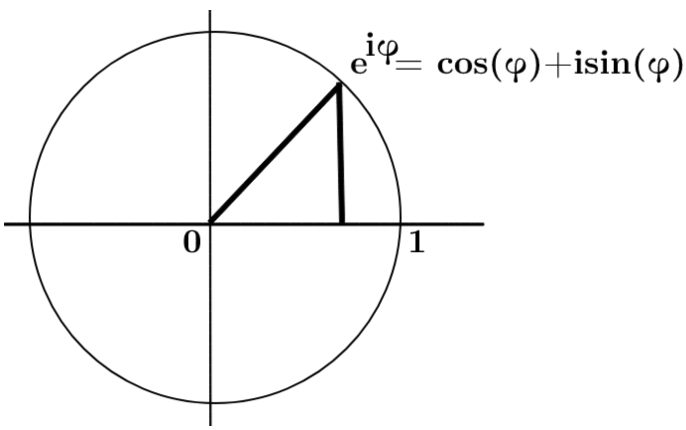
Hoja de problemas 1

1. Dar dos bases ortonormales de un espacio vectorial de dimensión tres (sobre $\mathbb{C}$), $B_1 = \{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ y $B_2 = \{|e_1'\rangle, |e_2'\rangle, |e_3'\rangle\}$ tal que $|\langle e_i|e_j'\rangle| = \frac{1}{\sqrt{3}}$ y $i, j = 1, 2, 3$.

$B_1 = \{|0\rangle, |1\rangle, |2\rangle\}$, $B_2 = \{\frac{1}{\sqrt{3}}\underbrace{(|0\rangle, |1\rangle, |2\rangle)}_{e_1'}, \frac{1}{\sqrt{3}}\underbrace{(\overbrace{e\,|0\rangle}^{i\alpha_1}, \overbrace{e\,|1\rangle}^{i\alpha_2}, \overbrace{e\,|2\rangle}^{i\alpha_3})}_{e_2'}, \frac{1}{\sqrt{3}}\underbrace{(\overbrace{e\,|0\rangle}^{i\beta_1}, \overbrace{e\,|1\rangle}^{i\beta_2}, \overbrace{e\,|2\rangle}^{i\beta_3})}_{e_3'}\}$

Para hacer que B_2 sea ortogonal $\langle e_1'|e_2'\rangle = \frac{1}{\sqrt{3}}(e^{i\alpha_1} + e^{i\alpha_2} + e^{i\alpha_3}) = 0 \Leftrightarrow e^{i\alpha_1} + e^{i\alpha_2} + e^{i\alpha_3} = 0$



$(1, 0, 0) \rightarrow \frac{1}{\sqrt{3}}(1, 1, 1)$

$(0, 1, 0) \rightarrow \frac{1}{\sqrt{3}}(1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2})$

- $\alpha_2 = 0$

  $\cos 0 + i \sin 0 = 1$, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$(0, 0, 1) \rightarrow \frac{1}{\sqrt{3}}(-1, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2})$

- $\alpha_3 = \pi$

  $\cos\pi + i\sin\pi = -1 + i * 0$, $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $\cos\frac{21\pi}{9} + i\sin\frac{21\pi}{9} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

2. Dar los autovalores y autovectores de la observable $X = \begin{pmatrix} 1 & 2-i \\ 2+1 & 2 \end{pmatrix}$.

<u>Autovalores</u>

$$X - \lambda I = \begin{bmatrix} 1-\lambda & 2-i \\ 2+i & 2-\lambda \end{bmatrix} \rightarrow det(X - \lambda I) = (1-\lambda)(2-\lambda) - (2+i)(2-i) = \lambda^2 - 3\lambda - 3$$

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4 * 1 * (-3)}}{2 * 1} = \frac{3 \pm \sqrt{9 + 12}}{2} = \frac{3 \pm \sqrt{21}}{2} \Rightarrow \left\{\frac{3 + \sqrt{21}}{2}, \frac{3 - \sqrt{21}}{2}\right\}$$

<u>Autovectores</u>

- $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $\left(X - \left(\dfrac{3+\sqrt{21}}{2}\right) I\right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2-i \\ 2+1 & 2 \end{pmatrix} - \begin{pmatrix} \dfrac{3+\sqrt{21}}{2} & 0 \\ 0 & \dfrac{3+\sqrt{21}}{2} \end{pmatrix} \to \begin{pmatrix} -1+\sqrt{21} & 2-i \\ 2+i & 1+\sqrt{21} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to$$

$$\to \begin{cases} \left(\dfrac{-1+\sqrt{21}}{2}\right) a + (2-i)b = 0 \\ (2+i)a + \left(\dfrac{1+\sqrt{21}}{2}\right) b = 0 \end{cases} \to \left(\begin{array}{cc|c} \left(\dfrac{-1+\sqrt{21}}{2}\right) & (2-i) & 0 \\ (2+i) & \left(\dfrac{1+\sqrt{21}}{2}\right) & 0 \end{array}\right) \overbrace{\sim}^{F_0=(2+i)F_0-\left(\frac{-1+\sqrt{21}}{2}\right)F_1}$$

$$\to \left(\begin{array}{cc|c} 0 & 0 & 0 \\ (2+i) & \left(\dfrac{1+\sqrt{21}}{2}\right) & 0 \end{array}\right) \overbrace{\sim}^{F_1(2-i)} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 5 & \left(\dfrac{1+\sqrt{21}}{2}\right)(2-i) & 0 \end{array}\right) \overbrace{\sim}^{\frac{F_1}{5}}$$

$$\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \dfrac{(1+\sqrt{21})(2-i)}{10} & 0 \end{array}\right) \to a + \frac{(1+\sqrt{21})(2-i)}{10} b = 0 \to a = -\frac{(1+\sqrt{21})(2-i)}{10} b$$

$$v_1 = \begin{pmatrix} -\dfrac{(1+\sqrt{21})(2-i)}{10} b \\ b \end{pmatrix} \overbrace{\to}^{b=1} \begin{pmatrix} -\dfrac{(1+\sqrt{21})(2-i)}{10} \\ 1 \end{pmatrix}$$

- $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, $\left(X - \left(\dfrac{3-\sqrt{21}}{2}\right) I\right) \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2-i \\ 2+1 & 2 \end{pmatrix} - \begin{pmatrix} \dfrac{3-\sqrt{21}}{2} & 0 \\ 0 & \dfrac{3-\sqrt{21}}{2} \end{pmatrix} \to \begin{pmatrix} \dfrac{-1-\sqrt{21}}{2} & 2-i \\ 2+i & \dfrac{1-\sqrt{21}}{2} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to$$

$$\to \begin{cases} \left(\dfrac{-1-\sqrt{21}}{2}\right) c + (2-i)d = 0 \\ (2+i)c + \left(\dfrac{1-\sqrt{21}}{2}\right) d = 0 \end{cases} \to \left(\begin{array}{cc|c} \left(\dfrac{-1-\sqrt{21}}{2}\right) & (2-i) & 0 \\ (2+i) & \left(\dfrac{1-\sqrt{21}}{2}\right) & 0 \end{array}\right) \overbrace{\sim}^{F_0=(2+i)F_0-\left(\frac{-1-\sqrt{21}}{2}\right)F_1}$$

$$\rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ (2+i) & \left(\dfrac{1-\sqrt{21}}{2}\right) & 0 \end{array}\right) \overbrace{\sim}^{F_1(2-i)} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 5 & \left(\dfrac{1-\sqrt{21}}{2}\right)(2-i) & 0 \end{array}\right) \overbrace{\sim}^{\frac{F_1}{5}}$$

$$\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \dfrac{(1-\sqrt{21})(2-i)}{10} & 0 \end{array}\right) \rightarrow c + \frac{(1-\sqrt{21})(2-i)}{10}d = 0 \rightarrow c = -\frac{(1-\sqrt{21})(2-i)}{10}d$$

$$v_2 = \begin{pmatrix} -\dfrac{(1-\sqrt{21})(2-i)}{10}d \\ d \end{pmatrix} \overbrace{\rightarrow}^{d=1} \begin{pmatrix} -\dfrac{(1-\sqrt{21})(2-i)}{10} \\ 1 \end{pmatrix}$$

3. Cuáles de los operadores siguientes son observables?

$$\sigma^x\sigma^z, \sigma^x \otimes \sigma^z, \sigma^x + \sigma^z, \sigma^x - \sigma^z, i\sigma^x$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Para saber si los operadores son observables debemos comparar el operador con el operador adjunto, si son iguales podremos decir que son observables.

- $\sigma^x\sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  $(\sigma^x\sigma^z)^* \Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overbrace{\Rightarrow}^{\text{conju.}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ no observable

- $\sigma^x \otimes \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

$$(\sigma^x \otimes \sigma^z)^* \Rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \overbrace{\Rightarrow}^{\text{conju.}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ observable}$$

- $\sigma^x + \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  $(\sigma^x + \sigma^z)^* \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \overbrace{\Rightarrow}^{\text{conju.}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ observable

- $\sigma^x - \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

  $(\sigma^x - \sigma^z)^* \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \overbrace{\Rightarrow}^{\text{conju.}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ observable

- $i\sigma^x = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

  $(i\sigma^x)^* \Rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \overbrace{\Rightarrow}^{\text{conju.}} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ no observable

4.

a) Dar los autovalores y los subespacios propios asociados de la observable $A_1 = \sigma^z_1 \otimes \sigma^z_2 \otimes \sigma^z_3$ y proponer un circuito cuántico para medirlo.

b) Misma pregunta para $A_2 = \sigma^z_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \sigma^z_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \sigma^z_3$.

c) Calcular $\langle\xi| A_1 |\xi\rangle$ y $\langle\xi| A_2 |\xi\rangle$ para $|\xi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

5.

a) Demostrar que el operador $U(t) = e^{-i\sigma^x t}$ es unitario

$U(t) = e^{-i\sigma^x t} = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}$, para que sea unitario $U * U^* = I$.

$U^* = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}$, $U * U^* = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\cos t * \cos t + (-i\sin t * i\sin t) = \cos^2 t - i^2 \sin^2 t = \cos^2 t + \operatorname{sen}^2 t = 1$

$\cos t * i\sin t - i\sin t * \cos t = 0$

$-i\sin t\cos t + \cos t * i\sin t = 0$

$-i\sin t * i\sin t + \cos t * \cos t = i^2 \sin^2 t + \cos^2 t = \sin^2 t + \cos^2 t = 1$

Se cumple $U * U^* = I$ por lo tanto $U(t)$ es unitario.

b) Sea $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Dar una expresión para $|\langle 0,0|\, U(t)_1 \otimes \mathbb{1}_2 \,|\phi^+\rangle|^2 \equiv v(t)$ en función de t y dar la gráfica de esta función para $t \in [0, 2\pi]$

Ayudas: Para cualquier endomorfismo X de un espacio vectorial, recordamos que se define e^X como $\sum_{k=0}^{\infty} \frac{X^k}{k!}$. Verifica que $(\sigma^x)^2 = \mathbb{1}$.