1 Port-Hamiltonian system without dissipation

flows:

$$f(t) := -\dot{x}(t)$$

efforts:

$$e(t) := \nabla H\big(x(t)\big)$$

dynamics:

$$-f(t) = J\big(x(t)\big)\, e(t) + G\big(x(t)\big)\, u(t)$$

with inputs $u(t)$

outputs:

$$y(t) = G^{\mathrm{T}}\big(x(t)\big)\, e(t)$$

2 Continuous collocation methods

2.1 Introduction

Solving the initial value problem

$$\dot{x}(t) = J\big(x(t)\big)\, e(t) + G\big(x(t)\big)\, u(t) \tag{1a}$$

$$x(0) = x_0 \tag{1b}$$

on some time interval $I = t \in [0, t_f]$ is in general not possible analytically.

The problem can be solved numerically by splitting the time interval I into K subintervals

$$I^k := [(k-1)\, h,\, kh] \ .$$

The index $k = 1, 2, \ldots, K$ identifies a paricular subinterval. The length of the subintervals has to be prescribed by chosing a time step h such that $t_f = hK$.

The true solution $x(t)$ (with $t \in I$) is then approximated by the numerical solution $\tilde{x}(t)$ which is defined picewise: On each subinterval, $\tilde{x}(t)$ (with $t \in I^k$) is a polynomial of degree s (in the variable t). More precisely, if $x(t) \in \mathbb{R}^N$ then $\tilde{x}(t)$ is a vector of N polynomials.

Like the time step h, the oder of the polynomial approximation is a parameter that has to be chosen by the user (to control the fidelity of the numerical method).

Since a polynomial of degree s is determined by $s + 1$ coefficients there are $s + 1$ (vectorial) unknowns per interval. The continuity of the solution between two intervals determines one (vectorial) unknown. For the first interval I^1, this unknown is determined by the initial condition (1b).

The remaining s unknowns are fixed by requiring that the differential equation has to hold at s collocation points $t_1^k, \ldots, t_s^k \in I^k$. The choice of these points within each subinterval is important for obtaining a good numerical approximation.

2.2 Details

By defining $t_0^k := (k-1)\,h$ and $t_{s+1}^k := hk$, each subinterval can be written as

$$I_k = \left[t_0^k, t_{s+1}^k\right] .$$

Then, the collocation points $t_1^k, \ldots, t_s^k$ can be written as $t_i^k := t_0^k + c_i\,h$ with coefficients $c_1, \ldots, c_s$ satisfying

$$\begin{aligned} &\forall\, i \in \{1, \ldots, s\} \;:\; 0 \le c_i \le 1 \\ &\forall\, i \in \{1, \ldots, s-1\} \;:\; c_i < c_{i+1} \,. \end{aligned}$$

Since on each subinterval the numerical solution $\tilde{x}(t)$ is given as a vector of polynomials of degree s, the time derivative $\dot{\tilde{x}}(t)$ is a vector of polynomials of degree $s-1$.

The numerical solution $\tilde{x}(t)$ with $t \in I^k$ can be written as

$$\tilde{x}(t) = \tilde{x}(t_0^k) + \int_{t_0^k}^{t} \dot{\tilde{x}}(u)\,\mathrm{d}u \,. \tag{2}$$

To make life easier, a local time coordinate $\tau^k \in [0,1]$ is defined on each subinterval I^k by requiring that $t = \big((k-1)+\tau\big)h$. Hence, $\tau^k = 0$ corresponds to t_0^k and $\tau^k = 1$ corresponds to t_{s+1}^k.

Based on this normalized time, the polynomial approximation of $\dot{\tilde{x}}$ can be defined in a unified manner for all k: On each subinterval, the vector $\dot{\tilde{x}}$ can be written as a linear combination

$$\dot{\tilde{x}}(\tau^k) := \sum_{i=1}^{s} \Big(-f_i^k\, l_i(\tau)\Big) \,.$$

More precisely, the n-th component of $\dot{\tilde{x}}$ is a vector in the vector space that is spanned by the s basis functions $l_1, \ldots, l_s$. Given such a basis, this vector is determined by the tuple consisting of the n-th component of each of the vectors $f_1, \ldots, f_s$.

With this change of time coordinate and a choice of basis, equation (2) can be written as

$$\begin{aligned} \tilde{x}(t_0^k + \tau^k\,h) &= \tilde{x}(t_0^k) + \int_0^{\tau} \sum_{j=1}^{s} \Big(-f_j^k\, l_j(\mu)\Big)\, h\,\mathrm{d}\mu \\ &= \tilde{x}(t_0^k) - h \sum_{j=1}^{s} \left(f_j^k \int_0^{\tau} l_j(\mu)\,\mathrm{d}\mu \right) . \end{aligned}$$

The collocation method is based on relating $\dot{\tilde{x}}$ and $\tilde{x}$ at the s collocation points $c_1, \ldots, c_s$ (in normalized time). Hence, it is convenient to chose the basis polynomials such that the n-th component of f_i expresses the value of the n-th component of $\dot{\tilde{x}}$ at $\tau = c_i$, i.e. $-f_i \stackrel{!}{=} \dot{\tilde{x}}(c_i)$.

These basis polynomials are the Lagrange polynomials

$$l_i(\tau) := \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{\tau - c_j}{c_i - c_j}$$

which have the desired property $l_i(c_j) = \delta_{ij}$ (Kronecker delta).

The value of the numerical solution $\tilde{x}$ at the beginning of each subinterval I^k shall be denoted by $x_0^k := \tilde{x}(t_0^k)$. This value is known because of the continuity of the solution between subintervals or because of the initial condition (1b), i.e. $x_0^1 = x_0$ and $x_0^k = x_{s+1}^{k-1}$ for $k > 1$.

By defining

$$b_j := \int_0^1 l_j(\mu)\,\mathrm{d}\mu$$

the numerical solution at the endpoint of a subinterval can be written as

$$x_{s+1}^k := \tilde{x}(t_{s+1}^k) = x_0^k - h \sum_{j=1}^{s} \left(b_j\, f_j^k \right).$$

It is the main goal of all computation done at a particular step k to obtain the value x_{s+1}^k which is determined by the result $x_{s+1}^{k-1} = x_0^k$ of the previous step and the coefficients $f_1^k, \ldots, f_s^k$.

These coefficients have to be determined by solving a (nonlinear) system of equations: Equation (1a) has to hold at the collocation points $t_1^k, \ldots, t_s^k$. Hence, the system of equation defines a relationship between the values of the derivative of the numerical solution at the collocation points given by the the unknowns $f_1^k, \ldots, f_s^k$ and the values of the numerical solution at the collocation points $\tilde{x}(t_1^k), \ldots, \tilde{x}(t_s^k)$ which in turn also depend on $f_1^k, \ldots, f_s^k$.

By defining

$$a_{ij} := \int_0^{c_i} l_j(\mu)\,\mathrm{d}\mu$$

the numerical solution $\tilde{x}$ can conveniently be expressed at the collocation points:

$$x_i^k := \tilde{x}(t_i^k) = x_0^k - h \sum_{j=1}^{s} \left(a_{ij}\, f_j^k \right)$$

for $i = 1, \ldots, s$.

The following definitions are introduced for convenience:

$$\begin{aligned} J_i^k &:= J(x_i^k) \\ e_i^k &:= \nabla H(x)|_{x=x_i^k} \\ G_i^k &:= G(x_i^k) \\ u_i^k &:= u(t_i^k) \end{aligned}$$

Then, the system of equations, which has to be solved at the k-th step, can be written as

$$f_i^k + J_i^k\, e_i^k + G_i^k\, u_i^k = 0 \quad \text{for } i = 1, \ldots, s\,. \tag{4}$$

In the general case, every term of the above equation, except for u_i^k, depends on (a subset of) the unknowns $f_1^k, \ldots, f_s^k$.

By introducing the block vectors and matrices

$$\begin{aligned} f^k &:= \left[\left(f_1^k\right)^{\mathrm{T}}, \ldots, \left(f_s^k\right)^{\mathrm{T}}\right]^{\mathrm{T}} \\ J^k &:= \mathrm{blockdiag}(J_1^k, \ldots, J_s^k) \\ e^k &:= \left[\left(e_1^k\right)^{\mathrm{T}}, \ldots, \left(e_s^k\right)^{\mathrm{T}}\right]^{\mathrm{T}} \\ G^k &:= \mathrm{blockdiag}(G_1^k, \ldots, G_s^k) \\ u^k &:= \left[\left(u_1^k\right)^{\mathrm{T}}, \ldots, \left(u_s^k\right)^{\mathrm{T}}\right]^{\mathrm{T}} \end{aligned}$$

the system of equations (4) can be written as

$$f^k + J^k\, e^k + G^k\, u^k = 0$$

2.3 Gauss-Legendre collocation

Gauss-Legendre collocation methods use the roots of the shifted Legendre polynomial

$$\frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\Big(\big(x\,(x-1)\big)^s\Big)$$

as coefficients $c_1, \ldots, c_s$.