MA2104 - Multivariable Calculus Suggested Solutions

(Semester 1, AY2023/2024)

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Question 1

Consider the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$.

(i) Find the value of $\frac{\partial z}{\partial x}$ at the point $(1, 0, -\frac{\sqrt{3}}{2})$ on the ellipsoid.

(ii) Find the tangent plane at the point $(1, 0, -\frac{\sqrt{3}}{2})$ on the ellipsoid.

(iii) Find the distance from the point $(-2, 0, 0)$ to the tangent plane in (ii).

Solution:

(i) Let $F(x, y, z) = \frac{x^2}{4} + y^2 + z^2$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{x}{2}}{2z} = -\frac{x}{4z}, \text{ so } \left.\frac{\partial z}{\partial x}\right|_{\left(1,0,-\frac{\sqrt{3}}{2}\right)} = -\frac{1}{-2\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}.$$

(ii) The normal vector is $\nabla F(x, y, z) = \langle \frac{x}{2}, 2y, 2z \rangle$. At the point $(1, 0, -\frac{\sqrt{3}}{2})$, the normal vector is

$$\mathbf{n} = \nabla F\left(1, 0, -\frac{\sqrt{3}}{2}\right) = \left\langle \frac{1}{2}, 0, -\sqrt{3} \right\rangle.$$

Therefore, the tangent plane is

$$\frac{1}{2}(x-1) + 0(y-0) - \sqrt{3}\left(z + \frac{\sqrt{3}}{2}\right) = 0 \Leftrightarrow \boxed{x - 2\sqrt{3}z = 4}.$$

(iii) We can take any point on the plane, e.g., $P = (1, 0, -\frac{\sqrt{3}}{2})$, and denote $Q = (-2, 0, 0)$. Then

$$\overrightarrow{PQ} = \left\langle -3, 0, \frac{\sqrt{3}}{2} \right\rangle.$$

The distance is

$$|\text{comp}_{\mathbf{n}}\overrightarrow{PQ}| = \frac{|\mathbf{n} \cdot \overrightarrow{PQ}|}{|\mathbf{n}|} = \frac{|\frac{1}{2} \cdot (-3) + 0 \cdot 0 + (-\sqrt{3} \cdot \frac{\sqrt{3}}{2})|}{\sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + (-\sqrt{3})^2}} = \boxed{\frac{6}{\sqrt{13}}}.$$

Question 2

You are standing directly above the point $(x, y) = (4, 3)$ on a surface whose height z is given by the equation $z = 2x^2 - y^2$.

(i) If you spill a glass of water on the surface, in which direction will it run off at the point $(x, y) = (4, 3)$? (Water will follow the steepest downhill path available.) Give your answer as a two-dimensional vector (i.e. only the x- and y-components are wanted).

(ii) What is the slope (vertical change divided by horizontal distance travelled) of the surface as you look directly towards the z-axis?

Solution:

(i) We have $\nabla z(x, y) = \langle 4x, -2y \rangle$. Thus the direction in which the water will run off will be $-\nabla z(4, 3) = \boxed{\langle -16, 6 \rangle}$.

(ii) The slope is given by the directional derivative in the direction $-\langle 4, 3 \rangle$, which equals $\nabla z(4, 3) \cdot \mathbf{u} = \langle 16, -6 \rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle = \boxed{-\frac{46}{5}}$.

Question 3

There is a triangular attic roof with vertices at $(3, 0, 0), (0, 4, 0)$ and $(0, 0, 2)$. You want to build a rectangular storage box with edges parallel to the axes, one corner at the origin and its diagonally opposite corner on the attic roof. Find the dimensions of the storage box with the largest volume that you can build.

Solution: Since the intersections of the plane where the roof lies on with the coordinate axes are $x = 3, y = 4$ and $z = 2$, the equation of the plane is $4x + 3y + 6z = 12$ and we arrive at the following constrained optimization problem:

Given a storage box with dimensions x, y, z, maximize the volume $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 4x + 3y + 6z = 12$, where $x, y, z \geq 0$.

The Lagrange multiplier equations are

$$\begin{cases} yz = 4\lambda \\ xz = 3\lambda \\ xy = 6\lambda \\ 4x + 3y + 6z = 12 \end{cases}$$

so we have $xyz = 4\lambda x = 3\lambda y = 6\lambda z$, assuming $\lambda \neq 0$, we have $y = \frac{4}{3}x$ and $z = \frac{2}{3}x$. Hence, $4x + 4x + 4x = 12 \Rightarrow x = 1$, so $y = \frac{4}{3}$ and $z = \frac{2}{3}$. If $\lambda = 0$, we get $xyz = 0$, which is the smallest possible volume. Thus, the dimensions that give us the largest volume is $\boxed{1 \times \frac{4}{3} \times \frac{2}{3}}$.

Question 4

Find the volume of the solid under the surface

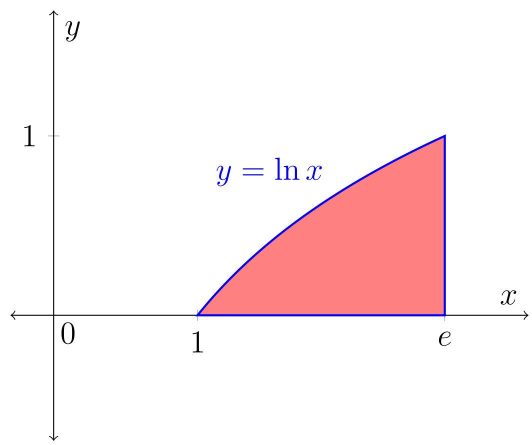
$$z = \frac{1}{\ln x}$$

and above the region D in the xy-plane given by $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1, e^y < x < e\}$.

Solution: We are asked to integrate

$$\int_0^1 \int_{e^y}^e \frac{1}{\ln x}\, dx\, dy.$$

We consider changing the order of integration. The region of integration is shown in the diagram below, where the top curve is $y = \ln x$ (or $x = e^y$) and the bottom curve is $y = 0$.



Changing the order of integration, the volume of the solid is

$$\int_0^1 \int_{e^y}^e \frac{1}{\ln x}\, dx\, dy = \int_1^e \int_0^{\ln x} \frac{1}{\ln x}\, dy\, dx = \int_1^e 1\, dx = \boxed{e - 1}.$$

Question 5

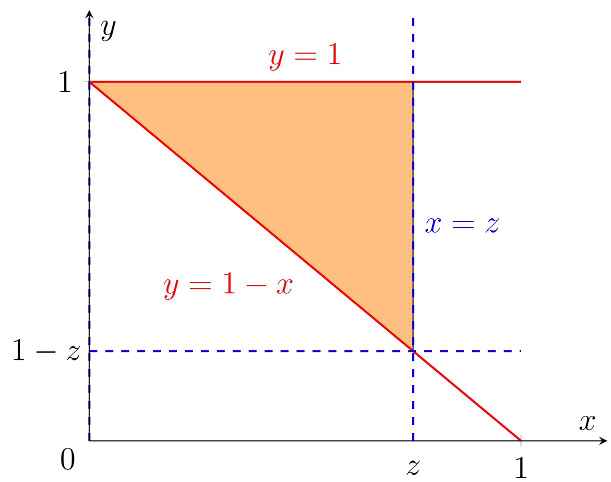
Find appropriate bounds for the triple integral

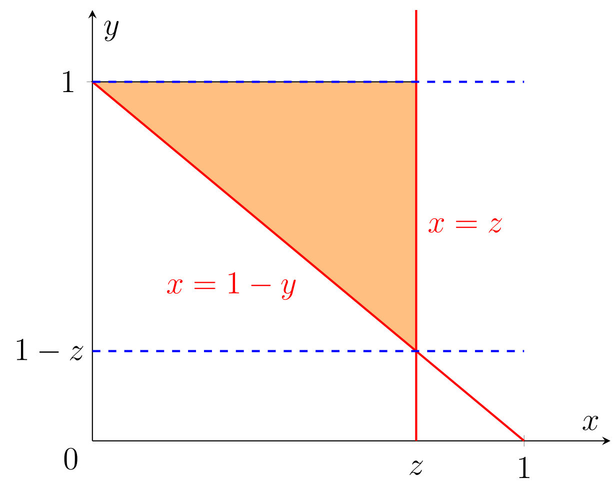
$$\int_0^1 \int_0^z \int_{1-x}^1 f(x,y,z)\,dy\,dx\,dz$$

if it is rewritten in the order

$$\int\int\int f(x,y,z)\,dx\,dy\,dz.$$

Solution: On a z-slice, the order of integration is switched as in the diagrams below:





Hence the bounds will be:

$$\boxed{\int_0^1 \int_{1-z}^1 \int_{1-y}^z f(x,y,z)\,dx\,dy\,dz}.$$

Question 6

Find the mass of the solid E that lies above $z = \sqrt{x^2+y^2}$ and between $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ if the density at any point is equal to its distance from the origin.

Solution: The cone $z = \sqrt{x^2+y^2}$ is given in spherical coordinates by the equation $\phi = \pi/4$, while the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ are given in spherical coordinates by $\rho = 1$ and $\rho = 2$ respectively. Hence the mass of the solid is

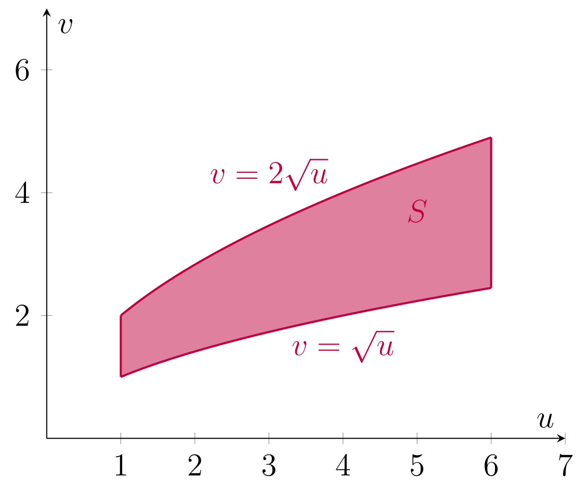
$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho\cdot\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta &= 2\pi\left(\int_0^{\pi/4}\sin\phi\,d\phi\right)\left(\int_1^2 \rho^3\,d\rho\right) = 2\pi\left[-\cos\phi\right]_0^{\pi/4}\left[\frac{\rho^4}{4}\right]_1^2 \\ &= 2\pi\left(-\frac{\sqrt{2}}{2}+1\right)\left(4-\frac{1}{4}\right) = \boxed{\frac{15\pi}{4}(2-\sqrt{2})}.\end{aligned}$$

Question 7

Using the change of variables $x = \frac{u}{v}, y = v$, evaluate $\iint_R xy\,dx\,dy$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 6$.

Solution: The region R is transformed under the change of variables bijectively onto the region S with boundary curves

$$\begin{aligned}
y = x \Rightarrow v = \frac{u}{v} \Rightarrow v = \sqrt{u} \quad &\text{(positive square root since } v = y > 0\text{)},\\
y = 4x \Rightarrow v = \frac{4u}{v} \Rightarrow v = 2\sqrt{u},&\\
xy = 1 \Rightarrow u = 1,&\\
xy = 6 \Rightarrow u = 6.&
\end{aligned}$$



The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

Since $v > 0$ on S, the Jacobian is positive on S. Therefore

$$\begin{aligned}
\int_R xy\,dx\,dy &= \int_1^6 \int_{\sqrt{u}}^{2\sqrt{u}} u \cdot \frac{1}{v}\,dv\,du = \int_1^6 [u \ln|v|]_{v=\sqrt{u}}^{v=2\sqrt{u}}\,du\\
&= \int_1^6 u(\ln 2\sqrt{u} - \ln\sqrt{u})\,du = \int_1^6 u \ln 2\,du\\
&= (\ln 2)\left[\frac{1}{2}u^2\right]_1^6 = \boxed{\frac{35}{2}\ln 2}.
\end{aligned}$$

Question 8

Show that the line integrals $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = \langle 2xe^y, (x^2+z^2)e^y, 2ze^y \rangle,$$

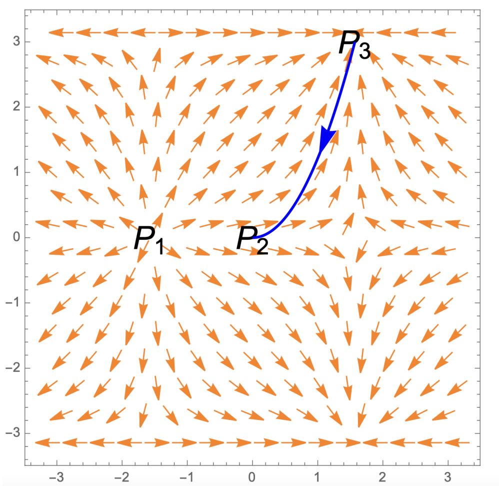
C_1 is the helix $r_1(t) = \langle \cos(\pi t), t, \sin(\pi t) \rangle$ for $0 \le t \le 1$ and C_2 is the line segment $r_2(s) = \langle 1-2s, s, 0 \rangle$ for $0 \le s \le 1$.

Solution: Since $\operatorname{curl} \mathbf{F} = \langle 2ze^y - 2ze^y, -(0-0), 2xe^y - 2xe^y \rangle = 0$ and $\mathbf{F}$ is defined on $\mathbb{R}^3$, which is open and simply connected, $\mathbf{F}$ is conservative (specifically, $\mathbf{F} = \nabla f$ where $f(x,y,z) = (x^2+z^2)e^y$). Notice that C_1 and C_2 have the same starting point $(1,0,0)$ and ending point $(-1,1,0)$. Therefore, by the fact that the line integral of a conservative vector field is path independent, we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Question 9

The following diagram shows the vector field plot of $\mathbf{F}$.



(a) Is $\operatorname{div} \mathbf{F}$ at point P_1 positive, zero, or negative? Explain your answer.

(b) Let C be the curve shown in the diagram with the given orientation. Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, zero, or negative? Explain your answer.

Solution:

(a) **Positive** since the arrows around P_1 point away from it, indicating that P_1 is a source, and the divergence at a source is positive.

(b) **Negative** since $\mathbf{F}$ points in roughly the opposite direction to the curve C, so the work done by a force field $\mathbf{F}$ along C (which is what the line integral computes) would be negative.

Question 10

Let C be the plane curve following these steps:

- from $(-3, 0)$ to $(3, 0)$ clockwise along the circle of radius 3 centered at the origin, then
- from $(3, 0)$ to $(2, 0)$ along the x-axis, then
- from $(2, 0)$ to $(-2, 0)$ anticlockwise along the circle of radius 2 centered at the origin, and finally
- from $(-2, 0)$ to $(-3, 0)$ along the x-axis.

Find the work done by the force field $\mathbf{F} = \langle y^2, x\rangle$ along the path C.

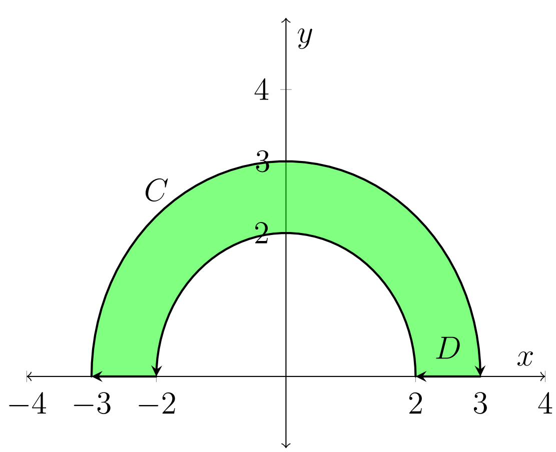
Solution: Let D be the region enclosed by the curve, then D is to the "right" of the oriented curve C. By Green's Theorem,

$$\int_C \langle y^2, x\rangle \cdot d\mathbf{r} = -\iint_D (1 - 2y)\, dA = -\iint_D 1\, dA + 2\iint_D y\, dA.$$

We know $\iint_D 1\, dA = \text{area}(D) = \frac{3^2\pi}{2} - \frac{2^2\pi}{2} = \frac{5\pi}{2}$ using geometry. Since D is given in polar coordinates by $0 \le \theta \le \pi, 2 \le r \le 3$,

$$\iint_D y\, dA = \int_0^\pi \int_2^3 r\sin(\theta)\, r\, dr\, d\theta = \left(\int_0^\pi \sin(\theta)\, d\theta\right)\left(\int_2^3 r^2\, dr\right) = 2 \cdot \frac{19}{3} = \frac{38}{3}.$$

Hence the work done is $\boxed{-\frac{5\pi}{2} + \frac{76}{3}}$.



Question 11

Consider the vector field $\mathbf{F}(x, y, z) = \langle x^2, -2xy, z\rangle$ and the following geometric objects:

- The surface S_1 is defined as the part of $z = x^2 + y^2$ below $z = 1$, with upwards orientation. The boundary of S_1 is a closed curve C.
- The surface S_2 is the planar disk that is bounded by C, with downwards orientation.
- The surfaces S_1 and S_2 together form a closed surface, call it S_3, with orientation inherited from S_1 and S_2.
- E is the solid region enclosed by S_3.

(i) Write down the Divergence Theorem as it applies to $\mathbf{F}, S_3$, and E (1st blank) and also relate the term involving S_3 to integrals over S_1 and S_2 (2nd blank) by filling in the blanks below:

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \boxed{} = \boxed{}$$

(ii) Compute the divergence of $\mathbf{F}$.

(iii) Evaluate $\displaystyle\iiint_E \operatorname{div}\mathbf{F}\,dV$ directly as a triple integral, without using the Divergence Theorem.

(iv) Provide

- the parametric representation of S_2 in terms of the parameters (x, y),
- the parameter domain, and
- a normal vector to S_2.

(v) Find $\displaystyle\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$.

(vi) Find $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$.

Solution:

(i) Since the orientation of S_3 points inward to E, this orientation is negative. So

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \boxed{-\iint_{S_3} \mathbf{F}\cdot d\mathbf{S}} = \boxed{-\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}}$$

(ii) $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(-2xy) + \dfrac{\partial}{\partial z}(z) = 2x - 2x + 1 = \boxed{1}$.

(iii) We use cylindrical coordinates as follows:

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_{x^2+y^2\le 1} \int_{x^2+y^2}^{1} 1\, dz\, dy\, dx = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1 (1-r^2) r\, dr = 2\pi \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1$$

$$= \boxed{\frac{\pi}{2}}.$$

(iv)
- The parametric representation is $\mathbf{r}(x,y) = \boxed{\langle x, y, 1\rangle}$.
- The parameter domain is $\boxed{\{(x,y) \mid x^2+y^2 \le 1\}}$.
- A normal vector is $\mathbf{r}_x \times \mathbf{r}_y = \boxed{\langle 0,0,1\rangle}$.

(v) Note that $\mathbf{r}_x \times \mathbf{r}_y$ is oriented upwards while S_2 is oriented downwards. Thus

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\iint_{x^2+y^2\le 1} \langle x^2, -2xy, 1\rangle \cdot \langle 0,0,1\rangle\, dy\, dx$$

$$= -\iint_{x^2+y^2\le 1} 1\, dy\, dx$$

$$= -(\text{area of disk of radius } 1)$$

$$= \boxed{-\pi}$$

(vi)

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = -\iiint_E \operatorname{div} \mathbf{F}\, dV - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\frac{\pi}{2} - (-\pi) = \boxed{\frac{\pi}{2}}.$$