Worked solutions for MA2101 23/24 S1 exam

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Note: These solutions were written in a great rush, so there may be many mistakes. Read with care! (I'm especially unsure about my answers to questions 5(c) and (d).)

Question 1

> *Let* $A = (a_{ij}) \in M_2(\mathbf{R})$ *be a real matrix, let* $P = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$*, and suppose*
>
> $$P^{-1}AP = D := \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$
>
> *Let* $y_i = y_i(x)$ $(i = 1, 2)$ *be differentiable functions in* x*. Solve the following system of differential equations:*
>
> $$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
>
> *Note: For the differential equation* $z'(x) + p(x)z(x) = q(x)$ *you may assume, without proof, that its general solution is given by* $z(x) = \frac{1}{\mu}(C + \int \mu q)$ *with* $\mu = e^{\int p}$*.*

It suffices to solve the system $Z' = DZ$, since then $Y = PZ$ would solve the original system $Y' = AY$. This new system is given by

$$\begin{cases} z_1'(x) = 2z_1(x) + z_2(x) \\ z_2'(x) = 2z_2(x), \end{cases}$$

from which we immediately have $z_2(x) = C_2 e^{2x}$. Substituting this into the first equation then yields

$$z_1'(x) = 2z_1(x) + C_2 e^{2x} \quad \text{or} \quad z_1'(x) + p(x)z_1(x) = q(x),$$

where $p(x) = -2$ and $q(x) = C_2 e^{2x}$. Computing $\mu(x) = \exp(\int p(x)\, dx) = e^{-2x}$, we deduce the general solution

$$\begin{aligned} z_1(x) &= \frac{1}{e^{-2x}} \left(\int e^{-2x} \cdot C_2 e^{2x}\, dx + C_1 \right) \\ &= e^{2x}(C_2 x + C_1). \end{aligned}$$

Now that we have solved the system $Z' = DZ$, it remains for us to convert this into a solution $Y = PZ$ of $Y' = AY$. The final solution is thus given by

$$\begin{cases} y_1(x) = z_1(x) = e^{2x}(C_2 x + C_1) \\ y_2(x) = z_1(x) + z_2(x) = e^{2x}(C_2 x + C_2 + C_1). \end{cases}$$

Question 2

> *Let $P_s[x]$ be the vector space over $\mathbf{R}$ of real polynomials of degree less than s, and let $B_s = (1, x, \ldots, x^{s-1})$ be the standard basis of $P_s[x]$. Given real numbers c and d, it is known that the map $T_{c,d}\colon P_3[x] \to P_4[x]$ defined by sending $f = f(x) = \sum_{i=0}^{2} a_i x^i$ to $xf(cx+d) = \sum_{i=0}^{2} x a_i (cx+d)^i$ is a linear transformation.*
>
> ***2(a)** Find the representation matrix $A_{c,d} := [T_{c,d}]_{B_3,B_4} \in M_{4,3}(\mathbf{R})$ in terms of c and d.*

We begin by computing how $T_{c,d}$ acts on the basis $B_3 = (1, x, x^2)$: We have

$$\begin{cases} T_{c,d}(1) & = & x \\ T_{c,d}(x) & = & x(cx+d) = dx + cx^2 \\ T_{c,d}(x^2) & = & x(cx+d)^2 = d^2x + 2cdx^2 + c^2x^3. \end{cases}$$

It follows that the representation matrix $A_{c,d} = [T_{c,d}]_{B_3,B_4}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & d & d^2 \\ 0 & c & 2cd \\ 0 & 0 & c^2 \end{pmatrix}.$$

> ***2(b)** Find conditions on c and d that are equivalent to $T_{c,d}$ being an injection. Justify your answer.*

The map $T_{c,d}$ is injective iff the matrix $A_{c,d}$ has a trivial null space. This is equivalent to having $\operatorname{rank}(A_{c,d}) = 3$, which happens iff the submatrix obtained by removing the zero row has nonzero determinant. Since this determinant is of a triangular matrix, it is the product of diagonal entries, which is c^3. Therefore $T_{c,d}$ is an injection if and only if $c^3 \neq 0$, if and only if $c \neq 0$.

> ***2(c)** Can you find c and d such that $T_{c,d}$ is a surjection? Justify your answer.*

No. The map $T_{c,d}$ is never a surjection, since the constant $1 \in P_4[x]$ is never in the image of $T_{c,d}$: Indeed, if $T_{c,d}(f) = xf(cx+d) = 1$, we would have $f(cx+d) = 1/x$ for nonzero x, which is impossible.

> ***2(d)** Suppose the matrix $A_{c,d}$ has rank equal to 2. Determine $\dim_{\mathbf{R}} \ker(T_{c,d})$. No justification is needed.*

We have $\dim_{\mathbf{R}} \operatorname{im}(T_{c,d}) + \dim_{\mathbf{R}} \ker(T_{c,d}) = \dim(B_3) = 3$ by the rank-nullity theorem, and $\operatorname{rank}(A_{c,d}) = \dim_{\mathbf{R}} \operatorname{im}(T_{c,d})$ more or less by definition. Thus, we have $\dim_{\mathbf{R}} \ker(T_{c,d}) = 1$.

> ***2(e)** Suppose the range $R(A_{c,d})$ of $A_{c,d}$ is spanned by two column vectors $Y_1 = (1,1,0,0)^t$ and $Y_2 = (0,0,1,1)^t$. Find polynomials $f_1, f_2 \in P_4[x]$ such that the range $R(T_{c,d})$ of $T_{c,d}$ is spanned by f_1 and f_2. No justification is needed.*

We can choose $f_1 = 1 + x$ and $f_2 = x^2 + x^3$, which are precisely the elements of $P_4[x]$ with basis representations (with respect to B_4) given by Y_1 and Y_2.

Question 3

> ***3(a)*** *Consider the matrix*
>
> $$A = \begin{pmatrix} 0 & -3 & -3 \\ -3 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}.$$
>
> ***3(a)(i)*** *Determine the characteristic polynomial $p_A(x)$.*

By definition, we have

$$\begin{aligned} p_A(x) = \det(xI_3 - A) &= \det \begin{pmatrix} x & 3 & 3 \\ 3 & x & 3 \\ 3 & 3 & x \end{pmatrix} \\ &= x^3 - 27x + 54 = (x-3)^2(x+6). \end{aligned}$$

> ***3(a)(ii)*** *Find all the eigenvalues λ_i of A.*

Recall that these are the roots of the characteristic polynomial. Thus we have an eigenvalue $\lambda_1 = 3$ with algebraic multiplicity $n_1 = 2$, as well as an eigenvalue $\lambda_2 = -6$ with algebraic multiplicity $n_2 = 1$.

> ***3(a)(iii)*** *Find a basis B_{λ_i} for each eigenspace $V_{\lambda_i}(A)$.*

Recall that $V_{\lambda_i}(A) := \ker(\lambda_i I - A)$. Let us first consider the case $\lambda_1 = 3$, so that

$$3I - A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Since all the columns are equal, we immediately obtain a basis $B_3 = (p_1, p_2)$ for $V_3(A)$, where $p_1 = (-1, 1, 0)^t$ and $p_2 = (-1, 0, 1)^t$.

Now for $\lambda_2 = -6$ we have

$$-6I - A = \begin{pmatrix} -6 & 3 & 3 \\ 3 & -6 & 3 \\ 3 & 3 & -6 \end{pmatrix},$$

and the observation that the columns add to zero yields the basis $B_3 = (p_3)$, where $p_3 = (1, 1, 1)^t$.

> ***3(a)(iv)*** *Find the minimal polynomial $m_A(x)$.*

The minimal polynomial and the characteristic polynomial have the same zeroes, and the minimal polynomial divides the characteristic polynomial, so there are only two possibilities for $m_A(x)$: It is either $(x-3)(x+6)$ or $(x-3)^2(x+6)$. Direct calculation shows that $(A-3)(A+6) = 0$, so we conclude that $m_A(x) = (x-3)(x+6)$.

> ***3(a)(v)*** *Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.*

The eigenspace bases we found in part (iii) form a matrix $P = (p_1, p_2, p_3)$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 6 \end{pmatrix} = \mathrm{diag}[3, 3, 6].$$

> ***3(b)*** *Can we choose the matrix P in part (a) to be an orthogonal matrix? Justify your answer.*

Yes. This is because the matrix A is symmetric and real, so the principal axis theorem implies the existence of an orthogonal matrix P with the desired properties.

> ***3(c)*** *Let G be a **diagonalizable** complex matrix. Suppose that the characteristic polynomial is given by $p_G(x) = (x-1)^2(x+1)$.*
>
> ***3(c)(i)*** *Find all possible sizes of the matrix G. Justify your answer.*

An $n \times n$ matrix has a characteristic polynomial of degree n, so G must be a 3×3 matrix.

> ***3(c)(ii)*** *Find all possible minimal polynomials $m_G(x)$ of G. Justify your answer.*

We have $m_G(x) = (x-1)(x+1)$ as a straightforward consequence of three facts: (1) $m_G(x)$ divides $p_G(x)$. (2) $m_G(x)$ has the same roots as $p_G(x)$. (3) The minimal polynomial of a diagonalizable matrix is such that all of its roots are of algebraic multiplicity one.

> ***3(d)*** *Is the matrix G is part (c) invertible? If so, find a polynomial $f(x)$ of degree at most two such that $G^{-1} = f(G)$.*

Yes. We have $m_G(G) = 0$ by definition of G, so that $(G-I)(G+I) = 0$. This can be rewritten as $G^2 = I$, so that $G^{-1} = G$. Thus we may set $f(x) = x$.

Question 4

> ***4(a)*** *State the definition of a positive-definite complex matrix.*

A positive-definite complex matrix is an $n \times n$ matrix A with complex entries that is self-adjoint ($A^* = A$) and satisfies $x^*Ax > 0$ for all $x \neq 0$.

[*Note:* The *adjoint* or *conjugate transpose* A^* of a complex matrix $A \in M_n(\mathbf{C})$ is defined by $A^* = (\overline{A})^t = (\overline{a}_{ji})$, where $\overline{a+bi} = a - bi$ denotes complex conjugation. The key property of the adjoint is that $\langle Av, w \rangle = \langle v, A^*w \rangle$. We care about positive-definite matrices in part because their eigenvalues are positive.]

> ***4(b)*** *Consider the matrix*
>
> $$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
>
> *Show that $(H^tHX)^t\overline{X} > 0$ for every **nonzero** complex column vector $X = (x_1, x_2, x_3)^t$, where $\overline{X} = (\overline{x}_1, \overline{x}_2, \overline{x}_3)^t$.*

Since H is real, we have $\overline{H} = H$. Thus

$$(H^tHX)^t\overline{X} = X^tH^tH^t\overline{X} = (HX)^t\overline{HX}.$$

Writing $Y = HX$, we see that $Y \neq 0$ because H is invertible and X is assumed to be nonzero. Then the conclusion follows from the fact that

$$Y^t\overline{Y} = \sum_{j=1}^{3} y_j\overline{y_j} = \sum_{j=1}^{3} |y_j|^2 > 0.$$

> **4(c)** *Is the H in part (b) diagonalizable? Is H^tH diagonalizable? Justify your answers.*

The matrix H is not diagonalizable because it is a nontrivial Jordan block. However, H^tH is real and symmetric, so it is diagonalizable by the principal axis theorem. (In fact, it is orthogonally diagonalizable.)

> **4(d)** *Let C be an invertible complex matrix. Show that C^*C is a positive-definite matrix. (You are not allowed to apply Remark 11.30 without giving a proof.)*

We compute

$$x^*(C^*C)x = (Cx)^*(Cx) = \sum_j \overline{y_j}y_j = \sum_j |y_j|^2 > 0,$$

where $y = Cx$ is nonzero because C is invertible and x is nonzero by definition.

> **4(e)** *Let A be a positive-definite complex matrix. Show that $A = G^*G$ for some invertible complex matrix G. (You are not allowed to apply Remark 11.30 without giving a proof, but you may use the principal axis theorem without proof.)*

By the principal axis theorem, $A = U^*DU$ for a unitary matrix U. Positive-definiteness then implies that the entries of D are positive and real, so that, writing $D = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$, we are permitted to define the matrix $\sqrt{D} := \operatorname{diag}[\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}]$. It follows that $A = G^*G$, where $G = \sqrt{D}U$.

Question 5

> **5(a)** *State the definition of a diagonalizable complex matrix.*

A diagonalizable complex matrix is an $n \times n$ matrix A with complex entries such that there exists an invertible $n \times n$ complex matrix P with $P^{-1}AP$ a diagonal matrix.

> **5(b)** *Let A_1 and A_2 be complex matrices in $M_n(\mathbf{C})$ such that $A_1A_2 = A_2A_1$. Show that there exists a common (column) eigenvector $X \neq 0$ such that $A_iX = \lambda_iX$ ($i = 1, 2$) for some eigenvalues λ_i of A_i.*

Suppose λ_1 is an eigenvalue of A_1, and let $v \in V_{\lambda_1}(A_1)$, so that $A_1v = \lambda_1v$. Then

$$A_1A_2v = A_2A_1v = A_2\lambda_1v = \lambda_1A_2v,$$

so that $A_2v \in V_{\lambda_1}(A_1)$. That is, the eigenspace $V_{\lambda_1}(A_1)$ is invariant under T_{A_2} (the linear transformation associated with the matrix A_2). The result then follows by taking an eigenvector X of the restricted linear transformation $T_{A_2}|V_{\lambda_1}(A_1)$.

> **5(c)** *Consider the complex diagonal matrix* $D = \mathrm{diag}[1, 2]$. *Find all complex matrices* $G \in M_2(\mathbf{C})$ *such that* $GD = DG$.

Since

$$GD = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} g_{11} & 2g_{12} \\ g_{21} & 2g_{21} \end{pmatrix}$$

and

$$DG = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ 2g_{21} & 2g_{22} \end{pmatrix},$$

so $g_{12} = g_{21} = 0$, and we conclude that $GD = DG$ iff G is a complex diagonal matrix.

> **5(d)** *Suppose that* $Q \in M_n(\mathbf{C})$ *is a complex matrix with* n ***distinct*** *eigenvalues* λ_1, …, λ_n. *Let* $A \in M_n(\mathbf{C})$ *be a matrix satisfying* $AQ = QA$. *Show that both* Q *and* A *are diagonalizable.*

Since Q has n distinct eigenvalues, it is diagonalizable, so our goal is to show that A is diagonalizable as well. By arguing as in part (b) for each eigenvalue λ_j, there exist vectors v_1, …, v_n such that $Qv_j = \lambda_j v_j$ and $Av_j = \mu_j v_j$. These vectors are linearly independent since they correspond to distinct eigenvalues of Q. It follows that the matrix $P = (v_1, \ldots, v_n)$ diagonalizes both Q and A: We have

$$P^{-1}QP = \mathrm{diag}[\lambda_1, \ldots, \lambda_n] \quad \text{and} \quad P^{-1}AP = \mathrm{diag}[\mu_1, \ldots, \mu_n].$$

> **5(e)** *Suppose* P *is an invertible matrix such that* $P^{-1}QP$ *is diagonal for the matrix* Q *from part (d). Can we say that* $P^{-1}AP$ *is also diagonal for the matrix* A *from part (d)? Justify your answer.*

Yes, we can. Following part (d), since the Q-eigenvectors v_1, …, v_n are independent, their eigenspaces $V_{\lambda_j}(Q)$ are in direct sum, so that $\mathbf{C}_c^n = \bigoplus_{j=1}^n V_{\lambda_j}(Q)$. But these are n eigenspaces summing to an n-dimensional space, so each must have dimension one. Thus $V_{\lambda_j}(Q) = \mathbf{C}v_j := \{av_j \mid a \in \mathbf{C}\}$ for all j.

Suppose $P^{-1}QP = D = \mathrm{diag}[d_1, \ldots, d_n]$. Then $QP = PD$, and if we write $P = (p_1, \ldots, p_n)$ where p_j is the jth column vector in the matrix P, it follows that $Qp_j = d_j p_j$. Since P is invertible, the vectors p_1, …, p_n are independent, and so the one-dimensionality of each Q-eigenspace implies that the set $\{d_1, \ldots, d_n\}$ of eigenvalues is equal to the set $\{\lambda_1, \ldots, \lambda_n\}$ of Q-eigenvalues from part (d). We may relabel the eigenvalues from part (d) so as to assume that $\lambda_j = d_j$ for all j; thus $p_j \in V_{\lambda_j}(Q) = \mathbf{C}v_j$, and so $v_j = c_j p_j$ for some nonzero c_j. Since $Av_j = \mu_j v_j$, it follows that $Ap_j = c_j \mu_j p_j$, so that $P^{-1}AP = \mathrm{diag}[c_1\mu_1, \ldots, c_n\mu_n]$, which completes the proof.