MA2108S - Mathematical Analysis I (S) Suggested Solutions

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Question 1

For each of the following statements, first answer whether it is true or false. If it is true, prove it. If it is false, find a counter example.

(a) Let (a_n) be a sequence in $\mathbb{R}$. Suppose that $\sum a_n$ is convergent, then $\sum a_n^2$ is also convergent.

(b) Let (a_n) and (b_n) be two convergent sequences in $\mathbb{R}$. Suppose that $\lim a_n \geq \lim b_n$, then there is an integer N such that $a_n \geq b_n$ for all $n > N$.

(c) Let X, Y and Z be three metric spaces. Suppose that $f : X \to Y$ is continuous and that $g : Y \to Z$ is uniformly continuous. Then $g \circ f$ is uniformly continuous.

(d) For a function $f : \mathbb{R} \to \mathbb{R}$, define the graph of f as a subset of $\mathbb{R}^2$ by

$$\text{Graph}(f) := \{(x, y) \in \mathbb{R}^2 : y = f(x) \text{ for some } x \in \mathbb{R}\}.$$

If f is continuous, then $\text{Graph}(f)$ is closed in $\mathbb{R}^2$.

Solution:

(a) False. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$ for all n. By the alternating series test, $\sum a_n$ converges, but $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

(b) False. Let $a_n = \frac{1}{2n}$ and $b_n = \frac{1}{n}$ for all n, then $\lim a_n = \lim b_n = 0$ but $a_n < b_n$ for all n.

(c) False. Let $X = Y = Z = \mathbb{R}$ and $f : x \mapsto x^2$ and $g : x \mapsto x$. Note that f is continuous and g is uniformly continuous, but $g \circ f : x \mapsto x^2$ is not uniformly continuous.

(d) True. Let $\{(x_n, f(x_n)\} \to (x, y) \in \mathbb{R}^2$ be a convergent sequence in $\text{Graph}(f)$. Then $x_n \to x$ and since f is continuous, $f(x_n) \to f(x)$. Hence, $y = f(x)$, so $(x, y) = (x, f(x)) \in \text{Graph}(f)$. Therefore, $\text{Graph}(f)$ is closed in $\mathbb{R}^2$.

Question 2

Determine whether the following series are convergent. Justify your answer.

a) $\sum \frac{1}{n^2}\left(2+\frac{3}{n}\right)$; b) $\sum \frac{2^n}{n^n}$; c) $\sum \cos\left(\frac{1}{n}\right)$.

Solution:

(a) Convergent. Since $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, $\sum \frac{1}{n^2}\left(2+\frac{3}{n}\right) = 2\sum \frac{1}{n^2} + 3\sum \frac{1}{n^3}$ converges.

(b) Convergent. For all $n \geq 4$, $4^n \leq n^n \Rightarrow \frac{2^n}{n^n} \leq \frac{1}{2^n}$, so $\sum \frac{2^n}{n^n} = \sum_{n\leq 3} \frac{2^n}{n^n} + \sum_{n\geq 4} \frac{2^n}{n^n} \leq \sum_{n\leq 3} \frac{2^n}{n^n} + \sum_{n\geq 4} \frac{1}{2^n}$ which converges to $\sum_{n\leq 3} \frac{2^n}{n^n} + \frac{1}{8}$, hence $\sum \frac{2^n}{n^n}$ converges.

(c) Divergent. As $n \to \infty$, $\frac{1}{n} \to 0$, and $\cos\left(\frac{1}{n}\right) \to 1 \neq 0$. Hence, $\sum \cos\left(\frac{1}{n}\right)$ diverges.

Question 3

Let (x_n) be a sequence in a metric space (X, d), we say that this sequence has *Property P* if $\sum d(x_n, x_{n+1})$ is convergent.

(a) If X is complete and (x_n) has Property P, then (x_n) is convergent.

(b) If every sequence with Property P converges, show that X is complete.

Solution:

(a) If X is complete and (x_n) has Property P, then since $\sum d(x_n, x_{n+1})$ is convergent, for any $\varepsilon > 0$, there exists an integer N such that

$$\sum_{i=N}^{\infty} d(x_i, x_{i+1}) < \varepsilon.$$

So for all $m > n \geq N$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < d(x_i, x_{i+1}) < \varepsilon.$$

Hence, (x_n) is a Cauchy sequence, and since X is complete, it must be convergent as well. $\square$

(b) Let (y_n) be any Cauchy sequence in X. Then we pick an strictly increasing sequence of indices (n_k) such that $d(y_{n_k}, y_{n_{k+1}}) < \frac{1}{2^k}$ for all integers $k \geq 0$. If we create a sequence (x_n) such that $x_k = y_{n_k}$ for all k, then $\sum d(x_n, x_{n+1}) = \sum d(y_{n_k}, y_{n_{k+1}}) < \sum \frac{1}{2^k} = 2$, so (x_n) has Property P and must converge. Since (x_n) is a convergent subsequence of (y_n), we have that (y_n) must be convergent, implying that X is complete. $\square$

Question 4

Let X be a subset of $\mathbb{R}^d$. Suppose that for every continuous $f : X \to \mathbb{R}$, we can find $\overline{x} \in X$ such that $f(\overline{x}) \geq f(x)$ for all $x \in X$. Show that X is compact.

Solution: If we let $f : x \to \|x\|$, we see that f is continuous, hence, f is bounded from above. If X is unbounded, then f is unbounded, but we know f cannot go lower than 0, so it must not be bounded from above, contradiction. Hence, X is bounded.

If X is not closed, let p be a limit point of X that is not in X. We let $g : x \to \frac{1}{\|x-p\|}$, which is continuous, hence, it must be bounded from above. Thus, $\{\|x - p\| \mid x \in X\}$ must have a nonzero infimum, contradicting that p is a limit point of X. Thus, X is closed.

Therefore, by the Heine-Borel Theorem, X is compact. $\square$