MA2001 AY23/24 Sem 2 Final

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Question 1

(a) Use Gaussian elimination to reduce the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ -1 & 0 & 1 & 3 & 2 & 1 \end{pmatrix}$$

to a row-echelon form. Indicate clearly the row operations used in each step.

(b) Let $\mathbf{u}_1 = (1,1,0,-1)$, $\mathbf{u}_2 = (0,1,2,0)$, $\mathbf{u}_3 = (1,2,2,1)$ and $\mathbf{v}_1 = (1,2,2,3)$, $\mathbf{v}_2 = (2,3,0,2)$, $\mathbf{v}_3 = (3,3,0,1)$. For each $i = 1,2,3$, use **(a)** to determine whether $\mathbf{v}_i \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. You do not need to explain your answers.

Solution.

(a) Working omitted. A row-echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In fact, this matrix is in reduced row-echelon form.

(b) Note that the matrix in **(a)** is

$$\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}.$$

By considering the red entries, we see that

$$\mathbf{v}_1 = -\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3.$$

By considering the blue entries, we see that $\mathbf{v}_2$ is not contained in the span of the $\mathbf{u}_i$'s. By considering the purple entries,

$$\mathbf{v}_3 = \mathbf{u}_1 - 2\mathbf{u}_2 + 2\mathbf{u}_3.$$

Hence, $\mathbf{v}_1, \mathbf{v}_3 \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. $\square$

Question 2

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \end{pmatrix}.$$

(a) Solve the linear system $\mathbf{Ax} = \mathbf{b}$.

(b) Is the linear system $\mathbf{Ax} = \mathbf{c}$ consistent? Justify your answer.

(c) Find a least squares solution to $\mathbf{Ax} = \mathbf{c}$.

(d) Use the result in **(c)** to find the projection of $\mathbf{c}$ onto the column space of $\mathbf{A}$.

Solution.

(a) Let $\mathbf{x} = (x, y, z)$. Then, by considering the RREF of $(\mathbf{A} \mid \mathbf{b})$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

So, $\mathbf{x} = (1, 1, -1)$.

(b) Let $\mathbf{c} = (x, y, z)$. Consider the RREF of $(\mathbf{A} \mid \mathbf{c})$ so

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The first row implies $x = 0$, the second row implies $y = 0$, and the third row implies $z = 0$. However, the fourth row implies $0 = 1$, which is a contradiction, so the system is inconsistent.

(c) Consider $\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{c}$, so

$$\mathbf{x} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

which is a least squares solution.

(d) Consider $\mathbf{Ax} = \mathbf{c}$. Let $\mathbf{p}$ be the projection of $\mathbf{c}$ onto the column space of $\mathbf{A}$. Recall that $\mathbf{u}$ is a least squares solution to $\mathbf{Ax} = \mathbf{c}$ if and only if $\mathbf{Au} = \mathbf{p}$. Hence, the projection is

$$\mathbf{p} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

□

Question 3

Let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix}.$$

It is known that the eigenvalues of $\mathbf{B}$ are 1 and -1.

(a) Find a basis for the eigenspace of $\mathbf{B}$ associated with the eigenvalue 1.

(b) Find a basis for the eigenspace of $\mathbf{B}$ associated with the eigenvalue -1.

(c) Write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D}$.

Solution.

(a) By considering $\mathbf{Bx} = \mathbf{x}$, where $\mathbf{x} = (x,y,z)$, we have $(\mathbf{B}-\mathbf{I})\,\mathbf{x} = \mathbf{0}$, so

$$\begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $x = 2y + z$. A basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(b) By considering $\mathbf{Bx} = -\mathbf{x}$, where $\mathbf{x} = (x,y,z)$, we have $(\mathbf{B}+\mathbf{I})\,\mathbf{x} = \mathbf{0}$, so

$$\begin{pmatrix} 3 & -2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving the system yields $x = y = z$. A basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(c) We have $\mathbf{B} = \mathbf{PDP}^{-}1$ so

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\square$

Question 4

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for $\mathbb{R}^3$. Define $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that

$$\mathbf{v}_1 = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3, \quad \mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2 + a\mathbf{u}_3, \quad \mathbf{v}_3 = \mathbf{u}_1 + b\mathbf{u}_2,$$

where a and b are constants.

(a) If T is a basis for $\mathbb{R}^3$, write down the transition matrix from T to S.

(b) Determine the values of a and b so that T is a basis for $\mathbb{R}^3$.

(c) Suppose S is orthonormal. Determine the values of a and b so that T is orthogonal.

Solution.

(a) Suppose the coordinate vector of $\mathbf{x}$ with respect to T is

$$[\mathbf{x}]_T = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \text{so} \quad \mathbf{x} = p\mathbf{v}_1 + q\mathbf{v}_2 + r\mathbf{v}_3.$$

Hence,

$$\mathbf{x} = (p+q+r)\,\mathbf{u}_1 + (-p-q+br)\,\mathbf{u}_2 + (p+aq)\,\mathbf{u}_3.$$

The transition matrix from T to S is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & b \\ 1 & a & 0 \end{pmatrix}.$$

(b) By the invertible matrix theorem,

$$\det \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & b \\ 1 & a & 0 \end{pmatrix} \neq 0 \quad \text{so} \quad (1-a)(1+b) \neq 0$$

Hence, $a \neq 1$ and $b \neq -1$.

(c) We have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \quad &\text{so} \quad \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + a\,\|\mathbf{u}_3\|^3 = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 = 0 \quad &\text{so} \quad \|\mathbf{u}_1\|^2 - b\,\|\mathbf{u}_2\|^2 = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 = 0 \quad &\text{so} \quad \|\mathbf{u}_1\|^2 - b\,\|\mathbf{u}_2\|^2 = 0 \end{aligned}$$

Since S is orthonormal, then $\|\mathbf{u}_i\| = 1$ so $a = -2$ and $b = 1$. $\square$

Question 5

Let $\mathbf{w}$ be a vector in $\mathbb{R}^n$. Define

$$V = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{w} = 0\}.$$

(a) Show that V is a subspace of $\mathbb{R}^n$.

(b) If $n = 3$ and $\mathbf{w} = (1, 1, -1)$, find an orthonormal basis for V.

Solution.

(a) Let $\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{w} = \mathbf{0} \cdot \mathbf{w} = 0$, so V is non-empty.

Next, let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Then, $\mathbf{u}_1 \cdot \mathbf{w} = 0$ and $\mathbf{u}_2 \cdot \mathbf{w} = 0$. So,

$$(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{w} = \mathbf{u}_1 \cdot \mathbf{w} + \mathbf{u}_2 \cdot \mathbf{w} = 0 + 0 = 0.$$

So, V is closed under addition. Lastly, let $k \in \mathbb{R}$. Then,

$$(k\mathbf{u}) \cdot \mathbf{w} = k(\mathbf{u} \cdot \mathbf{w}) = k \cdot 0 = 0.$$

So, V is closed under scalar multiplication. We conclude that V is a subspace of $\mathbb{R}^n$.

(b) Suppose $n = 3$ and $\mathbf{w} = (1, 1, -1)$. Then, let $\mathbf{u} = (u_1, u_2, u_3)$. Since $\mathbf{u} \cdot \mathbf{w} = 0$, then $u_1 + u_2 - u_3 = 0$. We first construct a basis for V, say

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

By the Gram-Schmidt process,

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is an orthonormal basis for V. □

Question 6

For each of the following statements, determine if it is true or false. Justify your answers.

(a) A non-homogeneous system of linear equations can have a trivial solution.

(b) For a square matrix $\mathbf{A}$, if λ is an eigenvalue of $\mathbf{A}$, then $a\lambda^2 + b\lambda + c$ is an eigenvalue of $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$, where a, b, c are real constants.

(c) For any positive integers n and m, there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\ker(T) = \{\mathbf{0}\}$.

(d) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}$, there exists a vector $\mathbf{u} \in \mathbb{R}^n$ such that every vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed as $\mathbf{v} = \mathbf{z} + a\mathbf{u}$ for some $\mathbf{z} \in \ker(T)$ and $a \in \mathbb{R}$.

Solution.

(a) False. Consider a non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$. Then, $\mathbf{x} = \mathbf{0}$ does not satisfy the mentioned equation.

(b) True. Suppose λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. Then, $\mathbf{Av} = \lambda\mathbf{v}$. As such,

$$\begin{aligned}\left(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}\right)\mathbf{v} &= a\mathbf{A}^2\mathbf{v} + b\mathbf{Av} + c\mathbf{v} \\ &= a\mathbf{A}\left(\mathbf{Av}\right) + b\lambda\mathbf{v} + c\mathbf{v} \\ &= a\mathbf{A}\left(\lambda\mathbf{v}\right) + b\lambda\mathbf{v} + c\mathbf{v} \\ &= a\lambda\left(\mathbf{Av}\right) + b\lambda\mathbf{v} + c\mathbf{v} \\ &= a\lambda^2\mathbf{v} + b\lambda\mathbf{v} + c\mathbf{v}\end{aligned}$$

which is equal to $\left(a\lambda^2 + b\lambda + c\right)\mathbf{v}$.

(c) False. Recall that

$$T : V \to W \text{ is injective} \quad \text{if and only if} \quad \dim(V) \le \dim(W) \quad \text{if and only if} \quad \ker(T) = \{\mathbf{0}\}.$$

To come up with a contradiction, we need $n > m$, so we can set $m = 1$ and $n = 2$. The matrix representation $\mathbf{A}$ has 1 row and 2 columns. Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{so} \quad \mathbf{Ax} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}.$$

Then, $\ker T = \left\{(x,y) \in \mathbb{R}^2 : x+y = 0\right\}$ so the kernel is $\neq \{\mathbf{0}\}$.

(d) True. We proceed with casework. If $T = 0$, i.e. T is the zero transformation, then $\ker(T) = \mathbb{R}^n$. To see why, choose $\mathbf{u} \neq \mathbf{0}$. Then for any $\mathbf{v}$, we can write

$$\mathbf{v} = \mathbf{v} + 0 \cdot \mathbf{u} \quad \text{where} \quad \mathbf{v} \in \ker T \text{ and } 0 \in \mathbb{R}.$$

On the other hand, if $T \neq 0$, then $\operatorname{rank} T = 1$ and $\operatorname{nullity} T = n-1$ by the rank-nullity theorem. Choose $\mathbf{u}$ so that $T(\mathbf{u}) = 1$. So,

$$\text{for any } \mathbf{v} \in \mathbb{R}^n \quad \text{set} \quad a = T(\mathbf{v}) \text{ and } \mathbf{z} = \mathbf{v} - a\mathbf{u}.$$

Thus,

$$T(\mathbf{z}) = T(\mathbf{v}) - aT(\mathbf{u}) = a - a \cdot 1 = 0,$$

so $\mathbf{z} \in \ker(T)$, and $\mathbf{v} = \mathbf{z} + a\mathbf{u}$. Hence every $\mathbf{v}$ decomposes as required. □

Question 7

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of order n such that $\mathbf{AB} = \mathbf{BA}$.

(a) Suppose $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- **(i)** For each $i = 1, 2, \ldots, n$, prove that $\dim(E_{\lambda_i}) = 1$.
- **(ii)** For each $i = 1, 2, \ldots, n$, prove that if $\mathbf{Bu}_i \neq 0$, then $\mathbf{Bu}_i$ is an eigenvector of $\mathbf{A}$ associated with λ_i.

(iii) Show that $\mathbf{B}$ is diagonalizable and there exists an invertible matrix $\mathbf{P}$ such that both $\mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{BP}$ are diagonal matrices.

(b) Let $n = 2$. Give an example of matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$ and $\mathbf{A}$ is diagonalizable while $\mathbf{B}$ is not.

Solution.

(a) **(i)** Since $\mathbf{A}$ is of order n and $\mathbf{A}$ has n distinct eigenvalues, then each eigenspace E_{λ_i} is at most one-dimensional, otherwise the order of $\mathbf{A}$ would be $> n$, which is a contradiction.

Next, the dimension of each E_{λ_i} is ≥ 1 since it contains an eigenvector $\mathbf{u}_i$ (by definition, $\mathbf{u}_i \neq \mathbf{0}$) such that $\mathbf{Au}_i = \lambda_i \mathbf{u}_i$.

Since $\dim\left(E_{\lambda_i}\right) \leq 1$ and $\dim\left(E_{\lambda_i}\right) \geq 1$, then the result follows.

(ii) From **(a)**, $\mathbf{Au}_i = \lambda_i \mathbf{u}_i$, so

$$\begin{aligned} \mathbf{BAu}_i &= \lambda_i \mathbf{Bu}_i \\ \mathbf{ABu}_i &= \lambda_i \mathbf{Bu}_i \quad \text{from the preamble} \end{aligned}$$

where we note that $\mathbf{Bu}_i \neq \mathbf{0}$. Since $\mathbf{A}\left(\mathbf{Bu}_i\right) = \lambda_i \left(\mathbf{Bu}_i\right)$, then the result follows.

(iii) From **(ii)**, we have $\mathbf{ABu}_i = \lambda_i \mathbf{Bu}_i$. By **(i)**, each eigenspace is one-dimensional, i.e. $E_{\lambda_i} = \operatorname{span}\{\mathbf{u}_i\}$. Geometrically, this is a line in $\mathbb{R}^n$, so $\operatorname{span}\{\mathbf{u}_i\} = \mu_i \mathbf{u}_i$ for some $\mu_i \in \mathbb{R}$. Thus, $\mathbf{Bu}_i = \mu_i \mathbf{u}_i$. Since $1 \leq i \leq n$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$. As such $\mathbf{B}$ has n distinct eigenvalues μ_i, which implies that it is diagonalizable.

Since $\mathbf{A}$ and $\mathbf{B}$ are diagonalizable, then there exists an invertible matrix $\mathbf{P}$ (this $\mathbf{P}$ is common to both $\mathbf{A}$ and $\mathbf{B}$ due to the result established in the first part of **(iii)**) and diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ such that

$$\mathbf{A} = \mathbf{PD}_1\mathbf{P}^{-1} \quad \text{and} \quad \mathbf{B} = \mathbf{PD}_2\mathbf{P}^{-1}.$$

Hence, $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}_1$ and $\mathbf{P}^{-1}\mathbf{BP} = \mathbf{D}_2$ which are diagonal matrices.

(b) To construct such an example, we motivate using the fact that diagonal matrices commute and are diagonalizable. So, choose $\mathbf{A} = \mathbf{D}$ for some diagonal matrix $\mathbf{D}$, i.e. $\mathbf{D} = \mathbf{I}$ and we can choose $\mathbf{B}$ to be non-diagonalizable. Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$\square$