MA2104 AY24/25 Sem 2 Final

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Question 1: Let

$$f(x,y,z) = z\sqrt{x^2+y^2}$$

Let E be the solid which is bounded by two cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and two planes $z=0$ and $z=6$. Evaluate

$$\iiint_E f(x,y,z)\ dV$$

Solution: By considering cylindrical coordinates, we let

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

such that we have

$$\begin{aligned}\iiint_E f(x,y,z) &= \int_0^{2\pi}\int_1^2\int_0^6 zr^2\ dzdrd\theta \\ &= 84\pi\end{aligned}$$

□

Question 2: Consider the surface

$$S := \{(x,y,z)\in\mathbb{R}^3 \mid z^2 = x^2+y^2, 1\le z\le 3\}$$

which is part of the cone $z^2=x^2+y^2$ between $z=1$ and $z=3$. Let

$$f(x,y,z) = e^{-x^2-y^2}$$

Evaluate

$$\iint_S f(x,y,z)\ dS$$

Solution: By considering cylindrical coordinates, we let

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r$$

such that we have

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, r)$$

By Jacobian transformation, we have

$$dS = |\mathbf{r}_r(r,\theta)\times\mathbf{r}_\theta(r,\theta)|\,drd\theta = \sqrt{2}r\,drd\theta$$

Hence

$$\begin{aligned}\iint_S f(x,y,z)\,dS &= \int_0^{2\pi}\int_1^3 e^{-r^2}\cdot\sqrt{2}r\,drd\theta \\ &= \sqrt{2}\pi(e^{-1}-e^{-9})\end{aligned}$$

□

Question 3:

a) Let C_1 be the positively oriented closed curve consisting of the upper half of the unit circle from $(-1,0)$ to $(1,0)$, and the line segments from $(1,0)$ to $(2,0)$, from $(2,0)$ to $(2,2)$, from $(2,2)$ to $(-2,2)$, from $(-2,2)$ to $(-2,0)$ and from $(-2,0)$ to $(-1,0)$. Evaluate

$$\int_{C_1} y(\cos x - 1)\,dx + \sin x\,dy$$

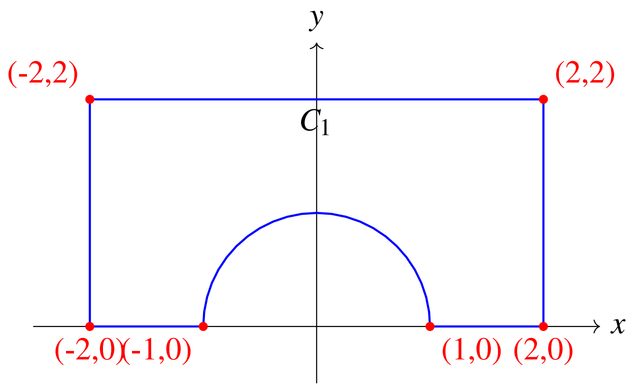
b) Recall that for a curve C with position vector $\mathbf{r}$, denote $d\mathbf{r} = \mathbf{T}ds$ where $\mathbf{T}$ is the unit tangent vector (with the given orientation) at a point of C. Let $\mathbf{F}(x,y) = \langle\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$. Let C_2 be the unit circle which is positively oriented (i.e. anticlockwise), and let C_3 be the positively oriented ellipse $\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{4}+\frac{y^2}{9}=1\}$. Evaluate

$$\int_{C_2}\mathbf{F}.d\mathbf{r} \quad \text{and} \quad \int_{C_2}\mathbf{F}.d\mathbf{r}$$

Solution:

(a) By Green's Theorem, we have

$$\int_{C_1} y(\cos x-1)\,dx+\sin x\,dy = \iint \cos x - (\cos x - 1)\,dA = \iint dA = \text{Area of } C_1$$



As such we have

$$\int_{C_1} y(\cos x - 1)\, dx + \sin x\, dy = (2)(4) - \frac{1}{2}\pi(1)^2 = 8 - \frac{\pi}{2}$$

$\square$

(b) Note that $\mathbf{F}$ is not defined at it origin which lies inside the circle. Therefore Green's Theorem cannot be applied here. Let

$$\mathbf{r}(t) = \langle \cos t, \sin t\rangle$$

then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t\rangle \qquad \text{and} \qquad \mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$$

As such

$$\int_{C_2} \mathbf{F}.d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} dt = 2\pi$$

(b) Similarly as (a). $\mathbf{F}$ is not defined at it origin which lies inside the circle. Therefore Green's Theorem cannot be applied here. Some may opt to parametrize the vector and will end up having

$$\int_0^{2\pi} \frac{6}{4\cos^2 t + 9\sin^2 t}\, dt$$

which is messy. Instead, we can observe that

$$|\mathbf{F}| = \frac{1}{r}$$

Hence it form a closed loops around the origin, the circulation around any simple closed curve enclosing the origin exactly once and oriented clockwise is 2π. Hence

$$\int_{C_3} \mathbf{F}.d\mathbf{r} = 2\pi$$

$\square$

Question 4: Recall that for a surface S, denote $d\mathbf{S} = \mathbf{n}\, dS$ where $\mathbf{n}$ is the unit normal vector (with the given orientation) at a point of S; for a curve C with position vector $\mathbf{r}$, denote $d\mathbf{r} = \mathbf{T}\, ds$ where $\mathbf{T}$ is the unit tangent (with the given orientation) at a point of C. Let $\mathbf{F}(x,y,z) = \langle z^2, -2x, y^5\rangle$

a) Calculate $\operatorname{curl}\mathbf{F}$

b) Let $S_1 := \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$ be the unit disk in the $xy-$plane of $\mathbb{R}^3$, which is upward pointing. Calculate

$$\iint_{S_1} \operatorname{curl}\mathbf{F}.d\mathbf{S}$$

c) Let $C_1 := \{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1\}$ be the positively oriented (i.e. anticlockwise) unit circle in the $xy-$plane of $\mathbb{R}^3$. Calculate

$$\int_{C_1} \mathbf{F}.d\mathbf{r}$$

d) Let $S_2 := \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0, x^2+y^2+z^2=1\}$ be the upper half of the unit sphere which is positively oriented (i.e. outward pointing). Calculate

$$\iint_{S_2} \operatorname{curl} \mathbf{F}.d\mathbf{S}$$

e) Let $\mathbf{G}(x,y,z) = \langle U,V,W\rangle$ where U,V,W have continuous partial derivatives in an open set $D \subset \mathbb{R}^3$. If $\mathbf{G}(x,y,z) = \nabla(H(x,y,z))$ for a function $H(x,y,z)$ defined in D, is it true that $\operatorname{curl}\mathbf{G} = \langle 0,0,0\rangle$? (Justify your answer). Conversely, if $\operatorname{curl}\mathbf{G} = \langle 0,0,0\rangle$ at every point in D, is it true that $\mathbf{G} = \nabla H(x,y,z)$ for some function H in the open set D?(No justification is needed)

Solution:

(a) It is straight forward that

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle 5y^4, -2z, -2\rangle$$

$\square$

(b) Since S_1 lies in $xy-$plane with upward normal, then

$$\mathbf{n} = \langle 0,0,1\rangle$$

as such

$$\iint_{S_1} \operatorname{curl}\mathbf{F}.d\mathbf{S} = -2\iint dA = -2\pi$$

(c) By Stokes' Theorem,

$$\int_{C_1} \mathbf{F}.d\mathbf{r} = \iint_{S_1} \operatorname{curl}\mathbf{F}.d\mathbf{S} = -2\pi$$

$\square$

(d) Let C_2 be the boundary of S_2. By Stokes' Theorem,

$$\iint_{S_2} \operatorname{curl}\mathbf{F}.d\mathbf{S} = \int_{C_2} \mathbf{F}.d\mathbf{r}$$

At $z=0$, $x^2+y^2=1$ which is the same circle as C_1. Therefore

$$\iint_{S_2} \operatorname{curl}\mathbf{F}.d\mathbf{S} = -2\pi$$

$\square$

(e) Suppose $\mathbf{G}(x,y,z) = \nabla(H(x,y,z))$, then $\mathbf{G}$ is conservative. Hence it is true that $\operatorname{curl}\mathbf{G} = \langle 0,0,0\rangle$. Conversely it is false as the domain needs to be simply connected but the context does not mention that the domain is simply connected Hence it is not necessarily true. $\square$

Question 5: Let $\mathbf{F}(x,y,z) = \frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

a) calculate $\operatorname{div}\mathbf{F}$

b) Let $S_a := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2\}$ be the sphere of radius $a > 0$ which is positively oriented (i.e outward pointing). Calculate $\iint_{S_a} \mathbf{F}.d\mathbf{S}$ [N.B you may use, without proof, the fact that the sphere S_a has area $4\pi a^2$]

c) Let S be the positively oriented surface which is bounded above by the paraboloid $z = 1-x^2-y^2$ and is bounded below by the paraboloid $z = -1+x^2+y^2$. Calculate $\iint_S \mathbf{F}.d\mathbf{S}$

Solution:

(a) We have

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{x^2+y^2+z^2-3x^2}{\sqrt{(x^2+y^2+z^2)^5}} + \frac{x^2+y^2+z^2-3y^2}{\sqrt{(x^2+y^2+z^2)^5}} + \frac{x^2+y^2+z^2-3z^2}{\sqrt{(x^2+y^2+z^2)^5}} \\ &= 0\end{aligned}$$

(b) Note that since $\mathbf{F}(x,y,z)$ is not defined at $(0,0,0)$, divergence theorem cannot be directly applied. Consider

$$\mathbf{n} = \frac{1}{a}\langle x,y,z\rangle$$

As such

$$\iint_{S_a} \mathbf{F}.d\mathbf{S} = \iint_{S_a} \frac{\mathbf{r}}{a^3}\cdot\frac{\mathbf{r}}{a}\,dS = \iint_{S_a} \frac{|\mathbf{r}|^2}{a^4}\,dS = \frac{1}{a^2}\iint_{S_a} dS = 4\pi$$

$\square$

(c) From (b) we know that $\iint_{S_a} \mathbf{F}.dS$ is independent of the radies a. Therefore

$$\iint_S \mathbf{F}.d\mathbf{S} = 4\pi$$

$\square$