Worked solutions for MA2101 15/16 S1 exam

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Note: This document was written in a great rush, so there may be many mistakes. Compared with the solutions for 23/24 S1, I am even less certain of my answers below, since they were prepared without the help of any other written solutions. Read with care!

Professor's note: If you use results in lecture notes or question sheets of tutorial assignments, state them clearly.

Question 1

> *Let $A \in M_2(\mathbf{R})$ be the following symmetric real matrix*
>
> $$A = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$
>
> *Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.*

We compute the characteristic polynomial

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda & -2 \\ -2 & \lambda - 3 \end{pmatrix} \\ = \lambda(\lambda - 3) - 4 = (\lambda + 1)(\lambda - 4), \end{aligned}$$

which tells us that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 4$. Let us find the eigenvectors associated with $\lambda_1 = -1$. We have

$$I - A = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix},$$

which yields the eigenvector $p_1 = \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$. As for $\lambda_2 = 4$, we have

$$4I - A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

which yields $p_2 = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. It follows that $\{p_1, p_2\}$ is a basis of eigenvectors for A, and we may normalize to get vectors $\frac{1}{\sqrt{5}}p_1$ and $\frac{1}{\sqrt{5}}p_2$ that form an orthonormal basis of eigenvectors for A. It follows that

$$P = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Question 2

> *Let* $A = (a_{ij}) \in M_2(\mathbf{R})$ *be a real matrix and let*
>
> $$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
>
> *such that*
>
> $$P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$
>
> *Let* $y_i = y_i(x)$ *(*$i = 1, 2$*) be differentiable functions in* x*. Solve the following system of differential equations:*
>
> $$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
>
> *Note: For the differential equation* $z'(x) + p(x)z(x) = q(x)$ *you may assume, without proof, that its general solution is given by* $z(x) = \frac{1}{\mu}(C + \int \mu q)$ *with* $\mu = e^{\int p}$*.*

We solve the modified system $Z' = DZ$ where $D = P^{-1}AP$, since this will yield a solution $Y = PZ$ to the original system $Y' = AY$. The modified system is given by

$$\begin{cases} z_1'(x) = 2z_1(x) + z_2(x) \\ z_2'(x) = 2z_2(x), \end{cases}$$

which immediately gives us $z_2(x) = C_2 e^{2x}$. The first equation then becomes

$$z_1'(x) = 2z_1(x) + C_2 e^{2x} \quad \text{or} \quad z_1'(x) + p(x)z_1(x) = q(x)$$

with $p(x) = -2$ and $q(x) = C_2 e^{2x}$, so that, setting $\mu(x) = \exp(\int -2\,dx) = e^{-2x}$, we get the general solution

$$z_1(x) = \frac{1}{e^{-2x}} \left(\int e^{-2x} \cdot C_2 e^{2x}\,dx + C_1 \right) = e^{2x}(C_2 x + C_1).$$

The last thing for us to do is then to substitute $Y = PZ$, which gives

$$\begin{cases} y_1(x) = z_1(x) + z_2(x) = e^{2x}(C_2 x + C_1 + C_2) \\ y_2(x) = z_1(x) = e^{2x}(C_2 x + C_1). \end{cases}$$

Question 3

> *Let* U *and* V *be vector spaces over a scalar field* F*, let* $T\colon U \to V$ *be a surjective linear transformation, and let* W *be a vector subspace of* V*.*
>
> *Warning: In this question,* $\dim U$ *and* $\dim V$ *may be infinite.*
>
> ***3(i)*** *Show that the preimage*
>
> $$T^{-1}(W) := \{u \in U \mid T(u) \in W\}$$
>
> *of* W *is a vector subspace of* U*.*

It suffices to prove that $c \in F$ and $u, u' \in T^{-1}(W)$ implies $cu + u' \in T^{-1}(W)$. By hypothesis this means that $Tu, Tu' \in W$, and so $T(cu + u') = cTu + Tu' \in W$, because W is itself a subspace. But this means that $cu + u' \in T^{-1}(W)$ as needed.

> ***3(ii)*** *Show that*
>
> $$\dim T^{-1}(W) + \dim V = \dim W + \dim U.$$

Note that $\dim T^{-1}(W) \leq \dim(U)$ and $\dim(W) \leq \dim(V)$ owing to the subspace relations.

We show that the quotient map $\phi\colon U/T^{-1}(W) \to V/W$ sending $u + T^{-1}(W)$ to $Tu + W$ is an isomorphism. If $\dim U = \infty$, then both sides of the desired identity are infinite and we are done; otherwise $\dim U < \infty$ implies $\dim V < \infty$ and we may apply the usual identities of the finite-dimensional setting such as $\dim(V/W) = \dim(V) - \dim(W)$ to conclude.

First we remark that the quotient map ϕ arises as the usual map $T\colon U \to V$ followed by the projection $V \to V/W$. We are permitted to descend to $U/T^{-1}(W)$ since $T(T^{-1}(W)) = W$ is the zero of V/W. Now the surjectivity of T implies that of the map $U \to V/W$ and consequently that of ϕ, so it remains for us to prove the injectivity of ϕ. To this end, suppose $\phi(u + T^{-1}(W)) = W$. Then $Tu + W = W$, so that $Tu \in W$. But this means that $u \in T^{-1}(W)$, so that $u + T^{-1}(W) = T^{-1}(W)$, the zero element of $U/T^{-1}(W)$, as needed. This completes the proof.

Question 4

> *Let $Q \in M_3(\mathbf{R})$ be an orthogonal real matrix of order 3. Let*
>
> $$p_Q(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$
>
> *be the characteristic polynomial for Q, where $\lambda_i \in \mathbf{C}$.*
>
> ***4(i)*** *Show that $\lambda_i^2 = 1$ for at least one of the λ_i.*

Since $p_Q(x)$ is a cubic polynomial with real coefficients, it has at least one real root λ_i. Then $\ker(Q - \lambda_i I)$ is a subspace of $\mathbf{R}^3$ since Q is a real matrix, and so we have a real nonzero eigenvector v satisfying $Qv = \lambda_i v$. It follows that

$$\lambda_i^2 v^t v = (Qv)^t (Qv) = v^t Q^t Q v = v^t v \neq 0,$$

and dividing both sides by $v^t v$ then gives the claim.

> ***4(ii)*** *Is it true that $\lambda_i^2 = 1$ for all i? If it is true, prove it; otherwise, provide a concrete counterexample.*

It is false. Consider rotating the xy-plane by 90 degrees while fixing the z-axis:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives a transformation with eigenvalues 1, i, and $-i$. But $i^2 \neq 1$.

Question 5

> *Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space. Let $T: V \to V$ be a linear operator and let T^* be the adjoint of T. Let W be a T^*-invariant vector subspace of V, and let*
>
> $$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$
>
> *be the orthogonal complement of W.*
>
> ***5(i)** Show that $W^\perp$ is a vector subspace of V.*

Let $c \in \mathbf{R}$ and v, $v' \in W^\perp$. We will show that $cv + v' \in W^\perp$. Indeed, we have $\langle v, w \rangle = \langle v', w \rangle = 0$ for all $w \in W$ by hypothesis, which means that

$$\langle cv + v', w \rangle = c\langle v, w \rangle + \langle v', w \rangle = c \cdot 0 + 0 = 0$$

for all $w \in W$ by hypothesis. This means that $cv + v' \in W^\perp$, which is what we wanted.

> ***5(ii)** Is $W^\perp$ a T-invariant subspace of V? If it is, prove it; otherwise, provide a concrete counterexample.*

Yes, it is. Suppose $\langle v, w \rangle = 0$ for all $w \in W$. Then $\langle Tv, w \rangle = \langle v, T^* w \rangle = 0$ for all $w \in W$, since W is T^* invariant.

> ***5(iii)** Is $W^\perp$ a T^*-invariant subspace of V? If it is, prove it; otherwise, provide a concrete counterexample.*

No. Consider $\mathbf{R}^2$ with the standard basis (e_1, e_2) and the usual inner product, and define T to be left-multiplication by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then T^* is represented as left-multiplication by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $W = \mathbf{R}e_1$, the x-axis, is T^*-invariant. But $W^\perp$, the y-axis, is not T^*-invariant.a

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Question 6

> *Let $A \in M_n(\mathbf{C})$ be a complex matrix of order $n \geq 9$ and let*
>
> $$f(x) = (x-1)^2(x-2)^3(x-3)^4.$$
>
> *Suppose that A is self-adjoint and $f(A) = 0$. Find all possible minimal polynomials $m_A(x)$ of A.*

Since A is self-adjoint, the principal axis theorem tells us that it is diagonalizable. This in turn implies that its minimal polynomial has distinct roots. By hypothesis the minimal polynomial divides $f(x)$, so we conclude that $m_A(x)$ can be any polynomial of degree at least one that divides $(x-1)(x-2)(x-3)$, or $m_0(x) = x$. Explicitly, $m_A(x)$ can be

$$\begin{gathered} x, \quad x-1, \quad x-2, \quad x-3, \\ (x-1)(x-2), \; (x-1)(x-3), \; (x-2)(x-3), \\ \text{or } (x-1)(x-2)(x-3). \end{gathered}$$

Question 7

> *Let $T\colon V \to V$ be a linear operator. For positive integers n, let $T^n := T \circ \cdots \circ T$ be the composition of n copies of T, and set*
>
> $$K_n = \ker(T^n).$$
>
> ***7(i)*** *Show that $K_m \subseteq K_{m+1}$ for all $m \geq 1$.*

Suppose $v \in K_m$. Then $T^m v = 0$, so $T^{m+1}v = TT^m v = T0 = 0$. But this means that $v \in K_{m+1}$.

> ***7(ii)*** *Show that*
>
> $$K_r = K_{r+1} = K_{r+2} = \cdots$$
>
> *for some $r \geq 1$, when V is finite dimensional.*

The dimensions $k_n = \dim(K_n)$ form a (weakly) increasing sequence $0 \leq k_1 \leq k_2 \leq \cdots$ by part (i). These dimensions are all bounded above by $\dim(V) < \infty$, so the set of indices i for which $k_i < k_{i+1}$ has size at most $\dim(V)$; consequently it is finite and so the k_i are all equal for i greater than the largest element of this set.

> ***7(iii)*** *If V is infinite dimensional, can one still say that $K_r = K_{r+1}$ for some $r \geq 1$? If so, prove it; otherwise, prove a concrete counterexample.*

No, we construct a counterexample. Consider sequence space $V = \mathbf{R}^{\mathbf{N}}$ with the left-shift map $T\colon (a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots)$. Then $\dim(\ker T^n) = n$.

Question 8

> *Let $A \in M_n(\mathbf{C})$ be a matrix of order $n \geq 2$. Let*
>
> $$p_A(x) = (x - \lambda_1) \ldots (x - \lambda_n)$$
>
> *be the characteristic polynomial of A, such that all λ_i are positive real numbers.*
>
> ***8(a)*** *When A is a real matrix, is A then a positive-definite matrix? If so, prove it; otherwise, provide a concrete counterexample.*

No. The real matrix

$$A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

has characteristic polynomial $(x-1)^2$, but $v^t A v = -1 < 0$ when $v = \binom{1}{1}$.

> ***8(b)*** *Suppose that A is a normal matrix. Prove that one can write:*
>
> ***8(b)(i)*** *$A = G^4$ for some self-adjoint matrix G;*

By the principal axis theorem, there exists unitary U such that $D = U^*AU$ is diagonal. In fact, D has positive entries, since the roots of $p_A(x)$ are all positive reals. With $D = \text{diag}[d_1, \ldots, d_n]$, we may define $D^\alpha = \text{diag}[d_1^\alpha, \ldots, d_n^\alpha]$, which gives

$$(UD^{1/4}U^*)^4 = U(D^{1/4})^4U^* = UDU^* = A.$$

Now $G = UD^{1/4}U^*$ satisfies $G^* = (U^*)^*(D^{1/4})^*U^* = G$, since $D^{1/4}$ is real; thus G is our desired self-adjoint matrix.

> *8(b)(ii) $A = H^*H$ for some invertible matrix H.*

We can set $H = D^{1/2}U^*$, so that

$$H^*H = UD^{1/2}D^{1/2}U^* = UDU^* = A.$$

It is easy to see that $(D^{1/2}U^*)^{-1} = UD^{-1/2}$, so H is invertible and we are done.