# MA2108S AY 22/23 Finals

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**Question 1(a)** Let $\{a_n\}$ be a sequence of real numbers. Suppose that $\sum a_n$ is absolutely convergent. For a real number $p \geq 1$, show that $\sum a_n^p$ is also absolutely convergent.

*Proof 1.* We shall first use induction to show that if $a_n \geq 0$ for each $n$, then $\sum a_n^p \leq (\sum a_n)^p$. We shall prove the base case of $n = 2$. (The statement is trivially true for $n = 1$.) We can safely assume that $a_1, a_2 > 0$ (if $a_1 = 0$ or $a_2 = 0$, we have the case of $n = 1$),

$$\begin{aligned} a_1^p + a_2^p &\leq (a_1 + a_2)^p \\ \iff 1 + \left(\frac{a_2}{a_1}\right)^p &\leq \left(1 + \frac{a_2}{a_1}\right)^p \end{aligned}$$

One can substitute $x = a_2/a_1$ and use differentiation to prove that the last inequality holds. As such, the case of $n = 2$ has been proven.

Now, suppose the statement holds for some $n = k$, where $k$ is some positive integer. Then,

$$\begin{aligned} \sum_{i=1}^{k+1} a_i^p &\leq \left(\sum_{i=1}^{k} a_i\right)^p + a_{k+1}^p \\ &\leq \left(\sum_{i=1}^{k+1} a_i\right)^p \quad \text{by case of } n = 2. \end{aligned}$$

With this property, we then have

$$\sum_{i=1}^{n} |a_i|^p \leq \left(\sum_{i=1}^{n} |a_i|\right)^p,$$

for all positive integers $n$. Since $\sum |a_n|$ is convergent, we have $(\sum_{i=1}^{n} |a_i|)^p$ to be a convergent sequence of nonnegative terms i.e. nondecreasing and bounded above. As $\sum_{i=1}^{n} |a_i|^p$ is a sum of nonnegative terms and is bounded above and therefore convergent as desired.

*Proof 2.* Since $\sum a_n$ is absolutely convergent, we must have $|a_n| \to 0$. Hence, for some natural number $N$, we have, for $n \geq N$,

$$|a_n| < 1 \implies |a_n^p| = |a_n|^p \leq |a_n|.$$

Since $\sum a_n$ is absolutely convergent and $|a_n^p| \leq |a_n|$ for $n \geq N$, by the comparison test, $\sum a_n^p$ is absolutely convergent.

**Question 1(b)** Suppose that $\sum a_n$ is convergent, is $\sum a_n^2$ necessarily convergent? If it has to be convergent, prove the statement. Otherwise, find a counterexample.

*Proof 1.* The statement is false. Let $a_n = (-1)^{n-1}\frac{1}{\sqrt{n}}$. It is well known that $\sum a_n^2$ diverges (harmonic series). We shall show that $\sum a_n$ does indeed converge. Let $m, n$ be positive integers $n < m$. We shall only consider the case where $m - n$ is odd (we can always increase $m$ by one if $m - n$ is even and the inequality below would still hold with minor modifications to the first line). Since $m - n$ is odd, we can pair the terms and have

$$\begin{aligned}
\left|\sum_{i=n}^{m} \frac{(-1)^{i-1}}{\sqrt{i}}\right| &= \left|\sum_{i=n}^{(m-n+1)/2} \left[\frac{(-1)^{i-1}}{\sqrt{i}} + \frac{(-1)^{i}}{\sqrt{i+1}}\right]\right| \\
&= \left|\sum_{i=n}^{(m-n+1)/2} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\right)\right| \\
&= \left|\sum_{i=n}^{(m-n+1)/2} \frac{1}{\sqrt{i}\sqrt{i+1}(\sqrt{i}+\sqrt{i+1})}\right| \\
&\leq \left|\sum_{i=n}^{(m-n+1)/2} \frac{1}{\sqrt{i}\sqrt{i}(\sqrt{i}+\sqrt{i})}\right| \\
&\leq \frac{1}{2}\left|\sum_{i=n}^{(m-n+1)/2} \frac{1}{i^{3/2}}\right|
\end{aligned}$$

As $\sum 1/n^{3/2}$ is known to be convergent, by the Cauchy criterion, for every $\varepsilon > 0$, there exists positive integers $n, m$ such that

$$\left|\sum_{i=n}^{(m-n+1)/2} \frac{1}{i^{3/2}}\right| < 2\varepsilon.$$

This proves that $\sum a_n$ is convergent and the proof is complete.

*Proof 2.* The statement is false. Let $a_n = (-1)^{n-1}\frac{1}{\sqrt{n}}$. Then

$$|a_{n+1}| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = |a_n|$$

and $|a_n| = \frac{1}{\sqrt{n}} \to 0$, so by the alternating series test, $\sum a_n$ converges, while $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

**Question 2** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with the following property: For each $\varepsilon > 0$, there is $R > 0$ such that

$$|f(x)| < \varepsilon \quad \text{for all } |x| > R.$$

For such a function, first answer whether the following statement are true or false. If the statement is true, prove it. Otherwise, find a counterexample.

a) The function $f$ must be bounded on $\mathbb{R}$.

*Proof.* Fix $\varepsilon = 1$. Then, there exists $R$ such that for all $|x| > R$, $|f(x)| < 1$. As f is continuous on the compact domain $[-R, R]$, there exists $a, b \in [-R, R]$ such that $f(a) \geq f(x)$ and $f(x) \leq f(x)$ for all $x \in [-R, R]$. Thus, $|f(x)| \leq \max\{|f(a)|, |f(b)|, 1\}$ and thus bounded.

b) The function $f$ must be uniformly continuous on $\mathbb{R}$.

*Proof.* Let $\varepsilon > 0$, then there exists $R$ such that for all $|x| > R$, $|f(x)| < \varepsilon/2$. As $[-R, R]$ is compact and $f$ is continuous, $f$ is thus uniformly continuous on $[-R, R]$.

Therefore, if $x, y \in [-R, R]$, then there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for $|x - y| < \delta_1$.
If $x, y \notin [-R, R]$, then $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \varepsilon$.
By continuity of $f$ at $x = R$, there exists $\delta_2 > 0$ such that for all $z$ where $|z - R| < \delta_2$, $|f(z) - f(R)| < \varepsilon/2$.
Similarly, if $x = -R$, there exists $\delta_3$ such that $|f(z) - f(-R)| < \varepsilon/2$ whenever $|z + R| < \delta_3$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, 2R\}$. Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x| \leq |y|$ wlog. If $x, y \in [-R, R]$, then $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon$. If $x, y \notin [-R, R]$, we are done. Since $|x| \leq |y|$, this leaves the case $|x| \leq R, |y| > R$.

Suppose $x, y \geq 0$. Then we have $x \leq R < y$. Since $|x - y| < \delta_2$, we have $|x - R| < \delta_2$ and $|y - R| < \delta_2$, so that

$$|f(x) - f(R)| < \frac{\varepsilon}{2},$$
$$|f(R) - f(y)| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(R)| + |f(y) - f(R)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

The case of $x, y \leq 0$ is similar, and so are the cases $x \geq 0, y \leq 0$ and $x \leq 0, y \geq 0$. This completes the proof that $f$ is uniformly continuous on $\mathbb{R}$.

c) There must be a point $\bar{x} \in \mathbb{R}$ such that

$$f(\bar{x}) \geq f(x) \quad \text{for all } x \in \mathbb{R}.$$

*Counterexample.* Consider $f(x) = -e^{x^2}$. When $|x| > \sqrt{\ln \varepsilon}$, we have $f(x) < \varepsilon$. Yet, $\sup_{x \in \mathbb{R}} f(x) = 0$ and there does not exist $\bar{x}$ such that $f(\bar{x}) = 0$.

**Question 3** A function $f : \mathbb{R} \to \mathbb{R}$ is convex if it has the following property:
For any $x, y \in \mathbb{R}$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

a) For a convex function $f : \mathbb{R} \to \mathbb{R}$, if it is differentiable at 0 with $f'(0) = 0$, show that $f(x) \geq f(0)$ for all $x \in \mathbb{R}$.
*Proof.* Let us consider the case when $x > 0$ (the case for $x < 0$ is similar). Let $0 < \varepsilon < x$. Using the definition of convex, we have

$$\begin{aligned} f(\varepsilon) &\leq \frac{\varepsilon}{x} f(x) + \left(1 - \frac{\varepsilon}{x}\right) f(0) \\ \frac{f(\varepsilon) - f(0)}{\varepsilon} &\leq \frac{f(x) - f(0)}{x} \end{aligned}$$

Using the definition of derivatives and taking limits as $\varepsilon \to 0^+$, we have

$$\frac{f(x) - f(0)}{x} \geq 0 \Rightarrow f(x) \geq f(0).$$

b) For a convex function $f : \mathbb{R} \to \mathbb{R}$, suppose that is is twice differentiable on $\mathbb{R}$ with a continuous second derivative $f''$, show that is derivative $f'$ is monotone.
*Proof.* We shall use the following result from (*Rudin 3rd Edition: Chapter 5 Q23*):
Let $f$ be convex on $(a, b)$ and $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Let $x, q, y \in \mathbb{R}$ and $x < q < y$. We shall now show that

$$f'(x) \leq \frac{f(q) - f(x)}{q - x}.$$

Let $r \in \mathbb{R}$ and $x < r < q$. Using the first result, we have

$$\begin{aligned} \frac{f(r) - f(x)}{r - x} &\leq \frac{f(q) - f(x)}{q - x} \\ \lim_{r \to x^+} \frac{f(r) - f(x)}{r - x} &\leq \frac{f(q) - f(x)}{q - x} \\ f'(x) &\leq \frac{f(q) - f(x)}{q - x} \end{aligned}$$

Using a similar argument, we have

$$f'(y) \geq \frac{f(y) - f(q)}{y - q}.$$

By applying the first result once more, we have

$$f'(x) \leq \frac{f(q) - f(x)}{q - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(q)}{y - q} \leq f'(y).$$

**Question 4** In this question, let $f$ be a Riemann integrable function on [0, 1] with $\int_0^1 f(x) = 1$. Define $F$ as

$$F(x) = \int_0^x f(t)\ dt \text{ for all } x \in [0,1]$$

a) Can we always find a point $x \in [0,1]$ such that $F(x) = \left(x - \frac{1}{2}\right)^2$ ?

*Solution.* Yes we can. Let

$$g(x) = F(x) - \left(x - \frac{1}{2}\right)^2.$$

Then, note that $g(0) = -\frac{1}{4} < 0$ and $g(1) = \frac{3}{4} > 0$. Since $F(x)$ is continuous on [0, 1] by the Fundamental Theorem of Calculus, we know $g(x)$ is continuous on [0,1], so by the Intermediate Value Theorem (IVT), there exists $x \in (0,1)$ such that $g(x) = 0$ i.e. $F(x) = \left(x - \frac{1}{2}\right)^2$.

b) If $f$ is continuous and $f(x) \neq 0$, show that $F$ is injective (one-to-one) on $(0,1)$.

*Solution.* Since $f(x) \neq 0$ and $f$ is continuous, by IVT, it is clear that either $f(x) < 0$ or $f(x) > 0$ for all $x \in [0,1]$. Since $\int_0^1 f(x)\ dx = 1$, we must then have $f(x) > 0$. We shall now show that $f$ is strictly increasing on $[0,1]$. Let $x, y \in [0,1]$ and $y > x$. Then,

$$\begin{aligned} F(y) - F(x) &= \int_0^y f(t)\ dt - \int_0^x f(t)\ dt \\ &= \int_x^y f(t)\ dt \\ &> 0 \end{aligned}$$

This proves that $F$ is strictly increasing on $[0,1]$ and thus injective on [0,1].

c) Under the same assumption in b), show that we can find a function $g : [0,1] \to [0,1]$ that is continuous on $[0,1]$ and differentiable on $(0,1)$ such that $g(F(x)) = x$ for all $x \in [0,1]$.

*Solution.* $F$ is injective on [0,1] as shown in (b). As $F(0) = 0$, $F(1) = 1$ and $F$ is continuous, it is clear by IVT that $F$ is surjective with range $[0,1]$. $F$ is thus invertible and there exists a function $g : [0,1] \to [0,1]$ such that $g(x) = F^{-1}(x)$.

To show that $g$ is continuous, we shall first prove a lemma.

*Lemma:* Let $h : K \to B$, where $K$ is compact and $h$ is continuous and invertible. Then, $h^{-1}$ is continuous.

*Proof.* Let $V$ be an open set in $K$. It suffices to show that $(h^{-1})^{-1}(V) = h(V)$ is open in $B$. To see that, note that $h(V^c \cap K)$ is compact (since $V^c$ is closed, $K$ is compact and $h$ is continuous), thus closed. As $h$ is bijective, $h(V) = [h(V^c)]^c = [h(V^c \cap K)]^c$, thus $h(V)$ is open, and the proof is complete.

Since $[0,1]$ is compact, we can use this lemma with $h = F$ to conclude that $g = F^{-1}$ is continuous on [0,1].

To show that $g$ is differentiable on $(0,1)$, let $y \in (0,1)$ and define a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in (0,1)$, $y_n \neq y$ for all $n \in \mathbb{Z}^+$ and $y_n \to y$ as $n \to \infty$. As $F$ is bijective, it follows that for each $n$, there is a unique $x_n$ such that $F(x_n) = y_n$, and a unique $x$ such that $F(x) = y$. Note that since $y_n \to y$ and $F^{-1}$ is continuous, we have $x_n \to x$. Note also that $F'(x) > 0$ on (0,1) since $F$ is strictly increasing. Now, we will show that the following limit exists:

$$\begin{aligned}
\lim_{r\to y}\frac{g(r)-g(y)}{r-y} &= \lim_{n\to\infty}\frac{g(y_n)-g(y)}{y_n-y} \quad \text{by continuity of } g \\
&= \lim_{n\to\infty}\frac{x_n-x}{F(x_n)-F(x)} \\
&= \lim_{n\to\infty}\left(\frac{F(x_n)-F(x)}{x_n-x}\right)^{-1} \\
&= \lim_{r\to x}\left(\frac{F(r)-F(x)}{r-x}\right)^{-1} \\
&= \frac{1}{F'(x)}.
\end{aligned}$$

This proves that $g$ is differentiable on $(0,1)$.