MA3110 Mathematical Analysis II - Solutions to the 2014/2015 Semester 1 Final Exam

1. (20 points) Answer **TRUE** or **FALSE** to each of the following questions. No explanation is necessary. Each question is worth 2.5 points.

(a) A sequence $\{f_n : [a,b] \to \mathbb{R}\}$ converges pointwise if and only if it converges uniformly.

False. For example, consider $\{f_n : [0,1] \to \mathbb{R}\}$ given by $f_n(x) = x^n$. The pointwise limit is

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

however the convergence is not uniform.

(b) $f : [a,b] \to \mathbb{R}$ is integrable if and only if $f^5 : [a,b] \to \mathbb{R}$ is integrable.

True. The statement follows from the composition theorem. The function x^5 is continuous on $[a,b]$ and so f is integrable implies that f^5 is integrable. The function $x^{1/5}$ is continuous on $[a,b]$ and so f^5 is integrable implies that $f = (f^5)^{1/5}$ is integrable.

(c) Let $f : (a,b) \to \mathbb{R}$ be a differentiable function. Then f is uniformly continuous if and only if f' is bounded.

False. For example, consider $f : (0,1) \to \mathbb{R}$ given by $f(x) = x^{1/2}$. Then f is uniformly continuous since it extends to a continuous function on $[0,1]$, however the derivative is $f'(x) = \frac{1}{2}x^{-1/2}$, which is not bounded on $(0,1)$.

(d) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

False. For example, consider $a_n = 2^{-n+(-1)^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \frac{2^{-n-2}}{2^{-n+1}} = \frac{1}{8} & n \text{ even} \\ \frac{2^{-n}}{2^{-n-1}} = 2 & n \text{ odd} \end{cases}$$

and so $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2$, however $a_n \leq 2 \cdot 2^{-n}$ for all n and so the series converges by comparison with a geometric series.

(e) If $\limsup_{n\to\infty} |a_n|^{1/n} > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

True. This is the root test (see for example Theorem 9.2.2 in the Bartle and Sherbert text).

(f) If $\{f_n : [a,b] \to \mathbb{R}\}$ is a sequence of differentiable functions converging uniformly to $f : [a,b] \to \mathbb{R}$ then $f_n' \to f'$ pointwise.

False. For example, consider $\{f_n : [0,1] \to \mathbb{R}\}$ given by $f_n(x) = \frac{x^n}{n}$. Then $\|f_n\| = \frac{1}{n}$ and so $f_n \to f \equiv 0$ uniformly, however $f_n'(x) = x^{n-1}$ which converges pointwise to

$$g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

Since $g \neq 0 = f'$ then the statement is false.

(g) If $f : \mathbb{R} \to \mathbb{R}$ is C^∞ then $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

False. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then $f^{(n)}(0) = 0$ for all n and so $f(x) \neq \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = 0$

(h) If a power series $\sum_{n=0}^{\infty} a_n x^n$ is pointwise convergent on a closed bounded interval $[a,b]$ then it is uniformly convergent.

True. Abel's theorem shows that the series is uniformly convergent on every closed subinterval of the interval of convergence.

2. (20 points) For each of the following statements, give a counterexample to show that the statement is false. To get full credit you must explain your counterexample. Each question is worth 5 points.

(a) Let $\{f_n : [a,b] \to \mathbb{R}\}$ be a sequence of integrable functions which converges pointwise to a function $f : [a,b] \to \mathbb{R}$. Then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx$$

Example. Consider the sequence of "spike" functions $\{f_n : [0,1] \to \mathbb{R}\}$ given by

$$f_n(x) = \begin{cases} n^2 x & x \in \left[0, \frac{1}{n}\right] \\ 2n - n^2 x & x \in \left[\frac{1}{n}, \frac{2}{n}\right] \\ 0 & x \in \left[\frac{2}{n}, 1\right] \end{cases}$$

Then $f_n \to 0$ pointwise, however we have

$$\lim_{n\to\infty} \int_0^1 f_n \, dx = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n \, dx$$

(b) Let $\{a_n\}$ be a sequence of nonzero real numbers. Then

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}|a_n|^{1/n}$$

Example. Consider the sequence

$$a_n = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

Then $\limsup_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty} 2^{1/n} = 1$, however

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 2 & n \text{ even} \\ \frac{1}{2} & n \text{ odd} \end{cases}$$

and so $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2 \neq 1 = \limsup_{n\to\infty}|a_n|^{1/n}$.

(c) If $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly then $\displaystyle\sum_{n=1}^{\infty}\|f_n\|$ converges.

Example. Consider $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & x \in [0,1] \setminus \left\{\frac{1}{n}\right\} \end{cases}$$

Then $\sum_{n=1}^{\infty} f_n$ converges pointwise to f with $f(\frac{1}{k}) = \frac{1}{k}$ for all $k \in \mathbb{N}$, and $f(x) = 0$ for all other $x \in [0,1]$. Note that $\sum_{n=1}^{\infty}\|f_n\| = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$.

Therefore $\displaystyle\sum_{n=1}^{\infty}\|f_n\|$ diverges, however $\|f - \sum_{k=1}^{n} f_k\| = \|\sum_{k=n+1}^{\infty} f_k\| = \frac{1}{n+1} \to 0$ and so $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions which are differentiable on all of $\mathbb{R}$ and suppose that $f(0) = g(0) = 0$. Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}$ exists if and only if $\displaystyle\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ exists.

Example. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x$. Then $\frac{f(x)}{g(x)} = x\sin\left(\frac{1}{x}\right)$ which converges to zero as $x \to 0$. Using the limit definition of derivative, we also have

$$f'(x) = \begin{cases} 0 & x = 0 \\ 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

and $g'(x) = 1$. Therefore $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0} f'(x)$ does not exist.

3. (20 points)

(a) (10 points) Use Taylor expansion to prove that $1 - \frac{y^2}{2} \le \cos y \le 1$ for all $y \in [-\pi, \pi]$.

Solution. We already know that $\cos y \le 1$ for all $y \in \mathbb{R}$.
Since $f(y) = \cos y$ is a C^∞ function on all of $[-\pi, \pi]$ then Taylor's Theorem applies. We have $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$ and $f^{(3)}(c) = \sin c$, and so

$$\cos y = 1 - \frac{y^2}{2} + \frac{\sin c}{3!} y^3 \quad \text{for some } c \text{ between } 0 \text{ and } y.$$

Since c and y have the same sign then $\sin c$ and y^3 also have the same sign for all $y \in [-\pi, \pi]$ and therefore $\frac{\sin c}{3!} y^3 \ge 0$ for all $y \in [-\pi, \pi]$. Therefore

$$\cos y = 1 - \frac{y^2}{2} + \frac{\sin c}{3!} y^3 \ge 1 - \frac{y^2}{2} \quad \text{for all } y \in [-\pi, \pi]$$

Alternatively, one can obtain the lower bound as follows. The first order Taylor polynomial with remainder is

$$\cos y = 1 - \frac{y^2}{2} \cos c.$$

Since $\cos c \le 1$ for all c, then $\cos y \ge 1 - \frac{y^2}{2}$.

(b) (10 points) Define $\{f_n : [-\pi, \pi] \to \mathbb{R}\}$ by

$$f_n(x) = \begin{cases} 0 & x = 0 \\ \frac{1 - \cos\left(\frac{x}{n}\right)}{x} & x \in [-\pi, \pi] \setminus \{0\} \end{cases}$$

Prove that $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[-\pi, \pi]$.

(You can use the result of the previous question even if you haven't solved it.)

Solution. Since $x \in [-\pi, \pi]$ then $\frac{x}{n} \in [-\pi, \pi]$ and so the previous part of the question shows that

$$0 \le \left| \frac{1 - \cos\left(\frac{x}{n}\right)}{x} \right| \le \left| \frac{x}{2n^2} \right|$$

Therefore the Squeeze Theorem shows that f_n is continuous at $x = 0$ and so f is continuous on the whole interval $[-\pi, \pi]$. The above inequality also shows that

$$0 \le \left| \frac{1 - \cos\left(\frac{x}{n}\right)}{x} \right| \le \frac{x}{2n^2} \le \frac{\pi}{2n^2} \quad \text{and so} \quad \|f_n\| \le \frac{\pi}{2n^2} \quad \text{for all } n.$$

Since $\sum_{n=1}^{\infty} \frac{\pi}{2n^2}$ converges by the p-series test then the original series converges uniformly by the Weierstrass M-test.

4. (10 points) Determine at which values of $x \in \mathbb{R}$ the following power series converge.

(a) (5 points) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\log n}$

Solution. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\log n}{\log(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1 \quad \text{(L'Hôpital's Rule)}$$

where we note that $\frac{\log n}{\log(n+1)}$ is the indeterminate form $\frac{\infty}{\infty}$. Therefore the radius of convergence is equal to 1.

At the endpoints $x = \pm 1$, we have

$$x = 1: \qquad \sum_{n=2}^{\infty} \frac{x^n}{\log n} = \sum_{n=2}^{\infty} \frac{1}{\log n} \quad \text{which diverges by comparison with} \quad \sum_{n=2}^{\infty} \frac{1}{n}$$

$$x = -1: \qquad \sum_{n=2}^{\infty} \frac{x^n}{\log n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} \quad \text{which converges by the Alternating Series Test.}$$

Therefore the interval of convergence is $[-1, 1)$.

(b) (5 points)

$$\sum_{n=0}^{\infty} a_n x^n, \quad \text{where} \quad a_n = \begin{cases} 1 & \text{if } n = 10^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Note that for all $n \in \mathbb{N}$ we have $\sup_{k\geq n} |a_n| = 1$ and hence $\sup_{k\geq n} |a_n|^{1/n} = 1$ also. Therefore

$$\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\sup_{k\geq n} |a_n|^{1/n} \right) = 1$$

and so the radius of convergence is equal to 1. If $x = 1$ then the terms in the series do **not** converge to zero and so the series $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n$ diverges by the n^{th} term test. If $x = -1$ then the n^{th} term test again shows that $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n a_n$ also diverges.

Therefore the interval of convergence is $(-1, 1)$.

5. (15 points) Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{\cos(kx)}{\log k}$.

(a) (5 points) Prove that it converges pointwise on $(0, 2\pi)$.

(b) (5 points) Prove that it converges uniformly on any closed subinterval of $(0, 2\pi)$.

(c) (5 points) Prove that it does not converge uniformly on $(0, 2\pi)$.

In this question you can use the fact that $\left|\sum_{k=1}^{n} \cos(kx)\right| \leq \dfrac{1}{\left|\sin \frac{x}{2}\right|}$ for all $x \in (0, 2\pi)$ and for all $n \in \mathbb{N}$.

Solution.

(a) Let $f_k(x) = \frac{1}{\log k}$ and $g_k(x) = \cos(kx)$. Since $\left|\sum_{k=1}^{n} \cos(kx)\right| \leq \dfrac{1}{\left|\sin \frac{x}{2}\right|}$ for all $x \in (0, 2\pi)$ then the partial sums

$$\sum_{k=1}^{n} g_k(x)$$

are bounded for each x. (Note that the bound is not uniform on $(0, 2\pi)$.)
Since $f_k(x)$ converges monotonically to zero, then we can apply Dirichlet's test for each $x \in (0, 2\pi)$ to show that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges pointwise.

(b) Let $[a, b]$ be a closed subinterval of $(0, 2\pi)$. Since $\frac{1}{\left|\sin \frac{x}{2}\right|}$ is continuous on the closed bounded interval $[a, b]$, then it is bounded above by a constant.
Therefore the partial sums $\sum_{k=1}^{n} g_k(x)$ are uniformly bounded on $[a, b]$ and $f_k(x)$ converges monotonically to zero. Moreover, the convergence of f_k is uniform, since $f_k(x)$ is a constant function with respect to x. Therefore Dirichlet's test shows that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges uniformly on $[a, b]$.

(c) Note that the functions $\frac{\cos(kx)}{\log k}$ are continuous on $[0, 2\pi]$ and therefore the partial sums $S_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{\log k}$ are also continuous on $[0, 2\pi]$. In particular $\sup_{x \in (0, 2\pi)} S_n(x) = \sup_{x \in [0, 2\pi]} S_n(x)$ is finite.
Since $S_n(0) = \sum_{k=1}^{n} \frac{1}{\log k}$ diverges to infinity as $n \to \infty$ then so does $\sup_{x \in (0, \pi)} S_n(x) = \sup_{x \in [0, 2\pi)} S_n(x)$. Therefore the pointwise limit is an unbounded function on $(0, 2\pi)$, and so the convergence cannot be uniform since the partial sums are bounded but the limit is unbounded (and so Cauchy's criterion fails).

6. (15 points) Recall the definition of Thomae's function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \cap [0, 1] \text{ with } \gcd(p, q) = 1 \end{cases}$$

Let $\{f_n : [0, 1] \to \mathbb{R}\}$ be the sequence given by $f_n(x) = f(x)^{1/n}$.

(a) (5 points) Prove that f_n converges pointwise to the Dirichlet function.

Solution. For each x we have

$$f_n(x) = \begin{cases} 0^{1/n} & \text{if}\, x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ \left(\frac{1}{q}\right)^{\frac{1}{n}} & \text{if}\, x = \frac{p}{q} \in \mathbb{Q} \cap [0,1]\, \text{with}\, \gcd(p,q) = 1 \end{cases}$$

Since $0^{1/n} \to 0$ as $n \to \infty$ and $\left(\frac{1}{q}\right)^{\frac{1}{n}} \to 1$ as $n \to \infty$, then f_n converges pointwise to the Dirichlet function given by

$$f(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 1 & x \in \mathbb{Q} \cap [0,1] \end{cases}$$

(b) (10 points) Show that f_n is integrable for all n. (You can use either the Riemann or Darboux integral.)

(Darboux integral solution) For each n, we want to show that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}$ of $[0,1]$ such that $\mathcal{U}(f_n;\mathcal{P}) - \mathcal{L}(f_n;\mathcal{P}) < 2\varepsilon$.

Let $\mathcal{P}$ be any partition, and let $\delta = \|\mathcal{P}\|$. Since the irrationals are dense then every subinterval $[x_{i-1}, x_i]$ contains a point with $f_n(x) = 0$. Therefore $\mathcal{L}(f;\mathcal{P}) = 0$.

Since $\frac{1}{q} < \varepsilon$ for all $q > \frac{1}{\varepsilon}$ then there exists a finite number $N_{\varepsilon,n}$ such that at most $N_{\varepsilon,n}$ values of x have $f_n(x) \geq \varepsilon$. Therefore there are at most $2N_{\varepsilon,n}$ subintervals $[x_{i-1}, x_i]$ where

$$\varepsilon \leq M_i = \sup_{x \in [x_{i-1}, x_i]} f_n(x) \leq 1.$$

Therefore, we see that

$$\mathcal{U}(f;\mathcal{P}) = \sum_{i=1}^{m} M_i(x_i - x_{i-1}) \leq 2N_{\varepsilon,n}\delta$$

If we choose $\delta < \frac{\varepsilon}{2N_{\varepsilon,n}}$ then we have $\mathcal{U}(f_n;\mathcal{P}) < \varepsilon$.

Therefore, we have shown that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}$ (any partition of size $\delta < \frac{\varepsilon}{2N_{\varepsilon,n}}$ will do) such that $\mathcal{U}(f;\mathcal{P}) - \mathcal{L}(f;\mathcal{P}) < \varepsilon$. Therefore f is integrable.

(Riemann integral) We want to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every tagged partition $\dot{\mathcal{P}}$ of $[0,1]$ with size $\|\dot{\mathcal{P}}\| < \delta$ we have $\left|\mathcal{S}(f_n;\dot{\mathcal{P}})\right| < 2\varepsilon$.

Again, there exists $N_{\varepsilon,n}$ such that $\varepsilon \leq f_n(x) \leq 1$ for at most $N_{\varepsilon,n}$ points and therefore there are at most $2N_{\varepsilon,n}$ subintervals $[x_{i-1}, x_i]$ where

$$\varepsilon \leq f_n(t_i) \leq 1.$$

Since $(x_i - x_{i-1}) < \delta$ for each i then we see that

$$0 \leq \mathcal{S}(f_n; \dot{\mathcal{P}}) = \sum_{i=1}^{m} f_n(t_i)(x_i - x_{i-1}) < \varepsilon + 2N_{\varepsilon,n}\delta$$

If we choose $\delta = \frac{\varepsilon}{2N_{\varepsilon,n}}$ then we have $0 \leq \mathcal{S}(f_n; \dot{\mathcal{P}}) < 2\varepsilon$.
Therefore, we have shown that for every $\varepsilon > 0$ there exists a $\delta > 0$ (choose $\delta = \frac{\varepsilon}{2N_{\varepsilon,n}}$) such that $\left|\mathcal{S}(f_n; \dot{\mathcal{P}})\right| < 2\varepsilon$ for all tagged partitions $\dot{\mathcal{P}}$ such that $\|\dot{\mathcal{P}}\| < \delta$.
Therefore f_n is integrable.