MA1100 AY24/25 Sem 1 Final

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Question 1

In each part, write only "**True**" or "**False**" as your answer; any other word will be regarded as a non-answer. No justification is needed

(a) Let P, Q and R be statements. Then

$$(P \wedge Q) \Rightarrow R \quad \text{and} \quad P \Rightarrow (Q \Rightarrow R) \quad \text{are logically equivalent.}$$

(b) Let $P(x)$ and $Q(x)$ be predicates with free variable x and universe $\mathcal{U}$. Then

$$(\exists x \in \mathcal{U})[P(x) \vee Q(x)] \quad \text{and} \quad [(\exists x \in \mathcal{U})P(x)] \wedge [(\exists x \in \mathcal{U})Q(x)] \quad \text{are logically equivalent.}$$

(c) For any sets X and Y, $\mathcal{P}(X - Y) = \mathcal{P}(X) - \mathcal{P}(Y)$.

(d) For any sets X, Y, Z and any functions $f : X \to Y$ and $g : Y \to Z$,

$$\text{if } g \circ f \text{ is injective} \quad \text{then} \quad f \text{ is injective.}$$

(e) Let $f : X \to Y$ be a function and let $I_X : X \to X$ be the identity function on X. If there exists a function $g : Y \to X$ such that $g \circ f = I_X$, then f is invertible.

(f) If $f : X \to Y$ is a function, then for all $A \subseteq Y$, we have $f[f^{-1}[A]] = A$

(g) If $f : X \to Y$ is a function, then

$$\text{for all } A, B \subseteq Y \quad \text{we have} \quad f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$$

(h) For each positive integer n, the number $n^2 + n + 41$ is a prime[1]

(i) For any prime numbers p and q, we have $\gcd(p, q) = p$ or $\gcd(p, q) = q$

(j) There are exactly 1022 surjective functions with domain $\mathbb{N}_{10}$ and codomain $\mathbb{N}_2$

[1] A fun fact is that this is known as Euler's prime formula. It is an interesting expression that generates prime numbers for consecutive integer values of n, though only up to a certain point.

Solution:

(a) **True**. By considering the truth table as shown, we have

P	Q	R	$P \wedge Q$	$Q \Rightarrow R$	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Therefore, there are logically equivalent.

(b) **False**. Let $\mathcal{U} = \{1\}$. Let

$$P(x) : x \text{ is even} \quad \text{and} \quad Q(x) : x \text{ is odd}.$$

So, $P(1)$ is false and $Q(1)$ is true. Then

$$(\exists x \in \mathcal{U})[P(x) \vee Q(x)] \text{ is true} \quad \text{but} \quad [(\exists x \in \mathcal{U})P(x)] \wedge [(\exists x \in \mathcal{U})Q(x)] \text{ is false}.$$

(c) **False**. Let $X = 1$ and $Y = 1$. We have $X - Y = \emptyset$. Hence $\mathcal{P}(X-Y) = \{\emptyset\}$ while $\mathcal{P}(X) - \mathcal{P}(Y) = \emptyset$. Note that $\emptyset \neq \{\emptyset\}$. Therefore, they are not equal.

(d) **True**. Suppose $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is injective, then $x_1 = x_2$. So f is injective.

(e) **False**. Let $X = \{1,2\}$ and $Y = \{a,b,c\}$. Define

$$f : X \to Y \quad \text{where} \quad f(1) = a \text{ and } f(2) = b.$$

Define

$$g : Y \to X \quad \text{where} \quad g(a) = 1 \text{ and } g(b) = 2 \text{ and } g(c) = 1.$$

Then, $(g \circ f)(1) = 1$ and $(g \circ f)(2) = 2$. So, there does not exist any element in X that maps to $c \in Y$, hence f is not surjective. So, f is not invertible.

(f) **False**. Let $X = \{1,2\}$ and $Y = \{a,b,c\}$. Define

$$f : X \to Y \quad \text{where} \quad f(1) = a \text{ and } f(2) = b.$$

Now let $A = \{b,c\} \subseteq Y$ such that we have $f^{-1}[A] = \{2\}$ but $f[f^{-1}[A]] = \{b\} \neq A$.

(g) **True**. We first prove $\subseteq$. Let $x \in f^{-1}[A \cap B]$. Then, $f(x) \in A \cap B$, thus $f(x) \in A$ and $f(x) \in B$. As such $x \in f^{-1}[A]$ and $x \in f^{-1}[B]$. It follows that $x \in f^{-1}[A] \cap f^{-1}[B]$.

We then prove $\supseteq$. Let $x \in f^{-1}[A] \cap f^{-1}[B]$ then $x \in f^{-1}[A]$ and $x \in f^{-1}[B]$. Thus $f(x) \in A$ and $f(x) \in B$. As such, $f(x) \in A \cap B$ which follows that $x \in f^{-1}[A \cap B]$.

(h) **False**. Consider $n = 40$, so $40^2 + 40 + 41 = 41^2$.

(i) **False**. We can choose p and q to be distinct primes, then p and q are coprime. Hence, $\gcd(p,q) = 1$.

(j) **True**. To see why, as every element in the domain $\mathbb{N}_{10}$ has 2 choices in the codomain $\mathbb{N}_2$, then there are 2^{10} total functions.

Next, note that a function fails to be onto exactly when it avoids one of the two target values entirely. Suppose the function never hits $1 \in \mathbb{N}_2$, then the function must map every element to 2. There is precisely $1^{10} = 1$ function. On the other hand, if the function never hits $2 \in \mathbb{N}_2$, it must map every element to 1. Again, there is only 1 function.

As there are no other ways to miss a value, there are exactly 2 functions are not onto. So, the number of surjections is $2^{10} - 2 = 1022$. $\square$

Question 2: Let $a_1 = 11$, $a_2 = 21$ and $a_{n+1} = 3a_n - 2a_{n-1}$ for all integers n with $n \geq 2$. Prove that for all positive integers n,

$$a_n = 5 \cdot 2^n + 1$$

Solution: We will prove this by strong induction.

For the base case, we have $a_1 = 5 \cdot 2^1 + 1 = 11$, so it is true. Next, suppose for all $1 \leq k \leq n$, we have $a_k = 5 \cdot 2^k + 1$. We will prove that $a_{k+1} = 5 \cdot 2^{k+1} + 1$. To deduce this, we have

$$\begin{aligned}
a_{k+1} &= 3a_k - 2_{k-1} \\
&= 3(5 \cdot 2^k + 1) - 2(5 \cdot 2^{k-1} + 1) \\
&= 15 \cdot 2^k + 3 - 5 \cdot 2^k - 2 \\
&= 10 \cdot 2^k + 1 \\
&= 5 \cdot 2^{k+1} + 1
\end{aligned}$$

By the principle of strong mathematical induction, a_n is true for all positive integers n. $\square$

Question 3

(i) Prove that the square of any integer has one of the forms $4k$ or $4k + 1$ where $k \in \mathbb{Z}$

(ii) Let a and b be two odd integers. Prove that $a^2 + b^2$ is not a perfect square.

Solution:

(i) We will proceed with casework.

First, suppose n is even. Then, there exists $m \in \mathbb{Z}$ such that $n = 2m$. So,

$$n^2 = (2m)^2 = 4m^2 = 4k \quad \text{where } k = m^2 \in \mathbb{Z}.$$

Next, suppose n is odd. Then, there exists $p \in \mathbb{Z}$ such that $n = 2p+1$. So,

$$n^2 = (2p+1)^2 = 4p^2 + 4p + 1 = 4(p^2 + p) + 1 = 4k + 1 \quad \text{where } k = p^2 + p \in \mathbb{Z}.$$

Combining both cases, the result follows.

(ii) Let $a = 2m+1$ and $b = 2n+1$ for some $m, n \in \mathbb{Z}$. Then

$$\begin{aligned} a^2 &= (2m+1)^2 = 4m^2 + 4m + 1 \\ b^2 &= (2n+1)^2 = 4n^2 + 4n + 1 \end{aligned}$$

so

$$a^2 + b^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4k + 2 \quad \text{where } k = m^2 + n^2 + m + n \in \mathbb{Z}.$$

By **(i)**, the square of any integer has one of the forms $4k$ or $4k+1$. Hence, $a^2 + b^2$ is not a perfect square. □

Question 4

For each $n \in \mathbb{Z}^+$, let

$$A_n = \left(1 - \frac{1}{n}, n\right) = \left\{x \in \mathbb{R} \mid 1 - \frac{1}{n} < x < n\right\}.$$

Find the sets

$$\bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} A_n$$

Justify you answers.

Solution. We claim that

$$\bigcup_{n=1}^{\infty} A_n = (0, \infty).$$

To prove $\subseteq$, first let $x \in \bigcup_{n=1}^{\infty} A_n$. Then, there exists $n \in \mathbb{Z}^+$ such that

$$1 - \frac{1}{n} < x < n$$

When $n = 1$, we have $0 < x < 1$. As $n \to \infty$, we have $x > 1$. As such, $0 < x$ which follows that $x \in (0, \infty)$

To prove the reverse inclusion $\supseteq$, let $x \in (0, \infty)$. Then for some $n \in \mathbb{Z}^+$, we can pick $n > \max\{x, \frac{1}{x}\}$ such that we have

$$x < n \qquad \text{or} \qquad \frac{1}{n} > x \implies -\frac{1}{n} < -x \implies 1 - \frac{1}{n} < 1 - x < x.$$

Hence,

$$1 - \frac{1}{n} < x < n \implies x \in \bigcup_{n=1}^{\infty} A_n.$$

We then claim that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Suppose for the sake of contradiction that the intersection is non empty. Then, there exists $x \in A_n$ such that

$$1 - \frac{1}{n} < x < n.$$

Since n is increasing, then $x < n$ is trivially satisfied for fixed x, since $n \to \infty$. As $n \to \infty$, the lower bound will be 1. As such $x \in (1, \infty)$. We have $A_1 = (0, 1)$ which is disjoint with $(1, \infty)$. Therefore, $x \notin A_1$ which is a contradiction. $\square$

Question 5

Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove that if f is surjective, then X has a subset Z such that the function $h : Z \to Y$ defined by

$$h(x) = f(x) \quad \text{for all } x \in Z$$

is a bijection.

Solution. Since f is surjective, then for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Define the set

$$Z = \{x \in X \mid y \in Y\} \subseteq X \quad \text{so} \quad h(x) = f(x) = y.$$

Suppose $h(x_1) = h(x_2)$, then $f(x_1) = f(x_2)$. which implies $y_1 = y_2$. Since f is surjective, then $x_1 = x_2$ which also shows that $h(x)$ is injective.

We then prove that h is surjective. Let $y \in Y$. Then, by the definition of Z, there exists $x \in Z$ such that $h(x) = y$. This shows that $h(x)$ is surjective. Since $h(x)$ is both injective and surjective, then h is a bijection. $\square$

Question 6

Let R be an equivalence relation on the set $\mathbb{N}_6 = \{1, 2, 3, 4, 5, 6\}$ such that

- $|[1]| < |[2]| < |[3]|$
- $(3, 4) \notin R$

(i) List all the elements in each equivalence class of R

(ii) List all the elements of R

Solution:

(i) The trick is to think of how the partition all 6 elements of $\mathbb{N}_6$ into 3 separate sets while separating elements 3 and 4. As such, let

$$[1] = \{1\}$$
$$[2] = \{2,4\}$$
$$[3] = \{3,5,6\}$$

(ii) Recall that an equivalence relation includes all reflexive, symmetric, transitive pairs. So,

$$R = \{(1,1),(2,2),(2,4),(4,2),(4,4),(3,3),(3,5),(5,3),(3,6),(6,3),(5,5),(5,6),(6,5),(6,6)\}$$

$\square$

Question 7

Let X and Y be two nonempty sets, and let $f : X \to Y$ be a surjective function. Let $\sim$ be the relation on X defined by, for all $x, y \in X$,

$$x \sim y \quad \text{if and only if} \quad f(x) = f(y)$$

(i) Prove that $\sim$ is an equivalence relation

(ii) Prove that $X/\sim$ is equinumerous with Y

Solution:

(i) We have

$$x \sim x \iff f(x) = f(x)$$

which is true for all $x \in X$. Hence $\sim$ is reflexive.

Suppose $x \sim y$, then

$$x \sim y \iff f(x) = f(y) \iff f(y) = f(x) \iff y \sim x$$

Hence $\sim$ is symmetric.

Suppose $x \sim y$ and $y \sim z$, then

$$f(x) = f(y) \quad \text{and} \quad f(y) = f(z)$$

so

$$f(x) = f(z) \iff x \sim z$$

Hence $\sim$ is transitive. Since $\sim$ is reflexive, symmetric and transitive, then $\sim$ is an equivalence relation.

(ii) We can define $\varphi : X/\sim \to Y$ by

$$\varphi([x]) = f(x)$$

Suppose $[x] = [x']$, then $x \sim x' \iff f(x) = f(x') \implies \varphi([x]) = \varphi([x'])$. This shows φ is a well-defined function.

Next, suppose $\varphi([x]) = \varphi([x'])$, then $f(x) = f(x') \implies x \sim x' \implies [x] = [x']$. This shows φ is injective.

Since it is given that f is surjective, then for every $y \in Y$, there exist $x \in X$ such that $f(x) = y$. Then $\varphi([x]) = f(x) = y$ so φ is surjective. Since φ is injective and surjective, it is a bijection. Hence $X/\sim$ is equinumerous with Y. $\square$

Question 8

Determine whether each of the following sets is finite, denumerable or uncountable. Justify your answers.

(i) $\mathbb{R} \setminus \mathbb{N}$

(ii) $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i + j \leq 100\}$

(iii) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{Q}\}$

Solution:

(i) It is known that $\mathbb{R}$ is uncountably infinite and $\mathbb{N}$ is countably infinite (also known as denumerable). We claim that in general, for any sets X and Y where $Y \subseteq X$,

$$X \text{ is uncountably infinite and } Y \text{ is denumerable} \quad \text{implies} \quad X \setminus Y \text{ is uncountable.}$$

In particular, $\mathbb{R} \setminus \mathbb{N}$ is uncountable. Suppose on the contrary that $X \setminus Y$ is countable. Then, because $X = Y \cup (X \setminus Y)$ (to be precise, this is a disjoint union, denoted by $\sqcup$), then X is the union of two countable sets, which is also countable. This leads to a contradiction.

(ii) Let $n \in \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i + j \leq 100\}$. Note that for each $n = 2, 3, \ldots, 100$, the number of pairs (i, j) such that $i + j = n$ is $n - 1$ because $1 \leq i, j \leq n$. Since $1 \leq i \leq n - 1$ and $j = n - i$, the total number of pairs is

$$\sum_{n=2}^{100} (n-1) = \sum_{k=1}^{99} k = \frac{99 \cdot 100}{2} = 4950$$

Since there are finitely many pairs, the desired set is finite.

(iii) Let the set be S. Note that

$$\text{for every } q \in \mathbb{Q} \text{ there exists } \pm\sqrt{q} \in \mathbb{R} \quad \text{such that} \quad S = \bigcup_{q \in \mathbb{Q}_{\geq 0}} \{-\sqrt{q}, \sqrt{q}\}.$$

Since $\mathbb{Q}_{\geq 0}$ is countable and $\{-\sqrt{q}, \sqrt{q}\}$, then as S is the countable union of countable sets, it follows that S is countable. $\square$