MA2001 AY24/25 Sem 1 Final

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Question 1

For each of the following systems of equations, say whether there is no solution, a unique solution, or infinitely many solutions (you do not need to provide a solution). Make sure to justify your answers.

(a)

$$\begin{aligned} x-y+z&=1\\ 2x-y+z&=4\\ 4x-3y+3z&=3 \end{aligned}$$

(b)

$$\begin{aligned} x+y&=4\\ 3x+y&=10\\ x-y&=2 \end{aligned}$$

(c)

$$\begin{aligned} 2w-2y+3z&=-1\\ -x-y+4z&=2\\ -w+x-10y&=-6 \end{aligned}$$

Solution.

(a) Consider

$$\begin{pmatrix} 1 & -1 & 1 & 1\\ 2 & -1 & 1 & 4\\ 4 & -3 & 3 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last row corresponds to $0x+0y+0z=1$, which is a contradiction, so the system has no solution.

(b) By considering the first and third equation, we have $(x,y)=(3,1)$. Substituting this into the second equation, we have $3\cdot 3+1\cdot 1=10$, which implies that the system is consistent and has a unique solution.

(c) Consider

$$\begin{pmatrix} 2 & 0 & -2 & 3 & -1\\ 0 & -1 & -1 & 4 & 2\\ -1 & 1 & -10 & 0 & -6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & \frac{25}{24} & -\frac{1}{8}\\ 0 & 1 & 0 & -\frac{85}{24} & -\frac{19}{8}\\ 0 & 0 & 1 & -\frac{11}{24} & \frac{3}{8} \end{pmatrix}.$$

This corresponds to

$$w+\frac{25}{24}z=-\frac{1}{8} \quad x-\frac{85}{24}z=-\frac{19}{8} \quad y-\frac{11}{24}z=\frac{3}{8}.$$

So, we can set z to be a free variable, which yields infinitely many solutions to the system. $\square$

Question 2

Which of the following sets are linearly independent? Justify all answers.

(a) $\{(1,1,1),(1,2,2),(1,2,3)\}$

(b) $\{(1,1,1,1),(1,0,-1,1),(-1,2,5,-1)\}$

(c) The solution set of $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is square and invertible.

(d) The solution set of $\mathbf{Ax}=\mathbf{0}$, where $\mathbf{A}$ is square and invertible.

Solution.

(a) Yes. The coefficient matrix is

$$\begin{pmatrix}1&1&1\\1&2&2\\1&2&3\end{pmatrix}$$

which has determinant 1. Since the determinant is non-zero, by the invertible matrix theorem, the set is a basis for $\mathbb{R}^3$. We conclude that the set is linearly independent.

(b) Yes. Consider

$$c_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}+c_3\begin{pmatrix}-1\\2\\5\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

This yields

$$\begin{pmatrix}1&1&-1\\1&0&2\\1&-1&5\\1&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The first and fourth rows imply $c_1+c_2-c_3=0$ and $c_1+c_2+c_3=0$ respectively, so $c_3=0$. Hence, $c_1=-c_2$. By considering the second row, $c_1=0$, so $c_2=0$. Substituting $c_1=c_2=c_3=0$ into the third equation, we see that the system is consistent. Hence, the only solution is the trivial one, so the three vectors are linearly independent in $\mathbb{R}^4$.

(c) Yes[1] Since $\mathbf{A}$ is an invertible matrix, then $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$, which is the only solution to the equation. A set containing one non-zero vector is linearly independent, so the result follows.

[1] We are under the assumption that $\mathbf{b}\neq\mathbf{0}$.

(d) No. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then, $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, but $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent since each vector is a scalar multiple of the other. □

Question 3

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}.$$

(a) Find a basis for the column space of $\mathbf{A}$.

(b) Find a basis for the row space of $\mathbf{A}$.

(c) Find a basis for the null space of $\mathbf{A}$.

Solution.

(a) We have

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, only the first and third columns are pivot columns. As such, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(b) From the RREF in **(a)**, a basis would be

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(c) From the RREF in **(a)**, suppose (w, x, y, z) is contained in the nullspace of $\mathbf{A}$. Then,

$$\begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, $w+3x-z=0$ and $y+z=0$. Setting z to be a free variable, we have $y=-z$ and $w=z-3x$. This implies that x is another free variable, so

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3x+z \\ x \\ -z \\ z \end{pmatrix} = x\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

A basis for the nullspace is

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

$\square$

Question 4

Let

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Find the least squares solution to $\mathbf{Mx} = \mathbf{b}$.

(b) Find the orthogonal projection of $\mathbf{b}$ onto the column space of $\mathbf{M}$.

(c) Show that for any matrix $\mathbf{X}$ with linearly independent columns, the matrix $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is invertible.

Solution.

(a) Let $\mathbf{x} = (x,y,z)$. Consider $\mathbf{M}^{\mathrm{T}}\mathbf{Mx} = \mathbf{M}^{\mathrm{T}}\mathbf{b}$, so

$$\begin{pmatrix} 7 & 3 & 6 \\ 3 & 2 & 2 \\ 6 & 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 11 \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 & 3 & 6 \\ 3 & 2 & 2 \\ 6 & 2 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 3 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

As such, the least squares solution is $\mathbf{x} = (1,-1,1)$.

(b) The orthogonal projection is

$$\mathbf{p} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}.$$

Alternatively, one can work out manually by using the Gram-Schmidt process.

(c) Suppose $\mathbf{X}$ is an $n \times n$ matrix. Let the columns of $\mathbf{X}$ be $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Then,

$$\mathbf{X} = \begin{pmatrix} \mathbf{c}_1 & \ldots & \mathbf{c}_n \end{pmatrix} \quad \text{so} \quad \mathbf{X}^{\mathrm{T}} = \begin{pmatrix} \mathbf{c}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{c}_n^{\mathrm{T}} \end{pmatrix}.$$

As such,

$$\mathbf{X}^{\mathrm{T}}\mathbf{X} = \begin{pmatrix} \mathbf{c}_1^{\mathrm{T}}\mathbf{c}_1 & \mathbf{c}_1^{\mathrm{T}}\mathbf{c}_2 & \ldots & \mathbf{c}_1^{\mathrm{T}}\mathbf{c}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_n^{\mathrm{T}}\mathbf{c}_1 & \mathbf{c}_n^{\mathrm{T}}\mathbf{c}_2 & \ldots & \mathbf{c}_n^{\mathrm{T}}\mathbf{c}_n \end{pmatrix}$$

The trick is to let $\mathbf{y} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and compute the quadratic form associated with $\mathbf{X}^{\mathrm{T}}\mathbf{X}$, i.e.

$$\mathbf{y}^{\mathrm{T}} \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right) \mathbf{y} = \left(\sum_{i=1}^{n} a_i \mathbf{c}_i\right)^{\mathrm{T}} \left(\sum_{j=1}^{n} a_j \mathbf{c}_j\right) = \sum_{1 \leq i,j \leq n} a_i a_j \mathbf{c}_i^{\mathrm{T}} \mathbf{c_j} = \left\| \sum_{i=1}^{n} a_i \mathbf{c}_i \right\|^2.$$

Because the columns $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are linearly independent, the only way the linear combination can be the zero vector is by taking all coefficients $a_i = 0$. Thus, $\mathbf{y} \neq \mathbf{0}$ implies that

$$\left\| \sum_{i=1}^{n} a_i \mathbf{c}_i \right\|^2 > 0.$$

So, $\mathbf{y}^{\mathrm{T}} \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right) \mathbf{y} > 0$, i.e. $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is positive-definite. Positive-definiteness forces all eigenvalues to be strictly positive, so 0 cannot be an eigenvalue. By the invertible matrix theorem, $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is invertible. □

Question 5

Consider the matrix

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{S}\mathbf{S}^{\mathrm{T}}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{S}^{\mathrm{T}}\mathbf{S}$.

(c) Prove that for any $m \times n$ matrix $\mathbf{B}$, all eigenvalues of $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ and of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ are non-negative.

(d) Prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ and $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ share the same non-zero eigenvalues.

Solution.

(a) We have

$$\mathbf{S}\mathbf{S}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

which is a diagonal matrix, so the eigenvalues of $\mathbf{S}\mathbf{S}^{\mathrm{T}}$ are 2 and 1. One can deduce that the corresponding eigenvectors are $(1, 0)$ and $(0, 1)$ respectively.

(b) We have

$$\mathbf{S}^{\mathrm{T}}\mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

which has

$$\text{eigenvalues } 0,1,2 \quad \text{and} \quad \text{corresponding respective eigenvectors } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Let λ be an eigenvalue of $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ with corresponding eigenvector $\mathbf{v}$. Then,

$$\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{v} = \lambda\mathbf{v} \quad \text{so} \quad \mathbf{v}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{v} = \lambda\mathbf{v}^{\mathrm{T}}\mathbf{v} = \lambda\,\|\mathbf{v}\|^2.$$

Thus, $(\mathbf{B}\mathbf{v})^{\mathrm{T}}(\mathbf{B}\mathbf{v}) = \lambda\,\|\mathbf{v}\|^2$. Note that $\mathbf{B}\mathbf{v}$ is a column vector, say $(v_1,\ldots,v_m) \in \mathbb{R}^m$, so

$$(\mathbf{B}\mathbf{v})^{\mathrm{T}}(\mathbf{B}\mathbf{v}) = v_1^2 + \ldots + v_m^2.$$

By definition, an eigenvector must be non-zero, so $\|\mathbf{v}\|^2 > 0$, but the sum of squares $v_1^2 + \ldots + v_m^2$ is ≥ 0, which forces $\lambda \geq 0$.

Similarly, let μ be an eigenvalue of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ with corresponding eigenvector $\mathbf{w}$. Then,

$$\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{w} = \mu\mathbf{w} \quad \text{so} \quad \mathbf{w}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{w} = \mu\,\|\mathbf{w}\|^2.$$

Thus, $\left(\mathbf{B}^{\mathrm{T}}\mathbf{w}\right)^{\mathrm{T}}\left(\mathbf{B}^{\mathrm{T}}\mathbf{w}\right) = \mu\,\|\mathbf{w}\|^2$. In a similar fashion, note that $\mathbf{B}^{\mathrm{T}}\mathbf{w}$ is a column vector, say $(w_1,\ldots,w_n) \in \mathbb{R}^n$, so

$$\left(\mathbf{B}^{\mathrm{T}}\mathbf{w}\right)^{\mathrm{T}}\left(\mathbf{B}^{\mathrm{T}}\mathbf{w}\right) = w_1^2 + \ldots + w_n^2.$$

By definition, an eigenvector must be non-zero, so $\|\mathbf{w}\|^2 > 0$, but the sum of squares $w_1^2 + \ldots + w_n^2$ is ≥ 0, which forces $\mu \geq 0$.

(d) Let $\lambda \neq 0$ be an eigenvalue of $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ with corresponding eigenvector $\mathbf{v}$, so $\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{v} = \lambda\mathbf{v}$, so

$$\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}$$

so λ is an eigenvalue of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$. Conversely, let $\mu \neq 0$ be an eigenvalue of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ with corresponding eigenvector $\mathbf{w}$, so $\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{w} = \lambda\mathbf{w}$, so

$$\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{w} = \mu\mathbf{B}^{\mathrm{T}}\mathbf{w}.$$

Thus, μ is an eigenvalue of $\mathbf{B}^{\mathrm{T}}\mathbf{B}$. We conclude that $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ and $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ share the same non-zero eigenvalues. $\square$

Question 6

Consider the vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$.

(a) Prove that there exists a linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ with } \ker(T) = \{\mathbf{0}\} \quad \text{if and only if} \quad m \geq n.$$

(b) Prove that there exists a linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ with } R(T) = \mathbb{R}^m \quad \text{if and only if} \quad m \leq n.$$

Solution.

(a) For the forward direction, suppose $\ker(T) = \{\mathbf{0}\}$. Then, T is injective, so nullity $(T) = 0$. By the rank-nullity theorem, rank $(T) = n$. Since the image of T is a subspace of $\mathbb{R}^m$, then $n \leq m$.

For the reverse direction, suppose $m \geq n$. For any vector in $\mathbb{R}^m$, we note that we can write it as $(y_1, \ldots, y_n, y_{n+1}, y_m)$. Define

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{where} \quad T((x_1, \ldots, x_n)) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

Then, The first n standard basis vectors of $\mathbb{R}^m$ form the columns of the matrix representation T. The basis vectors are linearly independent, so T is injective and $\ker(T) = \{\mathbf{0}\}$.

(b) For the forward direction, suppose $R(T) = \mathbb{R}^m$. Then, rank $(T) = m$. By the rank-nullity theorem, nullity $(T) = n - m$. Since the dimension of any subspace is ≥ 0, then $n - m \geq 0$, so $n \geq m$.

For the reverse direction, suppose $n \geq m$. Consider the linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{where} \quad T((x_1, \ldots, x_n)) = (x_1, \ldots, x_m).$$

Essentially, T projects the vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ to its first m coordinates, so T is surjective because for any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, there exists $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that the claim holds. Hence, $R(T) = \mathbb{R}^m$. $\square$

Question 7

State whether each statement is **TRUE** or **FALSE**. No justification is required.

(a) A square matrix with a 0 on its diagonal is necessarily singular.

(b) A system $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A} \mid \mathbf{b})$.

(c) The solution set of $\mathbf{Ax} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$.

(d) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ with $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(e) If $\mathbf{A}$ is an $m \times n$ matrix with $m > n$, the system $\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ always has a solution.

(f) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent eigenvectors of $\mathbf{A}$, then applying Gram–Schmidt to them yields orthogonal eigenvectors of $\mathbf{A}$.

(g) An $n \times n$ matrix with n distinct eigenvalues must be diagonalizable.

(h) Every square upper-triangular matrix is diagonalizable.

(i) Every square upper-triangular matrix is orthogonally diagonalizable.

(j) A linear transformation must send $\mathbf{0}$ to $\mathbf{0}$.

Solution.

(a) False. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is non-singular since it has non-zero determinant.

(b) True[2]. To see why, write the augmented matrix and perform Gaussian elimination as follows:

$$\left(\mathbf{A} \mid \mathbf{b}\right) \xrightarrow{\text{REF}} \begin{pmatrix} * & \dots & * & \mid & * \\ 0 & \ddots & * & \mid & * \\ \vdots & \ddots & \ddots & \mid & \vdots \\ 0 & \dots & 0 & \mid & k \end{pmatrix}.$$

If the last pivots occur only inside $\mathbf{A}$, then no row is of the form $(0\,0\,\ldots 0 \mid k)$ with $k \neq 0$. The augmented column does not create a new pivot and the ranks stay equal, i.e. the system is consistent. On the other hand, if such a row appears, the augmented column introduces an extra pivot, so $\operatorname{rank}(\mathbf{A} \mid \mathbf{b}) = \operatorname{rank}(\mathbf{A}) + 1$ and the system is inconsistent.

(c) False. The statement is true if and only if $\mathbf{b} = \mathbf{0}$.

(d) False. Let $\mathbf{u} = (1,0), \mathbf{v} = (0,1), \mathbf{w} = (0,2)$.

(e) True. $\mathbf{A}^{\mathrm{T}}\mathbf{b}$ lies in $C\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)$ and $C\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) = C\left(\mathbf{A}^{\mathrm{T}}\right)$. So, the system is always consistent.

(f) False. The Gram–Schmidt produces orthogonal vectors in the same span, but each new vector is a linear combination of several eigenvectors except in the special case where the vectors

[2] A fun fact is that this is known as the Rouché–Capelli theorem.

already belong to mutually orthogonal eigenspaces, the orthogonality step *destroys* the eigenvector property. To see why, let

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad \text{where} \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

So, the eigenvectors are $(1,0)$ and $(1,1)$. By the Gram-Schmidt process, we obtain the orthogonal set $\{(1,0),(0,1)\}$ but $(0,1)$ is not an eigenvector of $\mathbf{A}$.

(g) True. For every eigenvalue, its corresponding eigenspace is one-dimensional, so the matrix is diagonalisable.

(h) False. To come up with a counterexample, we can come up with a 2×2 upper triangular matrix with an eigenvalue of multiplicity 2 but its eigenspace is one-dimensional[3]. For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then, $\mathbf{A}$ has an eigenvalue $\lambda = 1$ of algebraic multiplicity 2. However, its corresponding eigenspace $\operatorname{span}\{(1,0)\}$ is one-dimensional, so $\mathbf{A}$ is not diagonalisable.

(i) False. By the spectral theorem, a square matrix is orthogonally diagonalisable if and only if it is symmetric. As a counterexample, one can come up with an upper triangular matrix that is not symmetric.

(j) True. By definition of a linear transformation, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$. Setting $\mathbf{x} = \mathbf{y} = \mathbf{0}$, we have $T(\mathbf{0}) = 2T(\mathbf{0})$, so $T(\mathbf{0}) = \mathbf{0}$. $\square$

[3] We say that the algebraic multiplicity of the eigenvalue is 2 but the geometric multiplicity is 1.