MA2104 AY19/20 Sem 1 Final

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Question 1

Suppose that a parametrized curve $\mathbf{r}(t)$ satisfies $\mathbf{r}'(t) = \langle 1, t, e^t \rangle$. Find the curvature κ and the unit normal $\mathbf{N}$ to the curve when $t = 0$.

Solution. Recall that the curvature of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is given by

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

We have

$$\mathbf{r}(t) = \left\langle t, \frac{t^2}{2}, e^t \right\rangle \quad \mathbf{r}'(t) = \left\langle 1, t, e^t \right\rangle \quad \mathbf{r}''(t) = \left\langle 0, 1, e^t \right\rangle.$$

So, $\mathbf{r}(0) = \langle 0, 0, 1 \rangle$, $\mathbf{r}'(t) = \langle 1, 0, 1 \rangle$, and $\mathbf{r}''(t) = \langle 0, 1, 1 \rangle$. So,

$$\kappa = \frac{\|\langle 1, 0, 1 \rangle \times \langle 0, 1, 1 \rangle\|}{\|\langle 1, 0, 1 \rangle\|^3} = \frac{\sqrt{6}}{4}.$$

For the second part, we first form the unit tangent vector, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 1, t, e^t \rangle}{\sqrt{1 + t^2 + e^{2t}}}.$$

So,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

Note that

$$\mathbf{T}'(t) = \frac{(1 + t^2 + e^{2t})\langle 0, 1, e^t \rangle - (t + e^{2t})\langle 1, t, e^t \rangle}{(1 + t^2 + e^{2t})^{3/2}}.$$

So,

$$\mathbf{T}'(0) = \frac{\langle -1, 2, 1 \rangle}{2\sqrt{2}} \quad \text{which implies} \quad \left\|\mathbf{T}'(0)\right\| = \frac{\sqrt{3}}{2}.$$

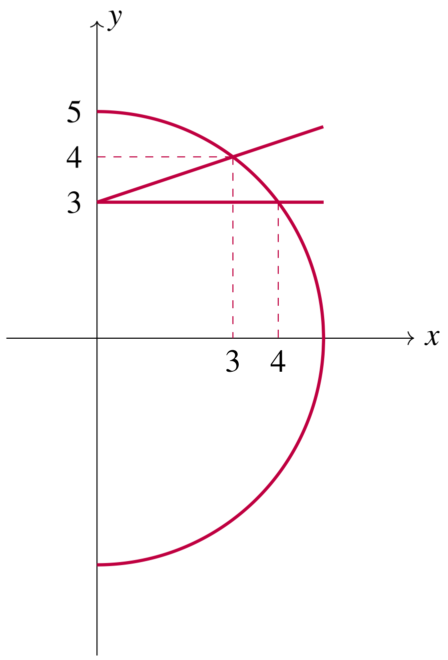
Consequently, $\mathbf{N}(t) = \frac{1}{\sqrt{6}}\langle -1, 2, 1\rangle$. $\square$

Question 2

Set up the following integral as an integral (or sum of integrals) with the order of integration "dydx"

$$\int_3^4 \int_{3y-9}^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Solution. Graphing out we have the following diagram:



Note that the region is

$$\left\{(x,y) \in \mathbb{R}^2 : 3y - 9 \le x \le \sqrt{25 - y^2} \text{ and } 3 \le y \le 4\right\}.$$

By considering $3y - 9 \le x$, we have $y \le \frac{x}{3} + 3$. Since $3 \le y$, then we have $3 \le y \le \frac{x}{3} + 3$. Conditioned to this, we have

$$3 \le \frac{x}{3} + 3 \le 4 \quad \text{so} \quad 0 \le x \le 3.$$

Next, by considering $x \le \sqrt{25 - y^2}$, we have $y^2 \le 25 - x^2$. Since $3 \le y \le 4$, then y is positive, so $3 \le y \le \sqrt{25 - x^2}$. Conditioned to this, we have

$$3 \le \sqrt{25 - x^2} \le 4 \quad \text{so} \quad 3 \le x \le 4.$$

To conclude, the integral is given by

$$\int_0^3 \int_3^{x/3+3} f(x,y)\mathrm{d}y\mathrm{d}x + \int_3^4 \int_3^{\sqrt{25-x^2}} f(x,y)\mathrm{d}y\mathrm{d}x$$

$\square$

Question 3

Evaluate the integral

$$\iiint_E z\mathrm{d}V$$

where E is the (solid) tetrahedron with vertices $(1,0,0)$, $(0,-1,0)$, $(0,1,0)$ and $(1,0,1)$.

Solution. Let $A(0,1,0)$, $B(0,-1,0)$, $C(1,0,0)$ and $D(1,0,1)$. The plane ABD is given by $z=x$ whereas the plane ABC is given by $x+y=1$ and the plane BCD is given by $x-y=1$. Thus we have

$$\begin{aligned}
\int_0^1 \int_0^x \int_{x-1}^{1-x} z\mathrm{d}y\mathrm{d}z\mathrm{d}x &= \int_0^1 \int_0^x z(2-2x)\mathrm{d}z\mathrm{d}x \\
&= \int_0^1 \frac{x^2}{2}(2-2x)\mathrm{d}x \\
&= \int_0^1 x^2 - x^3\mathrm{d}x \\
&= \frac{1}{12}
\end{aligned}$$

$\square$

Question 4

Let $\mathbf{F}$ be the two dimensional vector field given by

$$\mathbf{F}(x,y) = \langle ye^{xy} + y, xe^{xy} + x + 2y\rangle$$

(a) Find a potential function f for the vector field $\mathbf{F}$.

(b) Evaluate

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r},$$

where C is the upper half of the circle of radius 1 centered at the origin, traversed from the point $(1,0)$ to the point $(-1,0)$, followed by the line segment that runs from $(-1,0)$

to $(0,0)$.

Solution.

(a) We want to find some f such that $\nabla f = \mathbf{F}$. By

$$\begin{aligned} f_x = ye^{xy} + y &\Longrightarrow f = e^{xy} + xy + g(y) \\ f_y = xe^{xy} + x + 2y &\Longrightarrow f = e^{xy} + xy + y^2 + h(x) \end{aligned}$$

We can set $g(y) = y^2$ and $h(x) = 0$. This gives our potential function

$$f(x,y) = e^{xy} + xy + y^2.$$

(b) Since $\mathbf{F}$ is conservative, consider the line segment from $(0,0)$ to $(1,0)$, say ℓ. By the fundamental theorem of line integrals, we have

$$\int_{C+\ell} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

The parametrization of ℓ is given by $\langle t, 0 \rangle$ for $0 \leq t \leq 1$, so by the fundamental theorem of line integrals, we have

$$\int_\ell \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_\ell \nabla f \cdot \mathrm{d}\mathbf{r} = f(1,0) - f(0,0) = 0.$$

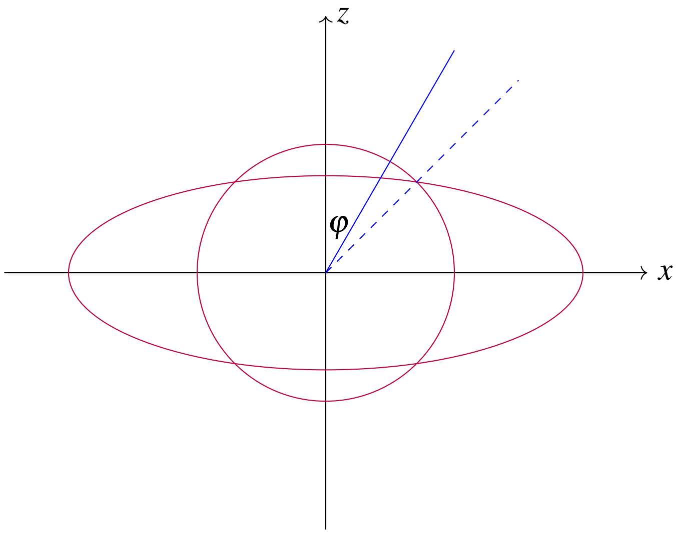
Hence we can conclude that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_\ell \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$ □

Question 5

Evaluate the given integral, where E is the solid region inside the sphere $x^2 + y^2 + z^2 = 1$, outside the ellipsoid $x^2 + y^2 + 7z^2 = 4$ and above the xy-plane.

$$\iiint_E \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \mathrm{d}V$$

Solution. We may use the spherical coordinates $\langle \rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi \rangle$. Notice that the graph is circular symmetry with respect to the z-axis. Consider the cross section along the xz-plane.



The line $z = x\cot\varphi$ intersects the ellipse at

$$\left(\sqrt{\frac{4}{1+7\cot^2\varphi}}, \sqrt{\frac{4\cot^2\varphi}{1+7\cot^2\varphi}}\right).$$

This point is $\sqrt{\frac{4}{1+6\cos^2\varphi}}$ away from the origin. When $\sqrt{\frac{4}{1+6\cos^2\varphi}} = 1$, we have $\varphi = \pi/4$. Hence the final integral is

$$\begin{aligned}
\iiint_E \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathrm{d}V &= \int_0^{2\pi}\int_0^{\pi/4}\int_{\sqrt{\frac{4}{1+6\cos^2\varphi}}}^1 \frac{\rho\cos\varphi}{(\rho^2)^{3/2}}\cdot\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\cos\varphi\sin\varphi\left(1-\sqrt{\frac{4}{1+6\cos^2\varphi}}\right)\mathrm{d}\varphi\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/4} 2\cos\varphi\sin\varphi\left(1-\sqrt{\frac{4}{7-6\sin^2\varphi}}\right)\mathrm{d}\varphi\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{1/2}\left(1-\sqrt{\frac{4}{7-6t}}\right)\mathrm{d}t\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\frac{1}{2}+\frac{2}{3}(2-\sqrt{7})\mathrm{d}\theta \\
&= \frac{\pi}{2}+\frac{2\pi}{3}(2-\sqrt{7})
\end{aligned}$$

which simplifies to $\frac{\pi}{6}\left(11-4\sqrt{7}\right)$. □

Remark: Alternatively, to obtain the bounds of integration in Question 5, by converting

$x^2+y^2+z^2=1$ to spherical coordinates, we have $\rho^2=1$. Since we are interested in the region inside this sphere, then $\rho \le 1$. Next, by converting $x^2+y^2+7z^2=4$ to spherical coordinates, we have $\rho^2+6\rho^2\cos^2\phi=4$ so $\rho \ge \frac{2}{\sqrt{1+6\cos^2\phi}}$. Hence,

$$\frac{2}{\sqrt{1+6\cos^2\phi}} \le \rho \le 1.$$

Next, to obtain the bounds for ϕ, we consider the intersection point of the sphere and the ellipsoid. Substituting $x^2+y^2=1-z^2$ into $x^2+y^2+7z^2=4$, we have $z^2=\frac{1}{2}$, so $z=\frac{1}{\sqrt{2}}$. So, $x^2+y^2=\frac{1}{2}$. Hence, $\tan\phi=\frac{z}{\sqrt{x^2+y^2}}=1$, so $\phi=\frac{\pi}{4}$. Consequently, $0\le\phi\le\frac{\pi}{4}$.

Question 6

Use the change of variables

$$u=\ln x+\ln y, \quad v=\frac{y}{x}$$

to evaluate the integral

$$\iint_D \frac{1}{y^2}\mathrm{d}x\mathrm{d}y$$

where d is the region in the first quadrant bounded by the curves $xy=e$, $xy=e^2$, $y=x$ and $y=2x$.

Solution. Note that $u=\ln(xy)$ so $xy=e^u$. Consequently, $x=\sqrt{\frac{e^u}{v}}$ and $y=\sqrt{e^u v}$, so the Jacobian determinant is given by

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{e^{u/2}}{2\sqrt{v}} & -\frac{e^{u/2}}{2\sqrt{v^3}} \\ \frac{e^{u/2}\sqrt{v}}{2} & \frac{e^{u/2}}{2\sqrt{v}} \end{vmatrix} = \frac{e^u}{2v}$$

and the bounds are given by $1\le u\le 2$ and $1\le v\le 2$. So the integral is given by

$$\iint_D \frac{1}{y^2}\mathrm{d}x\mathrm{d}y = \int_1^2\int_1^2 \frac{1}{e^u v}\left|\frac{e^u}{2v}\right|\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_1^2\int_1^2\frac{1}{v^2}\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_1^2\frac{1}{2}\mathrm{d}v = \frac{1}{4}.$$

□

Question 7

Consider the surface S that is the lateral surface of a cylinder, given by

$$x^2+y^2=1, \quad 0 \leq z \leq 1.$$

Endow S with the "outward" pointing normal, i.e., at any point $P(x,y,z)$ on S, the normal vector is $\mathbf{n}=\langle x,y,0\rangle$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z)=\left\langle \frac{\sin(y^3)}{y^3+z^3+1}, \frac{e^{x^2}}{x^4+z^4+1}, 1+z-2z^3 \right\rangle.$$

Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}.$$

Solution. By the divergence theorem,

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\cdot \mathrm{d}V = \iiint_E 1-6z^2 \mathrm{d}V.$$

Then, we can parametrize the cylinder (solid) by $\langle r\cos\theta, r\sin\theta, z\rangle$ with the bounds $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 1$. So, the above integral becomes

$$\int_0^{2\pi}\int_0^1\int_0^1 (1-6z^2) r \mathrm{d}z\mathrm{d}r\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^1 r\mathrm{d}r \cdot \int_0^1 1-6z^2 \mathrm{d}z = 2\pi\cdot\frac{1}{2}\cdot(-1) = -\pi.$$

□