MA2001 - Linear Algebra I
AY23/24 Sem 1 Suggested Solutions

Written by Thang Pang Ern
Audited by Sarji Elijah Bona

April 25, 2025

Question 1

Let $\mathbf{A}$ be a 4×5 matrix and $\mathbf{b}$ a 4×1 matrix. By applying Gauss-Jordan Elimination to

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array} \right)$$

the following series of elementary row operations were performed:

$$\begin{gathered} R_2 - 2R_1 \to R_2, \\ R_3 + R_1 \to R_3 \\ R_4 - R_2 \to R_4 \\ R_2 + R_3 \to R_2 \\ R_1 + R_2 \to R_1 \end{gathered}$$

and we obtained the reduced row-echelon form:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

(a) Find $\mathbf{A}$ and $\mathbf{b}$. (Indicate the elementary row operations used in each step.)

(b) Write down the solution set of the linear system $\mathbf{Ax} = \mathbf{b}$ explicitly.

Solution.

(a) We have

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2 \to R_1} \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - R_3 \to R_2} \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_4 + R_2 \to R_4} \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1 \to R_3} \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ -1 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 + 2R_1 \to R_2} \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 2 & -2 & -1 & -1 & 4 & 1 \\ -1 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

so

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 \\ 2 & -2 & -1 & -1 & 4 \\ -1 & 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(b) Based on the reduced row-echelon form, we have

$$\begin{aligned} x_1 - x_2 + x_5 &= 1 \\ x_3 &= 0 \\ x_4 - 2x_5 &= 1 \end{aligned}$$

So, $x_1 = 1 + x_2 - x_5$, $x_3 = 0$ and $x_4 = 1 + 2x_5$, which implies

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 + x_2 - x_5 \\ x_2 \\ 0 \\ 1 + 2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

To conclude, the solution set is

$$\left\{ \mathbf{v} \in \mathbb{R}^5 : \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, s, t \in \mathbb{R} \right\}. \qquad \square$$

Question 2

Let $\mathbf{B}$ be a square matrix of order n. Define

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{B}\mathbf{u} = \mathbf{B}^T\mathbf{u}\}.$$

(a) Show that V is a subspace of $\mathbb{R}^n$.

(b) Let $n = 4$ and

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 0 \end{pmatrix}.$$

(i) Use Gaussian Elimination to find a row-echelon form of $\mathbf{B}$ and hence write down a basis for the column space of $\mathbf{B}$.

(ii) Using the row-echelon form of $\mathbf{B}$ from part **(i)**, write down a basis for the column space of $\mathbf{B}^{\mathrm{T}}$.

(iii) Find a basis for V.

Solution.

(a) Note that $\mathbf{u} = \mathbf{0}$ satisfies $\mathbf{B}\mathbf{u} = \mathbf{B}^T\mathbf{u}$, so V is non-empty.

Next, let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Then,

$$\mathbf{B}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{B}\mathbf{u}_1 + \mathbf{B}\mathbf{u}_2 = \mathbf{B}^{\mathrm{T}}\mathbf{u}_1 + \mathbf{B}^{\mathrm{T}}\mathbf{u}_2 = \mathbf{B}^{\mathrm{T}}(\mathbf{u}_1 + \mathbf{u}_2)$$

so V is closed under addition.

Next, let $\alpha \in \mathbb{R}$ be arbitrary. Then,

$$\mathbf{B}(\alpha\mathbf{u}) = \alpha(\mathbf{B}\mathbf{u}) = \alpha\left(\mathbf{B}^{\mathrm{T}}\mathbf{u}\right) = \mathbf{B}^{\mathrm{T}}(\alpha\mathbf{u})$$

so V is closed under scalar multiplication.

(b) **(i)** We have

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 0 \end{pmatrix} \xrightarrow{-R_1+R_2\to R_2 \text{ and } -R_1+R_3\to R_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 0 \end{pmatrix}$$

$$\xrightarrow{-2R_2+R_3\to R_3 \text{ and } -2R_2+R_4\to R_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\xrightarrow{R_4\times(-1/6)\to R_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is a row-echelon form of $\mathbf{B}$. A basis for the column space of B is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 0 \end{pmatrix} \right\}.$$

(ii) This is precisely the same as finding a basis for the row space of $\mathbf{B}$, which is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(iii) Note that

$$\mathbf{B} - \mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{pmatrix}.$$

It suffices to find a basis for the null space of $\mathbf{B} - \mathbf{B}^{\mathrm{T}}$. Consider

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, $z = 0$, $2z = 0$, $3z = 0$ and $-w-2x-3y = 0$. These four equations imply $z = 0$ and $w = -2x-3y$. As such, we can write

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x-3y \\ x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We conclude that the desired basis is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\square$

Question 3

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1 = (0,1,0,1)$, $\mathbf{u}_2 = (-4,0,3,0)$ and $\mathbf{u}_3 = (3,-1,4,-1)$. Define $V = \operatorname{span}(S)$.

(a) Follow the Gram-Schmidt Process to transform S into an orthonormal basis for V.

(b) Find the projection of

$$b = (0,1,5,-1)$$

onto V.

(c) Find a least square solution to the system

$$\begin{pmatrix} 0 & -4 & 3 \\ 1 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix}.$$

Solution.

(a) We will apply the Gram-Schmidt process to transform $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis for $V = \operatorname{span}(S)$.

Set $\mathbf{v}_1 = \mathbf{u}_1$. Then,

$$\mathbf{v}_2 = \mathbf{u}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{u}_2) = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1.$$

Since $\mathbf{u}_2 \cdot \mathbf{v}_1 = 0$, then $\mathbf{v}_2 = \mathbf{u}_2$.

Then,

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{u}_3) - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{u}_3) \\ &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= \begin{pmatrix} 3 \\ -1 \\ 4 \\ -1 \end{pmatrix} - \frac{-2}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 0 \\ &= \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Lastly, we normalize $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to obtain

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix}.$$

We conclude that an orthonormal basis for V is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(b) We have

$$\operatorname{proj}_V(\mathbf{b}) = \sum_{i=1}^{3} \frac{\mathbf{b} \cdot \mathbf{e}_i}{\|\mathbf{e}_i\|^2} \mathbf{e}_i$$

which is equal to

$$\left(\begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix} \right) \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix}.$$

This simplifies to

$$\begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \end{pmatrix}.$$

(c) Let the coefficient matrix be $\mathbf{A}$. Then, the equation is of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$. By considering $\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$, we see that $\mathbf{x} = (4/5, 3/5, 4/5)$. $\square$

Question 4

Let

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) Show that the eigenvalues of $\mathbf{C}$ are $0, 1, 2$.

(b) Find a basis for the eigenspace of $\mathbf{C}$ associated with the eigenvalue 0.

(c) Find a basis for the eigenspace of $\mathbf{C}$ associated with the eigenvalue 1.

(d) Find a basis for the eigenspace of $\mathbf{C}$ associated with the eigenvalue 2.

(e) Write down two 4×4 matrices $\mathbf{P}$ and $\mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is an invertible matrix such that

$$\mathbf{P}^{-1}\mathbf{C}\mathbf{P} = \mathbf{D}.$$

Solution.

(a) We have

$$\mathbf{C}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 1 & -1 & -1\\ 1 & 1-\lambda & -1 & 1\\ 1 & 1 & -1-\lambda & 1\\ 0 & 0 & 0 & 2-\lambda\end{pmatrix}$$

so

$$\det(\mathbf{C}-\lambda\mathbf{I})=\lambda^2\left(\lambda^2-3\lambda+2\right)=\lambda^2(\lambda-1)(\lambda-2).$$

So, the eigenvalues are 0, 1, 2.

(b) When $\lambda=0$, we have

$$\mathbf{C}=\begin{pmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & 1\\ 1 & 1 & -1 & 1\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

One checks that a row-echelon form of $\mathbf{C}$ is

$$\begin{pmatrix}1 & 1 & -1 & -1\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Let (v_1,v_2,v_3,v_4) be an eigenvector corresponding to the eigenvalue 0. Then, $2v_4=0$, so $v_4=0$, and $v_1+v_2-v_3-v_4=0$, so $v_1+v_2=v_3$. As such, a basis for the eigenspace is

$$\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right\}.$$

(c) We have

$$\mathbf{C}-\mathbf{I}=\begin{pmatrix}0 & 1 & -1 & -1\\ 1 & 0 & -1 & 1\\ 1 & 1 & -2 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

A row echelon form of $\mathbf{C}-\mathbf{I}$ is

$$\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

So, a basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(d) We have

$$\mathbf{C} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A row-echelon form of $\mathbf{C} - 2\mathbf{I}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so we conclude that a basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(e)

$$\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

□

Question 5

Determine which of the following statements are true. Write down 'True' or 'False' in the boxes provided. Except part **(e)**, for each statement, if your answer is 'True', explain why the statement is always true; and if your answer is 'False', give a counter-example to justify that the statement is not always true.

(a) If a linear system $\mathbf{Ax} = \mathbf{b}$ has only one solution, then the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(b) If a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, then any linear system of the form $\mathbf{Ax} = \mathbf{b}$ has one solution.

(c) If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices of the same order, then $\mathbf{AB}$ is an orthogonal matrix.

(d) Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix}$$

be a square matrix, where $\mathbf{r}_i$ is the i-th row of $\mathbf{A}$. If $\mathbf{A}$ is invertible, then so is the matrix

$$\begin{pmatrix} \mathbf{r}_n \\ \mathbf{r}_{n-1} \\ \vdots \\ \mathbf{r}_1 \end{pmatrix}.$$

(e) There exists a 3×3 matrix $\mathbf{A}$ such that $\mathbf{A}^2 + \mathbf{A} = \mathbf{I}_3$ and 0 is an eigenvalue of $\mathbf{A}$.

Solution.

(a) True. If the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution, the matrix $\mathbf{A}$ is of full rank, which implies that its null space contains only the zero vector. The result follows.

(b) False. Consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which only has the trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^3$ but

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{has no solution.}$$

(c) True. Since $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, then $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$ and $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$. So,

$$(\mathbf{AB})^{\mathrm{T}}(\mathbf{AB}) = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(\mathbf{AB}) = \mathbf{B}^{\mathrm{T}}\mathbf{IB} = \mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}.$$

This implies $\mathbf{AB}$ is orthogonal.

(d) True. Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{r}_n \\ \mathbf{r}_{n-1} \\ \vdots \\ \mathbf{r}_1 \end{pmatrix}.$$

Observe that $\mathbf{B}$ can be obtained from $\mathbf{A}$ via a sequence of row swaps, i.e. $\mathbf{r}_1$ swaps with $\mathbf{r}_n$, $\mathbf{r}_2$ swaps with $\mathbf{r}_{n-1}$, and so on. Each of these row operations can be represented by some elementary matrix $\mathbf{E}$. As such, $\mathbf{B} = (\mathbf{E}_1 \ldots \mathbf{E}_i)\mathbf{A}$, which is a product of invertible matrices since each elementary matrix is invertible. So, $\mathbf{B}$ is invertible.

(e) False. We have $\mathbf{A}(\mathbf{A}+\mathbf{I}) = \mathbf{I}$, which implies $\mathbf{A}$ is invertible. However, 0 is an eigenvalue of $\mathbf{A}$, which contradicts the invertible matrix theorem. $\square$

Question 6

(a) Let $\mathbf{A}$ be a matrix. Prove that $\mathbf{A} = \mathbf{0}$ if and only if $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{0}$.

(b) Let T be a linear operator on $\mathbb{R}^n$. Suppose that $\operatorname{rank}(T) = \operatorname{rank}(T \circ T)$.

- **(i)** Prove that $\ker(T) = \ker(T \circ T)$.
- **(ii)** Prove that $\mathrm{R}(T) \cap \ker(T) = \{\mathbf{0}\}$.

(c) Find an example of a linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\mathrm{R}(S) = \ker(S)$. Can you find a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ with the same property? Justify your answers.

Solution.

(a) $(\implies)$ Suppose $\mathbf{A} = \mathbf{0}$. Then, $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{0} = \mathbf{0}$.

$(\impliedby)$ Suppose $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{0}$. Suppose $\mathbf{A}$ is an $m \times n$ matrix with entries a_{ij}. By considering the first row of $\mathbf{A}^{\mathrm{T}}$ and the first column of $\mathbf{A}$, we have

$$a_{11}^2 + a_{21}^2 + \ldots + a_{m1}^2 = 0.$$

Since the sum of squares is 0, it forces each square to be 0, and consequently, $a_{11} = a_{21} = \ldots = a_{m1} = 0$.

In general, by considering the j^{th} row of $\mathbf{A}^{\mathrm{T}}$ and the j^{th} column of $\mathbf{A}$, where $1 \leq j \leq n$, we have

$$a_{1j}^2 + a_{2j}^2 + \ldots + a_{mj}^2 = 0.$$

As such, $a_{1j} = a_{2j} = a_{mj} = 0$ for all $1 \leq j \leq n$. This forces $a_{ij} = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. So, $\mathbf{A} = \mathbf{0}$.

(b) **(i)** We first prove that

$$\ker(T) \subseteq \ker(T \circ T).$$

Suppose $\mathbf{v} \in \ker(T)$. Then, $T(\mathbf{v}) = \mathbf{0}$, so $T(T(\mathbf{v})) = T(\mathbf{0}) = \mathbf{0}$, so $\ker(T) \subseteq \ker(T \circ T)$.

Next, by the rank-nullity theorem, we have

$$n = \operatorname{rank}(T) + \dim(\ker(T)) \quad \text{and} \quad n = \operatorname{rank}(T \circ T) + \dim(\ker(T \circ T)).$$

Since $\operatorname{rank}(T) = \operatorname{rank}(T \circ T)$, then $\dim(\ker(T)) = \dim(\ker(T \circ T))$. Since $\mathbb{R}^n$ is a finite-dimensional vector space, it forces $\ker(T) \supseteq \ker(T \circ T)$. We conclude that $\ker(T) = \ker(T \circ T)$.

(ii) Suppose $\mathbf{w} \in \mathrm{R}(T) \cap \ker(T)$. Since $\mathbf{w} \in \mathrm{R}(T)$, then there exists $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{w}$. Also, since $\mathbf{w} \in \ker(T)$, then $T(\mathbf{w}) = \mathbf{0}$. So, $T^2(\mathbf{v}) = \mathbf{0}$, which implies $\mathbf{v} \in \ker(T \circ T)$.

Consequently, $\mathbf{v} \in \ker(T)$ by **(i)**. As such, $T(\mathbf{v}) = \mathbf{0}$, so $\mathbf{w} = \mathbf{0}$. We conclude that $\mathrm{R}(T) \cap \ker(T) = \{\mathbf{0}\}$.

(c) By the rank-nullity theorem, we have $\operatorname{rank}(S) + \operatorname{nullity}(S) = 4$, and because $\operatorname{rank}(S) = \operatorname{nullity}(S)$, then $\operatorname{rank}(S) = \operatorname{nullity}(S) = 2$. As such, any row-echelon form of the matrix representation of the linear transformation should contain 2 pivots. As such, we consider

$$S\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$$

We have $\mathrm{R}(S) = \operatorname{span}\{(1,0,0,0),(0,1,0,0)\}$ Also, $\ker(S) = \operatorname{span}\{(0,0,1,0),(0,0,0,1)\}$.

However, there does not exist any linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ with the same property, otherwise it would imply that $5 = 2\operatorname{rank}(T)$. However this is a contradiction as $\operatorname{rank}(T)$ must be a non-negative integer. $\square$