MA2108 - Mathematical Analysis I
AY24/25 Sem 1 Suggested Solutions

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Question 1

Find the following limits.

(a) $\lim_{n\to\infty} n^{3/2}\left(\sqrt{n+\sin\left(\frac{1}{n}\right)}-\sqrt{n}\right)$

(b) $\lim_{n\to\infty} \left(2^n+3^n\right)^{1/n}$

Solution.

(a) We have

$$\begin{aligned}
\lim_{n\to\infty} n^{3/2}\left(\sqrt{n+\sin\left(\frac{1}{n}\right)}-\sqrt{n}\right) &= \lim_{n\to\infty} \frac{n^{3/2}\left(n+\sin\left(\frac{1}{n}\right)-n\right)}{\sqrt{n+\sin\left(\frac{1}{n}\right)}+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{n^{3/2}\sin\left(\frac{1}{n}\right)}{\sqrt{n+\sin\left(\frac{1}{n}\right)}+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{n^{2}\sin\left(\frac{1}{n}\right)}{\sqrt{n^2+n\sin\left(\frac{1}{n}\right)}+n} \\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+\frac{\sin x}{x}}+\frac{1}{x}}\right) \quad \text{by letting } x=\frac{1}{n} \\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+\frac{\sin x}{x}}+\frac{1}{x}} \\
&= \lim_{x\to 0}\frac{1}{\sqrt{1+x\sin x}+1} \\
&= \frac{1}{2}
\end{aligned}$$

(b) Let the desired limit be S. Then,

$$\ln S = \lim_{n\to\infty}\frac{\ln\left(2^n+3^n\right)}{n}.$$

Using L'Hôpital's rule, we have

$$\ln S = \lim_{n\to\infty} \frac{2^n \ln 2 + 3^n \ln 3}{2^n + 3^n} = \lim_{n\to\infty} \frac{(2/3)^n \ln 2 + \ln 3}{(2/3)^n + 1} = \ln 3.$$

So, $S = 3$. $\square$

Question 2

Find the following limits.

(a) $\displaystyle \lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x$

(b) $\displaystyle \lim_{x\to 0} (2-\cos x)^{1/x^2}$

Solution.

(a) We have

$$\lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x = \lim_{x\to\infty} \left(1 + \frac{2}{x-1}\right)^x.$$

Recall the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

so we use the substitution

$$\frac{2}{x-1} = \frac{1}{u}.$$

As such, $x = 2u+1$ so

$$\lim_{x\to\infty} \left(1 + \frac{2}{x-1}\right)^x = \lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^{2u+1} = \lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^{2u} = \left[\lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^u\right]^2 = e^2.$$

(b) Let the desired limit be S. Then,

$$\ln S = \lim_{x\to 0} \frac{\ln(2-\cos x)}{x^2}.$$

Using L'Hôpital's rule, we have

$$\ln S = \lim_{x\to 0} \frac{\sin x}{2x(2-\cos x)} = \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{2-\cos x}\right) = \frac{1}{2}.$$

So, $S = \sqrt{e}$. $\square$

Question 3

Let a_n be a sequence satisfying the following conditions:

$$a_1 = \frac{5}{2} \quad \text{and} \quad a_{n+1} = 3 - \frac{4}{a_n^2}.$$

Prove that a_n converges and find the limit.

Solution. We first prove by strong induction that a_n is bounded below by 2. The base case is true. Suppose that the proposition holds for all $n \leq k$, where $k \in \mathbb{N}$. Then, $a_k^2 \geq 4$, so $3 - 4/a_k^2 \geq 2$. It implies that $a_{k+1} \geq 2$, so it follows that a_n is bounded below by 2.

We then prove that a_n is decreasing by strong induction, i.e. let $P(n)$ be the proposition that $a_{n+1} - a_n \leq 0$ for all $n \in \mathbb{N}$. Then, $a_2 - a_1 = -0.14 \leq 0$ so the base case $P(1)$ is true. Assume that the proposition holds for all $n \leq k$. Then, we wish to prove that $P(k+1)$ is true. So,

$$a_{k+1} - a_k = 3 - \frac{4}{a_k^2} - a_k = \frac{3a_k^2 - 4 - a_k^3}{a_k^2} = -\frac{(a_k - 2)^2 (a_k + 1)}{a_k^2}.$$

Since a_n is bounded below by 2, then $a_k + 1 \geq 3$. As $a_k^2, (a_k - 2)^2 \geq 0$ with $a_k \neq 0$, this implies $a_{k+1} - a_k \leq 0$, so it follows that a_n is decreasing.

As a_n is decreasing and bounded below, by the monotone convergence theorem, a_n converges. Suppose the limit is L. Then, L satisfies the equation $L = 3 - 4/L^2$. As such, $L^3 - 3L^2 + 4 = 0$. By the factor theorem, $L = 2$ is a root of this equation. The other roots are $L = 2$ and $L = -1$. However, as $a_n \geq 2$, it implies that a_n converges to 2. □

Question 4

Is the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$$

convergent or divergent? Make a claim and prove it.

Solution. The series is convergent. We shall use the alternating series test. Define

$$a_n = \frac{\sqrt{n}}{n+1}.$$

Then, it suffices to prove that

$$a_n \text{ is decreasing} \quad \text{and} \quad \lim_{n \to \infty} a_n = 0.$$

For the first claim, we have

$$a_{n+1} - a_n = \frac{\sqrt{n+1}}{n+2} - \frac{\sqrt{n}}{n+1} = \frac{(n+1)^{3/2} - (n+2)\sqrt{n}}{(n+1)(n+2)}.$$

As the denominator is always positive, it suffices to prove that the numerator is negative. We have

$$\begin{aligned} (n+1)^{3/2} < (n+2)\sqrt{n} \quad & \text{if and only if} \quad (n+1)^3 < n(n+2)^2 \\ & \text{if and only if} \quad n^3 + 3n^2 + 3n + 1 < n^3 + 4n^2 + 4n \\ & \text{if and only if} \quad 1 < n^2 + n \end{aligned}$$

Since $n^2 + n > 1$ holds for all $n \geq 1$ (consider index of the sequence a_n), the first claim follows, i.e. a_n is decreasing.

The second claim on the limit of a_n is clear as

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}=\lim_{n\to\infty}\sqrt{\frac{n}{(n+1)^2}}=0.$$

By the alternating series test, the series converges. $\square$

Question 5

Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that

(i) $f(x)\in[0,1]$ for all $x\in[0,1]$

(ii) $|f(x)-f(y)|<|x-y|$ for all $x,y\in[0,1]$ where $x\neq y$

Prove that there exists a unique $x_0\in(0,1]$ such that

$$f(x_0)=\frac{1-x_0}{x_0}.$$

Solution. We first prove the existence claim. Define

$$g(x)=f(x)-\frac{1-x}{x}\quad\text{with domain }(0,1].$$

Then, $g(1)=f(1)\geq 0$ and $g(\varepsilon)=f(\varepsilon)-(1-\varepsilon)/\varepsilon$, where $\varepsilon\in(0,1)$ is a positive number. Then, $f(\varepsilon)\to-\infty$ as $\varepsilon\to 0^+$. By the intermediate value theorem, there exists $x_0\in(0,1]$ such that $g(x_0)=0$. Equivalently, $f(x_0)=(1-x_0)/x_0$.

We then prove the uniqueness claim. Suppose on the contrary that there exist distinct $x_0,x_1\in[0,1]$ such that

$$f(x_0)=\frac{1-x_0}{x_0}\quad\text{and}\quad f(x_1)=\frac{1-x_1}{x_1}.$$

Then,

$$|f(x_0)-f(x_1)|=\left|\frac{1-x_0}{x_0}-\frac{1-x_1}{x_1}\right|=\left|\frac{x_1-x_0}{x_0x_1}\right|=\frac{1}{|x_0x_1|}\cdot|x_0-x_1|.$$

Since $x_0,x_1\in[0,1]$, then $1/|x_0x_1|$ is not bounded above, contradicting **(ii)**. It follows that x_0 is unique. $\square$

Question 6

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous at 0 and for any $x,y\in\mathbb{R}$,

$$f(x+y)=f(x)+f(y).$$

Prove that f is continuous everywhere.

Solution. Since f is continuous at $x=0$, then $g(x)=f(x+a)=f(x)+f(a)$ is continuous at $x=-a$. It follows that $f(x)=g(x)-f(a)$ is continuous at $x=-a$. Since $a\in\mathbb{R}$ is arbitrary, then f is continuous everywhere. $\square$

Question 7

Let a_n be a sequence such that

$$a_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \quad \text{diverges.}$$

Solution. We shall prove the contrapositive statement, i.e.

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \text{ converges} \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

So,

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_N/\left(1+a_N\right) \leq 1/2$, so $a_N \leq 1$. It follows that

$$\frac{a_N}{2} \leq \frac{a_N}{a_{N+1}}.$$

By applying the comparison test to a_N and a_N/a_{N+1}, it follows that the sum of a_n converges. $\square$

Question 8

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ such that

$$\left|f(x) - f(y)\right| \leq |x-y|^2 \quad \text{for any } x, y \in \mathbb{R}.$$

Prove that f is a constant function. That is, for any $x, y \in \mathbb{R}$, $f(x) = C$ for some $C \in \mathbb{R}$.

Solution. We have

$$\left|\frac{f(x) - f(y)}{x-y}\right| \leq |x-y|.$$

Letting x tend to y, by first principles, f is differentiable at x and $f'(x) = 0$, so f is a constant. $\square$