# MA2101S - Linear Algebra II (S) Suggested Solutions

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## Question 1

Let $F$ be a field, and let $V$ and $W$ be $F$-vector spaces (not necessarily finite dimensional). Let $T : V \to W$ be an $F$-linear map, and let $T^* : W^* \to V^*$ denote the transpose (or dual) of $T$, obtained by setting $T^*(g) := g \circ T$ for any $g \in W^*$.

(a) Show that $T$ is surjective if and only if $T^*$ is injective.

(b) Show that $T$ is injective if and only if $T^*$ is surjective.

**Solution:**

(a) If $T$ is surjective, let $f, g \in W^*$ such that $T^*(f) = T^*(g)$. Then for all $v \in V$, $f(T(v)) = g(T(v))$. Since $(f - g)(T(v)) = 0$ for all $v \in V$ and $T$ is surjective, $(f - g)(w) = 0$ for all $w \in W$, implying $f = g$ and $T^*$ is injective.

If $T$ is not surjective, then $\mathrm{Im}(T) \subsetneq W$. Extend a basis $\beta$ of $\mathrm{Im}(T)$ to a basis $\beta \cup \beta'$ of $W$. Let $f \in W^*$ satisfying $f(w_i) = 0\ \forall w_i \in \beta$ and $f(w_j) = 1\ \forall w_j \in \beta'$. Note that $T^*(f) = 0$, but $f \neq 0$, so $T^*$ is not injective. $\square$

(b) If $T$ is injective, note that there exists a linear map $S : \mathrm{Im}(T) \to V$ such that $S \circ T = \mathrm{id}_V$. Let $U$ be a subspace of $W$ such that $W = \mathrm{Im}(T) \oplus U$. Fix $f \in V^*$, and let $g \in W^*$ satisfying $g(u) = 0\ \forall u \in U$ and $g(w) = f(S(w))\ \forall w \in \mathrm{Im}(T)$. Then $(T^*(g))(v) = (g \circ T)(v) = g(T(v)) = f(S(T(v))) = f(v)\ \forall v \in V \Rightarrow T^*(g) = f$, hence, $T^*$ is surjective.

If $T^*$ is surjective, then $\mathrm{Im}(T^*) = V^*$. For any $f \in V^*$, there exists $g \in W^*$ such that $f = T^*(g) = g \circ T$. So for all $v \in \mathrm{Ker}(T)$, $f(v) = 0$. Assume that there exists nonzero $v \in \mathrm{Ker}(T)$, then we can construct $f \in V^*$ such that $f(v) = 1$, contradiction. Thus, $\mathrm{Ker}(T) = \{0\}$, implying that $T$ is injective. $\square$

# Question 2

Let $F$ be a field, let $V$ be a finite dimensional $F$-vector space, and let $T \in \mathcal{L}(V)$ be an $F$-linear operator on $V$. Show that there exists a vector $v \in V$ with the following property: for any polynomial $f \in F[t]$, if $f(T)v = 0$ for in $V$, then $f(T) = 0$ in $\mathcal{L}(V)$.

**Solution:**
Let

$$p_T(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \dots (\phi_k(t))^{m_k}$$

be the minimal polynomial of $T$ where $\phi_1(t), \phi_2(t), \dots, \phi_k(t)$ are distinct irreducible monic polynomials and $m_1, m_2, \dots, m_k$ are positive integers.
Then by the Primary Decomposition Theorem, we have

$$V = \bigoplus_{i=1}^{k} W_i,$$

where $W_i = \mathrm{Ker}(\phi_i(t)^{m_i})$ for all $i \in \{1, 2, \dots, k\}$, each of them being $T$-invariant.

For each $i \in \{1, 2, \dots, k\}$, let $u_i \in W_i \setminus \mathrm{Ker}(\phi_i(t)^{m_i - 1})$, and let $v = \displaystyle\sum_{i=1}^{k} u_i$.

Let $f$ be any polynomial such that $f(T)v = 0$. Then $\displaystyle\sum_{i=1}^{k} f(T)u_i = 0$, implying $f(T)u_i = 0$ for all $i \in \{1, 2, \dots, k\}$. Note that for all $i \in \{1, 2, \dots, k\}$, the $T$-annihilator of $u_i$ is $\phi_i(t)^{m_i}$, so $\phi_i(t)^{m_i} \mid f(t)$. Thus, $p_T(t) \mid f(t)$, implying that $f(T) = 0$. $\square$

# Question 3

Let $V$ be a 7-dimensional $\mathbb{C}$-vector space, and let $T \in \mathcal{L}(V)$ be a linear operator on $V$, with Jordan canonical form

$$\begin{pmatrix} 2 & 1 & 0 & & & & \\ 0 & 2 & 1 & & & & \\ 0 & 0 & 2 & & & & \\ & & & 2 & 1 & & \\ & & & 0 & 2 & & \\ & & & & & 3 & \\ & & & & & & 3 \end{pmatrix}.$$

For each eigenvalue $\lambda$ of $T$, we let $E_\lambda$ and $K_\lambda$ denote respectively the $\lambda$-eigenspace and the $\lambda$-generalized eigenspace of $T$, and we let $T|_{K_\lambda}$ denote the restriction of $T$ to $K_\lambda$.

Determine, with as little computation as possible,

(a) the characteristic polynomial of $T$;

and for each eigenvalue $\lambda$ of $T$:

(b) the dimensions $\dim(E_\lambda)$ and $\dim(K_\lambda)$;

(c) the smallest $p \in \mathbb{Z}_{>0}$ such that $K_\lambda = \operatorname{Ker}(T|_{K_\lambda} - \lambda)^p$;

(d) the dimensions $\dim \operatorname{Ker}(T|_{K_\lambda} - \lambda)$, $\dim \operatorname{Ker}(T|_{K_\lambda} - \lambda)^2$, and $\dim \operatorname{Ker}(T|_{K_\lambda} - \lambda)^3$.

**Solution:**

(a) the characteristic polynomial of the JCF of $T = \mathbf{(t-2)^5(t-3)^2}$

(b) $\dim(E_2) =$ number of Jordan blocks of eigenvalue $2 = \mathbf{2}$
$\dim(E_3) =$ number of Jordan blocks of eigenvalue $3 = \mathbf{2}$
$\dim(K_2) =$ number of times 2 appears in the diagonal of JCF $= \mathbf{5}$
$\dim(K_3) =$ number of times 3 appears in the diagonal of JCF $= \mathbf{2}$

(c) smallest $p$ for $K_2 =$ the size of the largest Jordan block corresponding to value $2 = \mathbf{3}$
smallest $p$ for $K_3 =$ the size of the largest Jordan block corresponding to value $3 = \mathbf{1}$

(d) $\dim \operatorname{Ker}(T|_{K_2} - 2\,\mathrm{id}_V) = \dim(E_2) = \mathbf{2}$
$\dim \operatorname{Ker}(T|_{K_3} - 3\,\mathrm{id}_V) = \dim(K_3) = \mathbf{2}$
There are 2 Jordan blocks of at least size 2 and eigenvalue 2, so
$\dim \operatorname{Ker}(T|_{K_2} - 2\,\mathrm{id}_V)^2 = \dim(E_2) + 2 = \mathbf{4}$
$\dim \operatorname{Ker}(T|_{K_3} - 3\,\mathrm{id}_V)^2 = \dim(K_3) = \mathbf{2}$
$\dim \operatorname{Ker}(T|_{K_2} - 2\,\mathrm{id}_V)^3 = \dim(K_2) = \mathbf{5}$
$\dim \operatorname{Ker}(T|_{K_3} - 3\,\mathrm{id}_V)^3 = \dim(K_3) = \mathbf{2}$ $\square$

# Question 4

Let $A \in \mathbb{M}_5(\mathbb{C})$ denote the $5 \times 5$ matrix

$$A := \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Determine a Jordan canonical form $J \in \mathbb{M}_5(\mathbb{C})$ of $A$, as well as an invertible matrix $Q \in \mathrm{GL}_5(\mathbb{C})$ such that $Q^{-1}AQ = J$.

**Solution:** Note that 2 is the only eigenvalue of $A$. We see that

$$A - 2I = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (A-2I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\dim \mathrm{Ker}(A-2I) = 3$, $\dim \mathrm{Ker}(A-2I)^2 = 4$, and $\dim \mathrm{Ker}(A-2I)^3 = 5$, so a JCF of $A$ contains 1 Jordan block of size 3 and 2 Jordan blocks of size 1.

Note that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathrm{Ker}(A-2I)^3 \setminus \mathrm{Ker}(A-2I)^2$ and $(A-2I)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, and

$(A-2I)^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and we see that $\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathrm{Ker}(A-2I)$.

Hence, $J$ is a JCF of $A$ and $Q^{-1}AQ = J$ where

$$\boxed{J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}.$$

$\square$

# Question 5

Determine a list of as many entries $A \in \mathbb{M}_8(\mathbb{R})$ as possible satisfying:

(i) the characteristic polynomial of each matrix is $(t-1)^4(t^2+3)^2$,

(ii) each matrix $A$ satisfies $(A-1)^2(A^2+3)^2 = 0$ in $\mathbb{M}_8(\mathbb{R})$,

(iii) no two matrices in the list are similar to each other over $\mathbb{R}$ (i.e. they are pairwise not $\mathrm{GL}_8(\mathbb{R})$-conjugate to each other).

**Solution:** The minimal polynomial of $A$ can only be $(t-1)(t^2+3), (t-1)^2(t^2+3)$, $(t-1)(t^2+3)^2$, or $(t-1)^2(t^2+3)^2$. We list all rational canonical forms satisfying (i), (ii):

*Case 1:* The minimal polynomial of $A$ is $(t-1)(t^2+3)$.
The only possible RCF is that with invariant factors $t-1 \mid t-1 \mid (t-1)(t^2+3) \mid (t-1)(t^2+3)$.

*Case 2:* The minimal polynomial of $A$ is $(t-1)^2(t^2+3)$.
There are two possible RCFs, one with invariant factors $(t-1)^2(t^2+3) \mid (t-1)^2(t^2+3)$ and another with invariant factors $t-1 \mid (t-1)(t^2+3) \mid (t-1)^2(t^2+3)$.

*Case 3:* The minimal polynomial of $A$ is $(t-1)(t^2+3)^2$.
The only possible RCF is that with invariant factors $t-1 \mid t-1 \mid t-1 \mid (t-1)(t^2+3)^2$.

*Case 4:* The minimal polynomial of $A$ is $(t-1)^2(t^2+3)^2$.
There are two possible RCFs, one with invariant factors $(t-1)^2 \mid (t-1)^2(t^2+3)^2$ and another with invariant factors $t-1 \mid t-1 \mid (t-1)^2(t^2+3)^2$.

These 6 RCFs give us the following list (the blank spaces are to be filled with 0s):

$$\begin{pmatrix} 0 & -3 &&&&&& \\ 1 & 0 &&&&&& \\ && 0 & -3 &&&& \\ && 1 & 0 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}, \begin{pmatrix} 0 & -3 &&&&&& \\ 1 & 0 &&&&&& \\ && 0 & -3 &&&& \\ && 1 & 0 &&&& \\ &&&& 0 & -1 && \\ &&&& 1 & 2 && \\ &&&&&& 0 & -1 \\ &&&&&& 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -3 &&&&&& \\ 1 & 0 &&&&&& \\ && 0 & -3 &&&& \\ && 1 & 0 &&&& \\ &&&& 0 & -1 && \\ &&&& 1 & 2 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -9 &&&& \\ 1 & 0 & 0 & 0 &&&& \\ 0 & 1 & 0 & -6 &&&& \\ 0 & 0 & 1 & 0 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -9 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & -6 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ & & & & 0 & -1 & & \\ & & & & 1 & 2 & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -9 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & -6 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ & & & & 0 & -1 & & \\ & & & & 1 & 2 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}.$$

Since two similar matrices must have the same RCF, this list satisfies condition (iii) and there can only be at most 6 entries in such a list. $\square$