MA2101S - Linear Algebra II (S) Suggested Solutions

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Question 1

Given a linear operator β on a vector space U, define $T_\beta(U)$ as follows:

$$T_\beta(U) := \{u \in U \mid \dim(\langle u \rangle_\beta) < \infty\}.$$

You may assume without proof that $\dim(\langle u \rangle_\beta) < \infty$ if and only if $f(\beta)(u) = 0_U$ for some nonzero $f(x) \in F[x]$.

Let α be a linear operator on a vector space V.

(a) Show that $T_\alpha(V)$ is a vector subspace of V.

(b) Show that $T_\alpha(V)$ is α-invariant.

(c) Let $\widetilde{V} = V/T_\alpha(V)$, and let $\widetilde{\alpha}$ be the linear operator on $\widetilde{V}$ defined by

$$\widetilde{\alpha}(v + T_\alpha(V)) = \alpha(v) + T_\alpha(V) \quad \text{for all } v \in V.$$

Prove that

$$T_{\widetilde{\alpha}}(\widetilde{V}) = \{0_{\widetilde{V}}\}.$$

Solution:

(a) Note that $\dim(\langle 0_V \rangle_\alpha) = \dim(\{0_V\}) = 0 < \infty$, so $0_V \in T_\alpha(V)$, so $T_\alpha(V)$ is non-empty. Let $v_1, v_2 \in T_\alpha(V)$, then there exist $f(x), g(x) \in F[x]$ such that $f(\alpha)(v_1) = 0_V$ and $g(\alpha)(v_2) = 0_V$. Then $(fg)(\alpha)(v_1 + v_2) = f(\alpha)(g(\alpha)(v_1 + v_2)) = f(\alpha)(g(\alpha)(v_1)) = g(\alpha)(f(\alpha)(v_1)) = g(\alpha)(0_V) = 0_V$, implying $v_1 + v_2 \in T_\alpha(U)$.

Now, let $\lambda \in F$ and $v \in T_\alpha(V)$, if $\lambda \neq 0$, then $\langle \lambda v \rangle_\alpha = \langle v \rangle_\alpha$, so $\lambda v \in T_\alpha(V)$. If $\lambda = 0$, $\lambda v = 0_V \in T_\alpha(V)$. Therefore, $T_\alpha(V)$ is a vector subspace of V. □

(b) Let $v \in T_\alpha(V)$. Then there exists $f(x) \in F[x]$ such that $f(\alpha)(v) = 0_V$. So $f(\alpha)(\alpha(v)) = (\alpha \circ f(\alpha))(v) = \alpha(0_V) = 0_V$. Hence, $\dim(\langle \alpha(v) \rangle_\alpha) < \infty$ and $\alpha(v) \in T_\alpha(V)$. □

(c) Obviously, $0_{\widetilde{V}} \in T_{\widetilde{\alpha}}(\widetilde{V})$. Assume for some $v \in V$ that $v + T_\alpha(V) \in T_{\widetilde{\alpha}}(\widetilde{V})$. So there exists $f(x) \in F[x]$ such that $f(\widetilde{\alpha})(v + T_\alpha(V)) = 0_{\widetilde{V}} = T_\alpha(V)$. Note that $f(\widetilde{\alpha})(v + T_\alpha(V)) = f(\alpha)(v) + T_\alpha(V)$, so this implies $f(\alpha)(v) \in T_\alpha(v)$. There exists $g(x) \in F[x]$ such that $g(\alpha)(f(\alpha)(v)) = (g \circ f)(\alpha)(v) = 0$. Thus, $v \in T_\alpha(V)$, implying $T_{\widetilde{\alpha}}(\widetilde{V}) \subseteq \{T_\alpha(V)\} = \{0_{\widetilde{V}}\}$. Therefore, $T_{\widetilde{\alpha}}(\widetilde{V}) = \{0_{\widetilde{V}}\}$. □

Question 2

Let α be a linear operator on a vector space V, and let U be an α-invariant vector subspace of V. Let $\widetilde{\alpha}$ be the linear operator on V/U defined by $\widetilde{\alpha}(v+U) = \alpha(v) + U$ for all $v \in V$. Suppose that

$$V/U = \bigoplus_{i\in I} \langle v_i + U\rangle_{\widetilde{\alpha}},$$

where for each $i \in I$, $\langle v_i + U\rangle_{\widetilde{\alpha}}$ is infinite-dimensional. (Note: I may be an infinite set.)

(a) Prove that for each $i \in I$, $\langle v_i\rangle_\alpha$ is infinite-dimensional.

(b) Show further that

$$V = U \oplus \bigoplus_{i\in I} \langle v_i\rangle_\alpha.$$

(You may assume without proof that $\langle v\rangle_\alpha$ is infinite-dimensional if and only if $f(\alpha)(v) \neq 0_V$ for all nonzero $f(x) \in F[x]$.)

Solution:

(a) Assume for some $i \in I$, $\langle v_i\rangle_\alpha$ is finite-dimensional, i.e., there exists $f(x) \in F[x]$ such that $f(\alpha)(v_i) = 0_V$. Then $f(\widetilde{\alpha})(v_i + U) = f(\alpha)(v_i) + U = 0_V + U = U = 0_{V/U}$, which contradicts $\langle v_i + U\rangle_{\widetilde{\alpha}}$ being infinite-dimensional. Thus, for each $i \in I$, $\langle v_i\rangle_\alpha$ is infinite-dimensional. $\square$

(b) Let $q : V \to V/U$ such that $q(v) = v + U$ for all $v \in V$. Then $q(f(\alpha)(v)) = f(\alpha)(v) + U = f(\widetilde{\alpha})(v + U)$ for any $f(x) \in F[x], v \in V$. We see for any $v \in V$, $v + U = \sum_{i\in I} f_i(\widetilde{\alpha})(v_i + U) = \sum_{i\in I} q(f_i(\alpha)(v_i)) = \sum_{i\in I} f_i(\alpha)(v_i) + U$, so $v - \sum_{i\in I} f_i(\alpha)(v_i) = u \in U$, implying $v = u + \sum_{i\in I} f_i(\alpha)(v_i) \in U + \sum_{i\in I}\langle v_i\rangle_\alpha$. Hence, $V = U + \sum_{i\in I}\langle v_i\rangle_\alpha$.

To show the sum is direct, we show $U \cap \sum_{\imath\in I}\langle v_i\rangle_\alpha = \{0_V\}$. Suppose $u \in U \cap \sum_{i\in I}\langle v_i\rangle_\alpha$. Then $u = \sum_{j\in J} f_j(\alpha)(v_j)$ for some finite $J \subseteq I$. So $q(u) = q(\sum_{j\in J} f_j(\alpha)(v_j)) = \sum_{j\in J} q(f_j(\alpha)(v_j)) \Rightarrow 0_{V/U} = \sum_{j\in J} f_j(\widetilde{\alpha})(v_j + U)$. Since $f_j(\widetilde{\alpha})(v_j + U) \in \langle v_j + U\rangle_{\widetilde{\alpha}}$ for some $j \in I$, we have $f_j(\widetilde{\alpha})(v_j + U) = 0_{V/U}$ for all $j \in J$ by the directness of $V/U = \bigoplus_{i\in I}\langle v_i + U\rangle_{\widetilde{\alpha}}$. But for each $i \in I$, since $\langle v_i + U\rangle_{\widetilde{\alpha}}$ is infinite-dimensional, f_j must be 0 for all $j \in J$. Hence, $u = \sum_{j\in J} f_j(\alpha)(v_j) = 0_V$ and $U \cap \sum_{\imath\in I}\langle v_i\rangle_\alpha = \{0_V\}$.

Therefore, $V = U \oplus \bigoplus_{i\in I}\langle v_i\rangle_\alpha$. $\square$

Question 3

Let α be a linear operator on a vector space V, and suppose that $V = \bigoplus_{j=1}^{n} \langle v_j \rangle_\alpha$ for some $v_1, \ldots, v_n \in V$ with $\dim(\langle v_j \rangle_\alpha) = \infty$ for all j.

(a) Prove that $\langle v \rangle_\alpha$ is infinite-dimensional for all $v \in V \setminus \{0_V\}$.

Let W be an α-invariant vector subspace of V, and let

$$V' = \bigoplus_{i=2}^{n} \langle v_i \rangle_\alpha.$$

(b) By considering the set $\Sigma := \{f(x) \in F[x] \mid f(\alpha)(v_1) \in W + V'\}$, or otherwise, show that there exists $f(x) \in F[x]$ such that

$$\{w + V' \mid w \in W\} = \langle f(\alpha)(v_1) + V' \rangle_{\widetilde{\alpha}_{V'}}.$$

Here, $\widetilde{\alpha}_{V'} : V/V' \to V/V'$ is defined by $\widetilde{\alpha}_{V'}(v + V') = \alpha(v) + V'$ for all $v \in V$.

(c) Let $U = W \cap V'$. Show that the following statements are equivalent:

(i) $f(x) = 0_{F[x]}$;

(ii) $W \subseteq V'$;

(iii) $U = W$.

(d) Assume first that $W \not\subseteq V'$. Let $w_1 \in W$ such that $f(\alpha)(v_1) + V' = w_1 + V'$. Show that:

(i) $w_1 \neq 0_V$;

(ii) $W/U = \langle w_1 + U \rangle_{\widetilde{\alpha}_U}$ (here $\widetilde{\alpha}_U : V/U \to V/U$ is defined by $\widetilde{\alpha}_U(v + U) = \alpha(v) + U$ for all $v \in V$);

(iii) $W = U \oplus \langle w_1 \rangle_\alpha$. (Hint: Use Question 2.)

(e) Now, disregard the assumption in (d) (so W may or may not be a subset of V'). Prove by induction on n, or otherwise, that

$$W = \bigoplus_{j=1}^{m} \langle w_j \rangle_\alpha$$

for some nonzero $w_1, \ldots, w_m \in W$ with $m \leq n$.

Solution:

(a) Assume there exists nonzero $v \in V$ such that $\langle v \rangle_\alpha$ is finite-dimensional. So there exists nonzero $f(x) \in F[x]$ satisfying $f(\alpha)(v) = 0_V$. Note that $v = \sum_{j=1}^{n} w_j$ where $w_j \in \langle v_j \rangle_\alpha$ for each $j \in \{1, 2, \ldots, n\}$. So $0_V = f(\alpha)(v) = f(\alpha)\left(\sum_{j=1}^{n} w_j\right) = \sum_{j=1}^{n} f(\alpha)(w_j)$. Since

$\bigoplus_{j=1}^{n}\langle v_j\rangle_\alpha$ is direct, $f(\alpha)(w_j) = 0_V$ for all $j \in \{1, 2, \ldots, n\}$. Also $v \neq 0_V$, so there must exist $k \in \{1, 2, \ldots, n\}$ such that $w_k \neq 0_V$. Then there exists $p(x) \in F[x]$ such that $p(\alpha)(v_k) = w_k \neq 0_V$, which implies $p(x) \neq 0$. Hence, $(f \circ p)(\alpha)(v_k) = f(\alpha)(w_k) = 0_V$. Since $(f \circ p)(x) \neq 0$, this contradicts $\langle v_k\rangle_\alpha$ being infinite-dimensional. Therefore, $\langle v\rangle_\alpha$ is infinite-dimensional for all $V \setminus \{0_V\}$.

(b) Consider the set $\Sigma := \{f(x) \in F[x] \mid f(\alpha)(v_1) \in W + V'\}$. Note that $0_{F[x]} \in \Sigma$ because $0_V \in W + V'$, so Σ is non-empty.

Now let $f(x) \in \Sigma$ with the least degree. Then $f(\alpha)(v_1) \in W + V'$ and since $W + V'$ is α-invariant, we have $h(\alpha)(f(\alpha)(v_1)) \in W + V'$ for all $h(x) \in F[x]$. Hence,

$$\begin{aligned}\langle f(\alpha)(v_1) + V'\rangle_{\widetilde{\alpha}_{V'}} &= \{h(\widetilde{\alpha}_{V'})(f(\alpha)(v_1) + V') \mid h(x) \in F[x] \\ &= \{h(\alpha)(f(\alpha)(v_1)) + V' \mid h(x) \in F[x]\} \subseteq \{w + V' \mid w \in W\}.\end{aligned}$$

On the other hand, if $w \in W$, then $w = \sum_{i=1}^{n} g_i(\alpha)(v_1)$ where $g_1(\alpha)(v_1) \in W + V'$, so $g_1(x) \in \Sigma$. Then $g_1(x) = q(x)f(x) + r(x)$ where $\deg r < \deg f$ or $r(x) = 0$. So $r(\alpha)(v_1) = (g_1(\alpha) - q(\alpha)f(\alpha))(v_1) \in W + V'$ since $g_1(x), f(x) \in \Sigma$ and $W + V'$ is α-invariant. Hence, $r(x) \in \Sigma$, and by the minimality of $f(x)$, we must have $r(x) = 0$. Thus, $g_1(x) = q(x)f(x)$, and therefore,

$$\begin{aligned}w + V' = g_1(\alpha)(v_1) + V' &= q(\alpha)(f(\alpha)(v_1)) + V' \\ &= q(\widetilde{\alpha}_{V'})(f(\alpha)(v_1) + V') \in \langle f(\alpha)(v_1) + V'\rangle_{\widetilde{\alpha}_{V'}}.\end{aligned}$$

Thus, $\{w + V' \mid w \in W\} = \langle f(\alpha)(v_1) + V'\rangle_{\widetilde{\alpha}_{V'}}$. □

(c) If (i) is true, $\{w + V' \mid w \in W\} = \langle 0_{V/V'}\rangle_{\widetilde{\alpha}_{V'}} = \{0_{V/V'}\}$. For all $w \in W$, $w \in V'$, hence, $W \subseteq V'$ and (ii) is true. If (ii) is true, then if $f(x) \neq 0_{F[x]}$, then $\langle w_1 + V'\rangle_{\widetilde{\alpha}_{V'}} = \{0_{V/V'}\}$ for some non-zero $w_1 \in \langle v_1\rangle_\alpha$. Since $V = \langle v_1\rangle_\alpha \oplus V'$, we have $w_1 \notin V'$, hence, $w_1 + V' \neq 0_{V/V'}$, contradiction. Thus, $f(x) = 0_{F[x]}$, and (i) is true. Hence, (i) $\Leftrightarrow$ (ii).

Obviously, $W \subseteq V' \Leftrightarrow W \cap V' = W$, so (ii) $\Leftrightarrow$ (iii). Therefore, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). □

(d) (i) If $w_1 = 0_V$, then $f(\alpha)(v_1) \in V'$. Since $f(\alpha)(v_1) \in \langle v_1\rangle_\alpha$ and $V = \langle v_1\rangle_\alpha \oplus V'$, we have $f(\alpha)(v_1) = 0$. Since $\langle v_1\rangle_\alpha$ is infinite-dimensional, we must have $f(x) = 0$. But part (c) showed this implies $W \subseteq V'$, contradiction. Thus, $w_1 \neq 0_V$. □

(ii) Note that for all $g(x) \in F[x]$, $g(\widetilde{\alpha}_U)(w_1 + U) = g(\alpha)(w_1) + U \in W/U$ since $w_1 \in W$ and W is α-invariant, so $\langle w_1 + U\rangle_{\widetilde{\alpha}_U} \subseteq W/U$. Now, for any $w \in W$, from part (b), there exists $h(x) \in F[x]$ such that $w + V' = h(\widetilde{\alpha}_{V'})(f(\alpha)(v_1) + V') = h(\widetilde{\alpha}_{V'})(w_1 + V') = h(\alpha)(w_1) + V'$. Then $h(\alpha)(w_1) - w \in V'$ and $h(\alpha)(w_1) - w \in W$, hence, $h(\alpha)(w_1) - w \in U$. Thus, $w + U = h(\alpha)(w_1) + U$, implying $W/U \subseteq \langle w_1 + U\rangle_{\widetilde{\alpha}_U}$ and $W/U = \langle w_1 + U\rangle_{\widetilde{\alpha}_U}$. □

(iii) If there exists $g(x) \in F[x]$ such that $g(\widetilde{\alpha}_U)(w_1 + U) = 0_{V/U}$, then $g(\alpha)(w_1) \in U \subseteq V'$. So $g(\alpha)(f(\alpha)(v_1)) + V' = g(\widetilde{\alpha}_{V'})(f(\alpha)(v_1) + V') = g(\widetilde{\alpha}_{V'})(w_1 + V') = g(\alpha)(w_1) + V' = 0_{V/U}$, implying $g(x)f(x) = 0$. Since $f(x) \neq 0_{F[x]}$ from $W \nsubseteq V'$ and part (c), we have $g(x) = 0_{F[x]}$. Therefore, $\langle w_1 + U\rangle_{\widetilde{\alpha}_{V'}}$ is infinite-dimensional, and by Question 2, we have $W = U \oplus \langle w_1\rangle_\alpha$. □

(e) We induct on n.
Base case: $n = 1$.
We have $V = \langle v_1 \rangle_\alpha$ for non-zero v_1. Let $W \subseteq V$ be α-invariant. If $W = \{0_V\}$, then it is a direct sum of 0 α-cyclic subspaces.
If $W \neq \{0_V\}$, then $V' = \{0_V\}$ and by part (b), there exists $f(x) \in F[x]$ such that $\langle f(\alpha)(v_1) + V' \rangle_{\widetilde{\alpha}_{V'}} = \{w + V' \mid w \in W\}$. If we let $w_1 = f(\alpha)(v_1)$, then there exists $w \in W$ such that $w_1 + V' = w + V' \Leftrightarrow w_1 = w$, so $w_1 \in W$ and $\langle w_1 \rangle_\alpha \subseteq W$.
For any $w \in W$, there exists $g(x) \in F[x]$ such that $w + V' = g(\alpha)(w_1) + V' \Leftrightarrow w = g(\alpha)(w_1) \in \langle w_1 \rangle_\alpha$. Therefore, $W = \langle w_1 \rangle_\alpha$ for non-zero w_1. We can express W as a direct sum of $m \leq n$ α-cyclic subspaces.

Induction step: Assume it is true for $n = k$.
Let $V = \bigoplus_{j=1}^{k+1} \langle v_j \rangle_\alpha$ and $V' = \bigoplus_{j=2}^{k+1} \langle v_j \rangle_\alpha$. If $W \subseteq V'$, then by the induction hypothesis, we can express W as the direct sum of at most k α-cyclic subspaces.
If $W \nsubseteq V'$, then by part (d), there exists non-zero $w_1 \in W$ such that $W = \langle w_1 \rangle_\alpha \oplus U$ where $U = W \cap V'$. Since U is an α-invariant vector subspace of V', by the induction hypothesis, we can express U as the direct sum of at most k α-cyclic subspaces.
Thus, W can be expressed as the direct sum of at most $k + 1$ α-cyclic subspaces.

Therefore by induction, for all n, W can be expressed as the direct sum of at most n α-cyclic subspaces. □

Question 4

Let V be a vector space, and denote the vector space of linear operators on V by $L(V, V)$. Let $\mathcal{A}$ be a vector subspace of $L(V, V)$, and let $f : \mathcal{A} \to F$ be a function. Suppose that

$$U := \{v \in V \mid \alpha(v) = f(\alpha)v \ \forall \alpha \in \mathcal{A}\} \neq \{0_V\}.$$

(a) Prove that:

(i) U is a vector subspace of V, and is α-invariant for all $\alpha \in \mathcal{A}$;

(ii) f is linear.

Let β be a linear operator on V, and assume that $\alpha \circ \beta - \beta \circ \alpha \in \mathcal{A}$ (but possibly $\alpha \circ \beta, \beta \circ \alpha \notin \mathcal{A}$) for all $\alpha \in \mathcal{A}$.

(b) Let $w \in U$ and assume that $\dim(\langle w \rangle_\beta) = k \in \mathbb{Z}^+$. Let $\mathcal{B}_0 = \emptyset$, and for each $1 \leq i \leq k$, let $\mathcal{B}_i = \{w, \beta(w), \ldots, \beta^{i-1}(w)\}$. You may assume without proof that $\mathcal{B}_k$ is a basis for $\langle w \rangle_\beta$.

(i) Show, by induction on i or otherwise, that

$$\alpha(\beta^i(w)) - f(\alpha)\beta^i(w) \in \operatorname{span}(\mathcal{B}_i)$$

for all $\alpha \in \mathcal{A}$ and $i \in \{0, 1, \ldots, k-1\}$.

(ii) Deduce that, for each $\alpha \in \mathcal{A}$, $\langle w \rangle_\beta$ is α-invariant, and write down all the information you can infer from (i) about the matrix representing the restricted linear operator $\alpha|_{\langle w \rangle_\beta}$ with respect to $\mathcal{B}_k$.

(iii) Hence, or otherwise, show that $f(\alpha \circ \beta - \beta \circ \alpha) = 0_F$ for all $\alpha \in \mathcal{A}$ when the characteristic of F does not divide k.

(Hint: $\alpha \circ \beta - \beta \circ \alpha \in \mathcal{A}$, and $\gamma \circ \delta - \delta \circ \gamma$ has zero trace whenever γ and δ are linear operators acting on the same finite-dimensional space.)

(c) Using (b)(iii), or otherwise, show that U is β-invariant when V is finite-dimensional and F has characteristic zero.

Solution:

(a) (i) Note that for any $\alpha \in \mathcal{A}$, we have $\alpha(0_V) = 0_V = f(\alpha) \cdot 0_V$, so $0_V \in U$ and U is non-empty. For any $v_1, v_2 \in U$, we have $\alpha(v_1) = f(\alpha)v_1$ and $\alpha(v_2) = f(\alpha)v_2$ for all $\alpha \in \mathcal{A}$, which implies $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2) = f(\alpha)v_1 + f(\alpha)v_2 = f(\alpha)(v_1 + v_2)$ for all $\alpha \in \mathcal{A}$, hence, $v_1 + v_2 \in U$. For any $v \in U$ and $\lambda \in F$, we have $\alpha(\lambda v) = \lambda\alpha(v) = \lambda f(\alpha)v = f(\alpha) \cdot (\lambda v)$ for all $\alpha \in \mathcal{A}$, hence, $\lambda v \in U$. Thus, U is a vector subspace of V. For any $\alpha \in \mathcal{A}$ and $v \in U$, we have $\beta(\alpha(v)) = \beta(f(\alpha)v) = f(\alpha)\beta(v) = f(\alpha)f(\beta)v = f(\beta)\alpha(v)$ for all $\beta \in \mathcal{A}$, implying $\alpha(v) \in U$. Therefore, U is an α-invariant vector subspace of V for all $\alpha \in \mathcal{A}$. $\square$

(ii) For any $\alpha, \beta \in \mathcal{A}$, $\lambda \in F$, and $v \in U \setminus \{0_V\}$, we have $f(\alpha + \beta)v = (\alpha + \beta)(v) = \alpha(v) + \beta(v) = f(\alpha)v + f(\beta)v \Rightarrow f(\alpha + \beta) = f(\alpha) + f(\beta)$, and $f(\lambda\alpha)v = (\lambda\alpha)(v) = \lambda\alpha(v) = \lambda f(\alpha)v \Rightarrow f(\lambda\alpha) = \lambda f(\alpha)$. Thus, f is linear. $\square$

(b) (i) **Base case:** $i = 0$.
For all $\alpha \in \mathcal{A}$, $\alpha(\beta^0(w)) - f(\alpha)\beta^0(w) = \alpha(w) - f(\alpha)w = 0_V \in \operatorname{span}(\mathcal{B}_0)$ since $w \in U$.
Induction step: Assume for all $\alpha \in \mathcal{A}$, $\alpha(\beta^i(w)) - f(\alpha)\beta^i(w) \in \operatorname{span}(\mathcal{B}_i)$ for some $i \in \{0, 1, \dots, k-2\}$.
Then

$$\begin{aligned}\alpha(\beta^{i+1}(w)) &= (\alpha \circ \beta)(\beta^i(w)) = (\beta \circ \alpha)(\beta^i(w)) + (\alpha \circ \beta - \beta \circ \alpha)(\beta^i(w)) \\ &= \beta(\alpha(\beta^i(w))) + f(\alpha \circ \beta - \beta \circ \alpha)\beta^i(w)\end{aligned}$$

and

$$\alpha(\beta^{i+1}(w)) - f(\alpha)\beta^{i+1}(w) = \beta(\alpha(\beta^i(w)) - f(\alpha)\beta^i(w)) + f(\alpha \circ \beta - \beta \circ \alpha)\beta^i(w).$$

Since $\alpha(\beta^i(w)) - f(\alpha)\beta^i(w) \in \operatorname{span}(\mathcal{B}_i)$, we have $\beta(\alpha(\beta^i(w)) - f(\alpha)\beta^i(w)) \in \operatorname{span}(\mathcal{B}_{i+1})$; also, $\beta^i(w) \in \mathcal{B}_{i+1}$, thus, $\alpha(\beta^{i+1}(w)) - f(\alpha)\beta^{i+1}(w) \in \operatorname{span}(\mathcal{B}_{i+1})$.
Therefore by induction on i, we have $\alpha(\beta^i(w)) - f(\alpha)\beta^i(w) \in \operatorname{span}(\mathcal{B}_i)$ for all $\alpha \in \mathcal{A}$ and $i \in \{0, 1, \dots, k-1\}$. $\square$

(ii) From part (i), for all $\alpha \in \mathcal{A}$ and $i \in \{0, 1, \dots, k-1\}$, we have $\alpha(\beta^i(w)) \in \langle w \rangle_\beta$ since $f(\alpha)\beta^i(w) \in \langle w \rangle_\beta$ and $\operatorname{span}(\mathcal{B}_i) \subseteq \langle w \rangle_\beta$. Since $\langle w \rangle_\beta = \operatorname{span}(\mathcal{B}_k)$, it is α-invariant.
The matrix $[\alpha|_{\langle w \rangle_\beta}]_{\mathcal{B}_k}$ is an upper-triangular $k \times k$ matrix with all diagonal entries being $f(\alpha)$. $\square$

(iii) Since $\alpha \circ \beta - \beta \circ \alpha \in \mathcal{A}$, the matrix $[\alpha \circ \beta - \beta \circ \alpha|_{\langle w \rangle_\beta}]_{\mathcal{B}_k}$ is an upper-triangular $k \times k$ matrix with all diagonal entries being $f(\alpha \circ \beta - \beta \circ \alpha)$, therefore, the trace of $\alpha \circ \beta - \beta \circ \alpha$ restricted to $\langle w \rangle_\beta$ is equal to $k \cdot f(\alpha \circ \beta - \beta \circ \alpha)$. This must equal 0_F, so we have $f(\alpha \circ \beta - \beta \circ \alpha) = 0_F$ since $\operatorname{char} F$ does not divide k. $\square$

(c) If V is finite-dimensional and F has characteristic zero, then for all $v \in U$ and $\alpha \in \mathcal{A}$, $(\alpha \circ \beta - \beta \circ \alpha)(v) = f(\alpha \circ \beta - \beta \circ \alpha)v = 0_V$, hence, $(\alpha \circ \beta)(v) = (\beta \circ \alpha)(v)$. Then $\alpha(\beta(v)) = \beta(\alpha(v)) = \beta(f(\alpha)v) = f(\alpha)\beta(v)$. Thus, $\beta(v) \in U$, implying that U is β-invariant. $\square$

Question 5

Let V be a vector space equipped with a nondegenerate symmetric bilinear form ϕ.

(a) Let $v \in V \setminus \{0_V\}$. Show that there exists a vector subspace W of V with $\dim(W) \leq 2$ such that $v \in W$ and $\phi|_{W \times W}$ is nondegenerate.

Now suppose that V is **infinite**-dimensional, and let U be a **finite**-dimensional vector subspace of V.
For any $X \subseteq V$, define $X^\perp := \{v \in V \mid \phi(v, x) = 0_F \ \forall x \in X\}$.

(b) Show by induction on $\dim(U)$, or otherwise, that there is a finite-dimensional vector subspace W of V with $U \subseteq W$ such that $\phi|_{W \times W}$ is nondegenerate.

(You may assume without proof that if X is a finite-dimensional vector subspace of V such that $\phi|_{X \times X}$ is nondegenerate, then $\phi|_{X^\perp \times X^\perp}$ is also nondegenerate.)

(c) Show further that $U^\perp = (U^\perp \cap W) \oplus W^\perp$.

(d) Hence, or otherwise, show that $(U^\perp)^\perp = (U^\perp \cap W)^\perp \cap W$.

(e) Deduce that $(U^\perp)^\perp = U$.

Solution:

(a) If $\phi(v, v) \neq 0_F$, then let $W = \operatorname{span}(\{v\})$, which has dimension 1.
If $\phi(v, v) = 0_F$, since ϕ is nondegenerate, there exists $u \in V \setminus \{0_V\}$ such that $\phi(u, v) \neq 0_F$. So we let $W = \operatorname{span}(\{v, u\})$, which has dimension at most 2, and that $\phi|_{W \times W}$ is nondegenerate. $\square$

(b) We induct on $\dim(U)$.
Base case: $\dim(U) = 0$.
Then $U = \{0_V\}$. We can let $W = U$ and $\phi|_{W \times W}$ is trivially nondegenerate.

Inductive step: Assume the statement holds for $\dim(U) \leq n$ where $n \in \mathbb{N}$.
Let $U \subseteq V$ have dimension $n + 1$. Let U' be an n–dimensional subspace of U. By the induction hypothesis, there exists a finite-dimensional subspace $W' \supseteq U$ such that $\phi|_{W' \times W'}$ is nondegenerate. If $U \subseteq W'$, then we can let $W = W'$. Otherwise, there exists $u \in U \setminus W'$. Since $V = W' \oplus W'^\perp$, $u = w_0 + w_1$ for some unique $w_0 \in W'$

and $w_1 \in W'^{\perp}$. Since $\phi|_{W' \times W'}$ is nondegenerate, $\phi|_{W'^{\perp} \times W'^{\perp}}$ must be nondegenerate. So by part (a), there must exist finite-dimensional X such that $w_1 \in X \subseteq W'^{\perp}$ and $\phi|_{X \times X}$ is nondegenerate. Now, let $W = W' + X$. Since $W' \cap W'^{\perp} = \{0_V\}$ and $X \subseteq W'^{\perp}$, we have $W = W' \oplus X$. So $u = w_0 + w_1 \in W' + X = W$ and $U = U' + \operatorname{span}(\{u\}) \subseteq W' + X = W$. Also, if there exists $w = w' + x$ such that $\phi(w, v) = 0_F$ for all $v \in W$, then $\phi(w' + x, w'_1 + x_1) = 0_F$ for all $w'_1 \in W', x_1 \in X$, implying $\phi(w', w'_1) + \phi(w', x_1) + \phi(x, w'_1) + \phi(x, x_1) = \phi(w', w'_1) + \phi(x, x_1) = 0_F$ for all $w'_1 \in W', x_1 \in X$. Hence, $\phi(w', w'_1) = 0_F$ and $\phi(x, x_1) = 0_F$ for all $w'_1 \in W', x_1 \in X$. But since $\phi|_{W' \times W'}$ and $\phi|_{X \times X}$ are both nondegenerate, this forces $w' = 0_V, x = 0_V$, and $w = 0_V$. Thus, $\phi|_{W \times W}$ is nondegenerate and W satisfies the conditions.

By induction, we are done. $\square$

(c) If $v \in W^{\perp}$, then $\phi(v, w) = 0$ for all $w \in W$. But $U \subseteq W$, so $\phi(v, u) = 0$ for all $u \in U$. Hence, $v \in U^{\perp}$, implying $W^{\perp} \subseteq U^{\perp}$. Thus, $U^{\perp} = V \cap U^{\perp} = (W^{\perp} \oplus W) \cap U^{\perp} = (W^{\perp} \cap U^{\perp}) \oplus (W \cap U^{\perp}) = W^{\perp} \oplus (W \cap U^{\perp})$. $\square$

(d) We prove the following claims:
Claim 1: For any subspaces A, B of V, $(A + B)^{\perp} = A^{\perp} \cap B^{\perp}$.
Proof: Let $v \in (A + B)^{\perp}$. Then for all $a \in A, b \in B$, we have $\phi(v, a + b) = 0_F$. But we can just set $a = 0_V$ or $b = 0_V$, giving us $\phi(v, a) = 0_F$ for all $a \in A$ and $\phi(v, b) = 0_F$ for all $b \in B$, implying $v \in A^{\perp} \cap B^{\perp}$ and $(A + B)^{\perp} \subseteq A^{\perp} \cap B^{\perp}$.

Now let $w \in A^{\perp} \cap B^{\perp}$. Each element c of $A + B$ can be expressed as $c = a + b$ for some $a \in A, b \in B$. But $\phi(w, a) = \phi(w, b) = 0_F$, so $\phi(w, c) = \phi(w, a) + \phi(w, b) = 0_F$. Hence, $w \in (A + B)^{\perp}$ and $A^{\perp} \cap B^{\perp} \subseteq (A + B)^{\perp}$. Therefore, $(A + B)^{\perp} = A^{\perp} \cap B^{\perp}$ and the claim is proven. $\blacksquare$

Claim 2: $(W^{\perp})^{\perp} = W$.
Proof: Note $\phi|_{W^{\perp} \times W^{\perp}}$ is nondegenerate and $V = W \oplus W^{\perp}$.
If $w \in W$, then for all $u \in W^{\perp}$, we must have $\phi(w, u) = 0_F$. Hence, $w \in (W^{\perp})^{\perp}$.
If $w \in (W^{\perp})^{\perp}$, then $w = w' + u$ for some $w' \in W$ and $u \in W^{\perp}$. It follows that $\phi(w, u') = 0_F$ for all $u' \in W^{\perp}$, so $\phi(w', u') + \phi(u, u') = 0_F$ for all $u' \in W^{\perp}$. But $\phi(w', u') = 0_F$, so $\phi(u, u') = 0_F$ for all $u' \in W^{\perp}$. This forces $u = 0_V$, so $w \in W$ and $(W^{\perp})^{\perp} \subseteq W$. Therefore, $(W^{\perp})^{\perp} = W$ and the claim is proven. $\blacksquare$

Since $U^{\perp} = (U^{\perp} \cap W) + W^{\perp}$ from part (c), by Claims 1 and 2, we have $(U^{\perp})^{\perp} = (U^{\perp} \cap W)^{\perp} \cap (W^{\perp})^{\perp} = (U^{\perp} \cap W)^{\perp} \cap W$. $\square$

(e) Note that W is a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form $\psi = \phi|_{W \times W}$. For any $X \subseteq W$, we have $X^{\perp_\psi} = \{w \in W \mid \psi(w, x) = 0_F \, \forall x \in X\} = X^{\perp} \cap W$. Then $(U^{\perp_\psi})^{\perp_\psi} = (U^{\perp_\psi})^{\perp} \cap W = (U^{\perp} \cap W)^{\perp} \cap W = (U^{\perp})^{\perp}$. Since $(U^{\perp_\psi})^{\perp_\psi} = U$ from W being finite-dimensional, we have $(U^{\perp})^{\perp} = U$. $\square$