MA2116 / ST2131 Final AY2425 Sem 1

Format: Examplify Secure, No Internet, Backward Navigation Enabled. Open book (printed material only). No calculators, but Examplify's Excel tool, scientific and graphing calculators are enabled. Phrasing may not be exactly the same as the actual exam.

Solutions may not be fully correct.

Question 1

Each part in this question is worth 1 mark.

A man and a woman agree to meet at a location at 12pm. The man arrives at the location at a time normally distributed with mean 11:50am and standard deviation 15 minutes. The woman arrives at the location at a time normally distributed with mean 12pm and standard deviation 10 minutes.

(a) What is the probability that the first person to arrive waits at least 5 minutes for the second person?

(b) What is the probability that the man arrives first?

Answer.

Let M and W denote the number of minutes after 12pm when the man and the woman arrive respectively. Then,

$$M \sim \mathrm{N}(-10, 15^2), \quad W \sim \mathrm{N}(0, 10^2) \implies M - W \sim \mathrm{N}(-10, 15^2 + 10^2)$$

as we can reasonably assume that M and W are independent.

Thus, by our calculator of choice, we obtain:

(a) $P\{|M - W| \geq 5\} = 1 - P\{-5 < M - W < 5\} \approx 81.19\%$,

(b) $P\{M < W\} = P\{M - W < 0\} \approx 71.05\%$. $\square$

Question 2

Each part in this question is worth 2 marks.

A box contains

- 3 coins of type A, each biased with a probability of 90% showing heads,
- 2 coins of type B, each biased with a probability of 60% showing heads,
- 1 coin of type C, each biased with a probability of 20% showing heads.

We pick one coin from the box and toss it 4 times.

(a) What is the probability that we see exactly 2 heads in the 4 tosses?

(b) Given that we see exactly 2 heads in the first 4 tosses, what is the probability that a coin of type B was selected?

We see exactly 2 heads in the first 4 tosses. Then, we toss the coin another 6 times.

(c) What is the expected number of heads we see in the 10 tosses?

(d) What is the probability that we observe at least 7 heads in all 10 tosses?

Answer.

Let A, B and C be the events where coin A, B and C are selected respectively, and let X_n be the number of heads seen in n tosses. Then, conditioned on the type of coin selected, $X_n \sim \text{Binom}(n, p)$ where p is the probability of heads of the selected coin.

(a) By the law of total probability,

$$P\{X_4 = 2\} = P\{X_4 = 2 \mid A\} \cdot P\{A\} + P\{X_4 = 2 \mid B\} \cdot P\{B\} + P\{X_4 = 2 \mid C\} \cdot P\{C\}$$

$$= \left[\binom{4}{2} 0.9^2 (1 - 0.9)^2\right] \cdot \frac{1}{2} + \left[\binom{4}{2} 0.6^2 (1 - 0.6)^2\right] \cdot \frac{1}{3}$$

$$+ \left[\binom{4}{2} 0.2^2 (1 - 0.2)^2\right] \cdot \frac{1}{6} = 16.51\%$$

Note: in the exam, we would just use a calculator to evaluate $P\{X_4 = 2 \mid A\}$.

*Parts **(b)** and **(c)** are on the next page.*

(b) By Bayes' Formula,

$$P\{B \mid X_4 = 2\} = \frac{P\{X_4 = 2 \mid B\} \cdot P\{B\}}{P\{X_4 = 2\}} = \frac{\left[\binom{4}{2} 0.6^2 (1 - 0.6)^2\right] \cdot \frac{1}{3}}{0.1651} = \frac{1152}{1651} \approx 69.78\%.$$

(c) In a similar manner to **(b)**, we may evaluate $P\{A \mid X_4 = 2\}$ and $P\{C \mid X_4 = 2\}$:

$$P\{A \mid X_4 = 2\} = \frac{243}{1651}, \quad P\{B \mid X_4 = 2\} = \frac{1152}{1651}, \quad P\{C \mid X_4 = 2\} = \frac{256}{1651}.$$

By the law of total expectation, we have

$$\begin{aligned}
\mathrm{E}[X_{10} \mid X_4 = 2] &= \mathrm{E}[(X_{10} \mid X_4 = 2) \mid A] \cdot P\{A \mid X_4 = 2\} \\
&\quad + \mathrm{E}[(X_{10} \mid X_4 = 2) \mid B] \cdot P\{B \mid X_4 = 2\} \\
&\quad + \mathrm{E}[(X_{10} \mid X_4 = 2) \mid C] \cdot P\{C \mid X_4 = 2\} \\
&= (\mathrm{E}[X_6 \mid A] + 2) \cdot P\{A \mid X_4 = 2\} + (\mathrm{E}[X_6 \mid B] + 2) \cdot P\{B \mid X_4 = 2\} \\
&\quad + (\mathrm{E}[X_6 \mid C] + 2) \cdot P\{C \mid X_4 = 2\} \\
&= (6 \cdot 0.9 + 2) \cdot \frac{243}{1651} + (6 \cdot 0.6 + 2) \cdot \frac{1152}{1651} + (6 \cdot 0.2 + 2) \cdot \frac{256}{1651} \\
&= \frac{45343}{8255} \approx 5.493.
\end{aligned}$$

*Part **(d)** is on the next page.*

(d) By the law of total probability, we have

$$\begin{aligned}
P\{X_{10} \geq 7 \mid X_4 = 2\} &= P\{(X_{10} \geq 7 \mid X_4 = 2) \mid A\} \cdot P\{A \mid X_4 = 2\} \\
&\quad + P\{(X_{10} \geq 7 \mid X_4 = 2) \mid B\} \cdot P\{B \mid X_4 = 2\} \\
&\quad + P\{(X_{10} \geq 7 \mid X_4 = 2) \mid C\} \cdot P\{C \mid X_4 = 2\} \\
&= P\{X_6 \geq 5 \mid A\} \cdot P\{A \mid X_4 = 2\} + P\{X_6 \geq 5 \mid B\} \cdot P\{B \mid X_4 = 2\} \\
&\quad + P\{X_6 \geq 5 \mid C\} \cdot P\{C \mid X_4 = 2\} \\
&\approx 29.34\%.
\end{aligned}$$

We use a calculator to compute $P\{X_6 \geq 5 \mid A\}$ and similar expressions in B and C. □

Question 3

Each part in this question is worth 1 mark.

There are 5 different colors of coupons. Each time a coupon is collected, it is equally likely to be any of the 5 different colors.

(a) What is the probability that exactly 8 coupons are collected until all 5 colors are collected (In other words, the 8th coupon collected is the last color collected)?

(b) What is the expected number of coupons collected in order to get all 5 colors?

(c) What is the expected number of different colors collected when 10 coupons are collected?

(d) If 10 coupons are collected, what is the expected number of changeovers? A changeover occurs when the most recently collected coupon has a different color from the previous one. For example, if the coupons collected are 123444321, then there are 6 changeovers.

Answer.

Parts **(a)**, **(b)** and **(c)** are done with formulae proved in lecture.

(a) Let T denote the number of coupons collected until one obtains all 5 colors. Then,

$$P\{T > n\} = \sum_{i=1}^{4} (-1)^{i+1} \cdot \binom{5}{i} \cdot \left(\frac{5-i}{5}\right)^n.$$

Using this result and $P\{T = n\} = P\{T > n-1\} - P\{T > n\}$ with $n = 8$, we get

$$P\{T = 8\} = \frac{336}{3125} \approx 10.75\%.$$

(b) From lecture, we have

$$\mathrm{E}[T] = 5 \sum_{i=1}^{5} \frac{1}{i} = \frac{137}{12} \approx 11.42.$$

*Parts **(c)** and **(d)** are on the next page.*

(c) Let D_n be the number of different colors collected when n coupons are collected. Then,

$$\mathrm{E}[D_{10}] = 5\left[1 - \left(1 - \frac{1}{5}\right)^{10}\right] \approx 4.463.$$

(d) Let Y denote the number of changeovers and I_k be an indicator variable for whether the k^{th} coupon was a changeover. Then,

$$Y = I_2 + \cdots + I_{10} \implies \mathrm{E}[Y] = \sum_{k=2}^{10} \mathrm{E}[I_k] = \sum_{k=2}^{10} P\{I_k = 1\} = 9 \cdot \frac{4}{5} = 7.2. \qquad \square$$

Question 4

Each part in this question is worth 4 marks.

An urn contains

- 4 red balls,
- 5 blue balls, and
- 6 white balls.

We draw 8 balls from the urn without replacement. Let X and Y denote the number of red and blue balls drawn respectively.

(a) Find $\mathrm{E}[XY]$.

(b) Find $\rho(X, Y)$, the correlation coefficient between X and Y.

Answer.

(a) Arbitrarily number the red balls from 1 to 4 and the blue balls from 1 to 5 and define

$$X_i = \begin{cases} 1 & i^{\text{th}} \text{ red ball is chosen,} \\ 0 & \text{otherwise,} \end{cases}, \qquad Y_j = \begin{cases} 1 & j^{\text{th}} \text{ blue ball is chosen,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, 4$ and $j = 1, \ldots, 5$.Then, for any pair i, j,

$$\mathrm{E}[X_i Y_j] = P\left\{X_i Y_j = 1\right\} = P\left\{X_i = 1 \text{ and } Y_j = 1\right\} = \frac{\binom{13}{6}}{\binom{15}{8}} = \frac{4}{15}$$

so

$$\mathrm{E}[XY] = \sum_{i=1}^{4} \sum_{j=1}^{5} \mathrm{E}[X_i Y_j] = 20 \cdot \frac{4}{15} = \frac{16}{3} \approx 5.333.$$

*Part **(b)** is on the next page.*

(b) First note that X and Y are hypergeometric with parameters $(8, 15, 4)$ and $(8, 15, 5)$ respectively. If Z is hypergeometric with parameters (n, N, m), then

$$\mathrm{E}[Z] = \frac{nm}{N}, \qquad \mathrm{Var}(Z) = \frac{nm}{N}\left(1 - \frac{m}{N}\right)\left(1 - \frac{n-1}{N-1}\right).$$

Thus, we have

$$\mathrm{E}[X] = \frac{32}{15}, \qquad \mathrm{E}[Y] = \frac{8}{3}, \qquad \mathrm{Var}(X) = \frac{176}{225}, \qquad \mathrm{Var}(Y) = \frac{8}{9}.$$

Thus,

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \approx -0.4264. \qquad \square$$

Question 5

Each part in this question is worth 2 marks.

A six-sided fair die is rolled. Let N be the value of the die. Let P be the probability of a biased coin showing heads, where P is chosen randomly between 0 and 1. We flip the coin N times. Let X be the number of heads we see in the N flips.

(a) Find $P\{X = 5\}$.

(b) Find $P\{X = 6\}$.

Answer.

We shall find $P\{X = x\}$ in general.

$$\begin{aligned}
P\{X = x\} &= \sum_{n=1}^{6} P\{X = x \mid N = n\} \cdot P\{N = n\} \\
&= \sum_{n=1}^{6} \int_0^1 P\{(X = x \mid N = n) \mid P = p\} \cdot f_P(p)\,\mathrm{d}p \cdot \frac{1}{6} \\
&= \sum_{n=1}^{6} \frac{1}{6} \int_0^1 \binom{n}{x} p^x (1-p)^{n-x}\,\mathrm{d}p \\
&= \sum_{n=\max\{x,1\}}^{6} \frac{1}{6} \cdot \frac{n!}{x! \cdot (n-x)!} \cdot \frac{x! \cdot (n-x)!}{(n+1)!} \\
&= \sum_{n=\max\{x,1\}}^{6} \frac{1}{6(n+1)}
\end{aligned}$$

Thus,

(a) $P\{X = 5\} = \dfrac{1}{36} + \dfrac{1}{42} = \dfrac{13}{252} \approx 5.159\%$,

(b) $P\{X = 6\} = \dfrac{1}{42} \approx 2.381\%$. □

Question 6

Each part in this question is worth 2 marks.

Let X and Y be random variables with the joint distribution density function

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P\left\{\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{2}\right\}$.

(b) Find $P\left\{\frac{1}{4} < Y < \frac{1}{2} \mid X = \frac{1}{2}\right\}$.

(c) Find $\mathrm{Cov}(X, Y)$.

Answer.

We first find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^x 8xy\,\mathrm{d}y = 4x^3,\ x \in (0,1), \qquad f_Y(y) = \int_y^1 8xy\,\mathrm{d}x = 4y - 4y^3,\ y \in (0,1).$$

(a) Since $x \in (1/4, 1/2)$ and $y = 1/2 \implies x < y \implies f(x,y) = 0$, this probability is 0.

(b) We have, conditioned on $X = 1/2$, $f_{Y|X}(y) = \dfrac{f(1/2, y)}{f_X(1/2)} = 8y$. Thus,

$$P\left\{\frac{1}{4} < Y < \frac{1}{2} \mid X = \frac{1}{2}\right\} = \int_{1/4}^{1/2} 8y\,\mathrm{d}y = \frac{3}{4} = 75\%.$$

(c) We have:

$$\mathrm{E}[X] = \int_0^1 x f_X(x)\,\mathrm{d}x = \int_0^1 4x^4\,\mathrm{d}x = \frac{4}{5},$$

$$\mathrm{E}[Y] = \int_0^1 y f_Y(y)\,\mathrm{d}y = \int_0^1 4y^2 - 4y^4\,\mathrm{d}y = \frac{8}{15},$$

$$\mathrm{E}[XY] = \int_0^1 \int_0^x xy f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x 8x^2y^2\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{8}{3}x^5\,\mathrm{d}x = \frac{4}{9}.$$

Thus, $\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = \dfrac{4}{225} \approx 0.01778.$ □

Question 7

Each part in this question is worth 4 marks.

The random variables X and Y are jointly continuous with joint density

$$f(x,y) = \begin{cases} 8/x^3y^5 & \text{if } x \geq 1,\ y \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P\left\{\frac{1}{2} < XY < 2 \text{ and } \frac{1}{2} < X/Y < 2\right\}$.

(b) Find $P\{2 < XY < 4 \text{ and } 1 < X/Y < 2\}$.

Answer.

Let $U = XY$ and $V = X/Y$. We first find the joint density g of U and V.

The Jacobian of this transformation is

$$J(u,v) = \begin{vmatrix} y & x \\ 1/y & -x/y^2 \end{vmatrix} = -\frac{2x}{y}.$$

Thus,

$$g(u,v) = f(x,y)|J(u,v)|^{-1} = \frac{8}{x^3y^5} \cdot \frac{y}{2x} = \frac{4}{u^4}.$$

To obtain its domain, we have $x \geq 1$ and $y \geq 1 \implies u = xy \geq 1$. Then,

$$uv = x^2 \geq 1 \implies v \geq \frac{1}{u}, \qquad \frac{u}{v} = y^2 \geq 1 \implies v \leq u.$$

In total, we have

$$g(u,v) = \begin{cases} 4/u^4 & \text{if } u \geq 1 \text{ and } 1/u \leq v \leq u \\ 0 & \text{otherwise.} \end{cases}$$

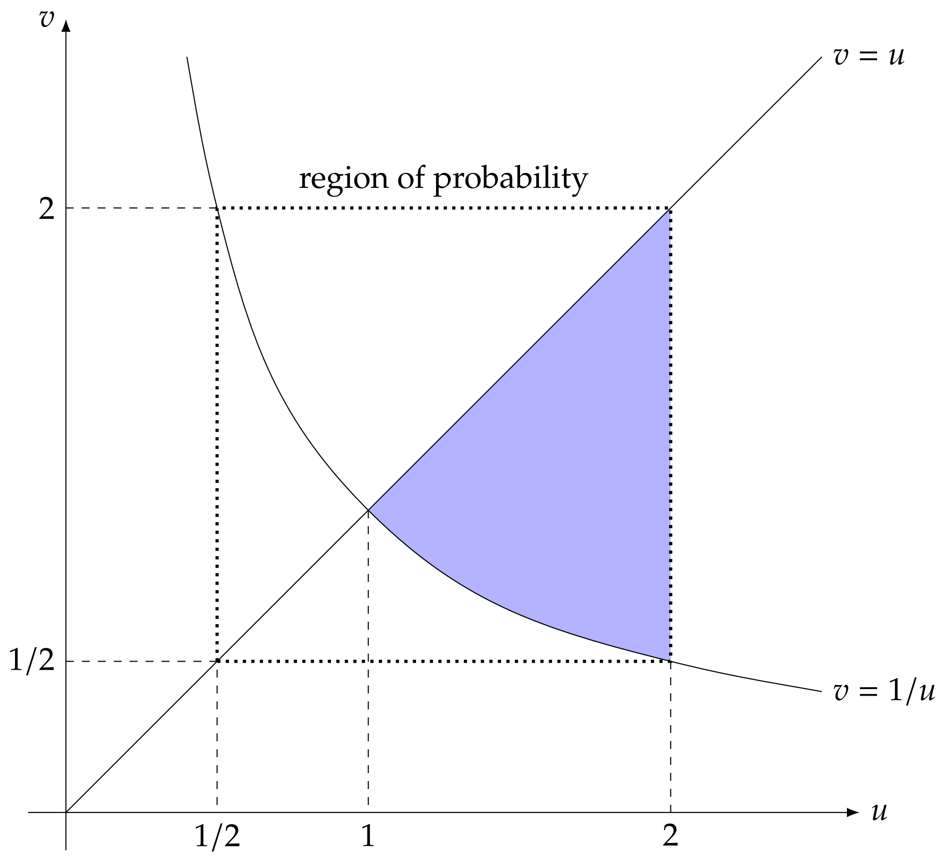
*Parts **(a)** and **(b)** are on the next page.*

(a) With reference to a graph (below), we have

$$P\left\{\frac{1}{2} < XY < 2 \text{ and } \frac{1}{2} < X/Y < 2\right\} = P\left\{\frac{1}{2} < U < 2 \text{ and } \frac{1}{2} < V < 2\right\}$$

$$= \int_1^2 \int_{1/u}^u \frac{4}{u^4}\,\mathrm{d}v\,\mathrm{d}u$$

$$= \int_1^2 \frac{4}{u^3} - \frac{4}{u^5}\,\mathrm{d}u = \frac{9}{16} = 56.25\%.$$



(b) The region $2 < u < 4$ and $1 < v < 2$ is entirely within the domain of g. Thus,

$$P\{2 < XY < 4 \text{ and } 1 < X/Y < 2\} = P\{2 < U < 4 \text{ and } 1 < V < 2\}$$

$$= \int_2^4 \int_1^2 \frac{4}{u^4}\,\mathrm{d}v\,\mathrm{d}u$$

$$= \int_2^4 \frac{4}{u^4}\,\mathrm{d}u = \frac{7}{48} \approx 14.56\%. \qquad \square$$

Question 8

Each part in this question is worth 2 marks.

X and Y are independent uniform random variables on $[0, 1]$. Find

(a) $P\{2 < 5X + 3Y < 4\}$,

(b) $P\{4 < 5X + 3Y < 6\}$.

Answer.

We first find the density function f of $5X + 3Y$. Define $U = X$ and $V = 5X + 3Y$ and let g be the joint density function of U and V. The Jacobian of this transformation is

$$\begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} = 3.$$

Thus, $g(u, v) = \dfrac{1}{3}$. To find the domain, we note $u = x \in [0, 1]$. Then,

$$v = 5u + 3y \text{ and } y \in [0, 1] \implies v \in [5u, 5u + 3].$$

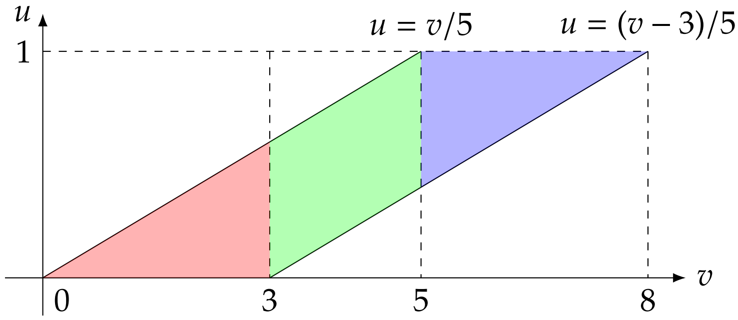
Thus,

$$g(u, v) = \begin{cases} 1/3 & \text{if } 0 \leq u \leq 1 \text{ and } 5u \leq v \leq 5u + 3 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, by referencing a graph (on the next page),

$$f(v) = \int_{\mathbb{R}} g(u, v)\, du = \begin{cases} \displaystyle\int_0^{v/5} \frac{1}{3}\, du & \text{if } 0 \leq v < 3 \\ \displaystyle\int_{(v-3)/5}^{v/5} \frac{1}{3}\, du & \text{if } 3 \leq v < 5 \\ \displaystyle\int_{(v-3)/5}^{1} \frac{1}{3}\, du & \text{if } 5 \leq v \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

This is continued on the next page.



We solve these integrals to get

$$f(v) = \int_{\mathbb{R}} g(u, v)\, \mathrm{d}u = \begin{cases} \dfrac{v}{15} & \text{if } 0 \le v < 3 \\ \dfrac{1}{5} & \text{if } 3 \le v < 5 \\ \dfrac{8-v}{15} & \text{if } 5 \le v \le 8 \\ 0 & \text{otherwise.} \end{cases}$$

(a) We have

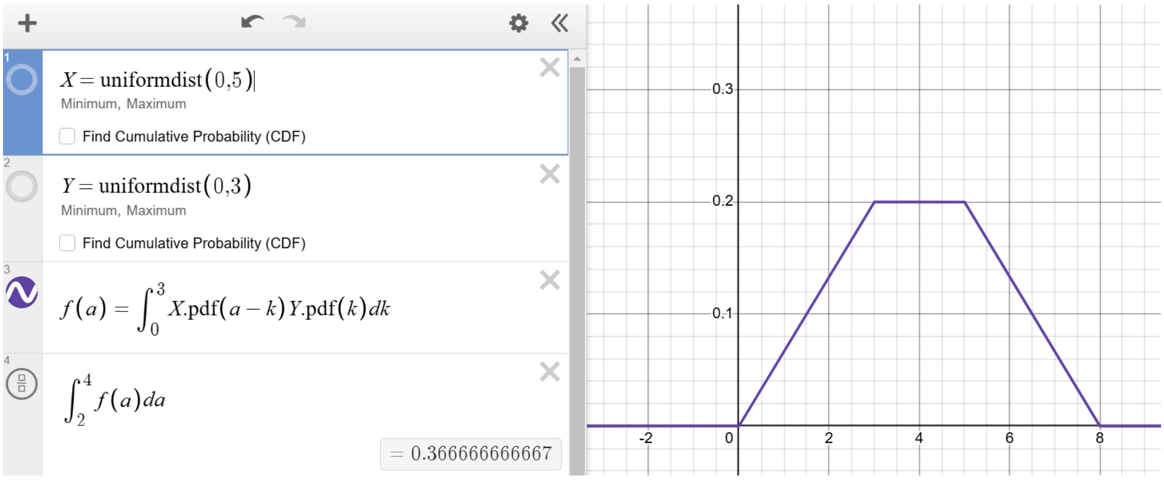
$$P\{2 < 5X + 3Y < 4\} = \int_2^3 \frac{v}{15}\, \mathrm{d}v + \int_3^4 \frac{1}{5}\, \mathrm{d}v = \frac{11}{30} \approx 36.67\%.$$

*Part **(b)** is on the next page.*

(b) We may notice, by graphing or otherwise, that $f(v)$ is symmetric about $v = 4$. Thus,

$$P\{4 < 5X + 3Y < 6\} = P\{2 < 5X + 3Y < 4\} \approx 36.67\%.$$

We may also tap into abit of Desmos magic (available via the graphic calculator):



This gives us both answers almost immediately to be 36.67%. $\square$

Question 9

This question is worth 2 marks.

A breath-analyser test reports a sensitivity of 98% and a specificity of 95% of detecting drink-driving (blood alcohol content exceeding the legal limit). The prevalence of drink-driving in the population is 2%. A driver is stopped at random and given the breath-analyser test. The test gives a positive result. What is the probability that the driver was drink-driving?

Answer.

Let D be the event that the chosen driver was drink-driving and C be the event that the test gives a positive result. Then, we are given:

$$P\{D\} = 0.02, \qquad P\{C \mid D\} = 0.98, \qquad P\{C^c \mid D^c\} = 0.95.$$

Thus, by Bayes' Formula,

$$P\{D \mid C\} = \frac{P\{C \mid D\} \cdot P\{D\}}{P\{C \mid D^c\} \cdot P\{D^c\}} = \frac{0.98 \cdot 0.02}{0.98 \cdot 0.02 + 0.05 \cdot 0.98} = \frac{2}{7} \approx 28.57\%. \qquad \square$$

Question 10

This question is worth 4 marks.

A, B and C are independent uniform random variables on $[0, 1]$.

What is the probability that the equation $Ax^2 + Bx + C = 0$ has **no real solutions**?

Answer.

The problem is equivalent to finding $P\{B^2 - 4AC < 0\} = P\{B^2 < 4AC\}$.

We first find the density f_{B^2} of B^2. Let the distribution of B^2 be F_{B^2}. Then,

$$F_{B^2}(t) = P\left\{B^2 \le t\right\} = P\left\{B \le \sqrt{t}\right\} = \begin{cases} 0 & \text{if } t < 0, \\ \sqrt{t} & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Thus,

$$f_{B^2}(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_{B^2}(t) = \begin{cases} \dfrac{1}{2\sqrt{t}} & \text{if } 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we find the distribution F_{AC} of AC. By the law of total probability,

$$\begin{aligned} F_{AC}(t) = P\{AC \le t\} &= \int_0^1 P\{AC \le t \mid C = c\} \cdot f_C(c)\,\mathrm{d}c \\ &= \int_0^1 P\{A \le t/c\}\,\mathrm{d}c \\ &= \int_0^t 1\,\mathrm{d}c + \int_t^1 \frac{t}{c}\,\mathrm{d}c \qquad (*) \\ &= t - t\ln t, \end{aligned}$$

where the bounds in $(*)$ are explained by:

$$c \in [0, t] \implies t/c > 1 \implies F_A(t/c) = 1 \text{ and } c \in [t, 1] \implies t/c \in [0, 1] \implies F_A(t/c) = t/c.$$

This is continued on the next page.

Thus, again by the law of total probability, for $t \in [0, 1]$,

$$
\begin{aligned}
P\left\{B^2 < 4AC\right\} &= 1 - P\left\{4AC < B^2\right\} \\
&= 1 - \int_0^1 P\left\{4AC < B^2 \mid B^2 = t\right\} \cdot f_{B^2}(t)\,\mathrm{d}t \\
&= 1 - \int_0^1 P\left\{AC < t/4\right\} \cdot f_{B^2}(t)\,\mathrm{d}t \\
&= 1 - \int_0^1 \left(\frac{t}{4} - \frac{t}{4}\ln\frac{t}{4}\right) \cdot \frac{1}{2\sqrt{t}}\,\mathrm{d}t \\
&= \frac{1}{36}(31 - 3\ln 4) \approx 74.56\%. \qquad \square
\end{aligned}
$$

Question 11

Each part in this question is worth 2 marks.

We are given a coin. Its probability of showing heads is assumed to be normally distributed on $[0, 1]$. We toss the coin 4 times and see exactly 3 heads.

(a) What would be our best prediction on the number of heads occurring when the coin is tossed another 4 times?

(b) What is the mean square error of this prediction?

We toss the coin another 4 times and see exactly 1 heads.

(c) What would be our best prediction on the number of heads occurring if the coin is tossed another 4 times? (Note that our very first observation still applies.)

Answer.

(a) Let X denote the probability of showing heads and N be the number of heads observed in 4 tosses. Then, the conditional density of X given that $N = 3$ is

$$f_{X|N}(x \mid 3) = \frac{P\{N = 3 \mid X = x\}}{P\{N = 3\}} f(x) = cx^3(1 - x), \quad 0 \leq x \leq 1$$

where c is a constant independent of x. We find c by integrating:

$$1 = \int_0^1 f_{X|N}(x \mid 3)\,\mathrm{d}x = \frac{c}{20} \implies c = 20.$$

Thus, our best prediction is simply $\mathrm{E}[N]$. By the law of total expectation,

$$\begin{aligned} \mathrm{E}[N] &= \int_0^1 \mathrm{E}[N \mid X = x] \cdot f_{X|N}(x \mid 3)\,\mathrm{d}x \\ &= \int_0^1 4x \cdot 20x^3(1 - x)\,\mathrm{d}x \\ &= \frac{80 \cdot 4! \cdot 1!}{6!} = \frac{8}{3} \approx 2.667. \end{aligned}$$

*Parts **(b)** and **(c)** are on the next page.*

(b) Our mean square error is $\mathrm{Var}(N)$. We first find $\mathrm{E}[N^2]$ by the law of total expectation.

$$\begin{aligned}
\mathrm{E}[N^2] &= \int_0^1 \mathrm{E}[N^2 \mid X = x] \cdot f_{X|N}(x \mid 3)\,\mathrm{d}x \\
&= \int_0^1 4x(1+3x) \cdot 20x^3(1-x)\,\mathrm{d}x \\
&= \int_0^1 240x^5(1-x)\,\mathrm{d}x + \int_0^1 80x^4(1-x)\,\mathrm{d}x \\
&= \frac{240 \cdot 5!}{7!} + \frac{80 \cdot 4!}{6!} = \frac{176}{21}.
\end{aligned}$$

Thus,

$$\mathrm{Var}(N) = \mathrm{E}[N^2] - (\mathrm{E}[N])^2 = \frac{176}{21} - \left(\frac{8}{3}\right)^2 = \frac{80}{63} \approx 1.270.$$

(c) Let N' be the number of heads seen in the second set of 4 tosses. Let the previous conditional density of X be Y. Then, the conditional density of Y given that $N' = 1$ is

$$\begin{aligned}
f_{Y|N'}(y \mid 1) &= \frac{P\{N' = 1 \mid Y = y\}}{P\{N' = 1\}} f_Y(y) \\
&= \frac{\binom{4}{1} y(1-y)^3}{P\{N' = 1\}} cy^3(1-y) \\
&= dy^4(1-y)^4, 0 \le y \le 1
\end{aligned}$$

where d is a constant independent of y. We find d by integrating:

$$1 = \int_0^1 f_{Y|N'}(y \mid 1)\,\mathrm{d}y = \frac{d}{630} \implies d = 630.$$

Thus, by the law of total expectation,

$$\begin{aligned}
\mathrm{E}[N] &= \int_0^1 \mathrm{E}[N \mid Y = y] \cdot f_{Y|N'}(y \mid 1)\,\mathrm{d}y \\
&= \int_0^1 4y \cdot 630y^4(1-y)^4\,\mathrm{d}y \\
&= \frac{4 \cdot 630 \cdot 5! \cdot 4!}{10!} = 2.
\end{aligned}$$

□

Question 12

Each part in this question is worth 1 mark.

A sample of radioactive substance has been observed to emit on average 0.8642 many α-particles per second.

(a) What is the probability that we observe at least 8 α-particles emitted in a 5 second interval?

(b) What is the probability that we wait at least 3 seconds before any α-particles are emitted?

(c) Given that no α-particles were observed in the last 3 seconds, what is the probability that we will see no α-particles emitted in the next 3 seconds?

(d) What is the shortest time interval we need to wait in order that the probability of the sample emitting any α-particle in that time interval is > 95%?

Answer.

Let X_t denote the number of α-particles emitted in a t second interval.

Then, $X_t \sim \text{Po}(\lambda t)$ where $\lambda = 0.8642$.

(a) $P\{X_5 \geq 8\} \approx 7.258\%$.

(b) $P\{X_3 = 0\} = e^{-3\lambda} \approx 7.483\%$.

Alternatively, let Y denote the amount of time we wait until an α-particle is emitted. Then, $Y \sim \text{Exp}(\lambda)$ and $P\{Y \geq 3\} \approx 7.483\%$.

(c) By the memoryless property:

$$P\{Y > 3 + 3 \mid Y > 3\} = P\{Y > 3\} \approx 7.843\%.$$

(d) We require that

$$P\{X_t \geq 1\} > 0.95 \implies P\{X_t = 0\} = e^{-\lambda t} < 0.05 \implies t > \frac{\ln 0.05}{-\lambda} \approx 3.4664$$

so the shortest amount of time to 4 significant figures is 3.467 seconds. □