Worked solutions for MA2101 17/18 S2 exam

Written 24 Nov 2024

Note: This document was written in a great rush, so there may be many mistakes. Compared with the solutions for 23/24 S1, I am even less certain of my answers below, since they were prepared without the help of any other written solutions. Read with care!

Question 1

> *Let v_1, …, v_4 be row vectors in the row $\mathbf{3}$-space $V = \mathbf{R}^3$. Form a matrix*
>
> $$A := (v_1^t, \ldots, v_4^t) \in M_{3,4}(\mathbf{R}).$$
>
> *Suppose that A is row-equivalent to the matrix*
>
> $$R = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
>
> ***1(a)** Find a subset $S \subseteq \{v_1, \ldots, v_4\}$ such that S is a basis of $\operatorname{span}\{v_1, \ldots, v_4\}$.*

The set $S = \{v_1, v_3\}$ works.

> ***1(b)** Express every vector v_i that is not in S as a linear combination of the vectors in S.*

We have $v_2 = 2v_1$ and $v_4 = 3v_1 + v_3$.

> ***1(c)** Find a basis of the row space $\operatorname{row}(A)$ for the matrix A.*

The basis with vectors $(1, 2, 0, 3)$ and $(0, 0, 1, 1)$ works.

> ***1(d)** Determine the dimension of the column space $\operatorname{col}(A)$ for the matrix A.*

The dimension of $\operatorname{col}(A)$ is two, the number of pivots in R.

> ***1(e)** Determine the nullity of the matrix A.*

By the rank-nullity theorem, the nullity of A is two.

Question 2

> *Let $A = (a_{ij}) \in M_2(\mathbf{R})$ be a real matrix such that for*
>
> $$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$
>
> *we have*

> $$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$
>
> *Let $y_i = y_i(x)$ ($i = 1, 2$) be differentiable functions in x. Solve the following system of differential equations:*
>
> $$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$
>
> *Note: For the differential equation $z'(x) + p(x)z(x) = q(x)$ you may assume, without proof, that its general solution is given by $z(x) = \frac{1}{\mu}(C + \int \mu q)$ with $\mu = e^{\int p}$.*

Write $D = P^{-1}AP$. It suffices to solve the new system $Z' = DZ$, since then $Y = PZ$ solves the original system $Y' = AY$. This new system is given by

$$\begin{cases} z_1'(x) = z_1(x) \\ z_2'(x) = -z_1(x) + z_2(x), \end{cases}$$

from which we immediately have $z_1(x) = C_1 e^x$. The second equation then becomes

$$z_2'(x) - z_2(x) = -z_1(x) \quad \text{or} \quad z_2'(x) + p(x)z_2(x) = q(x)$$

where $p(x) = -1$ and $q(x) = -z_1(x) = -C_1 e^x$. Since $\mu(x) = \exp(\int -1\, dx) = e^{-x}$, it follows that

$$z_2(x) = \frac{1}{e^{-x}} \left(\int e^{-x}(-C_1 e^x)\, dx + C_2 \right) = e^x (C_2 - C_1 x).$$

We may then substitute $Y = PZ$ to get

$$\begin{cases} y_1(x) = z_1(x) + 2z_2(x) = e^x(C_1 + 2C_2 - 2C_1 x) \\ y_2(x) = 3z_1(x) + 4z_2(x) = e^x(3C1 + 4C_2 - 4C_1 x). \end{cases}$$

Question 3

> *Let*
>
> $$V = P_3[x] = \left\{ f(x) = \sum_{i=0}^{2} a_i x^i \;\middle|\; a_i \in \mathbf{R} \right\}$$
>
> *and*
>
> $$W = M_2(\mathbf{R}) = \left\{ (b_{ij}) \mid b_{ij} \in \mathbf{R},\ 1 \leq i, j \leq 2 \right\}.$$
>
> *It is known that V and W are vector spaces over $\mathbf{R}$ with dimensions 3 and 4 respectively.*
>
> ***3(a)*** *State the definition of a linear transformation between two vector spaces over $\mathbf{R}$.*

Let V and W be vector spaces over $\mathbf{R}$. A function $T\colon V \to W$ is a linear transformation from V to W if $T(v + v') = T(v) + T(v')$ for all $v, v' \in V$, and $T(cv) = cT(v)$ for all $c \in \mathbf{R}$ and $v \in V$.

> ***3(b)*** *Construct an injective linear transformation $T_1\colon V \to W$. Justify your answer.*

Write $v_0 = 1$, $v_1 = x$, $v_2 = x^2$, $w_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $w_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $w_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $w_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then (v_0, v_1, v_2) is a basis for V, and $(w_{11}, w_{12}, w_{21}, w_{22})$ is a basis for W. We define $T_1(v_0) = w_{11}$, $T_1(v_1) = w_{12}$, and $T_1(v_2) = w_{21}$, extending to V by linearity. Then T_1 is injective, since we have mapped the basis of V to an independent subset of W.

> ***3(c)*** *Can you construct a surjective linear transformation $T_2 \colon V \to W$? Justify your answer.*

No, since $\dim(V) = 3 < 4 = \dim(W)$, and a surjective map from V to W would imply $\dim(V) \geq \dim(W)$.

Question 4

> *Let $T \colon V \to W$ be a linear transformation between vector spaces over a field F.*
>
> ***4(a)*** *State the definition of the kernel $\ker(T)$ of the map T.*

We have $\ker(T) = \{v \in V : Tv = 0_W\}$.

> ***4(b)*** *Show that for any $w_1 \in W$, the preimage*
>
> $$T^{-1}(w_1) := \{v \in V \mid T(v) = w_1\}$$
>
> *satisfies the following inequality of cardinalities:*
>
> $$|T^{-1}(w_1)| \leq |\ker(T)|.$$

If $|T^{-1}(w_1)| = 0$, there is nothing to show. Thus we may fix $v_0 \in T^{-1}(w_1)$. Then the translate $T^{-1}(w_1) - v_0 = \{v - v_0 : Tv = w_1\}$ is a subset of $\ker T$. Since V is a group with respect to its addition operation, translation is a bijection, and this gives the claim.

> ***4(c)*** *Suppose further that T is surjective. Can we have the following equality*
>
> $$|T^{-1}(w_1)| = |\ker(T)|$$
>
> *for all $w_1 \in W$? Justify your answer.*

Yes we can. Let V and W both be the trivial vector space $\{0\}$ over $\mathbf{R}$, and let $T \colon V \to W$ be the trivial map, which is trivially surjective. Then $|T^{-1}(0)| = |\ker(T)| = 1$.

Question 5

> *Let $A \in M_n(\mathbf{R})$ be a real matrix of size $n \times n$. Let A^t be the transpose of A.*
>
> ***5(a)*** *State the definition for a real square matrix to be orthogonal.*

A real square matrix Q is orthogonal if $Q^t Q = Q Q^t = I$.

> ***5(b)*** *Prove the existence of an orthogonal matrix $P \in M_n(\mathbf{R})$ such that*

> $$P^{-1}(A+A^t)P$$
>
> *is equal to a diagonal matrix D.*

Since $(A+A^t)^t = A^t + A = A + A^t$, it is symmetric. Since A is also real, it is self-adjoint, and so the principal axis theorem gives the existence of a matrix P with the desired properties.

> ***5(c)*** *If*
>
> $$A+A^t = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$
>
> *find the matrices P and D in part (b).*

Write $M = A + A^t$. We compute the characteristic polynomial

$$\det(\lambda I - M) = \det\begin{pmatrix} \lambda - 3 & 2 \\ 2 & \lambda - 3 \end{pmatrix} = (\lambda-3)^2 - 4 = (\lambda-1)(\lambda-5).$$

The eigenvector corresponding to $\lambda_1 = 1$ is easily seen to be $\binom{1}{1}$. As for $\lambda_2 = 5$, it is also easy to find $\binom{1}{-1}$. Thankfully these vectors are already orthogonal so we need only normalize to get $\frac{1}{\sqrt{2}}\binom{1}{1}$ and $\frac{1}{\sqrt{2}}\binom{1}{-1}$. This orthonormal eigenbasis then yields the desired matrices

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Question 6

> *Let $A \in M_4(\mathbf{C})$ be such that*
>
> $$(A - I_4)(A + I_4)(A - 2I_4) = 0.$$
>
> ***6(a)*** *State the definition for a complex square matrix to be diagonalizable over $\mathbf{C}$.*

A complex square matrix A is diagonalizable over $\mathbf{C}$ if there exists an invertible complex square matrix P such that $P^{-1}AP$ is a diagonal matrix.

> ***6(b)*** *State the definition of a monic polynomial $f(x)$.*

A polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is monic if $a_n = 1$.

> ***6(c)*** *Is A diagonalizable over $\mathbf{C}$? Justify your answer.*

Yes. Since the minimal polynomial of A divides $(x-1)(x+1)(x-2)$ by hypothesis, it must have distinct roots, and this is implies that A is diagonalizable over $\mathbf{C}$.

> ***6(d)*** *Show that A is invertible, and construct a monic polynomial $g(x)$ of degree three such that $A^{-1} = g(A)$.*

Since $(A - I_4)(A + I_4)(A - 2I_4) = 0$, we also have

$$(A - I_4)(A + I_4)(A - 2I_4)\Big(A + cI_4\Big) = 0.$$

This is equivalent to

$$A \cdot -\frac{1}{2c}\Big((2-c)I_4 - (2c+1)A + (c-2)A^2 + A^3\Big) = I_4,$$

so we may set $c = -1/2$ to obtain

$$A\Big(A^3 - \frac{5}{2}A^2 + \frac{5}{2}I_4\Big) = I_4,$$

so $A^{-1} = g(A)$, where $g(x) = x^3 - \frac{5}{2}x^2 + \frac{5}{2}$ is our desired monic polynomial of degree three.

Question 7

> **7(a)** *State the definition for a complex square matrix to be unitary.*

A complex square matrix is unitary if $AA^* = I$.

> **7(b)** *State the definition for a complex square matrix to be normal.*

A complex square matrix is normal if $AA^* = A^*A$.

> **7(c)** *Is every orthogonal (real) matrix a normal matrix? Justify your answer.*

Yes, since an orthogonal real matrix satisfies $AA^t = I = A^tA$, and $A^t = A^*$ since real numbers are unaffected by complex conjugation.

> **7(d)** *Let $A \in M_n(\mathbf{R})$ be an orthogonal matrix. Show that there exists a unitary matrix $U \in M_n(\mathbf{C})$ such that*
>
> $$U^{-1}AU = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$$
>
> *with modulus $|\lambda_i| = 1$ for all i.*

Since A is normal by part (c), the principal axis theorem gives the existence of unitary U such that $U^{-1}AU = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$. Thus it remains to show that $|\lambda_i| = 1$ for all i. Indeed, these are the eigenvalues of A, so given λ_i, we consider nonzero v satisfying $Av = \lambda_i v$. We then have

$$|\lambda_i|^2 v^*v = (\lambda_i v)^*(\lambda_i v) = (Av)^*(Av) = v^*A^*Av = v^*v,$$

so $|\lambda_i| = 1$ as needed.

> **7(e)** *Conversely, let $C \in M_n(\mathbf{R})$ be a real matrix and let $V \in M_n(\mathbf{C})$ be a unitary matrix. Show that if*
>
> $$V^{-1}CV = \mathrm{diag}[\mu_1, \ldots, \mu_n]$$
>
> *with $|\mu_i| = 1$ for all i, then C is orthogonal.*

Let $D = \mathrm{diag}[\mu_1, \ldots, \mu_n]$. We compute

$$
\begin{aligned}
C^tC &= C^*C && (1)\\
&= (VDV^{-1})^*(VDV^{-1}) && (2)\\
&= (V^{-1})^*D^*V^*VDV^{-1} && (3)\\
&= (V^*)^*D^*DV^{-1} && (4)\\
&= VV^{-1} && (5)\\
&= I,
\end{aligned}
$$

where each step is justified as follows: (1) C is real, so $C^* = C^t$. (2) Since $D = V^{-1}CV$, we have $C = VDV^{-1}$. (3) We have $(AB)^* = B^*A^*$. (4) Since V is unitary, we have $V^{-1} = V^*$ and $V^*V = I$. (5) We have $(V^*)^* = I$ and $D^*D = \mathrm{diag}[|\mu_1|^2, \ldots, |\mu_n|^2]$, so that $D^*D = I$ by the hypothesis $\mu_i| = 1$.

Question 8

> *8(a) State the definition for a real square matrix to be positive definite.*

A real square matrix A is positive definite if it is symmetric ($A^t = A$) and satisfies $x^tAx > 0$ for all $x \in \mathbf{R}^n_c \setminus \{0\}$.

> *8(b) Is every positive-definite real matrix invertible? Justify your answer.*

Yes. Suppose contrapositively that A is not invertible. Then $Ax = 0$ for some nonzero x, so $x^tAx = 0$ as needed.

> *8(c) Find a real matrix $A \in M_2(\mathbf{R})$ such that A is positive definite, but A is not a diagonal matrix. Justify your answer.*

The symmetric matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ works, since it has all positive eigenvalues. More directly, we have $x^tAx = x_1^2 + (x_1 + x_2)^2 + x_2^2 > 0$ for nonzero x.