MA3110 MATHEMATICAL ANALYSIS II
FINAL EXAM (2015/2016 SEMESTER 1)

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Question 1 (24 points)

(a) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable at a given $c \in \mathbb{R}$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$u \in (c-\delta, c] \text{ and } v \in [c, c+\delta) \quad \text{then} \quad \left|g(v) - g(u) - (v-u)g'(c)\right| \leq \varepsilon(v-u).$$

(b) Let $f : (-1,1) \longrightarrow \mathbb{R}$ be infinitely differentiable on $(-1,1)$, $f(0) = 1$ and

- $\left|f^{(n)}(x)\right| \leq n!$ for every $x \in (-1,1)$ and for every $n \in \mathbb{N}$,
- $f'\left(\frac{1}{m+1}\right) = 0$ for every $m \in \mathbb{N}$.

(i) Find the value of $f^{(n)}(0)$ for each $n \in \mathbb{N}$.

(ii) Determine the value of $f(x)$ for every $x \in (-1,1)$. Justify your answers.

Solution.

(a) Since g is differentiable at $c \in \mathbb{R}$, then

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c} \frac{g(c) - g(x)}{c - x}$$

exists. This implies that both the right-hand and left-hand limits exist, i.e.

$$\lim_{x \to c^-} \frac{g(x) - g(c)}{x - c} = g'(c) = \lim_{x \to c^+} \frac{g(c) - g(x)}{c - x}.$$

Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that if $c - x < \delta_1$, then

$$\frac{g(x) - g(c)}{x - c} - g'(c) \leq \left|\frac{g(x) - g(c)}{x - c} - g'(c)\right| < \varepsilon.$$

Using the same $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $x - c < \delta_2$, then

$$\frac{g(c) - g(x)}{c - x} - g'(c) \leq \left|\frac{g(c) - g(x)}{c - x} - g'(c)\right| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Let $u \in (c-\delta, c]$ and $v \in [c, c+\delta)$ so that $c - u < \delta$ and $v - c < \delta$ respectively. So,

$$-\varepsilon < \frac{g(c) - g(u)}{c - u} - g'(c) < \varepsilon \quad \text{and} \quad -\varepsilon < \frac{g(v) - g(c)}{v - c} - g'(c) < \varepsilon \quad \text{respectively.}$$

As such,

$$-\varepsilon(c-u) < g(c) - g(u) - g'(c)(c-u) < \varepsilon(c-u) \quad \text{and} \quad -\varepsilon(v-c) < g(v) - g(c) - g'(c)(v-c) < \varepsilon(v-c).$$

Adding both inequalities, we get

$$-\varepsilon(v-u) < g(v) - g(u) - g'(c)(v-u) < \varepsilon(v-u).$$

so that

$$\left|g(v) - g(u) - (v-u)g'(c)\right| \leq \varepsilon(v-u).$$

(b) **(i)** We will show that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Since f is infinitely differentiable, each $f^{(n)}$ is continuous. In particular, $f'(0) = 0$ since

$$0 = \lim_{m \to \infty} f'\left(\frac{1}{m+1}\right) = f'(0).$$

Since $f'(0)=0$ and $f'(1/m+1)=0$ for each $m\in\mathbb{N}$, by the mean value theorem, there exists $a_m^{(2)}\in(0,1/(m+1))$ such that $f''\left(a_m^{(2)}\right)=0$. Moreover,

$$\lim_{m\to\infty} a_m^{(2)}=0.$$

Since f'' is continuous, then $f''(0)=0$.

Now suppose that $f^{(n-1)}(0)=0$ and there exists a sequence $\left\{a_m^{(n-1)}\right\}_{m=1}^{\infty}$ such that $f^{(n-1)}\left(a_m^{(n-1)}\right)=0$ for each m and $a_m^{(n-1)}\to 0$ as $m\to\infty$. By a similar construction as above, by the mean value theorem, we can find a sequence $\left\{a_m^{(n)}\right\}_{m=1}^{\infty}$ such that $0<a_m^{(n)}<a_m^{(n-1)}$ for each m and $a_m^{(n)}\to 0$ and $f^{(n)}\left(a_m^{(n)}\right)=0$ for each $m\in\mathbb{N}$. By the continuity of $f^{(n)}$, we have

$$0=\lim_{m\to\infty} f^{(n)}\left(a_m^{(n)}\right)=f^{(n)}(0).$$

Hence, by induction, $f^{(n)}(0)$ for each $n\in\mathbb{N}$.

(ii) Since f is infinitely differentiable on $(-1,1)$, by Taylor's theorem, there exists $c\in(0,x)$ such that

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}=1+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

But

$$0\le|f(x)-1|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right|\le\left|x^{n+1}\right|.$$

Since $|x|<1$, then

$$\lim_{n\to\infty}\left|x^{n+1}\right|=0$$

so that by squeeze theorem, $|f(x)-1|=0$ for any $x\in(0,1)$. Therefore, $f(x)=1$ for all $x\in[0,1)$. By a similar argument, we see that $f(x)=1$ for all $x\in(-1,0]$. Therefore, $f(x)=1$ for all $x\in(-1,1)$. □

Question 2 (26 points).

(a) Let h be a bounded and integrable function on $[a,b]$.

(i) Using the Riemann Integrability Criterion, prove that the function h^3 is Riemann integrable on $[a,b]$.

(ii) Suppose that h is continuous on $[a,b]$ and $\int_a^b h^2=0$. Prove that $h(x)=0$ for all $x\in[a,b]$.

(b) Using Riemann integrals of suitably chosen functions, evaluate the following limit:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\sqrt{k}}{\sum_{k=1}^{n}\sqrt{n+k}}$$

Solution.

(a) **(i)** Since h is bounded on $[a,b]$, then there exists $M\in\mathbb{R}^+$ such that $|h(x)|\le M$[1] for all $x\in[a,b]$. This implies that $\left|h^2(x)\right|\le M^2$ for all $x\in[a,b]$. Let $\varepsilon>0$. Since h is integrable, there exists $n\in\mathbb{N}$ and a partition P of $[a,b]$ given by

$$P=\{a=x_0<x_1<\cdots<x_n=b\}$$

where

$$x_k=x_0+\frac{k(b-a)}{n}$$

such that

$$U(h,P)-L(h,P)<\frac{\varepsilon}{3M^2}$$

[1]It is worth noting that we considered bounding h^2 instead tackling the question directly by bounding h^3 — the reason will be apparent in just a bit when we consider some nice trick when dealing with $\left|h^3(u)-h^3(v)\right|$ for any $u,v\in[x_{k-1},x_k]$.

Let

$$\begin{aligned} m_k(h,P) &= \inf_{x\in[x_{k-1},x_k]} h(x) \\ M_k(h,P) &= \sup_{x\in[x_{k-1},x_k]} h(x). \end{aligned}$$

Observe that for any $u,v\in[x_{k-1},x_k]$, we have

$$\begin{aligned} \left|h^3(u)-h^3(v)\right| &= \left|h^3(u)-h^2(u)h(v)+h^2(u)h(v)-h^3(v)\right| \\ &\le \left|h^3(u)-h^2(u)h(v)\right|+\left|h^2(u)h(v)-h^3(v)\right| \\ &\le \left|h^2(u)\right|\left|h(u)-h(v)\right|+\left|h(v)\right|\left|h^2(u)-h^2(v)\right| \\ &= \left|h^2(u)\right|\left|h(u)-h(v)\right|+\left|h(v)\right|\left|h^2(u)-h(u)h(v)+h(u)h(v)-h^2(v)\right| \\ &\le \left|h^2(u)\right|\left|h(u)-h(v)\right|+\left|h(v)\right|\left|h(u)\right|\left|h(u)-h(v)\right|+\left|h^2(v)\right|\left|h(u)-h(v)\right| \\ &\le 3M^2\left|h(u)-h(v)\right| \\ &\le 3M^2\left(M_k(h,P)-m_k(h,P)\right). \end{aligned}$$

This implies that

$$M_k\left(h^3,P\right)-m_k\left(h^3,P\right)\le 3M^2\left(M_k(h,P)-m_k(h,P)\right).$$

Putting everything together,

$$\begin{aligned} U\left(h^3,P\right)-L\left(h^3,P\right) &= \sum_{k=1}^{n}\left(M_k\left(h^3,P\right)-m_k\left(h^3,P\right)\right)\cdot\frac{b-a}{n} \\ &\le \sum_{k=1}^{n}3M^2\left(M_k(h,P)-m_k(h,P)\right)\cdot\frac{b-a}{n} \\ &= 3M^2\left(U(h,P)-L(h,P)\right) \\ &< \varepsilon \end{aligned}$$

so h^3 is integrable on $[a,b]$.

(ii) Suppose that $h(x)\neq 0$ for some $x\in[a,b]$. Without loss of generality, suppose that $h(c)>0$ for some $c\in(a,b)$. Then there exists $\delta>0$ such that $h(x)>0$ for all $x\in(c-\delta,c+\delta)$ with $c+\delta<b$ and $c-\delta>a$. Since h is continuous, then h^2 is also continuous on $[a,b]$ and in particular, h^2 is continuous on $[c-\delta,c+\delta]$. Hence, h^2 attains the absolute maximum and absolute minimum on $[a,b]$. Let

$$\begin{aligned} M &= \max_{x\in[c-\delta,c+\delta]} h^2(x)>0 \\ m &= \min_{x\in[c-\delta,c+\delta]} h^2(x)>0. \end{aligned}$$

This implies that $(h(x))^2>0$ for all $x\in(c-\delta,c+\delta)$. Now,

$$0=\int_a^b h^2=\int_a^{c-\delta}h^2+\int_{c-\delta}^{c+\delta}h^2+\int_{c+\delta}^{b}h^2\ge\int_{c-\delta}^{c+\delta}h^2>m(2\delta)>0$$

which is a contradiction. Therefore, h is identically zero on $[a,b]$.

(b) We have

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\sqrt{k}}{\sum_{k=1}^{n}\sqrt{n+k}}=\lim_{n\to\infty}\frac{\frac{1}{n^{3/2}}\sum_{k=1}^{n}\sqrt{k}}{\frac{1}{n^{3/2}}\sum_{k=1}^{n}\sqrt{n+k}}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^{n}\sqrt{k/n}}{\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+(k/n)}}.$$

We evaluate the numerator and denominator separately. Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}=\int_0^1\sqrt{x}\,dx=\frac{2}{3}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+\frac{k}{n}}=\int_0^1\sqrt{1+x}\,dx=\frac{2^{3/2}}{3/2}-\frac{1}{3/2}=\frac{2}{3}\left(2^{3/2}-1\right)$$

then

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\sqrt{k}}{\sum_{k=1}^{n}\sqrt{n+k}}=\frac{2/3}{\frac{2}{3}\left(2^{3/2}-1\right)}=\frac{1}{2^{3/2}-1}.$$

□

Question 3 (30 points)

(a) Suppose that the sequence $\{f_n\}$ of functions converges uniformly to a function f on a non-empty set E and each $\{f_n\}$ is a bounded function on E (i.e., there exists a positive constant M_n such that $|f_n(x)|\le M_n$ for every $x\in E$).

(i) Prove that f is bounded on E.

(ii) Determine if the sequence $\{f_n\}$ is uniformly bounded on E.

(b) For every $n\in\mathbb{N}$, let $g_n(x)=\frac{1}{1+n^3x}$, $x>0$. For each of the following, state clearly the known results that you are using in obtaining your answers.

(i) Show that the series $\sum_{n=1}^{\infty}g_n(x)$ converges for every $x>0$.

(ii) Determine if the series $\sum_{n=1}^{\infty}g_n$ converges uniformly on the interval $[r,\infty)$, where $r>0$.

(iii) Determine if the series $\sum_{n=1}^{\infty}g_n$ converges uniformly on the interval $(0,\infty)$.

(iv) Determine if the series $\sum_{n=1}^{\infty}g_n'$ converges uniformly on $(0,\infty)$.

(v) Let $g(x)=\sum_{n=1}^{\infty}g_n(x)$, $x>0$. Show that g is differentiable on $(0,\infty)$ and

$$g'(x)=-\sum_{n=1}^{\infty}\frac{n^3}{\left(1+n^3x\right)^2},x>0.$$

Solution.

(a) **(i)** Let $\varepsilon=1>0$. Since $\{f_n\}$ converges uniformly to a function on E, there exists $N\in\mathbb{N}$ such that for all $x\in E$ and $n\ge N$, $|f_n(x)-f(x)|<1$. In particular, $|f_N(x)-f(x)|<1$, or equivalently, $|f(x)-f_N(x)|<1$. This implies that

$$-1-M_N<-1+f_N(x)<f(x)<1+f_N(x)\le 1+M_N.$$

Let $M=\max\{M_1,M_2,\ldots,M_N,M_{N+1}\}$. Then we see that $|f(x)|\le 1+M$. So, f is bounded on E.

(ii) Yes, the sequence $\{f_n\}$ is uniformly bounded on E. Let $\varepsilon=1>0$. Since $\{f_n\}$ converges uniformly to a function on E, there exists $N\in\mathbb{N}$ such that for all $x\in E$ and $n\ge N$, $|f_n(x)-f(x)|<1$. This implies that

$$-1-M<-1+f(x)<f_n(x)<1+f(x)\le 1+M$$

for all $n\ge N$. Let $K=\max\{1+M,M_1,\ldots,M_n\}$. Then we see that $|f_n(x)|\le K$ for all $n\in\mathbb{N}$.

(b) **(i)** Let $x>0$. Then

$$g_n(x)=\frac{1}{1+n^3x}\le\frac{1}{n^3x}.$$

Since the series

$$\sum_{n=1}^{\infty}\frac{1}{n^3x}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

converges, then by the comparison test, the series $\sum_{n=1}^{\infty}g_n(x)$ converges for every $x>0$.

(ii) Yes, the series $\sum_{n=1}^{\infty} g_n$ converges uniformly on the interval $[r,\infty)$, where $r>0$. Note that for any $x \geq r$, we have

$$|g_n(x)| = g_n(x) = \frac{1}{1+n^3x} \leq \frac{1}{n^3r}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{rn^3}$$

converges, then by the Weierstrass M-test, $\sum_{n=1}^{\infty} g_n$ converges uniformly on the interval $[r,\infty)$, where $r>0$.

(iii) No, the series $\sum_{n=1}^{\infty} g_n$ does not converge uniformly on the interval $(0,\infty)$. Since

$$\sup_{x\in(0,\infty)} |g_n(x)| = \sup_{x\in(0,\infty)} \left|\frac{1}{1+n^3x}\right| = \sup_{x\in[0,\infty)} \left|\frac{1}{1+n^3x}\right| \geq g_n(0) = 1$$

then $g_n \not\to 0$ uniformly on $(0,\infty)$. Therefore, the series $\sum_{n=1}^{\infty} g_n$ does not converge uniformly on the interval $(0,\infty)$.

(iv) No, the series $\sum_{n=1}^{\infty} g_n'$ does not converge uniformly on $(0,\infty)$. Note that

$$g_n'(x) = -\frac{n^3}{(1+n^3x)^2}.$$

Since

$$\sup_{x\in(0,\infty)} \left|g_n'(x)\right| = \sup_{x\in(0,\infty)} \left|-\frac{n^3}{(1+n^3x)^2}\right| = \sup_{x\in[0,\infty)} \left|-\frac{n^3}{(1+n^3x)^2}\right| \geq \left|g_n'(0)\right| = n^3 \not\to 0$$

then g_n' does not converge uniformly to 0 on $(0,\infty)$. Therefore, $\sum_{n=1}^{\infty} g_n'$ does not converge uniformly on $(0,\infty)$

(v) Let $y>0$. We will show that g is differentiable at y. Let $m\in\mathbb{N}$ be such that $y\in[1/m,m]$. Note that the series $\sum_{n=1}^{\infty} g_n(y)$ converges on $[1/m,m]\subseteq(0,\infty)$ by **(bi)**. Next, for any $x\in[1/m,m]$,

$$0 \leq \frac{n^3}{(1+n^3x)^2} \leq \frac{n^3}{(1+(1/m)n^3)^2} \leq \frac{1}{(1/m)^2n^3}$$

and $\sum_{n=1}^{\infty} \frac{1}{(1/m)^2n^3}$ converges. By the Weierstrass M-test, $\sum_{n=1}^{\infty} g_n'$ converges uniformly on $[1/m,m]$. Therefore, $\sum_{n=1}^{\infty} g_n$ converges uniformly to a differentiable function g on $[1/m,m]$, and

$$g'(x) = -\sum_{n=1}^{\infty} \frac{n^3}{(1+n^3x)^2}$$

Since $y>0$ is arbitrary, then g is differentiable on $(0,\infty)$. □

Question 4 (20 points)

(a) Consider the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{(2n+1)(2n-1)}.$$

(i) Find its radius of convergence and the set of all x for which the series converges.

(ii) Find a closed form of its sum function on its interval of convergence.
(Hint: $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$.)

(iii) Evaluate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)(2n-1)}.$$

Justify your answer.

(b) Let h be defined by $h(x)=x^2e^{x^3}$ for $x\in\mathbb{R}$, and let $h^{(n)}$ be the n-th order derivative of h. Find the values of $h^{(2015)}(0)$ and $h^{(2016)}(0)$. Justify your answers.

Solution.

(a) **(i)** We shall use the Cauchy-Hadamard formula. We have

$$\limsup_{n\to\infty}\left|\frac{(-1)^{n-1}}{(2n+1)(2n-1)}\right|^{1/n}=\limsup_{n\to\infty}\frac{1}{(2n+1)^{1/n}(2n-1)^{1/n}}=1$$

since

$$\lim_{n\to\infty}(2n+1)^{1/n}=\lim_{n\to\infty}e^{\frac{1}{n}\ln(2n+1)}=1 \quad\text{and}\quad \lim_{n\to\infty}(2n-1)^{1/n}=\lim_{n\to\infty}e^{\frac{1}{n}\ln(2n-1)}=1.$$

Hence, the radius of convergence is 1 and the series converges for all $x\in(-1,1)$.

Next, for the endpoints $x=\pm1$, we use the alternating series test. Let

$$x_n=\frac{1}{(2n+1)(2n-1)}.$$

We see that $x_n\geq 0$ for all $n\in\mathbb{N}$, the sequence $\{x_n\}_{n=1}^{\infty}$ is a decreasing sequence, and $x_n\to 0$ as $n\to\infty$. By the alternating series test, the series converges at ±1. We conclude that the interval of convergence is $[-1,1]$.

(ii) Note that

$$\frac{1}{(2n+1)(2n-1)}=\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}.$$

Then we can write

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{(2n+1)(2n-1)}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2(2n-1)}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2(2n+1)}.$$

Since

$$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2},$$

then for any $x\in(-1,1)$, there exists $r>0$ such that $x\in[-r,r]$, where $r<1$, so that

$$\int_0^x\sum_{n=1}^{\infty}(-1)^{n-1}t^{2n-2}\,dt=\sum_{n=1}^{\infty}\int_0^x(-1)^{n-1}t^{2n-2}\,dt=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

$$\int_0^x\sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt=\sum_{n=0}^{\infty}\int_0^x(-1)^n t^{2n}\,dt=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2n+1}.$$

This implies that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2(2n-1)}=\frac{1}{2}\int_0^x\frac{1}{1+t^2}\,dt=\frac{1}{2}\tan^{-1}x$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2(2n+1)}=x-\frac{1}{2}\int_0^x\frac{1}{1+t^2}\,dt=x-\frac{1}{2}\tan^{-1}x$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{(2n+1)(2n-1)}&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2(2n-1)}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2(2n+1)}\\&=\frac{x^2}{2}\tan^{-1}x-x+\frac{1}{2}\tan^{-1}x.\end{aligned}$$

(iii) When $x=1$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)(2n-1)}=\frac{1}{2}\tan^{-1}1-1+\frac{1}{2}\tan^{-1}1=\frac{1}{2}\cdot\frac{\pi}{4}-1+\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{4}-1.$$

(b) We start by writing $h(x)$ in its power series form. Since

$$e^{x^3}=\sum_{k=0}^{\infty}\frac{(x^3)^k}{k!}=\sum_{k=0}^{\infty}\frac{x^{3k}}{k!},$$

multiplying by x^2 gives

$$h(x) = x^2 e^{x^3} = \sum_{k=0}^{\infty} \frac{x^{2+3k}}{k!}.$$

In a Taylor series expansion

$$h(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the nth derivative at 0 is given by $h^{(n)}(0) = n! a_n$. From our series for $h(x)$, the coefficient a_n is nonzero only if n is of the form $3k+2$ (with $k \geq 0$), in which case

$$a_{3k+2} = \frac{1}{k!}.$$

Thus,

$$h^{(3k+2)}(0) = (3k+2)! \cdot \frac{1}{k!}.$$

For $h^{(2015)}(0)$, we need $2015 = 3k+2$. Solving for k, we have $k = 671$, so

$$h^{(2015)}(0) = \frac{(2015)!}{671!}.$$

However, 2016 cannot be written as $3k+2$ since 2014 is not divisible by 3. Hence, $h^{(2016)}(0) = 0$. $\square$