NATIONAL UNIVERSITY OF SINGAPORE

MA3110 Mathematical Analysis II

(Semester 2 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with help sheet) examination.
6. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.
7. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Use Taylor's Theorem to prove that for $x \in (0, \pi)$,

$$x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

(b) Suppose that the function f is infinitely differentiable on $(-1, 1)$ and there are constants $A > 0$ and $B > 0$ such that

$$|f^{(n)}(x)| \leq A\frac{n!}{B^n} \quad \text{for all } x \in (-1, 1) \text{ and all } n \in \mathbb{N}.$$

Prove that there exists $\delta > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{for } x \in (-\delta, \delta).$$

Question 2 [25 marks]

(a) Using the Riemann integral of a suitably chosen function, find the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n}\right).$$

(b) Let $G(x) = \displaystyle\int_x^{x^2} \sqrt{1 + t^2 + t^4}\, dt$ for $x \in \mathbb{R}$. Find a formula for $G'(x)$.

(c) Let the function $f : [0, \infty) \to \mathbb{R}$ be such that the improper integral $\displaystyle\int_0^{\infty} f(x)\, dx$ converges. Prove that for every $\varepsilon > 0$, there exists $M > 0$ such that

$$\left|\int_b^a f(x)\, dx\right| < \varepsilon \quad \text{whenever } a > b > M.$$

Question 3 [25 marks]

(a) For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{1}{1+x^{2n}}, \quad x \in [-1,1].$$

(i) Find $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [-1,1]$.

(ii) Prove that for each $0 < r < 1$, $f_n \to f$ uniformly on $[-r,r]$.

(iii) Does (f_n) converge uniformly on $[-1,1]$? Justify your answer.

(iv) Find $\displaystyle\lim_{n\to\infty} \int_0^{1/2} \frac{1}{1+x^{2n}}\, dx$.

(b) Let (g_n) be a sequence of functions on $[a,b]$ such that each g_n is continuous on $[a,b]$ and is differentiable on (a,b) and there exists $M > 0$ such that

$$|g_n'(x)| \leq M \quad \text{for all } x \in (a,b) \text{ and for all } n \in \mathbb{N}.$$

(i) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|g_n(x) - g_n(y)| < \varepsilon \text{ whenever } x, y \in [a,b],\ |x-y| < \delta \text{ and } n \in \mathbb{N}.$$

(ii) If (g_n) converges pointwise on $[a,b]$, then does (g_n) necessarily also converge uniformly on $[a,b]$? Justify your answer.

Question 4 [10 marks]

Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{\frac{x}{n}}$, $x \in \mathbb{R}$.

(i) Show that for each $r > 0$, the series converges uniformly on $[-r,r]$.

(ii) Is f is differentiable on $\mathbb{R}$? Justify your answer.

Question 5 [20 marks]

(a) Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+2}$.

(i) Find its radius of convergence and the set E of all x at which the series converges.

(ii) Find a close form of its sum function on E.

(b) Let $f(x) = (x-1)^3 e^{-2x}$, $x \in \mathbb{R}$.

(i) Find the Taylor series of f about $x = 1$.

(ii) Find the value of $f^{(8)}(1)$.

(You may assume that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.)

END OF PAPER

Solutions

1. (a) Let $f(x) = \sin x$. Then

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{(4)}(x) = \sin x,\ f^{(5)}(x) = \cos x,\ f^{(6)}(x) = -\sin x$$

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1.$$

Let $x \in (0, \pi)$. By Taylor's Theorem, there exist $c_1, c_2 \in (0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(c_1)}{4!}x^4 = x - \frac{x^3}{3!} + \frac{\sin(c_1)}{4!}x^4$$

and

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\frac{f^{(6)}(c_2)}{6!}x^6 = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{\sin(c_2)}{6!}x^6.$$

Since $c_1, c_2 \in (0, \pi)$, $\sin(c_1) > 0$ and $\sin(c_2) > 0$. Hence,

$$f(x) = x - \frac{x^3}{3!} + \frac{\sin(c_1)}{4!}x^4 > x - \frac{x^3}{3!}$$

and

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\sin(c_2)}{6!}x^6 < x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

(b) Let $x \in (-1, 1)$. Then for each $n \in \mathbb{N}$, by Taylor's Theorem,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}x^{n+1} \quad \text{for some } c_n \in (0, x).$$

Now

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c_n)}{(n+1)!}x^{n+1}\right| \le \frac{A\frac{(n+1)!}{B^{n+1}}}{(n+1)!}|x|^{n+1} = A\left(\frac{|x|}{B}\right)^{n+1}.$$

Let $\delta = \min(1, B)$, and let $x \in (-\delta, \delta)$. Then $x \in (-1, 1)$ and $|x| < B$, so that $\frac{|x|}{B} < 1$ and $\left(\frac{|x|}{B}\right)^{n+1} \to 0$. Hence,

$$|R_n(x)| \le A\left(\frac{|x|}{B}\right)^{n+1} \to 0.$$

It follows from this that

$$\begin{aligned}
f(x) &= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k + R_n(x)\right) \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k + \lim_{n\to\infty} R_n(x) \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k + 0 \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k.
\end{aligned}$$

2. (a) Let $f(x) = \cos x$. For each $n \in \mathbb{N}$, let

$$P = \left\{\frac{k\pi}{2n} \,:\, 0 \le k \le n\right\} \quad \text{and} \quad \xi^{(n)} = \left(\frac{\pi}{2n}, \frac{2\pi}{2n}, ..., \frac{n\pi}{2n}\right).$$

Since $\|P_n\| = \dfrac{\pi}{2n} \to 0$,

$$S(f, P_n)(\xi^{(n)}) = \sum_{k=1}^{n} f\left(\frac{k\pi}{2n}\right)\cdot\frac{\pi}{2n} = \frac{\pi}{2n}\sum_{k=1}^{n}\cos\left(\frac{k\pi}{2n}\right) \to \int_0^{\pi/2}\cos x\,dx = \sin\frac{\pi}{2} = 1.$$

So

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos\left(\frac{k\pi}{2n}\right) = \frac{2}{\pi}.$$

(b) By the Fundamental Theorem of Calculus and the chain rule,

$$\begin{aligned}
G'(x) &= \frac{d}{dx}\left(\int_0^{x^2}\sqrt{1+t^2+t^4}\,dt - \int_0^{x}\sqrt{1+t^2+t^4}\,dt\right) \\
&= 2x\sqrt{1+x^4+x^8} - \sqrt{1+x^2+x^4}.
\end{aligned}$$

(c) Let $F(x) = \displaystyle\int_0^x f(t)\,dt$, $x \ge 0$, and let $A = \displaystyle\int_0^\infty f(x)\,dx = \lim_{x\to\infty} F(x)$.

Let $\varepsilon > 0$. There exists $M > 0$ such that

$$x > M \Longrightarrow |F(x) - A| < \frac{\varepsilon}{2}.$$

Then for $a > b > M$,

$$\begin{aligned}
\left|\int_b^a f(x)\,dx\right| &= |F(a) - F(b)| \\
&\le |F(a) - A| + |A - F(b)| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

3. (a) (i) For $x \in (-1, 1)$,

$$f(x) = \lim_{n\to\infty} \frac{1}{1 + x^{2n}} = \lim_{n\to\infty} \frac{1}{1+0} = 1.$$

For $x = \pm 1$,

$$f(x) = \lim_{n\to\infty} \frac{1}{1+1} = \frac{1}{2}.$$

So

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ \frac{1}{2} & x = \pm 1. \end{cases}$$

(ii) For $x \in [-r, r]$, $|x| \le r$,

$$|f_n(x) - f(x)| \;=\; \left|\frac{1}{1+x^{2n}} - 1\right| = \frac{x^{2n}}{1+x^{2n}} \le x^{2n} \le r^{2n}.$$

Since $0 < r < 1$, $r^{2n} \to 0$. It follows that

$$\|f_n - f\|_{[-r,r]} \le r^{2n} \to 0.$$

(iii) Note that each f_n is continuous on $[-1, 1]$ but f is discontinuous at $x = \pm 1$. So (f_n) does not converge uniformly on $[-1, 1]$.

Alternatively, let $x_n = \left(\frac{1}{2}\right)^{\frac{1}{2n}}$, $n \in \mathbb{N}$. Then $x_n \in (0, 1)$ and

$$|f_n(x_n) - f(x_n)| = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3} \quad \forall n \in \mathbb{N}.$$

(iv) Since $f_n \to f$ uniformly on $[0, 1/2]$,

$$\lim_{n\to\infty} \int_0^{1/2} \frac{1}{1+x^{2n}}\, dx = \int_0^{1/2} \lim_{n\to\infty} \frac{1}{1+x^{2n}}\, dx = \int_0^{1/2} 1\, dx = \frac{1}{2}.$$

(b) (i) Let $\varepsilon > 0$. Take $\delta = \varepsilon/M$. Then for $x, y \in [a, b]$ with $|x - y| < \delta$ and $n \in \mathbb{N}$, by the Mean Value Theorem, there exists a point c between x and y such that

$$g_n(x) - g_n(y) = g_n'(c)(x - y)$$

so that

$$|g_n(x) - g_n(y)| = |g_n'(c)||x - y| \le M|x - y| < M\delta = \varepsilon.$$

(ii) Yes. Let $g(x) = \lim_{n\to\infty} g_n(x)$ for $x \in [a, b]$ and let $\varepsilon > 0$. By (i), there exists $\delta > 0$ such that

$$|g_n(x) - g_n(y)| < \frac{\varepsilon}{3} \text{ whenever } x, y \in [a, b],\ |x - y| < \delta \text{ and } n \in \mathbb{N}.$$

By letting $n \to \infty$, we also obtain

$$|g(x) - g(y)| \leq \frac{\varepsilon}{3} \text{ whenever } x, y \in [a, b], |x - y| < \delta \text{ and } n \in \mathbb{N}. \quad (*)$$

Next, choose a partition $P = \{x_0, x_1, ..., x_N\}$ of $[a, b]$ with $\|P\| < \delta$. Let $K \in \mathbb{N}$ be such that

$$n \geq K \Longrightarrow |g_n(x_i) - g(x_i)| < \frac{\varepsilon}{3} \quad \text{for } i = 1, 2, ..., N. \quad (**)$$

We now let $x \in [a, b]$ and $n \geq K$. Then $x \in [x_{i-1}, x_i]$ for some $1 \leq i \leq N$, and so $|x - x_i| < \delta$. It follows from this, $(*)$ and $(**)$ that

$$\begin{aligned} |g_n(x) - g(x)| &\leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g(x_i)| + |g(x_i) - g(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

4. (i) For each $n \in \mathbb{N}$, let $f_n(x) = \dfrac{(-1)^n}{\sqrt{n}} e^{\frac{x}{n}}$. Then $f = \displaystyle\sum_{n=1}^{\infty} f_n$.

First, we note that the series

$$\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by the Alternating Series Test.

Next, we consider the series

$$\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{\frac{x}{n}} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} e^{\frac{x}{n}}$$

For each $n \in \mathbb{N}$ and $x \in [-r, r]$,

$$\left| \frac{(-1)^n}{n^{3/2}} e^{\frac{x}{n}} \right| = \frac{1}{n^{3/2}} e^{\frac{x}{n}} \leq \frac{1}{n^{3/2}} e^{\frac{r}{n}} \leq \frac{e^r}{n^{3/2}}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{e^r}{n^{3/2}} = e^r \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a multiple of the p-series with $p = 3/2 > 1$, so it converges. By the Weiestrass M-test, the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[-r, r]$. By Theorem 8.3.5, the series converges uniformly to f on $[-r, r]$ and f is differentiable on $[-r, r]$.

(ii) Since f is differentiable on $[-r, r]$ for all $r > 0$, it is differentiable on $\mathbb{R}$.

5. (a) (i) Since

$$\lim_{n\to\infty}\left|\frac{1/(n+3)}{1/(n+2)}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+3}\right| = 1,$$

the radius of convergence is 1.

At $x = 1$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+2} = \sum_{n=1}^{\infty}\frac{1}{n} - 1$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+2}$ also diverges.

At $x = -1$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$. It converges by the Alternating Series Test.

So the series converges on $E = [-1, 1)$.

(ii) Let $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n+2}$ for $x \in [-1, 1)$. Then for $x \in (-1, 1)$,

$$x^2 f(x) = \sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2} = \sum_{n=0}^{\infty}\int_0^x t^{n+1}\,dt = \int_0^x\left(\sum_{n=0}^{\infty} t^{n+1}\,dt\right) = \int_0^x \frac{t}{1-t}\,dt$$

$$= \int_0^x\left(\frac{1}{1-t} - 1\right)dt = -\ln(1-t) - t\big|_0^x = -\ln(1-x) - x.$$

So for $x \in (-1, 1)$ with $x \neq 0$,

$$f(x) = -\frac{\ln(1-x)}{x^2} - \frac{1}{x}.$$

We also have

$$f(0) = \sum_{n=0}^{\infty}\frac{x^n}{n+2}\bigg|_{x=0} = \frac{1}{2}.$$

Moerover, since $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n+2}$ converges at $x = -1$, by Abel's Theorem,

$$f(-1) = \lim_{x\to -1}\left(-\frac{\ln(1-x)}{x^2} - \frac{1}{x}\right) = -\ln 2 + 1.$$

So

$$f(x) = \begin{cases} -\dfrac{\ln(1-x)}{x^2} - \dfrac{1}{x} & x \in [-1, 1)\backslash\{0\} \\ \dfrac{1}{2} & x = 0. \end{cases}$$

(b) (i) For $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= (x-1)^3 e^{-2x} = \frac{1}{e^2}(x-1)^3 e^{2(1-x)} \\ &= \frac{1}{e^2}(x-1)^3 \sum_{n=0}^{\infty} \frac{\{2(1-x)\}^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{e^2 n!}(x-1)^{n+3} \\ &= \sum_{m=3}^{\infty} \frac{(-1)^{m+1} 2^{m-3}}{e^2 (m-3)!}(x-1)^m. \end{aligned}$$

This series is the Taylor's Series for f about $x = 1$.

(ii) Take $m = 8$. Then

$$\frac{f^{(8)}(1)}{8!} = \frac{f^{(m)}(1)}{m!} = \frac{(-1)^{m+1} 2^{m-3}}{e^2 (m-3)!} = \frac{(-1)^9 2^5}{e^2 (5!)},$$

so that

$$f^{(8)}(1) = 8! \times \frac{-2^5}{e^2(5!)} = 8 \times 7 \times 6 \times \frac{-32}{e^2} = -\frac{10752}{e^2}.$$