# MA2108 MATHEMATICAL ANALYSIS I
# FINAL EXAM (2019/2020 SEMESTER II)

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**Question 1 (10 points).** Let $a_1 \geq 0$, $a_{n+1} = \dfrac{3(1+a_n)}{3+a_n}$, $n = 1, 2, \ldots$.

**(i)** Prove that $(a_n)$ converges.
**(ii)** Find the limit.

*Solution.*

**(i)** We first show that $0 \leq a_n \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_n$) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$) by induction on $n \in \mathbb{Z}^+$. Note that

$$a_{n+1} = \frac{3(1+a_n)}{3+a_n} = 3 - \frac{6}{3+a_n}.$$

When $n = 1$, if $0 \leq a_1 \leq \sqrt{3}$, then

$$1 = 3 - \frac{6}{3+0} \leq a_2 \leq 3 - \frac{6}{3+\sqrt{3}} = \sqrt{3}.$$

When $n = 1$, if $a_1 \geq \sqrt{3}$, then we see that

$$\sqrt{3} = 3 - \frac{6}{3+\sqrt{3}} \leq a_2.$$

Suppose that for $n = 1, 2, \ldots, k-1$, we have $0 \leq a_n \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_n$) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$). When $n = k$ and $0 \leq a_{k-1} \leq \sqrt{3}$, then

$$1 = 3 - \frac{6}{3+0} \leq a_k \leq 3 - \frac{6}{3+\sqrt{3}} = \sqrt{3}.$$

When $n = k$ and $\sqrt{3} \leq a_{k-1}$, then

$$\sqrt{3} = 3 - \frac{6}{3+\sqrt{3}} \leq a_k.$$

Hence, $0 \leq a_n \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_n$) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$).

Next, we show by induction on $n \in \mathbb{Z}^+$ that $(a_n)$ is monotonically increasing (resp. monotonically decreasing) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$). When $n = 1$, we have

$$a_2 = \frac{3(1+a_n)}{3+a_n} = \frac{\left(\sqrt{3}-a_1\right)\left(\sqrt{3}+a_1\right)}{3+a_1} \begin{cases} \geq 0 & \text{if } 0 \leq a_1 \leq \sqrt{3} \\ \leq 0 & \text{if } a_1 \geq \sqrt{3}. \end{cases}$$

Suppose that for $n = 1, 2, \ldots, k-1$, $(a_n)$ is monotonically increasing (resp. monotonically decreasing) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$). When $n = k$, we have

$$a_k = \frac{3(1+a_{k-1})}{3+a_{k-1}} = \frac{\left(\sqrt{3}-a_{k-1}\right)\left(\sqrt{3}+a_{k-1}\right)}{3+a_{k-1}} \begin{cases} \geq 0 & \text{if } 0 \leq a_{k-1} \leq \sqrt{3} \\ \leq 0 & \text{if } a_{k-1} \geq \sqrt{3}. \end{cases}$$

so that $(a_n)$ is monotonically increasing (resp. monotonically decreasing) when $0 \leq a_1 \leq \sqrt{3}$ (resp. $\sqrt{3} \leq a_1$).

Now, note that if $0 \leq a_1 \leq \sqrt{3}$, then $(a_n)$ is a bounded, monotonically increasing sequence of real numbers. By the monotone convergence theorem, $(a_n)$ converges. If $a_1 \geq \sqrt{3}$, then $(a_n)$ is a bounded, monotonically increasing sequence of real numbers. By the monotone convergence theorem, $(a_n)$ converges. If $a_1 = \sqrt{3}$, then $(a_n)$ is the constant sequence $\sqrt{3}$. Thus, in all cases, $(a_n)$ converges.

**(ii)** Let

$$L = \lim_{n\to\infty} a_n.$$

Then $3+a_n \to 3+L$, $3(1+a_n) \to 3(1+L)$ and $a_{n+1} \to L$ as $n \to \infty$. This implies that

$$L = \frac{3(1+L)}{3+L}$$

and so $L = \sqrt{3}$ since $a_n \geq 0$ for all $n \in \mathbb{Z}^+$ and so $L \geq 0$. □

**Question 2 (10 points).** Let $(a_n)$ be a sequence in $\mathbb{R}$.

**(i)** Prove that if

$$\lim_{n\to\infty} a_n = a,$$

then

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a.$$

**(ii)** Suppose that the sequence

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

converges, can we deduce that $(a_n)$ converges[1]? Justify your answer.

*Solution.*

**(i)** Let $\varepsilon > 0$. Then there exists $K > 0$ such that for $n \geq K$, $|a_n - a| < \varepsilon/2$. Now, let

$$n > \max\left\{2K, \frac{2K \cdot \max\{|a_1-a|, \ldots, |a_K-a|\}}{\varepsilon}\right\}.$$

Then

$$\begin{aligned}
\left|\frac{a_1+a_2+\cdots+a_n}{n} - a\right| &\leq \left|\frac{a_1-a}{n}\right| + \cdots + \left|\frac{a_K-a}{n}\right| + \cdots + \left|\frac{a_n-a}{n}\right| \\
&\leq \frac{K}{n}\max_{i=1,\ldots,n}|a_i-a| + \sum_{i=K+1}^{n}\frac{\varepsilon}{2n} \\
&\leq \frac{K}{n}\max_{i=1,\ldots,n}|a_i-a| + \sum_{i=K+1}^{n}\frac{\varepsilon}{2(n-K)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a.$$

**(ii)** No. Let $a_n = (-1)^n$ for each $n \in \mathbb{Z}^+$. Then

$$\frac{a_1+a_2+\cdots+a_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

so that $\frac{a_1+a_2+\cdots+a_n}{n} \to 0$ as $n \to \infty$. But $(a_n)$ is not a convergent sequence. □

[1] Interested readers can look up Cesàro summation.

**Question 3 (15 points).** Let $(x_n)$ and $(y_n)$ be two bounded sequences in $\mathbb{R}$.

**(i)** Prove that

$$\liminf x_n + \liminf y_n \leq \liminf (x_n + y_n).$$

**(ii)** Suppose there exists an $N \in \mathbb{N}$ such that when $n > N$, one has $x_n \leq y_n$. Prove that

$$\liminf x_n \leq \liminf y_n.$$

*Solution.*

**(i)** Since $(x_n)$ and $(y_n)$ are bounded sequences in $\mathbb{R}$, then there exist $M_x, M_y \in \mathbb{R}$ such that $|x_n| \leq M_x$ and $|y_n| \leq M_y$ for all $n \in \mathbb{Z}^+$. Let $z_n = x_n + y_n$. Then $z_n$ is a bounded sequence since

$$|z_n| = |x_n + y_n| \leq |x_n| + |y_n| \leq M_x + M_y.$$

Since $(z_n)$ is a bounded sequence, then by the Bolzano-Weierstrass theorem, there exists a subsequence $(z_{n_k})$ of $(z_n)$ such that $z_{n_k}$ converges to some $z_0 \in \mathbb{R}$. Write $z_{n_k} = x_{n_k} + y_{n_k}$. Then $(x_{n_k})$ and $(y_{n_k})$ are bounded subsequences of $(x_n)$ and $(y_n)$ respectively. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\left(x_{n_{k_\ell}}\right)$ of $(x_{n_k})$. Consider the subsequence $\left(y_{n_{k_\ell}}\right)$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\left(y_{n_{k_{\ell_m}}}\right)$ of $\left(y_{n_{k_\ell}}\right)$. Now, $\left(x_{n_{k_{\ell_m}}}\right)$ is a subsequence of the convergent sequence $\left(x_{n_{k_\ell}}\right)$, and so $\left(x_{n_{k_{\ell_m}}}\right)$ is also convergent. The subsequence $\left(z_{n_{k_{\ell_m}}}\right)$ is a subsequence of the convergent sequence $(z_{n_k})$ and so $\left(z_{n_{k_{\ell_m}}}\right)$ is also convergent.

$$z_0 = \lim_{k\to\infty} z_{n_k} = \lim_{m\to\infty} z_{n_{k_{\ell_m}}} = \lim_{m\to\infty} \left(x_{n_{k_{\ell_m}}} + y_{n_{k_{\ell_m}}}\right) = x_0 + y_0 \geq \inf S(x_n) + \inf S(y_n).$$

Since $(z_{n_k})$ was an arbitrary convergent subsequence of $(z_n)$, then

$$\liminf (x_n + y_n) = \inf S(z_n) \geq \inf S(x_n) + \inf S(y_n) = \liminf x_n + \liminf y_n.$$

**(ii)** We may assume that $x_n \leq y_n$ since the subsequential limit of any sequence does not depend on the first (finite) $K$ terms. Since $(x_n)$ and $(y_n)$ are bounded sequences in $\mathbb{R}$, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $(y_{n_k})$ of $(y_n)$, which converges to some $y_0 \in \mathbb{R}$. Consider the corresponding subsequence $(x_{n_k})$ of $(x_n)$, and so $(x_{n_k})$ is also a bounded sequence. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\left(x_{n_{k_\ell}}\right)$ of $(x_{n_k})$ which converges to some $x_0 \in \mathbb{R}$. The subsequence $\left(y_{n_{k_\ell}}\right)$ is a subsequence of the convergent sequence $(y_{n_k})$ and so $\left(y_{n_{k_\ell}}\right)$ also converges to the same $y_0 \in \mathbb{R}$. Now,

$$\liminf x_n \leq x_0 = \lim_{\ell\to\infty} x_{n_{k_\ell}} \leq \lim_{\ell\to\infty} y_{n_{k_\ell}} = \lim_{k\to\infty} y_{n_k} = y_0$$

Since $(y_{n_k})$ was an arbitrary convergent subsequence of $(y_n)$, then

$$\liminf x_n \leq \liminf y_n.$$

$\square$

**Question 4 (15 points).**

**(i)** Use definition to prove that

$$\lim_{x\to+\infty} \frac{3x^2 + 2x - 1}{x^2 - 2} = 3.$$

**(ii)** Find

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x + \sqrt{1+\sin^2 x} - 1}.$$

**(iii)** Suppose

$$\lim_{x\to 1} \frac{3x^2 + bx + c}{x^2 - 1} = 2.$$

Find $b$ and $c$.

*Solution.*

**(i)** Let $\varepsilon > 0$ and let

$$K = \max\left\{\sqrt{2}+2, \frac{10+\varepsilon\sqrt{2}}{\varepsilon}\right\}.$$

Note that

$$\begin{aligned}\left|\frac{3x^2+2x-1}{x^2-2}-3\right| &= \left|\frac{2x+5}{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}\right| \\ &\le \left|\frac{2x}{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}\right| + \left|\frac{5}{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}\right| \quad \text{by triangle inequality} \\ &\le \left|\frac{2x}{x\left(x-\sqrt{2}\right)}\right| + \left|\frac{5}{x-\sqrt{2}}\right| \\ &= \left|\frac{2}{x-\sqrt{2}}\right| + \left|\frac{5}{x-\sqrt{2}}\right|.\end{aligned}$$

Now for any $x > K$, we have

$$\begin{aligned} x > K > \frac{4+\varepsilon\sqrt{2}}{\varepsilon} &\iff \frac{2}{x-\sqrt{2}} < \frac{\varepsilon}{2} \\ x > K > \frac{10+\varepsilon\sqrt{2}}{\varepsilon} &\iff \frac{5}{x-\sqrt{2}} < \frac{\varepsilon}{2}.\end{aligned}$$

Hence,

$$\left|\frac{3x^2+2x-1}{x^2-2}-3\right| \le \left|\frac{2}{x-\sqrt{2}}\right| + \left|\frac{5}{x-\sqrt{2}}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{x\to+\infty}\frac{3x^2+2x-1}{x^2-2} = 3.$$

**(ii)** Note that for any $x \in \mathbb{R}$, we have $\sqrt{1+\sin^2 x} \ge 1$. Then

$$0 \le \left|\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x+\sqrt{1+\sin^2 x}-1}\right| \le \left|\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x-1}\right|.$$

Since

$$\lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0}\left|\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x-1}\right| = \left|\frac{0}{-1}\right| = 0,$$

then by squeeze theorem,

$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x+\sqrt{1+\sin^2 x}-1} = 0.$$

**(iii)** Note that

$$\frac{3x^2+bx+c}{x-1} = 3x+(b+3)+\frac{b+c+3}{x-1}.$$

Then

$$\lim_{x\to 1}\frac{3x^2+bx+c}{x-1} = \lim_{x\to 1}\left(\left(\frac{3x^2+bx+c}{x^2-1}\right)(x+1)\right) = \lim_{x\to 1}\left(\frac{3x^2+bx+c}{x^2-1}\right)\cdot\lim_{x\to 1}(x+1) = 4.$$

In order for

$$\lim_{x\to 1}\frac{3x^2+bx+c}{x-1} = 4$$

we must have $b+c+3=0$, or equivalently, $b=-c-3$. This implies that

$$4 = \lim_{x\to 1}(3x+b+3) = 6+b$$

so that $b = -2$. This implies that $c = -1$. $\square$

**Question 5 (15 points).** Prove that the function $f(x) = \sqrt{x(x-1)}$ is uniformly continuous on $[1, +\infty)$.

*Solution.* Let $\varepsilon > 0$. We first show that $f(x)$ is uniformly continuous on $[2, +\infty)$. Let $x, y \in [2, +\infty)$. Choose $\delta_1 = \frac{2}{3}\varepsilon$, so that if $|x - y| < \delta_1$, then

$$
\begin{aligned}
|f(x) - f(y)| &= \left|\sqrt{x(x-1)} - \sqrt{y(y-1)}\right| \\
&= \left|\frac{x(x-1) - y(y-1)}{\sqrt{x(x-1)} + \sqrt{y(y-1)}}\right| \\
&\leq \left|\frac{x(x-1) - y(y-1)}{\sqrt{(x-1)^2} + \sqrt{(y-1)^2}}\right| \\
&= \left|\frac{x(x-1) - y(y-1)}{(x-1) + (y-1)}\right| \\
&= |x - y| \left|\frac{x + y - 1}{x + y - 2}\right| \\
&= |x - y| \left|1 + \frac{1}{x + y - 2}\right| \\
&\leq |x - y| \cdot \frac{3}{2} < \delta_1 = \varepsilon.
\end{aligned}
$$

Hence, $f(x)$ is uniformly continuous on $[2, +\infty)$. On the other hand, since $f(x)$ is continuous on $[1, 10]$, then $f(x)$ is uniformly continuous on $[1, 10]$ as $[1, 10]$ is a compact interval (or closed and bounded if you wish by the Heine-Borel theorem)[2]. Now let $\delta = \min\{\delta_1, 1/10\}$. Then for any $x, y \in [1, +\infty)$ satisfying $|x - y| < \delta$, we must have $x, y \in [1, 10]$ or $x, y \in [2, +\infty)$. Therefore, $f(x)$ is uniformly continuous on $[1, +\infty)$. $\square$

**Question 6 (15 points).**

**(i)** Let $f$ be an increasing function on $[a, b]$ and $f([a, b]) = [f(a), f(b)]$. Prove that $f$ is continuous on $[a, b]$.

**(ii)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Suppose for any two rational numbers $r_1, r_2, r_1 < r_2$, we have $f(r_1) < f(r_2)$. Prove that $f$ is strictly increasing on $\mathbb{R}$.

*Solution.*

**(i)** Suppose on the contrary that $f$ is not continuous at some $c \in [a, b]$.

- **Case 1:** Suppose $c$ is not an end point, so $c \in (a, b)$. Let

$$L_1 = \sup\{f(x) : x \in (a, c)\} \quad \text{and} \quad L_2 = \inf\{f(x) : x \in (c, b)\}.$$

Since $f$ is increasing, then $L_1 \leq f(c) \leq L_2$. Without loss of generality, we may assume that $L_1 < f(c) \leq L_2$. Then there exists $y \in (L_1, f(c)) \subseteq [f(a), f(b)]$. This implies that there exists $d \in (a, b)$ such that $f(d) = y$. If $d < c$, then by the definition of $L_1$ and that $f$ is increasing, we must have $f(d) \leq L_1$. But this means that $f(d) \leq L_1 < f(d)$, which is a contradiction. If $d > c$, since $f$ is increasing, then $f(c) < f(d)$, contradicting that $f(d) \leq f(c) < f(d)$. Thus, $c = d$, so that $f(c) = f(d)$. But since $y = f(d) \in (L_1, f(c))$, then $f(d) < f(c)$, which is a contradiction. Thus, $f$ must be continuous at $c$ when $c$ is not an end point.

- **Case 2:** Suppose $c$ is an end point. Say $c = a$ and $f$ are not continuous at $a$. Let

$$L_3 = \inf\{f(x) : x \in (a, b]\}.$$

Then $L_3 > f(a)$. Pick $y \in (f(a), L_3) \subseteq [f(a), f(b)]$. Then there exists $d \in [a, b]$ such that $f(d) = y$. If $d > a$, then $L_3 \leq f(d)$. By the definition of $y$, we have $f(a) < f(d) < L_3 \leq f(d)$, which is a contradiction. This means that we must have $d = a$. But this means that $f(a) < y = f(d) = f(a)$, which is a contradiction. Thus, $f$ is continuous at $c = a$. If $f$ is not continuous at $b$, then we also get a contradiction using a similar argument.

Therefore, $f$ is continuous on $[a, b]$.

[2]There is a result, known as the Heine-Cantor theorem, which states that if $f : K \to \mathbb{R}$ where $K$ is a compact subset of $\mathbb{R}$, then $f$ is uniformly continuous.

**(ii)** We argue by contradiction. Suppose there exist real numbers $x < y$ such that $f(x) \geq f(y)$. There are two cases to consider.

- **Case 1:** Suppose $f(x) = f(y)$. Since $f$ is continuous on the closed interval $[x, y]$, by the extreme value theorem, $f$ attains a minimum and a maximum on $[x, y]$. As $f(x) = f(y)$ and $f$ is non-constant on any interval, the only possibility is that

$$f(x) = f(y) = \min\{f(z) : z \in [x, y]\} = \max\{f(z) : z \in [x, y]\}.$$

Thus, $f$ is be constant on $[x, y]$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ (or any closed sub-interval of $\mathbb{R}$), there exist $r_1, r_2 \in [x, y]$ with $r_1 < r_2$ such that $f(r_1) = f(r_2)$. This contradicts our assumption that for any two rationals with $r_1 < r_2$, it holds that $f(r_1) < f(r_2)$.

- **Case 2:** Suppose $f(x) > f(y)$. Define $\varepsilon = f(x) - f(y) > 0$. By the intermediate value theorem, since $f$ is continuous on $[x, y]$, it attains every value between $f(x)$ and $f(y)$. In particular, there exist $c, d \in [x, y]$ where $c < d$ such that

$$f(c) = f(x) - \frac{\varepsilon}{3} \quad \text{and} \quad f(d) = f(x) - \frac{2\varepsilon}{3}.$$

Again, using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < c$ and $d < r_2 < y$. The continuity of $f$ ensures that if we choose $r_1$ close enough to $c$, then

$$f(r_1) > f(c) = f(x) - \frac{\varepsilon}{3},$$

and if we choose $r_2$ close enough to $d$, then

$$f(r_2) < f(d) = f(x) - \frac{2\varepsilon}{3}.$$

Thus,

$$f(r_1) > f(x) - \frac{\varepsilon}{3} > f(x) - \frac{2\varepsilon}{3} > f(r_2).$$

However, this means that we have found two rational numbers $r_1 < r_2$ such that $f(r_1) > f(r_2)$, contradicting the hypothesis that for any two rational numbers $r_1 < r_2$ we have $f(r_1) < f(r_2)$. $\square$

Here is an alternative solution to Question 6**(ii)**.

*Solution.* We first observe that $f$ cannot be the constant function on any non-empty interval. Let $a, b \in \mathbb{R}$ with $a < b$. If $f(a) < f(b)$, then we are done.

We claim that if $f(a) > f(b)$, then there exists rational numbers $r_1, r_2$ such that $f(r_1) > f(r_2)$. To see why this holds, let

$$\varepsilon = \frac{f(a) - f(b)}{4}.$$

Then $f(a) > f(a) - \varepsilon > f(b) + \varepsilon > f(b)$. By the intermediate value theorem, there exists $d_1 \in (a, b)$ such that $f(d_1) = f(a) - \varepsilon$. Then there exists $\delta_1 > 0$ such that for all $x \in (a, a + \delta_1)$, $f(x) \geq f(d_1)$. By the intermediate value theorem, there exists $d_2 \in (a, b)$ such that $f(d_2) = f(b) + \varepsilon$. Then there exists $\delta_2 > 0$ such that for all $x \in (b - \delta_2, b)$, $f(x) \leq f(d_2)$. Note that $(a, a + \delta_1) \cap (b - \delta_2, b) = \emptyset$. By the Archimedian property for real numbers, there exists rational numbers $r_1, r_2$ such that $r_1 \in (a, a + \delta_1)$ and $r_2 \in (b - \delta_2, b)$. This implies that

$$f(r_1) \geq f(d_1) > f(d_2) \geq f(r_2)$$

which proves the claim. To tackle the problem, we shall consider two cases.

- **Case 1:** $f(a) = f(b)$. Since $f$ is continuous, then $f$ attains absolute maximum and absolute minimum on $[a, b]$. This means that there exist $c_{\min}, c_{\max} \in [a, b]$ such that $f(c_{\max}) \geq f(x) \geq f(c_{\min})$ for all $x \in [a, b]$. By claim 1, we must have $a \leq c_{\min} < c_{\max} \leq b$. If $c_{\max} < b$, and $f(c_{\max}) = f(b)$, then since $f$ cannot be constant on any interval, there exists $y \in (c_{\max}, b)$ such that $f(y) < f(c_{\max})$, and by the claim, we get a contradiction. If $c_{\max} < b$ and $f(c_{\max}) > f(b)$, by the claim, we also get a contradiction. Hence, $c_{\max} = b$. If $a < c_{\min}$, and $f(b) = f(a) > f(c_{\min})$, by the claim, we get a contradiction. If $a < c_{\min}$ and $f(b) = f(a) = f(c_{\min})$, then $f$ is constant on $[a, b]$, which is a contradiction. Hence, $a = c_{\min}$. But this implies that on $[a, b]$,

$$f(c_{\min}) = f(a) = f(b) = f(c_{\max})$$

so that $f$ is constant on $[a,b]$, which is a contradiction.

- Suppose $f(a) > f(b)$. By the claim, we will get a contradiction. Hence, we must have $f(a) < f(b)$ and so $f$ is increasing on $\mathbb{R}$. □

**Question 7 (10 points).** Let $f$ be continuous on $[0,1]$ and $f(0) = f(1)$. Prove that for any positive integer $n$, there exists $\xi \in [0,1]$ such that

$$f\left(\xi + \frac{1}{n}\right) = f(\xi).$$

*Solution.* Let

$$g(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

Then we observe that the following sum telescopes:

$$\sum_{k=0}^{n-1} g\left(\frac{k}{n}\right) = f(1) - f(0) = 0$$

If there exists $0 \le k \le n-1$ such that $g(k/n) = 0$, then

$$f\left(\frac{k}{n} + \frac{1}{n}\right) = f\left(\frac{k}{n}\right).$$

If for all $0 \le k \le n-1$, $g(k/n) \ne 0$, then there exists $0 \le m_1 \ne m_2 \le n-1$ such that $g(m_1/n) > 0$ and $g(m_2/n) < 0$. By the intermediate value theorem, there exists $\xi \in [a,b]$ such that $g(\xi) = 0$, or equivalently,

$$f\left(\xi + \frac{1}{n}\right) = f(\xi).$$

□

**Question 8 (10 points).** Let $f : (a, +\infty) \longrightarrow \mathbb{R}$ be a function such that it is bounded in any interval $(a,b)$ and

$$\lim_{x\to+\infty} [f(x+1) - f(x)] = A.$$

Prove that

$$\lim_{x\to+\infty} \frac{f(x)}{x} = A.$$

*Solution.* Let $\varepsilon > 0$. Since

$$\lim_{x\to+\infty} [f(x+1) - f(x)] = A$$

then

$$\begin{aligned}
&\lim_{x\to+\infty} (f(x+M) - f(x)) \\
&= \lim_{x\to+\infty} (f(x+M) - f(x+M-1) + f(x+M-1) - f(x+M-2) + \cdots + f(x+1) - f(x)) \\
&= MA.
\end{aligned}$$

There exists $K_1$ such that for all $x > K_1$,

$$|f(x+M) - f(x) - MA| < \varepsilon.$$

Since $f$ is bounded in any open interval, for any $[N, N+1]$, there exists $K_N \in \mathbb{R}$ such that $|f(x)| \le K_N$ for all $x \in [N, N+1]$. Next, for any $y > N$, there exists an integer $M$ such that $y = x + M$, where $x \in [N, N+1]$ so that

$$\begin{aligned}
|f(y) - yA| &= |f(x+M) - MA + MA - yA| \\
&\le |f(x+M) - MA| + |(M-y)A| \\
&< M\varepsilon + K + (N+1)|A| \\
&\le (y-N)\varepsilon + K + (N+1)|A| \\
&= y\varepsilon + C,
\end{aligned}$$

where $C = -N\varepsilon + K + (N+1)|A|$ is independent of $y$. So

$$\left|\frac{f(y)}{y} - A\right| \leq \varepsilon + \frac{C}{y}.$$

Choose $N_2 > N$ such that for all $y > N_2$, $C/y < \varepsilon$. Hence,

$$\left|\frac{f(y)}{y} - A\right| \leq \varepsilon + \frac{C}{y} < 2\varepsilon.$$

Therefore,

$$\lim_{x \to +\infty} \frac{f(x)}{x} = A.$$

□