MA3110
Solutions to Final Exam

1. (18 marks)
Answer TRUE or FALSE to each of the following questions. No explanation is necessary.

(a) If f is differentiable on $[a, b]$, then f is integrable on $[a, b]$.

(b) If f is integrable on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f \quad x \in [a, b]$$

is differentiable on $[a, b]$.

(c) If (f_n) and (g_n) both converge uniformly on a set E, then $(f_n g_n)$ also converges uniformly on E.

(d) If $f_n \to f$ uniformly on $[a, b]$ and for each n, f_n is differentiable on $[a, b]$, then f is differentiable on $[a, b]$.

(e) If (f_n) is a sequence of functions defined on a set E and $\sum_{n=1}^{\infty} \|f_n\|_E$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

(f) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R where $0 < R < \infty$, then $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $(-R, R)$.

Solutions: (a) TRUE. Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$. By Theorem 7.2.3, f is integrable on $[a, b]$.

(b) FALSE. Page 16 of Chapter 7 gives the following counter example: Let $f(x) = -1$ for $-1 \le x < 0$ and $f(x) = 1$ for $0 \le x \le 1$. Then $F(x) = |x|$ for $x \in [-1, 1]$ and $F'(0)$ does not exist.

(c) FALSE. See Question 6 of Tutorial 7.

(d) FALSE. See Question 4 of Tutorial 8.

(e) TRUE. This follows from the Weiestrass M-test by taking $M_n = \|f_n\|_E$.

(f) FALSE. The geometric series $\sum_{n=0}^{\infty} x^n$ has radius of convergence 1, but by Question H2 of Tutorial 9, $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$.

2. (22 marks)

(a) Using the Riemann integral of a suitably chosen function, find the limit

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{k=1}^{n} \sqrt{n+k}.$$

(b) Let f and g be continuous functions on $[a,b]$ such that $f(x) > 0$ and $g(x) > 0$ for all $x \in [a,b]$.

(i) Prove that for any $c \in (a,b]$, $\displaystyle\int_a^c f(x)\,dx > 0$.

(ii) Let

$$h(x) = e^{-x}\left(\int_a^x f(t)\,dt\right)\left(\int_x^b g(t)\,dt\right) \quad \text{for each } x \in [a.b].$$

Prove that

$$e^x(h(x) + h'(x)) = f(x)\int_x^b g(t)\,dt - g(x)\int_a^x f(t)\,dt.$$

(iii) Prove that there exists $d \in (a,b)$ such that

$$\frac{f(d)}{\int_a^d f(t)\,dt} - \frac{g(d)}{\int_d^b g(t)\,dt} = 1.$$

Solutions: (a) Let $f(x) = \sqrt{1+x}$. For each $n \in \mathbb{N}$, let

$$P_n = \left\{\frac{k}{n} : 0 \le k \le n\right\} \quad \text{and} \quad \xi^{(n)} = \left(\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right).$$

Then P_n is a partition of $[0,1]$, and

$$\frac{1}{n^{3/2}} \sum_{k=1}^{n} \sqrt{n+k} = \sum_{k=1}^{n} \sqrt{\frac{n+k}{n}} \cdot \frac{1}{n} = \sum_{k=1}^{n} \sqrt{1+\frac{k}{n}} \cdot \frac{1}{n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\Delta x_k = S(f, P_n)(\xi^{(n)}).$$

Since $\|P_n\| = \dfrac{1}{n} \to 0$,

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{k=1}^{n} \sqrt{n+k} = \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^1 = \frac{2}{3}(2\sqrt{2}-1).$$

(b) (i) **Method 1:** Let $c \in (a, b]$. Since f is continuous on $[a, b]$, it is continuous on $[a, c]$. By the Extreme-Value Theorem, there exist $x_0 \in [a, c]$ such that

$$f(x) \geq f(x_0) > 0 \quad \forall x \in [a, c].$$

Then

$$\int_a^c f(x)\,dx \geq \int_a^c f(x_0)\,dx = f(x_0)(c - a) > 0.$$

Method 2: Let $F(x) = \displaystyle\int_a^x f(t)\,dt$ for $x \in [a, b]$. Since f is continuous on $[a, b]$, by the Fundamental Theorem of Calculus I, $F'(x) = f(x) > 0$ for $x \in [a, b]$. So F is strictly increasing on $[a, b]$. It follows that for any $c \in (a, b]$, $F(a) < F(c)$ so that

$$\int_a^c f(x)\,dx = F(c) - F(a) > 0.$$

(ii) We have

$$e^x h(x) = \left(\int_a^x f(t)\,dt\right)\left(\int_x^b g(t)\,dt\right).$$

By the product rule and the Fundamental Theorem of Calculus I,

$$e^x(h(x) + h'(x)) = e^x h(x) + e^x h'(x) = f(x)\int_x^b g(t)\,dt - g(x)\int_a^x f(t)\,dt.$$

(iii) Since $\displaystyle\int_a^a f(t)\,dt = 0$ and $\displaystyle\int_b^b g(t)\,dt = 0$, $h(a) = 0$ and $h(b) = 0$. By Rolle's Theorem, there exists $d \in (a, b)$ such that $h'(d) = 0$. By (ii),

$$\left(\int_a^d f(t)\,dt\right)\left(\int_d^b g(t)\,dt\right) = e^d h(d) = e^d(h(d) + h'(d)) = f(d)\int_d^b g(t)\,dt - g(d)\int_a^d f(t)\,dt.$$

Now by (i), $\displaystyle\int_a^d f(t)\,dt > 0$. Similarly, we have $\displaystyle\int_d^b g(t)\,dt > 0$.

Dividing all the terms of the above equation by $\left(\displaystyle\int_a^d f(t)\,dt\right)\left(\displaystyle\int_d^b g(t)\,dt\right)$, we obtain

$$1 = \frac{f(d)}{\int_a^d f(t)\,dt} - \frac{g(d)}{\int_d^b g(t)\,dt}.$$

3. (20 marks)

(a) For each $n \in \mathbb{N}$, let
$$f_n(x) = \frac{x + x^n}{2 + x^n} \quad x \in [0, 1].$$

(i) Find $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0, 1]$.

(ii) Prove that for each $a \in (0, 1)$, $f_n \to f$ uniformly on $[0, a]$.

(iii) Does (f_n) converge uniformly on $[0, 1]$? Justify your answer.

(iv) Find $\lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{x + x^n}{2 + x^n}\, dx$.

(b) Let (g_n) be a sequence of functions on $[a, b]$ with the following properties:

(i) For each $n \in \mathbb{N}$, g_n is increasing on $[a, b]$.

(ii) The sequence (g_n) converges pointwise to a function g on $[a, b]$.

(iii) The function g is continuous on $[a, b]$.

Prove that $g_n \to g$ uniformly on $[a, b]$.

Solutions: (a) (i) For $x \in [0, 1)$, $x^n \to 0$, so that
$$f(x) = \lim_{n\to\infty} \frac{x + x^n}{2 + x^n} = \frac{x + 0}{2 + 0} = \frac{x}{2}.$$
For $x = 1$, $f_n(1) = \frac{1 + 1}{2 + 1} = \frac{2}{3} \to \frac{2}{3}$. Thus
$$f(x) = \begin{cases} \dfrac{x}{2} & x \in [0, 1) \\[2ex] \dfrac{2}{3} & x = 1. \end{cases}$$

(ii) For $x \in [0, a]$, $|x| \le a$ and
$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{x + x^n}{2 + x^n} - \frac{x}{2} \right| = \left| \frac{2x + 2x^n - 2x - x^{n+1}}{2(2 + x^n)} \right| \\ &= \left| \frac{2x^n - x^{n+1}}{2(2 + x^n)} \right| \le \frac{2x^n + x^{n+1}}{4} \le \frac{1}{4}(2a^n + a^{n+1}). \end{aligned}$$
It follows that
$$\|f_n - f\|_{[0,a]} \le \frac{1}{4}(2a^n + a^{n+1}) \to 0.$$

(iii) Note that each f_n is continuous on $[0, 1]$ but f is not continuous at $x = 1$. So (f_n) does not converge uniformly on $[0, 1]$.

(iv) Since $f_n \to f$ uniformly on $[0, \frac{1}{2}]$,

$$\lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{x + x^n}{2 + x^n}\, dx = \int_0^{\frac{1}{2}} \lim_{n\to\infty} \frac{x + x^n}{2 + x^n}\, dx = \int_0^{\frac{1}{2}} \frac{x}{2}\, dx = \frac{1}{16}.$$

(b) First we claim that g is increasing on $[a, b]$. In fact, for $a \le s < t \le b$, $g_n(s) \le g_n(t)$ for all $n \in \mathbb{N}$, so that

$$g(s) = \lim_{n\to\infty} g_n(s) \le \lim_{n\to\infty} g_n(t) = g(t).$$

We now let $\varepsilon > 0$. Since g is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. So there exists $\delta > 0$ such that

$$x, y \in [a, b],\ |x - y| < \delta \Longrightarrow |g(x) - g(y)| < \frac{\varepsilon}{2}.$$

Let $P = \{x_0, x_1, ..., x_m\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. For each $0 \le j \le m$, since $\lim_{n\to\infty} g_n(x_j) = g(x_j)$, there exits $K_j \in \mathbb{N}$ such that

$$n \ge K_j \Longrightarrow |g_n(x_j) - g(x_j)| < \frac{\varepsilon}{2}.$$

Let $K = \max(K_0, K_1, ..., K_m)$. Then

$$n \ge K \Longrightarrow |g_n(x_j) - g(x_j)| < \frac{\varepsilon}{2} \quad \forall 0 \le j \le m.$$

Claim: $n \ge K \Longrightarrow |g_n(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.
Let $x \in [a, b]$. Then $x \in [x_{i-1}, x_i]$ for some $1 \le i \le m$. For $n \ge K$, we have

$$g(x_{i-1}) - \frac{\varepsilon}{2} < g_n(x_{i-1}) \le g_n(x) \le g_n(x_i) < g(x_i) + \frac{\varepsilon}{2}.$$

Since g is also increasing on $[a, b]$, so

$$g(x_{i-1}) \le g(x) \le g(x_i).$$

It follows that

$$-\varepsilon = -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} < g(x_{i-1}) - g(x_i) - \frac{\varepsilon}{2} \le g_n(x) - g(x) \le g(x_i) - g(x_{i-1}) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

4. (20 marks)

Let $f(x) = \sum_{n=1}^{\infty} \frac{|x|}{x^2 + n^2}$, $x \in \mathbb{R}$.

(i) Show that for each $r > 0$, the series converges uniformly on $[-r, r]$.

(ii) Prove that f is continuous on $\mathbb{R}$.

(iii) Prove that f is differentiable on $(0, \infty)$.

(iv) Does $f'(0)$ exist? Justify your answer.

Solutions: For each $n \geq 0$, let $f_n(x) = \frac{|x|}{x^2 + n^2}$, $x \in \mathbb{R}$. Then $f = \sum_{n=1}^{\infty} f_n$.

(i) For $n \in \mathbb{N}$ and $x \in [-r, r]$,

$$|f_n(x)| = \frac{|x|}{x^2 + n^2} \leq \frac{r}{n^2}.$$

So

$$\|f_n\|_{[-r,r]} \leq \frac{r}{n^2} \quad \forall n \in \mathbb{N}.$$

Since the series $\sum_{n=1}^{\infty} \frac{r}{n^2} = r \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Weiestrass M-test, the series $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[-r, r]$.

(ii) For any $r > 0$, since the series $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[-r, r]$ and each f_n is continuous on $[-r, r]$, by Corollary 8.3.3, f is continuous on $[-r, r]$. Since $\mathbb{R} = \bigcup_{r>0} [-r, r]$, f is continuous on $\mathbb{R}$.

(iii) For $n \in \mathbb{N}$ and $x > 0$, we have

$$f_n'(x) = \frac{n^2 - x^2}{(x^2 + n^2)^2}$$

so that

$$|f_n'(x)| \leq \left| \frac{n^2 - x^2}{(x^2 + n^2)^2} \right| \leq \frac{n^2 + x^2}{(x^2 + n^2)^2} = \frac{1}{x^2 + n^2} \leq \frac{1}{n^2}.$$

Thus,

$$\|f_n'\|_{(0,\infty)} \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Weiestrass M-test, the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $(0, \infty)$.

It follows that for any $0 < a < b$, the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$. Hence by Theorem 8.3.5, f is differentiable on $[a, b]$. Since $(0, \infty) = \bigcup_{0<a<b} [a, b]$, f is differentiable on $(0, \infty)$.

(iv) We claim that $f'(0)$ does not exist.

First note that

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0} \frac{\sum_{n=1}^{\infty} \frac{|h|}{h^2+n^2}}{h} = \lim_{h\to 0} \frac{|h|}{h} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2}. \quad (*)$$

Let $g(h) = \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2}$, $h \in \mathbb{R}$. Then since $\frac{1}{h^2 + n^2} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$, by the Weiestrass M-test, the series $g(h) = \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2}$ converges uniformly on $\mathbb{R}$. So g is continuous on $\mathbb{R}$, and in particular at 0. Thus

$$\lim_{h\to 0} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2} = \lim_{h\to 0} g(h) = g(0) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Using this and $(*)$, we obtain

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0} \frac{h}{h} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2} = \lim_{h\to 0} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} > 0$$

and

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0} \frac{-h}{h} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2} = -\lim_{h\to 0} \sum_{n=1}^{\infty} \frac{1}{h^2 + n^2} = -\sum_{n=1}^{\infty} \frac{1}{n^2} < 0.$$

Hence $f'(0)$ does not exist.

5. (20 marks)

(a) Consider the power series $\sum_{n=1}^{\infty}(-1)^n n x^{2n}$.

(i) Find its radius of convergence and the set E of all $x \in \mathbb{R}$ at which the series converges.

(ii) Find a close form of its sum function on E.

(b) Let g be a function on $\mathbb{R}$ such that

$$g(x) = \frac{\cos(x^2) - 1}{x^4} \quad \text{for all } x \neq 0,$$

and g is continuous at $x = 0$.

(i) Find $g(0)$.

(ii) Show that $g^{(n)}(0)$ exists for all $n \in \mathbb{N}$.

(iii) Find the values of $g^{(5)}(0)$, $g^{(6)}(0)$, $g^{(7)}(0)$ and $g^{(8)}(0)$.

(You may assume that $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.)

(a) (i) Note that

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{2n+2}}{(-1)^n n x^{2n}}\right| = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)x^2 = x^2.$$

By the ratio test, the series $\sum_{n=0}^{\infty}(-1)^n n x^{2n}$ converges if $x^2 < 1$ (equivalently $|x| < 1$) and diverges if $x^2 > 1$ (equivalently $|x| > 1$). Hence, the radius of convergence is 1.

At $x = \pm 1$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n$, which clearly diverges.

So $\sum_{n=1}^{\infty}(-1)^n n x^{2n}$ converges on $E = (-1, 1)$.

(ii) Let $f(x) = \sum_{n=1}^{\infty}(-1)^n n x^{2n}$ for $x \in (-1, 1)$. Then for $x \in (-1, 1)$,

$$\begin{aligned} f(x) &= \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n 2n x^{2n-1} = \frac{x}{2}\sum_{n=1}^{\infty}\frac{d}{dx}((-1)^n x^{2n}) = \frac{x}{2}\frac{d}{dx}\sum_{n=0}^{\infty}(-x^2)^n \\ &= \frac{x}{2}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{x}{2}\left(\frac{-2x}{(1+x^2)^2}\right) = -\frac{x^2}{(1+x^2)^2}. \end{aligned}$$

(b)(i) By the L'Hospital Rule,

$$\begin{aligned} g(0) &= \lim_{x\to 0}\frac{\cos(x^2)-1}{x^4} = \lim_{x\to 0}\frac{-2x\sin(x^2)}{4x^3} = \lim_{x\to 0}\frac{-\sin(x^2)}{2x^2} \\ &= \lim_{x\to 0}\frac{-2x\cos(x^2)}{4x} = \lim_{x\to 0}\frac{-\cos(x^2)}{2} = -\frac{1}{2}. \end{aligned}$$

(ii) For $x \neq 0$,

$$g(x) = \frac{\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!} - 1}{x^4} = \frac{\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}}{x^4} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{4(n-1)}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{4n}}{(2n+2)!}.$$

Note that

$$\left.\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{4n}}{(2n+2)!}\right|_{x=0} = -\frac{1}{2} = g(0).$$

Thus,

$$g(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{4n}}{(2n+2)!} \quad \forall x \in \mathbb{R}. \quad (**)$$

Since g is a convergent power series on $\mathbb{R}$, it is infinitely differentiable on $\mathbb{R}$.
In particular, $g^{(n)}(0)$ exists for all $n \in \mathbb{N}$.

(iii) The series $(**)$ is the Maclaurin series of g. Let (a_n) be such that $g(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in \mathbb{R}$.

Then by Corollary 9.2.4, for each $n \in \mathbb{N}$, $a_n = \frac{g^{(n)}(0)}{n!}$ so that $g^{(n)}(0) = n!a_n$. Using this, we obtain

$$\begin{aligned} g^{(5)}(0) &= 5!a_5 = 0 \\ g^{(6)}(0) &= 6!a_6 = 0 \\ g^{(7)}(0) &= 7!a_7 = 0 \\ g^{(8)}(0) &= 8!a_8 = 0 = 8! \times \left(-\frac{1}{6!}\right) = -56. \end{aligned}$$