MA1100 AY24/25 Sem 2 Final

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Question 1: In each part, determine whether the statement is true or false. Write **True** or **False** as your answer in the box next to the statement. No justification is needed.

a) Let P, Q and R be statements. Then $(P \implies Q) \implies R$ and $P \implies (Q \implies R)$ are logically equivalent

b) Let $P(x)$ and $Q(x)$ be predicates with free variable x and universe $\mathcal{U}$. Then $(\forall x \in \mathcal{U})[P(x) \wedge Q(x)]$ and $\{(\forall x \in \mathcal{U})P(x)\} \wedge \{(\forall x \in \mathcal{U})Q(x)\}$ are logically equivalent

c) For any sets A, B and C, if $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$.

d) If $f : X \to Y$ is a function and Z is a proper subset of Y such that for all $x \in X$, $f(x) \in Z$, then Z is the range of f.

e) For any sets X, Y, Z and any functions $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is surjective, then f is surjective

f) If $f : X \to Y$ is a function, then for all $A \subseteq X$, we have $f^{-1}[f[A]] = A$

g) If $a, b \in \mathbb{Z}$ with a not zero, then $gcd(a,b) = gcd(a, a+b)$

h) If X and Y are uncountable sets, then there exists a bijection $f : X \to Y$

i) There does not exist a surjection $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

j) There are exactly 720 surjective functions with domain $\mathbb{N}_6$ and codomain $\mathbb{N}_6$

Solution:

(a) By truth table, we have

P	Q	R	$P \implies Q$	$Q \implies R$	$(P \implies Q) \implies R$	$P \implies (Q \implies R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	**F**	**T**
F	F	T	T	T	T	T
F	F	F	T	T	**F**	**T**

Therefore the statement is **False**. $\square$

(b) For the forward direction, suppose for all $x \in \mathcal{U}$, we have $P(x)$ and $Q(x)$ to be true. As such $\{(\forall x \in \mathcal{U})P(x)\} \wedge \{(\forall x \in \mathcal{U})Q(x)\}$ is also true.
For the reverse direction, suppose for all $x \in \mathcal{U}$, $P(x)$ is true and for all $x \in \mathcal{U}$, $Q(x)$ is true. Therefore $(\forall x \in \mathcal{U})[P(x) \wedge Q(x)]$ is also true. Hence the statement is **True** $\square$

(c) Consider $A = \{1,2\}$, $B = \{1\}$ and $C = \{2\}$. In this case $B \cup C = \{1,2\}$ which implies that $A \subseteq B \cup C$ but neither $A \subseteq B$ or $A \subseteq C$. Thus the statement is **False**. $\square$

(d) Since $Z \subseteq Y$ and for all $x \in X$, we have $f(x) \in Z$, then by definition $R_f = \{f(x) \mid x \in X\}$ which implies that $Z = R_f$. Therefore, the statement is **True**. $\square$

(e) Consider $X = \{1\}$, $Y = \{a,b\}$ and $Z = \{\gamma\}$, then we can define the followings

- $f(1) = a$
- $g(a) = \gamma$ and $g(b) = \gamma$

This implies that $g \circ f(1) = \gamma$ is surjective, but f is not surjective onto Y. Therefore the statement is **False**. $\square$

(f) Consider $A = \{1\}$ and $f(x) = x^2$, then we have

- $f(1) = 1$

This shows that $f[A] = \{1\}$. However, $f^{-1}[f[A]] = \{1,-1\} \neq A$. Therefore the statement is **False**. $\square$

(g) Suppose we have $\gcd(a, a+b)$, by Bezout Lemma, we have $x, y \in \mathbb{Z}$ such that

$$\gcd(a, a+b) = ax + (a+b)y = ax + ay + by = a(x+y) + by$$

Since $x+y$ is the sum of integers, by Bezout Lemma, we have

$$a(x+y)+by = \gcd(a,b)$$

Therefore

$$\gcd(a,a+b) = \gcd(a,b)$$

The statement is **True**. $\square$

(h) Consider $X = \mathbb{R}$ and $Y = \mathcal{P}(\mathbb{R})$. Both are uncountable but they do not have the same cardinality. hence there do not exist a bijection $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$. The statement is **False** $\square$

(i) Consider Cantor's Tuple Bijection. Bijection exists Hence there exist a surjection. The statement is **False**.

(j) Since $|\mathbb{N}_6| = 6$, then the number of surjective functions is give by

$$6! = 720$$

Therefore the statement is **True**. $\square$

Question 2: Let $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ be defined by, for all $(m,n) \in \mathbb{Z}\times\mathbb{Z}$,

$$f(m,n) = (2m-5n, m-2n)$$

i) Prove that f is injective

ii) Determine whether f is invertible and find its inverse function if it exists. Justify your answers.

Solution:
(i) Let $(m_1,n_1),(m_2,n_2) \in \mathbb{Z}\times\mathbb{Z}$ such that we have

$$f(m_1,n_1) = f(m_2,n_2)$$
$$(2m_1-5n_1, m_1-2n_1) = (2m_2-5n_2, m_2-2n_2)$$

By equality of ordered pairs, we have

$$2m_1-5n_1 = 2m_2-5n_2 \qquad \text{and} \qquad m_1-2n_1 = m_2-2n_2$$

From the first equation, we have

$$2(m_1-m_2) = 5(n_1-n_2)$$

From the second equation, we have

$$m_1-m_2 = 2(n_1-n_2)$$

Substituting the second equation to the first equation yields

$$\begin{aligned}4(n_1-n_2) &= 5(n_1-n_2)\\ n_1-n_2 &= 0\\ n_1 &= n_2\end{aligned}$$

Similarly from the second equation, we have

$$m_1-m_2=0 \implies m_1=m_2$$

This shows that $(m_1,n_1)=(m_2,n_2)$ as such f is injective. $\square$

(ii) For f to be invertible, f has to be a bijection. Since we have proven that f is injective, we will determine whether f is surjective. f is surjective if for all $(a,b) \in Z \times \mathbb{Z}$, there exist $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ such that

$$f(m,n)=(a,b)$$

We can choose $(m,n)=(5b-2a,2b-a)$ such that we have

$$f(m,n)=(2(5b-2a)-5(2b-a),5b-2a-2(2b-a))=(10b-4a-10b+5a,5b-2a-4b+2a)=(a,b)$$

Therefore f is surjective. Since f is both injective and surjective, f is bijective as such f is invertible with the inverse function to be

$$f^{-1}(m,n)=(5n-2m,2n-m)$$

$\square$

Question 3: Prove that if X and Y are nonempty finite sets and $|X|<|Y|$, then there does not exist a surjection $f: X \to Y$

Solution: Define $|X|=m$ and $|Y|=n$ where $m,n\in\mathbb{Z}$ and $m<n$. Since $|X|=m$ implies that X has only m elements, then set $R_f=\{f(x) \mid x\in X\}$ can have at most m elements. Therefore $|f[X]|\leq m$. Suppose for the sake of contradiction that there exist a surjection $f: X\to Y$, then $R_f=Y$ which implies $|R_f|=n$. Therefore $m=n$ which is a contradiction. $\square$

Question 4: Let R be an equivalence relation on the set $\mathbb{N}_7=\{1,2,3,4,5,6,7\}$ such that

- R has exactly 4 distinct equivalence classes;
- $|[1]|=|[2]|=1$
- $(3,4)\in\mathbb{R}$ and $(3,5)\notin R$
- $[4]\neq[6]$ and $[6]\cap[7]\neq\emptyset$

i) List all the elements in each equivalence class of R

ii) List all the elements of R

Solution:
(i) We have

- $[1] = \{1\}$
- $[2] = \{2\}$
- $[3] = [4] = \{3,4\}$
- $[5] = [6] = [7] = \{5,6,7\}$

$\square$

(ii) We have

$R = \{(1,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(5,7),(6,5),(6,6),(6,7),(7,5),(7,6),(7,7)\}$

$\square$

Question 5: Let $\sim$ be the relation on $\mathbb{R}^2$ defined by, for all $(a,b),(c,d) \in \mathbb{R}^2$,

$$(a,b) \sim (c,d) \iff |a|+|b| = |c|+|d|$$

i) Prove that $\sim$ is an equivalence relation.

ii) Give a geometrical description of the equivalence class $[(1,0)]$ as a subset of $\mathbb{R}^2$

iii) Find a bijection $g : \mathbb{R}^2/\sim \to [0,\infty)$.

Solution:
(i) Consider

$$(a,b) \sim (a,b) \iff |a|+|b| = |a|+|b|$$

which is trivially true. Hence $\sim$ is reflexive

Next Consider

$$(a,b) \sim (c,d) \iff |a|+|b| = |c|+|d| \iff |c|+|d| = |a|+|b| \iff (c,d) \sim (a,b)$$

Therefore $\sim$ is symmetric

Lastly, we can consider $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ then we have

$$(a,b) \sim (c,d) \iff |a|+|b| = |c|+|d| \qquad \text{and} \qquad (c,d) \sim (e,f) \iff |c|+|d| = |e|+|f|$$

Note that

$$|a|+|b|=|e|+|f| \iff (a,b)\sim(e,f)$$

Therefore $\sim$ is transitive

In conclusion, since $\sim$ is reflexive, symmetric and transitive, $\sim$ is an equivalence relation $\square$

(ii) We have

$$[(1,0)] = \{(x,y)\in\mathbb{R}^2 \mid |x|+|y|=1\}$$

This is the set of all points in $\mathbb{R}^2$ whose Manhattan norm(magnitude) is 1. This describes a diamond centered at the origin with vertices at $(1,0),(0,1),(-1,0),(0,-1)$ which are the boundaries of the unit ball. $\square$

(iii) We have

$$[(a,b)] = \{(x,y)\in\mathbb{R}^2 \mid |x|+|y|=|a|+|b|\}$$

As such we can define

$$g([(a,b)]) = |a|+|b|$$

If $(a,b)\sim(c,d)$, then $|a|+|b|=|c|+|d|$ which implies that $g([a,b])=g([c,d])$. As such the function is well defined.
Suppose $g([a,b])=g([c,d])$, then $|a|+|b|=|c|+|d|$ which implies $(a,b)\sim(c,d)$ thus $[a,b]=[c,d]$.
Hence g is injective
For any $r\in[0,\infty)$, we can choose $[(a,b)]=[(r,0)]$ such that $g([(r,0)])=|r|+|0|=r$. As such g is surjective.

In conclusion,

$$g([(a,b)]) = |a|+|b|$$

is the bijection $g:\mathbb{R}^2/\sim\to[0,\infty)$ $\square$

Question 6: Let $c,m\in\mathbb{Z}^+$, and let $d=\gcd(c,m)$, $r=\frac{c}{d}$ and $s=\frac{m}{d}$

i) Prove that r and s are relatively prime

ii) Prove that if $a,b\in\mathbb{Z}$ and $ca\equiv cb \mod m$, then $a\equiv b \mod s$

Solution:
(i) By Bezout Lemma, we can consider

$$\gcd(r,s) = rx+sy$$

where $x, y \in \mathbb{Z}$. Then we have

$$\gcd(r,s) = rx + sy = \frac{cx}{d} + \frac{my}{d} = \frac{cx+my}{d} = \frac{d}{d} = 1$$

Hence r and s are relatively prime. $\square$

(ii) Since we have $ca \equiv cb \mod m$, then $m \mid ca - cb$. As such there exist $k \in \mathbb{Z}$ such that

$$\begin{aligned} ca - cb &= mk \\ rd(a-b) &= sdk \\ r(a-b) &= sk \implies s \mid r(a-b) \end{aligned}$$

Since s and r are relatively prime, then $s \mid (a-b)$ which implies that $a \equiv b \mod s$ $\square$

Question 7: Let $\mathbb{N} = \{1, 2, 3, ...\}$ denote the set of all positive integers and let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$. Let

$$X = \{S \in \mathcal{P}(\mathbb{N}) \mid S \text{ is finite}\}$$

that is, X is the set of all finite subsets of $\mathbb{N}$. Prove that X is denumerable.

Solution: Since S is finite, then we can let $|S| = n$ where $n \in \mathbb{N}$. X_n will then be the set of all subsets of $\mathbb{N}$ of size exactly n. Since it is known that $\mathbb{N}$ is countable and $X_n \subseteq \mathbb{N}$, then X_n is also countable. Now define

$$X = \bigcup_{n=0}^{\infty} X_n$$

As such X is a union of countable sets with implies that X is countable. Since $n \in \mathbb{N}$ is infinite, then X is countable infinite which implies that X is denumerable. $\square$

Question 8: Let $\mathbb{R}^+$ denote the set of all positive real numbers.

i) For each $n \in \mathbb{Z}^+$, let

$$A_n = \left(\frac{1}{n}, \infty\right) = \left\{x \in \mathbb{R} \mid x > \frac{1}{n}\right\}$$

Prove that

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{R}^+$$

ii) Prove that if B is a subset of $\mathbb{R}^+$ and B is uncountable, then there exists $t \in \mathbb{R}^+$ such that the set

$$\{x \in B \mid x > t\}$$

is uncountable.

Solution:

(i) For the forward inclusion, we can let $x \in \bigcup_{n=1}^{\infty} A_n$, then there exist $n \in \mathbb{Z}^+$ such that $x \in A_n$. Hence

$$x > \frac{1}{n} > 0 \implies x \in \mathbb{R}^+$$

Therefore

$$\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{R}^+$$

For the reverse inclusion, we can let $x \in \mathbb{R}^+$, then $x > 0$. Note that since $\lim_{n\infty} \frac{1}{n} = 0$, there exist $n \in \mathbb{Z}^+$ such that

$$x > \frac{1}{n} > 0 \implies x \in A_n \qquad \text{for some } n \in \mathbb{Z}^+$$

Therefore

$$x \in \bigcup_{n=1}^{\infty} A_n \implies \mathbb{R}^+ \subseteq \bigcup_{n=1}^{\infty} A_n$$

In conclusion, we have proven

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{R}^+$$

$\square$

(ii) Suppose for the sake of contradiction that for all $t \in \mathbb{R}^+$, the set $\{x \in B \mid x > t\}$ is countable and since $B \subseteq \mathbb{R}^+$, then

$$B = B \cap \mathbb{R}^+ = B \cap \left(\bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} (B \cap A_n)$$

Let $t = \frac{1}{n}$ and $n \in \mathbb{Z}^+$ such that

$$B \cap A_n = \left\{ x \in B \mid x > \frac{1}{n} \right\}$$

From our assumption, $B \cap A_n$ is countable.Note that the union of all countable sets is countable, Hence $\bigcup_{n=1}^{\infty} (B \cap A_n)$ is countable. However, the equality suggest that B is also countable which is a contradiction. Therefore there exists $t \in \mathbb{R}^+$ such that the set is uncountable. $\square$