MA3209 MATHEMATICAL ANALYSIS III
FINAL EXAM (2016/2017 SEMESTER 1)

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Question 1 (18 points).

(a) Let (X, d_X) and (Y, d_Y) be metric spaces. Define $d : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, \; x_1, x_2 \in X, y_1, y_2 \in Y.$$

Show that d is a metric on $X \times Y$.

(b) Let $C[0,1]$ be the metric space of all continuous real-valued functions on $[0,1]$, equipped with the uniform metric d_∞. Let $g \in C[0,1]$ and

$$S = \{f \in C[0,1] : f(x) < g(x) \text{ for all } x \in [0,1]\}.$$

Is the set S open in $C[0,1]$? Justify your answer.

Solution.

(a) We verify the properties of a metric space:

- For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, since d_X, d_Y are metrics, then $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$, so that

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \geq 0.$$

- Suppose that $d((x_1, y_1), (x_2, y_2)) = 0$. Then we must have $d_X(x_1, x_2) = 0 = d_Y(y_1, y_2)$, which implies that $x_1 = x_2$ and $y_1 = y_2$. Hence, $(x_1, y_1) = (x_2, y_2)$. On the other hand, if $x_1 = x_2$ and $y_1 = y_2$, then $d_X(x_1, x_2) = 0 = d_Y(y_1, y_2)$, so that

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0.$$

- Since d_X, d_Y are metrics on X and Y respectively, then $d_X(x_1, x_2) = d_X(x_2, x_1)$ and $d_Y(y_1, y_2) = d_Y(y_2, y_1)$, so that

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = \max\{d_X(x_2, x_1), d_Y(y_2, y_1)\} = d((x_2, y_2), (x_1, y_1)).$$

- For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, since

$$\begin{aligned}
d_X(x_1, x_2) &\leq d_X(x_1, x_3) + d_X(x_3, x_2) \leq \max\{d_X(x_1, x_3) + d_X(x_3, x_2), d_Y(y_1, y_3) + d_Y(y_3, y_2)\} \\
d_Y(y_1, y_2) &\leq d_Y(y_1, y_3) + d_Y(y_3, y_2) \leq \max\{d_X(x_1, x_3) + d_X(x_3, x_2), d_Y(y_1, y_3) + d_Y(y_3, y_2)\}
\end{aligned}$$

then

$$\begin{aligned}
\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} &\leq \max\{d_X(x_1, x_3) + d_X(x_3, x_2), d_Y(y_1, y_3) + d_Y(y_3, y_2)\} \\
&\leq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} + \max\{d_X(x_3, x_2), d_Y(y_3, y_2)\}
\end{aligned}$$

where we used that $\max\{a+b, c+d\} \leq \max\{a, c\} + \max\{b, d\}$, since $a + b \leq \max\{a, c\} + \max\{b, d\}$ and $c + d \leq \max\{a, c\} + \max\{b, d\}$. Hence,

$$d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_3, y_3)) + d((x_3, y_3), (x_2, y_2)).$$

(b) Yes. Let $f \in S$. Let

$$r_f = \frac{1}{2} \inf_{x \in [0,1]} |f(x) - g(x)| = \frac{1}{2} \min_{x \in [0,1]} |f(x) - g(x)| > 0.$$

Take any function h in the open ball centred at f of radius r_f, denoted by $B(f, r_f)$. This means that for all $x \in [0,1]$, we have $|h(x) - f(x)| < r_f$. So, $h(x) < f(x) + r_f$. By definition, for every $x \in [0,1]$, we have

$$r_f < \frac{g(x) - f(x)}{2} \quad \text{so} \quad h(x) < f(x) + \frac{g(x) - f(x)}{2} = \frac{f(x) + g(x)}{2}.$$

Since $\frac{1}{2}(f(x) + g(x)) < f(x)$, then $h(x) < g(x)$, so $h \in S$. Hence, S is open in $C[0,1]$. $\square$

Question 2 (20 points). Let (X,d) be a metric space.

(a) For $x \in X$ and $r > 0$, denote by $B[x,r]$ the closed ball $\{y \in X : d(y,x) \le r\}$ in X. Suppose that whenever $\{B[x_n,r_n]\}_{n=1}^{\infty}$ is a sequence of closed balls in X satisfying the conditions

$$B[x_n,r_n] \supseteq B[x_{n+1},r_{n+1}], n = 1,2,\ldots, \text{ and } \lim_{n\to\infty} r_n = 0,$$

then the intersection $\bigcap_{n=1}^{\infty} B[x_n,r_n]$ is non-empty. Prove that (X,d) is complete.

(b) **(i)** Let $x \in X$. Prove that the singleton set $\{x\}$ is nowhere dense in X if and only if x is an accumulation point of X.

(ii) Suppose that (X,d) is complete and every x in X is an accumulation point of X. Prove that X is uncountable.

Solution.

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X,d). So, there exists N_1 such that for all $n,m \ge N_1$, $d(x_n,x_m) < 2^{-3}$. In particular, $d(x_{N_1},x_m) < 2^{-3}$. By applying the definition of a Cauchy sequence again, there exists $N_2 \ge N_1$ such that for all $n,m \ge N_2$, $d(x_n,x_m) < 2^{-4}$. In particular, $d(x_{N_2},x_m) < 2^{-4}$. Going in this manner, we can define $r_k = 2^{1-k}$. As such, we obtain the following:

- an increasing sequence of natural numbers $N_1 \le N_2 \le N_3 \le \ldots$,
- a subequence $\{x_{N_k}\}_{k=1}^{\infty}$
- a sequence of closed balls $\{B[x_{N_k},r_k]\}_{k=1}^{\infty}$.

We first show that $B[x_{N_k},r_k] \supseteq B\left[x_{N_{k+1}},r_{k+1}\right]$. Let $z \in B\left[x_{N_{k+1}},r_{k+1}\right]$. Then

$$\begin{aligned} d(z,x_{N_k}) &\le d\left(z,x_{N_{k+1}}\right) + d\left(x_{N_{k+1}},x_{N_k}\right) \quad \text{by the triangle inequality} \\ &\le r_{k+1} + 2^{-k-2} \\ &= 2^{-k} + 2^{-2-k} \\ &= \frac{5}{2^{k+2}} \\ &< \frac{2}{2^k} \\ &= r_k \end{aligned}$$

which implies that $z \in B[x_{N_k},r_k]$ for each $k \in \mathbb{N}$. By assumption,

$$\bigcap_{k=1}^{\infty} B[x_{N_k},r_k] \ne \emptyset.$$

Let $x \in \bigcap_{k=1}^{\infty} B[x_{N_k},r_k]$, so that $d(x,x_{N_k}) \le r_k = 2^{1-k}$ for each k. We will show that $x_n \to x$ as $n \to \infty$. Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ such that $2^{1-K} < \varepsilon/2$. Then for any $n \ge N_K$,

$$d(x_n,x) \le d(x_n,x_{N_K}) + d(x_{N_K},x) \le r_K + r_K = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and so $x_n \to x$ as $n \to \infty$. Thus, (X,d) is complete.

(b) **(i)** For the forward direction, suppose that x is not an accumulation point of X. Then there exists $r > 0$ such that $B(x,r) \cap X \setminus \{x\} = \emptyset$. This implies that $B(x,r) \subseteq \{x\}$, and so $B(x,r) = \{x\}$. This implies that $B(x,r) \subseteq \{x\} \subseteq \overline{\{x\}}$, so that x is an interior point of $\overline{\{x\}}$, contradicting that $\operatorname{int}\left(\overline{\{x\}}\right) = \emptyset$. Thus, if $\{x\}$ is nowhere dense in X, then x is an accumulation point of X.

For the reverse direction, suppose that x is an accumulation point of X. Let $G \subseteq X$ be an open set. If $x \notin G$, then for any open ball $B \subseteq G$, we have $B \cap \{x\} = \emptyset$. If $x \in G$, since G is an open set, then there exists $r_1 > 0$ such that $B(x,r_1) \subseteq G$. Since x is an accumulation point, then $B(x,r_1) \cap (X \setminus \{x\}) \ne \emptyset$. Let $y \in B(x,r_1) \cap (X \setminus \{x\})$, so that $y \ne x$ and $d(x,y) > 0$. Since $y \in B(x,r_1)$, there exists $r_2 > 0$ such that $B(y,r_2) \subseteq B(x,r_1)$. Let $r_3 = d(x,y)/4 > 0$ and $r_4 = \min\{r_2,r_3\} > 0$. We will show that $B(y,r_4) \subseteq G$ and $x \notin B(y,r_4)$. From the definition of r_1,r_2,r_4, we have

$$B(y,r_4) \subseteq B(y,r_2) \subseteq B(y,r_1) \subseteq G.$$

If $x \in B(y, r_4)$, then

$$d(x,y) < r_4 \leq r_3 = \frac{d(x,y)}{4} < d(x,y)$$

which is a contradiction. Therefore, $x \notin B(y, r_4)$. Hence, $B(y, r_4) \subseteq G$ such that $B(y, r_4) \cap \{x\} = \emptyset$ and so $\{x\}$ is nowhere dense in X.

(ii) Suppose on the contrary that X is countable. Then, we can enumerate the elments of X, i.e. $X = \{x_1, x_2, \ldots\}$. For each $n \in \mathbb{N}$, define $U_n = X \setminus \{x_n\}$. Since every point in X is an accumulation point, then each x_n is not isolated, i.e. in every open neighbourhood around x_n, there is some other point from X. In other words, removing one point x_n does not *affect* the density of the remaining set. Hence, each U_n is dense in X.

Recall the Baire category theorem, which states that in a complete metric space, the countable intersection of dense open sets is also dense. Since (X,d) is complete, then by the Baire category theorem, $\bigcap_{n=1}^{\infty} U_n$ is dense in (X,d). But this means $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$, which is a contradiction since $x_i \notin \bigcap_{n=1}^{\infty} U_n$ for all i. Therefore, X is uncountable. □

Question 3 (20 points). Let (X,d) be a metric space.

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X converging to some $x \in X$. Is the set

$$\{x_n : n \in \mathbb{N}\} \cup \{x\}$$

compact? Justify your answer.

(b) Let (X,d) be compact, and let $T : X \longrightarrow X$ be an isometry:

$$d(T(x), T(y)) = d(x,y)$$

for all $x, y \in X$. Determine if T is surjective. Justify your answer.

Solution.

(a) Yes, the set $S = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. Let $\mathcal{G}$ be an open cover for S. Since $x \in S$, there exists an open set $U \in \mathcal{G}$ such that $x \in U$. Let $r > 0$ be such that $B(x,r) \subseteq U$. Since $x_n \to x$ as $n \to \infty$, for a given $\varepsilon = r > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $x_n \in B(x,r)$. Since $\mathcal{G}$ is an open cover for S, then there exist open sets $U_1, U_2, \ldots, U_{M-1}$ such that $x_i \in U_i$ for $i = 1, 2, \ldots, M-1$. Then $\{U_1, \ldots, U_{M-1}, U\} \subseteq \mathcal{G}$ is a finite subcover for S. Hence, S is compact.

(b) Yes, T is surjective. Suppose on the contrary that T is not surjective. Then there exists $a \in X$ such that $T(x) \neq a$ for all $x \in X$. Since (X,d) is compact, then (X,d) is sequentially compact. Consider a sequence $\{T^n(a)\}_{n=0}^{\infty}$. Then there exists a convergent subsequence $\{T^{n_k}(a)\}_{k=0}^{\infty}$, which converges to some $b \in X$. Since $\{T^{n_k}(a)\}_{k=0}^{\infty}$ is convergent, then $\{T^{n_k}(a)\}_{k=0}^{\infty}$ is Cauchy. Let $\varepsilon > 0$. Then there exists $K_1 \in \mathbb{N}$ such that for all $k, \ell \geq K_1$, $d(T^{n_k}(a), T^{n_\ell}(a)) < \varepsilon/2$. Next, since $T^{n_k}(a) \to b$ as $k \to \infty$, then there exists $K_2 \in \mathbb{N}$ such that for all $k \geq K_2$, $d(T^{n_k}(a), b) < \varepsilon/2$. Now, choose $K = \max\{K_1, K_2\}$. Then for all $k, \ell \geq K$, we have

$$\begin{aligned}
d(a, T(b)) &= d\left(T^{n_k}(a), T^{n_k+1}(b)\right) \quad \text{since } T \text{ is an isometry} \\
&\leq d\left(T^{n_k}(a), T^{n_k+n_\ell+1}(a)\right) + d\left(T^{n_k+n_\ell+1}(a), T^{n_k+1}(b)\right) \\
&= d\left(T^{n_k}(a), T^{n_k+n_\ell+1}(a)\right) + d\left(T^{n_\ell}(a), b\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $d(a, T(b)) = 0$, contradicting that $T(x) \neq a$ for all $x \in X$. Therefore, T is surjective. □

Question 4 (12 points). Let (X,d) be a metric space. Prove that the following conditions are equivalent:

(1) X is connected.

(2) Every continuous function $f : X \longrightarrow \{1,2\}$ is a constant function, where $\{1,2\}$ is a subspace of $\mathbb{R}$ with the usual metric.

Solution. We first prove (1) implies (2). Suppose that X is connected but $f : X \longrightarrow \{1,2\}$ is continuous but non-constant. Since $\{1,2\}$ is a subspace of $\mathbb{R}$ with the usual metric, then $\{1\}$ and $\{2\}$ are open sets in $\mathbb{R}$ under the usual metric since $\{1\} = B(1,1/2)$ and $\{2\} = B(2,1/2)$. Let $G = f^{-1}(\{1\})$ and $H = f^{-1}(\{2\})$. Since f is continuous, then G and H are open in (X,d). Moreover, $G \cap H = \emptyset$ since f is a function. For any $x \in X$, $f(x) = 1$ or $f(x) = 2$, this implies that $x \in G$ or $x \in H$, and so $X = G \cup H$. The existence of such a partition (G,H) contradicts that X is connected.

We then prove (2) implies (1) using contradiction. Suppose X is disconnected, then there exist non-empty open sets $G, H \subseteq X$ such that $G \cap H = \emptyset$ and $X = G \cup H$. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in G \\ 2 & \text{if } x \in H \end{cases}$$

which is continuous but non-constant on X. To justify continuity, note that G and H are open in X so the pre-images of the open sets $\{1\}$ and $\{2\}$ are open in $\{1,2\}$. f being non-constant is clear as it takes on two values — 1 and 2. This is a contradiction. So, X must be connected. $\square$

Question 5 (14 points).

(a) Let $f : [a,\infty) \longrightarrow [a,\infty)$ be a differentiable function such that

$$\sup\left\{\left|f'(x)\right| : a < x < \infty\right\} < 1.$$

Prove that f has a unique fixed point in $[a,\infty)$.

(b) **(i)** Show that the function $g(x) = (1+\sqrt{x})^{1/3}$ is a contraction mapping on the interval $[1,\infty)$.

(ii) Deduce that the equation $x^6 - 2x^3 - x + 1 = 0$ has a root in $[1,\infty)$.

Solution.

(a) Note that for any $x \in (a,\infty)$,

$$f'(x) = \lim_{y\to x} \frac{f(x)-f(y)}{x-y} < 1 \quad \text{since } \sup\left|f'(x)\right| < 1,$$

so there exists $M \in (0,1)$ such that $|f'(x)| \leq M < 1$. In particular, since f is differentiable, for any $x,y \in [a,\infty)$ with $x \neq y$, by the mean value theorem, there exists $c \in (x,y)$ such that

$$f'(c) = \frac{f(x)-f(y)}{x-y} = M < 1$$

so that $|f(x)-f(y)| \leq M|x-y|$. This shows that f is a contraction mapping in $([a,\infty), \text{usual metric})$. Moreover, $([a,\infty), \text{usual metric})$ is complete. By Banach's fixed point theorem, f has a fixed point in $[a,\infty)$.

(b) **(i)** We have

$$\begin{aligned} |g(x)-g(y)| &= \left|(1+\sqrt{x})^{1/3} - (1+\sqrt{y})^{1/3}\right| \\ &= \left|\frac{(1+\sqrt{x}) - (1+\sqrt{y})}{(1+\sqrt{x})^{2/3} + (1+\sqrt{x})^{1/3}(1+\sqrt{y})^{1/3} + (1+\sqrt{y})^{2/3}}\right| \\ &\leq \left|\frac{\sqrt{x}-\sqrt{y}}{1+1+1}\right| \\ &= \frac{1}{3}\left|\sqrt{x}-\sqrt{y}\right| \\ &= \frac{1}{3}\left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \\ &\leq \frac{1}{6}|x-y| \end{aligned}$$

so that g is a contraction mapping on the interval $[1,\infty)$.

(ii) Let $g(x) = (1+\sqrt{x})^{1/3}$. By **(bi)**, since g is a contraction mapping on $([1,\infty), \text{usual metric})$, and $[1,\infty)$ is complete, then g has a unique fixed point in $[1,\infty)$. This implies that there exists $\alpha \in [1,\infty)$ such that

$g(\alpha) = \alpha$, or equivalently, $1 + \sqrt{\alpha} = \alpha^3$. This is equivalent to

$$\alpha^6 - 2\alpha^3 - \alpha + 1 = 0.$$

Hence, the equation $x^6 - 2x^3 - x + 1 = 0$ has a root α in $[1, \infty)$. □

Question 6 (16 points).

(a) Using the definition of differentiability, show that the function $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$F(x,y,z) = \left(x^2 + y^2 + z^2, xy + yz\right)$$

is differentiable at the point $(1,0,2)$. Find the Jacobian matrix $F'(1,0,2)$.

(b) Using implicit function theorem, show that the system of equations

$$\begin{aligned} x^2 - y^2 - z^2 - 2 &= 0 \\ x - y + z - 2 &= 0 \end{aligned}$$

and the condition $(x,y,z) = (2,1,1)$ determine a continuously differentiable function $g(x) = (y,z)$ near the point $x = 2$. Find $g'(2)$.

Solution.

(a) First, note that

$$F'(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ y & x+z & y \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

Then,

$$\begin{aligned}
&\lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\|F(1+h_1, h_2, 2+h_3) - F(1,0,2) - Ah\|}{\|h\|} \\
= &\lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\left\| \begin{pmatrix} (1+h_1)^2 + h_2^2 + (2+h_3)^2 \\ (1+h_1)h_2 + h_2(2+h_3) \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 2h_1 + 4h_3 \\ 3h_2 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\|} \\
= &\lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\left\| \begin{pmatrix} h_1^2 + h_2^2 + h_3^2 + 2h_1 + 4h_3 - 2h_1 - 4h_3 \\ 3h_2 + h_1h_2 + h_2h_3 - 3h_2 \end{pmatrix} \right\|}{\sqrt{h_1^2 + h_2^2 + h_3^2}}
\end{aligned}$$

which simplifies to

$$
\begin{aligned}
&= \lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\left\| \begin{pmatrix} h_1^2+h_2^2+h_3^2 \\ h_1h_2+h_2h_3 \end{pmatrix} \right\|}{\sqrt{h_1^2+h_2^2+h_3^2}} \\
&\le \lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\left|h_1^2+h_2^2+h_3^2\right| + |h_2|\,|h_1+h_3|}{\sqrt{h_1^2+h_2^2+h_3^2}} \\
&\le \lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{\left|h_1^2+h_2^2+h_3^2\right| + |h_2|\,|h_1| + |h_2|\,|h_3|}{\sqrt{h_1^2+h_2^2+h_3^2}} \\
&\le \lim_{(h_1,h_2,h_3)\to(0,0,0)} \sqrt{h_1^2+h_2^2+h_3^2} + |h_1| + |h_3|
\end{aligned}
$$

which is bounded above 0. Hence, F is differentiable at $(1,0,2)$. The Jacobian matrix is given by

$$F'(1,0,2) = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix}.$$

(b) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$f(x,y,z) = \left(x^2-y^2-z^2-2, x-y+z-2\right) = (f_1, f_2).$$

Then

$$
\begin{aligned}
f'(x,y,z) &= \begin{pmatrix} 2x & -2y & 2z \\ 1 & -1 & 1 \end{pmatrix} \\
f'(2,1,1) &= \begin{pmatrix} 4 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}.
\end{aligned}
$$

Moreover, the matrix

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$$

is invertible. Note that f is continuously differentiable on $\mathbb{R}^2$ since each component of f is a polynomial in x,y,z. Now $f(2,1,1) = (0,0)$. By the implicit function theorem, there exists an open set $W \subseteq \mathbb{R}$ with $2 \in W$ and a continuously differentiable function $g : W \longrightarrow \mathbb{R}^2$ such that $g(2) = (1,1)$ and $f(x, g(x)) = (0,0)$ for all $x \in W$. Furthermore,

$$g'(2) = -f_{(y,z)}(2,1,1)^{-1} f_x(2,1,1) = -\begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}.$$

$\square$