MA3110: Solutions to Final Exam of Semester 1 2019/20

1. **Differentiation and Riemann Integration (20 points total)**

   (a) (10 points) Let f be a differentiable function on $[0,1]$. Using the Fundamental Theorem of Calculus and Taylor's Theorem or otherwise, prove that there exists $c \in (0,1)$ such that
   $$\int_0^1 f(x)\,\mathrm{d}x = f(0) + \frac{1}{2} f'(c).$$

   (b) (10 points) By considering the integral of a suitably defined function, find the value of
   $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}.$$

   Solution:

   (a) Consider the anti-derivative $F(x) = \int_0^x f(t)\,\mathrm{d}t$. Since f is differentiable on $[0,1]$, the anti-derivative is twice differentiable on $[0,1]$. Applying Taylor's theorem to F on the interval $[0,1]$, there exists a $c \in (0,1)$ such that
   $$F(1) = F(0) + F'(0)(1-0) + \frac{F''(c)}{2}(1-0)^2$$
   This is the required statement.
   Note that the following steps constitute a common mistake: By Taylor's theorem (or the mean value theorem), for every $x \in (0,1]$, there exists a $c \in (0,x)$ such that
   $$f(x) = f(0) + f'(c)x. \qquad (*)$$
   Now integrate both sides from 0 to 1 to "get"
   $$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f(0)\,\mathrm{d}x + \int_0^1 f'(c)x\,\mathrm{d}x = f(0) + \frac{f'(c)}{2}.$$
   Why? This is because c in Eqn. (*) *depends on* x. Thus, it should be written as
   $$f(x) = f(0) + f'(c_x)x, \quad \text{for some } c_x \in (0,x).$$
   We can integrate the LHS of the above to get $\int_0^1 f(x)\,\mathrm{d}x$ but $f'(c_x)x$ is not straightforward to integrate. It is certainly not the simple polynomial $f'(c)x$. This is the essence of the mean value theorem for integrals!

   (b) Let $f : [0,1] \to \mathbb{R}$ be defined as $f(x) = (1+x)^{-1/2}$. Define the partition $P_n = \{k/n : 0 \le k \le n\}$ and the representative points $\xi^{(n)} = (1/n, 2/n, \ldots, n/n)$. Then note that $\|P_n\| = \frac{1}{n} \to 0$ so
   $$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{1+k/n}} = S(f,P_n)(\xi^{(n)}).$$
   Hence,
   $$\lim_{n\to\infty} S(f,P_n)(\xi^{(n)}) \to \int_0^1 (1+x)^{-1/2}\,\mathrm{d}x = 2\sqrt{2} - 2.$$

2. **Interchange of Limit and Improper Integral (15 points total)**

Let (f_n) be a sequence of functions defined on $[0, \infty)$ such that (f_n) converges uniformly to f on intervals of the form $[0, b]$ for all $b \geq 0$. Furthermore, assume that the improper integrals

$$\int_0^\infty f \quad \text{and} \quad \int_0^\infty f_n \quad \text{converge} \quad \forall\, n \in \mathbb{N}.$$

(a) (5 points) Show, via an example, that there exists (f_n) satisfying the above hypotheses such that

$$\lim_{n\to\infty} \int_0^\infty f_n \neq \int_0^\infty f.$$

(b) (10 points) Now, in addition to the assumption that (f_n) converges uniformly to f on intervals of the form $[0, b]$ for all $b \geq 0$, we assume the following:

- There exists a function $g : [0, \infty) \to \mathbb{R}$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in [0, \infty)$;
- The improper integral
$$\int_0^\infty |g| \quad \text{converges.}$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty f.$$

Thus, under the condition of the existence of such a g, one can interchange the limit and improper integral operations. This is a simplified form of the *dominated convergence theorem*.

Note: Sorry in the exam paper I wrote "$f_n(x) \leq g(x)$". It should really be "$|f_n(x)| \leq g(x)$". But this error did not materially change students' performances in the actual final exam.

Solution:

(a) Let $f_n(x) = 1$ if $n \leq x < n + 1$ and 0 on elsewhere on $[0, \infty)$. Then, for fixed $b \geq 0$, $f_n \to f \equiv 0$ uniformly on all intervals of the form $[0, b]$. This is because if we take $n + 1 \geq b$, then $f_n = 0$ on $[0, b]$ so $\lim_{n\to\infty} \|f_n - f\|_{[0,b]} = 0$. However,

$$\int_0^\infty f = 0 \quad \text{but} \quad \int_0^\infty f_n = 1 \quad \forall\, n \in \mathbb{N}.$$

(b) Fix $b \geq 0$. By a property of the improper integral ($\int_0^\infty h = \lim_{L\to\infty} \int_0^L h = \int_0^b h + \lim_{L\to\infty} \int_b^L h = \int_0^b h + \int_b^\infty h$),

$$\int_0^\infty (f_n - f) = \int_0^b (f_n - f) + \int_b^\infty (f_n - f).$$

By the triangle inequality,

$$\left|\int_0^\infty (f_n - f)\right| \leq \underbrace{\left|\int_0^b (f_n - f)\right|}_{=:A} + \underbrace{\left|\int_b^\infty (f_n - f)\right|}_{=:B}.$$

Now we notice that since $f_n \leq g$, $f = \lim_{n\to\infty} f_n \leq g$. Hence, the term B satisfies

$$B = \lim_{c\to\infty} \left|\int_b^c (f_n - f)\right| \leq \lim_{c\to\infty} \int_b^c |f_n - f| \leq \lim_{c\to\infty} \int_b^c 2|g| = 2\int_b^\infty |g|.$$

Fix $\epsilon > 0$. Since $\int_0^\infty |g|$ converges, there exists $b > 0$ such that

$$\int_b^\infty |g| < \frac{\epsilon}{4}.$$

With this choice of $b > 0$, we have $B < \epsilon/2$. For this fixed $b > 0$, since $f_n \to f$ uniformly on $[0, b]$,

$$\lim_{n\to\infty} \int_0^b f_n = \int_0^b f.$$

In other words, there exists a $K = K_\epsilon \in \mathbb{N}$ such that

$$\forall\, n \geq K \quad \Longrightarrow \quad A = \left| \int_0^b (f_n - f) \right| < \frac{\epsilon}{2}.$$

Putting the estimates for A and B together, we obtain that

$$\forall\, n \geq K \quad \Longrightarrow \quad \left| \int_0^\infty (f_n - f) \right| < \epsilon.$$

This implies by the definition of limit that

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty f.$$

3. **Uniform Continuity of Series of Functions (15 points total)**

   Consider the function

$$f(x) = \sum_{n=1}^\infty \frac{\sin(nx)}{n^2(n+1)}, \quad x \in \mathbb{R}.$$

   (a) (5 points) Prove that f is continuous on $\mathbb{R}$.

   (b) (5 points) Prove that f' exists on $\mathbb{R}$ and show that the set $\{|f'(x)| : x \in \mathbb{R}\}$ is bounded.

   (c) (5 points) Prove that f is uniformly continuous on $\mathbb{R}$.

   Solution:

   (a) Let $f_n(x) = \frac{\sin(nx)}{n^2(n+1)}$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$\|f_n\|_{\mathbb{R}} = \frac{1}{n^2(n+1)} =: M_n$$

   Since $\sum_n M_n < \infty$, by the Weierstrass M-test, the series $f(x) = \sum_n f_n(x)$ converges uniformly. Since f_n is continuous, so is f.

   (b) The derivative of the constituent functions are

$$f_n'(x) = \frac{\cos(nx)}{n(n+1)}, \quad x \in \mathbb{R}.$$

   We can check that

$$\|f_n'\|_{\mathbb{R}} = \frac{1}{n(n+1)} =: M_n'.$$

   Then since

$$\sum_{n=1}^\infty M_n' = \sum_{n=1}^\infty \frac{1}{n(n+1)} = \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 < \infty$$

by the Weierstrass M-test, the series $\sum_n f_n'$ converges uniformly. Furthermore, $f(0) = \sum_n f_n(0)$ clearly converges. Hence, by the differentiable limit theorem for any $r > 0$, $\sum_n f_n$ converges uniformly to a differentiable function f on $[-r, r]$ and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x), \quad x \in [-r, r].$$

In particular, f is differentiable on $\mathbb{R}$. Now for any $x \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$-1 \leq -\sum_{n=1}^{m} M_n' \leq -\sum_{n=1}^{m} |f_n'(x)| \leq \sum_{n=1}^{m} f_n'(x) \leq \sum_{n=1}^{m} |f_n'(x)| \leq \sum_{n=1}^{m} M_n' \leq 1$$

In other words, $|\sum_{n=1}^{m} f_n'(x)| \leq 1$ for every $m \in \mathbb{N}$ and $x \in \mathbb{R}$. Hence, for every $x \in \mathbb{R}$,

$$|f'(x)| = \lim_{m\to\infty} \left| \sum_{n=1}^{m} f_n'(x) \right| \leq \lim_{m\to\infty} 1 = 1$$

(c) Fix $\epsilon > 0$. Then for any $x, y \in \mathbb{R}$ with $|x - y| < \epsilon$, we have

$$|f(x) - f(y)| = |f'(c)||x - y| \leq |x - y| < \epsilon$$

where the equality follows from the mean value theorem and c is between x and y. Hence, f is uniformly continuous on $\mathbb{R}$.

4. **From Pointwise to Uniform Convergence (10 points total)**

Assume that (f_n) is a sequence of differentiable functions on $[a, b]$ with the following properties:

- The sequence (f_n) converges pointwise to f on $[a, b]$;
- The pointwise limit f is continuous;
- The sequence of derivatives (f_n') is uniformly bounded, i.e., there exists an $M < \infty$ such that

$$\sup \left\{ \|f_n'\|_{[a,b]} : n \in \mathbb{N} \right\} \leq M.$$

Prove that (f_n) converges uniformly to f on $[a, b]$.

Solution: Fix $\epsilon > 0$. Since f is continuous on $[a, b]$, there exists a $\delta > 0$ such that for all $x, y \in [a, b]$ with

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \epsilon/3. \tag{1}$$

We may choose $\delta < \epsilon/(3M)$. Let $P = \{x_0 = a, x_1, \ldots, x_{N-1}, x_N = b\}$ be a partition of $[a, b]$ such that $\|P\| < \delta$. Since (f_n) converges to f pointwise on $[a, b]$, there exists a $K \in \mathbb{N}$ such that

$$n \geq K \quad \Longrightarrow \quad |f_n(x_i) - f(x_i)| < \epsilon/3 \quad \forall\, i = 0, 1, \ldots, N. \tag{2}$$

Fix $x \in [a, b]$. Let x_i be the point in the partition P that is closest to x. Note that $|x_i - x| \leq \|P\| < \delta$. Now for $n \geq K$, consider

$$|f_n(x) - f(x)| \leq \underbrace{|f_n(x) - f_n(x_i)|}_{=:A} + \underbrace{|f_n(x_i) - f(x_i)|}_{=:B} + \underbrace{|f(x_i) - f(x)|}_{=:C}.$$

Then $B < \epsilon/3$ because of (2) and $C < \epsilon/3$ because of (1). For A, using the mean value theorem on the differentiable function f_n,

$$f_n(x) - f_n(x_i) = f_n'(c)(x - x_i) \quad \Longrightarrow \quad A = |f_n(x) - f_n(x_i)| \leq |f_n'(c)||x - x_i| \leq M\delta < \epsilon/3.$$

Thus, we conclude that for every $x \in [a, b]$ and

$$n \geq K \quad \Longrightarrow \quad |f_n(x) - f(x)| < \epsilon.$$

5. **Radius/Interval of Convergence (10 points total)**

Suppose that the following two functions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad g(x) = e^{f(x)} = \sum_{n=0}^{\infty} b_n x^n$$

have positive radii of convergence. Prove that for each $n \in \mathbb{N}$,

$$b_n = \frac{1}{n} \sum_{k=1}^{n} k\, a_k\, b_{n-k}.$$

Solution: Let R_1 and R_2 be the radii of convergence of $\sum_n a_n x^n$ and $\sum_n b_n x^n$ respectively. Then $R := \min\{R_1, R_2\} > 0$ by assumption. Then on $(-R, R)$,

$$\begin{aligned} \sum_{n=0}^{\infty} n b_n x^n &= x \cdot \frac{\mathrm{d}}{\mathrm{d}x} e^{f(x)} = e^{f(x)} \cdot x \cdot f'(x) \\ &= \left(\sum_{n=0}^{\infty} b_n x^n \right) \left(\sum_{n=1}^{\infty} n a_n x^n \right) \\ &= \sum_{n=0}^{\infty} \left(n a_n b_0 + (n-1) a_{n-1} b_1 + \ldots + a_1 b_{n-1} \right) x^n \end{aligned}$$

By uniqueness of power series,

$$n b_n = \sum_{k=1}^{n} k a_k b_{n-k}$$

which concludes the proof.

6. **Sum Function (15 points total)**

Define the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{n+2}.$$

(a) (5 points) Find the radius of convergence and the set E of all $x \in \mathbb{R}$ for which the series converges.

(b) (10 points) Find a closed form formula for $f(x)$ on E.

Solution:

(a) By the ratio test,

$$\lim_{n\to\infty} \left| \frac{x^{5(n+1)}}{n+3} \cdot \frac{n+2}{x^{5n}} \right| = |x|^5$$

Thus, the series converges if $|x|^5 < 1$. Equivalently $|x| < 1$. Thus, the radius of convergence is 1. It is easy to check that at $x = -1$, the series converges (by the alternating series test). At $x = 1$, the series does not converge. Thus, $E = [-1, 1)$.

(b) Consider the function

$$g(x) = x^{10} f(x) = \sum_{n=0}^{\infty} \frac{x^{5n+10}}{n+2}.$$

Then for $x \in (-1, 1)$, we have

$$g(x) = \sum_{n=0}^{\infty} \frac{5n+10}{n+2} \int_0^x t^{5n+9}\,\mathrm{d}t = 5\int_0^x \sum_{n=0}^{\infty} t^{5n+9}\,\mathrm{d}t = 5\int_0^x \frac{t^9}{1-t^5}\,\mathrm{d}t$$
$$= 5\int_0^x -t^4 + \frac{t^4}{1-t^5}\,\mathrm{d}t = -x^5 - \ln(1-x^5).$$

Thus, for $x \in (-1, 1) \setminus \{0\}$,

$$f(x) = -x^{-5} - x^{-10}\ln(1-x^{-5}).$$

For $x = 0$, it is clear from the definition of $f(x)$ that $f(0) = 1/2$. At $x = -1$, since $\sum_{n=0}^{\infty}(-1)^{5n}/(n+2)$ converges, by Abel's theorem,

$$f(-1) = \lim_{x\to(-1)^+} -x^{-5} - x^{-10}\ln(1-x^{-5}) = 1 - \ln 2.$$

Thus, in summary,

$$f(x) = \begin{cases} -x^{-5} - x^{-10}\ln(1-x^{-5}) & x \in [-1, 1) \setminus \{0\} \\ 1/2 & x = 0 \end{cases}.$$

7. **Maclaurin series (15 points total)**

(a) (7 points) Let

$$f(x) = \cos(x^3).$$

Find the Maclaurin representation of f about 0. Hence find the values of $f^{(2016)}(0)$ and $f^{(2019)}(0)$ where $f^{(k)}$ is the k-th derivative of f. You can leave your answer in terms of factorials of integers.

(b) (8 points) Let

$$f(x) = \int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t.$$

Find the Maclaurin series of f and determine the radius of convergence. Define the value of the integrand at 0 to be its limit at 0.

Solution:

(a) We know from the power series representation of $\cos y$ that

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots.$$

By the uniqueness of power series representation,

$$f(x) = \cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \dots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}$$

This is the Maclaurin series representation of f. Now

$$\frac{f^{(2016)}(0)}{2016!} = \text{coefficient of } x^{2016} = \frac{(-1)^{336}}{(2\cdot 336)!} \implies f^{(2016)}(0) = \frac{2016!}{672!}$$

Next, since 2019 is not an integer multiple of 6,

$$f^{(2019)}(0) = 0.$$

(b) Note that

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots$$

so

$$\frac{1-e^{-t}}{t} = 1 - \frac{t}{2!} + \frac{t^2}{3!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n+1)!} \qquad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

Since by L'Hospital's rule $\lim_{t\to 0^+} \frac{1-e^{-t}}{t} = 1$, by defining the value of $\frac{1-e^{-t}}{t}$ at 0 as 1, the above expansion is still valid at $t = 0$. Thus, for all $x \in \mathbb{R}$,

$$\int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n+1)!}\,\mathrm{d}t = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^n}{(n+1)!}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!(n+1)}$$

The interchange of integral and infinite sum can be justified using uniform convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n+1)!}$ on $[0, x]$ and Weierstrass M-test. The radius of convergence is $+\infty$.