

# MA2001 - Linear Algebra I

## AY23/24 Sem 2 Suggested Solutions

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### Question 1

Let  $\mathbf{A}$  be a  $4 \times 5$  matrix and  $\mathbf{b}$  a  $4 \times 1$  matrix. By applying Gauss-Jordan Elimination to

$$\left( \mathbf{A} \mid \mathbf{b} \right)$$

the following series of elementary row operations were performed:

$$R_2 - 2R_1 \rightarrow R_2,$$

$$R_3 + R_1 \rightarrow R_3$$

$$R_4 - R_2 \rightarrow R_4$$

$$R_2 + R_3 \rightarrow R_2$$

$$R_1 + R_2 \rightarrow R_1$$

and we obtained the reduced row-echelon form:

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

- (a) Find  $\mathbf{A}$  and  $\mathbf{b}$ . (Indicate the elementary row operations used in each step.)
- (b) Write down the solution set of the linear system  $\mathbf{Ax} = \mathbf{b}$  explicitly.

*Solution.*

(a) We have

$$\begin{aligned}
 \left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) & \xrightarrow{R_1 - R_2 \rightarrow R_1} \left( \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
 & \xrightarrow{R_2 - R_3 \rightarrow R_2} \left( \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
 & \xrightarrow{R_4 + R_2 \rightarrow R_4} \left( \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right) \\
 & \xrightarrow{R_3 - R_1 \rightarrow R_3} \left( \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ -1 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right) \\
 & \xrightarrow{R_2 + 2R_1 \rightarrow R_2} \left( \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 1 & 1 \\ 2 & -2 & -1 & -1 & 4 & 1 \\ -1 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right)
 \end{aligned}$$

so

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 \\ 2 & -2 & -1 & -1 & 4 \\ -1 & 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(b) Based on the reduced row-echelon form, we have

$$\begin{aligned}
 x_1 - x_2 + x_5 &= 1 \\
 x_3 &= 0 \\
 x_4 - 2x_5 &= 1
 \end{aligned}$$

So,  $x_1 = 1 + x_2 - x_5$ ,  $x_3 = 0$  and  $x_4 = 1 + 2x_5$ , which implies

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 + x_2 - x_5 \\ x_2 \\ 0 \\ 1 + 2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

To conclude, the solution set is

$$\left\{ \mathbf{v} \in \mathbb{R}^5 : \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, s, t \in \mathbb{R} \right\}. \quad \square$$

## Question 2

Let  $\mathbf{B}$  be a square matrix of order  $n$ . Define

$$V = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{B}\mathbf{u} = \mathbf{B}^T\mathbf{u}\}.$$

- (a) Show that  $V$  is a subspace of  $\mathbb{R}^n$ .
- (b) Let  $n = 4$  and

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 0 \end{pmatrix}.$$

- (i) Use Gaussian Elimination to find a row-echelon form of  $\mathbf{B}$  and hence write down a basis for the column space of  $\mathbf{B}$ .
- (ii) Using the row-echelon form of  $\mathbf{B}$  from part (i), write down a basis for the column space of  $\mathbf{B}^T$ .
- (iii) Find a basis for  $V$ .

*Solution.*

- (a) Note that  $\mathbf{u} = \mathbf{0}$  satisfies  $\mathbf{B}\mathbf{u} = \mathbf{B}^T\mathbf{u}$ , so  $V$  is non-empty.

Next, let  $\mathbf{u}_1, \mathbf{u}_2 \in V$ . Then,

$$\mathbf{B}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{B}\mathbf{u}_1 + \mathbf{B}\mathbf{u}_2 = \mathbf{B}^T\mathbf{u}_1 + \mathbf{B}^T\mathbf{u}_2 = \mathbf{B}^T(\mathbf{u}_1 + \mathbf{u}_2)$$

so  $V$  is closed under addition.

Next, let  $\alpha \in \mathbb{R}$  be arbitrary. Then,

$$\mathbf{B}(\alpha\mathbf{u}) = \alpha(\mathbf{B}\mathbf{u}) = \alpha(\mathbf{B}^T\mathbf{u}) = \mathbf{B}^T(\alpha\mathbf{u})$$

so  $V$  is closed under scalar multiplication.

(b) (i) We have

$$\begin{aligned}\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 0 \end{pmatrix} &\xrightarrow{-R_1+R_2 \rightarrow R_2 \text{ and } -R_1+R_3 \rightarrow R_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 0 \end{pmatrix} \\ &\xrightarrow{-2R_2+R_3 \rightarrow R_3 \text{ and } -2R_2+R_4 \rightarrow R_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} \\ &\xrightarrow{R_4 \times (-1/6) \rightarrow R_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

which is a row-echelon form of  $\mathbf{B}$ . A basis for the column space of  $B$  is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 0 \end{pmatrix} \right\}.$$

(ii) This is precisely the same as finding a basis for the row space of  $\mathbf{B}$ , which is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(iii) Note that

$$\mathbf{B} - \mathbf{B}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{pmatrix}.$$

It suffices to find a basis for the null space of  $\mathbf{B} - \mathbf{B}^T$ . Consider

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then,  $z = 0$ ,  $2z = 0$ ,  $3z = 0$  and  $-w - 2x - 3y = 0$ . These four equations imply  $z = 0$  and  $w = -2x - 3y$ . As such, we can write

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x - 3y \\ x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We conclude that the desired basis is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

□

### Question 3

Let  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ , where  $\mathbf{u}_1 = (0, 1, 0, 1)$ ,  $\mathbf{u}_2 = (-4, 0, 3, 0)$  and  $\mathbf{u}_3 = (3, -1, 4, -1)$ . Define  $V = \text{span}(S)$ .

- (a) Follow the Gram-Schmidt Process to transform  $S$  into an orthonormal basis for  $V$ .  
 (b) Find the projection of

$$\mathbf{b} = (0, 1, 5, -1)$$

onto  $V$ .

- (c) Find a least square solution to the system

$$\begin{pmatrix} 0 & -4 & 3 \\ 1 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix}.$$

*Solution.*

- (a) We will apply the Gram-Schmidt process to transform  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  into an orthonormal basis for  $V = \text{span}(S)$ .

Set  $\mathbf{v}_1 = \mathbf{u}_1$ . Then,

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{u}_2) = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1.$$

Since  $\mathbf{u}_2 \cdot \mathbf{v}_1 = 0$ , then  $\mathbf{v}_2 = \mathbf{u}_2$ .

Then,

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{u}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{u}_3) \\ &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\ &= \begin{pmatrix} 3 \\ -1 \\ 4 \\ -1 \end{pmatrix} - \frac{-2}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 0 \\ &= \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Lastly, we normalize  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to obtain

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix}.$$

We conclude that an orthonormal basis for  $V$  is  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ .

(b) We have

$$\text{proj}_V(\mathbf{b}) = \sum_{i=1}^3 \frac{\mathbf{b} \cdot \mathbf{e}_i}{\|\mathbf{e}_i\|^2} \mathbf{e}_i$$

which is equal to

$$\left( \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + \left( \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix} \right) \begin{pmatrix} -4/5 \\ 0 \\ 3/5 \\ 0 \end{pmatrix} + \left( \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \\ 0 \end{pmatrix}.$$

This simplifies to

$$\begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \end{pmatrix}.$$

(c) Let the coefficient matrix be  $\mathbf{A}$ . Then, the equation is of the form  $\mathbf{Ax} = \mathbf{b}$ . By considering  $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$ , we see that  $\mathbf{x} = (4/5, 3/5, 4/5)$ .  $\square$

#### Question 4

Let

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

- (a) Show that the eigenvalues of  $\mathbf{C}$  are 0, 1, 2.
- (b) Find a basis for the eigenspace of  $\mathbf{C}$  associated with the eigenvalue 0.
- (c) Find a basis for the eigenspace of  $\mathbf{C}$  associated with the eigenvalue 1.
- (d) Find a basis for the eigenspace of  $\mathbf{C}$  associated with the eigenvalue 2.
- (e) Write down two  $4 \times 4$  matrices  $\mathbf{P}$  and  $\mathbf{D}$ , where  $\mathbf{D}$  is a diagonal matrix and  $\mathbf{P}$  is an invertible matrix such that

$$\mathbf{P}^{-1} \mathbf{C} \mathbf{P} = \mathbf{D}.$$

*Solution.*

(a) We have

$$\mathbf{C} - \lambda \mathbf{I} = \begin{pmatrix} 1-\lambda & 1 & -1 & -1 \\ 1 & 1-\lambda & -1 & 1 \\ 1 & 1 & -1-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}$$

so

$$\det(\mathbf{C} - \lambda \mathbf{I}) = \lambda^2 (\lambda^2 - 3\lambda + 2) = \lambda^2 (\lambda - 1)(\lambda - 2).$$

So, the eigenvalues are 0, 1, 2.

(b) When  $\lambda = 0$ , we have

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

One checks that a row-echelon form of  $\mathbf{C}$  is

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let  $(v_1, v_2, v_3, v_4)$  be an eigenvector corresponding to the eigenvalue 0. Then,  $2v_4 = 0$ , so  $v_4 = 0$ , and  $v_1 + v_2 - v_3 - v_4 = 0$ , so  $v_1 + v_2 = v_3$ . As such, a basis for the eigenspace is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(c) We have

$$\mathbf{C} - \mathbf{I} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A row echelon form of  $\mathbf{C} - \mathbf{I}$  is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, a basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(d) We have

$$\mathbf{C} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A row-echelon form of  $\mathbf{C} - 2\mathbf{I}$  is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so we conclude that a basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(e)

$$\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

□

### Question 5

Determine which of the following statements are true. Write down ‘True’ or ‘False’ in the boxes provided. Except part (e), for each statement, if your answer is ‘True’, explain why the statement is always true; and if your answer is ‘False’, give a counter-example to justify that the statement is not always true.

- (a) If a linear system  $\mathbf{Ax} = \mathbf{b}$  has only one solution, then the homogeneous linear system  $\mathbf{Ax} = \mathbf{0}$  has only the trivial solution.
- (b) If a homogeneous linear system  $\mathbf{Ax} = \mathbf{0}$  has only the trivial solution, then any linear system of the form  $\mathbf{Ax} = \mathbf{b}$  has one solution.



(c) If  $\mathbf{A}$  and  $\mathbf{B}$  are orthogonal matrices of the same order, then  $\mathbf{AB}$  is an orthogonal matrix.

(d) Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix}$$

be a square matrix, where  $\mathbf{r}_i$  is the  $i$ -th row of  $\mathbf{A}$ . If  $\mathbf{A}$  is invertible, then so is the matrix

$$\begin{pmatrix} \mathbf{r}_n \\ \mathbf{r}_{n-1} \\ \vdots \\ \mathbf{r}_1 \end{pmatrix}.$$

(e) There exists a  $3 \times 3$  matrix  $\mathbf{A}$  such that  $\mathbf{A}^2 + \mathbf{A} = \mathbf{I}_3$  and 0 is an eigenvalue of  $\mathbf{A}$ .

*Solution.*

(a) True. If the system  $\mathbf{Ax} = \mathbf{b}$  has a unique solution, the matrix  $\mathbf{A}$  is of full rank, which implies that its null space contains only the zero vector. The result follows.

(b) False. Consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which only has the trivial solution  $\mathbf{x} = \mathbf{0} \in \mathbb{R}^3$  but

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ has no solution.}$$

(c) True. Since  $\mathbf{A}$  and  $\mathbf{B}$  are orthogonal, then  $\mathbf{A}^T \mathbf{A} = \mathbf{I}$  and  $\mathbf{B}^T \mathbf{B} = \mathbf{I}$ . So,

$$(\mathbf{AB})^T (\mathbf{AB}) = \mathbf{B}^T \mathbf{A}^T (\mathbf{AB}) = \mathbf{B}^T \mathbf{I} \mathbf{B} = \mathbf{B}^T \mathbf{B} = \mathbf{I}.$$

This implies  $\mathbf{AB}$  is orthogonal.

(d) True. Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{r}_n \\ \mathbf{r}_{n-1} \\ \vdots \\ \mathbf{r}_1 \end{pmatrix}.$$

Observe that  $\mathbf{B}$  can be obtained from  $\mathbf{A}$  via a sequence of row swaps, i.e.  $\mathbf{r}_1$  swaps with  $\mathbf{r}_n$ ,  $\mathbf{r}_2$  swaps with  $\mathbf{r}_{n-1}$ , and so on. Each of these row operations can be represented by some elementary matrix  $\mathbf{E}$ . As such,  $\mathbf{B} = (\mathbf{E}_1 \dots \mathbf{E}_i) \mathbf{A}$ , which is a product of invertible matrices since each elementary matrix is invertible. So,  $\mathbf{B}$  is invertible.

- (e) False. We have  $\mathbf{A}(\mathbf{A} + \mathbf{I}) = \mathbf{I}$ , which implies  $\mathbf{A}$  is invertible. However, 0 is an eigenvalue of  $\mathbf{A}$ , which contradicts the invertible matrix theorem.  $\square$

### Question 6

- (a) Let  $\mathbf{A}$  be a matrix. Prove that  $\mathbf{A} = \mathbf{0}$  if and only if  $\mathbf{A}^T \mathbf{A} = \mathbf{0}$ .
- (b) Let  $T$  be a linear operator on  $\mathbb{R}^n$ . Suppose that  $\text{rank}(T) = \text{rank}(T \circ T)$ .
- (i) Prove that  $\ker(T) = \ker(T \circ T)$ .
- (ii) Prove that  $\text{R}(T) \cap \ker(T) = \{\mathbf{0}\}$ .
- (c) Find an example of a linear transformation  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  such that  $\text{R}(S) = \ker(S)$ . Can you find a linear transformation  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  with the same property? Justify your answers.

*Solution.*

- (a) ( $\implies$ ) Suppose  $\mathbf{A} = \mathbf{0}$ . Then,  $\mathbf{A}^T \mathbf{A} = \mathbf{A}^T \mathbf{0} = \mathbf{0}$ .

( $\impliedby$ ) Suppose  $\mathbf{A}^T \mathbf{A} = \mathbf{0}$ . Suppose  $\mathbf{A}$  is an  $m \times n$  matrix with entries  $a_{ij}$ . By considering the first row of  $\mathbf{A}^T$  and the first column of  $\mathbf{A}$ , we have

$$a_{11}^2 + a_{21}^2 + \dots + a_{m1}^2 = 0.$$

Since the sum of squares is 0, it forces each square to be 0, and consequently,  $a_{11} = a_{21} = \dots = a_{m1} = 0$ .

In general, by considering the  $j^{\text{th}}$  row of  $\mathbf{A}^T$  and the  $j^{\text{th}}$  column of  $\mathbf{A}$ , where  $1 \leq j \leq n$ , we have

$$a_{1j}^2 + a_{2j}^2 + \dots + a_{mj}^2 = 0.$$

As such,  $a_{1j} = a_{2j} = \dots = a_{mj} = 0$  for all  $1 \leq j \leq n$ . This forces  $a_{ij} = 0$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . So,  $\mathbf{A} = \mathbf{0}$ .

- (b) (i) We first prove that

$$\ker(T) \subseteq \ker(T \circ T).$$

Suppose  $\mathbf{v} \in \ker(T)$ . Then,  $T(\mathbf{v}) = \mathbf{0}$ , so  $T(T(\mathbf{v})) = T(\mathbf{0}) = \mathbf{0}$ , so  $\ker(T) \subseteq \ker(T \circ T)$ .

Next, by the rank-nullity theorem, we have

$$n = \text{rank}(T) + \dim(\ker(T)) \quad \text{and} \quad n = \text{rank}(T \circ T) + \dim(\ker(T \circ T)).$$

Since  $\text{rank}(T) = \text{rank}(T \circ T)$ , then  $\dim(\ker(T)) = \dim(\ker(T \circ T))$ . Since  $\mathbb{R}^n$  is a finite-dimensional vector space, it forces  $\ker(T) \supseteq \ker(T \circ T)$ . We conclude that  $\ker(T) = \ker(T \circ T)$ .

(ii) Suppose  $\mathbf{w} \in \mathbf{R}(T) \cap \ker(T)$ . Since  $\mathbf{w} \in \mathbf{R}(T)$ , then there exists  $\mathbf{v} \in \mathbb{R}^n$  such that  $T(\mathbf{v}) = \mathbf{w}$ . Also, since  $\mathbf{w} \in \ker(T)$ , then  $T(\mathbf{w}) = \mathbf{0}$ . So,  $T^2(\mathbf{v}) = \mathbf{0}$ , which implies  $\mathbf{v} \in \ker(T \circ T)$ .

Consequently,  $\mathbf{v} \in \ker(T)$  by (i). As such,  $T(\mathbf{v}) = \mathbf{0}$ , so  $\mathbf{w} = \mathbf{0}$ . We conclude that  $\mathbf{R}(T) \cap \ker(T) = \{\mathbf{0}\}$ .

(c) By the rank-nullity theorem, we have  $\text{rank}(S) + \text{nullity}(S) = 4$ , and because  $\text{rank}(S) = \text{nullity}(S)$ , then  $\text{rank}(S) = \text{nullity}(S) = 2$ . As such, any row-echelon form of the matrix representation of the linear transformation should contain 2 pivots. As such, we consider

$$S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$$

We have  $\mathbf{R}(S) = \text{span}\{(1, 0, 0, 0), (0, 1, 0, 0)\}$  Also,  $\ker(S) = \text{span}\{(0, 0, 1, 0), (0, 0, 0, 1)\}$ .

However, there does not exist any linear transformation  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  with the same property, otherwise it would imply that  $5 = 2\text{rank}(T)$ . However this is a contradiction as  $\text{rank}(T)$  must be a non-negative integer.  $\square$