MA2101 AY17/18 SEM 1 SOLUTIONS

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Question 1. Let $A = \left(a_{ij}\right) \in M_2(\mathbb{C})$ be a complex matrix of size 2×2 and let

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{be an invertible matrix such that} \quad P^{-1}AP = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Let $y_i = y_i(x)$ $(i = 1, 2)$ be differentiable functions in x. Solve the following system of differential equations:

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Hint. You may assume the solution to the differential equation

$$z'(x) + p(x)y = q(x)$$

is given by

$$z = \frac{1}{\mu}\left(\int \mu q(x)\, dx + C\right) \quad \text{where} \quad \mu = e^{\int p(x)\, dx}.$$

Your solution should be of the form: $y_i = y_i(x)$ is a function in x with coefficients involving the entries a, b, c, d of the matrix P.

Solution. Let $PY = Z$, where $Z = (z_1, z_2)$ is a column vector. This implies that $Y' = PZ'$, and so it follows that

$$PZ' = AY = APZ.$$

Hence, $Z' = P^{-1}APZ$ or equivalently,

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

This gives us the following system of linear equations

$$\begin{aligned} z_1' &= z_1 + 2z_2 \\ z_2' &= -z_2. \end{aligned}$$

Solving the second equation yields $z_2 = a_1 e^{-x}$ for some $a_1 \in \mathbb{C}$. Then $z_1' - z_1 = 2a_1 e^{-x}$. Hence,

$$\begin{aligned} z_1 &= \frac{1}{e^{-x}}\left(\int e^{-x}\left(2a_1 e^{-x}\right)\,dx + a_2\right) \\ &= e^x\left(2a_1\left(-\frac{1}{2}e^{-2x}\right) + a_2\right) \\ &= -a_1 e^{-x} + a_2 e^x. \end{aligned}$$

Since $Z = PY$, then $Y = P^{-1}Z$ and so

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} -a_1 e^{-x} + a_2 e^x \\ a_1 e^{-x} \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} da_2 e^x - a_1\left(b+d\right)e^{-x} \\ -ca_2 e^x + a_1\left(a+c\right)e^{-x} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 &= da_2 e^x - a_1\left(b+d\right)e^{-x} \\ y_2 &= -ca_2 e^x + a_1\left(a+c\right)e^{-x} \end{aligned}$$

where $a_1, a_2 \in \mathbb{C}$. $\square$

Question 2. Consider the following real matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}.$$

- **(a)** Find the eigenvalues λ_i of A.
- **(b)** Find a basis B_i for each eigenspace $V_{\lambda_i}(A)$ of A.
- **(c)** Find an invertible matrix P such that $P^\top AP$ equals a diagonal matrix D; determine this D. Here $P^\top$ is the transpose of P.

Solution.

- **(a)** Let λ be an eigenvalue of A. The characteristic polynomial of A is

$$\det\begin{pmatrix} \lambda-2 & 2 \\ 2 & \lambda-5 \end{pmatrix} = \left(\lambda-2\right)\left(\lambda-5\right) - 4 = \lambda^2 - 7\lambda + 6 = \left(\lambda-1\right)\left(\lambda-6\right).$$

This implies that $1, 6$ are the eigenvalues of A.

(b) When $\lambda = 1$, we have

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

then $B_1 = \{(2, 1)\}$ is a basis for the eigenspace $V_1(A)$.

When $\lambda = 6$, we have

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

then $B_6 = \{(1, -2)\}$ is a basis for the eigenspace $V_6(A)$.

(c) Normalising the bases B_1 and B_6 we have

$$\widetilde{B_1} = \left\{ \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right\}$$

$$\widetilde{B_6} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \right\}$$

respectively. Let

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}.$$

Note that P is invertible since $\det(P) = -\frac{4}{5} - \frac{1}{5} = -1 \neq 0$. Then

$$P^T A P = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = D$$

$\square$

Question 3. Let V and W be vector spaces over a field F, and let $T : V \longrightarrow W$ be a surjective linear transformation. Let D be a linearly independent subset of W. For each $\mathbf{d}$ in D, fix a vector $\mathbf{c}_{\mathbf{d}}$ in V such that $T(\mathbf{c}_{\mathbf{d}}) = \mathbf{d}$, and denote

$$C = \{\mathbf{c}_{\mathbf{d}} : \mathbf{d} \in D\}.$$

(a) Show that C is a linearly independent subset of V.
(b) Show that if the C above is a basis of V then T is an isomorphism.
Note. You are not supposed to assume D is a finite set.

Solution.

(a) Let $C' \subseteq C$ be a non-empty finite subset of C and write $C' = \{\mathbf{c}_{\mathbf{d}_1}, \ldots, \mathbf{c}_{\mathbf{d}_m}\}$ and suppose that

$$a_1\mathbf{c}_{\mathbf{d}_1} + \cdots + a_m\mathbf{c}_{\mathbf{d}_m} = \mathbf{0}.$$

Applying T to both sides, we get

$$\begin{aligned}\mathbf{0} &= T(\mathbf{0}) \\ &= T(a_1\mathbf{c}_{\mathbf{d}_1} + \cdots + a_m\mathbf{c}_{\mathbf{d}_m}) \\ &= a_1T(\mathbf{c}_{\mathbf{d}_1}) + \cdots + a_mT(\mathbf{c}_{\mathbf{d}_m}) \\ &= a_1\mathbf{d}_1 + \cdots + a_m\mathbf{d}_m\end{aligned}$$

which implies that $a_1 = \cdots = a_m = 0$ since D is a linearly independent subset of W. Hence, C' is a linearly independent (finite) subset of C. Since $C' \subseteq C$ is arbitrary, then C is a linearly independent subset of V.

(b) Suppose that C is a basis of V. By assumption, T is surjective. It suffices to show that T is injective. Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in V$ is such that

$$T(\mathbf{v}_1) = T(\mathbf{v}_2).$$

Since C is a basis for V, then there exists $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in F$ (where we allow some α_k, β_l to be zero) and $\mathbf{c}_{\mathbf{d}_1}, \ldots, \mathbf{c}_{\mathbf{d}_m} \in C$ such that

$$\begin{aligned}\mathbf{v}_1 &= \sum_{i=1}^{m} \alpha_i\mathbf{c}_{\mathbf{d}_i} \\ \mathbf{v}_2 &= \sum_{j=1}^{m} \beta_j\mathbf{c}_{\mathbf{d}_j}.\end{aligned}$$

This implies that

$$\begin{aligned}\mathbf{0} &= T\left(\sum_{i=1}^{m}(\alpha_i-\beta_i)\,\mathbf{c}_{\mathbf{d}_i}\right)\\ &= \sum_{i=1}^{m}(\alpha_i-\beta_i)\,T\left(\mathbf{c}_{\mathbf{d}_i}\right)\\ &= \sum_{i=1}^{m}(\alpha_i-\beta_i)\,\mathbf{c}_{\mathbf{d}_i}.\end{aligned}$$

Since C is a basis and in particular is linearly independent, then $\alpha_i = \beta_i$ for all $i = 1,\dots,m$. This implies that $\mathbf{v}_1 = \mathbf{v}_2$. Hence, T is injective and so T is an isomorphism. □

Question 4. Let V be a vector space of finite dimension n over a field F, and let B be a basis of V. Let

$$H = \mathrm{Hom}_F(V,V) = \{f : V \longrightarrow V : f \text{ is a linear transformation}\}.$$

(a) State the definition of an isomorphism between two vector spaces.

(b) State the definitions of addition $f+g$ and scalar multiplication αf for linear transformations f,g in H and scalar α in F, in the way that H becomes a vector space over F.
(You are **not** required to verify that H satisfies the axioms to be a vector space).

(c) Construct linear transformations

$$\begin{aligned}\varphi &: H \longrightarrow M_n(F),\\ \psi &: M_n(F) \longrightarrow H\end{aligned}$$

such that they are isomorphisms and inverse to each other. **Justify** your answers.

(d) What is $\dim_F H$? (You are **not** required to justify your answer).

Solution.

(a) Let V_1,V_2 be vector spaces over the same field F. A map $\varphi : V_1 \longrightarrow V_2$ is called a linear transformation from (or between) V_1 to V_2 if φ is compatible with the vector addition and scalar multiplication on V_1 and V_2 in the sense below:

$$\begin{aligned}\varphi(\mathbf{v}_1+\mathbf{v}_2) &= \varphi(\mathbf{v}_1)+\varphi(\mathbf{v}_2)\\ \varphi(a\mathbf{v}_1) &= a\varphi(\mathbf{v}_1)\end{aligned}$$

for all $\mathbf{v}_1,\mathbf{v}_2 \in V$ and $a \in F$. A linear transformation $\varphi : V_1 \longrightarrow V_2$ between vector spaces V_1,V_2 over the same field F is an isomorphism if φ is a bijection.

(b) For any $\mathbf{v} \in V$ and $\alpha \in F$, define

$$\begin{aligned}(f+g)(\mathbf{v}) &= f(\mathbf{v})+g(\mathbf{v}) \\ \alpha f(\mathbf{v}) &= f(\alpha \mathbf{v})\end{aligned}$$

(c) Define

$$\begin{aligned}\varphi : H &\longrightarrow M_n(F) \\ f &\mapsto [f]_{B,B}\end{aligned}$$

and

$$\begin{aligned}\psi : M_n(F) &\longrightarrow H \\ A &\mapsto f_A\end{aligned}$$

where f_A is the linear transformation from V to V such that $[f_A(\mathbf{u}_i)]_B$ is the i-th column of A.

We first show that φ and ψ are linear transformations. Let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, $f, g \in H$, and $k \in F$. Then

$$\begin{aligned}\varphi(f+g) &= [f+g]_{B,B} \\ &= \begin{pmatrix}[f(\mathbf{u}_1)+g(\mathbf{u}_2)]_B & \cdots & [f(\mathbf{u}_n)+g(\mathbf{u}_n)]_B\end{pmatrix} \\ &= \begin{pmatrix}[f(\mathbf{u}_1)]_B+[g(\mathbf{u}_2)]_B & \cdots & [f(\mathbf{u}_n)]_B+[g(\mathbf{u}_n)]_B\end{pmatrix} \\ &= [f]_B+[g]_B \\ &= \varphi(f)+\varphi(g) \\ \varphi(kf) &= [kf]_B \\ &= \begin{pmatrix}[kf(\mathbf{u}_1)]_B & \cdots & [kf(\mathbf{u}_n)]_B\end{pmatrix} \\ &= \begin{pmatrix}k[f(\mathbf{u}_1)]_B & \cdots & k[f(\mathbf{u}_n)]_B\end{pmatrix} \\ &= k[f]_B \\ &= k\varphi(f)\end{aligned}$$

so that φ is a linear transformation. Next, for any $\mathbf{u}_j \in B$,

$$\begin{aligned}
\psi(A+B)\left(\mathbf{u}_j\right) &= f_{A+B}\left(\mathbf{u}_j\right) \\
&= \sum_{i=1}^{n}\left(a_{ij}+b_{ij}\right)\mathbf{u}_i \\
&= \sum_{i=1}^{n} a_{ij}\mathbf{u}_i + \sum_{i=1}^{n} b_{ij}\mathbf{u}_i \\
&= f_A\left(\mathbf{u}_j\right) + f_B\left(\mathbf{u}_j\right) \\
&= \psi(A)\left(\mathbf{u}_j\right) + \psi(B)\left(\mathbf{u}_j\right)
\end{aligned}$$

$$\psi(kA)\left(\mathbf{u}_j\right) = f_{kA}\left(\mathbf{u}_j\right) = \sum_{i=1}^{n} ka_{ij}\mathbf{u}_i = k\sum_{i=1}^{n} a_{ij}\mathbf{u}_i = kf_A\left(\mathbf{u}_j\right) = k\psi(A)\left(\mathbf{u}_j\right)$$

which implies that ψ is a linear transformation.

Now, we show that φ and ψ are inverses of each other. Let $f \in H$, $\mathbf{u}_i \in B$ and $\mathbf{e}_j \in F^n$. Then

$$\begin{aligned}
\psi\circ\varphi(f)\left(\mathbf{u}_i\right) &= \psi\left([f]_B\right)\left(\mathbf{u}_i\right) = \begin{pmatrix}\mathbf{u}_1 & \dots & \mathbf{u}_n\end{pmatrix}\left[f\left(\mathbf{u}_i\right)\right]_B = f\left(\mathbf{u}_i\right) \\
\varphi\circ\psi(A)\left(\mathbf{e}_j\right) &= \varphi\left(f_A\right)\left(\mathbf{e}_j\right) = \left[f_A\right]_B\left(\mathbf{e}_j\right) = A\mathbf{e}_j
\end{aligned}$$

so that $\varphi\circ\psi = \mathrm{id}_{M_n(F)}$ and $\psi\circ\varphi = \mathrm{id}_{\mathrm{Hom}_F(V,V)}$. Therefore, φ and ψ are isomorphisms and inverse to each other.

(d) Since $H \cong M_n(F)$ and $\dim_F M_n(F) = n^2$, then $\dim_F H = n^2$. $\square$

Question 5. Let $(V, \langle,\rangle)$ be a complex innner product space of dimension n, and let $T : V \longrightarrow V$ be a linear transformation.

(a) State the definition, i.e., the characteristic property, of the adjoint linear operator $T^* : V \longrightarrow V$ of T.

(b) State the definition of an orthonormal basis of V.

In (c) and (d) below, $B = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ is an orthonormal basis of V. Assume $T\left(\mathbf{w}_1, \dots, \mathbf{w}_n\right) = (\mathbf{w}_1, \dots, \mathbf{w}_n)A$ for a complex matrix $A = \left(a_{ij}\right) \in M_n(\mathbb{C})$. For $\mathbf{w} \in V$, denote by $[\mathbf{w}]_B = (a_1, \dots, a_n)^\top$ the coordinate vector of $\mathbf{w}$ relative to the basis B, and $\overline{[\mathbf{w}]_B} = (\overline{a_1}, \dots, \overline{a_n})^\top$ its conjugate.

(c) Is it true that $\langle \mathbf{u}, T^*(\mathbf{v})\rangle = \left([\mathbf{v}]_B\right)^\top A^\top \overline{[\mathbf{u}]_B}$ for all $\mathbf{u}, \mathbf{v} \in V$? If it is true prove it; if it is false, provide a **concrete** counterexample.

(d) Is it true that $\langle \mathbf{u}, T^*(\mathbf{v})\rangle = \left([\mathbf{u}]_B\right)^\top A^\top \overline{[\mathbf{v}]_B}$ for all $\mathbf{u}, \mathbf{v} \in V$? If it is true prove it; if it is false, provide a **concrete** counterexample.

Note. You are not supposed to assume V is the column space or $(V, \langle , \rangle)$ is the standard inner product column space.

Solution.

(a) Let $T : V \longrightarrow V$ be a linear operator on an n-dimensional inner product space V over a field F ($F = \mathbb{R}$, or $F = \mathbb{C}$). A linear operator $T^* : V \longrightarrow V$ such that

$$\langle T((\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, T^*(\mathbf{v})\rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$ is called an adjoint linear operator of T.

(b) Let $(V, \langle , \rangle)$ be a real or complex inner product space. A basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is called an orthonormal basis of the inner product space V (relative to the inner product $\langle , \rangle$) if it satisfies the following two conditions:
  (a) **Orthogonality:** for all $i \neq j$, we have: $\mathbf{v}_i \perp \mathbf{v}_j$, i.e., $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$.
  (b) **Normalized:** for all i, we have $\|\mathbf{v}_i\| = 1$, i.e., $\langle \mathbf{v}_i, \mathbf{v}_i\rangle = 1$. Namely, $\mathbf{v}_i$ is a **unit** vector.

(c) False. Let $V = \mathbb{C}^2$ over $\mathbb{C}$. Let $\mathbf{u} = \begin{pmatrix} 3i & 4 \end{pmatrix}^\top$, $\mathbf{v} = \begin{pmatrix} 7 & -5i \end{pmatrix}^\top$, and $A = \begin{pmatrix} 11 & 13 \\ -7i & 3 \end{pmatrix}$.

Let $\langle , \rangle$ be the standard inner product on $\mathbb{C}^2$ and $T = L_A$ be the left-multiplication by A. Let B be the standard orthonormal basis on $\mathbb{C}^2$ over $\mathbb{C}$. Then

$$\langle \mathbf{u}, T^*(\mathbf{v})\rangle = \left\langle \begin{pmatrix} 3i \\ 4 \end{pmatrix}, \begin{pmatrix} 11 & 7i \\ 13 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ -5i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 3i \\ 4 \end{pmatrix}, \begin{pmatrix} 112 \\ 91 - 15i \end{pmatrix} \right\rangle = 364 - 276i.$$

On the other hand,

$$([\mathbf{v}]_B)^\top A^\top \overline{[\mathbf{u}]_B} = \begin{pmatrix} 7 & -5i \end{pmatrix} \begin{pmatrix} 11 & -7i \\ 13 & 3 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = -195 - 487i.$$

Hence, $\langle \mathbf{u}, T^*(\mathbf{v})\rangle \neq ([\mathbf{v}]_B)^\top A^\top \overline{[\mathbf{u}]_B}$

(d) True. Let $\mathbf{u}=a_1\mathbf{w}_1+\cdots+a_n\mathbf{w}_n$, $\mathbf{v}=b_1\mathbf{w}_1+\cdots+b_n\mathbf{w}_n$, where $a_1,\ldots,a_n,b_1,\ldots,b_n\in\mathbb{C}$. Then

$$\begin{aligned}
\langle \mathbf{u}, T^*(\mathbf{v})\rangle &= \left\langle \sum_{i=1}^n a_i\mathbf{w}_i, T^*\left(\sum_{j=1}^n b_j\mathbf{w}_j\right)\right\rangle = \left\langle \sum_{i=1}^n a_i\mathbf{w}_i, \sum_{j=1}^n b_j T^*(\mathbf{w}_j)\right\rangle \\
&= \left\langle \sum_{i=1}^n a_i\mathbf{w}_i, \sum_{j=1}^n b_j \sum_{k=1}^n \overline{a_{jk}}\mathbf{w}_k\right\rangle \\
&= \sum_{i=1}^n\sum_{j=1}^n a_i\overline{b_j}\left\langle \mathbf{w}_i, \sum_{k=1}^n \overline{a_{jk}}\mathbf{w}_k\right\rangle \\
&= \sum_{i=1}^n\sum_{j=1}^n a_i\overline{b_j}\left\langle \mathbf{w}_i, \overline{a_{ji}}\mathbf{w}_i\right\rangle \\
&= \sum_{i=1}^n\sum_{j=1}^n a_i a_{ji}\overline{b_j}\left\langle \mathbf{w}_i, \mathbf{w}_i\right\rangle \\
&= \left([\mathbf{u}]_B\right)^\top A^\top \overline{[\mathbf{v}]_B}
\end{aligned}$$

□

Question 6.

(a) State the definition of the conjugate $\overline{\lambda}$ for the complex number $\lambda=a+bi$, with a,b real numbers.

(b) State the definition of a unitary matrix.

In (c) - (d) below, assume $U\in M_n(\mathbb{C})$ is a unitary matrix, and $Y=(y_1,\ldots,y_n)^\top$ is a non-zero column vector such that $UY=\lambda Y$ for some complex number λ. Let U^* be the adjoint linear operator of U and $\overline{Y}=(\overline{y_1},\ldots,\overline{y}_n)^\top$ the conjugate of Y.

(c) Is it true that $U^*Y=\overline{\lambda}Y$? If it is true prove it; if it is false, provide a **concrete** counterexample.

(d) Is it true that $U^*\overline{Y}=\overline{\lambda}\overline{Y}$? If it is true prove it; if it is false, provide a **concrete** counterexample.

Solution.

(a) Let $\lambda\in\mathbb{C}$ and write $\lambda=a+bi$, where $a,b\in\mathbb{R}$. Then $\overline{\lambda}=a-bi$.

(b) A complex matrix $A\in M_n(\mathbb{C})$ is unitary if $AA^*=I_n$, or equivalently, $A^*A=I_n$, where $A^*=\left(\overline{A}\right)^\top$ is the adjoint of the matrix A.

(c) True. Since $UY=\lambda Y$, then

$$Y^*Y=Y^*U^*UY=\lambda\overline{\lambda}Y^*Y=|\lambda|^2Y^*Y$$

so that $|\lambda|^2 = 1$. This implies that $1/\lambda = \overline{\lambda}$. Now, $UY = \lambda Y$ implies that $U^*UY = \lambda U^*Y$, so that $(1/\lambda)Y = U^*Y$. Since $1/\lambda = \overline{\lambda}$, then $U^*Y = \overline{\lambda}Y$.

(d) False. Let

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$Y = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Then, we see that

$$UY = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \begin{pmatrix} i \\ 1 \end{pmatrix} = \lambda Y$$

where $\lambda = \left(1/\sqrt{2}\right) + \left(i/\sqrt{2}\right)$. But

$$U^*\overline{Y} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{pmatrix} \neq \begin{pmatrix} -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \overline{\lambda}\overline{Y}.$$

□

Question 7. Let $\mathbf{A} \in M_3(\mathbb{C})$ be a complex matrix of size 3×3 such that $\mathbf{A}^3 = \mathbf{A}$.

(a) Is $\mathbf{A}$ diagonalizable? **Justify** your answer.

(b) Find **all** possible Jordan canonical forms $\mathbf{J}$ of $\mathbf{A}$. Justify your answers.

Solution.

(a) Yes (in fact this holds for any field F). To see why, recall that a matrix $\mathbf{A}$ is diagonalizable if and only if its minimal polynomial $m_{\mathbf{A}}(x)$ splits over $\mathbb{C}$ and has no repeated roots.

Note that the characteristic polynomial is $c_{\mathbf{A}}(x) = x^3 - x$. Since $m_{\mathbf{A}}(x) \mid c_{\mathbf{A}}(x)$, it forces $m_{\mathbf{A}}(x)$ to not have any repeated roots. Also, $m_{\mathbf{A}}(x)$ splits completely over $\mathbb{C}$, making $\mathbf{A}$ diagonalisable.

(b) Setting $c_{\mathbf{A}}(x) = 0$, we have $x = 0$ or 1 or -1. We shall consider

- **Case 1 (3 distinct Jordan blocks):** This means $\mathbf{A}$ is diagonalisable, so

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- **Case 2 (1 Jordan block of size 2):** This implies that the other Jordan block is of size 1 and it contains one of the other two eigenvalues that does not appear in the Jordan block of size 2. Hence, we have either of the following:

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

□

Question 8. Let $(V, \langle , \rangle)$ be a complex inner product space of dimension n. Let $T : V \longrightarrow V$ be a linear transformation and $T^* : V \longrightarrow V$ the adjoint linear operator of T.

(a) State the definition of a normal operator.

(b) Prove that if T is a normal operator then there is an orthonormal basis B such that the representation matrix $[T]_B$ relative to B is a diagonal matrix. Namely, prove one direction of the Principal Axis Theorem 10.31 (3) (or equivalently 10.31 (4)).
Note. If you use results in lecture notes or question sheets of tutorial assignments, state them clearly.

Solution.

(a) Let $T : V \longrightarrow V$ be a linear operator on an n-dimensional inner product space which is over a field F, and with an orthonormal basis B. Let T^* be the adjoint of T as in (5a). A linear operator T over a complex inner product space is normal if $TT^* = T^*T$.

(b) Suppose that T is normal. By the fundamental theorem of algebra, the characteristic polynomial of T splits in $\mathbb{C}$. By Schur's theorem, there exists an orthonormal basis $B = \{v_1, \dots, v_n\}$ for V such that $[T]_B$ is upper triangular.

Since $[T]_B$ is upper triangular, then v_1 is an eigenvector of T. Suppose that $v_1, \dots, v_{k-1}$ are eigenvectors of T. We will show that v_k is an eigenvector of T. It follows from induction on k that all the v_i's are eigenvectors of T. By assumption, since for any $j < k$, v_j is an eigenvector of T, then $T\left(v_j\right) = \lambda_j v_j$, where λ_j is an eigenvalue of T corresponding to the eigenvector v_j. Since $[T]_B$ is upper triangular, then we may write

$$T\left(v_k\right) = \sum_{i=1}^{k} \left\langle T\left(v_k\right), v_i \right\rangle v_i.$$

But for any $i \neq k$,

$$\left\langle T\left(v_k\right), v_i \right\rangle = \left\langle v_k, T^*\left(v_i\right) \right\rangle = \left\langle v_k, \overline{\lambda_i} v_i \right\rangle = \lambda_i \left\langle v_k, v_i \right\rangle = 0$$

so that

$$T\left(v_k\right)=\sum_{i=1}^{k}\left\langle T\left(v_k\right),v_i\right\rangle v_i=\left\langle T\left(v_k\right),v_k\right\rangle v_k=\lambda_k v_k$$

where $\lambda_k=\left\langle T\left(v_k\right),v_k\right\rangle$. This implies that v_k is an eigenvector of T, with corresponding eigenvalue λ_k. By induction, all the $v_i\in B$ are eigenvalues of T, and so $[T]_B$ is indeed diagonal. $\square$