IMDEN

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Homework 8

Exercice 3.1.

1) We have

$$\mathbb{E}_{v,u}\left[\|F^\lambda(v)-u\|^2\right] = \mathbb{E}_{v,u}\left[\|\lambda F(v)+(1-\lambda)v-u\|^2\right]$$

$$= \mathbb{E}_{v,u}\left[\|\lambda(F(v)-u)+(1-\lambda)(v-u)\|^2\right]$$

$$= \lambda^2\,\mathbb{E}_{v,u}\left[\|F(v)-u\|^2\right] + (1-\lambda)^2\,\mathbb{E}_{v,u}\left[\|v-u\|^2\right] + 2\lambda(1-\lambda)\,\mathbb{E}\left[\langle F(v)-u \mid v-u\rangle\right]$$

2) We also have

$$\mathbb{E}_{v,u}\left[\langle F(v)-u \mid v-u\rangle\right] = \mathbb{E}_u\left[\mathbb{E}_v\left[\langle F(v)-u \mid v-u\rangle \mid u\right]\right]$$

and

$$\mathbb{E}_v\left[\langle F(v)-u \mid v-u\rangle \mid u\right]$$

$$= \sum_x \mathbb{E}_{v_x, v_{\bar{x}}}\left[\langle \underbrace{F(v)_x}_{\text{(indep of } v_x)} - u_x \mid u_x - v_x\rangle \mid u\right],\ x \text{ patch or pixel}$$

and $\mathbb{E}_{v_x,v_{\bar{x}}}\left[\langle F(v)_x-u_x \mid u_x-v_x\rangle \mid u\right]$

$$= \mathbb{E}_{v_{\bar{x}}}\left[\mathbb{E}_{v_x}\left[\langle F(v)_x-u_x \mid u_x-v_x\rangle \mid u, v_{\bar{x}}\right] \mid u\right]$$

$$= \mathbb{E}_{v_{\bar{x}}}\left[\langle F(v)_x-u_x \mid u_x-\mathbb{E}_{v_x}[v_x \mid u, v_{\bar{x}}]\rangle \mid u\right]$$

$$= \mathbb{E}_{\bar{v}_x}\left[\langle F(v)_x-u_x \mid \underbrace{u_x-\mathbb{E}_{v_x}[v_x \mid u]}_{0}\rangle \mid u\right]$$

$$= 0$$

ie $\boxed{\mathbb{E}_{v,u}\left[\langle F(v)-u \mid v-u\rangle\right]=0}$

3) In the end, minimising the MSE leads to minimising

$$\lambda^2 \mathbb{E}_{v,u}\left[\|F(v)-u\|^2\right] + (1-\lambda)^2 \mathbb{E}_{u,v}\left[\|v-u\|^2\right]$$

This is a polynome of order 2 of the form $\lambda^2 a + (1-\lambda)^2 b = P(\lambda)$

ie $P'(\lambda) = 2\lambda a - 2(1-\lambda) b$.

$= 2\left[\lambda(a+b) - b\right]$

The λ^* minimising the MSE is:

$$\lambda^* = \frac{\mathbb{E}_{u,v}\left[\|v-u\|^2\right]}{\mathbb{E}_{u,v}\left[\|F(v)-u\|^2\right] + \mathbb{E}_{u,v}\left[\|v-u\|^2\right]}.$$

Finally, $\mathbb{E}_{u,v}\left[\|v-u\|^2\right] = \mathbb{E}_u\left[\mathbb{E}_v\left[\|v-u\|^2 \mid u\right]\right]$

$= \mathbb{E}_u\left[\mathbb{E}_v\left[\|v - \mathbb{E}[v|u]\|^2 | u\right]\right]$

bc unbiased estimator.

This leads to

$$\boxed{\lambda^* = \frac{\mathbb{E}_u\left[\mathbb{V}[v|u]\right]}{\mathbb{E}_{u,v}\left[\|F(v)-u\|^2\right] + \mathbb{E}_u\left[\mathbb{V}[v|u]\right]}}$$

4) The assumptions of prop 3.4 are verified. Thus

$$\mathbb{E}_v\left[\|F(v)-v\|^2\right] = \mathbb{E}_{v,u}\left[\|F(v)-u\|^2\right] + \mathbb{E}_u\left[\mathbb{V}[v|u]\right]$$

and

$$\lambda^* = \frac{\mathbb{E}_u\left[\mathbb{V}[v|u]\right]}{\mathbb{E}_v\left[\|F(v)-v\|^2\right]}.$$

By definition $R_{NLS}(F) = \mathbb{E}_v\left[\|F(v)-v\|^2\right]$.

ie

$$\boxed{\lambda^* = \frac{d\sigma^2}{R_{NLS}(F)}}$$

Exercice 3.2:

Let v be the noisy version of u, and $\hat{u}(v)$ an estimator of u.

We have

$$\begin{aligned}
&\mathbb{E}_v\left[\|\hat{u}(v) - \mathbb{E}_v[\hat{u}(v)|u]\|^2 \,|\, u\right] \\
&= \mathbb{E}_v\left[\|\hat{u}(v) - u - \mathbb{E}_v[\hat{u}(v)|u] + u\|^2 \,|\, u\right] \\
&= \mathbb{E}_v\left[\|\hat{u}(v) - u\|^2 \,|\, u\right] \\
&\qquad + \mathbb{E}_v\Big[\underbrace{\|\mathbb{E}_v[\hat{u}(v)|u] - u\|^2}_{\text{doesn't depend on } v.} \,|\, u\Big] \\
&\qquad - \mathbb{E}_v\left[2\langle \hat{u}(v) - u \,|\, \mathbb{E}_v[\hat{u}(v)|u] - u\rangle \,|\, u\right] \\
&= \mathbb{E}_v\left[\|\hat{u}(v) - u\|^2 \,|\, u\right] \\
&\qquad + \|\mathbb{E}_v[\hat{u}(v)|u] - u\|^2 \\
&\qquad - \underbrace{2\langle \mathbb{E}_v[\hat{u}(v)|u] - u \,|\, \mathbb{E}_v[\hat{u}(v)|u] - u\rangle}_{2\|\mathbb{E}_v[\hat{u}(v)|u] - u\|^2} \\
&= \mathbb{E}_v\left[\|\hat{u}(v) - u\|^2 \,|\, u\right] - \|\mathbb{E}_v[\hat{u}(v)|u] - u\|^2
\end{aligned}$$

In the end:

$$\boxed{\mathbb{E}_v\left[\|\hat{u}(v) - u\|^2 \,|\, u\right] = \|\mathbb{E}_v[\hat{u}(v)|u - u\|^2 + \mathbb{E}_v\left[\|\hat{u}(v) - \mathbb{E}_v[\hat{u}(v)|u]\|^2 \,|\, u\right].}$$