IMDEN

le 12/10

Homework 2:

Exercice 5.1:

let n: dimension of initial image
k: dimension of image at lower scale
u_0: initial image
u_1: lower scale image
DCT_n^{iso}: isometric DCT at scale n
DCT_k^{iso}: isometric DCT at scale k.
ZP_k: the padding operator keep the top left $k \times k$ coef.

To go from u_0 to u_1, we need to:

i) transform u_0 to the frequency space ie

$$u' = DCT_n^{iso}(u_0)$$

ii) pad the resulting image, ie keep the lowest frequency.

$$u'' = ZP_k(u') = ZP_k(DCT_n^{iso}(u_0)).$$

iii) we then adjust the scale of the image. To do so, we multiply the current image by $\frac{k}{n} = \sqrt{\frac{k \times k}{n \times n}} = \sqrt{\frac{Numpix(layer)}{Numpix(input)}}$

We have:

$$u''' = \sqrt{\frac{k \times k}{n \times n}}\, u''$$

iv) The last step is to transform the current image from frequency space to real image space. ie

$$u_1 = DCT_k^{-1\,iso}(u''')$$

$$\boxed{u_1 = \sqrt{\frac{k \times k}{n \times n}}\, IDCT_k^{iso}(ZP_k(DCT_n^{iso}(u_0)))}$$

The final image u_1 contains a white gaussian uniform noise ie

$$u_1 = \tilde{u}_1 + n_1 \quad , n_1 \sim \mathcal{N}(0, \sigma_1^2)$$

and

$$Var(u_1) = \sigma_1^2 = \frac{k \times k}{n \times n} Var\left(IDCT_k^{iso} (ZP_k (DCT_n^{iso}(u_0)) \right)$$

$$= \frac{k \times k}{n \times n} \sigma_0^2$$

ie

$$\sigma(u_1) = \sqrt{\frac{Numpix(layer)}{Numpix(input)}} \sigma_0$$

The standard deviation has been reduced by a factor $\sqrt{\frac{Numpix(layer)}{Numpix(input)}}$.

Exercise 6.1:

Let $(n(i))_{i \in \mathbb{Z}}$ a sequence of white noise with σ^2 variance.

$a(j)$, $j \in [-m; m]$ a filter of size $2m$

$*$ the convolution product.

Let's compute the variance of $a * n(i)$

For a pixel $i \in \mathbb{Z}$,

$$a * n(i) = \sum_j n(i-j) \, a(j)$$

ie $Var[a * n(i)] = \sum_j a^2(j) \, Var[n(i-j)]$

because noises are independent

ie $\boxed{Var[a * n(i)] = \sigma^2 \sum_j a^2(j)}$

We suppose $\sum_j a(j) = 1$. We also have $a(j) \geq 0 \ \forall j \in [-m; m]$

So, for any $j \in [-m; m]$

$a^2(j) \leq a(j)$ ie $\sum_j a^2(j) \leq 1$

we have $\boxed{Var[a * n(i)] \leq Var[n(i)]}$

The convolution helps at reducing the noise by a fixed factor.

Now let's look at the optimal filter weights. We want

$$a = \underset{\substack{a \\ \sum a(i)=1}}{\arg\min} \sum_{i=-m}^{m} a^2(i)$$

This is a continuous coercive convex cost function with linear inequality constraint ie the problem is convex.
Let's apply KKT conditions ie
For the solution $\hat{a}$, there exists $\mu \in \mathbb{R}$ such that for $i \in [-m;m]$

$$2\hat{a}(i) + \mu = 0 \quad \text{and} \quad \sum_{i=-m}^{m} \hat{a}(i) = 1$$

ie $$\hat{a}(i) = -\frac{\mu}{2} \quad \text{and} \quad \sum_i a(i) = 1$$

ie $$\hat{a}(i) = \frac{-\mu}{2} \quad \text{and} \quad -(2m+1)\frac{\mu}{2} = 1$$

ie $$\boxed{\hat{a}(i) = \frac{1}{2m+1}, \quad \forall i \in [-m;m]}$$