IMDEN

le 20/10

Homework 3

Exercise 8.1

Let P the ideal patch
$\tilde{P}$ the observed patch
N the noise patch
$C_{\tilde{P}}$ the covariance matrix of the patches similar to $\tilde{P}$
C_P the covariance matrix of the patches similar to P.

We assume that the patches and noise are independent.
We assume that if $\tilde{P}'$ is a similar patch of $\tilde{P}$ then, P' is a similar patch to P.

For any $\tilde{P}_i, \tilde{P}_j \in \tilde{q}$, the set of similar patch to $\tilde{P}$,

$$\begin{aligned}
Cov(\tilde{P}_i, \tilde{P}_j) &= Cov_{\tilde{P}}(i,j) \\
&= Cov(P_i + N_i, P_j + N_j) \\
&\quad \text{with } \tilde{P} = P + N \\
&= \mathbb{E}[(P_i + N_i - \mathbb{E}[P_i])(P_j + N_j - \mathbb{E}[P_j])] \\
&\quad \text{bc } \mathbb{E}[N] = 0 \\
&= \mathbb{E}[(P_i - \mathbb{E}[P_i])(P_j - \mathbb{E}[P_j])] \\
&\quad + \underbrace{\cancel{\mathbb{E}[(P_i - \mathbb{E}[P_i]) N_j]}}_{=0} \\
&\quad + \underbrace{\cancel{\mathbb{E}[(P_j - \mathbb{E}[P_j]) N_i]}}_{=0} \quad \text{bc noise and patch are indep.} \\
&\quad + \mathbb{E}[N_i N_j] \\
&= Cov(P_i, P_j) + \sigma^2 \delta_{ij}
\end{aligned}$$

In the end,

$$\boxed{C_{\tilde{P}} = C_P + \sigma^2 I}$$

We define $\bar{P}$ as the center of similar patches to P.

$$\mathbb{E}[\tilde{q}] = \sum_{\tilde{P} \in \tilde{q}} \tilde{P}\, \mathbb{P}(\tilde{P}) = \sum_{P \in q} (P+N)\, \mathbb{P}(P+N), \text{ N noise}$$

$$= \sum_{P \in q} P\, \mathbb{P}(P) \quad \text{bc } P, N \text{ indep.}$$

$$= \mathbb{E}[q] = \bar{P}$$

ie $\boxed{\mathbb{E}[\tilde{q}] = \bar{P}}$

Exercice 8.3

We want to compare:

$$\hat{P}_1 = \bar{P} + [C_{\hat{p}} - \sigma^2 I]\, C_{\hat{p}}^{-1} (\hat{P} - \bar{P})$$
$$\hat{P}_2 = \bar{P}' + C_p'\, [C_p' + \sigma^2 I]^{-1} (\hat{P} - \bar{P}')$$

Let G_i the eigen vector of $C_{\hat{p}}$, with eigen value $\mu_i > 0$
ie G_i the eigen vector of $C_{\hat{p}}^{-1}$, with eigen value $\frac{1}{\mu_i}$ (psd)
So

$$\hat{P}_1 = \bar{P} + [C_{\hat{p}} - \sigma^2 I]\, C_{\hat{p}}^{-1} (\hat{P} - \bar{P})$$
$$= \sum_{i=1}^{n} \langle \bar{P} + [C_{\hat{p}} - \sigma^2 I]\, C_{\hat{p}}^{-1} (\hat{P} - \bar{P}), G_i \rangle G_i$$

bc $\{G_i\}$ ortho basis.

$$= \sum_{i=1}^{n} \langle \bar{P}, G_i \rangle G_i + \langle \hat{P} - \bar{P}, ([C_{\hat{p}} - \sigma^2 I]\, C_{\hat{p}}^{-1})^T G_i \rangle G_i$$

$$= \sum_{i=1}^{n} \langle \bar{P}, G_i \rangle G_i + \langle \hat{P} - \bar{P}, C_{\hat{p}}^{-1} [C_{\hat{p}} - \sigma^2 I]\, G_i \rangle G_i$$

bc $C_{\hat{p}}$ and $[C_{\hat{p}} - \sigma^2 I]$ symmetric

$$= \sum_{i=1}^{n} \langle \bar{P}, G_i \rangle G_i + \frac{\mu_i - \sigma^2}{\mu_i} \langle \hat{P} - \bar{P}, G_i \rangle G_i$$

ie $$\boxed{\hat{P}_1 = \sum_{i=1}^{n} \frac{\sigma^2}{\mu_i} \langle \bar{P}, G_i \rangle G_i + \sum_{i=1}^{n} \underbrace{\frac{\mu_i - \sigma^2}{\mu_i}}_{= a(i) = \frac{\mu_i - \sigma^2}{\mu_i}} \langle \hat{P}, G_i \rangle G_i}$$

The second part of the formula is the Wiener filtering!

Let H_i the eigen vector of $C_{p'}$, with eigen value μ_i'
We have

$$\hat{P}_2 = \bar{P}' + C_p' [C_p' + \sigma^2 I]^{-1} (\hat{P} - \bar{P}')$$
$$= \sum_{i=1}^{n} \langle \bar{P}' + C_p' [C_p' + \sigma^2 I]^{-1} (\hat{P} - \bar{P}_1) \mid H_i \rangle H_i$$

bc $\{H_i\}$ ortho basis.

$$= \sum_{i=1}^{n} \langle \bar{P}_1 \mid H_i \rangle + \frac{\mu_i'}{\mu_i' + \sigma^2} \langle \hat{P} - \bar{P}_1 \mid H_i \rangle H_i$$

$$\boxed{\hat{P}_2 = \sum_{i=1}^{n} \frac{\sigma^2}{\mu_i' + \sigma^2} \langle \bar{P}' | H_i \rangle H_i + \sum_{i=1}^{n} \underbrace{\frac{\mu_i'}{\mu_i' + \sigma^2}}_{b(i) = \frac{\mu_i'}{\mu_i' + \sigma^2}} \langle \hat{P} | H_i \rangle H_i}$$

The second part is the Wiener filtering!
Thus, the two steps Bayesian method corresponds to a Wiener and oracular method. However, there is a small difference.
In this case, we denoise the patch $\tilde{P}$, but add a weighted average patch $\frac{\sigma^2}{\mu_i'} \bar{P}$ or $\frac{\sigma^2}{\mu_i' + \sigma^2} \bar{P}'$.
We keep patch $\bar{P}$ coordinates with the smallest μ_i elongation.
This allows not to erase the high frequency pixels.

Exercise 8.4:
Recall Fubini-Tonelli's theorem:

- $P \mapsto \int |\mathbb{P}(\tilde{P})\, \mathbb{P}(P|\tilde{P})\, \|P - \hat{P}\|^2 dP < +\infty$
(because finite number of patch)
- $\tilde{P} \mapsto \int |\mathbb{P}(\tilde{P})\, \mathbb{P}(P|\tilde{P})\, \|P - \hat{P}\|\, d\tilde{P} < +\infty$
(because finite number of patch).

Then $\iint \mathbb{P}(\tilde{P})\, \mathbb{P}(P|\tilde{P})\, \|P - \hat{P}\|^2\, dP\, d\tilde{P}$

$$= \iint \mathbb{P}(\tilde{P})\, \mathbb{P}(P|\tilde{P})\, \|P - \hat{P}\|^2\, d\tilde{P}\, dP.$$

Formula 8.16 gives:

$$MSE = \int \mathbb{P}(P) \int \mathbb{P}(\tilde{P}|P)\, \|P - \hat{P}\|\, d\tilde{P}\, dP.$$

$$= \iint \mathbb{P}(\tilde{P}, P)\, \|P - \hat{P}\|^2\, dP\, d\tilde{P} \quad \text{Fubini Thm}$$

$$= \iint \mathbb{P}(P|\tilde{P}) \times \mathbb{P}(\tilde{P})\, \|P - \hat{P}\|^2\, dP\, d\tilde{P} \quad \text{Bayes formula}$$

$$\boxed{MSE = \int \mathbb{P}(\tilde{P}) \int \mathbb{P}(P|\tilde{P})\, \|P - \hat{P}\|^2\, dP\, d\tilde{P}}$$

Fubini Thm.

Exercice 8.5:

Let

$$\hat{P} = \underset{\hat{P}}{\arg\min}\ MMSE(\hat{P}) = \underset{\hat{P}}{\arg\min} \underbrace{\int \mathbb{P}(P|\tilde{P})(P-\hat{P})^2 dP}_{\text{convex function}}$$

$$\frac{\partial MMSE(\hat{P})}{\partial \hat{P}} = -2 \int \mathbb{P}(P|\tilde{P})(P-\hat{P})\, dP = 0$$

ie $\int \mathbb{P}(P|\tilde{P})\, P\, dP = \int \mathbb{P}(P|\tilde{P})\, \hat{P}\, dP$

$$= \hat{P} \int \mathbb{P}(P|\tilde{P})\, dP$$

$$= \hat{P}\ \frac{\int \mathbb{P}(P,\tilde{P})\, dP}{\mathbb{P}(\tilde{P})}$$

$$= \hat{P}\ \frac{\mathbb{P}(\tilde{P})}{\mathbb{P}(\tilde{P})}$$

So $\hat{P} = \int \mathbb{P}(P|\tilde{P})\, P\, dP$

ie $\boxed{\hat{P} = E[P|\tilde{P}]}$

This is the Bayesian estimator on MSE!