IMDEN

Homework 1 :

Exercice 1 :

Let $X \sim P(\lambda)$. We have

- $\mathbb{E}[X] = \sum_{k=1}^{+\infty} k \, p(X=k)$

$= \sum_{k=1}^{+\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda e^{\lambda}$

ie $\boxed{\mathbb{E}[X] = \lambda}$

- $\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

and $\mathbb{E}[X^2] = \sum_{k=1}^{\infty} \frac{k^2 \lambda^k e^{-\lambda}}{k!}$, and $k^2 = k(k-1) + k$

$= \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!} + \underbrace{\sum_{k=1}^{\infty} \frac{k \lambda^k e^{-\lambda}}{k!}}_{\lambda}$

$= \lambda^2 + \lambda$

so $\boxed{\sigma^2(X) = \mathbb{V}[X] = \lambda}$

Exercice 2 :

Let $i \in \mathbb{N}^*$, $x_i \sim P(\lambda_i)$ independents.

Let $\lambda = \sum_{i=1}^{n} \lambda_i$. Let's show $\sum x_i \sim P(\lambda)$.

Reasoning by recurrence :

- $n = 1$: ok.
- $\forall n > 1$. Let $\alpha = \sum_{i=1}^{n-1} \lambda_i$, ie $\alpha + \lambda_n = \lambda$ and $y = \sum_{i=1}^{n} x_i$

Assume that $\sum_{i=1}^{n-1} x \sim P(\alpha)$.

So for any $k \in \mathbb{N}$,

$P(y = k) = P\left(\sum_{i=1}^{n-1} x_i + x_n = k\right)$

$= P\left(\bigcup_{t=0}^{k} \left\{\sum_{i=1}^{n-1} x_i = t, x_n = k-t\right\}\right)$

$= \sum_{t=0}^{k} P\left(\sum_{i=1}^{n-1} x_i = t, x_n = k-t\right)$ disjoint union

$= \sum_{t=0}^{k} \frac{\alpha^t e^{-\alpha}}{t!} \frac{\lambda_n^{k-t} e^{-\lambda_n}}{(k-t)!}$ by indep.

$= \sum_{t=0}^{k} \frac{\alpha^t \lambda_n^{k-t}}{t!(k-t)!} e^{-\lambda}$

ie $\mathbb{P}(y=k) = \dfrac{e^{-\lambda} \alpha^n}{k!}$

ie $\boxed{\mathbb{P}(y=k) = \dfrac{\exp\left[-\sum_{i=1}^{n} \lambda_i\right] \times \left(\sum_{i=1}^{n} \lambda_i\right)^n}{k!}}$

Exercise 3:

Let u be an ideal image, n noise such that $n \sim \mathcal{N}(0,1)$

Let $\tilde{u}$ be a noisy image ie

$$\tilde{u} = u + g(u)\, n$$

1) Let's apply VST

Let f be a smooth function. Taylor approximation provides

$$f(\tilde{u}) = f(u) + f'(u)\, g(u)\, n$$

and $g(u) = \sqrt{u}$ for $u \sim \mathcal{P}$.

ie $f(\tilde{u}) = f(u) + f'(u)\sqrt{u}\, n$.

In order to have $f(\tilde{u})$ have a uniform standard deviation independent of u, we need

$$f'(u) \sim \frac{1}{\sqrt{u}} \quad \text{ie} \quad f(u) \sim \sqrt{u}$$

This assumes $f(\tilde{u}) = f(u) + \alpha n$

2) We then denoise this image $f(\tilde{u})$ ie we now have $\hat{u}$ so that $\hat{u} = f(u)$.

3) We apply the inverse of VST to $\hat{u}$ ie

$$f^{-1}(\hat{u}) = u.$$

We have retrieved u!

Exercice 5:

Let's prove Thm 4.2.

Let • $\tilde{U}$ be noisy image

• U original image

• $(G_i)_{i \leq M}$ an orthonormal basis in image space of dim M.

• $D\tilde{U} = \sum_{i=1}^{m} a_i \langle \tilde{U} | G_i \rangle G_i$

We want to solve $\operatorname{argmin} \mathbb{E}[\|U - D\tilde{U}\|^2]$ (P)

• This problem is equivalent to solving

$$\underset{a_i}{\operatorname{argmin}} \ \mathbb{E}[\|U - D\tilde{U}\|^2]$$

This is a convex problem, Thus it admits a solution.

• We have

$$\mathbb{E}[\|U - D\tilde{U}\|^2] = \mathbb{E}[\|U\|^2] + \mathbb{E}[\|D\tilde{U}\|^2] - 2\mathbb{E}[\langle U | D\tilde{U} \rangle]$$

a) $\mathbb{E}[\|D\tilde{U}\|^2] = \mathbb{E}\left[\sum_{i=1}^{m} a_i^2 \langle \tilde{U} | G_i \rangle^2\right]$

$$= \sum_{i=1}^{m} a_i^2 \, \mathbb{E}[(\langle U | G_i \rangle + \langle N | G_i \rangle)^2]$$

$$= \sum_{i=1}^{m} a_i^2 (\langle U | G_i \rangle^2 + \sigma^2)$$ bc $\mathbb{E}[N] = 0$, bc U deterministic

b) $\mathbb{E}[\langle U | D\tilde{U} \rangle] = \mathbb{E}\left[\sum_{i=1}^{m} a_i \langle \tilde{U} | G_i \rangle \langle U | G_i \rangle\right]$

$$= \mathbb{E}\left[\sum_{i=1}^{m} a_i \langle U | G_i \rangle^2 + \langle N | G_i \rangle \times \langle U | G_i \rangle\right]$$

$$= \sum_{i=1}^{m} a_i \langle U | G_i \rangle^2$$ bc U deterministic, $\mathbb{E}[N] = 0$.

So $\mathbb{E}[\|U - D\tilde{U}\|^2] = \|U\|^2 + \sum_{i=1}^{m} a_i^2 (\langle U | G_i \rangle^2 + \sigma^2)$

$$- 2 \sum_{i=1}^{m} a_i \langle U | G_i \rangle^2$$

• Finding a minimum D^{opt} is like finding minimum $\hat{a}_i$, $\forall i$

• The problem is convex. We only need $\frac{\partial \mathbb{E}[\|U - D\tilde{U}\|^2]}{\partial a_i}(\hat{a}_i) = 0$

ie $2\hat{a}_i(\langle U|G_i\rangle^2 + \sigma^2 - 2a_i\langle U|G_i\rangle^2 = 0$

ie

$$\boxed{\hat{a}_i = \frac{\langle U|G_i\rangle^2}{\langle U|G_i\rangle^2 + \sigma^2} \quad \forall i \leq n}$$

- We have

$$U = \sum_{i=1}^{n} \langle U|G_i\rangle G_i$$

ie $$U - O^{\text{inf}}\tilde{U} = \sum_{i=1}^{n} \left(\langle U|G_i\rangle - \hat{a}_i \langle \tilde{U}|G_i\rangle\right) G_i$$

ie $$\|U - O^{\text{inf}}\tilde{U}\|^2 = \sum_{i=1}^{n} \left(\langle U|G_i\rangle^2 + \hat{a}_i^{\,2} \langle \tilde{U}|G_i\rangle^2 - 2\hat{a}_i\langle \tilde{U}|G_i\rangle\langle U|G_i\rangle\right)$$

ie $$\mathbb{E}\left[\|U - O^{\text{inf}}\tilde{U}\|^2\right] = \sum_{i=1}^{n} \langle U|G_i\rangle^2 + \hat{a}_i^2(\langle U|G_i\rangle^2 + \sigma^2)$$
$$2\hat{a}_i\langle U|G_i\rangle^2$$

$$= \sum_{i=1}^{n} \langle U|G_i\rangle^2 (1 - \hat{a}_i)^2 + \hat{a}_i^2\sigma^2$$

$$= \sum_{i=1}^{n} \frac{\langle U|G_i\rangle^2 \sigma^2}{\langle U|G_i\rangle^2 + \sigma^2}$$

ie $$\boxed{\mathbb{E}\left[\|U - O^{\text{inf}}\tilde{U}\|^2\right] = \sum_{i=1}^{n} \frac{\langle U|G_i\rangle^2\sigma^2}{\langle U|G_i\rangle^2 + \sigma^2}}$$

Exercice 6 :

let's restrict a_i in $\{0;1\}$

let's show that $MSE \leq \sum \min(\langle U|G_i\rangle^2, c\sigma^2)$ for $c \geq 1$.

Recall:

We previously proved

$$MSE = \sum_{i=1}^{m} \langle U|G_i\rangle^2 (1-a_i)^2 + a_i^2\sigma^2$$

- if $c = 1$, then
  - if $a_i = 1$ ie $\langle U|G_i\rangle \geq \sigma^2$

    we have $\langle U|G_i\rangle^2 (1-a_i)^2 + a_i^2\sigma^2 = \sigma^2$
    $= \min(\langle U|G_i\rangle^2, c\sigma^2)$
  - if $a_i = 0$ ie $\langle U|G_i\rangle < \sigma^2$

    we have $\langle U|G_i\rangle^2 (1-a_i)^2 + a_i^2\sigma^2 = \langle U|G_i\rangle^2$
    $= \min(\langle U|G_i\rangle^2, c\sigma^2)$

  In either case,

  $\langle U|G_i\rangle^2 (1-a_i)^2 + a_i^2\sigma^2 = \min(\langle U|G_i\rangle^2, c\sigma^2)$

  ie $\boxed{MSE = \sum \min(\langle U|G_i\rangle^2, c\sigma^2)}$

- if $c > 1$:
  - if $a_i = 1$, $\langle U|G_i\rangle^2 (1-a_i)^2 + a_i^2\sigma^2 < \min(\langle U|G_i\rangle^2, c\sigma^2$
  - if $a_i = 0$

  In either case:

  $\boxed{MSE \leq \sum \min(\langle U|G_i\rangle^2, c\sigma^2)}$

Exercice 7:

Let $DCT: X \longmapsto Y$, in $\mathbb{R}^N$

such that $Y_k = \alpha_k 2 \sum_{j=0}^{N-1} X_j \cos\left[\pi\left(j+\frac{1}{2}\right)\frac{k}{N}\right]$

Let's show that DCT is an isometry.

We have $DCT: X \longmapsto AY$

with $A_{ij} = 2\alpha_i \cos\left(\pi\left(j+\frac{1}{2}\right)\frac{k}{N}\right)$

We show that $A^TA = I$

$$(A^TA)_{ij} = \sum_{k=0}^{N-1} A_{ki} A_{kj}$$

$$= \sum_{k=0}^{N-1} 4\alpha_k^2 \cos\left[\pi\left(j+\frac{1}{2}\right)\frac{k}{N}\right] \cos\left[\pi\left(i+\frac{1}{2}\right)\frac{k}{N}\right]$$

$$= 2 \times Re\left[\sum_{k=0}^{N-1} \alpha_k \left(e^{i(i+j+1)\pi\frac{k}{N}} + e^{i(i-j)\pi\frac{k}{N}}\right)\right]$$

$$= 2Re\left[\frac{1}{2N} + \sum_{k=1}^{N-1} \frac{1}{2N}\left(e^{i(i+j+1)\pi\frac{k}{N}} + e^{i(i-j)\pi\frac{k}{N}}\right)\right]$$

If $i \neq j$:

$$(A^TA)_{ij} = Re\left[\frac{1}{N} + \frac{1}{N}\left[\frac{e^{i(i+j+1)\frac{\pi}{N}} - e^{i(i+j+1)\pi}}{1-e^{i(i+j+1)\frac{\pi}{N}}} + \frac{e^{i(i-j)\frac{\pi}{N}} - e^{i(i-j)\pi}}{1-e^{i(i-j)\frac{\pi}{N}}}\right]\right]$$

- if $i+j+1$ is impair: $e^{i\pi(i+j+1)} = -1$ $\quad e^{i\pi(i-j)} = 1$.

ie $(A^TA)_{ij} = \frac{1}{N} Re\left[1 + \frac{e^{i(i+j+1)\frac{\pi}{N}} + 1}{1 - e^{i(i+j+1)\frac{\pi}{N}}} - 1\right]$

$$= \frac{1}{N} Re\left[\frac{e^{i(i+j+1)\frac{\pi}{2N}}}{e^{i(i+j+1)\frac{\pi}{2N}}} \frac{\left(e^{i(i+j+1)\frac{\pi}{2N}} + e^{-i(i+j+1)\frac{\pi}{2N}}\right)}{e^{-i(i+j+1)\frac{\pi}{2N}} - e^{i(i+j+1)\frac{\pi}{2N}}}\right]$$

$$= \frac{1}{N} Re\left[\frac{\cos\left[(i+j+1)\frac{\pi}{2N}\right]}{\sin\left[(i+j+1)\frac{\pi}{2N}\right]} \times i\right]$$

ie $(A^TA)_{ij} = 0$.

• if $i+j+1$ is pair: ie $e^{i(i+j+1)\pi} = 1$, $e^{i(i-j)\pi} = -1$

$$\text{ie } (A^T A)_{ij} = \frac{1}{N} \operatorname{Re}\left[\frac{\cos\left[(i-j)\frac{\pi}{2N}\right]}{\sin\left[(i-j)\frac{\pi}{2N}\right]} \times i \right]$$

$$\text{ie } (A^T A)_{ij} = 0$$

If $i=j$:

$$(A^T A)_{ii} = \frac{1}{N} \operatorname{Re}\left[1 + \sum_{k=1}^{N-1} \left(e^{(2i+1)\frac{i\pi k}{N}} + 1 \right) \right]$$

$$= \frac{1}{N} \operatorname{Re}\left[N + \frac{e^{(2i+1)\frac{i\pi}{N}} - e^{(2i+1)i\pi}}{1 - e^{(2i+1)\frac{i\pi}{N}}} \right]$$

$$= 1 + \frac{1}{N} \operatorname{Re}\left[\frac{\cos\left[\frac{\pi}{N}(2i+1)\right]}{\sin\left[\frac{\pi}{N}(2i+1)\right]} \times (-i) \right]$$

$$= 1$$

ie $\boxed{A^T A = Id.}$ ie DCT is an isometry

Let's show that $IDCT : X \longmapsto BY$ is isometric
with $B_{ij} = 2\beta_j \cos\left(\pi\left(i+\frac{1}{2}\right)\frac{j}{N}\right)$

We see that $B = A^T$ and A^T is an isometry.
ie IDCT is an isometry

It also shows that $A^{-1} = A^T = B$
ie IDCT is the inverse transform of DCT

Exercice 8:

Let's consider:

$$(P) \quad \underset{\sum \alpha_k = 1}{\text{argmin}} \quad \sum_k \alpha_k^2 \, \mathbb{E}\left[(P_k - \mathbb{E}[P_k])^2 \right]$$

This problem is convex

- affine constraint $g(\alpha) = \sum \alpha_k - 1 = 0$
- convex cost function, continuous $f(\alpha)$.
- $C = \{ \alpha \in \mathbb{R}^n, \sum \alpha_k = 1 \}$ is convex compact set

So, it exist $\hat{\alpha} \in C$, $f(\hat{\alpha}) = \underset{\sum \alpha_k = 1}{\min} \sum \alpha_k^2 \, \mathbb{E}\left[(P_k - \mathbb{E}[P_k])^2 \right]$

We apply KKT conditions:

it exists $\mu \in \mathbb{R}$ such that:

$$\nabla f(\hat{\alpha}) + \mu \nabla g(\hat{\alpha}) = 0$$

We use the fact that $f(\alpha) = \sum_k \alpha_k^2 \sigma_k^2$, and so:

we have:

$$\boxed{\exists \mu \in \mathbb{R}, \quad 2 \alpha_k \sigma_k^2 + \mu = 0 \quad , \forall k}$$

Exercice 4.9:

For a patch X_k: $\mathbb{V}ar[X_k] = \mathbb{E}\left[(X_k - \mathbb{E}[X_k])^2 \right]$

ie $\sigma_k^2 = \sum \sigma^2 a_k^2$

a_k is obtained through wiener algorithm,

ie $a_k = \rho_{pk}$

(4.21) states $\alpha_k = \dfrac{\sigma_k^{-2}}{\sum \sigma_k^{-2}}$ and we have

$$\sigma_k^2 = \sigma^2 \sum a_k^2 = \sigma^2 \| \rho_{pk} \|^2$$

$$\boxed{\text{ie} \quad \alpha_k = \frac{\sigma_k^{-2}}{\sum \sigma_k^{-2}} = \frac{\| \rho_{pk} \|^2}{\sum_k \| \rho_{pk} \|^2} \quad , \forall k}$$