Examples in Graduate Mathematics

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1 Topology

1.1 Simple Example of Poincare's Torus Bundles

We will consider a simple example of a 3-manifold studied by Poincare; see Stillwell [1] for more information. Consider $\mathbb{R}^3$ under the following equivalencies:

$$\begin{cases} (x,y,z) \sim (x+1,y,z), \\ (x,y,z) \sim (x,y+1,z), \\ (x,y,z) \sim (x+y,y,z+1). \end{cases} \tag{1}$$

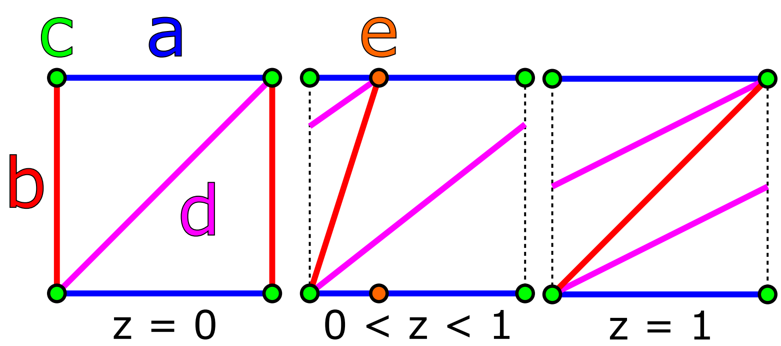
Our manifold $M = \mathbb{R}^3/\sim$, the quotient of $\mathbb{R}^3$ by these equivalencies. A fundamental domain is provided by the unit cube $\{0 \leq x,y,z \leq 1\}$ (ignoring issues of which sides to include if we are absolutely insistent as to uniqueness of representation of equivalence classes).

We see that every horizontal slice $\{z = c\}$ is a 2-dimensional torus. Furthermore, the slice $\{z = 0\}$ and $\{z = 1\}$ is given by a glide transformation where the transformation $T(x,y) = (x+y,x)$.

We will compute the fundamental group $\pi_1(M)$ of this manifold by using two triangulations. The first will be a very improper triangulation, and the second will be much more proper although a little more complex.

An Improper Triangulation

First let's consider an improper triangulation/delta complex on M. Pictured below are slices for this triangulation on the unit cube for $z = 0$, some $0 < z < 1$, and $z = 1$.



Why is this triangulation improper? It is due to the fact that the line segment d cuts across the face represented by blue in each slice. Now, this doesn't mean it is invalid. It is just not a proper delta-complex; it is more properly described as a CW-complex.

Let's compute $\pi_1(M)$ using this triangulation. First, we find the generators of the fundamental group that come from this triangulation. We need to find a minimal spanning tree inside the 1-complex of our triangulation. Let us root the tree at the vertex that is the corner of $\{z =\}$ of the fundamental domain; recall that all four corners are equivalent.

Next, we see that all of the 1-cells are attached to the root of the tree. Furthermore, every one cell takes the root back to itself. Therefore, our minimal spanning tree just conists of a single vertex, the root of the tree. Therefore, every 1-cell is a generator for $\pi_1(M)$. So we know that $\pi_1(M)$ is generated by a, b, c, d, and e.

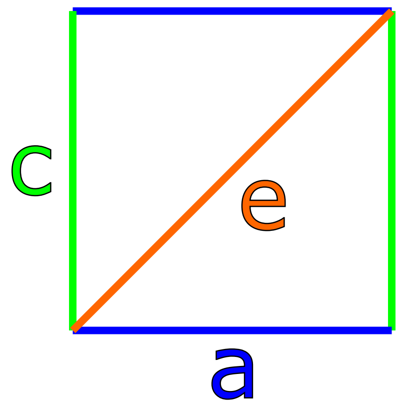
Now let us find the relations between these generators that come from the faces (i.e. 2-cells) of our triangulation.

The horizontal 2-cells in the slice $\{z = 0\}$ give

$$d \sim ab \sim ba. \tag{2}$$

This is expected as every horizontal slice is just a 2-dimensional torus.

Next, let us investigate the relations due to the vertical faces. So we look at each color in the $\{0 < z < 1\}$ to see where the faces are. Note, that the blue color is a little misleading as e cuts across this face. So let us look at this face explictly.



The blue face gives us

$$e \sim ac \sim ca. \tag{3}$$

Note that we also have that a commutes with c.

The red vertical face gives

$$beb^{-1}c^{-1} \sim 1. \tag{4}$$

This allows us to solve for e in terms of b and c,

$$e \sim b^{-1}cb. \tag{5}$$

Finally, the fuschia vertical face gives

$$ded^{-1}c^{-1} \sim 1. \tag{6}$$

Given that $d \sim ab$, $e \sim b^{-1}cb$, and a commutes with both b and c, we get

$$\begin{aligned} 1 &\sim abb^{-1}cbb^{-1}a^{-1}c^{-1}, && (7)\\ &\sim aca^{-1}c^{-1}, && (8)\\ &\sim 1. && (9) \end{aligned}$$

Therefore, the fuschia face hasn't given us anything new.

Our relations are thus

$$\begin{cases} d \sim ab \sim ba, \\ e \sim b^{-1}cb, \\ e \sim ac \sim ca. \end{cases} \tag{10}$$

So we see that the true generators are a, b, and c. Thus we see that our relations may be described more succinclty as

$$\begin{cases} ab \sim ba, \\ ac \sim ca, \\ cb \sim abc. \end{cases} \tag{11}$$

The last relation tells us how to deal with changing the order of b and c; in the process, we pick up an a.

So let us see what happens when we take the product of powers of a, b, and c. We have

$$a^{p_1} b^{q_1} c^{r_1} a^{p_2} b^{q_2} c^{r_2} \sim a^{p_1+p_2} b^{q_1} c^{r_1} b^{q_2} c^{r_2}, \tag{12}$$

$$\sim a^{p_1+p_2+r_1 q_2} b^{q_1+q_2} c^{r_1+r_2}. \tag{13}$$

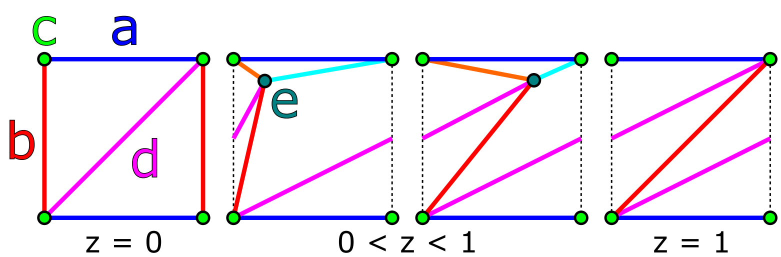
So we see that we can represent $\pi_1(M)$ by a semi-direct product

$$\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}, \tag{14}$$

where $\phi_r(p,q) = (p + rq, q) = T^r(p,q)$, $a = (1,0,0)$, $b = (0,1,0)$, and $c = (0,0,1)$.

A More Proper Triangulation

Next, let us remark on a more proper triangulation that would also let us compute $\pi_1(M)$.



Now, instead of having a 1-cell e running across a face, we separate it by a region that resembles a triangular cylinder with pinched top and bottom. Using this triangulation will give us the same result for $\pi_1(M)$.

References

[1] John Stillwell. Poincare and the early history of 3-manifolds. *Bulletin of the American Mathematical Society*, 2012.