Examples in Graduate Mathematics

Matthew McGonagle

September 3, 2018

1 Topology

1.1 Simple Example of Poincare's Torus Bundles

We will consider a simple example of a 3-manifold studied by Poincare; see Stillwell [1] for more information. Consider $\mathbb{R}^3$ under the following equivalencies:

$$\begin{cases} (x, y, z) \sim (x+1, y, z), \\ (x, y, z) \sim (x, y+1, z), \\ (x, y, z) \sim (x+y, y, z+1). \end{cases} \tag{1}$$

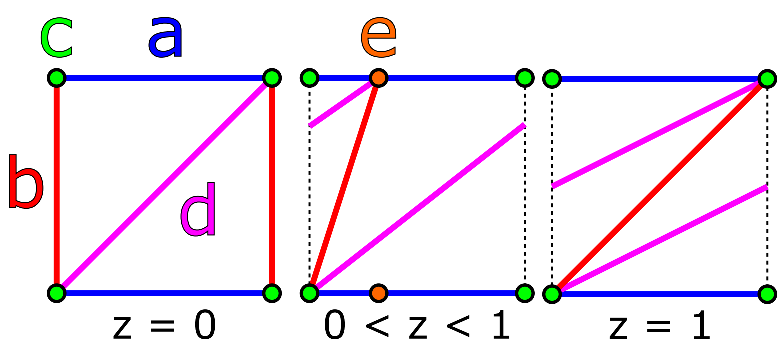
Our manifold $M = \mathbb{R}^3 / \sim$, the quotient of $\mathbb{R}^3$ by these equivalencies. A fundamental domain is provided by the unit cube $\{0 \leq x, y, z \leq 1\}$ (ignoring issues of which sides to include if we are absolutely insistent as to uniqueness of representation of equivalence classes).

We see that every horizontal slice $\{z = c\}$ is a 2-dimensional torus. Furthermore, the slice $\{z = 0\}$ and $\{z = 1\}$ is given by a glide transformation where the transformation $T(x, y) = (x + y, x)$.

We will compute the fundamental group $\pi_1(M)$ of this manifold by using two triangulations. The first will be a very improper triangulation, and the second will be much more proper although a little more complex.

An Improper Triangulation

First let's consider an improper triangulation/delta complex on M. Pictured below are slices for this triangulation on the unit cube for $z = 0$, some $0 < z < 1$, and $z = 1$.



Why is this triangulation improper? It is due to the fact that the line segment d cuts across the face represented by blue in each slice. Now, this doesn't mean it is invalid. It is just not a proper delta-complex; it is more properly described as a CW-complex.

Let's compute $\pi_1(M)$ using this triangulation. First, we find the generators of the fundamental group that come from this triangulation. We need to find a minimal spanning tree inside the 1-complex of our triangulation. Let us root the tree at the vertex that is the corner of $\{z =\}$ of the fundamental domain; recall that all four corners are equivalent.

Next, we see that all of the 1-cells are attached to the root of the tree. Furthermore, every one cell takes the root back to itself. Therefore, our minimal spanning tree just conists of a single vertex, the root of the tree. Therefore, every 1-cell is a generator for $\pi_1(M)$. So we know that $\pi_1(M)$ is generated by a, b, c, d, and e.

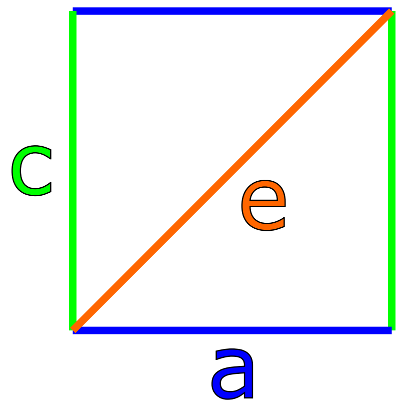
Now let us find the relations between these generators that come from the faces (i.e. 2-cells) of our triangulation.

The horizontal 2-cells in the slice $\{z = 0\}$ give

$$d \sim ab \sim ba. \tag{2}$$

This is expected as every horizontal slice is just a 2-dimensional torus.

Next, let us investigate the relations due to the vertical faces. So we look at each color in the $\{0 < z < 1\}$ to see where the faces are. Note, that the blue color is a little misleading as e cuts across this face. So let us look at this face explictly.



The blue face gives us

$$e \sim ac \sim ca. \tag{3}$$

Note that we also have that a commutes with c.

The red vertical face gives

$$beb^{-1}c^{-1} \sim 1. \tag{4}$$

This allows us to solve for e in terms of b and c,

$$e \sim b^{-1}cb. \tag{5}$$

Finally, the fuschia vertical face gives

$$ded^{-1}c^{-1} \sim 1. \tag{6}$$

Given that $d \sim ab$, $e \sim b^{-1}cb$, and a commutes with both b and c, we get

$$\begin{aligned} 1 &\sim abb^{-1}cbb^{-1}a^{-1}c^{-1}, && (7)\\ &\sim aca^{-1}c^{-1}, && (8)\\ &\sim 1. && (9) \end{aligned}$$

Therefore, the fuschia face hasn't given us anything new.

Our relations are thus

$$\begin{cases} d \sim ab \sim ba, \\ e \sim b^{-1}cb, \\ e \sim ac \sim ca. \end{cases} \tag{10}$$

So we see that the true generators are a, b, and c. Thus we see that our relations may be described more succinclty as

$$\begin{cases} ab \sim ba, \\ ac \sim ca, \\ cb \sim abc. \end{cases} \tag{11}$$

The last relation tells us how to deal with changing the order of b and c; in the process, we pick up an a.

So let us see what happens when we take the product of powers of a, b, and c. We have

$$a^{p_1} b^{q_1} c^{r_1} a^{p_2} b^{q_2} c^{r_2} \sim a^{p_1+p_2} b^{q_1} c^{r_1} b^{q_2} c^{r_2}, \tag{12}$$

$$\sim a^{p_1+p_2+r_1 q_2} b^{q_1+q_2} c^{r_1+r_2}. \tag{13}$$

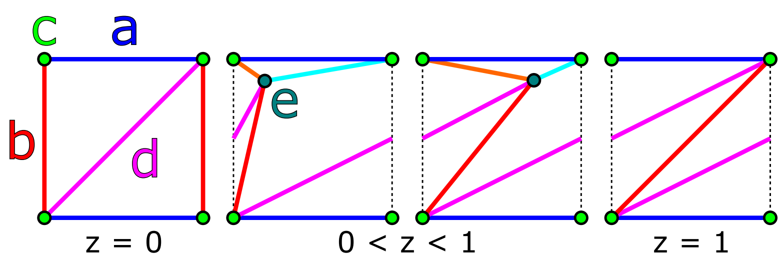
So we see that we can represent $\pi_1(M)$ by a semi-direct product

$$\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}, \tag{14}$$

where $\phi_r(p,q) = (p+rq, q) = T^r(p,q)$, $a = (1,0,0)$, $b = (0,1,0)$, and $c = (0,0,1)$.

A More Proper Triangulation

Next, let us remark on a more proper triangulation that would also let us compute $\pi_1(M)$.



Now, instead of having a 1-cell e running across a face, we separate it by a region that resembles a triangular cylinder with pinched top and bottom. Using this triangulation will give us the same result for $\pi_1(M)$.

1.2 Fundamental Group of a Quotient

The Setup

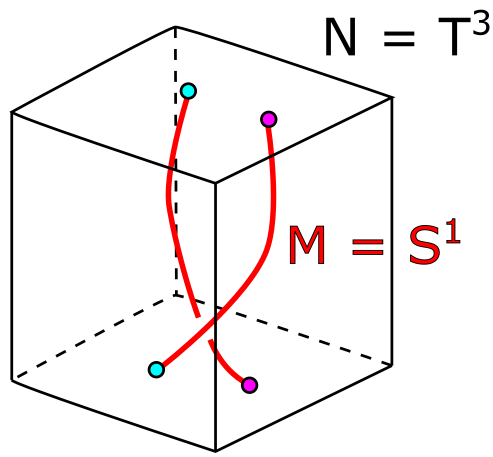
Here we give an example of a manifold N and a sub-manifold M embedded in N such that the fundamental group of the quotient, $\pi_1(N/M)$ is NOT homomorphic to $\pi_1(N)/\pi_1(M)$. In particular, it matters how M is embedded in N.

For our example, $N = T^3$, the 3-torus which can be viewed as the fundamental cube $\{0 \le x, y, z \le 1\}$ under the identifications

$$\begin{cases} (x,y,z) \equiv (x+1,y,z), \\ (x,y,z) \equiv (x,y+1,z), \\ (x,y,z) \equiv (x,y,z+1). \end{cases} \tag{15}$$

The effect is to identify opposite faces.

We will embed $M = S^1$ in a nontrivial way; we let M wrap around twice in the z-direction without crossing itself. See the figure below.

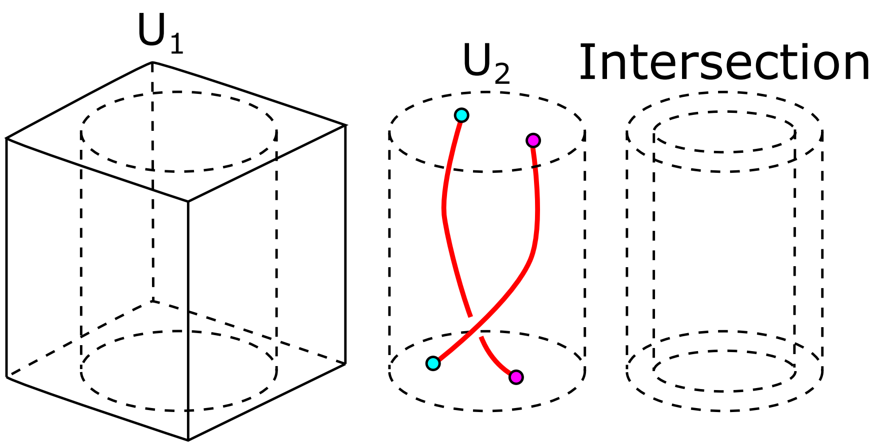


The Problem

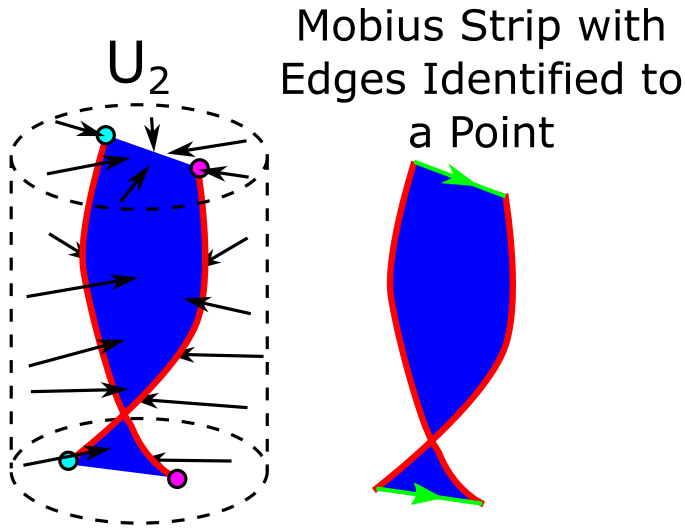
Compute the fundamental group of the quotient N/M as described above.

The Solution

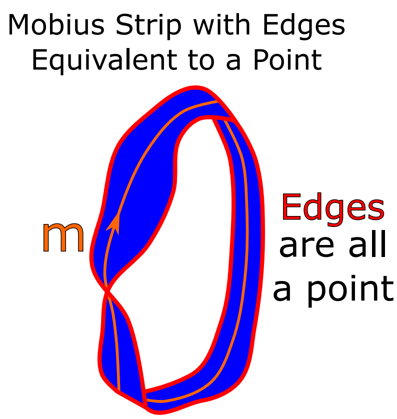
To compute $\pi_1(N/M)$, we will use the Seifert-van Kampen theorem. We cover N by two open sets U_1 U_2. The first open set U_1 consists of everything in N outside a closed torus enclosing M. The open set U_2 is simply an open torus containing M. Their intersection is simply an open solid annular torus. See the figure below.



Let us first consider how to compute fundamental group for the neighborhood containing M, $\pi_1(U_2)$. We see that U_2 deformation retracts onto a mobius strip whose edges are M; for the quotient, this is a mobius strip with its edges identified as one single point, which we will call P. See the figure below.



We need to compute $\pi_1(P)$. For this, we can again apply Seifer-van Kampen. First, consider the meridian curve m that runs along the waist of the mobius strip, see the figure below.



Let V_1 be a neighborhood of the meridian curve m. Then let V_2 be a neighborhood of the edges such that V_1 and V_2 cover P. Note that the twist in the Mobius strip gives us that $V_1 \cap V_2$ is connected.

Now, V_2 deformation retracts to the edges of the Mobius strip, which for P is just a single point. Therefore, V_2 is simply connected, i.e. $\pi_1(V_2) = \{1\}$. Next, we note that V_1 deformation retracts to the meridian m, which is an S^1.

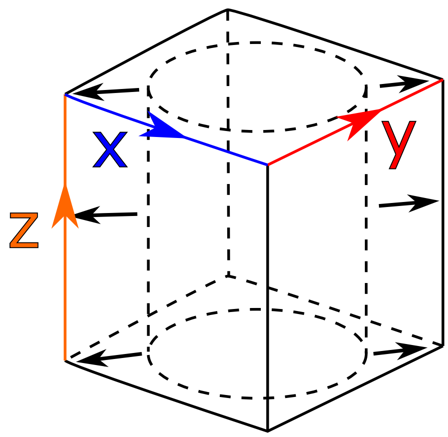
Therefore, $\pi_1(V_1)$ is just the free group generated by m, i.e. $\pi_1(V_1) = (m)$..

Next, observe that $V_1 \cap V_2$ is in fact a cylindrical strip; therefore $\pi_1(V_1 \cap V_2)$ is generated by the loop going around it. Now we see that such a loop covers m twice; so being a little loose with notation, $\pi_1(V_1 \cap V_2) = (m^2)$.

Therefore, by Seifert-van Kampen, we have that $\pi_1(U_2) = \pi_1(P) = (m)/(m^2) = \mathbb{Z}_2$.

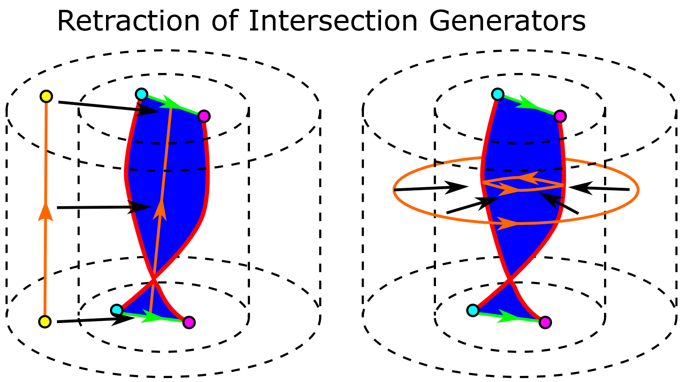
Next, consider U_1. We see that U_1 deformation retracts onto $(S^1 \vee S^1) \times S^1$. We denote the generators by x, y and z; each denotes its usual direction. See the figure below. So $\pi_1(U_1) = ((x) \star (y)) \times (z)$.

Deformation Retract of U_1



Finally, we need to consider $U_1 \cap U_2$. It is pretty clear that $\pi_1(U_1 \cap U_2)$ is generated by a horizontal counter-clockwise circle and a vertical upwards loop. Also, it is clear that the horizontal circle is equivalent to $xyx^{-1}y^{-1}$ inside U_1, and the upward loop is equivalent to z inside U_1. We need to consider their equivalents inside U_2.

It isn't hard to see that the vertical loop retracts onto the meridian m; however, note that it doesn't retract onto the edges. The horizontal circle retracts onto a loop that double backs on itself, and so up to homotopy is trivial. See the figure below.



So we see that the normal group generated by $i_\star(U_1)i_\star(U_2)^{-1}$ is generated by $xyx^{-1}y^{-1}$ and zm^{-1}. By Seifer-van Kampen, we have

$$\pi_1(N/M) = \frac{[((x) \star (y)) \times (z)] \star [(m)/(m^2)]}{(xyx^{-1}y^{-1}, zm^{-1})}, \tag{16}$$

$$= \frac{(x) \times (y) \times (z)}{(z^2)}, \tag{17}$$

$$= \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2. \tag{18}$$

References

[1] John Stillwell. Poincare and the early history of 3-manifolds. *Bulletin of the American Mathematical Society*, 2012.