Quantum Game Theory

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Game Theory
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Abstract

1. From Classical to Quantum Game Theory

Game theory is the study of mathematical models of strategic interaction among rational decision-makers. Such decision-makers are players that have to follow fixed rules and have interests in all the possible outcomes, which do not depend only on the choices they individually make but also from the decisions took by the other contenders. At the very beginning, this discipline was mainly focused on the contest of economic theory, until its first formalization provided by Von Neumann and Morgenstern. Subsequently much work was done in this field, transforming it in a mature discipline, that is now used in several areas such as social sciences, political sciences, biology and engineering.
However, in the middle of 90's, physicists started applying the rules of quantum mechanics to classical information theory, giving way to birth of a new discipline that nowadays is known as quantum information theory. Some of them, moreover, started thinking also about a possible recast of classical game theory using quantum probability amplitudes, and hence studying the effect of quantum superposition, interference and entanglement on the agents' optimal strategies.
There are several reasons for which quantizing games may be interesting:

- classical game theory is a well defined discipline in applied mathematics with applications in economy, psychology, biology,... and is based on probability to a large extend, and so there is a fundamental interest in generalizing it to the domain of quantum probabilities;
- a lot of games require a "nature choice" or even a "nature player", but under this new perspective, such nature is ruled by quantum mechanics;
- if the "selfish gene" theory proposed by Dawkins to justify the selfishness behavior of individuals, even when they live in groups, is reality, than someone can speculate about it bringing adversarial games back to a molecular level in which quantum mechanics dictate the rules [4];
- there is an intimate connection between game theory and quantum communication: whenever a player passes his decision to another one, he is basically transferring information, and so in a quantum world it is legitimate to start thinking about quantum information transferred.

Moreover, it will be shown that in most of the cases, a quantum description of a system provides advantages over the classical situation.
The paper starts analyzing zero-sum games, considering just two adversarial players, and will proceed reformulating several "canonical" problems under this new perspective, showing also that the extension of game theory to the quantum world will be quite "natural" and not even so complex.

2. Spin-Flip Game

In this section it will be provided an example of a game in which a "quantum player" can, in some situations, achieve better results than a classical one [5].
Let's start considering a modern version of a classic game theory problem, the penny-flip game, in which, instead of a coin, players use spin-1/2 particles (e.g. electrons) and distinguish between two possible outcomes depending on the spin state of such system, that can be either up or down. The game starts taking the particle initially set in a spin-up state, $|\uparrow\rangle$, then the players, Alice (A) and Bob (B), alternatively (A first, then B, then again A), can decide to flip the spin or do nothing.
Always remember that this is no more a classical but a quantum system, and so its actual state at any time will not be just "only spin-up" or "only spin-down", but a quantum superposition of the two. This means that, in the end, if the particle is in a generic state $a|\uparrow\rangle + b|\downarrow\rangle$, with $|a|^2 + |b|^2 = 1$, $a, b \in \mathbb{C}$, a (quantum projective) measure on it will return just "spin-up" or "spin-down" with probabilities $|a|^2$ and $|b|^2$ respectively.
A wins if, at the end, the resulting spin is up, while B wins if it is down. This is a two-players zero sum static game of incomplete information, meaning that none of the contenders will know which strategy his opponent will choose, and the final payoffs will be equal and opposite. The problem can be analyzed using a proper bi-matrix of payoffs, in which the columns represent B's strategy, that is just a single ac-

tion selected from the available set $\{F, N\}$ where F stands for "flip" and N for "no flip", while the rows represent A's strategies, that are instead pairs of actions (because A has two turns).

$$
\text{Player A} \quad
\begin{array}{c|cc}
 & \multicolumn{2}{c}{\text{Player B}} \\
 & N & F \\
\hline
NN & 1,-1 & -1,1 \\
NF & -1,1 & 1,-1 \\
FN & -1,1 & 1,-1 \\
FF & 1,-1 & -1,1
\end{array}
$$

The entries of the matrix are, as the name suggests, the payoffs for players A and B respectively, at the end of the game.

For a better understanding of the problem, it can be useful to construct a manageable mathematical framework: a quantum system made up only by one spin-1/2 particle is defined in an Hilbert space $\mathcal{H}$ of dimension 2, and the most natural basis that we can construct for such space is made just by the two spin's eigenstates $\{|\uparrow\rangle, |\downarrow\rangle\}$. Using the $\mathbb{C}^2$ representation:

$$
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Pure strategies, instead, has to be defined as unitary operators over $\mathcal{H}$, i.e. 2x2 matrices in $SU(2)$, acting by left multiplication on the vectors representing the state of the spin. In particular, the strategies F and N described before, can be represented as:

$$
\hat{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{N} = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

A generic *mixed strategy*, so a linear combination of F and N with probabilities p and $1-p$ respectively, can be constructed consequently:

$$
\hat{m} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}
$$

As last clarification, it may be useful to introduce the representation of quantum states via density matrices: in this case the initial state of the problem considered can be written as

$$
|\psi_0\rangle = |\uparrow\rangle \longrightarrow \rho_0 = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

In order to exploit this representation, pure strategies U have now to be applied as contractions over density matrices, while mixed strategies are, as always, just convex combination of them:

$$
\rho_{pure} = U\rho_0 U^\dagger \qquad \rho_{mixed} = pU_1\rho U_1^\dagger + (1-p)U_2\rho U_2^\dagger
$$

Back to the original problem, suppose that only Alice has studied quantum mechanics and can apply these new quantum strategies, while Bob didn't and so is limited to classical mixtures. First move is of player A, that decides to apply the following operator to the spin in the initial state

$$
U_1 \equiv U(a,b) = \begin{pmatrix} a & b \\ \bar{b} & -\bar{a} \end{pmatrix} \qquad \rho_1 = U_1\rho_0 U_1^\dagger = \begin{pmatrix} a\bar{a} & ab \\ \bar{a}\bar{b} & b\bar{b} \end{pmatrix}
$$

Next is player B's turn, that selects a generic mixed strategy:

$$
\begin{aligned}
\rho_2 &= p\hat{F}\rho_1\hat{F}^\dagger + (1-p)\hat{N}\rho_1\hat{N}^\dagger = \\
&= \begin{pmatrix} pb\bar{b} + (1-p)a\bar{a} & p\bar{a}\bar{b} + (1-p)ab \\ pab + (1-p)\bar{a}\bar{b} & pa\bar{a} + (1-p)b\bar{b} \end{pmatrix}
\end{aligned}
$$

Given the density matrix of a final state, the expected payoffs of that configuration can be computed considering the probability that the outcome of a projective measure in such state returns $|\uparrow\rangle$ or $|\downarrow\rangle$. Since A has bet on "spin-up" and B on "spin-down" the final utility values will be:

$$
payoff_A = p_\uparrow \cdot (+1) + p_\downarrow \cdot (-1) = -payoff_B
$$

where $p_\uparrow$ and $p_\downarrow$ are given by the diagonal entries of ρ (formally they can be computed as expectation values: $p_\uparrow = \langle\uparrow|\,\rho\,|\uparrow\rangle$, $p_\downarrow = \langle\downarrow|\,\rho\,|\downarrow\rangle$).

Considering just this restricted two-moves game, its Nash equilibrium can be computed by reasoning on the quantities in the last density matrix ρ_2:

- setting $p = 1/2$, both diagonal terms of ρ_2 will be equal to $1/2$, meaning that Bob's strategy would lead him an expected payoff of 0, independently of Alice's actions;
- if A were to employ any strategy for which $a\bar{a} \neq b\bar{b}$, B could instead obtain an expected payoff of $|a\bar{a} - b\bar{b}| > 0$ just by setting $p = 0, 1$ to whether $b\bar{b} > a\bar{a}$, or the reverse;
- similarly, if Bob were to choose a mixed strategy with $p \neq 1/2$, Alice could maximize her payoff up to $|2p-1|$ by setting $a = 1$ when $p < 1/2$ or $b = 1$ in the other case.

Thus, the mixed/quantum equilibria for this restricted game are pairs

$$
NE = \left\{ U(a,b),\ \left(\frac{1}{2}\hat{F} + \frac{1}{2}\hat{N}\right) \right\} \qquad \text{with } a\bar{a} = \frac{1}{2} = b\bar{b}
$$

and the outcome is the same as if both players utilize optimal mixed strategies $(1/2, 1/2)$.

But Alice has another move at her disposal (U_3), which again transforms the state of the electron by conjugation as $\rho_3 = U_3\rho_2 U_3^\dagger$. If a particular operation is considered (in quantum information this is the *Hadamard Gate*)

$$
U_1 = U\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
$$

the first application puts the electron in a quantum superposition of both up and down states, $|\psi_1\rangle = 1/\sqrt{2}\,(|\uparrow\rangle + |\downarrow\rangle)$, which is therefore invariant under any mixed strategy selected by Bob during his turn; the fact is that Alice's last move inverts her first action projecting the system into the state $\rho_3 = |\uparrow\rangle\langle\uparrow|$, and so a measure over the electron will return, with probability 1, "spin-up" at the end of the game. Since A can do no better than winning with probability 1, this is an optimal quantum strategy for her, and all the pairs

$$
\left\{ p\hat{F} + (1-p)\hat{N},\ U\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ U\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}
$$

are mixed/quantum equilibria of the game, with final payoffs of $(1, -1)$ for the two players.
The main purpose for this game being illustrated, is to teach that quantum mechanics can, in some scenarios, offer profitable strategies over the classical ones.

3. Formalization of a Quantum Game

Basically, any quantum system that can be manipulated by at least one party and where the utility of the moves can be reasonably defined, quantified and ordered, may be conceived as a quantum game.
The simpler way to fully specify a game in classical game theory is using its *normal form* Γ, i.e. listing players, strategies and payoffs. For a quantum game, a couple of additional terms have to be given, then the extension is straightforward:

$$\Gamma = \{\mathcal{H}, \rho, (M_i)_{i=1,..,N}, (S_i)_{i \in M}, (u_i)_{i \in M}\}$$

where $\mathcal{H}$ is the Hilbert space of the physical system, $\rho \in \mathcal{S}(\mathcal{H})$ is the initial state ($\mathcal{S}(\mathcal{H})$ is the associated state space), M_i is the set of N players while S_i and u_i are the available quantum strategies and payoffs associated to each of that players.
A generic *quantum strategy* $s_A \in S_A$ is a collection of admissible quantum operations, that are maps from $\mathcal{S}(\mathcal{H})$ onto itself and usually are supposed to be positive-defined and trace preserving operators. In many two-person problems, like the previous example $\mathcal{S}(\mathcal{H}) = SU(2)$.
As anticipated in the first section, the extension of game theory to the quantum world is quite "natural", since the only thing to do is to re-adapt some definitions. In particular, the concepts of *dominant strategy*, *Nash equilibrium* and *Pareto optimality* are formally identical but extended to include also quantum strategies.
A full description of all the results found in these last 20 years in the field of quantum game theory would be very huge, and would be quite off topic respect to the initial purposes of this paper. For this reason, a deeper analysis will be reserved to a particular, and simple, class of games, that involve two players, each of them with a finite set of strategies, playing a static game of complete information. These are basically the foundations of classical game theory, since problems like these can be generalized and applied to much more complex tasks.
Stick to the example seen before, there is another result coming from classical game theory, that is worth to recall: according to Von Neumann, not every two-person zero-sum finite game has an equilibrium in the set of pure strategies, but there is always an equilibrium at which each player follows a mixed strategy. By definition any other strategy deviating from the equilibrium would lead to a smaller expected payoff, and so it should never be chosen by a rational player. But this is not what shown instead in the quantum spin-flip game. The answer to this problem was found by A. Meyer, one of the initiators of quantum game theory, that was able to derive three interesting theorems (the proof can be found in the original article [3]).

Theorem 1. *There is always a mixed/quantum equilibrium for a two-person zero-sum game, at which the expected payoff for the player utilizing a quantum strategy is at least as great as his expected payoff with an optimal mixed strategy*

Of course, the more interesting question is for which games there is a quantum strategy which improves upon the optimal mixed strategy, and another one is what happens if both players utilize quantum strategies.

Theorem 2. *A two-person zero-sum game need not have a quantum/quantum equilibrium.*

Here it is possible to recognize the equivalent classical result, in which several games (like odd/even or rock/paper/scissor for example) lack of pure strategy Nash equilibria. But, at the same time, this result suggests to look for the analogue of Von Neumann's theorem about mixed strategies.

Theorem 3. *A two-person zero-sum game always has a mixed quantum/mixed quantum equilibrium.*

3.1. Quantum 2 x 2 Games

In traditional 2x2 games, usually there are two contenders that can choose a single move in a given finite set of available strategies. In order to construct a physical quantum model for these kind of problems, it is not enough to construct *superpositions* of actions just as linear combinations of them (like for mixed strategies), but a further step is needed: it is necessary to produce an *entanglement* between various moves. Entanglement is a peculiar quantum mechanics' concept without classical equivalent: if the game that is being studied shows this phenomenon, than the two states composing the system (one for each player) cannot be anymore defined independently one of the other, and any action/modification/measure on the first will inevitably have repercussions also on the second.
Always keep in mind that a classical game has to be a subset of its quantum version, and so anyone should be able to re-derive it as a particular case.
The quantum formulation proceeds by assigning the possible outcomes of the classical strategies D and C (usually *defect* and *cooperate*) represented by two basis vectors $|D\rangle$ and $|C\rangle$ in the Hilbert space of the two-state system (i.e. qubits). At each instance, the state of the game is described by a vector in the tensor product space $\mathcal{H} \otimes \mathcal{H}$, which is spanned by the classical game basis $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, $|DD\rangle$, where the first and second entry refer to Alice's and Bob's qubit, respectively. A generic game can be represented by a simple quantum circuit (1, [1]):

- In the first part of the system, the game is built taking two classical strategies (represented as separated qubits) to which a gate $\hat{J}$ is applied: this operator generates the entanglement between the two qubits, effectively introducing non-classical effects in the game (without using it, the problem would be equivalent to the classical one, when solved using mixed strategies); the game's initial state vector is now $|\psi_0\rangle = \hat{J}\,|CC\rangle$ (the initial choice of $|C\rangle$ or $|D\rangle$ is meaningless, as long as the other operations are "tuned" correctly).

- $\hat{U}_A$ an $\hat{U}_B$ are the quantum strategies effectively selected by Alice and Bob: as anticipated in the spin-flip example, these have to be represented by unitary trace-preserving operators, most of the time in $\mathcal{H} = SU(2)$. In a more general treatment, each player should be allowed to use any local operation that quantum mechanics provide, that is applicable to the problem; the fact is that an eventual analysis of all the possible cases would probably fill up an entire book and would not be even necessary, since the solutions found in $SU(2)$ are usually better (and simpler) than all the others.
- The last part of the circuit consists in a disentangling gate $\tilde{J}$, that finally separates the players' states, allowing a measurement (projective measure in the original basis $\{|C\rangle, |D\rangle\}$). Then it's just about consulting the payoff's table.
- Just a couple of remarks about $\hat{J}$ and $\tilde{J}$:
  - $\hat{J}$ is a unitary operator, symmetric with respect to the interchange of the two players, and known by both of them and so it is common knowledge;
  - since the classical game was required to be a subset of the quantum one, necessarily $\hat{J}$ commutes with the direct product of any pair of classical moves:
    $$\left[\hat{J}, \hat{C}\otimes\hat{C}\right] = \left[\hat{J}, \hat{D}\otimes\hat{D}\right] = \left[\hat{J}, \hat{C}\otimes\hat{D}\right] = 0$$
    Moreover: $\tilde{J} = \hat{J}^\dagger$.

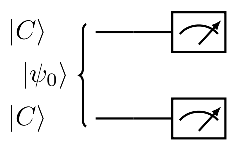

Figure 1: Quantum Circuit of a generic 2x2 game

The final part of the system consists, as said, by a couple of two-channels detectors, each of them labeled by $\sigma = C, D$, representing the outcome of the measure. The final state of the game prior to detection is given by

$$|\psi_f\rangle = \hat{J}^\dagger \left(\hat{U}_A \otimes \hat{U}_B\right) \hat{J}\,|CC\rangle$$

Then, given a certain joint strategy selected by Alice and Bob, the final expected payoff of each of them can be computed with

$$\langle i_\$ \rangle = \sum_{\sigma\sigma'} P_{\sigma\sigma'} |\langle\psi_f|\sigma\sigma'\rangle|^2$$

where $\sigma\sigma'$ represents the possible outcomes of the detector (CC, CD, DC and DD), and $P_{\sigma\sigma'}$ is the corresponding entry of player i in the payoff matrix.
If the actions' space is restricted to $SU(2)$, then each quantum strategy can be written as function of 2 parameters:

$$\hat{U}(\theta,\phi) = \begin{pmatrix} e^{i\phi}\cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & e^{-i\phi}\cos\theta/2 \end{pmatrix} \qquad \theta\in[0,\pi], \phi\in[0,\frac{\pi}{2}]$$

And also classical strategies have this kind of representation in which

$$\hat{C} = \hat{U}(0,0), \qquad \hat{D} = \hat{U}(\pi,0)$$

while $\hat{U}(\theta,0)$ is the set including all classical mixtures.
To conclude this section, exploiting quantum information together with abelian group theory, an explicit form of the entangling operator $\hat{J}$ is provided, and the nice thing is that this form is a function of a parameter γ that can be seen a sort of "measure of entanglement":

$$\hat{J} = \exp\left(-i\gamma\hat{D}\otimes\hat{D}/2\right) \qquad \gamma\in[0,\frac{\pi}{2}]$$

For a separable game $\gamma = 0$, and the joint probabilities $P_{\sigma\sigma'}$ factorize for all possible pairs of strategies. A different situation holds for $\gamma = \pi/2$, that corresponds to states that are maximally entangled, and so the game played is as far as possible from its classical counterpart.

4. Quantum Prisoners' Dilemma

In this section will be analyzed and simulated a canonical non-zero sum game usually known as the prisoners' dilemma. In these kind of games, in contrast to zero-sum games, the two players no longer appear in strict opposition to each other, but may rather benefit from mutual cooperation.
Note: all the plots shown in this section have been realized in Python: the original code is contained in a Jupyter Notebook that can be found on Github [6]. It may be useful to give a look there if interested in a computational implementation of this problem and/or a deeper overview of it.
There are two players, Alice and Bob, that are thieves caught by the police, which are being questioned, and can choose among two strategies, Mum (collaborating, C) or Fink (defecting, D). The payoffs are distributed according table 1 and, as usually, the objective of each player is to maximize his or her individual utility.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} C & D \end{array} \\ \text{Player A} & \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{bmatrix} \end{array} \qquad (1)$$

Analyzing the problem using classical game theory, one can conclude that the best strategy to be played is found at the Nash Equilibrium, i.e. playing $\{DD\}$. The "dilemma" in the name of the game stands exactly for this: $\{DD\}$ is a strategy that is Pareto dominated by $\{CC\}$, so both players have, at least from a theoretical point of view, the possibility of obtaining an higher payoff simply unilaterally changing their own strategies, but the common rationality instead force them to play D.
But when the problem is instead reformulated in the quantum context [2], it can be proven that :

- there exists a particular pair of quantum strategies which always gives high reward and is a Nash equilibrium;

- there exists a particular quantum strategy which always gives a positive reward if played against any classical strategy.

According to the formalism presented in the previous section, assuming that each player is able to use quantum strategies, than the available actions at the beginning of the game are represented by operators in $SU(2)$ (unitary and trace-preserving 2x2 matrices). Moreover, in order to properly construct a quantum game, also a dose of entanglement, adjusted by the parameter γ, is needed.
From the numerical simulation of the problem it is possible to assert that all the Nash equilibrium strategies are of the type $\{\hat{U}(\pi, \phi_A), \hat{U}(\pi, \phi_B)\}$ and the corresponding utility values are $[1, 1]$. But remember that $\hat{U}(\pi, \phi)$ is nothing but the old pure strategy of defection $\hat{D}$. A further confirmation of this comes from figure 2, that shows the expected payoffs for player A as function of θ_A and θ_B (since the values of ϕ are not influent, and the equivalent heatmap for B is identical but reflected): as expected, if one player goes for $\theta \to 0$ and the other for $\theta \to \pi$ the game falls in a classical situation like $\{CD\}$ or $\{DC\}$, in which the utility is high for someone and low for the other, but this is not a NE. From the right plot, it is obvious that if A selects $\hat{D}$ his utility is maximized regardless of the opponent's strategy, and the same holds for B. This means that $\{\hat{D}, \hat{D}\} \equiv \hat{D} \otimes \hat{D}$ is an equilibrium in dominant strategies. Indeed separable games ($\gamma = 0$), manifest the same behavior as their classical counterpart, bringing also to the same results.

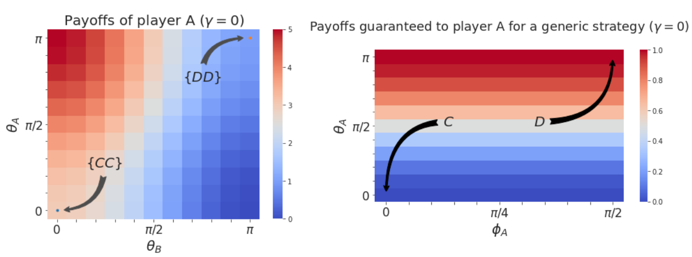

Figure 2: Utilities of the contender A in a game with $\gamma = 0$, as function of the strategies chosen (left). Minimum payoff guaranteed to A for a generic strategy of B (right).

The situation is entirely different for a maximally entangled game, with $\gamma = \pi/2$: this time, any pair of strategies chosen will have no counterpart in the classical domain, providing new and interesting solutions. To be precise, one can always re-derive the original prisoners' dilemma if both players choose actions of the type $\hat{U}(\pi/2, 0)$. From the numerical analysis of the problem [6], one can clearly observe that there are no dominant strategies this time, and this means that $\{\hat{D}\hat{D}\}$ is no longer an equilibrium in dominant strategies. Furthermore, such NE is replaced by a new joint strategy of best replies (from which no rational player would deviate), i.e. $\{\hat{Q}\hat{Q}\}$.

$$\hat{Q} \equiv \hat{U}(0, \frac{\pi}{2}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

This is the new unique equilibrium of the game, and guarantees a final payoff of 3 for both players; in fact

$$payoff_A(\hat{Q}, \hat{U}_B) = \cos^2\left(\frac{\theta}{2}\right)\left(3\sin^2\phi + \cos^2\phi\right) \leq 3 \quad \forall \theta, \phi$$

and the same holds for Bob.
Using the same representation as before, according to the payoff plot 3 of player B, notice that, assuming that the first player chooses this new optimal strategy $\hat{Q}$, then there is no choice for the second that would improve his payoff rather than $\hat{Q}$ as well.

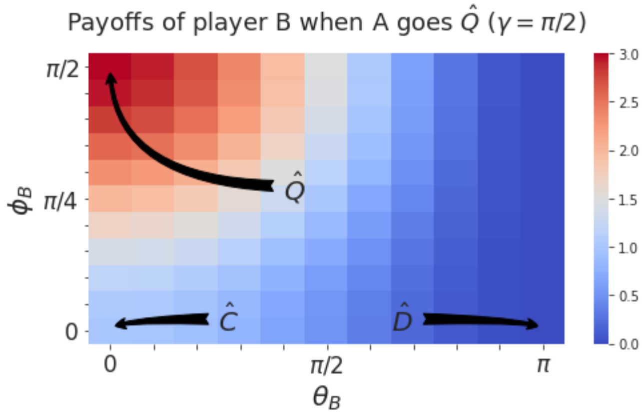

Figure 3: Utilities of player B when A selects the optimal strategy $\hat{Q}$.

Moreover, $\hat{Q} \otimes \hat{Q}$ is not just a Nash equilibrium but has also the property of being *Pareto Optimal*, meaning that deviating from this pair of strategies it is not possible to increase the payoff of one player without reducing the utility of the other. The solution of the classical game was not Pareto optimal: the only strategy with this property was the mutual cooperation, but such choice was not holdable by rational players without considering a multistage game (with carrot and stick or grim trigger approaches). And this is the first advantage provided by the quantum transposition of the game since, in a certain sense, the dilemma has just been solved.

4.1. Quantum player vs Classical opponent

What if the game was instead unfair like the spin-flip example presented in the second section, in the sense that quantum strategies are accessible only to one player?
Suppose that Alice has studied quantum mechanics, so her state space is still $SU(2)$, while Bob is limited to select actions characterized by $\phi = 0$: in this case A is well advised to play the so called *miracle move* (according to Eisert et al. [2])

$$\hat{M} \equiv \hat{U}(\pi/2, \pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}$$

that guarantees the following payoffs

$$payoff_A\left(\hat{M}, \hat{U}(\theta, 0)\right) \geq 3 \quad payoff_B\left(\hat{M}, \hat{U}(\theta, 0)\right) \leq \frac{1}{2}$$

The comparison among the possible combinations of actions is shown in figure 4.

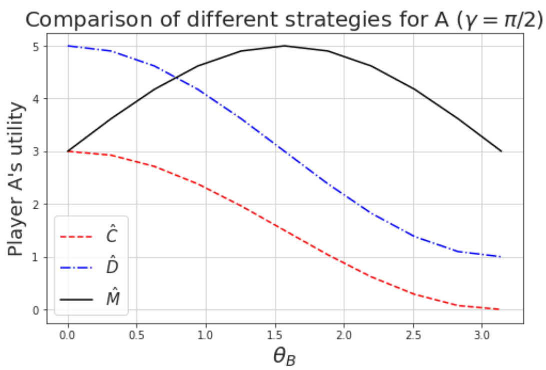

Figure 4: Final payoffs for different strategies selected by Alice, as function of the parameter θ_B.

So player A (rationally) will play "always $\hat{M}$", since this strategy will give her a utility of at least 3, regardless of the strategy chosen by B, which is inevitably in a position of disadvantage. This approach certainly outperforms "tit-for-tat", at least in these kind of asymmetric problems.
Finally, it may be interesting to study also the effective payoff of the quantum player as a function of the degrees of entanglement of the game, i.e. the parameter γ. The minimal expected payoff accessible by Alice is given by

$$m = \max_{\hat{U}_A \in SU(2)} \min_{\hat{U}_B \in \{\hat{C}, \hat{D}\}} payoff_A \left(\hat{U}_A, \hat{U}_B \right)$$

and the plot realized is shown in figure 5.

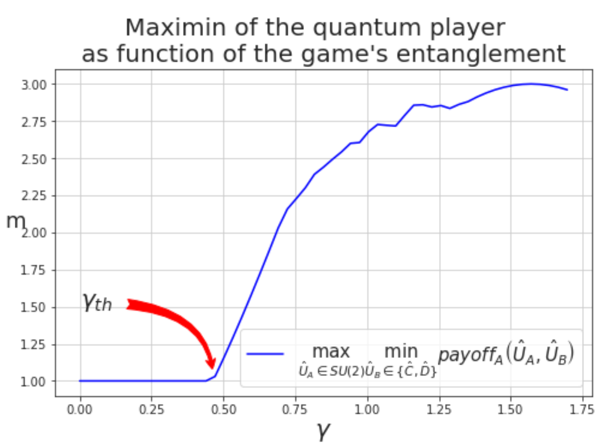

Figure 5: Payoffs guaranteed to player A as function of the entanglement measure γ.

It is immediately possible to notice that under a certain threshold, in term of entanglement degrees, that is found to be around $\gamma_{th} = \arcsin\left(1/\sqrt{5}\right) \simeq 0.464$, the forced choice is playing $\hat{F}$; after such threshold, near to the maximum entanglement among the states, then the best reply is always the miracle move. An interesting fact, not shown by the graph because of the limited precision, is that the "strategy swap", from $\hat{F}$ to $\hat{Q}$ is much more discontinuous than how it appears: this is a clue that a phenomenon known as *quantum phase transition* has occurs around γ_{th}.

4.2. Quantum Battle of Sexes

Another game that may be interesting to reformulate in the quantum world is the Battle of Sexes, mainly because the steps are the same as of the Prisoners' Dilemma, and the only thing that effectively change is the payoff matrix. This time Alice and Bob are partners that agreed to meet at the cinema to watch a film: A would like to see a romance (R) movie, while B prefer a sci-fi (S), but anyway each of them would make a compromise in order to avoid going alone. The problem is again a static game of complete information with the payoff matrix shown in 2.

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cc} R & S \end{array} \\ \text{Player A} \begin{array}{c} S \\ S \end{array} & \begin{bmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{bmatrix} \end{array} \tag{2}$$

Applying classical game theory in order to find the equilibria of the problem is quite straightforward and confirm the existence of 3 NE: two pure strategy NE, i.e. $\{\hat{R}\hat{R}\}$ and $\{\hat{S}\hat{S}\}$, and one mixed strategy NE in $(2/3, 1/3)$.

5. none

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