Nash Equilibrium in the Quantum Battle of Sexes Game

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Abstract

We investigate Nash Equilibrium in the quantum Battle of Sexes Game. We find the game has infinite Nash Equilibria and all of them leads to the asymmetry result. We also show that there is no unique but infinite Nash Equilibrium in it if we use the quantizing scheme proposed by Eisert et al and the two players are allowed to adopt any unitary operator as his/her strategies.

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I. INTRODUCTION

Game theory is a very useful and important branch of mathematics because it can solve many problems in economics, social science and biology[1]. Recently, people are very interested in what would happen when a classical game is extended into the quantum domain. They did lead to new sight into the nature of information[2, 3, 4] and quantum algorithms[5].Also some marvelous results are found. Meyer quantized the PQ Game - a coin tossing game[6] -and found out that one player could increase his expected payoff by implementing a quantum strategy against his classical opposite. J.Eisert, M.Wilkens and M.Lewenstein[7] investigated the Prisoner's Dilemma in the quantum world. They found a unique Nash Equilibrium, which is different from the classical one, and the dilemma could be solved if the two players applied quantum strategies and both players were better satisfied than in the classical world.

Not only 2-player games, but also multi-player games have been shown that better quantum strategies exist[8]. With all these achievement, it is natural to think that any classical game has a quantum version with better solution (payoff) or even think that the quantized game definitely has a unique Nash Equilibrium, which is better than classical one too.

Recently, Luca Marinatto and Tullio Weber studied the Battle of Sexes Game[9]. In that paper they proposed a unique Nash Equilibrium and the dilemma was moved. However, Benjamin argued that the quantum Battle of the Sexes game does not in fact have a unique solution[10]. We carefully study this problem and find out their approach was more like a classical one with classical possibilities (denoted by "p" and "q"). In this paper, we first fully quantize this game and find out that the quantum Sexes Game is much more complicated than the classical one. In this game, there are virtually infinite Nash Equilibrium rather than unique, and the quantum payoff is no better than the classical one. Further more, we prove that for nontrivial 2-player quantum games with the scheme proposed in Ref[7], if both players can apply any possible pure strategy in the whole SU(2), there is virtually no Nash Equilibrium.

II. QUANTIZATION OF THE BATTLE OF SEXES GAME

Our physical model for quantizing the Battle of Sexes Game is presented by Eisert[7]. Here we depicted it with quantum logic gates showed in Figure 1.

We send each player a classical 2-state system(a bit) in the zero state. U_A and U_B belong to the strategy space SU(2). We describe it by

$$U = \begin{pmatrix} e^{i\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} & ie^{i\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} \\ ie^{-i\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} & e^{-i\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} \end{pmatrix}, \ \varphi,\psi,\theta \in (-\pi,\pi) \tag{1}$$

We restrict it with $\psi = \varphi$, which will be proved reasonable later, then U should be such as

$$U = \begin{pmatrix} e^{i\varphi}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2} \end{pmatrix} \tag{2}$$

The payoff matrix for the Battle of sexes game is given as following.

		Bob	Bob
		O	T
Alice	O	(α,β)	(γ,γ)
Alice	T	(γ,γ)	(β,α)

In the matrix, the first entry in the parenthesis denotes the payoff of Alice and the second of Bob, and in this game $\alpha > \beta > \gamma$.

We start the game with the initial state $|\Psi_{in}\rangle = J|OO\rangle$, and the outcome state of the process is $|\Psi_{out}\rangle = J^+U_A \otimes U_B J|OO\rangle$.

So the payoff function is

$$\begin{cases} \$_A = \alpha P_{OO} + \beta P_{TT} + \gamma(P_{OT} + P_{TO}) \\ \$_B = \alpha P_{TT} + \beta P_{OO} + \gamma(P_{TO} + P_{OT}) \end{cases} \tag{3}$$

$P_{\sigma\tau}$ is the probability of the different outcome after measurement. It is described by

$$P_{\sigma\tau} = \left| \langle\sigma\tau| J^+U_A \otimes U_B J|OO\rangle \right|^2 \tag{4}$$

where $\sigma,\tau \in \{O,T\}$.

A. Non-Entangled Situation

The non-entangle condition is $\delta = 0$, hence $P_{\sigma\tau}$ can be expressed by

$$
\begin{cases}
P_{OO} = \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} \\
P_{TT} = \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\
P_{OT} = \cos^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\
P_{TO} = \sin^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2}
\end{cases}
\tag{5}
$$

where A denotes Alice and B denotes Bob.

According to the definition of Nash Equilibrium, we have the following inequation.

$$
\begin{cases}
\$_A(S_A^*, S_B^*) \geqslant \$_A(S_A, S_B^*) \\
\$_B(S_A^*, S_B^*) \geqslant \$_B(S_A^*, S_B)
\end{cases}
\forall S_A \in SU(2), \forall S_B \in SU(2)
\tag{6}
$$

The * denotes Nash Equilibrium.

We find three profiles of Nash Equilibrium.

1. $\{\theta_A^* = \theta_B^* = 0\}$. It means that Alice and Bob both use the identical strategy $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The payoff for Alice is α and for Bob is β .

2. $\{\theta_A^* = \theta_B^* = \pi\}$. It means that Alice and Bob both use the flip operation $i\sigma_x$, which is equivalent to a NOT-Gate. The payoff for Alice is β and for Bob is α .

3. $\{\theta_A^* = \arcsin\sqrt{\frac{\beta-\gamma}{\alpha+\beta-2\gamma}}, \theta_B^* = \arcsin\sqrt{\frac{\alpha-\gamma}{\alpha+\beta-2\gamma}}\}$. The payoff is $\$_A = \$_B = \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$.

We can see when $\delta = 0$, which is the non-entangle condition, the whole procedure includes the classical Battle of the Sexes Game and has no novel characters exceeding it. Also, if $\varphi = 0$, but $\delta \neq 0$, the game appears completely classical and the result is the same as when $\delta = 0$.

B. Max-Entangled Strategy

If $\delta = \frac{\pi}{2}$, which is the max-entangle condition, and $\varphi \neq 0$, the situation is quite different from the previous one and the result is some kind strange.

Now the $P_{\sigma\tau}$'s explicit expression are

$$
\begin{cases}
P_{OO} = \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} \cos^2(\varphi_A + \varphi_B) \\
P_{TT} = [\sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} - \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \sin(\varphi_A + \varphi_B)]^2
\end{cases}
\tag{7}
$$

because of $P_{TO} + P_{OT} = 1 - (P_{OO} + P_{TT})$, the payoff is given by

$$
\begin{cases}
\$_A = (\alpha - \gamma)P_{OO} + (\beta - \gamma)P_{TT} + \gamma \\
\$_B = (\beta - \gamma)P_{OO} + (\alpha - \gamma)P_{TT} + \gamma
\end{cases}
\tag{8}
$$

After simply calculation, we find out the strategy profile $\{S_A = S_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, which is Nash Equilibrium in the classical game, is no longer Nash Equilibrium in the quantum field. Since it is easy to see if Bob unilaterally changes his strategy to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, because the initial state is $|OO\rangle$, the outcome state should be $|TT\rangle$. That is to say, Bob can increase his payoff (form β to α) by unilaterally changing his own strategy.

We find infinite Nash Equilibria. Some of them are listed as follows.

	θ_A^*	θ_B^*	$\varphi_A^* + \varphi_B^*$	$\$_A$	$\$_B$
(1)	π	π	$\frac{\pi}{2}$	β	α
(2)	$-2\arcsin\sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$	$2\arcsin\sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$	$\frac{\pi}{2}$	β	α
(3)	$2\arcsin\sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$	$-2\arcsin\sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$	$\frac{\pi}{2}$	β	α

The interesting thing is that the payoff for Alice and Bob is always the same for any different Nash Equilibrium. Actually any Nash Equilibrium leads to the same asymmetrical result. This is due to the internal asymmetry lying in the game itself (the Battle of the Sexes Game is an asymmetric game). Alice always gets β and Bob always gets α. From the matrix we can see that strategy profile (2) and (3) cannot be distinguished. Further more, they cannot be analogized by the classical counterpart.

C.   Reason For Restricted Strategic Space

Here we prove that in general 2-player quantum games, if the players can apply any possible pure strategy in the whole $SU(2)$, there is virtually no Nash Equilibrium. Benjamin[10] pointed out for any $U_A \in SU(2)$,$U_B \in SU(2)$, we can always find a matrix $U \in SU(2)$ which satisfies

$$U_A \otimes U_B J \left| OO \right\rangle = I \otimes (UU_B)J \left| OO \right\rangle \tag{9}$$

that is to say, if Alice's strategy is given constant, Bob could make the outcome state be any eigenstate by unilaterally changing his own strategy.

If the profile $\{S_A^*, S_B^*\}$ is Nash Equilibrium. Because of equation (8), we can always find an $S_A^{'} \in SU(2)$, which satisfies $\$_A(S_A^{'}, S_B^*) = (\$_A)_{MAX}$, hence we have $\$_A(S_A^*, S_B^*) \geqslant \$_A(S_A^{'}, S_B^*) = (\$_A)_{MAX}$ (Because of the definition of Nash Equilibrium).

But in fact, it is always true that $\$_A(S_A^*, S_B^*) \leqslant (\$_A)_{MAX}$, hence

$$\$_A(S_A^*, S_B^*) = (\$_A)_{MAX} \tag{10}$$

Following the similar process, we can prove

$$\$_B(S_A^*, S_B^*) = (\$_B)_{MAX} \tag{11}$$

so

$$\begin{cases} (\$_A)_{MAX} = \$_A(S_A^*, S_B^*) = (\$_A)_{00} P_{00} + (\$_A)_{01} P_{01} + (\$_A)_{10} P_{10} + (\$_A)_{11} P_{11} \\ (\$_B)_{MAX} = \$_B(S_A^*, S_B^*) = (\$_B)_{00} P_{00} + (\$_B)_{01} P_{01} + (\$_B)_{10} P_{10} + (\$_B)_{11} P_{11} \end{cases} \tag{12}$$

where $P_{\sigma\tau} = \left| \left\langle \sigma\tau \right| J^+ U_A \otimes U_B J \left| OO \right\rangle \right|^2$, and $\sigma, \tau \in \{0, 1\}$.

But for any $\sigma, \tau \in \{0, 1\}$,

$$(\$_A)_{\sigma\tau} + (\$_B)_{\sigma\tau} \leqslant (\$_A)_{MAX} + (\$_B)_{MAX} \tag{13}$$

which is actually four inequations. For a nontrivial game, these inequations cannot be equations at the same time for any set of $\{\sigma, \tau\}$, at least one of them must be "true" inequation. Therefore, when we add the two equations in (12) together with inequation (13), we have

$$\begin{aligned} (\$_A)_{MAX} + (\$_B)_{MAX} &< [(\$_A)_{MAX} + (\$_B)_{MAX}](P_{00} + P_{01} + P_{10} + P_{11}) \\ &= (\$_A)_{MAX} + (\$_B)_{MAX} \end{aligned} \tag{14}$$

where $P_{00} + P_{01} + P_{10} + P_{11} = 1$. This inequation could never be true. Hence in general 2-player quantum games, if the players can apply any possible pure strategy in the whole $SU(2)$, there's virtually no Nash Equilibrium. This is the reason why we restrict the strategy space to be a subset of the whole $SU(2)$.

III.   CONCLUSION

In conclusion we have fully quantized the Battle of the Sexes Game. The classical process of the game is included in the non-entanglement situation. In particular the interesting thing is that, in the entangled situation, we find infinite Nash equilibrium in the quantum game and the payoff is no better than it's classical version. We then prove that in most general 2-player static quantum games, there is no Nash equilibrium if both players can apply any possible pure strategy in the whole $SU(2)$. But this proving is no longer true while extending into static multi-player quantum games. It seems that multi-player quantum games may have Nash Equilibrium with the whole $SU(2)$ as strategy space. So it is interesting to study multi-player games and the next work is in preparation.

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IV.   FIGURECAPTION:

The setup of 2-player quantum game. Quantum gate J is encoding gate and J^+ is decoding gate, each of them is composed of a single rotation gate and a CNOT gate . Strategy moves of Alice and Bob are associated with unitary operators U_A and U_B. J is a unitary operator which is known to both players and symmetric with respect to the interchange of the two players.