Remark On Quantum Battle of The Sexes Game

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Recently Quantum *Battle of The Sexes Game* has been studied by Luca Marinatto and Tullio Weber. Yet some important problems exist in their scheme. Here we propose a new scheme to quantize *Battle of The Sexes Game*, and this scheme will truly remove the dilemma that exists in the classical form of the game.

Introduction

Quantum game and quantum strategies is a new born field. Although many novel features have been discovered by researchers in the previous works[1, 2, 3, 4, 5], some problems also exist. In a recent article of Luca Marinatto and Tullio Weber[6], they proposed a scheme to quantize the famous *Battle of The Sexes Game*. In the usual exposition of this game, Alice and Bob are trying to decide where to go on Saturday night. Alice wants to attend the Opera, while Bob prefers to watch TV. And both players would be happier to spend the evening together rather than apart. Both players want to maximize their individual payoff. Table (1) indicates the payoffs of Alice and Bob. The first entry in the parenthesis refers to Alice's payoff, and the second to Bob's. To satisfy the preferences of the two players, the condition $\alpha > \beta > \gamma$ is imposed.

Bob:
$$O$$
 Bob: T
Alice: O (α, β) (γ, γ)
Alice: T (γ, γ) (β, α)

There are two Nash equilibrium (O, O) and (T, T) existing in the classical form of the game. Since there is no transfer of information between two players, they face the dilemma in choosing between two stable solutions. If the mismatch situation occurs, that is if one player chooses strategy O and the other chooses strategy T, then the payoff to both players is γ and it is the worst situation.

Luca Marinatto and Tullio Weber[6] give an Hilbert structure to the strategic spaces of the players, so allowing the existence of linear combinations of classical strategies to be interpreted according to usual formalism of orthodox quantum mechanics. They applied this formalism to the study of the *Battle of the Sexes Game*. The result is that if both players are allowed to play entangled quantum strategies, the game has a unique solution.

This article attracts the attention of S.C. Benjamin and he made a comment [7] on it. In his comment two observations were made: "Firstly, the overall quantization scheme is fundamentally very similar to the scheme proposed by Eisert et al.[2]-the similarity is non-obvious because of the very different use of the word 'strategy' in the two approaches. Secondly, we argue that the quantum Battle of the Sexes Game does not in fact have a unique solution, hence the players are still subject to a dilemma." In the work of Luca Marinatto and Tullio Weber[6], the initial strategy is set to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and 'tactics' of the players are limited to a probabilistic choice between applying the identity I and $C = \sigma_x$. So Benjamin pointed out "this is a severe restriction on the full range of quantum mechanically possible manipulation". In the scheme of Luca Marinatto and Tullio Weber[6], p^* and q^* are used to represent the probability of Alice's and Bob's choosing σ_x respectively. The authors get two profile tactics which satisfy the maximum expected payoff of the players. One is $(p^* = 0, q^* = 0)$ and the other is $(p^* = 1, q^* = 1)$. Since the final entangled quantum 'strategy' remains unchanged in either case, so this is the unique solution in the game. However, Benjamin wrote that "it seems to us that a clear dilemma remains for the players". Because there are two pairs of tactics $(p^* = 0, q^* = 0)$ and $(p^* = 1, q^* = 1)$, without knowing the choice of the other player, the players face the dilemma to choose between them, Benjamin continue, "If the tactics are mismatched, i.e. if either $(p^* = 0, q^* = 1)$ or $(p^* = 1, q^* = 0)$ are adopted, then the worst situation will occur. This is almost the same problem faced by the players in the traditional game."

In replying these remarks, Luca Marinatto & Tullio Weber outlined some topical points of their work which they hold to be crucial for a better comprehension of their new approach to the quantum theory of games[8]. Regarding terminology, they think "the choice of calling 'strategies' the quantum states instead of the 'operators' used to manipulate them, is very natural and quite consistent with the spirit of the classical game theory". As far as the choice of the tactics set is concerned, they wrote that "obviously the class of allowed manipulations could be enlarged, but in our paper we did not take care of this possibility since our minimal choice was enough to reproduce intact the classical results when considering only factorizable strategies and to obtain the disappearance of the dilemma when resorting to entangled strategies". As regards the possibility of the occurrence of the mismatch situation, they consider it from the perspective of practical operation. They think that "since both the choices will eventually lead to the same final strategy", "it is therefore apparent that both players, knowing this fact, will decide doing nothing on their strategy". They have two reasons for their decision. One is that "doing nothing could be considered cheaper than doing something". The other is "Which player will in fact decide to refuse a certain gain, without prospects of a better gain and running the risk of incurring a loss?".

The reply of Luca Marinatto and Tullio Weber sounds reasonable from the perspective of their paper. Yet we consider the terminology 'strategy' as the operation of a player on his state. The process of obtaining initial state is not included in the strategic space. According to the concept of the static game, the strategic moves of the players are local operations, and each player has no information about the strategies of other player. Since the process of obtaining initial state $\frac{1}{\sqrt{2}}(|OO\rangle + |TT\rangle)$ from $|OO\rangle$ must be a global operation and every player clearly knows it, so this process could not be called as strategy. Furthermore we think that in the scheme of Luca Marinatto &

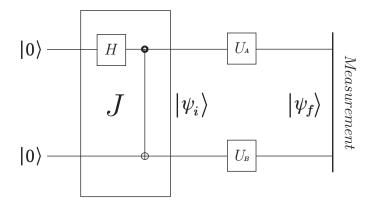


FIG. 1: The setup of Battle of Sexes Game.

Tullio Weber the probability p^* and q^* , which are used to represent the probability of Alice's and Bob's choosing σ_x respectively, are classical probabilities. Since σ_x and I are equivalent to classical pure strategies "flip" and "not-flip" respectively, the operation of the player could be considered as the classical mixture of classical strategies.

In this paper, we propose a new scheme in which there is no restriction on the quantum mechanically possible manipulations of the players, and the strategy space is the complete set of SU(2). Here the terminology 'strategy' is consistent with the one used by Eisert et al.[2]. And the operation of a player is the probabilistic mixture of the pure quantum strategies. We show that if both players resort to mixed quantum strategy, although there are still many profiles of Nash equilibria, the possible tactic mismatch will have no influence on the players. So the dilemma which exists in the traditional $Battle\ of\ the\ Sexes\ Game$ is removed.

I. THE QUANTIZATION OF BATTLE OF THE SEXES GAME

Our physical model for quantizing the game is similar to the delicate scheme introduced by Eisert et al.[2, 9]. Figure 1 indicates the flow of information in the *Battle of the Sexes Game*. It is composed of a source of two bits, one bit for each player. $|O\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents the state of going to Opera and $|T\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents the state of watching TV. The state of the game is described by a vector in the tensor product space which is spanned by the basis $|OO\rangle$

TV. The state of the game is described by a vector in the tensor product space which is spanned by the basis $|OO\rangle$, $|OT\rangle$, $|TO\rangle$ and $|TT\rangle$, where the first entry refers to Alice' state and the second to Bob'. Gate J is a unitary operator and consist of a Hadamard gate and a C-NOT gate. So the prepared state after gate J is

$$|\psi_i\rangle = J|OO\rangle = \frac{1}{\sqrt{2}}(|OO\rangle + |TT\rangle)$$
 (2)

So the density matrix of the initial state is

$$\rho_i = |\psi_i\rangle \langle \psi_i| \tag{3}$$

The strategies of Alice and Bob are denoted by unitary operator U_A and U_B . Alice and Bob independently operate on her/his qubit with U_A and U_B respectively. U_A and U_B are chosen from the strategy space S. Considering the possible strategies of the players, the strategy space S should be all of SU(2), and the general expression of the unitary operator is

$$U(\theta, \phi, \psi) = \begin{pmatrix} e^{i(\phi+\psi)/2} \cos\frac{\theta}{2} & ie^{i(\phi-\psi)/2} \sin\frac{\theta}{2} \\ ie^{-i(\phi-\psi)/2} \sin\frac{\theta}{2} & e^{-i(\phi+\psi)/2} \cos\frac{\theta}{2} \end{pmatrix}$$
(4)

with $-\pi \leqslant \theta \leqslant \pi, -\pi \leqslant \phi \leqslant \pi, -\pi \leqslant \psi \leqslant \pi$. We set f as the probability of the operator U which will be chosen by the players. Naturally f is a probability density function with respect to U. It is obvious that $f(U) \geqslant 0$ and $\int_{SU(2)} f(U) \, dU = 1$. Here we use the invariance of the Haar measure, assumed to be normalized so that the volume

of SU(2) is 1. The probabilities of Alice choosing U_A and Bob choosing U_B are denoted by $f_A(U_A)$ and $f_B(U_B)$ respectively. So after the strategy moves of Alice and Bob, the final density matrix

$$\rho_f = \int_A \int_B f_A f_B \left(U_A \otimes U_B \right) \rho_i \left(U_A \otimes U_B \right)^+ dU_A dU_B \tag{5}$$

The payoff operators for Alice and Bob are

$$\begin{cases} \hat{\$}_A = \alpha |OO\rangle \langle OO| + \beta |TT\rangle \langle TT| + \gamma (|OT\rangle \langle OT| + |TO\rangle \langle TO|) \\ \hat{\$}_B = \beta |OO\rangle \langle OO| + \alpha |TT\rangle \langle TT| + \gamma (|OT\rangle \langle OT| + |TO\rangle \langle TO|) \end{cases}$$
(6)

The expected payoffs of Alice and Bob are mean values of these operators and hence are functional of f_A and f_B .

$$\bar{\$}_{A}(f_{A}, f_{B}) = Tr\left(\rho_{f}\hat{\$}_{A}\right)$$

$$= \int_{A} \int_{B} f_{A}f_{B} \left[Tr\left(\left(U_{A} \otimes U_{B}\right)\rho_{i}\left(U_{A} \otimes U_{B}\right)^{+}\hat{\$}_{A}\right)\right] dU_{A}dU_{B}$$

$$= \int_{A} \int_{B} f_{A}f_{B}\$_{A}\left(U_{A}, U_{B}\right) dU_{A}dU_{B}$$

$$\bar{\$}_{B}\left(f_{A}, f_{B}\right) = Tr\left(\rho_{f}\hat{\$}_{B}\right)$$

$$= \int_{A} \int_{B} f_{A}f_{B} \left[Tr\left(\left(U_{A} \otimes U_{B}\right)\rho_{i}\left(U_{A} \otimes U_{B}\right)^{+}\hat{\$}_{B}\right)\right] dU_{A}dU_{B}$$

$$= \int_{A} \int_{B} f_{A}f_{B}\$_{B}\left(U_{A}, U_{B}\right) dU_{A}dU_{B}$$

$$= \int_{A} \int_{B} f_{A}f_{B}\$_{B}\left(U_{A}, U_{B}\right) dU_{A}dU_{B}$$

$$(8)$$

Where $\$_A(U_A, U_B)$ and $\$_B(U_A, U_B)$ are the payoff functions of Alice and Bob when they adopt the pure quantum strategies U_A and U_B respectively, the expression of $\$_A$ and $\$_B$ are as follows:

$$\$_A = \alpha P_{OO} + \beta P_{TT} + \gamma \left(P_{OT} + P_{TO} \right) = (\alpha - \gamma) P_{OO} + (\beta - \gamma) P_{TT} + \gamma \tag{9}$$

$$\$_B = \alpha P_{OO} + \beta P_{TT} + \gamma \left(P_{OT} + P_{TO} \right) = (\beta - \gamma) P_{OO} + (\alpha - \gamma) P_{TT} + \gamma \tag{10}$$

Since Alice and Bob adopt the pure quantum strategies, here $P_{\sigma\tau} = |\langle \sigma\tau | (U_A \otimes U_B) | \psi_i \rangle|^2$ is the joint probability of the final state collapses into the basis $|\sigma\tau\rangle$.

$$P_{OO} = P_{TT} = \frac{1}{4} \left(1 + \cos \theta_A \cos \theta_B - \cos \left(\psi_A + \psi_B \right) \sin \theta_A \sin \theta_B \right) \tag{11}$$

From the expression of P_{OO} and P_{TT} , we have

$$\$_{A}(U_{A}, U_{B}) = \$_{B}(U_{A}, U_{B})$$

$$= \frac{\alpha + \beta - 2\gamma}{4} (1 + \cos\theta_{A}\cos\theta_{B} - \cos(\psi_{A} + \psi_{B})\sin\theta_{A}\sin\theta_{B}) + \gamma$$

$$= \frac{\alpha + \beta + 2\gamma}{4} + \frac{\alpha + \beta - 2\gamma}{4} (\cos\theta_{A}\cos\theta_{B} - \cos(\psi_{A} + \psi_{B})\sin\theta_{A}\sin\theta_{B})$$
(12)

Substitute equation (12) into equations (7,8), we can obtain expressions of $\bar{\$}_A$ and $\bar{\$}_B$ as funtionals of f_A^* and f_B^* . Now we investigate Nash Equilibrium in this game. The definition of Nash Equilibrium $\{s_A^*, s_B^*\}$ for pure strategy in two-player games can be expressed as the following inequalities

$$\begin{cases} \$_{A}\left(s_{A}^{*}, s_{B}^{*}\right) \geqslant \$_{A}\left(s_{A}, s_{B}^{*}\right) \\ \$_{B}\left(s_{A}^{*}, s_{B}^{*}\right) \geqslant \$_{B}\left(s_{A}^{*}, s_{B}\right) \end{cases} \forall s_{A}, s_{B} \in SU(2)$$

$$(13)$$

When extended to the mixed strategies $(f_A \text{ and } f_B)$, the Nash Equilibrium profile $\{f_A^*, f_B^*\}$ can be expressed as follows

$$\begin{cases} \bar{\$}_A (f_A^*, f_B^*) \geqslant \bar{\$}_A (f_A, f_B^*) \\ \bar{\$}_B (f_A^*, f_B^*) \geqslant \bar{\$}_B (f_A^*, f_B) \end{cases} \forall f_A \geqslant 0, \forall f_B \geqslant 0, \int_A f_A dU_A = \int_B f_B dU_B = 1.$$
 (14)

It is obvious that $f_A^* \ge 0$, $f_B^* \ge 0$, $\int_A f_A^* dU_A = \int_B f_B^* dU_B = 1$. By using calculus of variations to the equation (14), we obtain the following equations:

$$\begin{cases}
\left(\frac{\delta \bar{\$}_A(f_A, f_B^*)}{\delta f_A}\right)_{f_A = f_A^*} = \lambda_A \\
\left(\frac{\delta \bar{\$}_B(f_A^*, f_B)}{\delta f_B}\right)_{f_B = f_B^*} = \lambda_B
\end{cases}$$
(15)

where λ_A and λ_B are constant. From equation (7,8), the left hand of the equation (15) can be rewritten as follows

$$\begin{cases}
\left(\frac{\delta \bar{\$}_{A}(f_{A}, f_{B}^{*})}{\delta f_{A}}\right)_{f_{A} = f_{A}^{*}} = \int_{B} f_{B}^{*}(U_{B}) \,\$_{A}(U_{A}, U_{B}) \, dU_{B} \\
\left(\frac{\delta \bar{\$}_{B}(f_{A}^{*}, f_{B})}{\delta f_{B}}\right)_{f_{B} = f_{B}^{*}} = \int_{A} f_{A}^{*}(U_{A}) \,\$_{B}(U_{A}, U_{B}) \, dU_{A}
\end{cases} \tag{16}$$

From above calculation, we get the following equation.

$$\begin{cases} \lambda_{A} = \frac{\delta \bar{\$}_{A}(f_{A}, f_{B}^{*})}{\delta f_{A}} = \int_{B} f_{B}^{*}(U_{B}) \,\$_{A}(U_{A}, U_{B}) \, dU_{B} \\ \lambda_{B} = \frac{\delta \bar{\$}_{B}(f_{A}^{*}, f_{B})}{\delta f_{B}} = \int_{A} f_{A}^{*}(U_{A}) \,\$_{B}(U_{A}, U_{B}) \, dU_{A} \end{cases}$$
(17)

Here, the left hand of equation λ_A (or λ_B) is a constant independent of U_A and U_B , but the right hand of equation (17) generally depends on U_A (or U_B). The equation has o solution without the guarantee of $f_B^*(U_B)$ (or $f_A^*(U_A)$) that there is no $U_A(U_B)$ in the result of the integral of $\$_A(U_A, U_B)$ (or $\$_B(U_A, U_B)$). On the other hand, any $f_A^*(U_A)$ and $f_B^*(U_B)$ which meet the guarantee of $f_B^*(U_B)$ (or $f_A^*(U_A)$) will be solution of equation (17), and the profile $\{f_A^*, f_B^*\}$ will be one of the Nash Equilibria of the game. When Alice's mixed strategy is f_A^* and Bob's is f_B^* , neither of them can increase her/his payoff by unilaterally changing her/his own strategy.

The solutions of equation (17) are obviously infinite, that means there is no unique stable strategies. However, the possible tactic mismatch will have no influence on the players because for any f_A^* which satisfies equation(17) will yield

$$\bar{\$}_{A} (f_{A}^{*}, f_{B}^{*}) = \int_{A} \int_{B} f_{A}^{*} f_{B}^{*} \$_{A} (U_{A}, U_{B}) dU_{A} dU_{B}$$

$$= \int_{A} f_{A}^{*} \left[\int_{B} f_{B}^{*} \$_{A} (U_{A}, U_{B}) dU_{B} \right] dU_{A}$$

$$= \lambda_{A} \int_{A} f_{A}^{*} dU_{A}$$

$$= \lambda_{A}$$

$$= \lambda_{A}$$
(18)
$$= \lambda_{A}$$

So the payoff for Alice $(\$_A(f_A^*, f_B^*))$ is not related to her own strategy (f_A^*) but her opponent's strategy (f_B^*) . Similarly the payoff for Bob $(\$_B(f_A^*, f_B^*))$ is λ_B , which is not related to his own strategy (f_B^*) . It is apparent that for any combination of f_A^* and f_B^* which belongs to the stable strategies (solutions of equation(17)), the payoffs for alice is λ_A and for Bob is λ_B . Further more, through the following caculation we find that the payoffs of Alice and Bob is the same $(\lambda_A = \lambda_B)$. So the players will not worry about the occurrence of tactic mismatch. Then the dilemma in the classical *Battle of The Sexes Game* is removed.

From the expression of θ , ϕ , ψ in equation(4), the normalized invariance of the Haar measure is

$$dU(\theta, \phi, \psi) = \frac{1}{16\pi^2} |\sin \theta| \, d\theta d\phi d\psi \tag{20}$$

So the integration on the right hand of the equation (17) can be rewritten as integration with respect to θ , ϕ , ψ . For any f_A^* which satisfies the equation (17), we get $\int_A f_A^* (U_A) \$_B (U_A, U_B) dU_A$ being independent of U_B , hence $\int_A f_A^* (U_A) P_{\sigma\tau} (U_A, U_B) dU_A$ is independent of U_B . From the expression of $P_{\sigma\tau}$ (see equation (11)), after integrating

with respect to θ_A, ϕ_A, ψ_A , the values of the integration of the terms that contain θ_B, ϕ_B, ψ_B is zero. So we get

$$\int_{A} f_{A}^{*}(U_{A}) P_{\sigma\tau}(U_{A}, U_{B}) dU_{A}$$

$$= \int_{A} f_{A}^{*}(U_{A}) P_{OO}(U_{A}, U_{B}) dU_{A}$$

$$= \int_{A} f_{A}^{*}(U_{A}) P_{TT}(U_{A}, U_{B}) dU_{A}$$

$$= \int_{A} f_{A}^{*}(U_{A}) \frac{1}{4} dU_{A} = \frac{1}{4}$$
(21)

Repeating the same procedure, we obtain that $\int_B f_B^*(U_B) P_{\sigma\tau}(U_A, U_B) dU_B = \frac{1}{4}$. According to the payoff functions of $A_A(U_A, U_B)$ and $B_B(U_A, U_B)$ (see equation(12)), finally the payoff to both players is $\bar{A}_A = \bar{A}_B = \frac{\alpha + \beta + 2\gamma}{4}$. We list four of the simplest solutions(Nash Equilibria) in the following table:

(1)	$f_A^* = 1$	$f_B^* = 1$	$\bar{\$}_A = \bar{\$}_B = \frac{\alpha + \beta + 2\gamma}{4}$
(2)	$f_A^* = 1$	$f_B^* = \frac{2}{\pi \sin \theta_B }$	$\bar{\$}_A = \bar{\$}_B = \frac{\alpha + \beta + 2\gamma}{4}$
(3)	$f_A^* = \frac{2}{\pi \sin \theta_A }$	$f_B^* = 1$	$\bar{\$}_A = \bar{\$}_B = \frac{\alpha + \beta + 2\gamma}{4}$
(4)	$f_A^* = \frac{2}{\pi \sin \theta_A }$	$f_B^* = \frac{2}{\pi \sin \theta_B }$	$\bar{\$}_A = \bar{\$}_B = \frac{\alpha + \beta + 2\gamma}{4}$

Any strategy in the table is a stable solution.

In the classical Battle of The Sexes Game, there are two Nash Equilibria: (O,O) and (T,T). If the tactic mismatch occurs, the strategic profile becomes (O,T) or (T,O) and the payoffs turn out to be the worst-case payoff $\$_A = \$_B = \gamma$. In the quantum version of this game, there are more than one Nash Equilibrium appearing when players adopted the deterministic strategies (see in Ref[6]), and any tactic mismatch can lead to the worst-case payoffs. While when we propose that if the players adopt quantum mixed strategies, it can be guaranteed that the possible appearance of the tactic-mismatch will have no effect on the payoffs of the players, and the payoffs will always remain as $\frac{\alpha+\beta+2\gamma}{4}$.

II. CONCLUSION

In this paper, the problems existing in the previous quantum $Battle\ of\ The\ Sexes\ Game$ are studied in detail. We proposed a new scheme, applying mixed quantum strategies, to quantize this game. In our scheme, the players can choose their strategies in the all of SU(2). It means that there is no restriction on the quantum mechanically possible manipulations of the players. We show that if the players resort to the mixed quantum strategy, in the game the equilibria that they reach will be much more efficient than that in its classical version. Furthermore, the tactic-mismatch which is the difficulty faced by the players in the traditional game[7] has no effect on the payoffs of the players. Thus the dilemma which exists in the classical $Battle\ of\ the\ Sexes\ Game$ is truly removed. The scheme of mixed quantum strategy proves successful. We also hope that the scheme of mixed quantum strategy can be useful for the investigation of other quantum games.

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