

Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers

# UNIVERSITY OF BIRMINGHAM

**School of Computer Science**

**LM Mathematical Foundations of Artificial Intelligence (AI) and Machine Learning  
(ML)**

Main Summer Examinations 2023

Time allowed: 2 hours

[Answer all questions]

## Note

Answer ALL questions. Each question will be marked out of 20. The paper will be marked out of 60, which will be rescaled to a mark out of 100.

## Question 1

Consider the set  $\mathcal{L}$  of all polynomials over real numbers of the form

$$f(x) = a \cdot x^2 + b \cdot x^4 + c \cdot x^7, \quad x \in \mathbb{R},$$

where  $a, b, c \in \mathbb{R}$  are real coefficients. On  $\mathcal{L}$  we define a binary (pointwise) addition operation  $\oplus$ : for any  $f, g \in \mathcal{L}$ ,

$$(f \oplus g)(x) = f(x) + g(x),$$

and a scalar multiplication operation  $\odot$ : for any  $f \in \mathcal{L}$  and  $\alpha \in \mathbb{R}$ ,

$$(\alpha \odot f)(x) = \alpha \cdot f(x).$$

Here  $+$  and  $\cdot$  denote the usual addition and multiplication on real numbers, respectively.

- (a) Show that  $(\mathcal{L}, \oplus)$  is an Abelian group.

**[10 marks]**

- (b) Knowing that  $(\mathcal{L}, \oplus)$  is an Abelian group, show that  $(\mathcal{L}, \oplus, \odot)$  is a vector space.

**[10 marks]**

## Question 2

We are given three fixed probabilistic models over the space of gray-scale animal images  $\mathbf{x} \in \mathbb{R}^{n \times n}$  of  $n \times n$  pixels. In particular, distribution  $p_{cat}(\mathbf{x})$  is a model distribution of pictures of cats. Similarly, we have fixed model distributions  $p_{dog}(\mathbf{x})$  and  $p_{horse}(\mathbf{x})$  of images of dogs and horses, respectively.

We are also given a dataset  $\mathcal{D} = \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ ,  $\mathbf{x}^i \in \mathbb{R}^{n \times n}$ , of  $N$  unlabelled cat, dog, or horse images and we would like to construct a mixture model that would cluster (in a probabilistic manner) the dataset  $\mathcal{D}$  into likely images of cats, dogs and horses.

- (a) Formulate the mixture model and specify what are the free parameters. **[7 marks]**

- (b) Explain how the free parameters of our mixture model would be trained, given the data  $\mathcal{D}$ . **[6 marks]**

- (c) What modifications to the model would you suggest if we were told that it is possible that the dataset  $\mathcal{D}$  may also contain images of animals other than cats, dogs or horses? **[7 marks]**

**Question 3**

In certain real-world classification problems, predicting 0 instead of 1 carries a cost  $\alpha$ , and predicting 1 instead of 0 carries of cost  $\beta$ , where  $\alpha$  may not be equal to  $\beta$ .

- (a) Devise a classification loss function, depending on  $\alpha$  and  $\beta$ , that takes these differing costs into account, and recovers the 0-1 loss when  $\alpha/\beta = 1$ . Hint: This loss function definition should have 3 cases. **[4 marks]**
- (b) Give the sample complexity of learning a finite hypothesis class  $\mathcal{H}$ , in the agnostic setting, using your loss function from (a). **[8 marks]**
- (c) Is the Bayes-optimal classifier that corresponds to your loss function from (a) any different from the Bayes-optimal classifier that corresponds to the usual 0-1 loss? Justify your answer. **[8 marks]**

**Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so**

**Important Reminders**

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches must be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

**Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.**