UNIVERSITY OF BIRMINGHAM

School of Computer Science

Algorithms and Complexity

Main Summer Examinations 2023

Time allowed: 2 hours

[Answer all questions]

Turn Over

Note

Answer ALL questions. **There are four questions worth 8,12,15,15 marks each for a total of 50 marks.** You can use (without proof) anything that was taught in the module (slides, lectures, weekly assignments)

Question 1

Let $f : \{0,1\}^3 \to \{0,1\}$ be defined by $f(x,y,z) = 1$ just if $x = y$ or $y = z$.

(a) Give a truth table for f. **[3 marks]**

(b) Give a CNF computing f, and justify this. **[5 marks]**

Question 2

(a) What is the *depth* of a circuit? **[2 marks]**

(b) Show that, for any Boolean circuit C of depth d, there is a Boolean formula F of size $< 2^d$ computing the same Boolean function.

NB: for this question, the *size* of a formula is its number of connectives.

Hint: You should proceed by induction on the depth (or structure) of C. **[8 marks]**

(c) Suppose $L \subseteq \{0,1\}^*$ is computed by a family of Boolean circuits $(C_n)_{n=0}^{\infty}$ with each C_n having depth $O(\log n)$. Conclude from part (b) that L is also computed by a family of Boolean formulas $(F_n)_{n=0}^{\infty}$ with each F_n having size $O(n^k)$ for some k. **[2 marks]**

NB: you may use part (b) to answer this part, even if you did not solve it.

Question 3

The input of the SAT problem is a CNF formula with N variables and M clauses, and the question is whether there exists an assignment of variables which satisfies *all* the clauses. Recall that we have discussed in the module that the SAT problem is **NP**-hard.

(a) Show that there always exists an assignment of the variables in the SAT problem which satisfies *at least half* of the clauses. Design an $O(M \cdot N)$ time algorithm that finds such an assignment. **[7 marks]**

NB: For this question, you may assume that no clause contains both a variable x and its negation $\neg x$

(b) Show that the following problem is **NP**-complete:
Almost-Almost-SAT: Given a CNF formula with N variables and M clauses, is there an assignment of variables which satisfies *exactly* $M - 2$ of the clauses. **[8 marks]**

Question 4

You want to run for a total of exactly N miles in preparation for a race. Each day, you can do either a short run of exactly 1 miles, a medium run of exactly 6 miles, a long run of exactly 9 miles or an ultra-long run of exactly 15 miles. Let $OPT[N]$ be the minimum number of days needed to achieve your goal of running a total of exactly N miles.

(a) Let $\mathcal{G}$ be the greedy algorithm where you just do the longest run possible on each day consistent with achieving the goal. Find a value of N for which the greedy algorithm $\mathcal{G}$ requires strictly more days to achieve the goal than $OPT[N]$. Explain your answer. **[5 marks]**

(b) Design a dynamic programming algorithm which for each $N \geq 1$ runs in $O(N)$ time and computes $OPT[N]$. Explain both the correctness and running time of your algorithm. **[10 marks]**

End of Paper

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.