Formal Concept Analysis, Lattice Polinomials and Approximation

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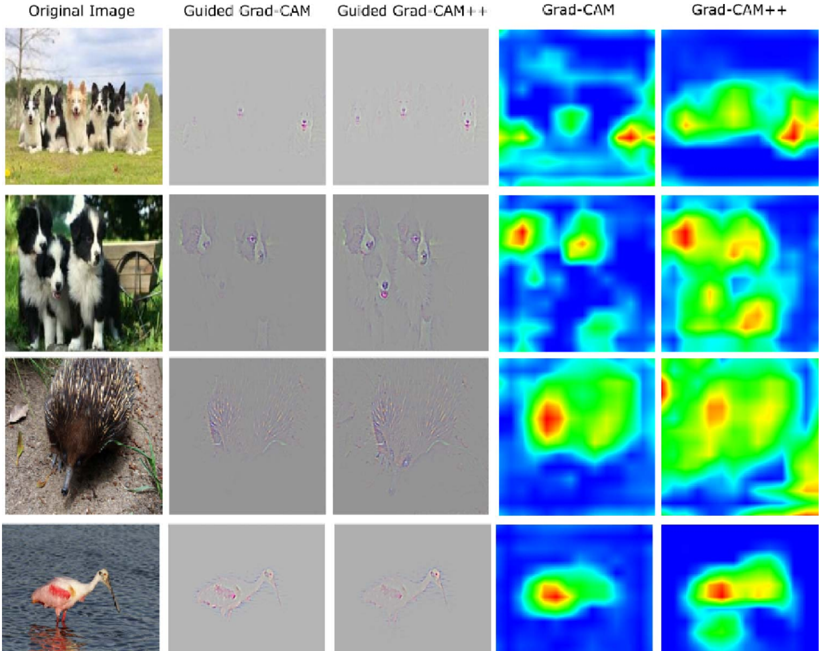
Why Formal Concept Analysis



Aggregation of Common Features

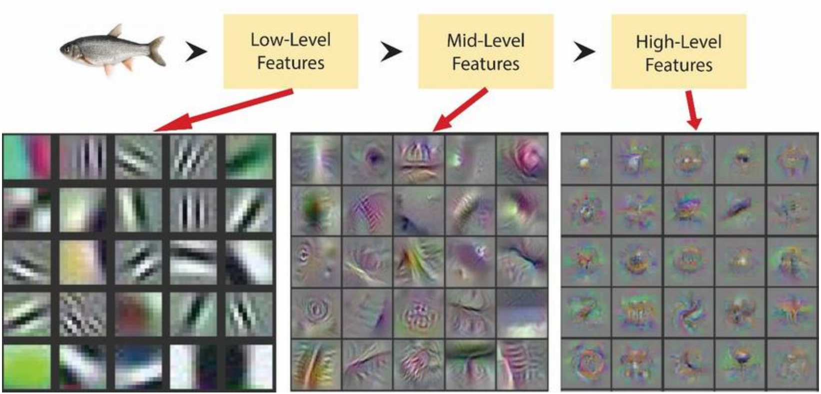


Figure: Hierarchical Aggregation of Features

Concept Partial Order

As it is defined in [1]:

- Formal Context is a triple (G, M, I) where G and M are sets and $I \subset G \times M$ is a relation betweeen them which is callsed incidence relation, G is called objects and M attributes
- For subset of objects $A \subset G$ define $A' = \{m \in M | (g, m) \in I \text{ for all } g \in A\}$
- For subset of attributes $B \subset M$ define $B' = \{g \in G | (g, m) \in I \text{ for all } m \in B\}$
- A concept of the context (G, M, I) is apair (A, B) where $A \subset G$, $B \subset M$ with properties $A' = B$ and $B' = A$
- For two concepts (A_1, B_1) and (A_2, B_2) define order $(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subset A_2 \iff B_1 \subset B_2$

Example of Concepts

	leathery peel	sweet	sour	juicy	contains seeds	made from milk	sandwich filling
banana	×	×			×		
goat cheese						×	×
ice cream		×				×	
kiwifruit		×		×	×		
lemon	×		×	×	×		
orange	×	×		×	×		
strawberry		×		×	×		
yoghurt						×	

Figure: Incidence Table of Fruits.

Hesse Diagram of Concepts

	s	t	u	v	w
A		×	×		
B	×			×	
C		×		×	×
D		×	×	×	×

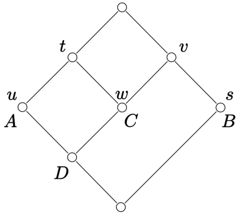


Figure: Cross-table and Hesse Diagram

Context with order might be visualized an Hesse diagram

Concept Lattice

Denote set of all concepts on context (G, M, I) as $\mathcal{B}(G, M, I)$ Then it is easy to proofe that $\mathcal{B}(G, M, I)$ is complete lattice with meet and join operations:

$$\bigvee_{j \in J} (A_j, B_j) = ((\bigcup_{j \in J} A_j)', \bigcap_{j \in J} B_j)$$

and

$$\bigwedge_{j \in J} (A_j, B_j) = (\bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)')$$

With this we can refine context or reduce it, or find appropriate attributes for class of objects for clusterization, classification, knowledge discovery, etc.

Formal Concept Analysis and Decision Trees

If we consider objects as in our dataset and attributes as features, we can define analysis of machine learning model and find similarities with formal concept analysis and machine learning architectures

- In [2] defined several methods for classification using Formal Concept Analysis
- In [3], [4], [5] Formal Concept Analysis is applied for decision tree classification and found different concept classification rules also conversion of lattice to the decision trees

Piecewise Apprixmation Functions

In [13] neural networks are defined as continous piecewise linear functions and representation of each finity pieces continous piecewise linear functions with neural netwoeks established. Which gives isomorphic functor $F : \mathbb{NN} \to \mathbb{CPWL}$ among categories of neural networks and continous piecewise linear functions.
For piecewise approximation defined on X as $\mathcal{F} : X \times P \to Y$ where for each $p \in P$ projection is basis:

$$\mathcal{F}|_{X \times \{p\}} = \varphi : X \to Y$$

we can define $\mathcal{B} \subset 2^X$ as pieces or batches for which approximation happens on one particular basis function.
Define $\mathcal{A} = (\mathcal{A}, \leq)$ partially ordered set and $\alpha : \mathcal{F} \times 2^X \to \mathcal{A}$ which maps each pair $(\varphi, B) = \varphi(B)$ to the single approximation $A \in \mathcal{A}$

- For each $B \in \mathcal{B}$ approximation happens on one particular basis function φ
- $\cup \mathcal{B} = X$

For instance linear regresion or neural network as piecewise universal approximator for l_2 or l_1 with $\mathcal{A} = \mathbb{R}_+$

Spetwise Approximation

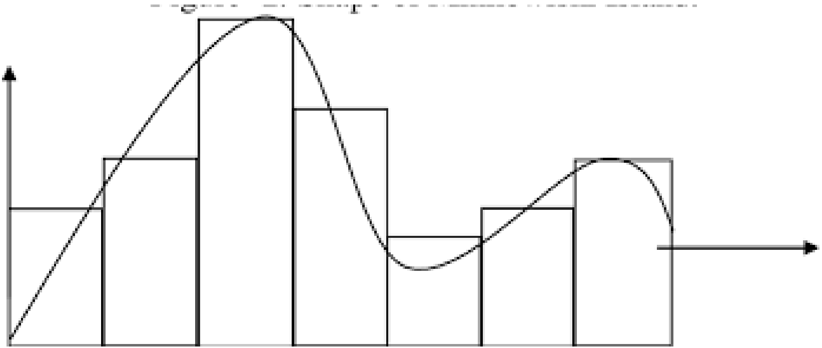
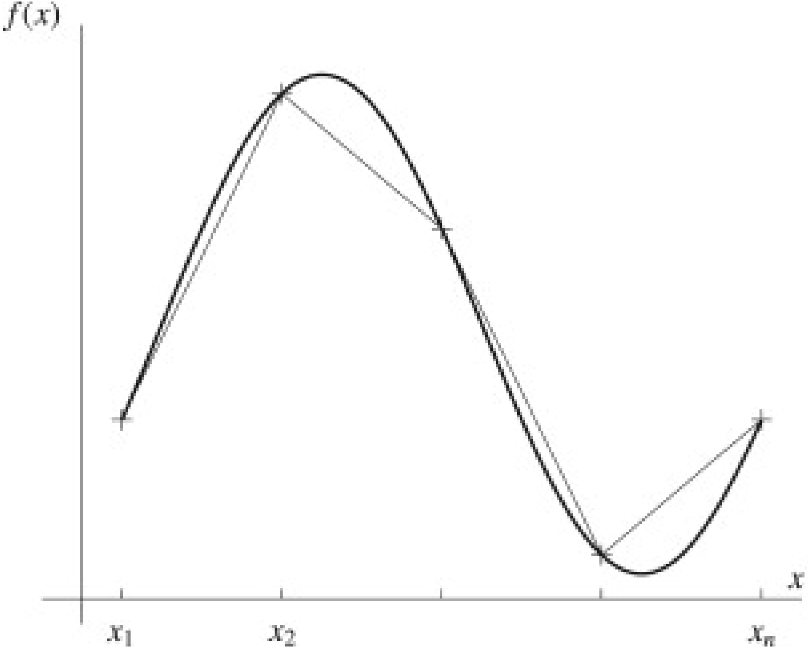


Figure: Step by step approximation (MLP).

Priecwise Linear Approximation



Sigmoid Approximation

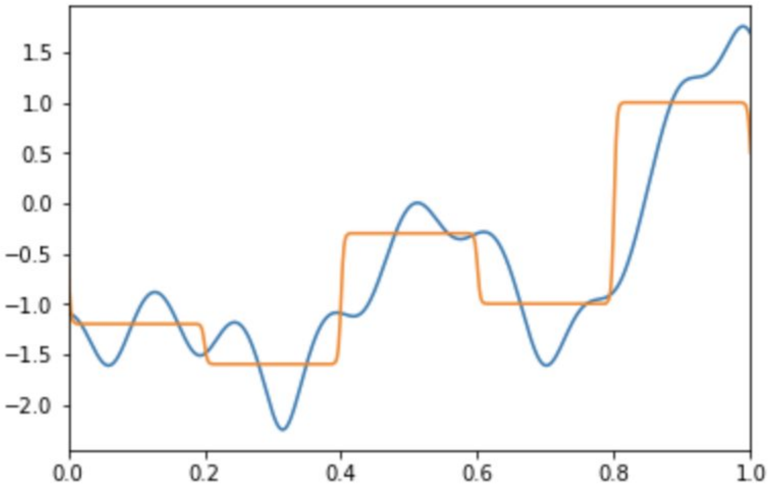


Figure: Piecewise continous nonlinear approximation (Sigmoid).

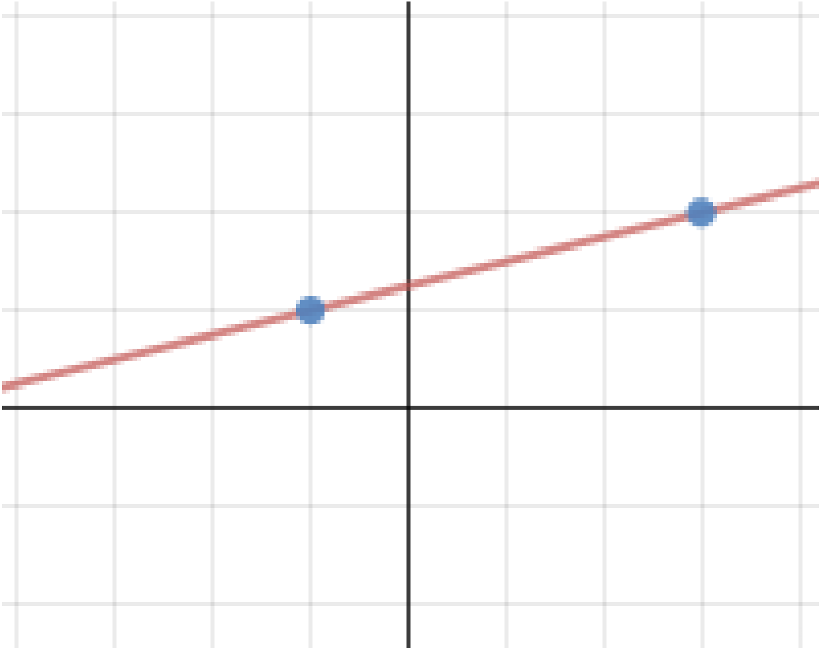
Piecewise linear / continous piecewise linear function as lattice polinomials

Consider pairs (φ, B) on batches and rder:

- $(\varphi, B) \leq (\varphi, D)$ if $\alpha(\varphi(B)) \leq \alpha(\varphi(D)) \iff B \subset D$
- $\varphi(x) \wedge \psi(y) = \mu(\{x, y\})$ where φ, ψ and μ are basis of $\mathcal{F}$

Of course $\wedge$ operation is defined with respect to $\mathcal{A}$ and it is more categorical limit which is unique up to isomorphism

Data Fitting on Two Items



Data Fitting on Two Items

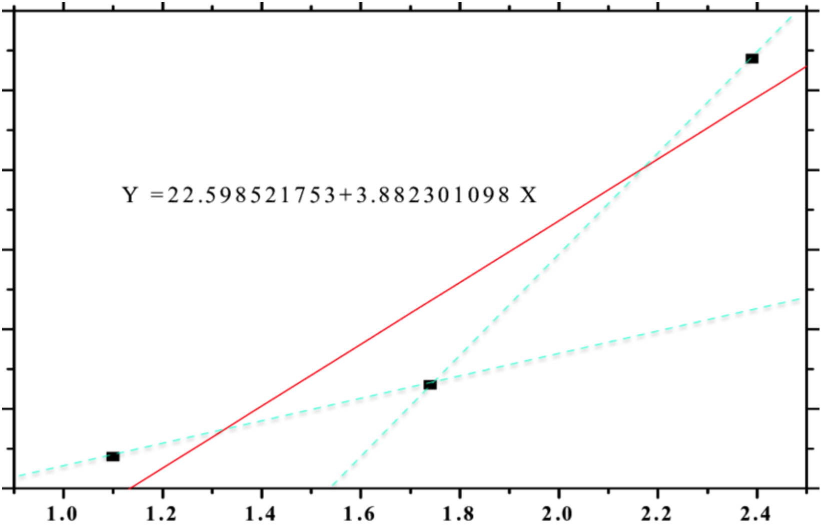


Figure: Approximation on three points.

The Best Approximation Batches

Consider pairs (φ, B) on batches and order:

- $\bigwedge_{\varphi \in \mathcal{F}} (\varphi(B))$
- $\bigvee_{B \in \mathcal{B}} (\varphi, B)$

This gives the best approximation batches $\mathcal{B}$ for $\mathcal{F}$ function and $\mathcal{A}$ approximation.

Lattice Polinomial and Formal Concept

So we can consider neural network, linear regression and multi-linear regressions as:

1. $K(B) = \bigwedge_{\varphi \in \mathcal{F}} \varphi(B)$. fitting on batch
2. $K(\mathcal{A}, \mathcal{F}) = \bigvee_{B \in \mathcal{B}} K(B)$ fitting on functions where $\varphi(x) \vee \psi(x) = \max(\alpha(\varphi(x)), \alpha(\psi(x)))$ as in [6], [7] and [8] but towards approximation.

Hierarchical Lattice Polinomials

If we define refinement of $\mathcal{B}^l$ by the B^{l+1}:

1. For each $B^l \in \mathcal{B}^l$ there exists $B^{l+1} \in \mathcal{B}^{l+1}$ such that $B^{l+1} \subset B^l$
2. For each $B^l \in \mathcal{B}^l$ we have $\bigcup_{B^{l+1} \subset B^l} B^{l+1} = B^l$

We can distinguish layer of the lattice polinomial:

$$K^{l+1}(\mathcal{A}) = \bigvee_{B^{l+1} \in \mathcal{B}^{l+1}} K^l(B^{l+1})$$

Representation Learning

For later layers we can investigate lattice polinomial of this layer:

$$K^{l+1}(\mathcal{A})$$

If we fix the layer l we can consider approximation / representation power of the appropriated polinomial

Further Work 1

Investigate lattice polinomial and learning objective along with natural topology / metric / (quasi) uniform structures on X and Y spaces

- Investigate model capacity by amount of batches and memorization capacity [12]
- Investigation if kernel machines can be represented as a lattice polinomials
- Compare lattice polinomials of different models and analysis of similarities [9], [10]
- Investigate overparametrization [11]
- Self-supervised representation learning and transfer learning as lattice polinomials [27], [28], [29], [30], [31], [32], [33], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46].
- Investigate (uniformly) continous property for each lattice polinomial
- Compact and locally compact structures

Further Work 2

- Completness and completion of X and Y spaces
- Tropical geometry of machine learning and deep learning [16], [17]
- Analysis on generalization capaciry and overfitting

Whole area for research.

Thank You

Questions

Thank You Angain

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