Unit 2: Differentiation

Linear approximation

```
10 * 0.4 + 150
```

154.

$$\Delta f \quad \approx \left. \frac{df}{dx}\right|_{x=a} \cdot \Delta x \qquad \text{for } \Delta x \text{ near } 0$$

$$f(x) \quad \approx f'(a)(x-a) + f(a) \quad \text{for } x \text{ near } a$$

```
D[ √x , x]
```

$$\frac{1}{2 \sqrt{x}}$$

$$\frac{1}{2 \sqrt{100}}$$

$$\frac{1}{20}$$

```
4 * %
```

$$\frac{1}{5}$$

```
f[x_] := x^2.5
f '[4] * -0.03 + f[4]
```

31.4

```
D[ √x , {x, 2}]
```

$$-\frac{1}{4 x^{3/2}}$$

$$-\frac{1}{4 \times 100^{3/2}}$$

$$-\frac{1}{4000}$$

```
Plot[ √x , {x, 50, 150}]
```
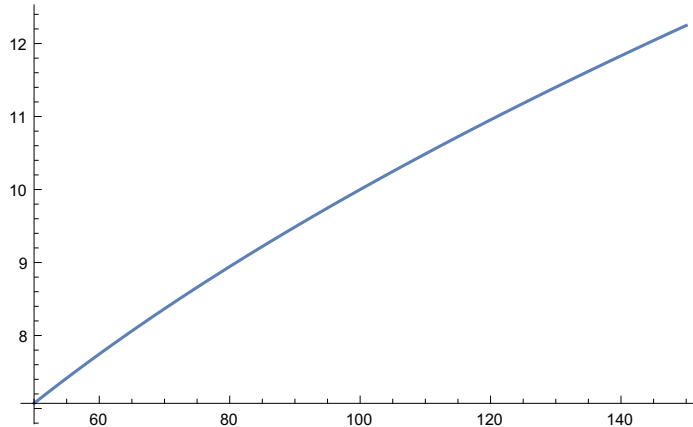


```
v[x_] := 4 π * x³ / 3
v '[r]
v '[10] * -0.03
```

$$4 \pi r^2$$

```
-37.6991
```

```
v '[10]
```

$$400 \pi$$

```
20 / 400
```

$$\frac{1}{20}$$

Product rule

If $h(x) = f(x) g(x)$, then

$$h'(x) = f(x) g'(x) + g(x) f'(x)$$

at all points where the derivatives $f'(x)$ and $g'(x)$ are defined.

```
D[ √x , x] * Cos[x] + √x * -Sin[x]
```

$$\frac{Cos[x]}{2 \sqrt{x}} - \sqrt{x} \, Sin[x]$$

```
100 * 0.4
40.
```

```
100 * - 0.01 + 3 * 0.4
```

```
0.2
```

$$f(x) = x^2 \, Sin[x] \, Cos[x]$$
$$f\,'[x] = 2 \, x * Sin[x] * Cos[x]$$
$$\quad + x^2 * -Cos[x] * Sin[x]$$
$$\quad + x^2 * Sin[x] * -Sin[x]$$

Quotient rule

$$\text{Limit}\left[\frac{f2 * g - f * g2}{t}, \, t \to 0\right]$$

Indeterminate

If $h(x) = \dfrac{f(x)}{g(x)}$ for all x, then

$$h'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

at all points where f and g are differentiable and $g(x) \neq 0$.

$$D\left[\frac{2 + Cos[X]}{x^2 + 1}, \, x\right]$$

$$-\frac{2 \, x \, (2 + Cos[X])}{(1 + x^2)^2}$$

$$\frac{d}{dx} \sin x \quad = \cos x$$

$$\frac{d}{dx} \cos x \quad = -\sin x$$

$$\frac{d}{dx} \tan x \quad = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx} \cot x \quad = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx} \sec x \quad = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

$$\frac{d}{dx} \csc x \quad = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$$

Chain rule

If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x)) g'(x)$$

at all points where the derivatives $f'(g(x))$ and $g'(x)$ are defined.

```
D[120 * t + 100, t]
```

120

```
3 * 0.01 + 9
```

9.03

```
4 * (9.03 - 9) + 5
```

5.12

```
Solve[5.12 == x * (2.01 - 2) + 5, x]
```

{{x → 12.}}

```
3 * 4 * (5 * 2)
```

120

Examples

$f[x\_] := \sqrt{x^3 + 2 x + 1}$

$f1[x\_] := \dfrac{1}{2 \sqrt{x^3 + 2 x + 1}} * (3 x^2 + 2)$

f1[1]

$\dfrac{5}{4}$

$g[x\_] := Sin[x^2 + 2 x]$

$g1[x\_] := -Cos[x^2 + 2 x] * (2 x + 2)$

g1[x]

$(2 + 2 x) Cos[2 x + x^2]$

Exercises

$f[x\_] := Sin\left[\dfrac{x}{x^2 + 1}\right]$

f1[x_] := Cos[u] * u '

$f2[x\_] := Cos\left[\dfrac{x}{x^2 + 1}\right] * \dfrac{(x^2 + 1) - (x * 2 * x)}{(x^2 + 1)^2}$

f2[x] == f '[x]

f2[x]

True

$\left(-\dfrac{2 x^2}{(1 + x^2)^2} + \dfrac{1}{1 + x^2}\right) Cos\left[\dfrac{x}{1 + x^2}\right]$

$k[x\_] := Cos[x] * Sin[x]^4$

$u := Sin[x]^4$

k1[x_] := -Sin[x] * u + Cos[x] * u '

inn := Sin[x]

$auss := x^4$

$k2[x\_] := -Sin[x] * Sin[x]^4 + Cos[x] * (4 * (Sin[x])^3 * Cos[x])$

k '[x] == k2[x]

k2[x]

True

$4 Cos[x]^2 Sin[x]^3 - Sin[x]^5$

Review

```
g '[f[3]] * f '[3];
g '[5] * 3;
4 * 3
```

```
f '[g[3]] * g ' (3);
f '[4] * 5;
-3 * 5
```

12

-15

```
p[i_] := 10 * i²
p '[6]
```

120

Implicit Differentiation

$$Solve\left[x^2 + y^2 == 25, \{x, y\}\right]$$

$$\left\{\left\{y \to -\sqrt{25 - x^2}\right\}, \left\{y \to \sqrt{25 - x^2}\right\}\right\}$$

$$D\left[\sqrt{25 - x^2}, x\right]$$

$$D\left[-\sqrt{25 - x^2}, x\right]$$

$$-\frac{x}{\sqrt{25 - x^2}}$$

$$\frac{x}{\sqrt{25 - x^2}}$$

```
x = -3;
```

$$-\frac{x}{\sqrt{25 - x^2}}$$

$$\frac{3}{4}$$

$$eclipse := x^4 - 3 x^2 + y^4 + y^2 + 2 x^2 y^2 == 0$$

$$D\left[2 y + Cos[x] + x^2 y + Sin[2 y] == 3, x\right]$$

$$2 x y - Sin[x] == 0$$

$y^3 + x^3 == 3 xy;$

$3 y^2 * (dy / dx) + 3 x^2 == 3 * y + (dy / dx) * x * 3;$

$(3 y^2 - 3 x) * (dy / dx) == 3 y - 3 x^2;$

$$\frac{dy}{dx} == \frac{3 y - 3 x^2}{3 y^2 - 3 x};$$

Practice

$y^2 == x;$

$$\frac{dy}{dx} == \frac{1}{2 y}$$

$$\frac{dy}{dx} == \frac{1}{2 y}$$

$x^2 + 4 x * y == 2 y^2 + 5;$

$$2 x + 4 y + 4 x * \frac{dy}{dx} == 4 y * \frac{dy}{dx};$$

$$2 x + 4 y == (4 y - 4 x) * \frac{dy}{dx};$$

$$\frac{2 x + 4 y}{4 y - 4 x} == \frac{dy}{dx}$$

$$\frac{2 x + 4 y}{-4 x + 4 y} == \frac{dy}{dx}$$

$u == Sin[y^2 + u];$

$$\frac{du}{dy} == Cos[y^2 + u] * \left(2 y + \frac{du}{dy}\right);$$

$$\frac{du}{dy} == Cos[y^2 + u] * 2 y + Cos[y^2 + u] * \frac{du}{dy};$$

$$(1 - Cos[y^2 + u]) * \frac{du}{dy} == Cos[y^2 + u] * 2 y;$$

$$\frac{du}{dy} == \frac{Cos[y^2 + u] * 2 y}{1 - Cos[y^2 + u]};$$

$$Simplify\left[\frac{Cos[y^2 + u] * 2 y}{1 - Cos[y^2 + u]}\right]$$

$$-\frac{2 y Cos[u + y^2]}{-1 + Cos[u + y^2]}$$

```
w^2 v^3 == w^3 v^2;

2 w * dw/dv * v^3 + w^2 * 3 v^2 == 3 w^2 * dw/dv * v^2 + w^3 * 2 v;

(2 w * v^3 - 3 w^2 * v^2) * dw/dv == w^3 * 2 v - w^2 * 3 v^2;

dw/dv == Simplify[ (w^3 * 2 v - w^2 * 3 v^2)/(2 w * v^3 - 3 w^2 * v^2) ]

dw/dv == (w (-3 v + 2 w))/(v (2 v - 3 w))

x * y == y^3;

y + dy/dx * x == 3 y^2 * dy/dx;

(3 y^2 - x) * dy/dx == y;

dy/dx == Simplify[ y/(3 y^2 - x) ]

dy/dx == - y/(x - 3 y^2)

dy/dx == -x/y;

dy/dx == (x * dy/dx - y)/y^2;

dy/dx == x * y^-2 * dy/dx - y^-1;

(1 - x * y^-2) * dy/dx == -y^-1;

dy/dx == Simplify[ -y^-1/(1 - x * y^-2) ]

dy/dx == y/(x - y^2)
```

Inverse functions

If g is a (full or partial) inverse of a function f, then

$$g'(x) = \frac{1}{f'(g(x))}$$

at all x where $f'(g(x))$ exists and is non-zero.

$\sqrt[3]{-2197}$

$13 (-1)^{1/3}$

```
Solve[6 x - 16 == 4, x]
```

$\left\{\left\{x \rightarrow \frac{10}{3}\right\}\right\}$

```
h[x_] := 3 - 2 / x
h[4]
y == 3 - 2 / x;
y - 3 == -2 / x;
1 / (y - 3) == x / -2;
(1 / (y - 3)) * -2 == x;
Simplify [(1 / (x - 3)) * -2]
```

$\text{h1[x\_] := } -\frac{2}{-3 + x}$

```
h1[5 / 2]
```

$\frac{5}{2}$

$-\frac{2}{-3 + x}$

4

```
Plot[ArcSin[x], {x, -π / 2, π / 2}]
Plot[ArcCos[x], {x, 0, π}]
Plot[ArcTan[x], {x, -π / 2, π / 2}]
ArcCos[-1]
ArcSin[-1]
ArcTan[-1]
Sin[ArcTan[3 / 4]]
```
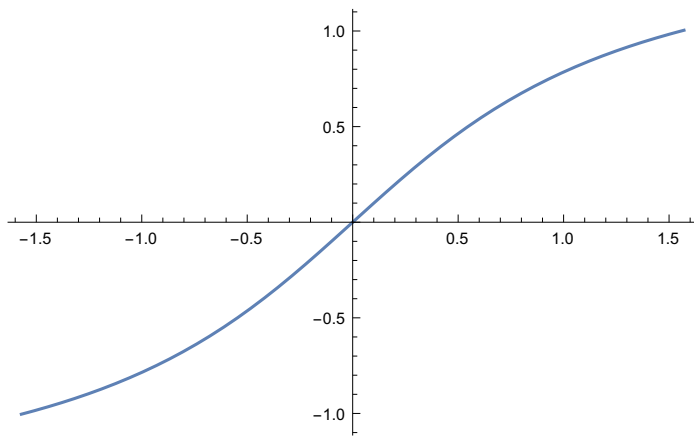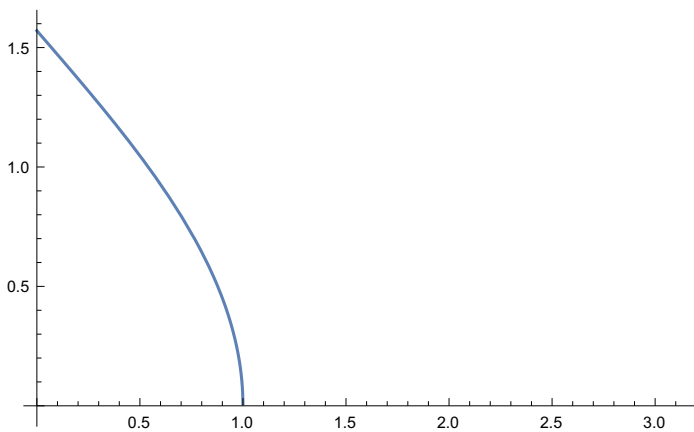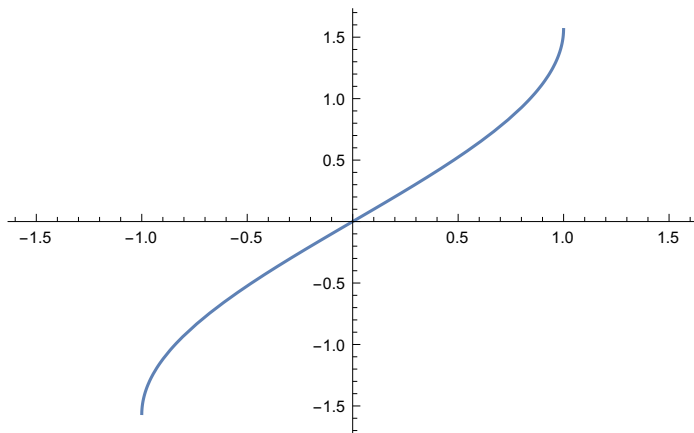
π

$-\dfrac{\pi}{2}$

$-\dfrac{\pi}{4}$

$\dfrac{3}{5}$

$y == \dfrac{5}{2} x - 3;$

$\text{Solve}\left[y == \dfrac{5}{2} x - 3, x\right]$

$\left\{\left\{x \to \dfrac{2(3+y)}{5}\right\}\right\}$

$h'[2] == \dfrac{1}{g'[h[2]]};$

$h'[2] == \dfrac{1}{g'[3/2]};$

$h'[2] == \dfrac{1}{4}$

$h'[2] == \dfrac{1}{4}$

$f[x\_] := -2 x^3 - 7 x + 5;$

$g[x\_] := \text{InverseFunction}[f][x]$

$g[5]$

$g'[5]$

··· **InverseFunction** : Inverse functions are being used . Values may be lost for multivalued inverses .

0

$-\dfrac{1}{7}$

$-\dfrac{1}{280}$

Derivatives of inverse functions using implicit differentiation

This is another way of finding the derivative of $\theta = \arcsin x$. The relationship between θ and x is given by

$$\sin \theta = x.$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\sin \theta = \frac{d}{dx}x$$

$$\cos \theta \frac{d\theta}{dx} = 1$$

$$\frac{d\theta}{dx} = \frac{1}{\cos \theta}$$

We know that

$$\cos^2\theta + x^2 = \cos^2\theta + \sin^2\theta = 1,$$

so $\cos\theta = \pm\sqrt{1 - x^2}$. Since θ must lie in $[-\pi/2, \pi/2]$, $\cos\theta$ is positive. Hence

$$\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1 - x^2}}.$$

We can similarly find the derivative of $\theta = \arccos x$. The relationship between θ and x is given by

$$\cos\theta = x.$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\cos\theta = \frac{d}{dx}x$$

$$-\sin\theta\frac{d\theta}{dx} = 1$$

$$\frac{d\theta}{dx} = -\frac{1}{\sin\theta}$$

We know that

$$\sin^2\theta + x^2 = \sin^2\theta + \cos^2\theta = 1,$$

so $\sin\theta = \pm\sqrt{1 - x^2}$. Since θ must lie in $[0, \pi]$, $\sin\theta$ is positive. Hence

$$\frac{d\theta}{dx} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1 - x^2}}.$$

D[ArcTan[x], x]

$$\frac{1}{1 + x^2}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1 + x^2}$$

```
InverseFunction [ArcTan[3 #] &][x]
```

$$\frac{\text{Tan}[x]}{3} \quad \text{if} \quad \boxed{condition} \;+$$

```
D[ArcTan[3 * #] , #] /. # → -1
```

$$\frac{3}{10}$$

```
D[#² * ArcCos[#], #] /. # → 1/2
```

$$-\frac{1}{2\sqrt{3}} + \frac{\pi}{3}$$
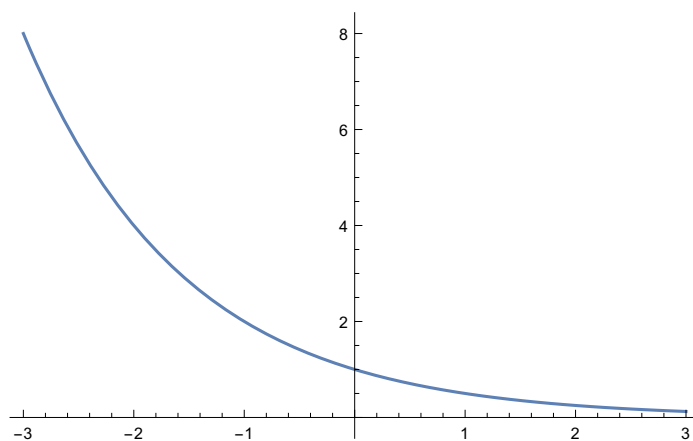
```
Re[N[ArcSin[2] + ArcCos[2]]]
```

```
1.5708
```

Exponential functions

Let a be a positive real number.

- $a^0 = 1$

- $a^1 = a$

- $a^m a^n = a^{m+n}$

- $(a^m)^n = a^{mn}$

- $a^{m/n} = \sqrt[n]{a^m}$

```
Plot[(1/2)ˣ, {x, -3, 3}]
```

The derivative of the exponential function is

$$\frac{d}{dx}a^x = M(a)\,a^x,$$

where the mystery number $M(a)$ is the slope of the tangent line at zero:

$$M(a) = \frac{d}{dx}a^x\bigg|_{x=0} = \lim_{\Delta x \to 0}\frac{a^{\Delta x} - 1}{\Delta x}.$$

```
e^4.5 * 0.05 - e^4.5;
90 * 0.05 + 90
94.5
```

```
0.5 a (e^x/a + e^-x/a) /. a → 2
Simplify[D[%, x]]
1. (e^-x/2 + e^x/2)
```

The base e is the unique real number so that $M(e) = \dfrac{d}{dx}e^x\bigg|_{x=0} = 1$. Then

$$\frac{d}{dx}e^x = e^x.$$

Logarithms

$\log_{10}(x)$ is the inverse function of 10^x.

The natural log, denoted $\ln(x)$, is the inverse function of e^x.

- $\ln e^x = x$

- $e^{\ln x} = x$

- $\ln(ab) = \ln(a) + \ln(b)$

- $\ln(a^b) = b\ln(a)$

Log[10, 1000]

Log[10, 0.000001]

3

-6.

Log[2]

Log[2]

N[Log[20]]

2.99573

N[Log[10^{100}]]

230.259

Plot[{e^x, Log[x]}, {x, -5, 5}]
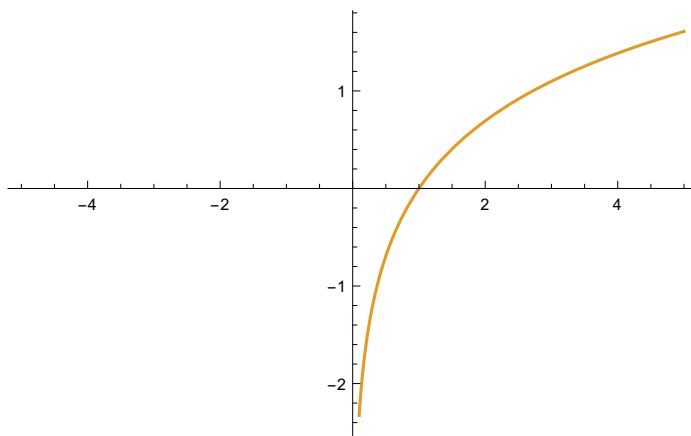
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

D[Log[3 * x], x]

$$\frac{1}{x}$$

For any positive constant a,

$$\frac{d}{dx}a^x = a^x \ln a.$$

```
D[Log[10, t], t]
```

$$\frac{1}{t \; Log[10]}$$

```
D[Log[ √x ], x];
D[Log[x^(1/2)], x];
D[1 / 2 * Log[x], x];
```

$$0 * Log[x] + \frac{1}{x} * \frac{1}{2};$$

$$\frac{1}{2\,x} \; /. \; x \to 1/4$$

```
2
```

```
D[x^Log[x], x]
```

$$2\,x^{-1+Log[x]} \; Log[x]$$

```
Limit[(1 + 1 / n)^n, n → ∞]
```

e

Differentiation methods

Method 1: Change to base e

To calculate $\left(f(x)^{g(x)} \right)'$,

- first change base $f(x)^{g(x)} = e^{g(x)\ln f(x)}$,

- then differentiate using the chain rule, and the derivatives of the exponential and natural logarithm functions.

Method 2: Logarithmic differentiation

To calculate u',

- first calculate $(\ln(u))'$,

- then use the formula

$$\frac{u'}{u} = (\ln(u))'$$

and solve for u'.

Homework 2 Part A

```
Limit[(Tan[x] - 1)/(x - π/4), x → π/4]
```

2

```
f[x_] := √(a + b * x)
```

```
f'[0] * (x) + f[0]
```

$$\sqrt{a} + \frac{b\,x}{2\,\sqrt{a}}$$

```
f2[x_] := 1/(a + b * x)
f2'[0] * x + f2[0]
```

$$\frac{1}{a} - \frac{b\,x}{a^2}$$

```
D[(x² + 2)², x];
2 (x² + 2) * 2 x
```

$$4\,x\,(2 + x^2)$$

```
V == 1/3 π * r² * h;
```

```
Dt[0 == 1/3 π * r² * h, h];
Solve[%, Dt[r, h]]
```

$$\left\{\left\{Dt[r, h] \to -\frac{r}{2\,h}\right\}\right\}$$

```
Dt[P * V¹¹¹ == 2 * 333 * 2, V];
Solve[%, Dt[P, V]]
```

$$\left\{\left\{Dt[P, V] \to -\frac{111\,P}{V}\right\}\right\}$$

```
Dt[c² == a² + b² - 2 * a * b * Cos[t], b, Constants → {t, c}]
Solve[%, Dt[a, b, Constants → {c, t}]]
```

0 == 2 b - 2 a Cos[t] + 2 a Dt[a, b, Constants → {c, t}] - 2 b Cos[t] Dt[a, b, Constants → {c, t}]

$$\left\{\left\{Dt[a, b, Constants → \{c, t\}] → \frac{b - a\,Cos[t]}{-a + b\,Cos[t]}\right\}\right\}$$

$D[x * e^x, x]$

$D[e^{-x^2}, x]$

$D[Log[x^2], x]$

$e^x + e^x \, x$

$-2 \, e^{-x^2} \, x$

$\dfrac{2}{x}$

$cosh[x\_] := \dfrac{e^x + e^{-x}}{2}$

$cosh\,'[x]$

$cosh\,''[x]$

$cosh\,'''[x]$

$\dfrac{1}{2} \, (-e^{-x} + e^x)$

$\dfrac{1}{2} \, (e^{-x} + e^x)$

$\dfrac{1}{2} \, (-e^{-x} + e^x)$

$D[Sin[3\,x]^2, x]$

$6 \, Cos[3\,x] \, Sin[3\,x]$

$D[Log[Cos[2\,x]], x]$

$-2 \, Tan[2\,x]$

$D[Tan[3 * x]^2, x]$

$6 \, Sec[3\,x]^2 \, Tan[3\,x]$

$D[Sin[ArcCos[x]], x]$

$-\dfrac{x}{\sqrt{1-x^2}}$

$V[t\_] := 2 * (10 - t)^{5/2}$

$N\left[\dfrac{V[5] - V[0]}{5 - 0}\right]$

$N[V\,'[301 / 60]]$

$N[V\,'[5]]$

-104.13

-55.6224

-55.9017

$\text{Simplify}\left[D\left[\left(\frac{48}{\pi} V\right)^{1/3}, V\right]\right]$

$$\frac{2 \left(\frac{2}{\pi}\right)^{1/3}}{3^{2/3} V^{2/3}}$$