Unit 2: Differentiation

Linear approximation

```
10 * 0.4 + 150
```

154.

$$\Delta f \approx \left.\frac{df}{dx}\right|_{x=a} \cdot \Delta x \qquad \text{for } \Delta x \text{ near } 0$$

$$f(x) \approx f'(a)(x-a) + f(a) \quad \text{for } x \text{ near } a$$

```
D[√x, x]
```

$$\frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{100}}$$

$$\frac{1}{20}$$

```
4 * %
```

$$\frac{1}{5}$$

```
f[x_] := x^2.5
f'[4] * -0.03 + f[4]
```

31.4

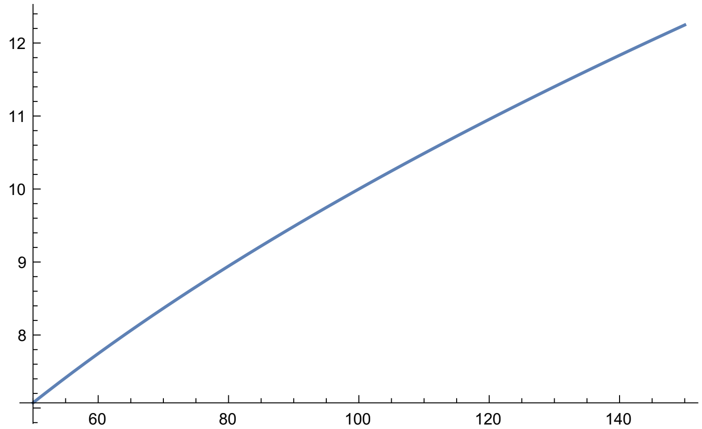
```
D[√x, {x, 2}]
```

$$-\frac{1}{4\,x^{3/2}}$$

$$-\frac{1}{4 \times 100^{3/2}}$$

$$-\frac{1}{4000}$$

```
Plot[√x , {x, 50, 150}]
```




```
v[x_] := 4 π * x^3 / 3
v'[r]
v'[10] * -0.03
```

$4 \pi r^2$

-37.6991

```
v'[10]
```

400π

```
20 / 400
```

$\frac{1}{20}$

Product rule

If $h(x) = f(x)\,g(x)$, then

$$h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$$

at all points where the derivatives $f'(x)$ and $g'(x)$ are defined.

```
D[√x , x] * Cos[x] + √x * -Sin[x]
```

$\frac{\text{Cos}[x]}{2\sqrt{x}} - \sqrt{x}\ \text{Sin}[x]$

```
100 * 0.4
```

40.

```
100 * -0.01 + 3 * 0.4
```

0.2

$f(x) = x^2 \, \mathrm{Sin}[x] \, \mathrm{Cos}[x]$

$f'[x] = 2\,x * \mathrm{Sin}[x] * \mathrm{Cos}[x]$

$\quad + x^2 * -\mathrm{Cos}[x] * \mathrm{Sin}[x]$

$\quad + x^2 * \mathrm{Sin}[x] * -\mathrm{Sin}[x]$

Quotient rule

$$\mathrm{Limit}\left[\frac{f2 * g - f * g2}{t}, t \to 0\right]$$

Indeterminate

If $h(x) = \dfrac{f(x)}{g(x)}$ for all x, then

$$h'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

at all points where f and g are differentiable and $g(x) \neq 0$.

$$D\left[\frac{2 + \mathrm{Cos}[X]}{x^2 + 1}, x\right]$$

$$-\frac{2\,x\,(2 + \mathrm{Cos}[X])}{(1 + x^2)^2}$$

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}\cot x &= -\frac{1}{\sin^2 x} = -\csc^2 x \\
\frac{d}{dx}\sec x &= \frac{\sin x}{\cos^2 x} = \sec x \tan x \\
\frac{d}{dx}\csc x &= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x
\end{aligned}$$

Chain rule

If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))\, g'(x)$$

at all points where the derivatives $f'(g(x))$ and $g'(x)$ are defined.

```
D[120 * t + 100 , t]
```

```
120
```

```
3 * 0.01 + 9
```

```
9.03
```

```
4 * (9.03 - 9) + 5
```

```
5.12
```

```
Solve[5.12 == x * (2.01 - 2) + 5, x]
```

```
{{x → 12.}}
```

```
3 * 4 * (5 * 2)
```

```
120
```

Examples

```
f[x_] := Sqrt[x^3 + 2 x + 1]
```

$$f1[x\_] := \frac{1}{2\sqrt{x^3 + 2x + 1}} * (3x^2 + 2)$$

```
f1[1]
```

$$\frac{5}{4}$$

```
g[x_] := Sin[x^2 + 2 x]
g1[x_] := -Cos[x^2 + 2 x] * (2 x + 2)
g1[x]
```

(2 + 2 x) Cos[2 x + x^2]

Exercises

$$f[x\_] := \mathrm{Sin}\left[\frac{x}{x^2+1}\right]$$

```
f1[x_] := Cos[u] * u '
```

$$f2[x\_] := \mathrm{Cos}\left[\frac{x}{x^2+1}\right] * \frac{(x^2+1) - (x*2*x)}{(x^2+1)^2}$$

```
f2[x] == f '[x]
f2[x]
```

True

$$\left(-\frac{2x^2}{(1+x^2)^2} + \frac{1}{1+x^2}\right) \mathrm{Cos}\left[\frac{x}{1+x^2}\right]$$

```
k[x_] := Cos[x] * Sin[x]^4
u := Sin[x]^4
k1[x_] := -Sin[x] * u + Cos[x] * u '
inn := Sin[x]
auss := x^4
k2[x_] := -Sin[x] * Sin[x]^4 + Cos[x] * (4 * (Sin[x])^3 * Cos[x])
k '[x] == k2[x]
k2[x]
```

True

$4\,\mathrm{Cos}[x]^2\,\mathrm{Sin}[x]^3 - \mathrm{Sin}[x]^5$

Review

```
g'[f[3]] * f'[3];
g'[5] * 3;
4 * 3

f'[g[3]] * g'(3);
f'[4] * 5;
-3 * 5
```

12

-15

```
p[i_] := 10 * i^2
p'[6]
```

120

Implicit Differentiation

```
Solve[x^2 + y^2 == 25, {x, y}]
```

$\left\{\left\{y \to -\sqrt{25-x^2}\right\}, \left\{y \to \sqrt{25-x^2}\right\}\right\}$

```
D[Sqrt[25 - x^2], x]
D[-Sqrt[25 - x^2], x]
```

$-\frac{x}{\sqrt{25-x^2}}$

$\frac{x}{\sqrt{25-x^2}}$

```
x = -3;
-x/Sqrt[25 - x^2]
```

$\frac{3}{4}$

```
eclipse := x^4 - 3 x^2 + y^4 + y^2 + 2 x^2 y^2 == 0
D[2 y + Cos[x] + x^2 y + Sin[2 y] == 3, x]
```

2 x y - Sin[x] == 0

$$y^3 + x^3 == 3\,xy;$$
$$3\,y^2 * (dy / dx) + 3\,x^2 == 3 * y + (dy / dx) * x * 3;$$
$$(3\,y^2 - 3\,x) * (dy / dx) == 3\,y - 3\,x^2;$$
$$\frac{dy}{dx} == \frac{3\,y - 3\,x^2}{3\,y^2 - 3\,x};$$

Practice

$$y^2 == x;$$
$$\frac{dy}{dx} == \frac{1}{2\,y}$$
$$\frac{dy}{dx} == \frac{1}{2\,y}$$

$$x^2 + 4\,x * y == 2\,y^2 + 5;$$
$$2\,x + 4\,y + 4\,x * \frac{dy}{dx} == 4\,y * \frac{dy}{dx};$$
$$2\,x + 4\,y == (4\,y - 4\,x) * \frac{dy}{dx};$$
$$\frac{2\,x + 4\,y}{4\,y - 4\,x} == \frac{dy}{dx}$$

$$\frac{2\,x + 4\,y}{-4\,x + 4\,y} == \frac{dy}{dx}$$

$$u == \mathrm{Sin}[y^2 + u];$$
$$\frac{du}{dy} == \mathrm{Cos}[y^2 + u] * \left(2\,y + \frac{du}{dy}\right);$$
$$\frac{du}{dy} == \mathrm{Cos}[y^2 + u] * 2\,y + \mathrm{Cos}[y^2 + u] * \frac{du}{dy};$$
$$(1 - \mathrm{Cos}[y^2 + u]) * \frac{du}{dy} == \mathrm{Cos}[y^2 + u] * 2\,y;$$
$$\frac{du}{dy} == \frac{\mathrm{Cos}[y^2 + u] * 2\,y}{1 - \mathrm{Cos}[y^2 + u]};$$
$$\mathrm{Simplify}\left[\frac{\mathrm{Cos}[y^2 + u] * 2\,y}{1 - \mathrm{Cos}[y^2 + u]}\right]$$
$$-\frac{2\,y\,\mathrm{Cos}[u + y^2]}{-1 + \mathrm{Cos}[u + y^2]}$$

$$w^2 v^3 == w^3 v^2;$$

$$2 w * \frac{dw}{dv} * v^3 + w^2 * 3 v^2 == 3 w^2 * \frac{dw}{dv} * v^2 + w^3 * 2 v;$$

$$(2 w * v^3 - 3 w^2 * v^2) * \frac{dw}{dv} == w^3 * 2 v - w^2 * 3 v^2;$$

$$\frac{dw}{dv} == \text{Simplify}\left[\frac{w^3 * 2 v - w^2 * 3 v^2}{2 w * v^3 - 3 w^2 * v^2}\right]$$

$$\frac{dw}{dv} == \frac{w (-3 v + 2 w)}{v (2 v - 3 w)}$$

$$x * y == y^3;$$

$$y + \frac{dy}{dx} * x == 3 y^2 * \frac{dy}{dx};$$

$$(3 y^2 - x) * \frac{dy}{dx} == y;$$

$$\frac{dy}{dx} == \text{Simplify}\left[\frac{y}{3 y^2 - x}\right]$$

$$\frac{dy}{dx} == -\frac{y}{x - 3 y^2}$$

$$\frac{dy}{dx} == \frac{-x}{y};$$

$$\frac{dy}{dx} == \frac{x * \frac{dy}{dx} - y}{y^2};$$

$$\frac{dy}{dx} == x * y^{-2} * \frac{dy}{dx} - y^{-1};$$

$$(1 - x * y^{-2}) * \frac{dy}{dx} == -y^{-1};$$

$$\frac{dy}{dx} == \text{Simplify}\left[\frac{-y^{-1}}{1 - x * y^{-2}}\right]$$

$$\frac{dy}{dx} == \frac{y}{x - y^2}$$

Inverse functions

If g is a (full or partial) inverse of a function f, then

$$g'(x) = \frac{1}{f'(g(x))}$$

at all x where $f'(g(x))$ exists and is non-zero.

```
∛-2197
```

$13(-1)^{1/3}$

```
Solve[6 x - 16 == 4, x]
```

$\left\{\left\{x \to \frac{10}{3}\right\}\right\}$

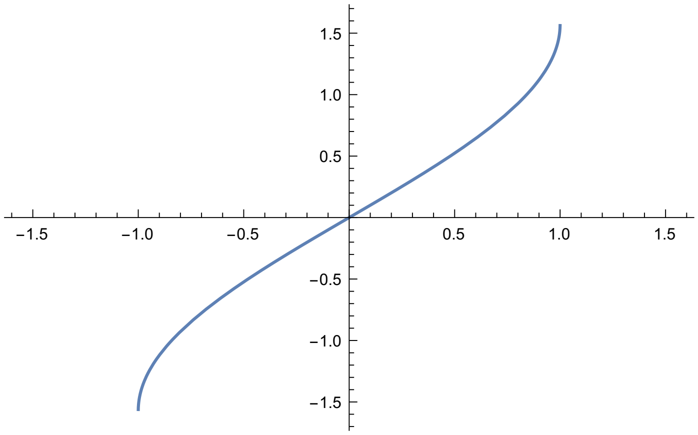
```
h[x_] := 3 - 2 / x
h[4]
y == 3 - 2 / x;
y - 3 == -2 / x;
1 / (y - 3) == x / -2;
(1 / (y - 3)) * -2 == x;
Simplify[(1 / (x - 3)) * -2]
h1[x_] := -2/(-3 + x)
h1[5 / 2]
```

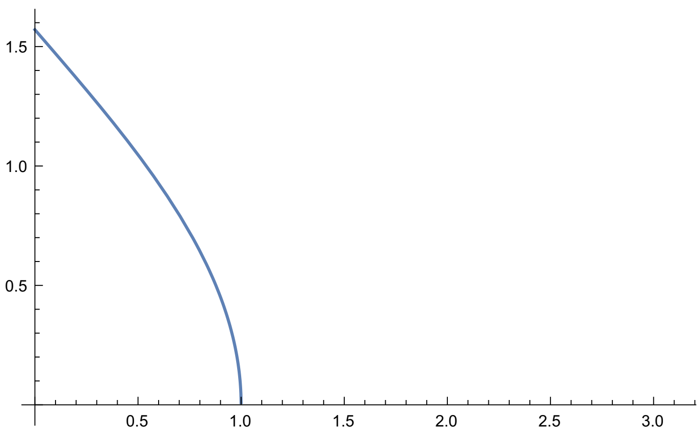
$\frac{5}{2}$

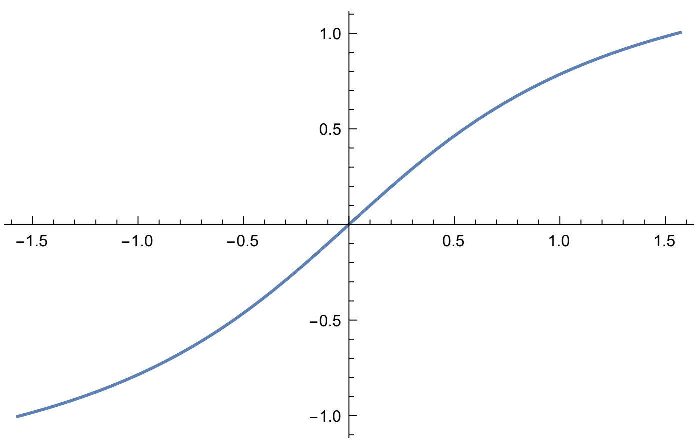
$-\frac{2}{-3+x}$

4

```
Plot[ArcSin[x], {x, -π / 2, π / 2}]
Plot[ArcCos[x], {x, 0, π}]
Plot[ArcTan[x], {x, -π / 2, π / 2}]
ArcCos[-1]
ArcSin[-1]
ArcTan[-1]
Sin[ArcTan[3 / 4]]
```







π

$-\frac{\pi}{2}$

$-\frac{\pi}{4}$

$\frac{3}{5}$

```
y == 5/2 x - 3;
Solve[y == 5/2 x - 3, x]
```

$\{\{x \to \frac{2(3+y)}{5}\}\}$

```
h'[2] == 1/g'[h[2]];
h'[2] == 1/g'[3/2];
h'[2] == 1/4
```

$h'[2] == \frac{1}{4}$

```
f[x_] := -2 x^3 - 7 x + 5;
g[x_] := InverseFunction [f][x]
g[5]
g'[5]
```

InverseFunction : Inverse functions are being used. Values may be lost for multivalued inverses.

0

$-\frac{1}{7}$

$-\frac{1}{280}$

Derivatives of inverse functions using implicit differentiation

This is another way of finding the derivative of $\theta = \arcsin x$. The relationship between θ and x is given by

$$\sin\theta = x.$$

Differentiating both sides with respect to x yields:

$$\begin{aligned}
\frac{d}{dx}\sin\theta &= \frac{d}{dx}x \\
\cos\theta\frac{d\theta}{dx} &= 1 \\
\frac{d\theta}{dx} &= \frac{1}{\cos\theta}
\end{aligned}$$

We know that

$$\cos^2\theta + x^2 = \cos^2\theta + \sin^2\theta = 1,$$

so $\cos\theta = \pm\sqrt{1-x^2}$. Since θ must lie in $[-\pi/2, \pi/2]$, $\cos\theta$ is positive. Hence

$$\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}.$$

We can similarly find the derivative of $\theta = \arccos x$. The relationship between θ and x is given by

$$\cos\theta = x.$$

Differentiating both sides with respect to x yields:

$$\begin{aligned} \frac{d}{dx}\cos\theta &= \frac{d}{dx}x \\ -\sin\theta\frac{d\theta}{dx} &= 1 \\ \frac{d\theta}{dx} &= -\frac{1}{\sin\theta} \end{aligned}$$

We know that

$$\sin^2\theta + x^2 = \sin^2\theta + \cos^2\theta = 1,$$

so $\sin\theta = \pm\sqrt{1-x^2}$. Since θ must lie in $[0, \pi]$, $\sin\theta$ is positive. Hence

$$\frac{d\theta}{dx} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-x^2}}.$$

```
D[ArcTan[x], x]
```

$$\frac{1}{1 + x^2}$$

$$\begin{aligned} \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\arccos x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\arctan x &= \frac{1}{1+x^2} \end{aligned}$$

```
InverseFunction [ArcTan[3 #] &][x]
```

$\boxed{\frac{\text{Tan}[x]}{3} \quad \text{if} \quad \textit{condition}}$

```
D[ArcTan[3 * #] , #] /. # → -1
```

$\frac{3}{10}$

```
D[#^2 * ArcCos[#], #] /. # → 1 / 2
```

$-\frac{1}{2\sqrt{3}} + \frac{\pi}{3}$

```
Re[N[ArcSin[2] + ArcCos[2]]]
```

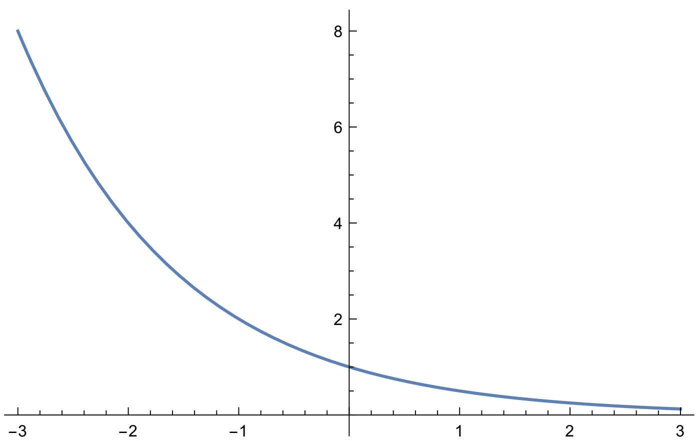
1.5708

Exponential functions

Let a be a positive real number.

- $a^0 = 1$
- $a^1 = a$
- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $a^{m/n} = \sqrt[n]{a^m}$

```
Plot[(1 / 2)^x, {x, -3, 3}]
```



The derivative of the exponential function is

$$\frac{d}{dx}a^x = M(a)\,a^x,$$

where the mystery number $M(a)$ is the slope of the tangent line at zero:

$$M(a) = \frac{d}{dx}a^x\bigg|_{x=0} = \lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}.$$

```
e^4.5 * 0.05 - e^4.5;
90 * 0.05 + 90
```

```
94.5
```

```
0.5 a (ⅇ^(x/a) + ⅇ^(-x/a)) /. a → 2
Simplify[D[%, x]]
```

```
1. (ⅇ^(-x/2) + ⅇ^(x/2))
```

The base e is the unique real number so that $M(e) = \frac{d}{dx}e^x\Big|_{x=0} = 1$. Then

$$\frac{d}{dx}e^x = e^x.$$

Logarithms

$\log_{10}(x)$ is the inverse function of 10^x.

The natural log, denoted $\ln(x)$, is the inverse function of e^x.

- $\ln e^x = x$
- $e^{\ln x} = x$
- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a^b) = b\ln(a)$

```
Log[10, 1000]
Log[10, 0.000001]
```

```
3
```

```
-6.
```

```
Log[2]
```

```
Log[2]
```

```
N[Log[20]]
```

```
2.99573
```