Unit 2: Differentiation

Linear approximation

In[ ]:=
```
10 * 0.4 + 150
```
Out[ ]=
```
154.
```

In[ ]:=

$$\Delta f \approx \left.\frac{df}{dx}\right|_{x=a} \cdot \Delta x \qquad \text{for } \Delta x \text{ near } 0$$

$$f(x) \approx f'(a)(x-a) + f(a) \quad \text{for } x \text{ near } a$$

Out[ ]=

$$\Delta f \approx \left.\frac{df}{dx}\right|_{x=a} \cdot \Delta x \qquad \text{for } \Delta x \text{ near } 0$$

$$f(x) \approx f'(a)(x-a) + f(a) \quad \text{for } x \text{ near } a$$

In[ ]:= $D\left[\sqrt{x}, x\right]$

Out[ ]= $\frac{1}{2\sqrt{x}}$

In[ ]:= $\frac{1}{2\sqrt{100}}$

Out[ ]= $\frac{1}{20}$

In[ ]:=
```
4 * %
```
Out[ ]= $\frac{1}{5}$

In[ ]:=
```
f[x_] := x^2.5
f'[4] * -0.03 + f[4]
```
Out[ ]=
```
31.4
```

In[ ]:= `D[√x, {x, 2}]`

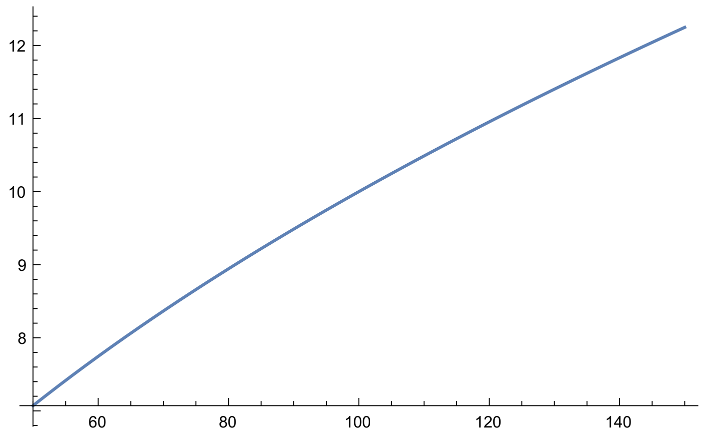
Out[ ]= $-\frac{1}{4\,x^{3/2}}$

In[ ]:= $-\frac{1}{4 \times 100^{3/2}}$

Out[ ]= $-\frac{1}{4000}$

In[ ]:= `Plot[√x, {x, 50, 150}]`

Out[ ]=



In[ ]:=
```
v[x_] := 4 π * x^3 / 3
v'[r]
v'[10] * -0.03
```

Out[ ]= $4\,\pi\,r^2$

Out[ ]= -37.6991

In[ ]:= `v'[10]`

Out[ ]= $400\,\pi$

In[ ]:= `20 / 400`

Out[ ]= $\frac{1}{20}$

Product rule

If $h(x) = f(x)\,g(x)$, then

In[ ]:= $$h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$$

at all points where the derivatives $f'(x)$ and $g'(x)$ are defined.

In[ ]:= `D[Sqrt[x], x] * Cos[x] + Sqrt[x] * -Sin[x]`

Out[ ]= $\frac{\text{Cos}[x]}{2\sqrt{x}} - \sqrt{x}\ \text{Sin}[x]$

In[ ]:= `100 * 0.4`

Out[ ]= `40.`

In[ ]:= `100 * -0.01 + 3 * 0.4`

Out[ ]= `0.2`

f (x) = x^2 Sin[x] Cos[x]

f '[x] = 2 x * Sin[x] * Cos[x]

+ x^2 * -Cos[x] * Sin[x]

+ x^2 * Sin[x] * -Sin[x]

Quotient Rule

In[ ]:= `Limit[(f2 * g - f * g2)/t, t → 0]`

Out[ ]= `Indeterminate`

If $h(x) = \frac{f(x)}{g(x)}$ for all x, then

In[ ]:= $$h'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

at all points where f and g are differentiable and $g(x) \neq 0$.

In[ ]:=
```
D[(2 + Cos[X])/(x^2 + 1), x]
```

Out[ ]= $-\frac{2\,x\,(2+\mathrm{Cos}[X])}{(1+x^2)^2}$

In[ ]:=

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}\cot x &= -\frac{1}{\sin^2 x} = -\csc^2 x \\
\frac{d}{dx}\sec x &= \frac{\sin x}{\cos^2 x} = \sec x \tan x \\
\frac{d}{dx}\csc x &= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x
\end{aligned}$$

Chain rule

If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))\,g'(x)$$

at all points where the derivatives $f'(g(x))$ and $g'(x)$ are defined.

In[ ]:=
```
D[120 * t + 100, t]
```

Out[ ]= 120

In[ ]:=
```
3 * 0.01 + 9
```

Out[ ]= 9.03

In[ ]:=
```
4 * (9.03 - 9) + 5
```

Out[ ]= 5.12

```
Solve[5.12 == x * (2.01 - 2) + 5, x]
```

Out[ ]= {{x → 12.}}

In[ ]:=
```
3 * 4 * (5 * 2)
```

Out[ ]= 120

Examples

In[ ]:=
```
f[x_] := √(x^3 + 2 x + 1)
f1[x_] := 1/(2 √(x^3 + 2 x + 1)) * (3 x^2 + 2)
f1[1]
```

Out[ ]= $\frac{5}{4}$

In[ ]:=
```
g[x_] := Sin[x^2 + 2 x]
g1[x_] := -Cos[x^2 + 2 x] * (2 x + 2)
g1[x]
```

Out[ ]= (2 + 2 x) Cos[2 x + x^2]

Exercises

In[ ]:=
```
f[x_] := Sin[x/(x^2 + 1)]
f1[x_] := Cos[u] * u '
f2[x_] := Cos[x/(x^2 + 1)] * ((x^2 + 1) - (x * 2 * x))/(x^2 + 1)^2
f2[x] == f '[x]
f2[x]
```

Out[ ]= True

Out[ ]= $\left(-\frac{2\,x^2}{(1+x^2)^2}+\frac{1}{1+x^2}\right)\mathrm{Cos}\left[\frac{x}{1+x^2}\right]$

In[ ]:=
```
k[x_] := Cos[x] * Sin[x]^4
u := Sin[x]^4
k1[x_] := -Sin[x] * u + Cos[x] * u '
inn := Sin[x]
auss := x^4
k2[x_] := -Sin[x] * Sin[x]^4 + Cos[x] * (4 * (Sin[x])^3 * Cos[x])
k '[x] == k2[x]
k2[x]
```

Out[ ]= True

Out[ ]= 4 $\mathrm{Cos}[x]^2\,\mathrm{Sin}[x]^3 - \mathrm{Sin}[x]^5$

Review

In[ ]:=
```
g'[f[3]] * f'[3];
g'[5] * 3;
4 * 3

f'[g[3]] * g' (3);
f'[4] * 5;
-3 * 5
```

Out[ ]= 12

Out[ ]= -15

In[ ]:=
```
p[i_] := 10 * i^2
p'[6]
```

Out[ ]= 120

Implicit Differentiation

In[2]:= Solve$[x^2 + y^2 == 25, \{x, y\}]$

Out[2]= $\{\{y \to -\sqrt{25 - x^2}\}, \{y \to \sqrt{25 - x^2}\}\}$

In[4]:= $D[\sqrt{25 - x^2}, x]$

$D[-\sqrt{25 - x^2}, x]$

Out[4]= $-\frac{x}{\sqrt{25 - x^2}}$

Out[5]= $\frac{x}{\sqrt{25 - x^2}}$

In[8]:= $x = -3;$

$-\frac{x}{\sqrt{25 - x^2}}$

Out[9]= $\frac{3}{4}$

In[64]:= eclipse := $x^4 - 3x^2 + y^4 + y^2 + 2x^2y^2 == 0$

$D[2y + \mathrm{Cos}[x] + x^2 y + \mathrm{Sin}[2y] == 3, x]$

Out[65]= $2xy - \mathrm{Sin}[x] == 0$

In[86]:=
$$y^3 + x^3 == 3\,xy;$$
$$3\,y^2 * (dy / dx) + 3\,x^2 == 3 * y + (dy / dx) * x * 3;$$
$$(3\,y^2 - 3\,x) * (dy / dx) == 3\,y - 3\,x^2;$$
$$\frac{dy}{dx} == \frac{3\,y - 3\,x^2}{3\,y^2 - 3\,x};$$

Practice

In[52]:=
$$y^2 == x;$$
$$\frac{dy}{dx} == \frac{1}{2\,y}$$

Out[53]=
$$\frac{dy}{dx} == \frac{1}{2\,y}$$

In[79]:=
$$x^2 + 4\,x * y == 2\,y^2 + 5;$$
$$2\,x + 4\,y + 4\,x * \frac{dy}{dx} == 4\,y * \frac{dy}{dx};$$
$$2\,x + 4\,y == (4\,y - 4\,x) * \frac{dy}{dx};$$
$$\frac{2\,x + 4\,y}{4\,y - 4\,x} == \frac{dy}{dx}$$

Out[82]=
$$\frac{2\,x + 4\,y}{-4\,x + 4\,y} == \frac{dy}{dx}$$

In[83]:=
$$u == \mathrm{Sin}[y^2 + u];$$
$$\frac{du}{dy} == \mathrm{Cos}[y^2 + u] * \left(2\,y + \frac{du}{dy}\right);$$
$$\frac{du}{dy} == \mathrm{Cos}[y^2 + u] * 2\,y + \mathrm{Cos}[y^2 + u] * \frac{du}{dy};$$
$$(1 - \mathrm{Cos}[y^2 + u]) * \frac{du}{dy} == \mathrm{Cos}[y^2 + u] * 2\,y;$$
$$\frac{du}{dy} == \frac{\mathrm{Cos}[y^2 + u] * 2\,y}{1 - \mathrm{Cos}[y^2 + u]};$$
$$\mathrm{Simplify}\left[\frac{\mathrm{Cos}[y^2 + u] * 2\,y}{1 - \mathrm{Cos}[y^2 + u]}\right]$$

Out[88]=
$$-\frac{2\,y\,\mathrm{Cos}[u + y^2]}{-1 + \mathrm{Cos}[u + y^2]}$$

In[91]:=

$$w^2 v^3 == w^3 v^2;$$

$$2 w * \frac{dw}{dv} * v^3 + w^2 * 3 v^2 == 3 w^2 * \frac{dw}{dv} * v^2 + w^3 * 2 v;$$

$$(2 w * v^3 - 3 w^2 * v^2) * \frac{dw}{dv} == w^3 * 2 v - w^2 * 3 v^2;$$

$$\frac{dw}{dv} == \text{Simplify}\left[\frac{w^3 * 2 v - w^2 * 3 v^2}{2 w * v^3 - 3 w^2 * v^2}\right]$$

Out[94]=

$$\frac{dw}{dv} == \frac{w(-3 v + 2 w)}{v(2 v - 3 w)}$$

$$x * y == y^3;$$

$$y + \frac{dy}{dx} * x == 3 y^2 * \frac{dy}{dx};$$

$$(3 y^2 - x) * \frac{dy}{dx} == y;$$

$$\frac{dy}{dx} == \text{Simplify}\left[\frac{y}{3 y^2 - x}\right]$$

Out[110]=

$$\frac{dy}{dx} == -\frac{y}{x - 3 y^2}$$

In[125]:=

$$\frac{dy}{dx} == \frac{-x}{y};$$

$$\frac{dy}{dx} == \frac{x * \frac{dy}{dx} - y}{y^2};$$

$$\frac{dy}{dx} == x * y^{-2} * \frac{dy}{dx} - y^{-1};$$

$$(1 - x * y^{-2}) * \frac{dy}{dx} == -y^{-1};$$

$$\frac{dy}{dx} == \text{Simplify}\left[\frac{-y^{-1}}{1 - x * y^{-2}}\right]$$

$$\frac{dy}{dx} == \frac{y}{x - y^2}$$