2004-10-05 STEP Practice: Problem 5 (2003.01.01)

$$\sum_{r=-1}^{n} r^2 = pn^3+qn^2+rn+s$$

$$\sum_{r=-1}^{-1} r^2 = [-1]^2 = 1 = -p+q-r+s$$

$$\sum_{r=-1}^{0} r^2 = [-1]^2+0^2 = 1 = s \leadsto -p+q-r=0 \quad ①$$

$$\sum_{r=-1}^{1} r^2 = [-1]^2+0^2+1^2 = 2 = p+q+r+1 \Longrightarrow p+q+r=1 \quad ②$$

$$\sum_{r=-1}^{2} r^2 = 1+0+1+4 = 6 = 8p+4q+2r+1 \Longrightarrow 8p+4q+2r=5 \quad ③$$

①+②: $2q=1 \Longrightarrow q=\frac{1}{2} \leadsto p+r=\frac{1}{2}$

③: $8p+2+2r=5 \Longrightarrow 8p+2r=3 \Longrightarrow 8p+2[\frac{1}{2}-p]=3 \Longrightarrow 8p+1-2p=3 \Longrightarrow 6p=2$

$p=\frac{1}{3} \leadsto r=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

$$\therefore \sum_{r=0}^{n} r^2 = \sum_{r=-1}^{n}[r^2] - 1 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n+1-1$$

$$= \frac{2}{6}n^3+\frac{3}{6}n^2+\frac{1}{6}n = \frac{1}{6}n[n^2+3n+\frac{1}{6}] = \frac{1}{6}n[n+1][2n+1]$$

$$\sum_{r=-2}^{n} r^3 = an^4+bn^3+cn^2+dn+e$$

$$\sum_{r=-2}^{0} r^3 = -8-1+0 = -9 = e$$

$$\sum_{r=-2}^{-1} r^3 = -8-1 = -9 = a-b+c-d-9 \Longrightarrow a-b+c-d=0 \quad ①$$

$$\sum_{r=-2}^{1} r^3 = -8-1+0+1 = -8 = a+b+c+d-9 \Longrightarrow a+b+c+d=1 \quad ②$$

$$\sum_{r=-2}^{-2} r^3 = -8 = 16a-8b+4c-2d-9 \Longrightarrow 16a-8b+4c+2d=1 \quad ③$$

$$\sum_{r=-2}^{2} r^3 = -8-1+0+1+8 = 0 = 16a+8b+4c+2d-9 \Longrightarrow 16a+8b+4c+d = 9 \quad ④$$

①+②: $2a+2c=1 \Longrightarrow 2c = 1-2a$

③+④: $32a+8c=10$

$32a+4[1-2a]=10$

$24a = 6$

$a = \frac{1}{4}$

$2c = 1-\frac{1}{2} = \frac{1}{2} \Longrightarrow c = \frac{1}{4}$

①: $\frac{1}{4} - b + \frac{1}{4} - d = 0 \Longrightarrow b+d = \frac{1}{2} \Longrightarrow 2b+2d = 1$ ⑤

③: $4 - 8b + 1 - 2d = 1 \Longrightarrow 8b+2d = 4$ ⑥

⑥−⑤: $6b = 3 \Longrightarrow b = \frac{1}{2}$

$2d = 1-2b = 0 \Longrightarrow d = 0$

$$\therefore \sum_{r=-2}^{n} r^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 - 9$$

$$\therefore \sum_{r=0}^{n} r^3 = \sum_{r=-2}^{n}[r^3] + 1 + 8 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 - 9 + 9$$

$$= \frac{1}{4}n^2[n^2+2n+1] = \frac{1}{4}n^2[n+1]^2$$