

2024-09-21 ST<sup>EP</sup> Practice Problem 1 (2004.2.1)

i)  $\sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0$

$$(\sqrt{3x^2+1} + \sqrt{x})^2 = (2x+1)^2$$

$$3x^2 + 1 + 2\sqrt{x(3x^2+1)} + x = 4x^2 + 4x + 1$$

$$(2\sqrt{x(3x^2+1)})^2 = (x^2 + 4x)^2$$

$$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$$

$$12x^3 + 4x = x^4 + 6x^3 + 9x^2$$

$$x^4 - 6x^3 + 9x^2 - 4x = 0$$

$$x \underbrace{(x^3 - 6x^2 + 9x - 4)}_{f(x)} = 0 \implies x = 0$$

$$f(1) = 1^3 - 6(1)^2 + 9(1) - 4 = 0$$

$\Rightarrow x-1$  is a factor of  $f(x)$  (Factor theorem)

$$f(x) = x^3 - 6x^2 + 9x - 4 = (x-1)(ax^2 + bx + c)$$

$$= (x-1)(x^2 - 5x + 4)$$

$$f(x) = (x-1)(x-1)(x-4)$$

$$\therefore \boxed{x=0, x=1, x=4}$$

$$\begin{aligned} -4 &= -c \implies c = 4 \\ x^3 &= ax^3 \implies a = 1 \\ -6x^2 &= bx^2 - x^2 \\ -6 &= b - 1 \\ -5 &= b \end{aligned}$$

Checking Solutions:

$$\left. \begin{array}{l} x=0 \quad \sqrt{3(0)^2+1} + \sqrt{0} - 2(0) - 1 = 0 \\ x=1 \quad \sqrt{3(1)^2+1} + \sqrt{1} - 2(1) - 1 = 0 \\ x=4 \quad \sqrt{3(4)^2+1} + \sqrt{4} - 2(4) - 1 = 0 \end{array} \right\} \text{Valid solutions}$$

ii)  $\sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0$

$$(\sqrt{3x^2+1} - 2\sqrt{x})^2 = (1-x)^2$$

$$3x^2 + 1 - 4\sqrt{x(3x^2+1)} + 4x = 1 - 2x + x^2$$

$$(-4\sqrt{x(3x^2+1)})^2 = (-2x^2 - 6x)^2$$

$$16x(3x^2+1) = 4x^4 + 24x^3 + 36x^2$$

$$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$$

$$12x^3 + 4x = x^4 + 6x^3 + 9x^2$$

$$x^4 - 6x^3 + 9x^2 - 4x = 0$$

$$x(x^3 - 6x^2 + 9x - 4) = 0$$

(via result obtained in part i)

$$\therefore x=0, x=1, x=4$$

Checking Solutions:

$$\left. \begin{array}{l} x = 0 \stackrel{o}{\circ} \sqrt{3(0)^2 + 1} - 2\sqrt{0} + 0 - 1 = 0 \\ x = 1 \stackrel{o}{\circ} \sqrt{3(1)^2 + 1} - 2\sqrt{1} + 1 - 1 = 0 \\ x = 4 \stackrel{o}{\circ} \sqrt{3(4)^2 + 1} - 2\sqrt{4} + 4 - 1 = 6 \end{array} \right\} \text{Valid solutions}$$
$$x \neq 4$$

(iii)  $\sqrt{3x^2 + 1} - 2\sqrt{x} - x + 1 = 0$

$$(\sqrt{3x^2 + 1} - 2\sqrt{x})^2 = (x - 1)^2$$

$$3x^2 + 1 - 4\sqrt{3x^2 + 1}x + 4x = x^2 - 2x + 1$$

$$(4\sqrt{3x^2 + 1}x)^2 = (-2x^2 - 6x)^2$$

$$16x(3x^2 + 1) = 4x^4 + 24x^3 + 36x^2$$

$$x(x-1)^2(x-4) = 0 \quad (\text{via result obtained in part ii})$$

$$\therefore x = 0, x = 1, x = 4$$

Checking Solutions:

$$\left. \begin{array}{l} x = 0 \stackrel{o}{\circ} \sqrt{3(0)^2 + 1} - 2\sqrt{0} - 0 + 1 = 2 \Rightarrow x \neq 0 \\ x = 1 \stackrel{o}{\circ} \sqrt{3(1)^2 + 1} - 2\sqrt{1} - 1 + 1 = 0 \\ x = 4 \stackrel{o}{\circ} \sqrt{3(4)^2 + 1} - 2\sqrt{4} - 4 + 1 = 0 \end{array} \right\} \text{Valid solutions}$$

Note:

From parts ii and iii, we can see that the solutions we obtain may not be valid, even if we utilised the correct method. In these problems, **extraneous** solutions emerged from squaring our equations. Thus, it is important to check our solutions by substituting them back into the original equation.