

2025-03-25 STEP Practice ° Trigonometric-Substitution Integrals

1)

i) $y = \arctan x$
 $y' = \frac{1}{1+x^2}$

ii) $y = \arctan\left(\frac{x}{a}\right)$
 $y' = \frac{1}{a} \cdot \frac{1}{1+\frac{x^2}{a^2}} = \frac{1}{a+\frac{x^2}{a}} = \frac{a}{a^2+x^2}$

2)

i) $\int \frac{1}{1+x^2} dx = \arctan x + C$

ii) $\int \frac{a}{a^2+x^2} dx = \arctan\left(\frac{x}{a}\right) + C$

3)

i) $I = \int \frac{3}{5+4x^2} dx$ Let $x = \frac{\sqrt{5}}{2} \tan \theta$
 $dx = \frac{\sqrt{5}}{2} \sec^2 \theta d\theta$
 $\theta = \arctan\left[\frac{2x}{\sqrt{5}}\right]$
 $= 3 \int \frac{\frac{\sqrt{5}}{2} \sec^2 \theta}{5+5 \tan^2 \theta} d\theta$
 $= \frac{3\sqrt{5}}{10} \int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta$
 $= \frac{3}{2\sqrt{5}} \int d\theta$
 $= \frac{3}{2\sqrt{5}} \theta + C$
 $= \frac{3}{2\sqrt{5}} \arctan\left[\frac{2x}{\sqrt{5}}\right] + C$

ii) $I = \int \frac{7}{4+3x^2} dx = \frac{7}{\sqrt{12}} \arctan\left[\frac{\sqrt{3}x}{2}\right] + C$

iii) $I = \int \frac{4}{6+5x^2} dx = \frac{4}{\sqrt{30}} \arctan\left[\frac{\sqrt{5}x}{\sqrt{6}}\right] + C$

4)

i) $y = \arcsin x$
 $y' = \frac{1}{\sqrt{1-x^2}}$

ii) $y = \arcsin\left(\frac{x}{a}\right)$
 $y' = \frac{1}{a} \cdot \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{1}{\sqrt{a^2-x^2}}$

5)

i) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

ii) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$

6)

i)
$$\begin{aligned} I &= \int \frac{5}{\sqrt{4-4x^2}} dx & \text{Let } x &= \frac{2}{3} \sin \theta \\ &= 5 \int \frac{\frac{2}{3} \cos \theta}{\sqrt{4-4\sin^2 \theta}} d\theta & dx &= \frac{2}{3} \cos \theta d\theta \\ &= \frac{10}{3} \int \frac{\cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta & \theta &= \arcsin\left(\frac{3x}{2}\right) \\ &= \frac{5}{3} \int d\theta \\ &= \frac{5}{3} \arcsin\left[\frac{3x}{2}\right] + C \end{aligned}$$

ii) $I = \int \frac{2}{\sqrt{7-6x^2}} dx = \frac{2 \cdot \frac{\sqrt{7}}{\sqrt{6}}}{\frac{\sqrt{7}}{\sqrt{6}}} \arcsin\left[\frac{\frac{\sqrt{7}x}{\sqrt{6}}}{\frac{\sqrt{7}}{\sqrt{6}}}\right] + C = \frac{2}{\sqrt{6}} \arcsin\left[\frac{\sqrt{7}x}{\sqrt{6}}\right] + C$