

2024-09-21 ST^{EP} Practice

Problem 1 (2004.2.1)

$$i) \sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0$$

$$(\sqrt{3x^2+1} + \sqrt{x})^2 = (2x+1)^2$$

$$3x^2 + 1 + 2\sqrt{x(3x^2+1)} + x = 4x^2 + 4x + 1$$

$$(2\sqrt{x(3x^2+1)})^2 = (x^2 + 4x)^2$$

$$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$$

$$12x^3 + 4x = x^4 + 6x^3 + 9x^2$$

$$x^4 - 6x^3 + 9x^2 - 4x = 0$$

$$x \underbrace{(x^3 - 6x^2 + 9x - 4)}_{f(x)} = 0 \implies x = 0$$

$$f(1) = (1)^3 - 6(1)^2 + 9(1) - 4 = 0$$

$\Rightarrow x-1$ is a factor of $f(x)$ (Factor theorem)

$$f(x) = x^3 - 6x^2 + 9x - 4 = (x-1)(ax^2 + bx + c)$$

$$= (x-1)(x^2 - 5x + 4)$$

$$f(x) = (x-1)(x-1)(x-4)$$

$$-4 = -c \implies c = 4$$

$$x^3 = ax^3 \implies a = 1$$

$$-6x^2 = bx^2 - x^2$$

$$-6 = b - 1$$

$$\underline{-5} = b$$

$$\therefore \boxed{x = 0, x = 1, x = 4}$$

Back Substitution:

$$\left. \begin{array}{l} x = 0 \\ x = 1 \\ x = 4 \end{array} \right\} \begin{array}{l} \sqrt{3(0)^2+1} + \sqrt{0} - 2(0) - 1 = 0 \\ \sqrt{3(1)^2+1} + \sqrt{1} - 2(1) - 1 = 0 \\ \sqrt{3(4)^2+1} + \sqrt{4} - 2(4) - 1 = 0 \end{array} \right\} \text{Valid solutions}$$

$$ii) \sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0$$

$$(\sqrt{3x^2+1} - 2\sqrt{x})^2 = (1-x)^2$$

$$3x^2 + 1 - 4\sqrt{x(3x^2+1)} + 4x = 1 - 2x + x^2$$

$$(-4\sqrt{x(3x^2+1)})^2 = (-2x^2 - 6x)^2$$

$$16x(3x^2+1) = 4x^4 + 24x^3 + 36x^2$$

$$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$$

$$12x^3 + 4x = x^4 + 6x^3 + 9x^2$$

$$x^4 - 6x^3 + 9x^2 - 4x = 0$$

$$x(x^3 - 6x^2 + 9x - 4) = 0$$

(via result obtained in part i)

$$\therefore x = 0, x = 1, x = 4$$

Back Substitution:

$$\left. \begin{array}{l} x = 0 : \sqrt{3(0)^2 + 1} - 2\sqrt{0} + 0 - 1 = 0 \\ x = 1 : \sqrt{3(1)^2 + 1} - 2\sqrt{1} + 1 - 1 = 0 \\ x = 4 : \sqrt{3(4)^2 + 1} - 2\sqrt{4} + 4 - 1 = 6 \end{array} \right\} \text{Valid solutions}$$

$$iii) \sqrt{3x^2 + 1} - 2\sqrt{x} - x + 1 = 0$$

$$(\sqrt{3x^2 + 1} - 2\sqrt{x})^2 = (x - 1)^2$$

$$3x^2 + 1 - 4\sqrt{x(3x^2 + 1)} + 4x = x^2 - 2x + 1$$

$$(-4\sqrt{x(3x^2 + 1)})^2 = (-2x^2 - 6x)^2$$

$$16x(3x^2 + 1) = 4x^4 + 24x^3 + 36x^2$$

$$x(x-1)^2(x+4) = 0 \quad (\text{via result obtained in part ii})$$

$$\therefore x = 0, x = 1, x = 4$$

Back Substitution:

$$\left. \begin{array}{l} x = 0 : \sqrt{3(0)^2 + 1} - 2\sqrt{0} - 0 + 1 = 2 \Rightarrow x \neq 0 \\ x = 1 : \sqrt{3(1)^2 + 1} - 2\sqrt{1} - 1 + 1 = 0 \\ x = 4 : \sqrt{3(4)^2 + 1} - 2\sqrt{4} - 4 + 1 = 0 \end{array} \right\} \text{Valid solutions}$$