2025-03-22 STEP Practice: Problem 42 (2012.01.05)

$$I_1 = \int_0^{\frac{\pi}{4}} \sin(2x)\ln(\cos x)\,dx = \int_0^{\frac{\pi}{4}} 2\sin x\cos x\ln(\cos x)\,dx$$

Let $u = \cos x$
$du = -\sin x\,dx$
$x \to 0 : u \to 1$
$x \to \frac{\pi}{4} : u \to \frac{\sqrt{2}}{2}$

$$= -2\int_1^{\frac{\sqrt{2}}{2}} u\ln u\,du = -2\left[\left[\ln u \cdot \frac{u^2}{2}\right]_1^{\frac{\sqrt{2}}{2}} - \int_1^{\frac{\sqrt{2}}{2}} \frac{u^2}{2}\cdot\frac{1}{u}\,du\right]$$

$$= -2\left[\left[\frac{1}{4}\ln\frac{\sqrt{2}}{2} - \overbrace{\frac{1}{2}\ln 1}^{0}\right] - \left[\frac{u^2}{4}\right]_1^{\frac{\sqrt{2}}{2}}\right] = -2\left[\frac{1}{4}\ln\frac{\sqrt{2}}{2} - \frac{1}{8} + \frac{1}{4}\right] = -\frac{1}{2}\ln\frac{1}{\sqrt{2}} - \frac{1}{4}$$

$$= -\frac{1}{2}\ln(2^{-\frac{1}{2}}) - \frac{1}{4} = \frac{1}{4}\ln 2 - \frac{1}{4} = \frac{1}{4}[\ln 2 - 1]$$

$$I_2 = \int_0^{\frac{\pi}{4}} \cos(2x)\ln(\cos x)\,dx = \left[\frac{1}{2}\sin(2x)\ln(\cos x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(2x)\cdot\frac{-\sin x}{\cos x}\,dx$$

$$= \frac{1}{2}\left[\overbrace{\sin\frac{\pi}{2}}^{1}\ln\left(\cos\frac{\pi}{4}\right) - \overbrace{\sin 0\ln(\cos 0)}^{0}\right] + \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin x\cos x\cdot\frac{\sin x}{\cos x}\,dx$$

$$= \frac{1}{2}\ln\frac{\sqrt{2}}{2} + \int_0^{\frac{\pi}{4}} \sin^2 x\,dx = -\frac{1}{4}\ln 2 + \int_0^{\frac{\pi}{2}} \left[\frac{1-\cos(2x)}{2}\right]dx$$

$$= -\frac{1}{4}\ln + \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{4}} = -\frac{1}{4}\ln x + \left[\left[\frac{\pi}{8} - \frac{1}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]\right]$$

$$= -\frac{1}{4}\ln 2 + \frac{\pi}{8} - \frac{1}{4} = \frac{1}{8}[\pi - 2\ln 2 - 2] = \frac{1}{8}[\pi - \ln 4 - 2]$$

$A\cos x + B\sin x \equiv R\cos(x+\alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

$A = R\cos\alpha$ ①

$B = -R\sin\alpha$ ②

①^2+②^2 $R^2[\cos^2\alpha + \sin^2\alpha] = A^2 + B^2$

$R = \sqrt{A^2+B^2}$

②/① $-\tan\alpha = \frac{B}{A}$

$\alpha = \arctan\left(-\frac{B}{A}\right) = -\arctan\left(\frac{B}{A}\right)$

$\therefore A\cos x + B\sin x \equiv \sqrt{A^2+B^2}\cos\left(x - \arctan\left(\frac{B}{A}\right)\right)$

$$I_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\cos(2x)+\sin(2x)]\ln(\cos x+\sin x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\cos(2x)+\sin(2x)]\ln(\sqrt{2}\cos(x-\tfrac{\pi}{4}))\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\ln(\sqrt{2}\cos(x-\tfrac{\pi}{4}))\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x)\ln(\sqrt{2}\cos(x-\tfrac{\pi}{4}))\,dx$$

$$= \ln\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\ln(x-\tfrac{\pi}{4})\,dx + \ln\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x)\,dx$$

$$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x)\ln(\cos(x-\tfrac{\pi}{4}))\,dx$$

$$\ln\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\,dx + \ln\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x)\,dx = \ln\sqrt{2}\left[\tfrac{1}{2}\sin(2x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \ln\sqrt{2}\left[-\tfrac{1}{2}\cos(2x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\ln\sqrt{2}[\sin\pi - \sin\tfrac{\pi}{2}] - \tfrac{1}{2}\ln\sqrt{2}[\cos\pi - \cos\tfrac{\pi}{2}] = -\tfrac{1}{2}\ln\sqrt{2} + \tfrac{1}{2}\ln\sqrt{2} = 0$$

$$\leadsto I_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\ln(\cos(x-\tfrac{\pi}{4}))\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x)\ln(\cos(x-\tfrac{\pi}{4}))\,dx$$

Let $u = x - \frac{\pi}{4} \leadsto dx = du$

$\cos(2x) = \cos(2u+\frac{\pi}{2}) = -\sin(2u)$

$\sin(2x) = \sin(2u+\frac{\pi}{2}) = \cos(2u)$

$x \to \frac{\pi}{4} : u \to 0 \,.\, x \to \frac{\pi}{2} : u \to \frac{\pi}{4}$.

$$\leadsto I_3 = \int_0^{\frac{\pi}{4}} \cos(2u)\ln(\cos u)\,du - \int_0^{\frac{\pi}{4}} \sin(2u)\ln(\cos u)\,du$$

$$= \tfrac{1}{8}[\pi - \ln 4 - 2] - \tfrac{1}{4}[\ln 2 - 1] = \tfrac{\pi}{8} - \tfrac{1}{4}\ln 2 - \tfrac{1}{4} - \tfrac{1}{4}\ln 2 + \tfrac{1}{4}$$

$$= \tfrac{\pi}{8} - \tfrac{1}{2}\ln 2$$