01-02-2025 **STEP Practice: Product and Quotient Rules**

Product Rule

$u(x)$ and $v(x)$ are functions of x.

Let $y(x) = u(x)v(x)$

$y = uv$ for brevity

Suppose x changes by a small amount δx.

u changes by δu

v changes by δv

y changes by δy

$\rightsquigarrow y + \delta y = [u+\delta u][v+\delta v]$

$\cancel{y} + \delta y = \cancel{uv} + u\delta v + v\delta u + \delta u \delta v$ $\quad$ δu and δv are small, so $\delta u \delta v$ can be ignored

$\delta y = u\delta v + v\delta u$

$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x}$

Let $\delta x \to 0$.

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

E.g. $y = x^5 \implies \frac{dy}{dx} = 5x^4$

$= x^2x^3 \implies \frac{dy}{dx} = x^2 \cdot 3x^2 + 2x \cdot x^3 = 3x^4 + 2x^4 = 5x^4$

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[e^x] = e^x$

E.g. $y = x^2 \sin x \implies \frac{dy}{dx} = 2x\sin x + x^2\cos x$

$y = [x^2+4]e^x \implies \frac{dy}{dx} = 2xe^x + [x^2+4]e^x = [x^2+2x+4]e^x$

$y = [x^2+5x+6]\tan x \implies \frac{dy}{dx} = [2x+5]\tan x + [2x^2+5x+6]\sec^2 x$

$y = [3x^2+2x-1]^9 e^x \implies \frac{dy}{dx} = 9[6x+2][3x^2+2x-1]^8 e^x + [3x^2+2x-1]^9 e^x$

$= e^x[3x^2+2x-1]^8[54x+18+3x^2+2x-1] = e^x[3x^2+2x-1]^8[3x^2+56x+17]$

13) $\frac{d}{dx}\left[x[2x+5]^7\right] = [2x+5]^7 + 7\cdot 2[2x+5]^6 x = [2x+5]^7 + 14x[2x+5]^6 = [2x+5]^6[2x+5+14x]$

$= [2x+5]^6[16x+5]$

14) $\frac{d}{dx}\left[x^2[2x+5]^7\right] = 2x[2x+5]^7 + 14x^2[2x+5]^6 = [2x+5]^6[2x[2x+5]+14x^2] = [2x+5]^6[18x^2+10x]$

15) $\frac{d}{dx}\left[x^4[x+3]^5\right] = 4x^3[x+3]^5 + 5x^4[x+3]^4 = x^3[x+3]^4[4x+12+5x] = x^3[x+3]^3[9x+12]$

16) $\frac{d}{dx}\left[x^4[x^2+3]^5\right] = 4x^3[x^2+3]^5 + x^4\cdot 10x[x^2+3]^4 = x^3[x^2+3]^4[4x^2+12+10x^2] = x^3[x^2+3]^4[14x^2+12]$

17) $\frac{d}{dx}\left[x^3[3-x^4]^5\right] = 3x^2[3-x^4]^5 + x^3\cdot 5\cdot -4x^3[3-x^4]^4 = 3x^2[3-x^4]^5 - 20x^6[3-x^4]^4 = x^2[3-x^4]^4[9-3x^4-20x^3]$

18) $\frac{d}{dx}\left[[x^2-7][x^4+1]^2\right] = 2x[x^4+1]^2 + 8x^3[x^4+1][x^2-7] = 2x[x^4+1][x^4+1+4x^2[x^2-7]] = 2x[x^4+1][5x^4-28x^2+1]$

19) $\frac{d}{dx}\left[[2x^2+1]^3[3x-1]^7\right] = 3\cdot 4x[2x^2+1]^2[3x-1]^7 + 7\cdot 3[3x-1]^6[2x^2+1]^3 = [2x^2+1]^2[3x-1]^6[12x[3x-1]+21[2x^2+1]]$

$= [2x^2+1]^2[3x-1]^6[78x^2-12x+21]$

20) $\frac{d}{dx}\left[[3x+5]^9[2x-1]^6\right] = 27[3x+5]^8[2x-1]^6 + 12[2x-1]^5[3x+5]^9 = [3x+5]^8[2x-1]^5[27[2x-1]+12[3x+5]]$

$= [3x+5]^8[2x-1]^5[90x+33]$

21) $\frac{d}{dx}\left[[4-x]^5[2x+1]^3\right] = -5[4-x]^4[2x+1]^3 + 6[2x+1]^2[4-x]^5 = [4-x]^4[2x+1]^2[-5[2x+1]+6[4-x]]$

$= [4-x]^4[2x+1]^2[19-16x]$

22) $\frac{d}{dx}\left[x\sqrt{x+3}\right] = \frac{d}{dx}\left[x[x+3]^{\frac{1}{2}}\right] = [x+3]^{\frac{1}{2}} - \frac{1}{2}x[x+3]^{-\frac{1}{2}}$

23) $\frac{d}{dx}\left[x^2\sqrt{x+3}\right] = \frac{d}{dx}\left[x^2[x+3]^{\frac{1}{2}}\right] = 2x[x+3]^{\frac{1}{2}} - \frac{1}{2}x^2[x+3]^{-\frac{1}{2}}$

24) $\frac{d}{dx}\left[[x+3]^4\sqrt{x}\right] = 4[x+3]^3x^{\frac{1}{2}} - \frac{1}{2}[x+3]^4x^{-\frac{1}{2}}$

<u>Quotient Rule</u>

$y = \frac{u}{v} = uv^{-1}$

$$\frac{dy}{dx} = v^{-1}\frac{du}{dx} \quad u\cdot -v^{-2}\frac{dv}{dx} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

E.g. $\frac{d}{dx}\left[\frac{x^3}{\sin x}\right] = \frac{3x^2\sin x - x^3\cos x}{\sin^2 x} = x^2\csc x[3 - x\tan x]$

$\frac{d}{dx}\left[\frac{\tan x}{e^x}\right] = \frac{e^x\sec^2 x - e^x\tan x}{e^{2x}} = e^{-x}[\sec^2 x - \tan x]$