2025-03-08 STEP Practice: Problem 89 (1998.01.06)

$a_1 = \cos x$ with $0 < x < \frac{\pi}{2}$. Let $b_1 = 1$.

$a_{n+1} = \frac{1}{2}[a_n + b_n]$

$b_{n+1} = [a_{n+1} b_n]^{\frac{1}{2}}$

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$
$\cos x = [\frac{1}{2}\cos(2x) + \frac{1}{2}]^{\frac{1}{2}}$

$\rightsquigarrow \cos(\frac{x}{2}) = [\frac{1}{2}\cos x + \frac{1}{2}]^{\frac{1}{2}}$
$\cos^2(\frac{x}{2}) = \frac{1}{2}\cos x + \frac{1}{2}$

$\rightsquigarrow \cos(\frac{x}{4}) = [\frac{1}{2}\cos(\frac{x}{2}) + \frac{1}{2}]^{\frac{1}{2}}$
$\cos^2(\frac{x}{4}) = \frac{1}{2}\cos(\frac{x}{2}) + \frac{1}{2}$

$\rightsquigarrow \cos(\frac{x}{8}) = [\frac{1}{2}\cos(\frac{x}{4}) + \frac{1}{2}]^{\frac{1}{2}}$
$\cos^2(\frac{x}{8}) = \frac{1}{2}\cos(\frac{x}{4}) + \frac{1}{2}$

$a_2 = \frac{1}{2}[a_1 + b_1] = \frac{1}{2}[\cos x + 1] = \frac{1}{2}\cos x + \frac{1}{2} = \cos^2(\frac{x}{2})$

$b_2 = [a_2 b_1]^{\frac{1}{2}} = [\frac{1}{2}\cos x + \frac{1}{2}]^{\frac{1}{2}} = \cos(\frac{x}{2})$

$$\begin{aligned} a_3 = \frac{1}{2}[a_2 + b_2] &= \frac{1}{2}[\cos^2(\tfrac{x}{2}) + \cos(\tfrac{x}{2})] \\ &= \cos(\tfrac{x}{2})[\tfrac{1}{2}\cos(\tfrac{x}{2}) + \tfrac{1}{2}] \\ &= \cos(\tfrac{x}{2})\cos^2(\tfrac{x}{4}) \end{aligned}$$

$$\begin{aligned} b_3 = [a_3 b_2]^{\frac{1}{2}} &= [\cos(\tfrac{x}{2})\cos^2(\tfrac{x}{4}) \cdot \cos(\tfrac{x}{2})]^{\frac{1}{2}} \\ &= [\cos^2(\tfrac{x}{2})\cos^2(\tfrac{x}{4})]^{\frac{1}{2}} \\ &= \cos(\tfrac{x}{2})\cos(\tfrac{x}{4}) \end{aligned}$$

$$\begin{aligned} a_4 = \frac{1}{2}[a_3 + b_3] &= \frac{1}{2}[\cos(\tfrac{x}{2})\cos^2(\tfrac{x}{4}) + \cos(\tfrac{x}{2})\cos(\tfrac{x}{4})] \\ &= \cos(\tfrac{x}{2})\cos(\tfrac{x}{4})[\tfrac{1}{2}\cos(\tfrac{x}{4}) + \tfrac{1}{2}] \\ &= \cos(\tfrac{x}{2})\cos(\tfrac{x}{4})\cos^2(\tfrac{x}{8}) \end{aligned}$$

$$\begin{aligned} b_4 = [a_4 b_3]^{\frac{1}{2}} &= [\cos^2(\tfrac{x}{2})\cos^2(\tfrac{x}{4})\cos^2(\tfrac{x}{8})]^{\frac{1}{2}} \\ &= \cos(\tfrac{x}{2})\cos(\tfrac{x}{4})\cos(\tfrac{x}{8}) \end{aligned}$$

Guess: $a_n = \cos(\frac{x}{2^{n-1}}) \prod_{i=1}^{n-1} \cos(\frac{x}{2^i})$, $b_n = \prod_{i=1}^{n-1} \cos(\frac{x}{2^i})$ for $n \geq 2$

These are valid if they satisfy they satisfy the recurrence equations.

Verification: $a_{n+1} = \cos(\frac{x}{2^n}) \prod_{i=1}^{n} \cos(\frac{x}{2^i}) = \cos^2(\frac{x}{2^n}) \prod_{i=1}^{n-1} \cos(\frac{x}{2^i}) = [\frac{1}{2}\cos(\frac{x}{2^{n-1}}) + \frac{1}{2}] \prod_{i=1}^{n-1} \cos(\frac{x}{2^i})$

$= \frac{1}{2}\cos(\frac{x}{2^{n-1}}) \prod_{i=1}^{n-1} \cos(\frac{x}{2^i}) + \frac{1}{2} \prod_{i=1}^{n-1} \cos(\frac{x}{2^i}) = \frac{1}{2}a_n + \frac{1}{2}b_n = \frac{1}{2}[a_n + b_n]$

$b_{n+1} = \prod_{i=1}^{n} \cos(\frac{x}{2^i}) = \cos(\frac{x}{2^n}) \prod_{i=1}^{n-1} \cos(\frac{x}{2^i}) = [\frac{1}{2}\cos(\frac{x}{2^{n-1}}) + \frac{1}{2}]^{\frac{1}{2}} b_n = [\frac{1}{2}\cos(\frac{x}{2^{n-1}}) b_n^2 + \frac{1}{2} b_n^2]^{\frac{1}{2}}$

$$b_{n+1} = \left[\frac{1}{2}a_n b_n + \frac{1}{2}b_n^2\right]^{\frac{1}{2}} = \left[b_n\left[\frac{1}{2}a_n + \frac{1}{2}b_n\right]\right]^{\frac{1}{2}} = \left[a_{n+1} b_n\right]^{\frac{1}{2}}$$

Thus, the general expression for a_n and b_n are correct.