04-11-2024 STEP Practice: Problem 13 (2003.3.6)

$$2\sin(\tfrac{1}{2}\theta)\cos(r\theta) = \sin[(r+\tfrac{1}{2})\theta] - \sin[(r-\tfrac{1}{2})\theta]$$

$\sin A + \sin B = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]$

$\sin A - \sin B = 2\cos\left[\frac{A+B}{2}\right]\sin\left[\frac{A-B}{2}\right]$

$\cos A + \cos B = 2\cos\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]$

$\cos A - \cos B = -2\sin\left[\frac{A+B}{2}\right]\sin\left[\frac{A-B}{2}\right]$

pf

$$\begin{aligned} RHS &= \sin[(r+\tfrac{1}{2})\theta] - \sin[(r-\tfrac{1}{2})\theta] \\ &= 2\cos\left[\frac{(r+\frac{1}{2})\theta + (r-\frac{1}{2})\theta}{2}\right]\sin\left[\frac{(r+\frac{1}{2})\theta - (r-\frac{1}{2})\theta}{2}\right] \\ &= 2\cos(r\theta)\sin(\tfrac{\theta}{2}) = \underline{\underline{LHS}} \end{aligned}$$

Q.E.D.

$$\begin{aligned}
r=a:&\quad 2\cos(a\theta)\sin(\tfrac{\theta}{2}) = \cancel{\sin[(a+\tfrac{1}{2})\theta]} - \sin[(a-\tfrac{1}{2})\theta] \\
r=a+1:&\quad 2\cos[(a+1)\theta]\sin(\tfrac{\theta}{2}) = \cancel{\sin[(a+\tfrac{3}{2})\theta]} - \cancel{\sin[(a+\tfrac{1}{2})\theta]} \\
r=a+2:&\quad 2\cos[(a+2)\theta]\sin(\tfrac{\theta}{2}) = \cancel{\sin[(a+\tfrac{5}{2})\theta]} - \cancel{\sin[(a+\tfrac{3}{2})\theta]} \\
&\quad \vdots \\
r=b-3:&\quad 2\cos[(b-3)\theta]\sin(\tfrac{\theta}{2}) = \cancel{\sin[(b-\tfrac{5}{2})\theta]} - \cancel{\sin[(b-\tfrac{7}{2})\theta]} \\
r=b-2:&\quad 2\cos[(b-2)\theta]\sin(\tfrac{\theta}{2}) = \cancel{\sin[(b-\tfrac{3}{2})\theta]} - \cancel{\sin[(b-\tfrac{5}{2})\theta]} \\
r=b-1:&\quad 2\cos[(b-1)\theta]\sin(\tfrac{\theta}{2}) = \sin[(b-\tfrac{1}{2})\theta] - \cancel{\sin[(b-\tfrac{3}{2})\theta]}
\end{aligned}$$

Method of differences for telescoping series.

$$\rightsquigarrow \sin(\tfrac{\theta}{2})[\cos(a\theta) + \cos((a+1)\theta) + \dots + \cos((b-1)\theta)] = \sin[(b-\tfrac{1}{2})\theta] - \sin[(a-\tfrac{1}{2})\theta]$$

$$\sin[(b-\tfrac{1}{2})\theta] - \sin[(a-\tfrac{1}{2})\theta] = 0$$

$$2\cos\left[\frac{(b-\frac{1}{2})\theta + (a-\frac{1}{2})\theta}{2}\right]\sin\left[\frac{(b-\frac{1}{2})\theta - (a-\frac{1}{2})\theta}{2}\right] = 0$$

$$2\cos\left[\frac{(a+b-1)\theta}{2}\right]\sin\left[\frac{(b-a)\theta}{2}\right] = 0$$

$$\cos\left[\frac{(a+b-1)\theta}{2}\right] = 0 \implies \frac{(a+b-1)\theta}{2} = \frac{\pi}{2} + \pi k,\ k\in\mathbb{Z}$$

$$\implies \theta = \frac{\pi + 2\pi k}{a+b-1} = \boxed{\frac{[2k+1]\pi}{a+b-1}}$$

OR:

$$\sin\left[\frac{(b-a)\theta}{2}\right] = 0 \implies \frac{(b-a)\theta}{2} = \pi k,\ k\in\mathbb{Z}$$

$$\implies \boxed{\theta = \frac{2\pi k}{b-a}}$$