2024-10-12 STEP Practice: Problem 6 (2010.01.01)

$$\underbrace{5x^2 + 2y^2 - 6xy + 4x - 4y}_{\text{2nd order expression in two variables: } x \text{ and } y} \equiv a[x-y+2]^2 + b[cx+y]^2 + d$$

represents conic section

$$[x-y+2][x-y+2] = x^2 - xy + 2x - yx + y^2 + 2x - 2y + 4$$
$$= \underbrace{x^2 - 2xy + y^2}_{\text{quadratic terms}} + \underbrace{4x - 4y}_{\text{linear terms}} + \underbrace{4}_{\text{constant term}}$$

$$RHS \equiv ax^2 - 2axy + ay^2 + 4ax - 4ay + 4a + bc^2x^2 + 2bcxy + by^2 + d$$
$$\equiv x^2[a + bc^2] + xy[2bc - 2a] + y^2[a+b] + 4ax - 4ay + 4a + d$$

Comparing Coefficients:

$a + bc^2 = 5$

$a + b = 2$

$2bc - 2a = -6$

$\left.\begin{array}{l} 4a = 4 \\ -4a = -4 \end{array}\right\}$ not independent

$4a + d = 0$

$\rightsquigarrow a = \frac{4}{4} = 1$

$\rightsquigarrow 4(1) + d = 0 \Rightarrow d = -4$

$\rightsquigarrow 1 + b = 2 \Rightarrow b = 1$

$\rightsquigarrow 2(1)c - 2(1) = -6 \Rightarrow 2c - 2 = -6 \Rightarrow c = -2$

Solve the simultaneous equations

Eq. 1) $5x^2 + 2y^2 - 6xy + 4x - 4y = 9$

Eq. 2) $6x^2 + 3y^2 - 8xy + 8x - 8y = 14$

Eq. 1 – Comparing Coefficients:

$a + bc^2 = 5$

$a + b = 2$

$2bc - 2a = -6$

$\left.\begin{array}{l} 4a = 4 \\ -4a = -4 \end{array}\right\}$ not independent

$4a + d = 0$

$\rightsquigarrow a = 1$

$\rightsquigarrow d = 0 - 4 = -4$

$\rightsquigarrow b = 2 - 1 = 1$

$\rightsquigarrow 2(1)c - 2(1) = -6 \Rightarrow 2c - 2 = -6 \Rightarrow c = -2$

Eq. 2 – Comparing Coefficients

$a + bc^2 = 6$

$a + b = 3$

$2bc - 2a = -8$

$\left.\begin{array}{l} 4a = 8 \\ -4a = -8 \end{array}\right\}$ not independent

$4a + d = -14$

$\rightsquigarrow a = \frac{8}{4} = 2$

$\rightsquigarrow b = 3 - 2 = 1$

$\rightsquigarrow 2(1)c - 2(2) = -8 \implies 2c - 4 = -8 \implies c = -2$

$\rightsquigarrow 4(2) + d = 0 \implies d = -8$

$\therefore$

Eq. 1) $[x - y + 2]^2 + [-2x + y]^2 - 4 = 9$ $\qquad p = [x-y+2]^2$

Eq. 2) $2[x - y + 2]^2 + [-2x + y]^2 - 8 = 14$ $\qquad q = [-2x+y]^2$

Eq. 1) $p + q - 4 = 9$

Eq. 2) $2p + q - 8 = 14$

$q = 13 - p$

$2p + 13 - p - 8 = 14$

$p + 5 = 14$

$\therefore p = 9$

$\therefore q = 13 - 9 = 4$

$\rightsquigarrow [x - y + 2]^2 = 9 \implies x - y + 2 = \pm 3$

$\vee\ [-2x + y]^2 = 4 \implies -2x + y = \pm 2$

(A) $x - y + 2 = 3 \implies x - y = 1$

(B) $x - y + 2 = -3 \implies x - y = -5$

(C) $-2x + y = 2$

(D) $-2x + y = -2$

4 combinations: A&C, A&D, B&C, B&D

(A) & (C):

$x = 1 + y$

$-2(1 + y) + y = 2$

$-2 - 2y + y = 2$

$-y = 4$

$y = -4$

$x = 1 + (-4) = -3 \rightsquigarrow x = -3 \wedge y = -4$ is a solution

(A) & (D):

$x = 1 + y$

$-2(1+y) + y = -2$

$-2 - 2y + y = -2$

$-y = 0$

$y = 0$

$x = 1 + 0 = 1 \rightsquigarrow x = 1 \wedge y = 0$ is a solution

(B) & (C):

$x = y - 5$

$-2(y-5) + y = 2$

$-2y + 10 + y = 2$

$-y = -8$

$y = 8$

$x = 8 - 5 = 3 \rightsquigarrow x = 3 \wedge y = 8$ is a solution

(B) & (D):

$x = y - 5$

$-2(y-5) + y = -2$

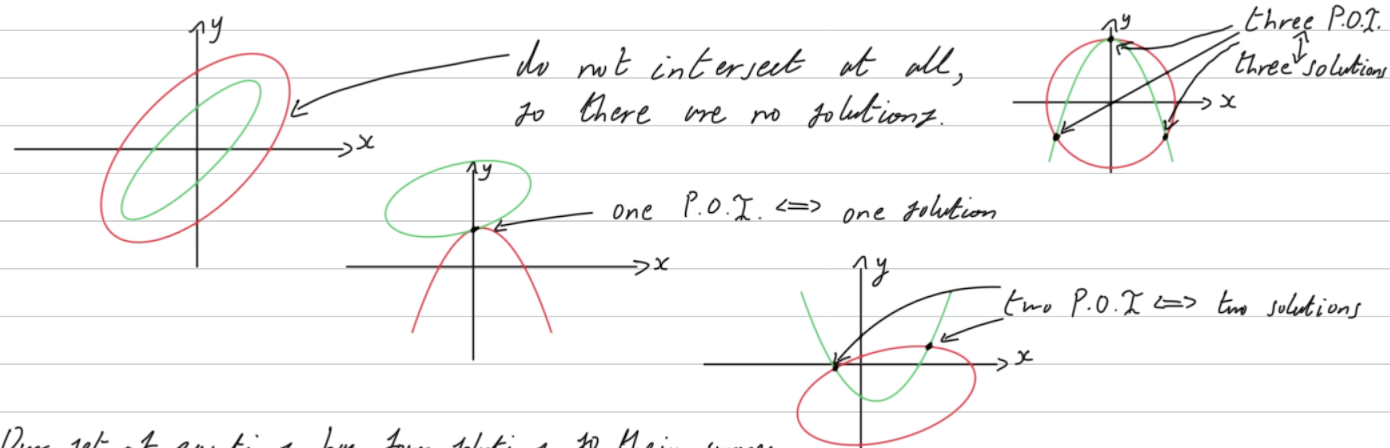
$-2y + 10 = -2$

$-y = -12$

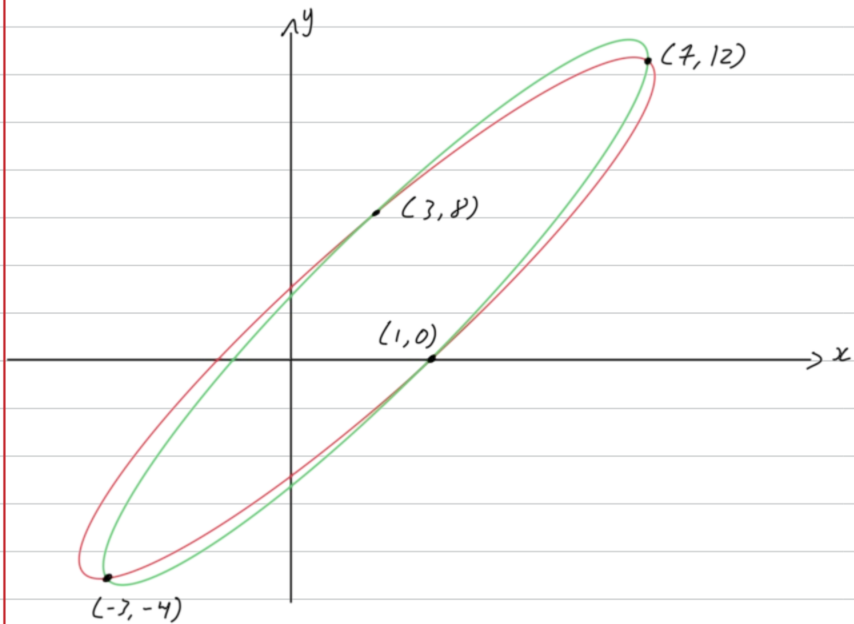
$y = 12$

$x = 12 - 5 = 7 \rightsquigarrow x = 7 \wedge y = 12$ is a solution

Two second order equations in two variables will have – at most – four solutions. If we graphed the two equations, there would be four points where the two curves intersect. However, they can have less than four solutions; you can even get no solutions. This is shown below:



Our set of equations has four solutions, so their curves will intersect at four points. It should look something like this:



Note: You need 5 points to specify a unique conic section.

Bezout's Theorem:

For two polynomial curves, of order m and n, there will be up to m·n points of intersection

can have fewer than m·n P.O.I.

Note:

From this problem, I have learnt that some STEP problems may be impossibly difficult without the information that is given to you at the start. This is a problem I would not have been able to solve - if I was not given a way to rewrite the equations. Also, these problems have reinforced the idea of STEP problems requiring you to learn something from one part in order to apply it to a later part of the problem.