2025-03-15 STEP Practice: Problem 86 (2016.03.06)

$A>0$. $A\sinh x + B\cosh x$ can be written as $\cosh(x+\gamma)$ provided we can find R and γ.

$$A\sinh x + B\cosh x \equiv R\cosh(x+\gamma)$$
$$\equiv R\cosh x\cosh\gamma + R\sinh x\sinh\gamma$$

$\leadsto A = R\sinh\gamma$, $B = R\cosh\gamma$

$$A\sinh x + B\cosh x = \frac{e^{x}[A+B] + e^{-x}[B-A]}{2}$$

$R^2\cosh^2\gamma - R^2\sinh^2\gamma = R^2 = B^2 - A^2$

$\frac{R\sinh\gamma}{R\cosh\gamma} = \tanh\gamma = \frac{A}{B}$

Since $B>A>0$, $B^2-A^2>0$ and $1>\frac{A}{B}>0$.
$A\sinh x + B\cosh >0$, so $R>0$ $\therefore R=\sqrt{B^2-A^2}$
$\gamma = \operatorname{arctanh}(\frac{B}{A})$

we can also use $B = R\cosh\gamma$, where $B>0$ and $\cosh\gamma>0$, so $R>0$.

If $B<-A$, then $B^2-A^2>0$ and $-1<\frac{A}{B}<0$.
$A\sinh x + B\cosh x < 0$ $\therefore R = -\sqrt{B^2-A^2}$
$\gamma = \operatorname{arctanh}(\frac{B}{A})$

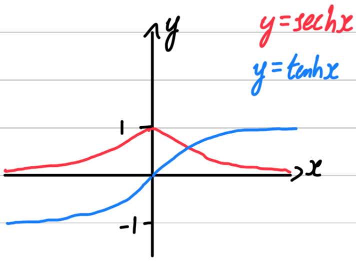
If $A=B$: $A\sinh x + B\cosh = A[\sinh x + \cosh x] = A\left[\frac{e^x-e^{-x}}{2} + \frac{e^x+e^{-x}}{2}\right] = Ae^x$

If $A=-B$: $A\sinh x + B\cosh x = A[\sinh x - \cosh x] = A\left[\frac{e^x-e^{-x}}{2} - \frac{e^x+e^{-x}}{2}\right] = -Ae^{-x} = Be^{-x}$

If $B<|A| \iff -A<B<A$, $B^2-A^2<0$.
We can use $R\sin(x+\gamma)$ instead.

$$A\sinh x + B\cosh x \equiv R\sinh(x+\gamma)$$
$$\equiv R\sinh x\cosh\gamma + R\cosh x\sinh\gamma$$

$\leadsto A = R\cosh\gamma$, $B = R\sinh\gamma$
$R^2 = A^2 - B^2$, $\tanh\gamma = \frac{B}{A}$
Since $B<|A|$, $A^2-B^2>0$ and $-1<\frac{B}{A}<1$.
Since $A = R\cosh\gamma$, $A>0$, and $\cosh\gamma>0$, then $R>0$. $\therefore R = \sqrt{A^2-B^2}$
$\gamma = \operatorname{arctanh}(\frac{B}{A})$



Two curves have the equation $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a>0$.

i) If and only if $y = \operatorname{sech}x$ and $y = a\tanh x + b$ intersect, then the x-coordinate of the P.O.I. satisfies

$\operatorname{sech}x = a\tanh x + b$

$\iff \cosh x \operatorname{sech}x = \cosh x[a\tanh x + 1]$

$\iff 1 = a\sinh x + b\cosh x$

$\iff 1 = \sqrt{b^2 - a^2}\cosh\left[x + \operatorname{arctanh}\left(\frac{a}{b}\right)\right] \quad \because b > a > 0$

$\iff \cosh\left[x + \operatorname{arctan}\left(\frac{a}{b}\right)\right] = \frac{1}{\sqrt{b^2-a^2}}$

$\therefore x = \pm \operatorname{arccosh}\left[\frac{1}{\sqrt{b^2-a^2}}\right] - \operatorname{arctanh}\left(\frac{a}{b}\right)$

since cosh is an even function

ii) For $a > b > 0$,

$a\sinh x + b\cosh x = 1$

$\iff \sqrt{a^2-b^2}\sinh\left[x + \operatorname{arctan}\left(\frac{b}{a}\right)\right] = 1$

$\therefore x = \operatorname{arcsinh}\left[\frac{1}{\sqrt{a^2-b^2}}\right] - \operatorname{arctanh}\left(\frac{b}{a}\right)$ No '$\pm$' since sinh is an odd function

iii) Note the following about $y = \operatorname{sech}x$.

- Lies entirely in $0 < y \leq 1$.
- Increasing for $x < 0$ and decreasing for $x > 0$.

Since $a > 0$, $y = a\tanh x + b$ is strictly increasing for all x.
If $b \leq 0$, then the curve lies in $y \leq 0$ for x for $x \leq 0$.
Thus, no intersections in $x \leq 0$ and at most one in $x > 0$ (one curve is strictly increasing for $x > 0$, whilst the other is strictly decreasing).
$\therefore b > 0$ is a necessary condition for two distinct intersection points.

If $b < a < 0$, and the curves intersect, there is exactly one point of intersection by (ii).
Thus $b \geq a$ is a necessary condition.
If $a = b$,
$a\sinh x + a\cosh x = 1$
$ae^x = 1$
$x = -\ln a$
We only get one point of intersection, so $b > a$ is a necessary condition.

By (i), if $b > a$ and the curves intersect, there will be two distinct points of intersection.
Thus, $b > a$ and $\frac{1}{\sqrt{b^2-a^2}} > 1$ are sufficient conditions for two distinct points of intersection.
The latter is so that arccosh is defined and gives two distinct roots.

Therefore, the necessary and sufficient conditions for two distinct points of intersection are
$b > a$ and $\sqrt{b^2-a^2} < 1$
$\iff b > a$ and $b < \sqrt{a^2+1}$
$\iff a < b < \sqrt{a^2+1}$.

iv) For the two curves to touch, the two points of intersection must coincide.
intersect tangentially
We find this at the boundary between two and zero points of intersection.
The necessary and sufficient conditions for two curves to touch are
$b > a$ and $\sqrt{b^2-a^2} = 1$
$\iff b > a$ and $b = \sqrt{a^2+1}$
$\iff b = \sqrt{a^2+1}$

In this case, the point at which they touch is given by

$$x = -\operatorname{arctanh}\left(\frac{a}{b}\right) = -\operatorname{arctanh}\left[\frac{a}{\sqrt{a^2+1}}\right] = \operatorname{arctanh}\left[-\frac{a}{\sqrt{a^2+1}}\right] \quad \because \text{arctanh is odd}$$

$$\Rightarrow y = a\tanh\left[\operatorname{arctanh}\left[-\frac{a}{\sqrt{a^2+1}}\right]\right] + b$$

$$= -\frac{a^2}{\sqrt{a^2+1}} + \sqrt{a^2+1}$$

$$= -\frac{a^2}{\sqrt{a^2+1}} + \frac{a^2+1}{\sqrt{a^2+1}}$$

$$= \frac{1}{\sqrt{a^2+1}}.$$

Notes

This is not a very good question. That is, if faced with this question in an exam, you would be better off skipping it. There is just too much to do within the 30 minutes you are supposed to spend on it. There are many cases to consider before you even start the first part of the question. You are required to be very familiar with the hyperbolic functions, their graphs, and their identities. Everything you do in the initial parts will need to be applied to obtain the final parts. When asked to provide necessary and sufficient conditions, you have to go back and bi-directional arrows to your prior work as you would not have a reason to do so previously. This is a difficult question, and most people who sat the exam did not do particularly well with this question – as is stated in the examiners' report. On one hand, doing this question shows you how important it is to be familiar with the graphs of functions and their properties; without such knowledge, you are doomed to fail. On the other hand, the more important thing to learn is that questions requiring you to evaluate many cases are often best avoided.