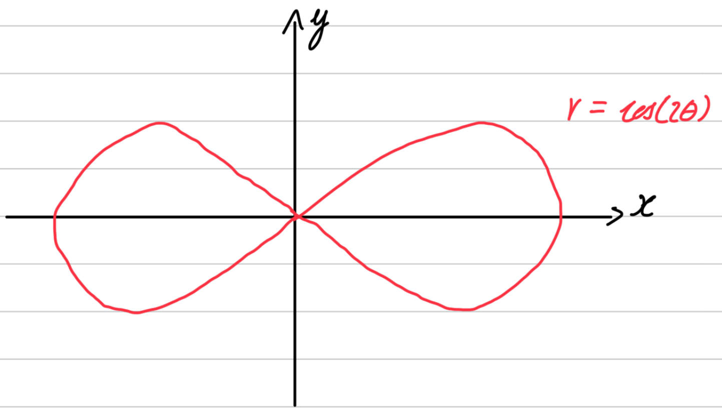
2025-03-29 STEP Practice: Problem 59 (1998.03.04)

Consider the curve $r = \cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$\sqrt{x^2+y^2} = \cos\left[\arctan\left[\frac{y}{x}\right]\right] = \frac{x}{\sqrt{x^2+y^2}} \rightsquigarrow x^2+y^2 = x \rightsquigarrow \left[x-\frac{1}{2}\right]^2 + y^2 = \frac{1}{4}$

$\therefore$ The equation represents a circle of radius $\frac{1}{2}$ and centre $(\frac{1}{2}, 0)$.

Consider the curve $r = \cos(2\theta)$ for $0 \le \theta \le 2\pi$.
$r \ge 0$ for $\theta \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi]$.
For $\theta \in [0, \frac{\pi}{4}]$, r goes from 1 to 0.
For $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$, r goes from 0 to 1 to 0.
For $\theta \in [\frac{7\pi}{4}, 2\pi]$, r goes from 0 to 1.



The curve $r = \cos(2n\theta)$ has $2n$ lobes evenly-spaced around the origin. Each lobe is 1 long and $\frac{\pi}{2n}$ wide. One lobe is centered on the x-axis.

The area of one lobe is given by: $\int_{-\frac{\pi}{4n}}^{\frac{\pi}{4n}} \frac{r^2}{2}\, d\theta$.

The total area enclosed by the curve:

$$A = 2n \times \int_{-\frac{\pi}{4n}}^{\frac{\pi}{4n}} \frac{\cos^2(2n\theta)}{2}\, d\theta = n\int_{-\frac{\pi}{4n}}^{\frac{\pi}{4n}} \left[\frac{\cos(4n\theta)+1}{2}\right] d\theta = n\left[\frac{\sin(4n\theta)}{8n} + \frac{\theta}{2}\right]_{-\frac{\pi}{4n}}^{\frac{\pi}{4n}}$$

$$= \cancel{n}\left[\frac{\sin(n\pi)}{8\cancel{n}} + \frac{\pi}{8\cancel{n}}\right] - \cancel{n}\left[\frac{\sin(-n\pi)}{8\cancel{n}} - \frac{\pi}{8\cancel{n}}\right] = 0 + \frac{\pi}{8} - 0 + \frac{\pi}{8} = \frac{\pi}{4}$$

$\therefore A$ is independent of n.

Consider the curve $r = \cos(3\theta)$ for $0 \le \theta \le 2\pi$.
$r \ge 0$ for $\theta \in [0, \frac{\pi}{6}] \cup [\frac{\pi}{2}, \frac{5\pi}{6}] \cup [\frac{7\pi}{6}, \frac{3\pi}{2}] \cup [\frac{11\pi}{6}, 2\pi]$.

30 90 150 210 270 330 360

For $\theta \in [0, \frac{\pi}{6}]$, r goes from 1 to 0.
For $\theta \in [\frac{\pi}{2}, \frac{5\pi}{6}]$, r goes from 0 to 1 to 0.
For $\theta \in [\frac{7\pi}{6}, \frac{3\pi}{2}]$, r goes from 0 to 1 to 0.
For $\theta \in [\frac{11\pi}{6}, 2\pi]$, r goes from 0 to 1.