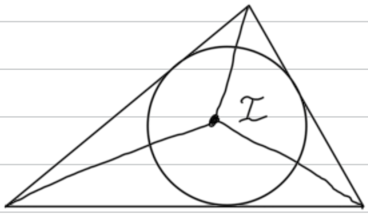
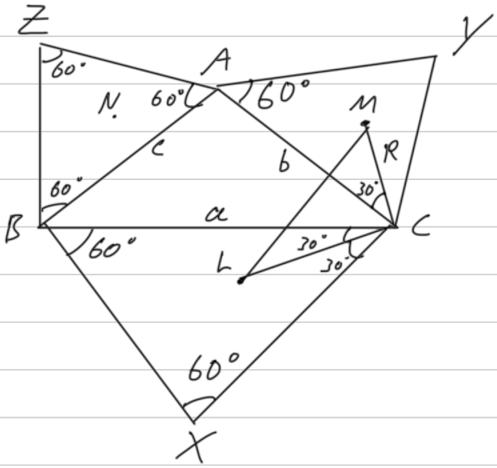
09-11-2024 STEP Practice: Problem 9 (2017.1.17)



The angle bisectors are concurrent at the in-centre. It is the centre of the in-circle, which is tangent to the sides of the triangle.



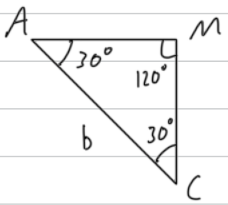
Because M is the centre of symmetry of triangle ACY, it is also the circumcentre of triangle ACY.

$\therefore$ $|CM|$ is the radius of the circumscribed circle.

M is also the in-centre so MC bisects angle ACY.

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

i)



$$\frac{|CM|}{\sin(30^\circ)} = \frac{b}{\sin(120^\circ)} \Rightarrow |CM| = \frac{\sin(30^\circ)b}{\sin(120^\circ)} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}b = \frac{b}{\sqrt{3}}$$ Q.E.D.

Similarly, $|LC| = \frac{a}{\sqrt{3}}$

we could easily find $|LC|$ manually, but the method is the exact same except for the side of triangle ABC we use.

ii) Apply the cosine rule to triangle LCM

$$\cos(L\hat{C}M) = \cos(C + A\hat{C}M + B\hat{C}L)$$
$$= \cos(C + 30^\circ + 30^\circ)$$
$$= \cos(C + 60^\circ)$$
$$= \cos C \cos(60^\circ) - \sin C \sin(60^\circ)$$
$$= \tfrac{1}{2}\cos C - \tfrac{\sqrt{3}}{2}\sin C$$

$$|LM|^2 = |MC|^2 + |CL|^2 - 2\cdot|MC|\cdot|CL|\cdot\cos(L\hat{C}M)$$
$$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - 2\cdot\frac{b}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}\cdot\cos(C+60^\circ)$$
$$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - \frac{2ab}{3}\cos(C+60^\circ)$$
$$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - \frac{2\cancel{ab}}{3}\left[\frac{a^2+b^2-c^2}{4\cancel{ab}} - \frac{\sqrt{3}\Delta}{\cancel{ab}}\right]$$
$$= \frac{b^2}{3} + \frac{a^2}{3} - \frac{a^2}{6} - \frac{b^2}{6} + \frac{c^2}{6} + \frac{2\sqrt{3}\Delta}{3}$$
$$6|LM|^2 = 2b^2 + 2a^2 - a^2 - b^2 + a^2 + 4\sqrt{3}\Delta$$

Apply the cosine rule to triangle ABC

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\therefore 6|LM|^2 = a^2+b^2+c^2+4\sqrt{3}\Delta$$ Q.E.D.

Similarly, $6|MN| = 6|LM|$
$$= a^2+b^2+c^2+4\sqrt{3}\Delta$$

Let $\Delta :=$ Area of triangle $= \frac{1}{2}ab\sin C$
$\rightsquigarrow \sin C = \frac{2\Delta}{ab}$

$\therefore$ Triangle LMN is equilateral.

$$\therefore \cos(L\hat{C}M) = \cos(C+60^\circ)$$
$$= \frac{a^2+b^2-c^2}{4ab} - \frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}\Delta}{ab}$$

Let $\Delta_* :=$ Area of triangle LMN
$$\Delta_* = \tfrac{1}{2}|LM|^2\cdot\sin(60^\circ) = \tfrac{\sqrt{3}}{4}|LM|^2$$
$$\Delta_* = \Delta \Rightarrow \tfrac{\sqrt{3}}{4}|LM|^2 = \Delta = \tfrac{\sqrt{3}}{24}\left[a^2+b^2+c^2+4\sqrt{3}\Delta\right]$$
$$24\Delta = \sqrt{3}[a^2+b^2+c^2] + 12\Delta$$
$$a^2+b^2+c^2 = \frac{12\Delta}{\sqrt{3}} = 4\sqrt{3}\Delta$$
$$a^2+b^2+c^2 = 4\sqrt{3}\Delta \Rightarrow |LM|^2 = \tfrac{1}{6}\cdot 8\sqrt{3}\Delta = \tfrac{4}{3}\sqrt{3}\Delta \Rightarrow \Delta_* = \frac{\sqrt{3}}{\cancel{4}}\cdot\frac{\cancel{4}}{3}\sqrt{3} = \Delta$$
$$\therefore \Delta_* = \Delta \iff a^2+b^2+c^2 = 4\sqrt{3}\Delta$$ Q.E.D.

iii) $$[a-b]^2 = -2ab[1-\cos(C-60^\circ)]$$
$$a^2 - 2ab + b^2 = -2ab[1-[\cos C\cos(60^\circ) + \sin C\sin(60^\circ)]]$$
$$= -2ab[1 - \tfrac{1}{2}\cos C - \tfrac{\sqrt{3}}{2}\sin C]$$
$$a^2 - \cancel{2ab} + b^2 = -\cancel{2ab} + ab\cos C - ab\sqrt{3}\sin C$$
$$a^2+b^2 = \cancel{ab}\cdot\frac{a^2+b^2-c^2}{2\cancel{ab}} + \cancel{ab}\sqrt{3}\cdot\frac{2\Delta}{\cancel{ab}}$$

$$1-\cos(C-60^\circ) = 0$$
$$\cos(C-60^\circ) = 1$$
$$C - 60^\circ = 0^\circ$$
$$\therefore C = 60^\circ$$

$$a^2+b^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{2} + 2\sqrt{3}\Delta$$
$$\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} = 2\sqrt{3}\Delta$$
$$\therefore a^2+b^2+c^2 = 4\sqrt{3}\Delta$$

Steps are reversible $\therefore$ conditions are equivalent. Q.E.D.

$$\Delta_* = \Delta \iff a^2+b^2+c^2 = 4\sqrt{3}\Delta \iff [a-b]^2 = -2ab[1-\cos(C-60^\circ)]$$
$$[a-b]^2 \geq 0 \ \wedge\ -2ab[1-\cos(C-60^\circ)] \leq 0 \because -2ab < 0 \ \wedge\ 1-\cos(C-60^\circ) \geq 0$$
$$\iff [a-b]^2 = 0 \ \wedge\ 1-\cos(C-60^\circ) = 0$$
$$\iff a = b \ \wedge\ C = 60^\circ$$
$\iff$ Triangle ABC is equilateral. Q.E.D.