Problem 32. **2013.01.02**

In this question $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$and $\lfloor -1.5 \rfloor = -2$.
The function f is defined, for $x \neq 0$, by

$$f(x) = \frac{\lfloor x \rfloor}{x}$$

(i) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $f(x) = \frac{7}{12}$. Solve also the equations $f(x) = \frac{17}{24}$ and $f(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $f(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $f(x) = c$ to have exactly n roots where $n \geq 1$.

Prerequisites.

1. Knowledge of the Greatest Integer function,
2. Graph sketching skills,
3. Experience of finding roots of equations and the graphical representation of the roots of an equation,
4. Some familiarity with inequalities and interval notation.

First Thoughts.

1. This problem requires the use of the greatest integer function with negative values, so some care will be needed. From our previous experience of this function we know that it generates discontinuities and that these discontinuities occur at integer values of x. We will need to consider very carefully how the function behaves either side of these discontinuities. As before, we need to work in a methodical piecewise way, possibly using a table to organise the work.
2. The fractions that appear in the question suggest that considerable care will be needed when choosing a scale for the y axis.
3. Part (iii) and the last part sound unfamiliar. Perhaps things will become clearer as the answer develops.
4. When the phrase "or otherwise" appears in a problem we are free to use any methods we like. However, experience tells us that the recommended method is almost invariably the shortest and the easiest, and in contrast alternative methods often prove to be more lengthy and of greater difficulty.

Before starting the solution you should know why I include this problem at this stage. I use it in order to have a context for talking about left and right hand limits as we approach a discontinuity and I do this as a part of building up experience of the limit concept. Another