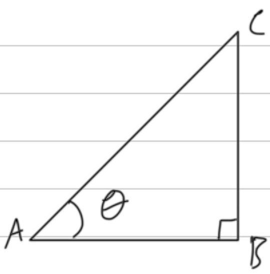
2024-10-19 STEP Practice: Problem 8 (2012.02.06)



$$AC^2 = AB^2 + BC^2$$

$$1 = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2}$$

$$= \left[\frac{AB}{AC}\right]^2 + \left[\frac{BC}{AC}\right]^2$$

$$= (\cos\theta)^2 + (\sin\theta)^2$$

$$= \cos^2\theta + \sin^2\theta$$

$$\therefore \boxed{1 \equiv \cos^2\theta + \sin^2\theta}$$

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{BC}{AC} \cdot \frac{AC}{AB}$$

$$= \sin\theta \cdot \frac{1}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$\boxed{\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}}$$

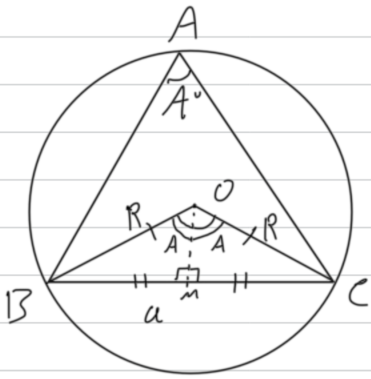
The Cosine Rule

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}\right\} \text{cyclic symmetry}$$

$$\left.\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \text{cosine rule for angles}$$

The Sine Rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \underline{\underline{2R}}$

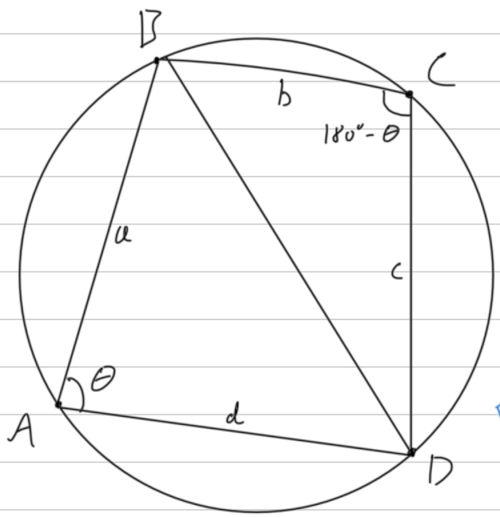


$\frac{BM}{BO} = \sin A$

$BM = R \sin A$

$\frac{a}{2} = R \sin A$

$$\left.\begin{aligned} \frac{a}{\sin A} &= 2R \\ \frac{b}{\sin B} &= 2R \\ \frac{c}{\sin C} &= 2R \end{aligned}\right\} \text{cyclic symmetry}$$



Area = Q

$\overbrace{\angle BAC}^{\theta} + \angle BCD = 180°$

$\Rightarrow \angle BCD = (180° - \theta)$

If $d = 0$, A and D coincide.
$\rightsquigarrow$ quad. becomes triangle.

From triangle ABD:

$BD^2 = AB^2 + AD^2 - (AD)(AB)\cos\theta$

$BD^2 = a^2 + d^2 - 2ad\cos\theta$

From triangle BCD:

$BD^2 = BC^2 + CD^2 - 2(BC)(CD)\overbrace{\cos(180° - \theta)}^{-\cos\theta}$

$= BC^2 + CD^2 + 2(BC)(CD)\cos\theta$

$BD^2 = b^2 + c^2 + 2bc\cos\theta$

$\therefore BD^2 = a^2 + d^2 - ad\cos\theta = b^2 + c^2 + 2bc\cos\theta$

$a^2 + d^2 - b^2 - c^2 = 2ad\cos\theta + 2bc\cos\theta$

$= 2\cos\theta[ad + bc]$

$$\therefore \cos\theta = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$$

Area of $\triangle BAD = \frac{1}{2} ad \sin\theta$

Area of $\triangle BCD = \frac{1}{2} bc \sin(180^\circ-\theta)$

$= \frac{1}{2} bc \sin\theta$

$\Rightarrow Q = \frac{1}{2}\sin\theta[ad+bc]$

$$\frac{2Q}{\sin\theta} = ad+bc$$

$$\sin\theta = \frac{2Q}{ad+bc}$$

$\cos^2\theta + \sin^2\theta \equiv 1$

$$\left[\frac{a^2+d^2-b^2-c^2}{2(ad+bc)}\right]^2 + \left[\frac{2Q}{ad+bc}\right]^2 = 1$$

$$\frac{[a^2+d^2-b^2-c^2]^2}{4[ad+bc]^2} + \frac{4 \cdot 4Q^2}{4 \cdot [ad+bc]^2} = 1$$

$$\frac{[a^2+d^2-b^2-c^2]^2 + 16Q^2}{4[ad+bc]^2} = 1$$

$$[a^2+d^2-b^2-c^2]^2 + 16Q^2 = 4[ad+bc]^2$$

$$16Q^2 = \underbrace{4[ad+bc]^2 - [a^2+d^2-b^2-c^2]^2}_{\text{diff. of two squares}} \qquad \text{Q.E.D.}$$

$a+b+c+d =:$ Perimeter

$s := \frac{a+b+c+d}{2}$ (semi-perimeter) $\longleftarrow$ Standard notation

$$\begin{aligned}
16Q^2 &= [2[ad+bc]+[a^2+d^2-b^2-c^2]][2[ad+bc]-[a^2+d^2-b^2-c^2]] \\
&= [a^2+2ad+d^2-[b^2+c^2-2bc]][[b^2+c^2+2bc]-[a^2+d^2-2ad]] \\
&= [[a+d]^2-[b-c]^2][[b+c]^2-[a-d]^2] \\
&= [[a+d+b-c][a+d-b+c]][[b+c+a-d][b+c-a+d]] \\
&= [a+d+b+c-2c][a+d+c+b-2b][b+c+a+d-2d][b+c+d+a-2a] \\
&= [2s-2c][2s-2b][2s-2a][2s-2d] \\
&= [s-c][s-b][s-a][s-d]\cdot 2\cdot 2\cdot 2\cdot 2
\end{aligned}$$

$$16Q^2 = 16[s-a][s-b][s-c][s-d]$$

$$\therefore Q^2 = [s-a][s-b][s-c][s-d] \qquad \text{Q.E.D.}$$

Let $d = 0 \rightsquigarrow$ A and D coincide

$$Q^2 = [s-0][s-a][s-b][s-c]$$
$$= s[s-a][s-b][s-c] \quad , \; s = \frac{a+b+c}{2}$$

Q is the area of a triangle

$$Q = \sqrt{s[s-a][s-b][s-c]}$$

Heron's Formula for the area of a triangle.

Note:

We took our general formula for the area of a cyclic quadrilateral, and we found the area of a triangle — this is a special case of our formula (where $d=0$). This STEP problem has shown the difficulty of problems, even if the knowledge required for it is nothing beyond GCSE level.