Problem 9. 2017.1.7

The triangle ABC has side lengths $|BC| = a, |CA| = b$ and $|AB| = c$. Equilateral triangles BXC, CYA and AZB are erected on the sides of the triangle ABC, with X on the other side of BC from A, and similarly for Y and Z. Points L, M and N are the centres of rotational symmetry of triangles BXC, CYA and AZB respectively.

(i) Show that $|CM| = \frac{b}{\sqrt{3}}$ and write down the corresponding expression for $|CL|$.

(ii) Use the cosine rule to show that

$$6|LM|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta,$$

where Δ is the area of triangle ABC. Deduce that LMN is an equilateral triangle. Show further that the areas of triangles LMN and ABC are equal if, and only if

$$a^2 + b^2 + c^2 = 4\sqrt{3}\Delta.$$

(iii) Show that the conditions

$$(a-b)^2 = -2ab(1 - \cos(C - 60°))$$

and

$$a^2 + b^2 + c^2 = 4\sqrt{3}\Delta$$

are equivalent.

Deduce that the areas of triangles LMN and ABC are equal if, and only if, ABC is equilateral.

Prerequisites.

You will need to know all the trigonometric formulae given above and be aware of the special geometric properties of an equilateral triangle. In addition you need to be aware that an "if and only if" proof requires you show that the proof works in both directions.

First Thoughts.

Draw a diagram.