23-11-2024 STEP Practice: Problem 22 (2011.02.07)

Geometric Progression

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \sum_{i=1}^{n} ar^{i-1}$

i) $a_n = \lambda^n + \mu^n$, $b_n = \lambda^n - \mu^n$, $\lambda = 1+\sqrt{2}$, $\mu = 1-\sqrt{2}$

$\sum_{r=1}^{n} b_r = \sum_{r=1}^{n} \lambda^r - \sum_{r=1}^{n} \mu^r$

$= \frac{[1-\lambda^n]}{1-\lambda} - \frac{[1-\mu^n]}{1-\mu} + \lambda^n - \mu^n$

$= \frac{[1-\lambda^n][1-\mu] - [1-\mu^n][1-\lambda]}{[1-\lambda][1-\mu]} + \lambda^n - \mu^n$

$= \frac{[1-\lambda^n][\cancel{1}-\cancel{1}+\sqrt{2}] - [1-\mu^n][\cancel{1}-\cancel{1}-\sqrt{2}]}{[\cancel{1}-\cancel{1}-\sqrt{2}][\cancel{1}-\cancel{1}+\sqrt{2}]} + \lambda^n - \mu^n$

$= \frac{\sqrt{2}[1-\lambda^n] + \sqrt{2}[1-\mu^n]}{-2} + \lambda^n - \mu^n$

$= -\frac{\sqrt{2}}{2}[1-\lambda^n + 1 - \mu^n] + \lambda^n - \mu^n$

$= -\frac{\sqrt{2}}{2}[2-\lambda^n-\mu^n] + \lambda^n - \mu^n$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}[\lambda^n + \mu^n] + \lambda^n - \mu^n$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}a_n + \lambda^n - \mu^n$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}[\lambda^n + \mu^n + \sqrt{2}\lambda^n - \sqrt{2}\mu^n]$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}[[1+\sqrt{2}]\lambda^n + [1-\sqrt{2}]\mu^n]$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}[\lambda \cdot \lambda^n + \mu \cdot \mu^n]$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}[\lambda^{n+1} + \mu^{n+1}]$

$= -\sqrt{2} + \frac{1}{\sqrt{2}}a_{n+1}$ Q.E.D

$b_n = \lambda^n - \mu^n$

$\sum_{i=0}^{n} \lambda^i = \sum_{i=1}^{n} \lambda^{i-1} + \lambda^n$

$= \frac{1\cdot[1-\lambda^n]}{1-\lambda} + \lambda^n$

$\sum_{i=0}^{n} \mu^i = \frac{1\cdot[1-\mu^n]}{1-\mu} + \mu^n$

$a_{n+1} = \lambda\lambda^n + \mu\mu^n$

$= [1+\sqrt{2}]\lambda^n + [1-\sqrt{2}]\mu^n$

$= \lambda^n + \sqrt{2}\lambda^n + \mu^n - \sqrt{2}\mu^n$

$\frac{1}{\sqrt{2}}a_{n+1} = \frac{1}{\sqrt{2}}a_n + \lambda^n - \mu^n$

$$\sum_{r=0}^{n} u_r = \frac{[1-\lambda^n]}{1-\lambda} + \frac{[1-\mu^n]}{1-\mu} + \lambda^n + \mu^n$$

$$= \frac{[1-\lambda^n][1-\mu] + [1-\mu^n][1-\lambda]}{[1-\lambda][1-\mu]} + \lambda^n + \mu^n$$

$$= \frac{\sqrt{2}[1-\lambda^n] - \sqrt{2}[1-\mu^n]}{-2} + \lambda^n + \mu^n$$

$$= -\frac{\sqrt{2}}{2}[\cancel{1} - \lambda^n - \cancel{1} + \mu^n] + \lambda^n + \mu^n$$

$$= \frac{\sqrt{2}}{2}[\lambda^n - \mu^n] + \lambda^n + \mu^n$$

$$= \frac{1}{\sqrt{2}}[\lambda^n - \mu^n + \sqrt{2}\lambda^n + \sqrt{2}\mu^n]$$

$$= \frac{1}{\sqrt{2}}[[1+\sqrt{2}]\lambda^n - [1-\sqrt{2}]\mu^n]$$

$$= \frac{1}{\sqrt{2}}[\lambda^{n+1} - \mu^{n+1}]$$

$$= \frac{1}{\sqrt{2}} b_{n+1}$$

$$\frac{1}{2}b_{n+1}^2 = \frac{1}{2}[b_n + \sqrt{2}a_n]^2$$

$$= \frac{1}{2}b_n^2 + \sqrt{2}a_n b_n + a_n^2$$

$$= \frac{1}{2}[\lambda^n - \mu^n]^2 + \sqrt{2}[\lambda^n + \mu^n][\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2$$

ii) *If n is odd:*

$$\sum_{m=0}^{2n}\sum_{r=0}^{m} u_r = \frac{1}{\sqrt{2}}\sum_{m=0}^{2n} b_{m+1} = \frac{1}{\sqrt{2}}\sum_{m=0}^{2n} b_m + \sqrt{2}a_m = \frac{1}{\sqrt{2}}\left[-\sqrt{2} + \frac{1}{\sqrt{2}}a_{2n+1} + \sqrt{2}\left[\frac{1}{\sqrt{2}}b_{2n+1}\right]\right]$$

$$= -1 + \frac{1}{2}a_{2n+1} + \frac{1}{\sqrt{2}}b_{2n+1} = -1 + \frac{1}{2}\lambda^{2n+1} + \frac{1}{2}\mu^{2n+1} + \frac{1}{\sqrt{2}}\lambda^{2n+1} - \frac{1}{\sqrt{2}}\mu^{2n+1}$$

$$= \lambda\mu + \frac{1}{2}\lambda\lambda^{2n} + \frac{1}{2}\mu\mu^{2n} + \frac{1}{\sqrt{2}}\lambda\lambda^{2n} - \frac{1}{\sqrt{2}}\mu\mu^{2n}$$

$$= \lambda\mu + \frac{1}{2}\lambda^{2n} + \frac{1}{2}\mu^{2n} + \frac{1}{\sqrt{2}}\lambda^{2n} - \frac{1}{\sqrt{2}}\mu^{2n} + \frac{1}{\sqrt{2}}\lambda^{2n} - \frac{1}{\sqrt{2}}\mu^{2n} + \lambda^{2n} + \mu^{2n}$$

$$= \lambda\mu + \frac{1}{2}[\lambda^n - \mu^n]^2 + \lambda^n\mu^n + \frac{1}{\sqrt{2}}[\lambda^n + \mu^n][\lambda^n - \mu^n] + \frac{1}{\sqrt{2}}[\lambda^n + \mu^n][\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2 - 2\lambda^n\mu^n$$

$$= \lambda\mu - \lambda^n\mu^n + \frac{1}{2}[\lambda^n - \mu^n]^2 + \sqrt{2}[\lambda^n + \mu^n][\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2$$

$$= \lambda\mu - \lambda^n\mu^n + \frac{1}{2}b_n^2 + \sqrt{2}a_n b_n + a_n^2$$

$$= \frac{1}{2}b_n^2 + \sqrt{2}a_n b_n + a_n^2 \quad \because \lambda^n\mu^n = (-1)^n = -1 = \lambda\mu \text{ if } n \text{ is odd.}$$

$$= \frac{1}{2}[b_n^2 + 2\sqrt{2}a_n b_n + 2a_n^2] = \frac{1}{2}[b_n + \sqrt{2}a_n]^2$$

$$\sum_{m=0}^{2n}\sum_{r=0}^{m} a_r = \tfrac{1}{2}b_{n+1}^2 \text{, if } n \text{ is odd.}$$ Q.E.D

If n is even:

$$\sum_{m=0}^{2n}\sum_{r=0}^{m} a_r = \lambda\mu - \lambda^n\mu^n + \tfrac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2$$

$$= \tfrac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2 + 2\lambda\mu \quad \because \lambda^n\mu^n = (-1)^n = 1 = -\lambda\mu \text{ if } n \text{ is even.}$$

$$= \tfrac{1}{2}b_{n+1}^2 - 2$$

$$\sum_{i=0}^{n} a_{2i+1} = \sum_{i=0}^{n} \lambda^{2i+1} + \mu^{2i+1}$$

$$= \sum_{i=0}^{n} \lambda[\lambda^2]^i + \mu[\mu^2]^i$$

$$= \sum_{i=1}^{n} \lambda[\lambda^2]^{i-1} + \mu[\mu^2]^{i-1} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \frac{\lambda[1-\lambda^{2n}]}{1-\lambda^2} + \frac{\mu[1-\mu^{2n}]}{1-\mu^2} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \frac{\lambda-\lambda^{2n+1}}{1-\lambda^2} + \frac{\mu-\mu^{2n+1}}{1-\mu^2} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \frac{[\lambda-\lambda^{2n+1}][1-\mu^2] + [\mu-\mu^{2n+1}][1-\lambda^2]}{[1-\lambda^2][1-\mu^2]} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \frac{[\lambda-\lambda^{2n+1}][2\sqrt{2}-2] + [\mu-\mu^{2n+1}][-2\sqrt{2}-2]}{[2\sqrt{2}-2][-2\sqrt{2}-2]} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \frac{-2\mu[\lambda-\lambda^{2n+1}] - 2\lambda[\mu-\mu^{2n+1}]}{4\lambda\mu} + \lambda^{2n+1} + \mu^{2n+1}$$

$$= -\tfrac{1}{2}[1-\lambda^{2n}] - \tfrac{1}{2}[1-\mu^{2n}] + \lambda^{2n+1} + \mu^{2n+1}$$

$$= \tfrac{1}{2}\lambda^{2n} + \tfrac{1}{2}\mu^{2n} + \lambda\lambda^{2n} + \mu\mu^{2n} - 1$$

$$= \tfrac{1}{2}b_n^2 + \lambda^n\mu^n + a_n^2 - 2\lambda^n\mu^n + \sqrt{2}a_nb_n + \lambda\mu$$

$$= \tfrac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2 - \lambda^n\mu^n + \lambda\mu$$

$$= \begin{cases} \tfrac{1}{2}b_{n+1}^2 - 2 & \text{if } n \text{ is even} \\ \tfrac{1}{2}b_{n+1}^2 & \text{if } n \text{ is odd} \end{cases}$$

iii) If n is even:

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1}$$

$$= \left[\tfrac{1}{\sqrt{2}}b_{n+1}\right]^2 - \left[\tfrac{1}{2}b_{n+1}^2 - 2\right]$$

$$= \tfrac{1}{2}b_{n+1}^2 - \tfrac{1}{2}b_{n+1}^2 + 2$$

$$= 2$$ Q.E.D.

If n is odd:

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1}$$

$$= \tfrac{1}{2}b_{n+1}^2 - \tfrac{1}{2}b_{n+1}^2$$

$$= 0$$