Problem 41. 2012.02.03

Show that, for any function f (for which the integrals exist),

$$\int_0^\infty f\left(x + \sqrt{1 + x^2}\right) dx = \frac{1}{2}\int_1^\infty \left(1 + \frac{1}{t^2}\right) f(t) dt .$$

Hence evaluate

$$\int_0^\infty \frac{1}{2x^2 + 1 + 2x\sqrt{x^2 + 1}} dx,$$

and, using the substitution $x = \tan\theta$,

$$\int_0^{\frac{\pi}{2}} \frac{1}{(1 + \sin\theta)^3} d\theta$$

Prerequisites.

You need to be quite confident about applying the method of Integration by Substitution.

First Thoughts.

There aren't any significant clues here. It looks as if the use of Integration by Substitution may be the thing to try, so let's see what happens.