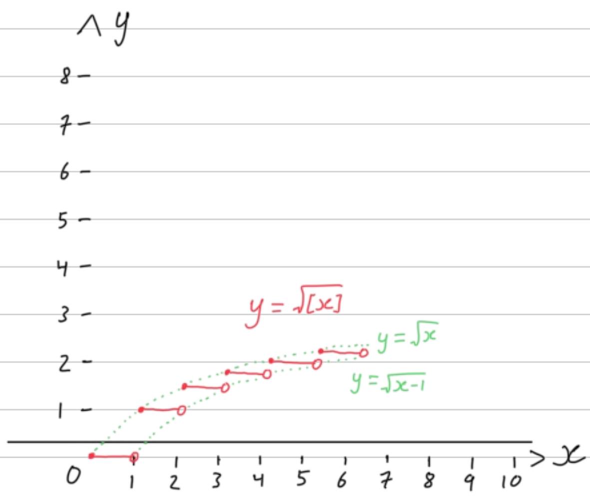
16-11-2024 STEP Practice: Problem 18 (2004.01.02)

$[\pi] = 3$, $[\sqrt{24}] = 4$, $[5] = 5$

$x \in$	$[x]$	$\sqrt{[x]}$	$2^{[x]}$
$[0,1)$	0	0	1
$[1,2)$	1	1	2
$[2,3)$	2	1.4	4
$[3,4)$	3	1.7	8
$[4,5)$	4	2	
$[5,6)$	5	2.2	

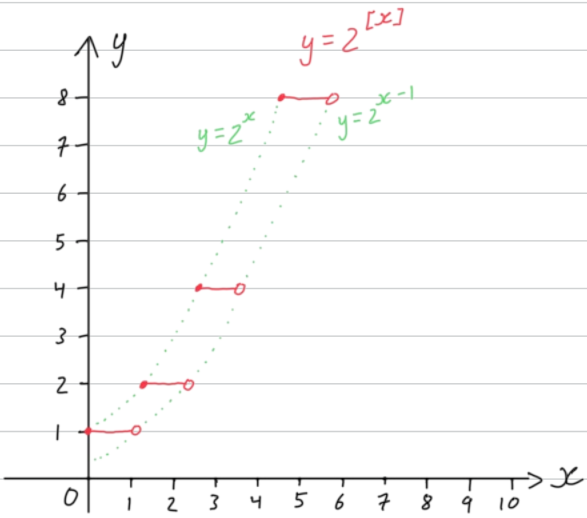
i)



$$\int_0^a \sqrt{[x]}\,dx = 1\cdot\sqrt{0} + 1\cdot\sqrt{1} + 1\cdot\sqrt{2} + 1\cdot\sqrt{3} + \ldots + 1\cdot\sqrt{a-1} = \sum_{r=0}^{a-1} \sqrt{r}, \quad a \in \mathbb{Z}^+.$$

Q.E.D.

ii)



$$\int_0^a 2^{[x]}\,dx = 1 + 2 + 4 + 8 + \ldots + 2^{a-1} = \sum_{r=1}^{a-1} 2^{r-1}$$

Geometric Progression
a: starting value (1)
r: common ratio (2)

General Geometric Progression

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

$rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$

$$\rightsquigarrow \int_0^a 2^{[x]}\,dx = \sum_{r=1}^{a-1} 2^{r-1} = \frac{1\cdot(1-2^a)}{1-2} = 2^a - 1, \quad a \in \mathbb{Z}^+.$$

Q.E.D.

(iii) $$\int_0^a 2^{[x]}\,dx = \int_0^{[a]} 2^{[x]}\,dx + \int_{[a]}^a 2^{[x]}\,dx = 2^a - 1 + \underbrace{2^a}_{\text{height}} \cdot \underbrace{(a-[a])}_{\text{width}} = \boxed{2^a(a-[a]+1)-1}$$

Note

This problem shows the usefulness of the geometrical interpretation of an integral for solving problems. Additionally, it shows how one can pretty easily solve problems which involve concepts that they are unfamiliar with — by using well-tested methods.