Derivation of Summation Formulae

The sum of positive integers, from 1 to n (inclusive) is given by a rather well-known formula.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

This can be proved pretty easily via induction.

pf $n=1$: $\sum_{i=1}^{n} i = \frac{1(1+1)}{2} = 1$ $\therefore$ the statement is true for $n=1$.

Suppose $\sum_{i=1}^{n} i$ is true for $n=k$. Then,

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 \quad \text{(properties of summations)}$$

$$= \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\therefore$ the statement is true for $n=k+1$, if it is true for $n=k$.
$\therefore$ the statement is true for $n \in \mathbb{Z}^+$.

The proof is very basic, and we can perform similar proofs for the formulae of the sum of i^j, where $j \in \mathbb{Z}^+$. However, how would we actually derive the formulae in the first place? Well, I found a very cool method.

Definition (falling factorial):
Let $x, n \in \mathbb{Z}$.
$x^{\underline{n}} := x(x-1)(x-2)\cdots(x-n+1)$

Definition (forward difference):
Let u_n be the nth term of a series
$\Delta u_n := u_{n+1} - u_n$

Discrete Fundamental Theorem of Calculus:

$$\sum_{i=n_0}^{n} \Delta u_i = u_i \Big|_{i=n_0}^{i=n+1} , \text{ and } \Delta \sum_{i=n_0}^{n} u_i = u_{n+1}$$

The truth of these statements is rather important to our derivations, so we will prove them.

pf 1)
$$\sum_{i=n_0}^{n} \Delta u_i = \sum_{i=n_0}^{n} u_{i+1} - u_i = \sum_{i=n_0}^{n} u_{i+1} - \sum_{i=n_0}^{n} u_i$$
$$= \sum_{i=n_0+1}^{n+1} u_i - \sum_{i=n_0}^{n} u_i = u_{n+1} - u_{n_0} + \cancel{\sum_{i=n_0}^{n} u_i} - \cancel{\sum_{i=n_0}^{n} u_i}$$
$$= u_{n+1} - u_{n_0} = \underline{\underline{u_i \Big|_{i=n_0}^{i=n+1}}} \quad \text{Q.E.D.}$$

pf 2)
$$\Delta \sum_{i=n_0}^{n} u_i = \sum_{i=n_0}^{n+1} u_i - \sum_{i=n_0}^{n} u_i = u_{n+1} + \cancel{\sum_{i=n_0}^{n} u_i} - \cancel{\sum_{i=n_0}^{n} u_i}$$
$$= \underline{\underline{u_{n+1}}} \quad \text{Q.E.D.}$$

Now that we have proved those statements, we have everything we need to derive some summation formulae. Firstly, we will show that the falling factorial has a nice property with the forward difference operator – the discrete version of the power rule.

$$\Delta x^{\underline{n}} = (x+1)^{\underline{n}} - x^{\underline{n}} = (x+1)(x)(x-1)\cdots(x-n+2) - x(x-1)(x-2)\cdots(x-n+1)$$
$$= x(x-1)\cdots(x-n+2)\cdot[(\cancel{x}+\cancel{1})-(\cancel{x}-n+\cancel{1})] = nx(x-1)\cdots(x-n+2)$$
$$= nx^{\underline{n-1}}$$

$x^{\underline{1}} = \overset{x-1+1}{x}$, $\Delta x^{\underline{2}} = 2x^{\underline{1}} = 2x$ $\qquad$ $a^{\underline{b}} = 0$ if $a<b$, $a(a-1)\cdots\overbrace{(a-b+1)}^{<1}$

$$\rightsquigarrow \sum_{i=1}^{n} i = \frac{1}{2}\sum_{i=1}^{n} \Delta i^{\underline{2}} = \frac{1}{2} i^{\underline{2}}\Big|_{i=1}^{i=n+1} = \frac{1}{2}\left[(n+1)^{\underline{2}} - \cancelto{0}{1^{\underline{2}}}\right]$$

$$= \frac{1}{2}(n+1)^{\underline{2}} = \underline{\frac{1}{2}(n+1)\overset{n+1-2+1}{n}}$$

$$x^{\underline{2}} = x(x-1) = x^2 - x \implies x^2 = x^{\underline{2}} + x^{\underline{1}}, \quad \Delta x^{\underline{3}} = 3x^{\underline{2}} \implies x^2 = \frac{1}{3}\Delta x^{\underline{3}} + \frac{1}{2}\Delta x^{\underline{2}}$$

$$\rightsquigarrow \sum_{i=1}^{n} i^2 = \frac{1}{3}\sum_{i=1}^{n} \Delta i^{\underline{3}} + \frac{1}{2}\sum_{i=1}^{n} \Delta i^{\underline{2}}$$

$$= \frac{1}{3} i^{\underline{3}}\Big|_{i=1}^{i=n+1} + \frac{n(n+1)}{2} = \frac{1}{3}\left[(n+1)^{\underline{3}} - \cancelto{0}{1^{\underline{3}}}\right] + \frac{n(n+1)}{2} = \frac{1}{3}(n+1)^{\underline{3}} + \frac{n(n+1)}{2}$$

$$= \frac{1}{3}(n+1)(n)\overbrace{(n-1)}^{n+1-3+1} + \frac{n(n+1)}{2} = \frac{2n(n+1)(n-1)}{6} + \frac{3n(n+1)}{6} = \frac{n(n+1)[\overbrace{2(n-1)}^{2n-2}+3]}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$x^{\underline{3}} = x(x-1)(x-2) = x(x^2-3x+2) = x^3-3x^2+2x$, $\Delta x^{\underline{4}} = 4x^{\underline{3}} \Rightarrow x^{\underline{3}} = \frac{1}{4}\Delta x^{\underline{4}}$

$$\Rightarrow x^3 = x^{\underline{3}} + 3x^2 - 2x = x^{\underline{3}} + 3(x^{\underline{2}} + x^{\underline{1}}) - 2x^{\underline{1}} = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} = \frac{1}{4}\Delta x^{\underline{4}} + \cancel{\tfrac{3}{3}}\Delta x^{\underline{3}} + \frac{1}{2}x^{\underline{2}}$$

$$\rightsquigarrow \sum_{i=1}^{n} i^3 = \frac{1}{4}\sum_{i=1}^{n} \Delta i^{\underline{4}} + \sum_{i=1}^{n} \Delta i^{\underline{3}} + \frac{1}{2}\sum_{i=1}^{n} \Delta i^{\underline{2}}$$

$$= \frac{1}{4} i^{\underline{4}}\Big|_{i=1}^{i=n+1} + (n+1)(n)(n-1) + \frac{n(n+1)}{2} = \frac{1}{4}\left[(n+1)^{\underline{4}} - \cancelto{0}{1^{\underline{4}}}\right] + (n+1)(n)(n-1) + \frac{n(n+1)}{2}$$

$$= \frac{1}{4}(n+1)^{\underline{4}} + (n+1)(n)(n-1) + \frac{n(n+1)}{2} = \frac{1}{4}(n+1)(n)(n-1)\overbrace{(n-2)}^{n+1-4+1} + \frac{4}{4}(n+1)(n)(n-1) + \frac{2n(n+1)}{4}$$

$$= \frac{1}{4}n(n+1)\left[(n-1)(n-2) + 4(n-1) + 2\right] = \frac{1}{4}n(n+1)\left[n^2-3n+2+4n-4+2\right] = \frac{1}{4}n(n+1)(n^2+n)$$

$$= \frac{1}{4}n(n+1)(n)(n+1) = \frac{1}{4}n^2(n+1)^2$$

We have derived the formulae for the sum of i, i^2, and i^3. It would be nice to have a formula for i^d. In order to do this, we need to be able to write the monomial x^n, where $n \in \mathbb{Z}^+$, as the sum of falling factorials.

$x^{\underline{1}} = x$

$x^{\underline{2}} = x(x-1) = x^2 - x$

$x^{\underline{3}} = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

$x^{\underline{4}} = (x^3-3x^2+2x)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$

$\vdots$

coefficient of x^n is always 1

coefficient of x is $(-1)^{n-1}(n-1)!$

$x^{\underline{n}}$ has n terms — degree n polynomial

$x^{\underline{n}} = x^{\underline{n-1}}(x-n+1)$

Other coefficients:

n	coefficient of x^{n-1}		
1	0	2	
2	-1	2	1
3	-3	3	1
4	-6	4	1
5	-10	5	1
6	-15	6	
7	-21		

$\boxed{-\frac{n^2}{2} - \frac{n}{2}}$

$n=5$: $x^{\underline{n}} = x^{\underline{4}}(x-4)$

$= (x^4 - 6x^3 + 11x^2 - 6x)(x-4)$

$x^{\underline{n-1}}$: $-4x^4 - 6x^4 = -10x^4$

Considering first term with x, and last term with $n-1$.

$x^{\underline{n}} = {}_n a_1 x^n + {}_n a_2 x^{n-1} + \dots + {}_n a_n$

Then:

$${}_n a_2 = \overbrace{{}_{n-1}a_1}^{1} \cdot (1-n) + {}_{n-1}a_2 \cdot 1$$

$$= {}_{n-1}a_2 - n + 1$$

$$ {}_n u_2 = {}_{n-1}u_2 - n + 1 = {}_{n-2}u_2 - (n-1) + 1 - n + 1 = {}_1\overset{0}{\widetilde{u}}_2 + n - \sum_{i=1}^{n+1} n - i = n - \sum_{i=1}^{n+1} n + \sum_{i=1}^{n+1} i $$

$$ = -n(n+1) + \frac{n(n+1)}{2} = -\frac{n(n+1)}{2} = -\frac{n^2}{2} - \frac{n}{2} $$

$$ {}_n u_3 = {}_{n-1}u_2 \cdot (1-n) + {}_{n-1}u_3 \cdot 1 = (1-n)\,{}_{n-1}u_2 + {}_{n-1}u_3 $$

$$ {}_n u_4 = {}_{n-1}u_3 \cdot (1-n) + {}_{n-1}u_4 \cdot 1 = (1-n)\,{}_{n-1}u_3 + {}_{n-1}u_4 $$

⋮

n	${}_n u_1$	${}_n u_2$	${}_n u_3$	${}_n u_4$	${}_n u_5$	. . . . . .
1	1	0	0	0	0	
2	1	-1	0	0	0	
3	1	-3	2	0	0	. . . . . .
4	1	-6	11	-6	0	
5	1	-10	35	-50	24	
6	1	-15	85	-225	274	
7	1	-21	175	-735	1624	. . . . . .
⋮			⋮		⋮	

These coefficients have a name, and another notation. They are Sterling numbers of the first kind.

$$ x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)\, x^k $$

That would mean that my notation for the coefficients would be able to be translated to the notation for the sterling numbers.

$$ s(n,k) = {}_n u_{n-k} $$

Additionally, we can now express the sum of i^j, where $j \in \mathbb{Z}^+$.

$$ x^{\underline{n}} = \overbrace{s(n,n)}^{1} x^n + s(n,n-1)\,x^{n-1} + \dots + s(n,1)\,x $$

$$ \Rightarrow x^n = x^{\underline{n}} - s(n,n-1)\,x^{n-1} - \dots - s(n,1)\,x $$

$$ x^{\underline{n-1}} = \overbrace{s(n-1,n-1)}^{1} x^{n-1} + s(n-1,n-2)\,x^{n-2} + \dots\ s(n-1,1)\,x $$

$$ \Rightarrow x^{n-1} = x^{\underline{n-1}} - s(n-1,n-2)\,x^{n-2} - \dots - s(n-1,1)\,x $$

$$ \Rightarrow x^{n-k} = x^{\underline{n-k}} - s(n-k,n-k-1)\,x^{n-k-1} - \dots - s(n-k,1)\,x $$

$$x^n - \overbrace{s(n,n)}^{1} x^n \qquad s(n,n-1)\left[x^{n-1} - \sum_{k=1}^{n-2} s(n-1,k)\, x^k\right]$$

$$\rightsquigarrow x^n = \underbrace{x^n - \sum_{k=1}^{n-1} s(n,k)\, x^k}_{} = x^n - \underbrace{s(n,n-1)\, x^{n-1}}_{} - \sum_{k=1}^{n-2} s(n,k)\, x^k$$

$$= x^n - s(n,n-1)\left[x^{n-1} - \underbrace{s(n-1,n-2)\, x^{n-2}}_{s(n-1,n-2)\left[x^{n-2} - \sum_{k=1}^{n-3} s(n-2,k)\, x^k\right]} - \sum_{k=1}^{n-3} s(n-1,k)\, x^k\right] - \sum_{k=1}^{n-2} s(n,k)\, x^k$$

$$x^n = \underbrace{x^n - s(n,n-1)\, x^{n-1} + s(n,n-1)\, s(n-1,n-2) - \ldots}_{}$$

Not sure about this $\longrightarrow$ $\displaystyle\sum_{i=0}^{n-1} \prod_{j=0}^{i} s(n-j-1, n-j)\, x^{n-i} (-1)^i + \cancel{\sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} \prod_{k=0} s(n-k, n-j)}$

I can't figure out what term would go there for the sums in my formula

The polynomial x^n can be expressed as the sum of falling factorials, and Sterling numbers of the second kind.

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$$

I was unable to find a formula to relate Sterling numbers of the first and second kind.