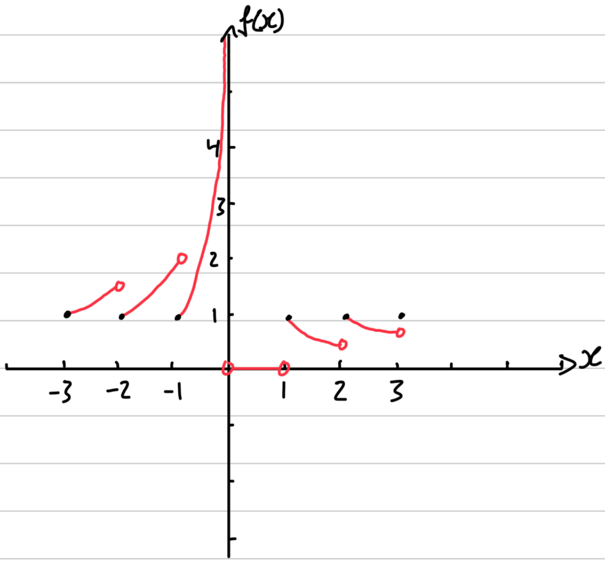
2025-02-08 STEP Practice: Problem 32 (2013.01.02)

$f(x) = \frac{\lfloor x \rfloor}{x} = 1 \quad \forall x \in \mathbb{Z} \setminus \{0\}$

$x = 0.5 : f(x) = \frac{\lfloor 0.5 \rfloor}{0.5} = 0 \rightsquigarrow f(x) = 0 \ \forall x \in (0,1) \because \frac{0}{x} = 0$

x	$\lfloor x \rfloor$	$\frac{\lfloor x \rfloor}{x}$
$[-3,-2)$	-3	$-\frac{3}{x}$
$[-2,-1)$	-2	$-\frac{2}{x}$
$[-1,0)$	-1	$-\frac{1}{x}$
0	0	undefined
$(0,1)$	0	0
$[1,2)$	1	$\frac{1}{x}$
$[2,3)$	2	$\frac{2}{x}$

i)



$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} -\frac{3}{x} = 1^+$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} -\frac{3}{x} = \left(\frac{3}{2}\right)^-$

and similarly at other points of discontinuity

ii) $f(x) = \frac{7}{12}$. $y = \frac{7}{12}$ only intersects the graph in the interval $[1,2)$, where $f(x) = \frac{1}{x}$

$\frac{1}{x} = \frac{7}{12} \implies x = \frac{12}{7}$

$f(x) = \frac{17}{24}$. $y = \frac{17}{24}$ intersects $f(x)$ once in $[1,2)$ and once in $[2,3)$.

$\frac{1}{x} = \frac{17}{24} \implies \frac{24}{17}$. $\frac{2}{x} = \frac{17}{24} \implies x = \frac{48}{17}$

$f(x) = \frac{4}{3}$. $y = \frac{4}{3}$ intersects $f(x)$ once in $[-1,0)$, $[-2,-1)$, and $[-3,-2)$ each.

$-\frac{1}{x} = \frac{4}{3} \implies x = -\frac{3}{4}$. $-\frac{2}{x} = \frac{4}{3} \implies x = -\frac{3}{2}$. $-\frac{3}{x} = \frac{4}{3} \implies x = -\frac{9}{4}$

iii) For $x \in [9,10)$, $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} \frac{9}{x} = \frac{9}{10}$. Note that $f(10) \neq \frac{9}{10}$ so $y = \frac{9}{10}$ does not intersect $f(x)$ in $[9,10)$

Greatest root of $f(x) = \frac{9}{10}$ is in $[8,9)$, where $f(x) = \frac{8}{x}$. $\frac{8}{x} = \frac{9}{10} \implies x = \frac{20}{9}$.

$f(x) = c$ has 1 root if $\frac{1}{2} < c \leq \frac{2}{3}$ or if $c \geq 2$

$f(x) = c$ has 2 roots if $\frac{2}{3} < c \leq \frac{3}{4}$ or if $\frac{3}{2} \leq c < 2$

In general, $f(x)=c$ has exactly n roots if $\frac{n}{n+1} < c \leq \frac{n+1}{n+2}$ or if $\frac{n+1}{n} \leq c < \frac{n}{n-1}$

$\frac{n}{n-1}$ is undefined for $n=1$. The conditions for it are given above.

c having n roots means c must be in the intervals given.
Hence, the condition is necessary and sufficient.