2024-09-21 # STEP Practice

Problem 1 (2004.2.1)

i) $\sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0$

$(\sqrt{3x^2+1} + \sqrt{x})^2 = (2x+1)^2$

$3x^2 + 1 + 2\sqrt{x(3x^2+1)} + x = 4x^2 + 4x + 1$

$(2\sqrt{x(3x^2+1)})^2 = (x^2+3x)^2$

$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$

$12x^3 + 4x = x^4 + 6x^3 + 9x^2$

$x^4 - 6x^3 + 9x^2 - 4x = 0$

$x\underbrace{(x^3 - 6x^2 + 9x - 4)}_{f(x)} = 0 \quad \rightsquigarrow x = 0$

$f(1) = (1)^3 - 6(1)^2 + 9(1) - 4 = 0$

$\Rightarrow x - 1$ is a factor of $f(x)$ (factor theorem)

$-4 = -c \Rightarrow c = 4$

$x^3 = ax^3 \Rightarrow a = 1$

$-6x^2 = bx^2 - x^2$

$-6 = b - 1$

$-5 = b$

$f(x) = x^3 - 6x^2 + 9x - 4 = (x-1)(ax^2 + bx + c)$

$= (x-1)(x^2 - 5x + 4)$

$f(x) = (x-1)(x-1)(x-4)$

$\therefore \boxed{x = 0,\ x = 1,\ x = 4}$

Back Substitution:

$x = 0$: $\sqrt{3(0)^2+1} + \sqrt{0} - 2(0) - 1 = 0$

$x = 1$: $\sqrt{3(1)^2+1} + \sqrt{1} - 2(1) - 1 = 0$

$x = 4$: $\sqrt{3(4)^2+1} + \sqrt{4} - 2(4) - 1 = 0$

} Valid solutions

ii) $\sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0$

$(\sqrt{3x^2+1} - 2\sqrt{x})^2 = (1-x)^2$

$3x^2 + 1 - 4\sqrt{x(3x^2+1)} + 4x = 1 - 2x + x^2$

$(-4\sqrt{x(3x^2+1)})^2 = (-2x^2 - 6x)^2$

$16x(3x^2+1) = 4x^4 + 24x^3 + 36x^2$

$4x(3x^2+1) = x^4 + 6x^3 + 9x^2$

$12x^3 + 4x = x^4 + 6x^3 + 9x^2$

$x^4 - 6x^3 + 9x^2 - 4x = 0$

$x(x^3 - 6x^2 + 9x - 4) = 0$

$x(x-1)^2(x-4) = 0$ (via result obtained in part i)

$\therefore x = 0,\ x = 1,\ x = 4$

Back Substitution:

$x = 0$: $\sqrt{3(0)^2+1} - 2\sqrt{0} + 0 - 1 = 0$
$x = 1$: $\sqrt{3(1)^2+1} - 2\sqrt{1} + 1 - 1 = 0$
} Valid solutions

$x = 4$: $\sqrt{3(4)^2+1} - 2\sqrt{4} + 4 - 1 = 6 \rightsquigarrow x \neq 4$

iii) $\sqrt{3x^2+1} - 2\sqrt{x} - x + 1 = 0$

$(\sqrt{3x^2+1} - 2\sqrt{x})^2 = (x-1)^2$
$3x^2 + 1 - 4\sqrt{x(3x^2+1)} + 4x = x^2 - 2x + 1$
$(-4\sqrt{x(3x^2+1)})^2 = (-2x^2 - 6x)^2$
$16x(3x^2+1) = 4x^4 + 24x^3 + 36x^2$
$x(x-1)^2(x-4) = 0$ (via result obtained in part ii)

$\therefore x = 0,\ x = 1,\ x = 4$

Back Substitution:

$x = 0$: $\sqrt{3(0)^2+1} - 2\sqrt{0} - 0 + 1 = 2 \rightsquigarrow x \neq 0$
$x = 1$: $\sqrt{3(1)^2+1} - 2\sqrt{1} - 1 + 1 = 0$
$x = 4$: $\sqrt{3(4)^2+1} - 2\sqrt{4} - 4 + 1 = 0$
} Valid solutions