

23-11-2024 STEP Practice: Problem 22 (2011.02.07)

Geometric Progression

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \sum_{i=1}^n ar^{i-1}$$

i) $a_n = \lambda^n + \mu^n$, $b_n = \lambda^n - \mu^n$, $\lambda = 1 + \sqrt{2}$, $\mu = 1 - \sqrt{2}$

$$\sum_{r=1}^n b_r = \sum_{r=1}^n \lambda^r - \sum_{r=1}^n \mu^r$$

$$= \frac{[1-\lambda^n]}{1-\lambda} - \frac{[1-\mu^n]}{1-\mu} + \lambda^n - \mu^n$$

$$= \frac{[1-\lambda^n][1-\mu] - [1-\mu^n][1-\lambda]}{[1-\lambda][1-\mu]} + \lambda^n - \mu^n$$

$$= \frac{[1-\lambda^n][\lambda-\lambda+\sqrt{2}] - [1-\mu^n][\lambda-\lambda-\sqrt{2}]}{[\lambda-\lambda-\sqrt{2}][\lambda-\lambda+\sqrt{2}]} + \lambda^n - \mu^n$$

$$= \frac{\sqrt{2}[1-\lambda^n] + \sqrt{2}[1-\mu^n]}{-2} + \lambda^n - \mu^n$$

$$= -\frac{\sqrt{2}}{2} [1-\lambda^n + 1-\mu^n] + \lambda^n - \mu^n$$

$$= -\frac{\sqrt{2}}{2} [2 - \lambda^n - \mu^n] + \lambda^n - \mu^n$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} [\lambda^n + \mu^n] + \lambda^n - \mu^n$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} a_n + \lambda^n - \mu^n$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} [\lambda^n + \mu^n + \sqrt{2}\lambda^n - \sqrt{2}\mu^n]$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} [(1+\sqrt{2})\lambda^n + (1-\sqrt{2})\mu^n]$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} [\lambda \cdot \lambda^n + \mu \cdot \mu^n]$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} [\lambda^{n+1} + \mu^{n+1}]$$

$$= -\sqrt{2} + \frac{1}{\sqrt{2}} a_{n+1}$$

Q.E.D.

$$b_n = \lambda^n - \mu^n$$

$$\sum_{i=0}^n \lambda^i = \sum_{i=1}^n \lambda^{i-1} + \lambda^0$$

$$= \frac{1 \cdot [1-\lambda^n]}{1-\lambda} + \lambda^0$$

$$\sum_{i=0}^n \mu^i = \frac{1 \cdot [1-\mu^n]}{1-\mu} + \mu^0$$

$$a_{n+1} = \lambda \lambda^n + \mu \mu^n$$

$$= [1+\sqrt{2}]\lambda^n + [1-\sqrt{2}]\mu^n$$

$$= \lambda^n + \sqrt{2}\lambda^n + \mu^n - \sqrt{2}\mu^n$$

$$\frac{1}{\sqrt{2}} a_{n+1} = \frac{1}{\sqrt{2}} a_n + \lambda^n - \mu^n$$

$$\begin{aligned}
\sum_{r=0}^n a_r &= \frac{[1-\lambda^n]}{1-\lambda} + \frac{[1-\mu^n]}{1-\mu} + \lambda^n + \mu^n \\
&= \frac{[1-\lambda^n][1-\mu] + [1-\mu^n][1-\lambda] + \lambda^n + \mu^n}{[1-\lambda][1-\mu]} \\
&= \frac{\sqrt{2}[1-\lambda^n] - \sqrt{2}[1-\mu^n]}{-2} + \lambda^n + \mu^n \\
&= -\frac{\sqrt{2}}{2} [\lambda^n - \mu^n] + \lambda^n + \mu^n \\
&= \frac{\sqrt{2}}{2} [\lambda^n - \mu^n] + \lambda^n + \mu^n \\
&= \frac{1}{\sqrt{2}} [\lambda^n - \mu^n + \sqrt{2}\lambda^n + \sqrt{2}\mu^n] \\
&= \frac{1}{\sqrt{2}} [(1+\sqrt{2})\lambda^n - (1-\sqrt{2})\mu^n] \\
&= \frac{1}{\sqrt{2}} [\lambda^{n+1} - \mu^{n+1}] \\
&= \frac{1}{\sqrt{2}} b_{n+1}
\end{aligned}$$

$$\frac{1}{2} b_{n+1}^2 = \frac{1}{2} [b_n + \sqrt{2}a_n]^2$$

$$= \frac{1}{2} b_n^2 + \sqrt{2} a_n b_n + a_n^2$$

$$= \frac{1}{2} [\lambda^n - \mu^n]^2 + \sqrt{2} [\lambda^n + \mu^n] [\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2$$

ii) If n is odd:

$$\begin{aligned}
\sum_{m=0}^{2n} \sum_{r=0}^m a_r &= \frac{1}{\sqrt{2}} \sum_{m=0}^{2n} b_{m+1} = \frac{1}{\sqrt{2}} \sum_{m=0}^{2n} b_m + \sqrt{2} a_m = \frac{1}{\sqrt{2}} [-\sqrt{2} + \frac{1}{\sqrt{2}} a_{2n+1} + \sqrt{2} [\frac{1}{\sqrt{2}} b_{2n+1}]] \\
&= -1 + \frac{1}{2} a_{2n+1} + \frac{1}{\sqrt{2}} b_{2n+1} = -2 + \frac{1}{2} \lambda^{2n+1} + \frac{1}{2} \mu^{2n+1} + \frac{1}{\sqrt{2}} \lambda^{2n+1} - \frac{1}{\sqrt{2}} \mu^{2n+1} \\
&= \lambda\mu + \frac{1}{2} \lambda\lambda^{2n} + \frac{1}{2} \mu\mu^{2n} + \frac{1}{\sqrt{2}} \lambda\lambda^{2n} - \frac{1}{\sqrt{2}} \mu\mu^{2n} \\
&= \lambda\mu + \frac{1}{2} \lambda^{2n} + \frac{1}{2} \mu^{2n} + \frac{1}{\sqrt{2}} \lambda^{2n} - \frac{1}{\sqrt{2}} \mu^{2n} + \frac{1}{\sqrt{2}} \lambda^{2n} - \frac{1}{\sqrt{2}} \mu^{2n} + \lambda^{2n} + \mu^{2n} \\
&= \lambda\mu + \frac{1}{2} [\lambda^n - \mu^n]^2 + \lambda^n \mu^n + \frac{1}{\sqrt{2}} [\lambda^n + \mu^n] [\lambda^n - \mu^n] + \frac{1}{\sqrt{2}} [\lambda^n + \mu^n] [\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2 - 2\lambda^n \mu^n \\
&= \lambda\mu - \lambda^n \mu^n + \frac{1}{2} [\lambda^n - \mu^n]^2 + \sqrt{2} [\lambda^n + \mu^n] [\lambda^n - \mu^n] + [\lambda^n + \mu^n]^2 \\
&= \lambda\mu - \lambda^n \mu^n + \frac{1}{2} b_n^2 + \sqrt{2} a_n b_n + a_n^2 \\
&= \frac{1}{2} b_n^2 + \sqrt{2} a_n b_n + a_n^2 \quad \because \lambda^n \mu^n = (-1)^n = -1 = \lambda\mu \text{ if } n \text{ is odd.} \\
&= \frac{1}{2} [b_n^2 + 2\sqrt{2} a_n b_n + 2a_n^2] = \frac{1}{2} [b_n + \sqrt{2} a_n]^2
\end{aligned}$$

$$\sum_{m=0}^{2n} \sum_{r=0}^m a_r = \frac{1}{2} b_{n+1}^2, \text{ if } n \text{ is odd. } \underline{\text{Q.E.D.}}$$

If n is even:

$$\sum_{m=0}^{2n} \sum_{r=0}^m a_r = \lambda\mu - \lambda^n\mu^n + \frac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2$$

$$= \frac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2 + 2\lambda\mu \because \lambda^n\mu^n = (-1)^n = 1 = -\lambda\mu \text{ if } n \text{ is even.}$$

$$= \frac{1}{2}b_{n+1}^2 - 2$$

$$\begin{aligned} \sum_{i=0}^n a_{2i+1} &= \sum_{i=0}^n \lambda^{2i+1} + \mu^{2i+1} \\ &= \sum_{i=0}^n \lambda[\lambda^2]^i + \mu[\mu^2]^i \\ &= \sum_{i=1}^n \lambda[\lambda^2]^{i-1} + \mu[\mu^2]^{i-1} + \lambda^{2n+1} + \mu^{2n+1} \\ &= \frac{\lambda[1-\lambda^{2n}]}{1-\lambda^2} + \frac{\mu[1-\mu^{2n}]}{1-\mu^2} + \lambda^{2n+1} + \mu^{2n+1} \\ &= \frac{\lambda - \lambda^{2n+1}}{1-\lambda^2} + \frac{\mu - \mu^{2n+1}}{1-\mu^2} + \lambda^{2n+1} + \mu^{2n+1} \end{aligned}$$

iii) If n is even:

$$\begin{aligned} \left(\sum_{r=0}^n a_r\right)^2 - \sum_{r=0}^n a_{2r+1} \\ &= \left[\frac{1}{\sqrt{2}}b_{n+1}\right]^2 - \left[\frac{1}{2}b_{n+1}^2 - 2\right] \\ &= \frac{1}{2}b_{n+1}^2 - \frac{1}{2}b_{n+1}^2 + 2 \\ &= 2 \quad \underline{\text{Q.E.D.}} \end{aligned}$$

If n is odd:

$$\begin{aligned} \left(\sum_{r=0}^n a_r\right)^2 - \sum_{r=0}^n a_{2r+1} \\ &= \frac{1}{2}b_{n+1}^2 - \frac{1}{2}b_{n+1}^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} &= \frac{[\lambda - \lambda^{2n+1}][1-\mu^2] + [\mu - \mu^{2n+1}][1-\lambda^2] + \lambda^{2n+1} + \mu^{2n+1}}{[1-\lambda^2][1-\mu^2]} \\ &= \frac{[\lambda - \lambda^{2n+1}][2\sqrt{2}-2] + [\mu - \mu^{2n+1}][-2\sqrt{2}-2] + \lambda^{2n+1} + \mu^{2n+1}}{[2\sqrt{2}-2][-2\sqrt{2}-2]} \\ &= \frac{-2\mu[\lambda - \lambda^{2n+1}] - 2\lambda[\mu - \mu^{2n+1}] + \lambda^{2n+1} + \mu^{2n+1}}{4\lambda\mu} \\ &= -\frac{1}{2}[1-\lambda^{2n}] - \frac{1}{2}[1-\mu^{2n}] + \lambda^{2n+1} + \mu^{2n+1} \end{aligned}$$

$$= \frac{1}{2}\lambda^{2n} + \frac{1}{2}\mu^{2n} + \lambda\lambda^{2n} + \mu\mu^{2n} - 1$$

$$= \frac{1}{2}b_n^2 + \lambda^n\mu^n + a_n^2 - 2\lambda^n\mu^n + \sqrt{2}a_nb_n + \lambda\mu$$

$$= \frac{1}{2}b_n^2 + \sqrt{2}a_nb_n + a_n^2 - \lambda^n\mu^n + \lambda\mu$$

$$= \begin{cases} \frac{1}{2}b_{n+1}^2 - 2 & \text{if } n \text{ is even} \\ \frac{1}{2}b_{n+1}^2 & \text{if } n \text{ is odd} \end{cases}$$