2025-04-05 STEP Practice: Trigonometric Substitution Integrals 2

1) $I = \int \frac{3}{5+2x+x^2}\,dx = 3\int \frac{1}{[x+1]^2+4}\,dx$ Let $x = 2\tan\theta - 1$, $dx = 2\sec^2\theta\,d\theta$

$= 3\int \frac{2\sec^2\theta}{4\tan^2\theta+4}\,d\theta = \frac{3}{2}\theta + C$

$= \frac{3}{2}\arctan\left[\frac{x+1}{2}\right] + C$

2) $I = \int \frac{4}{5-4x+2x^2}\,dx = 2\int \frac{1}{\frac{5}{2}-2x+x^2}\,dx = 2\int \frac{1}{[x-1]^2+\frac{3}{2}}\,dx$

$= \frac{2}{\sqrt{\frac{3}{2}}}\arctan\left[\frac{x-1}{\sqrt{\frac{3}{2}}}\right] + C = \frac{2\sqrt{6}}{3}\arctan\left[\frac{\sqrt{6}}{3}[x+1]\right] + C$

3) $I = \int \frac{7}{9-4x+x^2}\,dx = 7\int \frac{1}{[x-2]^2+5}\,dx = \frac{7}{\sqrt{5}}\arctan\left[\frac{x-1}{\sqrt{5}}\right] + C$

4) $I = \int \frac{5}{7+4x+3x^2}\,dx = \frac{5}{3}\int \frac{1}{[x+\frac{2}{3}]^2+\frac{17}{9}}\,dx = \frac{5}{3\sqrt{\frac{17}{9}}}\arctan\left[\frac{x+\frac{2}{3}}{\sqrt{\frac{17}{9}}}\right] + C$

$= \frac{5}{\sqrt{17}}\arctan\left[\frac{3x+2}{\sqrt{17}}\right] + C$

Consider the general integral $I = \int \frac{d}{ax^2+bx+c}\,dx$

$I = \frac{d}{a}\int \frac{1}{x^2+\frac{b}{a}x+\frac{c}{a}}\,dx = \frac{d}{a}\int \frac{1}{[x+\frac{b}{2a}]^2+\frac{4ac-b^2}{4a^2}}\,dx$

$I = \frac{d}{a\sqrt{\frac{4ac-b^2}{4a^2}}}\arctan\left[\frac{x+\frac{b}{2a}}{\sqrt{\frac{4ac-b^2}{4a^2}}}\right] + C$

$I = \frac{2d}{\sqrt{4ac-b^2}}\arctan\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right] + C$

This works perfectly fine for cases where $4ac-b^2 > 0$.
This represents the case where the quadratic has no real roots.
In such cases, the integrand is defined for all x.

Consider the case where $4ac - b^2 = 0$.
We cannot use the arctangent.

$$I = \frac{d}{a}\int \frac{1}{[x+\frac{b}{2a}]^2}\,dx = -\frac{d}{a[x+\frac{b}{2a}]} + C$$

In the case where $4ac - b^2 < 0$, we would get complex numbers in our expression. However, the indefinite integral can be used, when limits are added. Thus, obviously, our expression should only produce real numbers.

$$\sqrt{4ac-b^2} = i\sqrt{b^2-4ac}$$

$$I = \frac{2d}{i\sqrt{b^2-4ac}} \arctan\left[\frac{2ax+b}{i\sqrt{b^2-4ac}}\right] + C = \frac{-2di}{\sqrt{b^2-4ac}} \arctan\left[-i\frac{2ax+b}{\sqrt{b^2-4ac}}\right] + C$$

$$= \frac{2di}{\sqrt{b^2-4ac}} \arctan\left[i\frac{2ax+b}{\sqrt{b^2-4ac}}\right] + C$$

$$\arctan z = \frac{1}{2i}\ln\left[\frac{i-z}{i+z}\right] + k\pi \text{ for } k \in \mathbb{Z}.$$

$$I = \frac{d}{\sqrt{b^2-4ac}} \ln\left[\frac{i - i\frac{2ax+b}{\sqrt{b^2-4ac}}}{i + i\frac{2ax+b}{\sqrt{b^2-4ac}}}\right] + C$$

$$I = \frac{d}{\sqrt{b^2-4ac}} \ln\left[\frac{1 + \frac{2ax+b}{\sqrt{b^2-4ac}}}{1 - \frac{2ax+b}{\sqrt{b^2-4ac}}}\right] + C$$

$$I = \frac{d}{\sqrt{b^2-4ac}}\left[\ln[\sqrt{b^2-4ac} + 2ax + b] - \ln[\sqrt{b^2-4ac} - 2ax - b]\right] + C$$

Absolute value signs are not present as we are working with the complex numbers. In the definite integral, the imaginary parts will cancel. Putting absolute value signs will allow for the formula to be used without considering the logarithm of a negative number.