Problem 102. 2017.03.07

Show that the point T with coordinates

$$\left(\frac{a(1-t^2)}{1+t^2}, \frac{2bt}{1+t^2}\right) \qquad (*)$$

(where a and b are non-zero) lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(i) The line L is the tangent to the ellipse at T. The point (X,Y) lies on L, and $X^2 \neq a^2$. Show that

$$(a+X)bt^2 - 2aYt + b(a-X) = 0.$$

Deduce that if $a^2Y^2 > (a^2-x^2)b^2$, then there are two distinct lines through (X,Y) that are tangents to the ellipse. Interpret this result geometrically. Show, by means of a sketch, that the result holds also if $X^2 = a^2$.

(ii) The points P and Q are given by $(*)$, with $t=p$ and $t=q$, respectively. The tangents to the ellipse at P and Q meet at the point with coordinates (X,Y), where $X^2 \neq a^2$. Show that

$$(a+X)pq = a - X$$

and find an expression for $p+q$ in terms of a, b, X and Y.
Given that the tangents meet the y-axis at points $(0,y_1)$ and $(0,y_2)$, where $y_1+y_2=2b$, show that

$$\frac{X^2}{a^2}+\frac{Y}{b}=1.$$

Prerequisites.

Students need to be familiar with the basic processes of Coordinate Geometry to complete the preliminary part and the only additional requirements for part (i) is the ability to carry out implicit differentiation and to use the discriminant of a quadratic equation to establish the conditions for the equation to have real and distinct roots. Using the results relating the coefficients of a quadratic equation to the sum and the product of the roots is the only extra requirement for part (ii) and these minimal prerequisites make this a fairly straightforward problem with which to give this prior learning a good workout.

First Thoughts.

This is quite an inviting question as the preliminary part and part (i) seem quite straightforward. I can envisage what the graph sketch for part (i) should look like. Part (ii) looks as if it may have something to do with the sum and product of the roots of a quadratic equation. I don't like the look of the last expression which is so like the equation of the ellipse but at the same time is so unlike it.