2025-03-22 STEP Practice: Problem 83 (2009.03.03)

The function $f(t)$ is defined for $t \neq 0$ by

$$f(t) = \frac{t}{e^t - 1}.$$

i) $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$$\rightsquigarrow f(t) = \frac{t}{t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots} = \frac{1}{1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots}$$

$$\therefore \lim_{t \to 0} f(t) = \frac{1}{1 + 0 + 0 + \ldots} = 1$$

$$f'(t) = \frac{e^t - 1 - te^t}{[e^t - 1]^2} = \frac{e^t[1 - t] - 1}{[e^t - 1]^2}$$

Attempting to evaluate $\lim_{t \to 0} f'(t)$ by substituting $t = 0$ gives $\frac{0}{0}$, which is indeterminate.

$$\rightsquigarrow \lim_{t \to 0} f'(t) = \lim_{t \to 0} \frac{e^t[1 - t] - e^t}{2e^t[e^t - 1]} \quad \text{by L'Hôpital's Rule}$$

$$= \lim_{t \to 0} \frac{-t}{2[e^t - 1]} = -\frac{1}{2} \lim_{t \to 0} \frac{t}{e^t - 1} = -\frac{1}{2}$$

Alternatively,

$$f'(t) = \frac{[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots] - [t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \ldots]}{[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots]^2} = \frac{-\frac{1}{2}t^2 - \frac{2}{3!}t^3 - \frac{3}{4!}t^4 - \ldots}{[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots]^2}$$

$$= \frac{-\frac{1}{2} - \frac{2}{3!}t - \frac{3}{4!}t^2 - \ldots}{[1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots]^2}$$

$$\therefore \lim_{t \to 0} f'(t) = \frac{-\frac{1}{2} - 0 - 0 - \ldots}{[1 + 0 + 0 + \ldots]^2} = -\frac{1}{2}$$

ii) Let $g(t) = f(t) + \frac{1}{2}t = \frac{t + \frac{1}{2}t[e^t - 1]}{e^t - 1} = \frac{t[1 + \frac{1}{2}e^t - \frac{1}{2}]}{e^t - 1} = \frac{\frac{1}{2}t[e^t + 1]}{e^t - 1}$

$$y(-t) = \frac{-\frac{1}{2}t[e^{-t}+1]}{e^{-t}-1} = \frac{\frac{1}{2}t[e^{-t}+1]}{1-e^{-t}} = \frac{\frac{1}{2}te^{t}[e^{-t}+1]}{e^{t}-1} = \frac{\frac{1}{2}t[1+e^{t}]}{e^{t}-1} = y(t)$$

$\therefore y(t)$ is an even function.

iii) Let $h(t) = e^{t}[1-t]$.

$\lim_{t\to\infty} h(t) = -\infty$, $\lim_{t\to-\infty} h(t) = 0^{+}$

$h(t) = 0 \Rightarrow t = 1$

$h'(t) = e^{t} - e^{t} - te^{t} = -te^{t}$

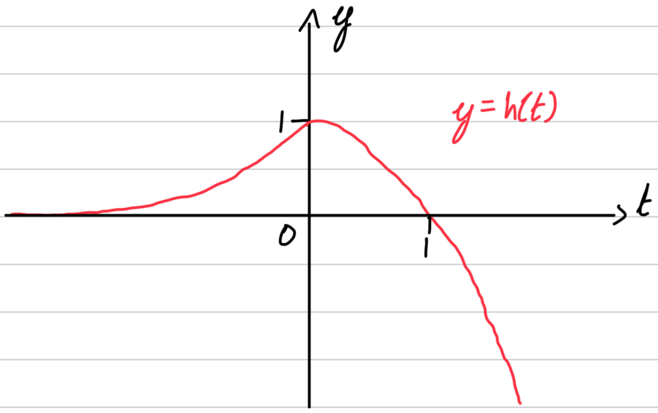
$h'(t) = 0 \Rightarrow t = 0$.

$h(0) = 1$.

$h''(t) = -e^{t} - te^{t}$

$h''(0) = -1 < 0$

$\therefore$ Maximum at $(0,1)$.



It can be seen that $h(t) = e^{t}[1-t] \leq 1$.

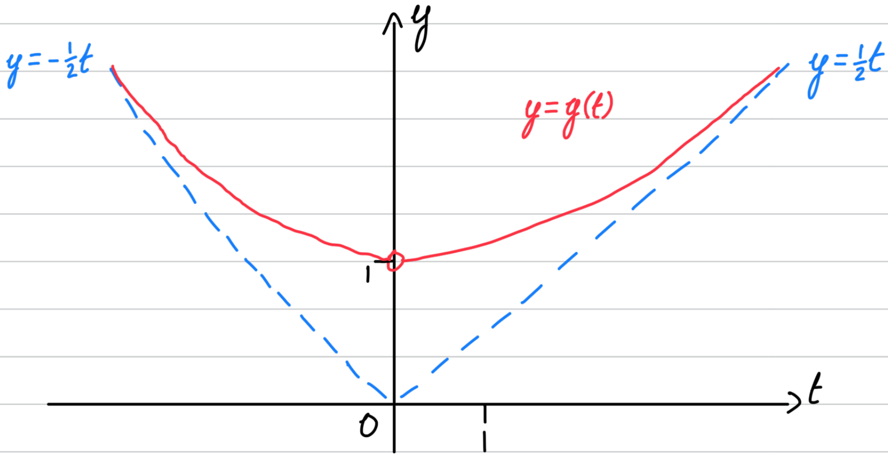
Thus, $e^{t}[1-t] - 1 \leq 0$.

$e^{t}[1-t] - 1 = 0$ only if $t = 0$.

$\therefore f'(t) = \frac{e^{t}[1-t]-1}{[e^{t}-1]^{2}} \neq 0$ for $t \neq 0$.

$\lim_{t\to\infty} y(t) = \infty$. The graph is symmetrical about the y-axis since $y(t)$ is even.

As $t \to \infty$: $y(t) = \frac{\frac{1}{2}t[e^{t}+1]}{e^{t}-1} \approx \frac{1}{2}t$.



Consider $y=f(t)=g(t)-\frac{1}{2}t$.

$\lim_{t\to\infty} f(t)=0^+$, $\lim_{t\to-\infty} f(t)=\infty$.

As $t\to-\infty$: $f(t)\approx-\frac{1}{2}t-\frac{1}{2}t=-t$.