2025-05-10 STEP Practice: Complex Numbers

Complex numbers are those of the form $(x,y) \in \mathbb{R}^2$ where
1) $(a,b)+(c,d) = (a+c, b+d)$
2) $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$
3) $c\cdot(a,b) = (ac, bc)$

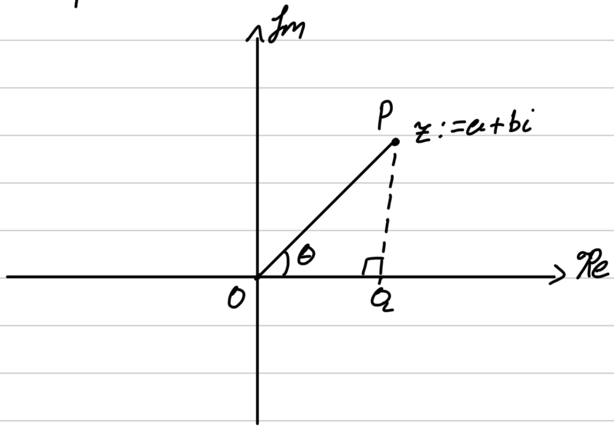
Complex numbers of the form $(a,0)$ are identified with the real number a.
Let $i := (0,1)$,
then $i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$

$(a,b) = a(1,0) + b(0,1) = a+bi$

$\mathrm{Re}: \mathbb{C} \to \mathbb{R}$ $\qquad$ $\mathrm{Im}: \mathbb{C} \to \mathbb{R}$
$\mathrm{Re}(z := a+bi) := a$ $\qquad$ $\mathrm{Im}(z := a+bi) := b$

$\mathbb{R} \subset \mathbb{C}$

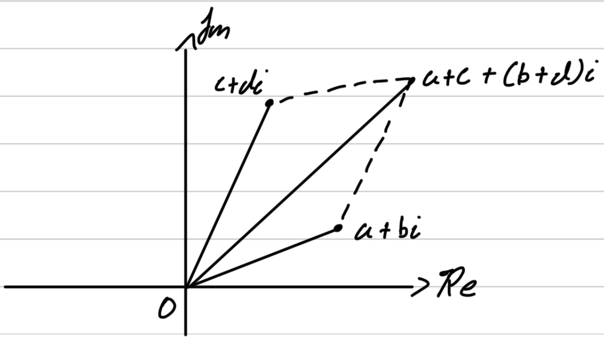
Polar Representation



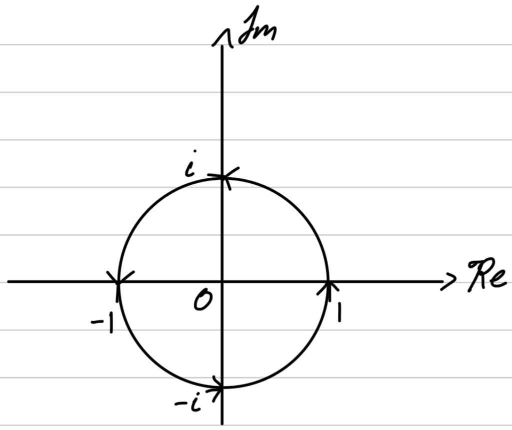
$OQ = r\cos\theta$
$PQ = r\sin\theta$

$z = r[\cos\theta + i\sin\theta]$

$OQ = a$, $PQ = b$, $OP = \sqrt{a^2+b^2}$
$|z| := \sqrt{a^2+b^2}$
$\mathrm{Arg}(z) := \theta$, where $-\pi < \theta \le \pi$
$\arg(z) := \theta + 2\pi k$, where $k \in \mathbb{Z}$



Addition represents a translation



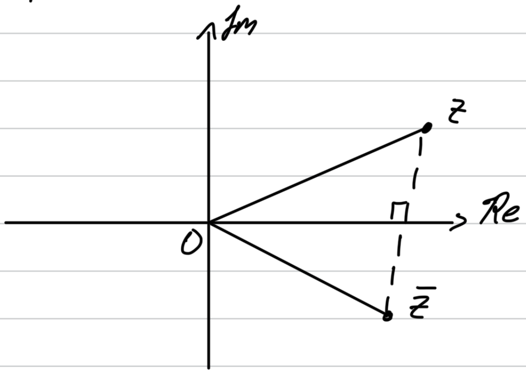
Multiplication by i represents an anticlockwise rotation by $\frac{\pi}{2}$ about O.

$z := r_1[\cos\theta + i\sin\theta]$
$w := r_2[\cos\varphi + i\sin\varphi]$

$$zw = r_1 r_2[\cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\sin\theta\cos\varphi - \sin\theta\sin\varphi]$$
$$= r_1 r_2[[\cos\theta\cos\varphi - \sin\theta\sin\varphi] + i[\sin\theta\cos\varphi + \cos\theta\sin\varphi]]$$
$$= r_1 r_2[\cos(\theta+\varphi) + i\sin(\theta+\varphi)]$$

Multiplication by a complex number represents a combination of a rotation about the origin and a dilation centered at the origin.

$\bar{z} := a - bi$ is the complex conjugate of $z := a + bi$.
It represents a reflection in the real axis.



$$e^z := \sum_{i=0}^{\infty} \frac{z^i}{i!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \dots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$= [1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots] + i[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots]$$

$$= \cos\theta + i\sin\theta$$ Euler's Formula

$\therefore z = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$\therefore e^{i\pi} + 1 \equiv 0$ Euler's Identity