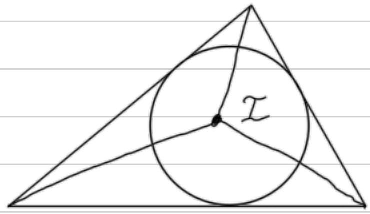
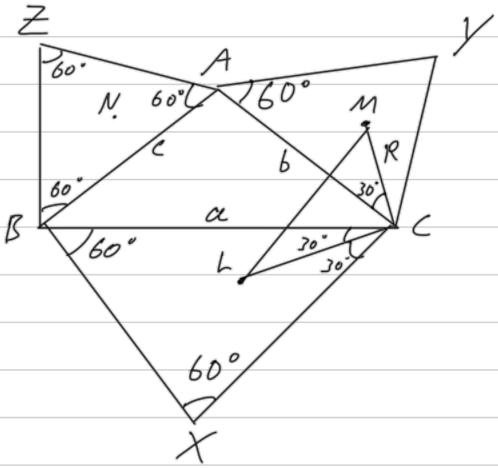
09-11-2024 STEP Practice: Problem 9 (2017.1.07)



The angle bisectors are concurrent at the in-centre. It is the centre of the in-circle, which is tangent to the sides of the triangle.



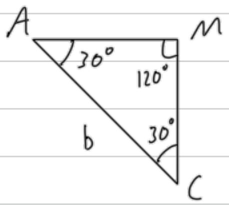
Because M is the centre of symmetry of triangle ACY, it is also the circumcentre of triangle ACY.

$\therefore |CM|$ is the radius of the circumscribed circle.

M is also the in-centre so MC bisects angle ACY.

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

i)



$$\frac{|CM|}{\sin(30^\circ)} = \frac{b}{\sin(120^\circ)} \Rightarrow |CM| = \frac{\sin(30^\circ) b}{\sin(120^\circ)} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} b = \frac{b}{\sqrt{3}}$$ Q.E.D.

Similarly, $|LC| = \frac{a}{\sqrt{3}}$

we could easily find $|LC|$ manually, but the method is the exact same except for the side of triangle ABC we use.

ii) Apply the cosine rule to triangle LCM

$|LM|^2 = |MC|^2 + |CL|^2 - 2 \cdot |MC| \cdot |CL| \cdot \cos(L\hat{C}M)$

$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - 2 \cdot \frac{b}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} \cdot \cos(C+60°)$

$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - \frac{2ab}{3}\cos(C+60°)$

$|LM|^2 = \frac{b^2}{3} + \frac{a^2}{3} - \frac{2\cancel{ab}}{3}\left[\frac{a^2+b^2-c^2}{4\cancel{ab}} - \frac{\sqrt{3}\Delta}{\cancel{ab}}\right]$

$= \frac{b^2}{3} + \frac{a^2}{3} - \frac{a^2}{6} - \frac{b^2}{6} + \frac{c^2}{6} + \frac{2\sqrt{3}\Delta}{3}$

$6|LM|^2 = 2b^2 + 2a^2 - a^2 - b^2 + c^2 + 4\sqrt{3}\Delta$

$\therefore 6|LM|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta$ Q.E.D.

$\cos(L\hat{C}M) = \cos(C + A\hat{C}M + B\hat{C}L)$
$= \cos(C + 30° + 30°)$
$= \cos(C+60°)$
$= \cos C \cos(60°) - \sin C \sin(60°)$
$= \frac{1}{2}\cos C - \frac{\sqrt{3}}{2}\sin C$

Apply the cosine rule to triangle ABC

$\cos C = \frac{a^2+b^2-c^2}{2ab}$

Let $\Delta :=$ Area of triangle $= \frac{1}{2}ab\sin C$

$\rightsquigarrow \sin C = \frac{2\Delta}{ab}$

$\therefore \cos(L\hat{C}M) = \cos(C+60°)$
$= \frac{a^2+b^2-c^2}{4ab} - \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}\Delta}{ab}$

Similarly, $6|MN| = 6|LM|$
$= a^2+b^2+c^2+4\sqrt{3}\Delta$

$\therefore$ Triangle LMN is equilateral.

Let $\Delta_* :=$ Area of triangle LMN

$\Delta_* = \frac{1}{2}|LM|^2 \cdot \sin(60°) = \frac{\sqrt{3}}{4}|LM|^2$

$\Delta_* = \Delta \Rightarrow \frac{\sqrt{3}}{4}|LM|^2 = \Delta = \frac{\sqrt{3}}{24}\left[a^2+b^2+c^2+4\sqrt{3}\Delta\right]$

$24\Delta = \sqrt{3}\left[a^2+b^2+c^2\right] + 12\Delta$

$a^2+b^2+c^2 = \frac{12\Delta}{\sqrt{3}} = 4\sqrt{3}\Delta$

$a^2+b^2+c^2 = 4\sqrt{3}\Delta \Rightarrow |LM|^2 = \frac{1}{6} \cdot 8\sqrt{3}\Delta = \frac{4}{3}\sqrt{3}\Delta \Rightarrow \Delta_* = \frac{\sqrt{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{3}\sqrt{3} = \Delta$

$\therefore \Delta_* = \Delta \iff a^2+b^2+c^2 = 4\sqrt{3}\Delta$ Q.E.D.

iii) $[a-b]^2 = -2ab[1-\cos(C-60°)]$

$a^2 - 2ab + b^2 = -2ab[1 - [\cos C\cos(60°) + \sin C \sin(60°)]]$

$= -2ab[1 - \frac{1}{2}\cos C - \frac{\sqrt{3}}{2}\sin C]$

$a^2 - \cancel{2ab} + b^2 = -\cancel{2ab} + ab\cos C - ab\sqrt{3}\sin C$

$a^2 + b^2 = \cancel{ab} \cdot \frac{a^2+b^2-c^2}{2\cancel{ab}} + \cancel{ab}\sqrt{3} \cdot \frac{2\Delta}{\cancel{ab}}$

$a^2+b^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{2} + 2\sqrt{3}\Delta$

$\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} = 2\sqrt{3}\Delta$

$\therefore a^2+b^2+c^2 = 4\sqrt{3}\Delta$

Steps are reversible $\therefore$ conditions are equivalent. Q.E.D.

$1 - \cos(C-60°) = 0$
$\cos(C-60°) = 1$
$C - 60° = 0°$
$\therefore C = 60°$

$\Delta_* = \Delta \iff a^2+b^2+c^2 = 4\sqrt{3}\Delta \iff [a-b]^2 = -2ab[1-\cos(C-60°)]$

$[a-b]^2 \geq 0 \wedge -2ab[1-\cos(C-60°)] \leq 0 \because -2ab < 0 \wedge 1-\cos(C-60°) \geq 0$

$\iff [a-b]^2 = 0 \wedge 1-\cos(C-60°) = 0$

$\iff a = b \wedge C = 60°$

$\iff$ Triangle ABC is equilateral. Q.E.D.