25-01-2025 STEP Practice: Limits

A function $f(x)$ is continuous if $f(a) = \lim_{x \to a} f(x)$

1) $\lim_{x \to a} k = k$ if a and k are constant.

2) $\lim_{x \to \infty} \frac{1}{x} = 0^+$. $\lim_{x \to -\infty} \frac{1}{x} = 0^-$.

3) $f(x) = \frac{1}{x}$. $\lim_{x \to 0^+} \frac{1}{x} = +\infty$. $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. $\rightsquigarrow$ infinite discontinuity at $x = 0$.

4) If $p(x)$ is any polynomial of degree n in which the coefficient of x^n is positive,
$\lim_{x \to \infty} p(x) = +\infty$. $\lim_{x \to -\infty} p(x) = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{cases}$

5) Let a be real and suppose that: $\lim_{x \to a} f(x) = l_1$, $\lim_{x \to a} g(x) = l_2$.

a) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l_1 + l_2$

b) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l_1 - l_2$

c) $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = l_1 l_2$

d) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l_1}{l_2}$ provided $l_2 \neq 0$

e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{l_1}$ provided $\sqrt[n]{l_1} \geq 0$ if n is even.

Note: it is not always possible to swap the order of operations with limits.

Problem 31.

i) $\lim_{x \to \infty} \frac{x}{x+1} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1} = 1$

ii) $\lim_{x \to 0} \frac{[x-5]^2 - 25}{x} = \lim_{x \to 0} \frac{[x \cancel{-5+5}][x-5-5]}{x} = \lim_{x \to 0} \frac{\cancel{x}[x-10]}{\cancel{x}} = \lim_{x \to 0} x - 10 = -10$

iii) $\lim_{x \to \infty} \frac{\sin x}{x} = 0 \because \sin x$ oscillates as $x \to \infty$ and $\frac{1}{x} \to 0$

iv) $\lim_{x \to \infty} [\tan^{-1} x]^{-1} = [\frac{\pi}{2}]^{-1} = \frac{2}{\pi}$. Note: The domain of $\tan^{-1} x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

v) $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x-3} = \lim_{x \to 3} \frac{[\sqrt{x+1} - 2][\sqrt{x+1} + 2]}{[x-3][\sqrt{x+1} + 2]} = \lim_{x \to 3} \frac{\cancel{x-3}}{[\cancel{x-3}][\sqrt{x+1} + 2]} = \lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

vi) $\lim\limits_{x \to \infty} \dfrac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \lim\limits_{x \to \infty} \dfrac{2 - \frac{1}{x^2} + 8\frac{1}{x^3}}{-5 + 7\frac{1}{x^4}} = -\dfrac{2}{5}$