

# Faulhaber's Formula

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## I Prerequisites

You need very little to understand everything that will be discussed. A decent number of new concepts will be introduced, but when I do introduce something, I plan on being thorough on it. The most important 'prerequisite' is the ability to understand new things, and to follow simple the simple logic that will be used to go from one equation to another.

Additionally, knowledge of basic calculus (derivatives, integrals, and the relationship between them) would certainly aid in one's appreciation for the results we will derive. Whilst someone could technically understand everything without it, I do not think that there would be as much value in doing so. Connections between that which is seemingly unrelated lies at the heart of what I deem to be interesting in mathematics – which happens to be most things. There is a very particular beauty in these connections. The more you know beforehand, the more you have to link new concepts to.

Summation notation is required to be able to read, and by extension, understand, the maths present here. As it is just shorthand for repeated addition, I do not consider it to be a 'new' piece of maths. You could pick it up via the simple examples given in the introduction, but prior experience is expected.

## II Introduction

The formula for the sum of the first  $n$  positive integers is something that many people learn fairly early in school.

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

The sum of the squares of the first  $n$  positive integers, as well as the sum of cubes, have reasonably well-known formulae.

$$\begin{aligned}\sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^n i^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

These formulae are good and all, but how do we prove them? We can take an algebraic approach, a geometric one, or we may simply use induction. The latter method will always work as long as we already have an algebraic expression for the sum in terms of  $n$ . What if we do not have the end result? Thus, the problem arises – deriving a formula for the sum of the  $k$ -th powers of the first  $n$  positive integers: Faulhaber’s formula.

$$S_k(n) = \sum_{i=1}^n i^k = 1^k + 2^k + \dots + n^k = ?$$

$S_k$  is just another shorthand introduced; the previous three sums can be referred to as  $S_1$ ,  $S_2$ , and  $S_3$ . There are several ways of deriving Faulhaber’s formula, and the method that I will use is not the most common one. Whether it is the ”best” way is subjective, but, to me, it is certainly the most interesting.

First, let us suppose that we have some sequence  $a$ , or equivalently,  $(a_n)$ . The  $n$ -th element of the sequence is denoted as  $a_n$ . Now, let us define the forward difference operator,  $\Delta$ , as follows.

$$\Delta_n a_n = a_{n+1} - a_n \tag{1}$$

Note that  $\Delta$  applies to the sequence  $a$ , and not to  $n$ -th element of the sequence. Therefore, you could write  $\Delta_n a_n$  as  $(\Delta a)_n$ , which may technically be more correct – but it is rather cumbersome to use.

The indefinite sum, or the antidifference, of a sequence can be defined in relation to the forward difference.

$$\Delta_n \sum_n a_n = a_n \quad (2)$$

That is, if  $\sum_n a_n = A_n$ , then

$$A_{n+1} - A_n = a_n. \quad (3)$$

Now, you may notice that this seems familiar. Basic knowledge of calculus allows one can spot a connection. We have ventured into the realm of discrete calculus, dealing with finite changes in quantities.

The forward difference is the discrete analogue to the derivative. Where the latter measures the instantaneous rate of change of a function at some point, the former measures the rate of change of a sequence between two points. Similarly, the antidifference is the discrete analogue to the antiderivative (indefinite integral).

Just as the fundamental theorem of calculus relates derivatives to integrals, we can construct a discrete version of it which relates forward differences to antidifferences.

$$\Delta_n \sum_{i=n_0}^n a_i = a_{n+1} \quad (4)$$

$$\sum_{i=n_0}^n \Delta_i a_i = a_{n+1} - a_{n_0} \quad (5)$$

The proof of these two states are rather simple, only requiring the use of the definitions we have laid out, and the properties of sums. The proof of the first statement is as follows.

$$\Delta_n \sum_{i=n_0}^n a_i = \sum_{i=n_0}^{n+1} a_i - \sum_{i=n_0}^n a_i$$

$$\begin{aligned}
&= a_{n+1} + \sum_{i=n_0}^n a_i - \sum_{i=n_0}^n a_i \\
&= a_{n+1}
\end{aligned}$$

The proof of the second statement is as follows.

$$\begin{aligned}
\sum_{i=n_0}^n \Delta_i a_i &= \sum_{i=n_0}^n (a_{i+1} - a_i) \\
&= \sum_{i=n_0}^n a_{i+1} - \sum_{i=n_0}^n a_i \\
&= \sum_{i=n_0+1}^{n+1} a_i - \sum_{i=n_0}^n a_i \\
&= a_{n+1} - a_{n_0} + \sum_{i=n_0}^n a_i - \sum_{i=n_0}^n a_i \\
&= a_{n+1} - a_{n_0}
\end{aligned}$$

The relationship between the definite sum and its indefinite counterpart follows from the second statement.

$$\sum_{i=n_0}^n a_i = \sum_{i=n_0}^n \Delta_i A_i = A_{n+1} - A_{n_0} \tag{6}$$

Whilst we will not mention indefinite sums again, I thought that there inclusion here would serve to further illustrate the connection between regular old calculus, and what we have started to formulate. The connections between various topics in mathematics is what, in my belief, leads to ones perception of its beauty. Many more connections are to be found as we continue on. There is one more definition that needs to be made (kind of). That is the falling factorial. There are a few different notations for it, and its rising counterpart (which we will not be mentioning again). The notation that I will be using is  $x^{\underline{k}}$ , which may be read as x to the falling k.

$$x^{\underline{k}} = x(x-1)(x-2) \cdots (x-k+1). \tag{7}$$

You can see the falling factorial as just a regular factorial, but with a specified end point. What we will do next makes a lot more sense if you see everything

we have done so far as creating this discrete version of calculus. The two most important concepts in calculus, the derivative and the integral, have already been translated over. Derivatives have work nicely with some functions. You can find the derivative of any polynomial very easily with the power rule. Now, it would only make sense for there to be some discrete version of the power rule. Turns out there is, and it involves the falling factorial.

$$\begin{aligned}
\Delta_x x^{\underline{k}} &= (x+1)^{\underline{k}} - x^{\underline{k}} \\
&= (x+1)x(x-1)\cdots(x-k+2) - x(x-1)(x-2)\cdots(x-k+1) \\
&= x(x-1)\cdots(x-k+2) \cdot [(x+1) - (x-k+1)] \\
&= kx(x-1)\cdots(x-k+2) \\
&= kx^{\underline{k-1}}
\end{aligned}$$

Once again, we have an eerily familiar result. Well, we best make use of it. It is important to always be clear on what our goal is. We have a sum to evaluate, and we have created some tools that resemble familiar concepts in calculus. If instead of a sum, we had an integral to evaluate, one way of doing so would be to express the integrand (everything inside of the integral) as the derivative of something else. Using the fact that derives and integrals are the opposite of each other, we could then 'cancel them out' to get our answer. This is the essence of our method. Since we know the forward difference of the falling factorial, we know the sum of it as well.

$$\begin{aligned}
\sum_{i=1}^n i^{\underline{k}} &= \sum_{i=1}^n \Delta_i \frac{i^{\underline{k+1}}}{k+1} \\
&= \frac{(n+1)^{\underline{k+1}}}{k+1} - \frac{1^{\underline{k+1}}}{k+1} \\
&= \frac{(n+1)^{\underline{k+1}}}{k+1}
\end{aligned}$$

A result we used above to simplify our expression was that  $k > x$  implies  $x^{\underline{k}} = 0$ . This is because  $x - k + 1$  will be less than or equal to 0 (thus one of the factors will always be 0). Since  $k + 1 > 1$  (the falling factorial doesn't make much sense if  $k$  is less than or equal to 0), we were able to apply this. Now, if we can express some sequence in terms of falling factorials, then we can evaluate the sum of that sequence. Let us evaluate

The formula for the sum of the first  $n$  integers can be derived as follows.

$$\begin{aligned}\sum_{i=1}^n i &= \sum_{i=1}^n i^1 \\ &= \frac{(n+1)^2}{2} \\ &= \frac{n(n+1)}{2}\end{aligned}$$

We can derive the formula for the sum of the squares of the first  $n$  integers. However, we need to find a way to express  $i^2$  in terms of falling factorials first.

$$\begin{aligned}i^2 &= i(i-1) = i^2 - i \\ \implies i^2 &= i^2 + i = i^2 + i^1\end{aligned}$$

The rest of the work simply involves using the "reverse power rule" and simplifying the expression we get.

$$\begin{aligned}\sum_{i=1}^n i^2 &= \sum_{i=1}^n (i^2 + i^1) \\ &= \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} \\ &= \frac{n(n+1)(n-1)}{3} + \frac{n(n+1)}{2} \\ &= \frac{2n(n+1)(n-1) + 3n(n+1)}{6} \\ &= \frac{n(n+1)[2(n-1) + 3]}{6} \\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

We can continue this for higher powers.

$$\begin{aligned}i^3 &= i(i-1)(i-2) = i^3 - 3i^2 + 2i \\ \implies i^3 &= i^3 + 3i^2 - 2i = i^3 + 3i^2 + i^1\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^n i^3 &= (i^3 + 3i^2 + i^1) \\
&= \frac{(n+1)^4}{4} + 3\frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} \\
&= \frac{n(n+1)(n-1)(n-2)}{4} + n(n+1)(n-1) + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(n-1)(n-2)}{4} + \frac{4n(n+1)(n-1)}{4} + \frac{2n(n+1)}{4} \\
&= \frac{n(n+1)(n-1)(n-2) + 4n(n+1)(n-1) + 2n(n+1)}{4} \\
&= \frac{n(n+1)[(n-1)(n-2) + 4(n-1) + 2]}{4} \\
&= \frac{n(n+1)(n^2 + n)}{4} \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

We could do this for any power of  $n$ . The amount of work would increase time, but the method is exactly the same. It would be nice if we could express the sum of some power of  $n$  in some compact formula. To generalise what we have done, first notice that expanding the falling factorial gives us a polynomial with certain coefficients.

$$\begin{aligned}
x^{\underline{1}} &= x \\
x^{\underline{2}} &= x^2 - x \\
x^{\underline{3}} &= x^3 - 3x^2 + 2x \\
x^{\underline{4}} &= x^4 - 6x^3 + 11x^2 - 6x \\
x^{\underline{5}} &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x \\
&\vdots
\end{aligned}$$

These coefficients are the Stirling numbers of the first kind,  $s(k, i)$ . One way of defining these numbers is via the above expansion.

$$x^{\underline{k}} = \sum_{i=1}^k s(k, i)x^i$$

The absolute value of these numbers are referred to as the unsigned Stirling numbers of the first kind, denoted  $\left[ \begin{smallmatrix} k \\ i \end{smallmatrix} \right]$ . They can be defined as the number of permutations of  $k$  elements with  $i$  disjoint cycles. For example,  $3! = 6$  is the number of permutations of three elements. One of these permutations is with three cycles:  $(1)(2)(3)$ . 1 is sent to 1, 2 is sent to 2, and 3 is sent to 3. Three of the permutations are with two cycles:  $(1)(23)$ ,  $(12)(3)$ , and  $(13)(2)$ . In the first one listed, 1 is sent to 1, 2 is sent to 3, and 3 is sent to 2. Two of the six permutations are with one cycle:  $(123)$  and  $(132)$ . Disjoint means that two cycles cannot have the same element.

Stirling numbers of the first kind and their unsigned counterparts are related via the parity of  $k - i$ .

$$s(k, i) = (-1)^{k-i} \left[ \begin{smallmatrix} k \\ i \end{smallmatrix} \right]$$

When we rearrange the polynomials we get in order to get  $x^k$  in terms of falling factorials, the coefficients of the falling factorials are important.

$$\begin{aligned} x &= x^1 \\ x^2 &= x^2 + x^1 \\ x^3 &= x^3 + 3x^2 + x^1 \\ x^4 &= x^4 + 6x^3 + 7x^2 + x^1 \\ x^5 &= x^5 + 10x^4 + 25x^3 + 15x^2 + x^1 \\ &\vdots \end{aligned}$$

These coefficients are the Stirling numbers of the second kind,  $S(k, i)$  or  $\left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\}$ . They can be defined via the above expansion.

$$x^k = \sum_{i=1}^k \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} x^i$$

They can also be defined as the number of ways to partition a set of  $k$  elements into  $i$  nonempty subsets. This representation of  $\left\{ \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \right\}$  to  $\left\{ \begin{smallmatrix} 4 \\ 4 \end{smallmatrix} \right\}$  are shown below. Each colour is its own subset and dots that are not within a colour are within their own subset.  $\left\{ \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \right\}$  is on the top of the diagram and  $\left\{ \begin{smallmatrix} 4 \\ 4 \end{smallmatrix} \right\}$  is on the bottom.

The sum of the  $k$ -th powers of the first  $n$  positive integers can be written in



terms of the Stirling numbers of the second kind and falling factorials.

$$\begin{aligned}
\sum_{i=1}^n i^k &= \sum_{i=1}^n \sum_{j=1}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} i^j \\
&= \sum_{j=1}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \sum_{i=1}^n i^j \\
&= \sum_{j=1}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \frac{(n+1)^{\underline{j+1}}}{j+1}
\end{aligned}$$

With a swap of index variables, we get Faulhaber's formula.

$$\sum_{i=1}^n i^k = \sum_{i=1}^k \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \frac{(n+1)^{\underline{i+1}}}{i+1} \tag{8}$$