## Problem 89.    1998.01.06

Let $a_1 = \cos x$ with $0 < x < {}^{\pi}/_2$ and let $b_1 = 1$. Given that

$$a_{n+1} = \frac{1}{2}(a_n + b_n)$$

$$b_{n+1} = (a_{n+1}b_n)^{1/2}$$

find $a_2$ and $b_2$ and show that

$$a_3 = \cos\frac{x}{2}\cos^2\frac{x}{4} \quad \text{and} \quad b_3 = \cos\frac{x}{2}\cos\frac{x}{4}$$

Guess general expressions for $a_n$ and $b_n$ (for $n \geq 2$) as products of cosines and verify that they satisfy the given equations.

### Prerequisites.

Students will need to be familiar with the double angle formulae and with $\Pi$ notation for products.

### First Thoughts.

Finding $a_2$, $b_2$, $a_3$, and $b_3$ should be reasonably easy but there is a clear need for some kind of half angle formula. I could probably manufacture one from an expression for $\cos 2\theta$.
Guessing the general form of something depends upon recognising the underlying pattern and in order to be sure to seize upon the correct pattern it is sometimes necessary to calculate more than just two terms of a sequence.