Problem 86. 2016.03.06

Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x + \gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x coordinates of the points of intersection can be written in the form

$$\pm\cosh^{-1}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \tanh^{-1}\frac{a}{b}$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y coordinate of the point of contact in terms of a.

Prerequisites.

You need to be secure in your knowledge of the identities concerning the hyperbolic functions and you need to be comfortable with adapting the process of finding the harmonic forms of $A\sin\theta + B\cos\theta$ to the needs of this problem. You will need to draw heavily upon your knowledge of the graphs of the hyperbolic functions and their inverses and take account of their domains and ranges.

First Thoughts.

Having been told what to do, all I need to do is actually do it. This is a clear case of adapting the work on trigonometry and applying it in the setting of hyperbolic functions. Since parts (i) and (ii) make clear that there are two cases to be considered the conditions that apply need careful thought.
Finding necessary and sufficient conditions for almost anything can be problematic. I don't expect this to be anything other than difficult!