2024-09-21

STEP Practice

Problem 1 (2004.2.1)

i) $\sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0$

$(\sqrt{3x^2+1} + \sqrt{x})^2 = (2x+1)^2$

$3x^2+1+2\sqrt{x(3x^2+1)} + x = 4x^2+4x+1$

$(2\sqrt{x(3x^2+1)})^2 = (x^2+3x)^2$

$4x(3x^2+1) = x^4+6x^3+9x^2$

$12x^3+4x = x^4+6x^3+9x^2$

$x^4-6x^3+9x^2-4x = 0$

$x\underbrace{(x^3-6x^2+9x-4)}_{f(x)} = 0 \rightsquigarrow x = 0$

$f(1) = (1)^3 - 6(1)^2 + 9(1) - 4 = 0$

$\Rightarrow$ $x-1$ is a factor of $f(x)$ (factor theorem)

$f(x) = x^3-6x^2+9x-4 = (x-1)(ax^2+bx+c)$

$= (x-1)(x^2-5x+4)$

$f(x) = (x-1)(x-1)(x-4)$

$-4 = -c \Rightarrow \underline{c = 4}$

$x^3 = ax^3 \Rightarrow \underline{a = 1}$

$-6x^2 = bx^2 - x^2$

$-6 = b - 1$

$\underline{-5 = b}$

$\therefore \boxed{x = 0,\ x = 1,\ x = 4}$

Checking Solutions:

$x = 0$: $\sqrt{3(0)^2+1} + \sqrt{0} - 2(0) - 1 = 0$

$x = 1$: $\sqrt{3(1)^2+1} + \sqrt{1} - 2(1) - 1 = 0$

$x = 4$: $\sqrt{3(4)^2+1} + \sqrt{4} - 2(4) - 1 = 0$

} Valid solutions

ii) $\sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0$

$(\sqrt{3x^2+1} - 2\sqrt{x})^2 = (1-x)^2$

$3x^2+1 - 4\sqrt{x(3x^2+1)} + 4x = 1 - 2x + x^2$

$(-4\sqrt{x(3x^2+1)})^2 = (-2x^2-6x)^2$

$16x(3x^2+1) = 4x^4+24x^3+36x^2$

$4x(3x^2+1) = x^4+6x^3+9x^2$

$12x^3+4x = x^4+6x^3+9x^2$

$x^4-6x^3+9x^2-4x = 0$

$x(x^3-6x^2+9x-4) = 0$

$x(x-1)^2(x-4) = 0$ (via result obtained in part i)

$\therefore\ x = 0,\ x = 1,\ x = 4$

Checking Solutions:

$$\left.\begin{aligned} x = 0&: \sqrt{3(0)^2+1} - 2\sqrt{0} + 0 - 1 = 0 \\ x = 1&: \sqrt{3(1)^2+1} - 2\sqrt{1} + 1 - 1 = 0 \end{aligned}\right\} \text{valid solutions}$$
$$x = 4: \sqrt{3(4)^2+1} - 2\sqrt{4} + 4 - 1 = 6 \rightsquigarrow x \neq 4$$

iii) $\sqrt{3x^2+1} - 2\sqrt{x} - x + 1 = 0$

$$\left(\sqrt{3x^2+1} - 2\sqrt{x}\right)^2 = (x-1)^2$$
$$3x^2+1 - 4\sqrt{x(3x^2+1)} + 4x = x^2 - 2x + 1$$
$$\left(-4\sqrt{x(3x^2+1)}\right)^2 = \left(-2x^2 - 6x\right)^2$$
$$16x(3x^2+1) = 4x^4 + 24x^3 + 36x^2$$
$$x(x-1)^2(x-4) = 0$$ (via result obtained in part ii)

$\therefore\ x = 0,\ x = 1,\ x = 4$

Checking Solutions:

$$x = 0: \sqrt{3(0)^2+1} - 2\sqrt{0} - 0 + 1 = 2 \rightsquigarrow x \neq 0$$
$$\left.\begin{aligned} x = 1&: \sqrt{3(1)^2+1} - 2\sqrt{1} - 1 + 1 = 0 \\ x = 4&: \sqrt{3(4)^2+1} - 2\sqrt{4} - 4 + 1 = 0 \end{aligned}\right\} \text{valid solutions}$$

Note:

From parts ii and iii, we can see that the solutions we obtain may not be valid, even if we utilised the correct method. In these problems, extraneous solutions emerged from squaring our equation. Thus, it is important to check our solutions by substituting them back into the original equation.