Problem 83. 2009.03.03.

The function $f(t)$ is defined for $t \neq 0$, by

$$f(t) = \frac{t}{e^t - 1}.$$

(i) By expanding e^t, show that $\lim_{t \to 0} f(t) = 1$. Find $f'(t)$ and evaluate $\lim_{t \to 0} f'(t)$.

(ii) Show that $f(t) + \frac{1}{2}t$ is an even function.

[Note: A function $g(t)$ is said to be even if $g(t) = g(-t)$.]

(iii) Show with the aid of a sketch that $e^t(1-t) \leq 1$ and deduce that $f'(t) \neq 0$ for $t \neq 0$. Sketch the graph of $f(t)$.

Prerequisites.

Clearly, you need to know that "By expanding e^t ..." means that you need to use a Maclaurin or Taylor series to represent e^t. The use of such a representation for evaluating limits is the primary reason for the inclusion of this problem. However, it also gives rise to a situation which is most easily resolved by using L'Hopital's rule and therefore affords an opportunity to introduce this device, if you are not already familiar with it.

L'Hopital's rule is used in evaluating limits when an indeterminate expression arises. Consider

$$\lim_{x \to 0} \frac{\sin x}{x},$$

which we looked in detail in Chapter 9. As x approaches zero we are faced with having to resolve the indeterminate expression $\frac{0}{0}$. Similarly in trying to resolve

$$\lim_{t \to \infty} \frac{e^t}{t}$$

we would need to make sense of $\frac{\infty}{\infty}$. These are examples of what is meant by "an indeterminate expression". L'Hopital's rule can sometimes help us to evaluate these and some similar expressions. There are a number of conditions that must be met if L'Hopital's rule is to be used correctly and with proper understanding. However, at this level it would not be appropriate to go into all the details. These would be covered in an undergraduate course in Analysis. At this level, all I can do is to give you a result that you can use blindly to obtain answers:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

where c is any real number, but can also be $\pm\infty$.

Normally, simply using a rule in this blind manner is something that I would never advocate. I do so here because there is one STEP problem that was set in the pre-1998 era where its use was necessary. I have used it in my solution as a further illustration but you should write an alternative solution as I have described in the reflection at the end of the problem.

If we apply L'Hopitals rule to the examples given above we get,

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1,$$

and

$$\lim_{t \to \infty} \frac{e^t}{t} = \lim_{t \to \infty} \frac{e^t}{1} = \infty.$$

For now that is as far as we can go.

<u>First Thoughts</u>.

The rule for f is a combination of simple polynomials and a transcendental function and these can often be quite tricky functions to sketch. In the sketch, the vertical axis, $t = 0$, will be an asymptote, so I will need the left and right hand limits of f as t approaches zero, as well as the limits as t approaches $\pm\infty$. The expansion for e^t will enable me to write the denominator as a polynomial which will enable me to find the required limit. It is possible that the same would be true when I come to evaluate $\lim_{t \to 0} f'(t)$.
Proving that $f(t) + \frac{1}{2}t$ is an even function is probably just a matter of applying the definition. As usual, the final part will depend upon the preceding work and I expect that the work on limits will be significant in establishing the "global" features of the graph. I would expect that some additional work will be necessary to establish local features. In particular, it seems likely that using derivatives to identify turning points will be necessary.