2024-09-28 # STEP Practice - Problem 4 (2003.01.02)

$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$, where $a \neq 0$ and $b \neq 0$

$\frac{1}{x} = \frac{a+b}{ab}$

$x = \frac{ab}{a+b}$

$x[a+b] = ab$

$x = a+b \implies [a+b]^2 = ab \implies ab \geq 0$
Also, $[a+b]^2 = ab \implies a^2+2ab+b^2 = ab$
$\implies a^2+b^2 = -ab \implies -ab > 0 \implies ab < 0$ since $a^2+b^2 > 0$ for $a \neq 0$ and $b \neq 0$

$x = a+b$ leads to a contradiction, $ab \geq 0$ and $ab < 0$
∴ There are no values of a and b for which $x = a+b$ will give a correct answer.

$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where $a \neq 0$, $b \neq 0$, and $c \neq 0$

$\frac{1}{x} = \frac{ab+ac+bc}{abc}$

$x = \frac{abc}{ab+ac+bc}$

$x[ab+ac+bc] = abc$

$x = a+b+c \iff [a+b+c][ab+ac+bc] = abc$
$\iff a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 3abc = abc$
$\iff a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 2abc = 0$
$\iff [a+b][a+c][b+c] = 0$
$\iff a+b = 0$ or $a+c = 0$ or $b+c = 0$

∴ The candidates' answer will only be correct if a, b, and c satisfy at least one of these equations.

Notes

This is another question which relies on you making the connection between the first part of the question and subsequent parts. The actual knowledge required here is not advanced at all. GCSE algebra, and basic knowledge (e.g. $x^2 \geq 0 \rightsquigarrow a^2+b^2 \geq 0$) is enough. The last step is not the easiest to notice. Experience with solving problems will aid in noticing factorisations of expressions. Here, you could work backwards a bit from what the question gives you to get a factored expression as your goal.