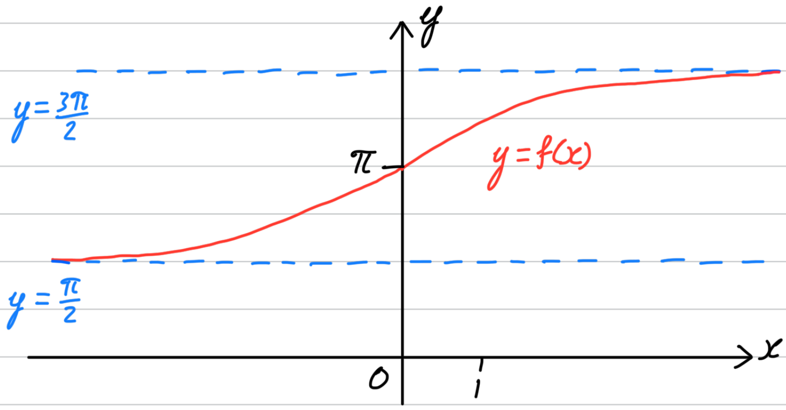
2025-03-22 STEP Practice: Problem 82 (2015.02.04)

i) The continuous function f is defined by
$\tan(f(x)) = x \quad (-\infty < x < \infty)$
and $f(0) = \pi$.

Note that f exhibits a property of inverse functions, but $f(x) \neq \arctan x$ since $\arctan 0 = 0$.
However, $\tan(x+\pi) = \tan x$.
$\leadsto f(x) = \pi + \arctan x$.



ii) The continuous function y is defined by
$\tan(y(x)) = \dfrac{x}{1+x^2} \quad (-\infty < x < \infty)$

and $y(0) = \pi$.

Thus, $y(x) = \pi + \arctan\left[\frac{x}{1+x^2}\right]$.

First, consider $y = \frac{x}{1+x^2}$.

$\lim_{x\to-\infty} y = 0^-$ and $\lim_{y\to\infty} y = 0^+$.

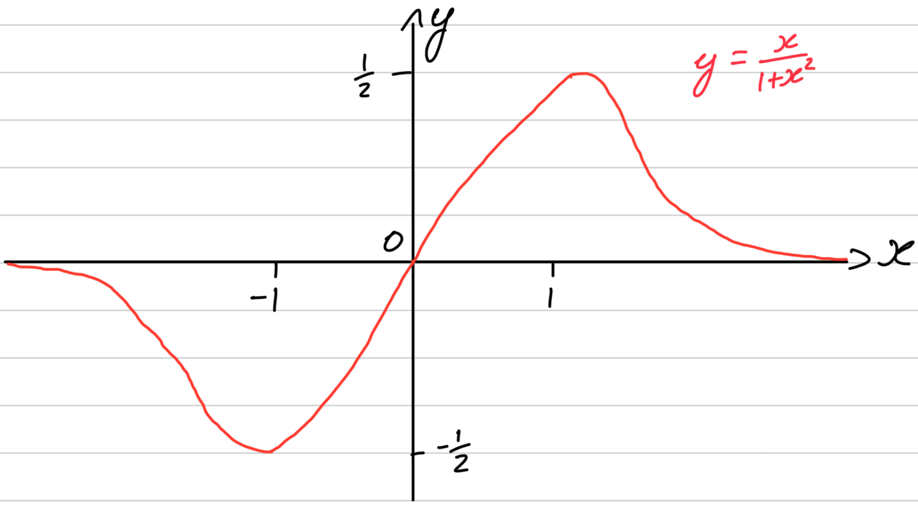
$y = 0 \Rightarrow x = 0$

$$\frac{dy}{dx} = \frac{1\cdot[1+x^2] - 2x\cdot x}{[1+x^2]^2} = \frac{1-x^2}{[1+x^2]^2}$$

$\frac{dy}{dx} = 0 \Rightarrow x = \pm 1$

Extrema at $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$.
Since there are only two extrema, and we know y approaches 0 from the top and bottom, for $x \to \infty$ and $x \to -\infty$ respectively, we know the nature of these extrema.

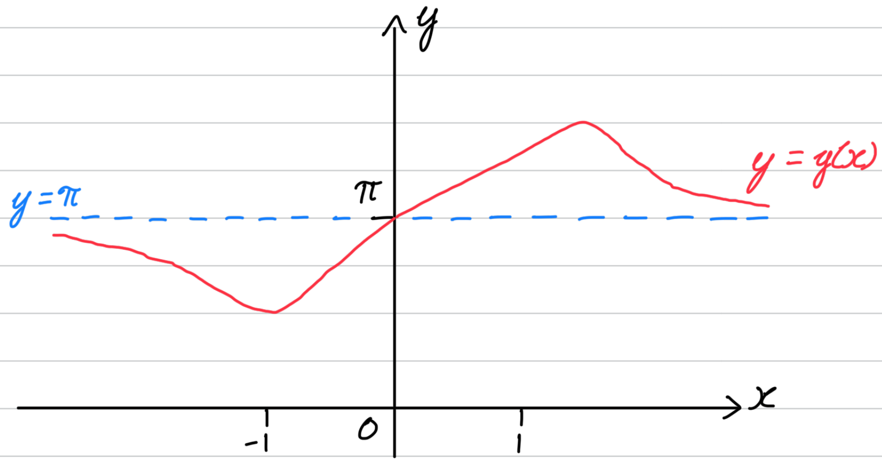
$(1, \frac{1}{2})$ is a maximum and $(-1, -\frac{1}{2})$ is a minimum.



$g(x) = f(x) \circ \frac{x}{1+x^2}$

f is continuous and increasing on its entire domain.
Thus $y = g(x)$ will mimick the shape of $y = f(x)$.

$\lim_{x \to -\infty} g(x) = \pi^-$ and $\lim_{x \to \infty} g(x) = \pi^+$.

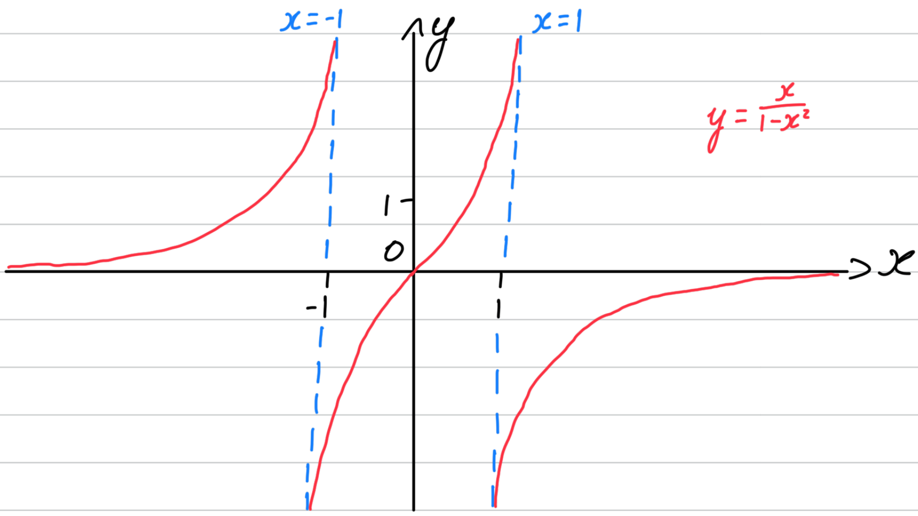


iii) The continuous function h is defined by
$\tan(h(x)) = \frac{x}{1-x^2} \quad (x \neq \pm 1)$

and $h(0) = \pi$.

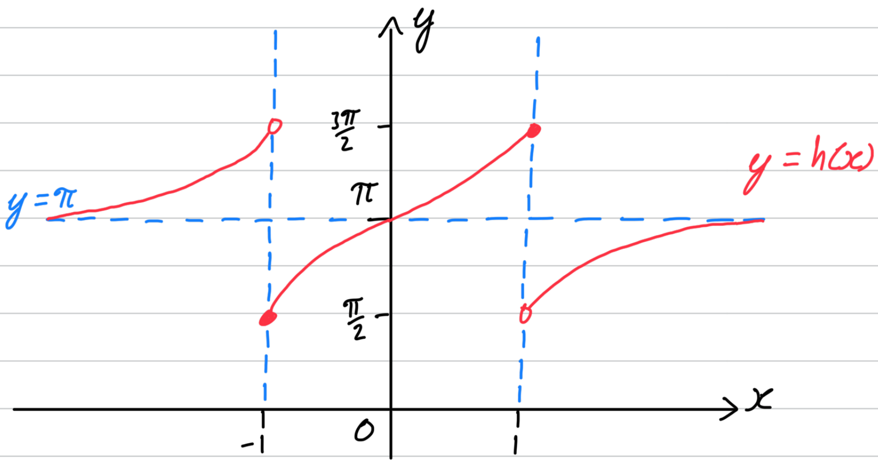
First, consider $y = \frac{x}{1-x^2}$.

$\lim_{x\to-\infty} y = 0^+$, $\lim_{x\to\infty} y = 0^-$, $\lim_{x\to-1^-} y = \infty$, $\lim_{x\to-1^+} y = -\infty$, $\lim_{x\to 1^-} y = \infty$, $\lim_{x\to 1^+} y = -\infty$.



From the information we have, and the previous cases, it is reasonable to propose

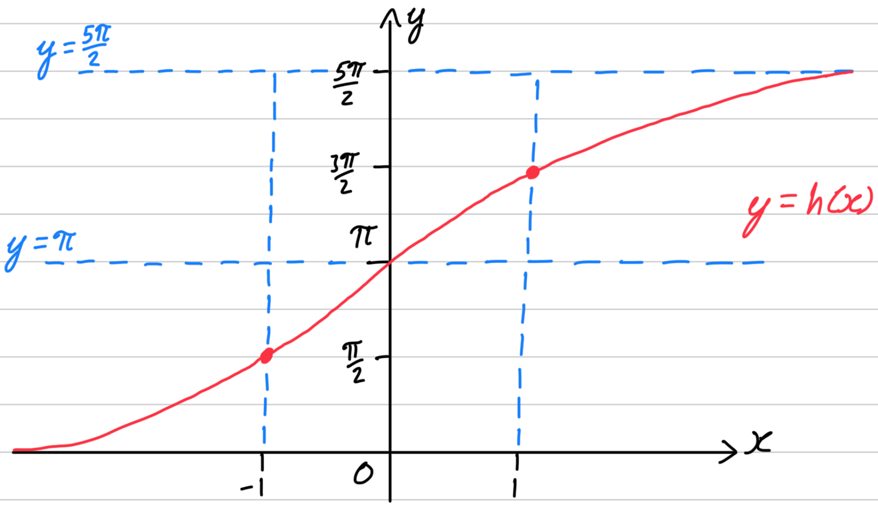
$$h(x) = \begin{cases} \pi + \arctan\left[\frac{x}{1-x^2}\right] & \text{for } x \neq \pm 1 \\ \frac{3\pi}{2} & \text{for } x = 1 \\ \frac{\pi}{2} & \text{for } x = -1 \end{cases}.$$



The problem is that the function we have graphed is not continuous.
Since the problem specifies that h is continuous, we have not drawn the correct graph.

$$h(x) = \begin{cases} \arctan\left[\frac{x}{1-x^2}\right] & \text{for } -\infty < x < -1 \\ \frac{\pi}{2} & \text{for } x = -1 \\ \arctan\left[\frac{x}{1-x^2}\right] + \pi & \text{for } -1 < x < 1 \\ \frac{3\pi}{2} & \text{for } x = 1 \\ \arctan\left[\frac{x}{1-x^2}\right] + 2\pi & \text{for } 1 < x < \infty \end{cases}$$

This definition ensures that h is continuous.
It works due to the periodicity of the tangent function.



Notes

There is plenty to learn from this question. Being familiar with your standard graphs, such as that of the arctangent function, is a given. You need to be strong with sketching unfamiliar graphs as well – this is best done by utilising a systematic approach. Global features such as asymptotes should be considered first, and then local features like intercepts and turning points. A good systematic approach will allow you to sketch the graphs of most functions, including the ones present in this problem. Additionally, this question raises the issue about inverse functions and many-to-one functions. Depending on the subset of the domain you choose, you will get a different inverse function. Typically, we pick one of the many possible inverse functions and call it the principal one. This is just convention, and the other definitions are perfectly valid, though perhaps not as useful. Familiarity with trigonometric functions and their periodicities is assumed knowledge, and so is the ability to differentiate to find turning points. These are tools that are frequently utilised in STEP questions.