2025-02-08 STEP Practice: Problem 33 (2002.01.02)

$f(x) = x^m[x-1]^n$, where $m,n \in \{k \in \mathbb{Z} \mid k > 1\}$

$f'(x) = mx^{m-1}[x-1]^n + nx^m[x-1]^{n-1} = x^{m-1}[x-1]^{n-1}[m[x-1]+nx]$

$y = f(x)$ has stationary points where $f'(x) = 0$

$f'(x) = 0 \implies x = 0$ or $x = 1$, $x = \frac{m}{m+n}$

Since $m > 1$ and $n > 1$, $0 < \frac{m}{m+n} < 1$ $\therefore$ $y = f(x)$ has a stationary point with $0 < x < 1$.

$$\begin{aligned} f''(x) &= [m-1]x^{m-2}[x-1]^{n-1}[m[x-1]+nx] + x^{m-1}[n-1][x-1]^{n-2}[m[x-1]+nx] + x^{m-1}[x-1]^{n-1}[m+n] \\ &= x^{m-2}[x-1]^{n-2}[[m-1][x-1][m[x-1]+nx] + x[n-1][m[x-1]+nx] + x[x-1][m+n]] \\ &= x^{m-2}[x-1]^{n-2}[x[m+n][m-1][x-1] - m[m-1][x-1] + x^2[m+n][n-1] - mx[n-1] + x[x-1][m+n]] \end{aligned}$$

$$\begin{aligned} f''\left(\tfrac{m}{m+n}\right) &= \left[\tfrac{m}{m+n}\right]^{m-2}\left[\tfrac{-n}{m+n}\right]^{n-2}\left[m[m-1]\left[\tfrac{-n}{m+n}\right] - m[m-1]\left[\tfrac{-n}{m+n}\right] + m[n-1]\left[\tfrac{m}{m+n}\right] - m[n-1]\left[\tfrac{m}{m+n}\right] + m\left[\tfrac{-n}{m+n}\right]\right] \\ &= \left[\tfrac{m}{m+n}\right]^{m-2}\left[\tfrac{-n}{m+n}\right]^{n-2}\left[\tfrac{-mn}{m+n}\right] \end{aligned}$$

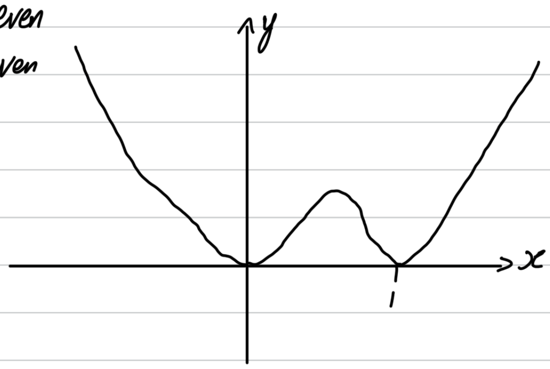
$\left[\frac{m}{m+n}\right]^{m-2} > 0$ $\because m > 0$ and $n > 0$. $\left[\frac{-mn}{m+n}\right] < 0$

n is even: $(-n)^{n-2} > 0 \rightsquigarrow \left[\frac{-n}{m+n}\right]^{n-2} > 0 \rightsquigarrow f''\left(\frac{m}{m+n}\right) < 0$ $\therefore$ the stationary point is a maximum.

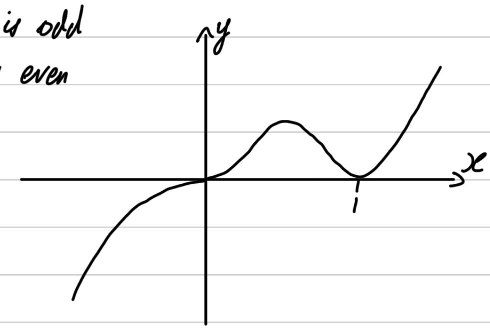
n is odd: $(-n)^{n-2} < 0 \rightsquigarrow \left[\frac{-n}{m+n}\right]^{n-2} < 0 \rightsquigarrow f''\left(\frac{m}{m+n}\right) > 0$ $\therefore$ the stationary point is a minimum.

$f(0) = 0$ $\quad f(1) = 0$ $\quad$ Curve crosses axis when power of factor is odd, just touches when even.
When the curve crosses the x-axis, it is an inflection point $\because f'(x) = 0$

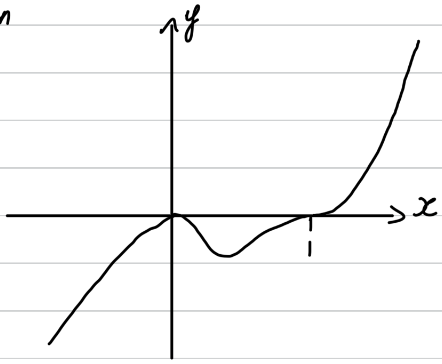
m is even
n is even



m is odd
n is even



m is even
n is odd



m is odd
n is odd