2025-04-05 STEP Practice: Problem 102 (2017.03.07)

$T: \left(\frac{a[1-t^2]}{1+t^2}, \frac{2bt}{1+t^2}\right)$ (*)

$$\frac{T_x^2}{a^2}+\frac{T_y^2}{b^2}=\frac{\cancel{a^2}[1-t^2]^2}{\cancel{a^2}[1+t^2]^2}+\frac{4\cancel{b^2}t^2}{\cancel{b^2}[1+t^2]}=\frac{[1-t^2]^2+4t^2}{[1+t^2]^2}=\frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}=1$$

$\therefore T$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$

$y'=y'(x,y)=-\frac{b^2x}{a^2y}$

L is tangent to ellipse at T.

$\rightsquigarrow L: y-T_y=[x-T_x]y'(T_x,T_y)$

$$y-\frac{2bt}{1+t^2}=\left[x-\frac{a[1-t^2]}{1+t^2}\right]\cdot -\frac{b^2\cdot\frac{a[1-t^2]}{1+t^2}}{a^2\cdot\frac{2bt}{1+t^2}}$$

$$y-\frac{2bt}{1+t^2}=\left[x-\frac{a[1-t^2]}{1+t^2}\right]\cdot -\frac{b[1-t^2]}{2at}$$

$$2ayt-\frac{4abt^2}{1+t^2}=-b[1-t^2]\left[x-\frac{a[1-t^2]}{1+t^2}\right]$$

$$\frac{4abt^2}{1+t^2}-b[1-t^2]\left[x-\frac{a[1-t^2]}{1+t^2}\right]-2ayt=0$$

$$\frac{4abt^2}{1+t^2}-bx[1-t^2]+\frac{ab[1-t^2]^2}{1+t^2}-2ayt=0$$

$$\frac{ab[4t^2+1-2t^2+t^4]}{1+t^2}-bx+bxt^2-2ayt=0$$

$$\frac{ab[1+t^2]^2}{1+t^2}-bx+bxt^2-2ayt=0$$

$$ab[1+t^2]-bx+bxt^2-2ayt=0$$

$$ab+abt-bx+bxt^2-2ayt=0$$

$[a+x]bt^2 - 2ayt + b[a-x] = 0$

Since (X,Y) lies on L,

$[a+X]bt^2 - 2aYt + b[a-X] = 0$ $\qquad (**)$

There are two distinct lines through (X,Y) if and only if $(**)$ has two distinct real roots.

$\iff D = 4a^2Y^2 - 4b^2[a+X][a-X] > 0$

$\iff 4a^2Y^2 - 4b^2[a^2-X^2] > 0$

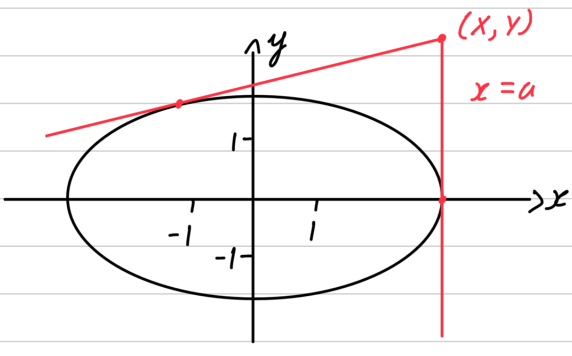
$\iff a^2Y^2 > [a^2-X^2]b^2$

$\iff a^2Y^2 > a^2b^2 - b^2X^2$

$\iff \frac{X^2}{a^2} + \frac{Y^2}{b^2} > 1 \quad \because a \neq 0$ and $b \neq 0$

$\therefore$ There are two distinct lines through (X,Y) that are tangents to the ellipse, if and only if (X,Y) lies outside of the ellipse.

If $X^2 = a^2$, $X = \pm a$. One of the tangents is vertical.
Additionally, $a^2Y^2 > 0 \Rightarrow Y \neq 0$.
E.g.



ii) $P: \left(\frac{a[1-p^2]}{1+p^2}, \frac{2bp}{1+p^2}\right)$, $Q: \left(\frac{a[1-q^2]}{1+q^2}, \frac{2bq}{1+q^2}\right)$

The tangents to the ellipse at P and Q have the equations meet at (X,Y).
Thus, $t=p$ and are solutions to $(**)$.

$\therefore pq = \frac{b[a-X]}{b[a+X]}$

$\therefore [a+X]pq = a-X$

Additionally, $p + q = \frac{2aY}{b[a+X]}$.

The tangents meet the y-axis at $(0, y_1)$ and $(0, y_2)$, where $y_1 + y_2 = 2b$.
Then,

$abp^2 - 2ay_1p + ab = 0$ and $abq^2 - 2ay_2q + ab = 0$

$bp^2 - 2y_1p + b = 0$ and $bq^2 - 2y_2q + b = 0$

$\Rightarrow \frac{bp^2+b}{2p} = y_1$ and $\frac{bq^2+b}{2q} = y_2$

$$\Rightarrow \frac{bp^2+b}{2p} + \frac{bq^2+b}{2q} = 2b$$

$$\frac{p^2+1}{p} + \frac{q^2+1}{q} = 4$$

$$p + q + \frac{p+q}{pq} = 4$$

$$\frac{2aY}{b[a+X]} + \frac{\frac{2aY}{b[a+X]}}{\frac{a-X}{a+X}} = 4$$

$$\frac{2aY}{b[a+X]} + \frac{2aY}{b[a-X]} = 4$$

$$\frac{a}{a+X} + \frac{a}{a-X} = \frac{2b}{Y}$$

$$\frac{a[a-X] + a[a+X]}{a^2 - X^2} = \frac{2b}{Y}$$

$$\frac{2a^2}{a^2-X^2} = \frac{2b}{Y}$$

$$\frac{a^2-X^2}{a^2} = \frac{Y}{b}$$

$$1 - \frac{X^2}{a^2} = \frac{Y}{b}$$

$$\therefore \frac{X^2}{a^2} + \frac{Y}{b} = 1$$