2025-03-15 STEP Practice: Problem 41 (2012.02.03)

Let $I = \int_0^\infty f(x+\sqrt{1+x^2})\,dx$

Let $t = x+\sqrt{1+x^2}$

$t^2 = x^2 + 2x\sqrt{1+x^2} + 1+x^2 = 1+2x^2+2x\sqrt{1+x^2} = 1+2x[x+\sqrt{1+x^2}] = 1+2xt$

$\rightsquigarrow x = \frac{t^2-1}{2t}$

$\frac{dx}{dt} = \frac{2t\cdot 2t - 2[t^2-1]}{4t^2} = \frac{4t^2-2t^2+2}{4t^2} = \frac{2t^2+2}{4t^2} = \frac{1}{2}\cdot\frac{t^2+1}{t^2} = \frac{1}{2}[1+\frac{1}{t^2}]\,dt$

$dx = \frac{1}{2}[1+\frac{1}{t^2}]\,dt$

$x\to\infty : t\to\infty$, $x \to 0 : t \to 0+\sqrt{1+0} = 1$

$\therefore I = \frac{1}{2}\int_1^\infty [1+\frac{1}{t^2}]f(t)\,dt$ (*)

Let $J = \int_0^\infty \frac{1}{2x^2+1+2x\sqrt{x^2+1}}\,dx$

Let $f(t) = \frac{1}{t^2}$. $f(x+\sqrt{1+x^2})$ is the integrand.

$= \frac{1}{2}\int_1^\infty [1+\frac{1}{t^2}]\cdot\frac{1}{t^2}\,dt$ via (*)

$= \frac{1}{2}\int_1^\infty [\frac{1}{t^2}+\frac{1}{t^4}]\,dt$

$= \frac{1}{2}\left[-\frac{1}{t}-\frac{1}{3t^3}\right]_1^\infty = \frac{1}{2}[0-[-\frac{1}{1}-\frac{1}{3}]] = \frac{1}{2}[1+\frac{1}{3}] = \frac{2}{3}$

Let $K = \int_0^{\frac{\pi}{2}} \frac{1}{[1+\sin\theta]^3}\,d\theta$

Let $x = \tan\theta$

$dx = \sec^2\theta\,d\theta = [1+x^2]\,d\theta$

$\theta = \arctan x$

$\theta \to 0 : x \to 0$

$\theta \to \frac{\pi}{2} : x \to \infty$

$\sin\theta = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$ $\rightsquigarrow 1+\sin\theta = \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$

$[1+\sin\theta]^3 = \frac{[x+\sqrt{1+x^2}]^3}{[1+x^2]\sqrt{1+x^2}}$

$$K = \int_0^\infty \frac{[1+x^2]\sqrt{1+x^2}}{[x+\sqrt{1+x^2}]^3} \cdot \frac{1}{1+x^2}\, dx$$

$$= \int_0^\infty \frac{\sqrt{1+x^2}}{[x+\sqrt{1+x^2}]^3}\, dx$$

Let $f(t) = \frac{t-x}{t^3} = \frac{t - \frac{t^2-1}{2t}}{t^3} = \frac{2t^2-t^2+1}{2t^4} = \frac{t^2+1}{2t^4} = \frac{1}{2t^2} + \frac{1}{2t^4}$

$$= \frac{1}{2}\int_1^\infty \left[1+\frac{1}{t^2}\right]\left[\frac{1}{2t^2}+\frac{1}{2t^4}\right] dt$$

via (*)

$$= \frac{1}{2}\int_1^\infty \left[\frac{1}{2t^2}+\frac{1}{2t^4}+\frac{1}{2t^4}+\frac{1}{2t^6}\right] dt$$

$$= \frac{1}{2}\int_1^\infty \left[\frac{1}{2t^2}+\frac{1}{t^4}+\frac{1}{2t^6}\right] dt$$

$$= \frac{1}{2}\left[-\frac{1}{2t}-\frac{1}{3t^3}-\frac{1}{10t^5}\right]_1^\infty$$

$$= 0 - \frac{1}{2}\left[-\frac{1}{2}-\frac{1}{3}-\frac{1}{10}\right]$$

$$= \frac{1}{4}+\frac{1}{6}+\frac{1}{20}$$

$$= \frac{15}{60}+\frac{10}{60}+\frac{3}{60}$$

$$= \frac{28}{60}$$

$$= \frac{7}{15}$$