Calculus I: One-Sided Limits

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In the previous section, we looked at two limits which do not exist; the reason for them not existing, however, were quite different. One of the limits were:

$$\lim_{t \to 0} \cos\left(\frac{\pi}{t}\right)$$

We saw that this limit does not exist as the function oscillates wildly as we approach $t = 0$ from either side. As the function does not settle on any one value, the limit does not exist. We had also considered the following limit.

$$\lim_{x \to 0} H(t), \quad H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

This limit does not exist either but for a different reason. From the left of $t = 0$, $H(t)$ approaches 0 while it approaches 1 from the right. As the function settles on two different values depending on which side of $t = 0$ we are looking at, the limit does not exist.

We will differentiate these two cases of limits that do not exist. This will be done with **one-sided limits** whose definitions are present below; these involve us looking at what a function approaches from only one side.

Definition 1. The limit of $f(x)$ is L as x approaches a from the right, written as

$$\lim_{x \to a^+} f(x) = L,$$

if $f(x)$ can be made to be close to L for all values of x close to a, with $x > a$.

Definition 2. The limit of $f(x)$ is L as x approaches a from the left, written as

$$\lim_{x \to a^-} f(x) = L,$$

if $f(x)$ can be made to be close to L for all values of x close to a, with $x < a$.

Notice that the change in notation here from normal limits is very slight. The only difference is that there is a superscripted sign after the a, under the "lim" part; right-handed limits have us go to a^+ while left-handed limits have us go to a^-. These tell us whether we are considering $x > a$ or $x < a$, the direction we approach the target value.