2025-02-22 STEP Practice: Problem 38 (2012.02.04)

$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$ for $|x|<1$

i) Let $n \in \mathbb{Z}$ and $n>1$. For any $k \in \mathbb{Z}^+$,

$k+1 > k$

$n^{k+1} > n^k$ $\because f(x) = n^x$ is a strictly increasing function for $n>1$

$(k+1)n^{k+1} > (k+1)n^k > kn^k$

$\therefore \frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}$ $\because f(x) = \frac{1}{x}$ is strictly decreasing for $x>0$

$\ln\left[1+\frac{1}{n}\right] = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \ldots (-1)^{k+1}\frac{1}{k \cdot n^k} + \ldots$ $\because 0 < \frac{1}{n} < 1 \Rightarrow \left|\frac{1}{n}\right| < 1$ for $n>1$

$= \frac{1}{n} + \left[\frac{1}{3n^3} - \frac{1}{2n^2}\right] + \left[\frac{1}{5n^5} - \frac{1}{4n^4}\right] + \ldots + \left[\frac{(-1)^{k+2}}{(k+1)n^{k+1}} - \frac{(-1)^{k+1}}{kn^k}\right] + \ldots$

From the above inequality, it follows that

$\frac{1}{(k+1)n^{k+1}} - \frac{1}{kn^k} < 0.$

Then, $\ln\left[1+\frac{1}{n}\right]$ equals $\frac{1}{n}$ plus an infinite number of terms, all of which are less than 0.

$\therefore \ln\left[1+\frac{1}{n}\right] < \frac{1}{n}$

$n\ln\left[1+\frac{1}{n}\right] < 1$ $\because n>1>0$

$\ln\left[\left[1+\frac{1}{n}\right]^n\right] < 1$

$e^{\ln\left[\left[1+\frac{1}{n}\right]^n\right]} < e$ $\because f(x) = e^x$ is strictly increasing

$\therefore \left[1+\frac{1}{n}\right]^n < e$

ii) $\ln\left[\frac{2y+1}{2y-1}\right] = \ln\left[\frac{1+\frac{1}{2y}}{1-\frac{1}{2y}}\right] = \ln\left(1+\frac{1}{2y}\right) - \ln\left(1-\frac{1}{2y}\right)$

$\ln\left(1+\frac{1}{2y}\right) = \frac{1}{2y} - \cancel{\frac{1}{2[2y]^2}} + \frac{1}{3[2y]^3} - \cancel{\frac{1}{4[2y]^4}} + \ldots + (-1)^{k+1}\frac{1}{k[2y]^k} + \ldots$ for $\left|\frac{1}{2y}\right| < 1 \Rightarrow \frac{1}{2y} < 1 \Rightarrow y > \frac{1}{2}$

$\ln\left(1-\frac{1}{2y}\right) = -\frac{1}{2y} - \cancel{\frac{1}{2[2y]^2}} - \frac{1}{3[2y]^3} - \cancel{\frac{1}{4[2y]^4}} + \ldots - \frac{1}{k[2y]^k} + \ldots$ for $y > \frac{1}{2}$

$\therefore \ln\left[\frac{2y+1}{2y-1}\right] = \frac{2}{2y} + \frac{2}{3[2y]^3} + \frac{2}{5[2y]^5} + \ldots \frac{2}{[2k-1][2y]^{2k-1}} + \ldots$ for $y > \frac{1}{2}$

$= \frac{1}{y} + \frac{2}{3[2y]^3} + \frac{2}{5[2y]^5} + \ldots \frac{2}{[2k-1][2y]^{2k-1}} + \ldots$

$> \frac{1}{y}$ $\because \frac{2}{[2k-1][2y]^{2k-1}} > 0$ for all $k \geq 1$ and $y > \frac{1}{2} > 0$ (all the terms are positive)

$\therefore \ln\left[\frac{2y+1}{2y-1}\right] > \frac{1}{y}$ for $y > \frac{1}{2}$

$$\left[1+\frac{1}{n}\right]^{n+\frac{1}{2}} > e > \left[1+\frac{1}{n}\right]^n$$

$y\ln\left[\frac{2y+1}{2y-1}\right] > 1 \quad \because y > \frac{1}{2} > 0$

$$\begin{array}{r} 1 \\ 2y-1 \overline{)\, 2y+1} \\ 2y-1 \\ \hline 2 \end{array}$$

$e^{y\ln\left[\frac{2y+1}{2y-1}\right]} > e$

$e^{\ln\left[\left[\frac{2y+1}{2y-1}\right]^y\right]} > e$

$\left[\frac{2y+1}{2y-1}\right]^y > e$

$\left[\frac{2y+1}{2y-1}\right]^y = \left[1+\frac{2}{2y-1}\right]^y = \left[1+\frac{1}{y-\frac{1}{2}}\right]^y$

Let $n = y - \frac{1}{2}$. Then, $y = n + \frac{1}{2}$. $y > \frac{1}{2} \Longrightarrow n > 0$

$\leadsto \left[\frac{2y+1}{2y-1}\right]^y = \left[1+\frac{1}{n}\right]^{n+\frac{1}{2}}$ for $n > 0$ $\leadsto$ statement true for any $n \in \mathbb{Z}^+$.

$\therefore \left[1+\frac{1}{n}\right]^{n+\frac{1}{2}} > e$ for all $n \in \mathbb{Z}^+$.

iii) From the previous results,

$\left[1+\frac{1}{n}\right]^{n+\frac{1}{2}} > e > \left[1+\frac{1}{n}\right]^n$ for all $n \in \mathbb{Z}^+$

$\left[1+\frac{1}{n}\right]^n\left[1+\frac{1}{n}\right]^{\frac{1}{2}} > e > \left[1+\frac{1}{n}\right]^n$

$\left[1+\frac{1}{n}\right]^{\frac{1}{2}} > \frac{e}{\left[1+\frac{1}{n}\right]^n} > 1$ inequality direction preserved $\because \left[1+\frac{1}{n}\right]^n > 0$ for $n \in \mathbb{Z}^+$

$\frac{1}{\left[1+\frac{1}{n}\right]^n} < \frac{\left[1+\frac{1}{n}\right]^n}{e} < 1$

$\lim_{n\to\infty} \frac{1}{\left[1+\frac{1}{n}\right]^n} < \lim_{n\to\infty} \frac{\left[1+\frac{1}{n}\right]^n}{e} < \lim_{n\to\infty} 1$ limit preserves inequality direction $\because$ inequality holds for all $n \in \mathbb{Z}^+$

$\lim_{n\to\infty} \left[1+\frac{1}{n}\right]^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$

$\lim_{n\to\infty} 1 = 0$

$$\therefore 1 < \lim_{n\to\infty} \frac{[1+\frac{1}{n}]^n}{e} < 1$$

$$\therefore \lim_{n\to\infty} \frac{[1+\frac{1}{n}]^n}{e} = 1$$ via the Squeeze Rule/Theorem

$$\lim_{n\to\infty} \frac{[1+\frac{1}{n}]^n}{e} = \frac{1}{e}\lim_{n\to\infty}[1+\tfrac{1}{n}]^n = 1$$

$$\therefore \lim_{n\to\infty}[1+\tfrac{1}{n}]^n = e$$

Notes

This is not an incredibly difficult question. The main challenge is spotting the first step. The method thereafter is not complicated. The algebraic manipulations are far simpler than what you may see in some other STEP questions. I have noticed that STEP questions that test you on a topic that is relatively advanced by A-level standards tend to be easier than questions that you can attempt with much less knowledge. Considering the fact that I have not completed an awful lot of STEP questions, compared to how many there are, this apparent pattern may not hold. It makes sense, however. More credit is awarded to being able understand a more advanced topic, and thus there is less emphasis on the application and nuance. In this problem, we are given the taylor series expansion for the function $\ln(1+x)$. We are given an interval for which this converges. We are not asked to do much with this, and it is thus not necessary to think about the details. One can link it to their knowledge of other topics, such as the conditions for the convergence of a geometric series or the binomial expansions. Considering such links is always a good thing to do, as it causes you to think more. It simply is not needed for the sake of completing this question, and thus should be at the back of your mind until you are done. Regardless, this question still contains things worth considering. I have explained my reasoning whenever I have applied a function to an inequality for the first time on the matter of whether the direction of the inequality is preserved or not. I find this to be an important thing to do so that it is clear as to why you see a sign flipped when you do, rather than it just being the result of an arbitrary set of rules. The last part of this question takes a bit of the purpose of STEP questions – to make connections – away. It still involves a nice application of the Squeeze Theorem, and it links Euler's number in its relation with the natural logarithm and its financial interpretation. I have spent – careful not to say wasted – quite some time on this question attempting to bruteforce it. Then, I am frustrated after seeing how simple the solution is. The only way to be more efficient with solving problems is to solve (a lot) more problems. Hence, that is what I am doing.