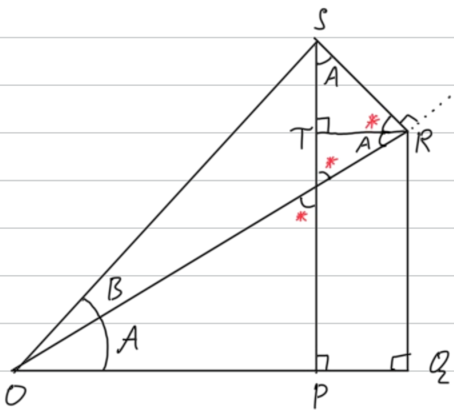
26-10-2024 STEP Practice: Problem 11 (2005.01.04)

The Compound Angle Formula



$* = 90° - A$

$\angle POS = A+B$

$\sin A = \frac{QR}{OR}$

$\cos A = \frac{TS}{SR}$

$\sin B = \frac{SR}{OS}$

$\cos B = \frac{OR}{OS}$

$$\sin(A+B) = \frac{opp.}{hyp.} = \frac{SP}{OS}$$

$$= \frac{PT+TS}{OS} = \frac{PT}{OS} + \frac{TS}{OS} = \frac{QR}{OS} + \frac{TS}{OS}$$

$$= \frac{OR\sin A}{OS} + \frac{SR\cos A}{OS}$$

$$= \frac{\cancel{OS}\cos B\sin A}{\cancel{OS}} + \frac{\cancel{OS}\sin B\cos A}{\cancel{OS}}$$

$$\therefore \sin(A+B) = \sin A\cos B + \cos A\sin B$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\Rightarrow \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \cdot \frac{\frac{1}{\cos A\cos B}}{\frac{1}{\cos A\cos B}}$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\rightsquigarrow \sin(A-B) = \sin A\cos(-B) + \cos A\sin(-B)$$
$$= \sin A\cos B - \cos A\sin B$$

$$\rightsquigarrow \cos(A-B) = \cos A\cos(-B) - \sin A\sin(-B)$$
$$= \cos A\cos B + \sin A\sin B$$

$$\rightsquigarrow \tan(A-B) = \frac{\tan A + \tan(-B)}{1 + \tan A\tan(-B)}$$
$$= \frac{\tan A - \tan B}{1 - \tan A\tan B}$$

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$
$$= 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$
$$= \cos^2\theta - \sin^2\theta$$

$$\tan(2\theta) = \tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$$
$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

$\rightsquigarrow \sin\theta = \mathrm{Im}(e^{i\theta})$

$$\sin(\theta+\varphi) = \mathrm{Im}(e^{i[\theta+\varphi]})$$
$$= \mathrm{Im}(e^{i\theta}e^{i\varphi})$$
$$= \mathrm{Im}([\cos\theta + i\sin\theta][\cos\varphi + i\sin\varphi])$$
$$= \mathrm{Im}(\cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\sin\theta\cos\varphi - \sin\theta\cos\varphi)$$
$$= \sin\theta\cos\varphi + \cos\theta\sin\varphi \quad \text{Q.E.D.}$$

$\rightsquigarrow \cos\theta = \mathrm{Re}(e^{i\theta})$

$$\rightsquigarrow \cos(\theta+\varphi) = \mathrm{Re}(e^{i[\theta+\varphi]})$$
$$= \mathrm{Re}(\cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\sin\theta\cos\varphi - \sin\theta\cos\varphi)$$
$$= \cos\theta\cos\varphi - \sin\theta\sin\varphi \quad \text{Q.E.D.}$$

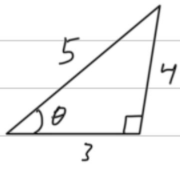
$$1 = \cos^2\theta + \sin^2\theta$$
$$\cos^2 = 1 - \sin^2\theta$$
$$\rightsquigarrow \cos(2\theta) = 1 - \sin^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta$$
$$\rightsquigarrow \cos(2\theta) = \cos^2\theta - 1 + \cos^2\theta$$
$$= 2\cos^2\theta - 1$$

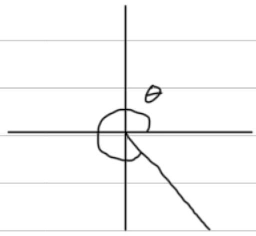
Note:

The formulae which allow us to convert between trig functions that are squared and those of double angles are very useful for trig. integrals.

E.g. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1-\cos(2\theta)}{2} \, d\theta = \frac{\pi}{4}$

a) $\cos\theta = \frac{3}{5}$, $\frac{3\pi}{2} \leq \theta \leq 2\pi$





$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2 \cdot -\frac{4}{5} \cdot \frac{3}{5}$

$\sin(2\theta) = -\frac{24}{25}$ $\qquad \sin\theta = -\frac{4}{5}$

Q.E.D.

$\cos(3\theta) = \cos(\theta + 2\theta) = \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta)$

$= \cos\theta\left[\cos^2\theta - \sin^2\theta\right] - \sin\theta\sin(2\theta)$

$= \frac{3}{5}\left[\left(\frac{3}{5}\right)^2 - \left(-\frac{24}{25}\right)\right] - \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right)$

$= \frac{3}{5}\left[\frac{4}{25} + \frac{24}{25}\right] - \frac{96}{125}$

$= \frac{3}{5} \cdot \frac{33}{125} - \frac{96}{125}$

$= \frac{99}{125} - \frac{96}{125}$

$= \frac{3}{125}$

b) $\tan(3\theta) = \tan(\theta + 2\theta) = \frac{\tan\theta + \tan(2\theta)}{1 - \tan\theta\tan(2\theta)} = \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta \cdot \frac{2\tan\theta}{1-\tan^2\theta}}$ $\quad \cdot 1 - \tan^2\theta$ / $\cdot 1 - \tan^2\theta$

$= \frac{(1-\tan^2\theta)\tan\theta + 2\tan\theta}{(1-\tan^2\theta) - \tan\theta \cdot 2\tan\theta}$

$= \frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ $\qquad$ Q.E.D.

$\tan(3\theta) = \frac{11}{2}$, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$\leadsto \frac{11}{2} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$11[1 - 3\tan^2\theta] = 2[3\tan\theta - \tan^3\theta]$

$11 - 33\tan^2\theta = 6\tan\theta - 2\tan^3\theta$

$2\tan^3\theta - 33\tan^2\theta - 6\tan\theta + 11 = 0$ $\qquad$ Let $u := \tan\theta$

$2u^3 - 33u^2 - 6u + 11 = 0$ $\qquad -22u - bu = -6u$

Let $f(u) := 2u^3 - 33u^2 - 6u + 11$ $\qquad u = 1 \qquad -22 - b = -6$

$f(\frac{1}{2}) = 0$ $\qquad c = -11 \qquad -b = 16 \Rightarrow b = -16$

$\leadsto f(u) = (2u-1)(au^2 + bu + c)$ $\qquad$ (Factor Theorem)

$= (2u-1)(u^2 - 16u - 11)$

$u = \frac{1}{2}, \; u = \dfrac{16 \pm \overbrace{\sqrt{16^2 + 44}}^{\sqrt{300} = 10\sqrt{3}}}{2} = 8 \pm 5\sqrt{3}$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$:

$1 \leq \tan\theta < \infty$

$\frac{1}{2} < 1, \; 8 - 5\sqrt{3} < 1$

$\therefore \tan\theta = 8 + 5\sqrt{3}$