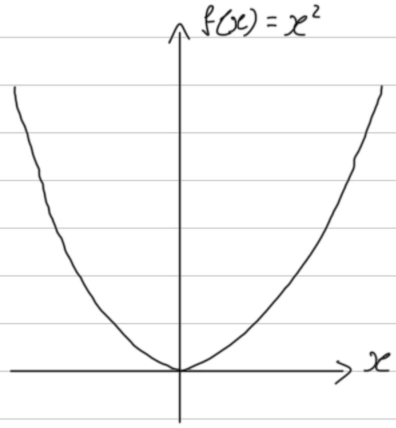
11-01-2024 **STEP Practice: Problem 20 (2013.01.08)**

Function – a mathematical structure consisting of:
- A domain
- A codomain
- A rule which describes how an element in the domain is mapped to an element of the codomain.

E.g. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
f is a many-to-one mapping.
If you change any part of its definition, you get a different mapping.



$g: [0, \infty) \to \mathbb{R}$
$g(x) = x^2$
g is a one-to-one mapping.
It is an injection.

$h: \mathbb{R} \to [0, \infty)$
$h(x) = x^2$
h is an onto function, because every element in the codomain is an image of an element in the domain.
It is a surjection.

If a function, f, is a one-to-one mapping and an onto function, it has an inverse function: f^{-1}.
f^{-1} maps every element in the codomain of f to some element in the domain.

$\sin(x)$ is not an one-to-one function, unless the domain is restricted, e.g. to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The inverse of $\sin(x)$ on this domain is $\arcsin(x)$.

i) $a: \mathbb{R} \to \mathbb{R}$ $\quad b: (0, \infty) \to \mathbb{R}$ $\quad c: \mathbb{R} \to \mathbb{R}$ $\quad d: [0, \infty) \to \mathbb{R}$
$a(x) = x^2$ $\quad b(x) = \ln x$ $\quad c(x) = 2x$ $\quad d(x) = \sqrt{x}$

$(c \circ b)(x) = c(b(x)) = 2\ln x$
Domain of $(c \circ b)(x)$ is the domain of $b(x)$: $(0, \infty)$.
Range of $b(x)$ is $\mathbb{R}$ $\rightsquigarrow$ Range of $(c \circ b)(x)$ is $\mathbb{R}$.

$(a \circ b)(x) = \ln^2 x$
Domain of $(a \circ b)(x)$ is the domain of $b(x)$: $(0, \infty)$.

Range of $(a \circ b)(x)$ is $[0, \infty)$.

$(d \circ a)(x) = \sqrt{x^2} = |x|$
Domain of $(d \circ a)(x)$ is the domain of $a(x)$: $\mathbb{R}$.
Range of $(d \circ a)(x)$ is $[0, \infty)$.

$(a \circ d)(x) = (\sqrt{x})^2 = x$
Domain of $(a \circ d)(x)$ is the domain of $d(x)$: $[0, \infty)$.
Range of $(a \circ d)(x)$ is $[0, \infty)$.

ii) $f: [1, \infty) \to \mathbb{R}$ $\quad g: \mathbb{R} \to \mathbb{R}$
$f(x) = \sqrt{x^2 - 1}$ $\quad g(x) = \sqrt{x^2 + 1}$

$(f \circ g)(x) = f(\sqrt{x^2+1}) = \sqrt{(\sqrt{x^2+1})^2 - 1} = \sqrt{x^2} = |x|$
Domain of fg is the domain of g: $\mathbb{R}$.
Range of fg is $[0, \infty)$.

$(g \circ f)(x) = g(\sqrt{x^2-1}) = \sqrt{(\sqrt{x^2-1})^2 + 1} = \sqrt{x^2} = |x|$
Domain of gf is the domain of f: $[1, \infty)$.
Range of gf is $[1, \infty)$.

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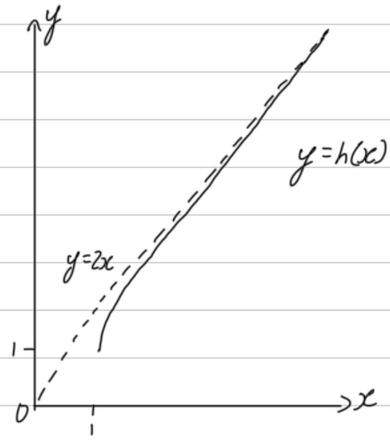
iii) $h(x) = x + \sqrt{x^2 - 1}$ for x $x \geq 1$
$\lim_{x \to \infty} h(x)$ diverges to ∞. For arbitrarily large x, $\sqrt{x^2-1} \lesssim x$.
$\rightsquigarrow h(x) \lesssim 2x \rightsquigarrow y = 2x$ is an asymptote.
$h(1) = 1$. $h'(x) = 1 + \frac{2x}{2\sqrt{x^2-1}} = 1 + \frac{x}{\sqrt{x^2-1}}$
$h'(1)$ DNE. $\lim_{x \to 1^+} h'(x)$ diverges to infinity $\rightsquigarrow$ tangent to $h(x)$ is vertical at $x = 1$.
$\lim_{x \to \infty} h'(x) = 2$. $\frac{x}{\sqrt{x^2-1}}$ approaches 1, but is always less $\rightsquigarrow h'(x) > 2 \ \forall x \geq 1$ $\therefore h(x)$ is an increasing function



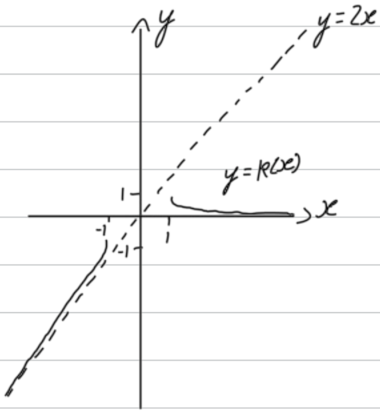
$k(x) = x - \sqrt{x^2-1}$ for $|x| \geq 1$

$k'(x) = 1 - \frac{x}{\sqrt{x^2-1}}$

$k(1) = 1$, $k(-1) = -1$

$\lim_{x\to\infty} k(x) = 0^+$ $\lim_{x\to-\infty} k(x)$ diverges to $-\infty$. $\rightsquigarrow$ x-axis is asymptote for $x \to \infty$.

$\lim_{x\to1^+} k'(x)$ diverges to $-\infty$. $\lim_{x\to-1^-} k'(x)$ diverges to ∞. $\lim_{x\to-\infty} k'(x) = 2^+$. $\rightsquigarrow$ $y = 2x$ is an asymptote for $x \to -\infty$.



Domain of $(k\circ h)(x)$ is the domain of $h(x)$: $[1,\infty)$.

Range of $(k\circ h)(x)$ is $(0,1]$.