2025-03-22 STEP Practice: Integration by Parts

1) $I = \int x\sin x\,dx = x \cdot -\cos x - \int 1 \cdot -\cos x\,dx = -x\cos x + \sin x + C$

2) $I = \int x\cos(2x)\,dx = x \cdot \frac{1}{2}\sin(2x) - \int 1 \cdot \frac{1}{2}\sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

3) $I = \int x\ln x\,dx = \ln x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx = \frac{1}{2}x^2\ln x - \frac{x^2}{4} + C$

4) $I = \int xe^x\,dx = x \cdot e^x - \int 1 \cdot e^x\,dx = xe^x - e^x + C$

5) $\frac{d}{dx}[e^{x^2}] = 2xe^{x^2}$

$I = \int x^3 e^{x^2}\,dx$ Let $u = x^2 \rightsquigarrow du = 2x\,dx$

$= \frac{1}{2}\int x^2 e^u\,du$

$= \frac{1}{2}\int ue^u\,du = \frac{1}{2}ue^u - \frac{1}{2}e^u + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$

6) $I = \int \arctan x\,dx = \arctan x \cdot x - \int \frac{x}{x^2+1}\,dx$ Let $u = x^2 \rightsquigarrow du = 2x\,dx$

$= x\arctan x - \frac{1}{2}\int \frac{1}{u+1}\,du$

$= x\arctan x - \frac{1}{2}\ln|x^2+1| + C$

7) $I = \int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Let $u = 1-x^2$
$du = -2x\,dx$

$= x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= x\arcsin x + \frac{1}{2} \cdot 2u^{\frac{1}{2}} + C$

$= x\arcsin x + \sqrt{1-x^2} + C$

8) $I = \int \ln(2x)\,dx = x\ln(2x) - \int x \cdot \frac{2}{2x}\,dx = x\ln(2x) - x + C$

9) $I = \int \arcsin(3x)\,dx = \frac{1}{3}[3x\arcsin(3x) + \sqrt{1-9x^2}] + C = x\arcsin(3x) + \frac{1}{3}\sqrt{1-9x^2} + C$

10) $I = \int \ln(x^2)\,dx = \int 2\ln x\,dx = 2x\ln x - 2x + C$

11) $I = \int x^3\cos(3x)\,dx$

$= \frac{1}{3}x^2\sin(3x) + \frac{2}{9}x\cos(3x) - \frac{2}{27}\sin(3x) + C$

D	I
$+\ x^2$	$\cos(3x)$
$-\ 2x$	$\frac{1}{3}\sin(3x)$
$+\ 2$	$-\frac{1}{9}\cos(3x)$
$-\ 0$	$-\frac{1}{27}\sin(3x)$

12) $I = \int x^3 e^x\,dx$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

D	I
$+\ x^3$	e^x
$-\ 3x^2$	e^x
$+\ 6x$	e^x
$-\ 6$	e^x
$+\ 0$	e^x

Trig-Sub Integrals

1) $I = \int \frac{2}{3+5x^2}\,dx$

Let $x = \sqrt{\frac{3}{5}}\tan\theta$

$dx = \sqrt{\frac{3}{5}}\sec^2\theta\,d\theta$

$\theta = \arctan\left[\frac{\sqrt{5}}{\sqrt{3}}x\right]$

$= \int \frac{2}{3+3\tan^2\theta}\cdot\frac{\sqrt{3}}{\sqrt{5}}\sec^2\theta\,d\theta$

$= \frac{2}{3}\cdot\frac{\sqrt{3}}{\sqrt{5}}\int\frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$

$= \frac{2\sqrt{3}}{3\sqrt{5}}\int d\theta = \frac{2\sqrt{3}}{3\sqrt{5}}\theta + C = \frac{2\sqrt{3}}{3\sqrt{5}}\arctan\left[\frac{\sqrt{5}}{\sqrt{3}}x\right] + C$

2) $I = \int\frac{4}{7+4x+x^2}\,dx = \int\frac{4}{[x+2]^2+3}\,dx$

Let $x = \sqrt{3}\tan\theta - 2$

$dx = \sqrt{3}\sec^2\theta\,d\theta$

$\theta = \arctan\left(\frac{x+2}{\sqrt{3}}\right)$

$= 4\int\frac{1}{3\tan^2\theta+3}\cdot\sqrt{3}\sec^2\theta\,d\theta$

$= \frac{4\sqrt{3}}{3}\theta + C = \frac{4\sqrt{3}}{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C$

A general result that may be useful to remember is

$\int\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$