Problem 38. 2012.02.04

In this question you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1}\frac{x^n}{n} + \dots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}$$

Hence show that

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

Deduce that

$$\left(1+\frac{1}{n}\right)^n < e.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that

$$\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y} \text{ for } y > \frac{1}{2}$$

Deduce that, for any positive integer n,

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to e.$$

Prerequisites.

The infinite series given at the start is the Taylor series expansion for the function $\ln(1+x)$. Notice that the condition $|x| < 1$ is the same convergence condition as for the infinite sum of a GP and for the convergence of Binomial expansions for fractional or negative indices. These issues will not trouble you in this question. What may trouble you is the inequality at the start and possibly the manipulation of limits at the end.