Introduction to Complex Numbers

Suppose $i = \sqrt{-1}$:

Paradox 1

$-1 = -1$

$\frac{-1}{1} = \frac{1}{-1}$ ⟵ obviously true

$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$ square root both sides

$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}}$ split square roots.

$\frac{i}{1} = \frac{1}{i}$ $i = \sqrt{-1}$

$i = \frac{1}{i}$

$i^2 = 1$ Multiply both sides by i

$(\cancel{\sqrt{}}\,{-1})^{\cancel{2}} = 1$ $i = \sqrt{-1}$

$-1 = 1$ ⟵ obviously false!

Note:

There was an error in our manipulation of our equation. In general, the following is not true in general when $a, b \notin \mathbb{R}$:

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Paradox 2

$1 = e^{2i\pi}$ (Via Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$)

$1 = e^{4i\pi}$ (Via Euler's formula)

$\Rightarrow e^{2i\pi} = e^{4i\pi}$ Both are equal to 1

$\cancel{\ln}(\cancel{e}^{2i\pi}) = \cancel{\ln}(\cancel{e}^{4i\pi})$ Taking the natural log of both sides

$2i\pi = 4i\pi$ Definition of natural logarithm, $\ln(x) = f(x)$ s.t. $f(e^x) = x$

$\therefore 2 = 4$ ⟵ obviously false!

Note:

In both of these cases, we have reached absurd conclusions when working with i. This time, it was the step where we cancelled out the exponential and the logarithm.

Negative numbers do not have square roots. The definition of i, where $i = \sqrt{-1}$, does not make sense. It does not say what i actually is. Instead, we can get a definition for i by defining the set of complex numbers, $\mathbb{C}$.

Definition (complex numbers):

The set of complex numbers $\mathbb{C}$ is the set of all ordered pairs $(x, y) \in \mathbb{R}^2$. Then, i can be defined to be $(0, 1) \in \mathbb{C}$. With this definition, $\mathbb{C}$ has the following operations:

- **Addition**: For $(a,b), (c,d) \in \mathbb{C}$, their sum is given by
  $$(a,b) + (c,d) := (a+c, b+d)$$
- **Multiplication**: For $(a,b), (c,d) \in \mathbb{C}$, their product is given by
  $$(a,b) \cdot (c,d) := (ac-bd, ad+bc)$$

Our definition is robust and without logical issues. We defined complex numbers using something that already exists (and we are familiar with), $\mathbb{R}$. Giving operations to $\mathbb{C}$ is not wrong, because we are simply defining functions
$+, \cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

These operations being called "addition" and "multiplication", respectively, does not mean that they will satisfy the same properties that those operations satisfy for real numbers. That must be proven using the definitions we already have. Therefore, this is not the best way to introduce the concept of complex numbers. Writing them in the form $x+iy$ is much easier for the purposes of algebraic manipulation.

Let us say that we have the number $2+3i$. What does that mean? We have defined complex numbers as an ordered pair of real numbers, and we know $i := (0,1)$. Thus, $2+3i$ would be $2+3(0,1)$. However, we have not defined multiplication or addition between a real number and a complex number. This problem may be solved via "abuse of notation". We identify the x-axis of $\mathbb{R}^2 = \mathbb{C}$ with $\mathbb{R}$. Whenever we see $(x,0)$, we view it as x. When we see x, we view it as $(x,0)$. This is not rigorous at all, but it makes algebra easier.

1) With our definitions, we can prove some important properties. Firstly, a key property of i is that $i^2 = -1$.

pf
$$\begin{aligned} i^2 = i \cdot i &= (0,1)\cdot(0,1) \\ &= (0\cdot 0 - 1\cdot 1, 0\cdot 1 - 1\cdot 0) \\ &= (0-1, 0-0) \\ &= (-1, 0) \end{aligned}$$
Q.E.D.

2) Let $a,b \in \mathbb{R}$. $a+bi = (a,b)$.

pf
$$\begin{aligned} a+bi &= (a,0) + (b,0)\cdot(0,1) \\ &= (a,0) + (b\cdot 0 - 0\cdot 1, b\cdot 1 - 0\cdot 0) \\ &= (a,0) + (0,b) \\ &= (a+0, 0+b) \\ &= (a,b) \end{aligned}$$
Q.E.D.

3) Let $a, b, c \in \mathbb{R}$. $a(b+ci) = ab + aci$ (scalar multiplication)

pf $a(b+ci) = a \cdot (b,c)$ (via result obtained in 2)
$= (a,0)\cdot(b,c)$
$= (a\cdot b - 0\cdot c, a\cdot c - 0\cdot b)$
$= (ab, ac)$
$= (ab, 0) + (0, ac)$ (via definition of addition)
$= (ab, 0) + (0, ac)\cdot(0,1)$ (via result obtained in 2)
$= ab + aci$

4) Let $u, v, w \in \mathbb{C}$.
i) $uv = vu$ (commutativity of multiplication)

pf $uv = (u_a v_a - u_b v_b, u_a v_b + u_b v_a)$
$= (v_a u_a - v_b u_b, v_b u_a + v_a u_b)$ (via the fact that $ab = ba$, where $a, b \in \mathbb{R}$)
$= (v_a u_a - v_b u_b, v_a u_b + v_b u_a)$ (via the fact that $a+b = b+a$, where $a, b \in \mathbb{R}$)
$= \underline{\underline{vu}}$ Q.E.D.

ii) $(uv)w = u(vw)$ (associativity of multiplication)

pf $(uv)w = (u_a v_a - u_b v_b, u_a v_b + u_b v_a)\cdot(w_a, w_b)$
$= (w_a(u_a v_a - u_b v_b) - w_b(u_a v_b + u_b v_a), w_b(u_a v_a - u_b v_b) + w_a(u_a v_b + u_b v_a))$
$= (u_a v_a w_a - u_b v_b w_a - u_a v_b w_b - u_b v_a w_b, u_a v_a w_b - u_b v_b w_b + u_a v_b w_a + u_b v_a w_a)$
$= (u_a v_a w_a - u_a v_b w_b - u_b v_a w_b - u_b v_b w_a, u_a v_a w_b + u_a v_b w_a + u_b v_a w_a - u_b v_b w_b)$
$= (u_a(v_a w_a - v_b w_b) - u_b(v_a w_b + v_b w_a), u_a(v_a w_b + v_b w_a) + u_b(v_a w_a - v_b w_b))$
$= (u_a, u_b)\cdot(v_a w_a - v_b w_b, v_a w_b + v_b w_a)$
$= \underline{\underline{u(vw)}}$ Q.E.D.

5) Let $u, v, w \in \mathbb{C}$. $u(v+w) = uv + uw$ (distributivity)

pf $u(v+w) = (u_a, u_b)\cdot((v_a, v_b) + (w_a, w_b))$
$= (u_a, u_b)\cdot(v_a + w_a, v_b + w_b)$
$= (u_a(v_a + w_a) - u_b(v_b + w_b), u_a(v_b + w_b) + u_b(v_a + w_a))$
$= (u_a v_a + u_a w_a - u_b v_b - u_b w_b, u_a v_b + u_a w_b + u_b v_a + u_b w_a)$
$= (u_a v_a - u_b v_b + u_a w_a - u_b w_b, u_a v_b + u_b v_a + u_a w_b + u_b w_a)$
$= (u_a v_a - u_b v_b, u_a v_b + u_b v_a) + (u_a w_a - u_b w_b, u_a w_b + u_b w_a)$
$= (u_a, u_b)\cdot(v_a, v_b) + (u_a, u_b)\cdot(w_a, w_b)$
$= \underline{\underline{uv + uw}}$ Q.E.D.

6) Let $z \in \mathbb{C}$. What does $-z$ mean? For real numbers, a number added to the

negative version of that number gives you 0. That is, $x + (-x) = 0$, where $x \in \mathbb{R}$.
Any negative number, $-x$, can be expressed as $-1 \cdot x$. Let us do this with complex numbers.
$z := (a,b)$, $-z := (-1,0)\cdot(a,b) = (-a,-b)$

Additionally, all real numbers x, except 0, have a reciprocal: $\frac{1}{x}$. The following statement holds true. $x \cdot \frac{1}{x} = 1$, where $x \in \mathbb{R}, x \neq 0$. We can attempt to use this property to define
$\frac{1}{z} := (c,d)$
$z \cdot \frac{1}{z} = (ac-bd, ad+bc) := (1,0)$
$\rightsquigarrow ac-bd = 1, \quad ad+bc = 0$ $\qquad ad = -bc$
$\Rightarrow ac = 1+bd \Rightarrow c = \frac{1+bd}{a}$ $\qquad c = -\frac{ad}{b}$
$\Rightarrow ad + b\left[\frac{1+bd}{a}\right] = 0$
$\Rightarrow ad + \frac{b+b^2d}{a} = 0$
$\Rightarrow a^2d + b + b^2d = 0$
$\Rightarrow d[a^2+b^2] + b = 0$
$\Rightarrow d = -\frac{b}{a^2+b^2}$
$\rightsquigarrow c = \cancel{-}\frac{a}{\cancel{b}} \cdot \cancel{-}\frac{\cancel{b}}{a^2+b^2}$
$\Rightarrow c = \frac{a}{a^2+b^2}$

$\therefore \frac{1}{z} := \left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)$

7) We have defined the negative of some complex number. Let $z, w \in \mathbb{C}$. We can define subtraction now. First, $z := (a,b)$, and $w := (c,d)$.
$- : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. $z - w := (a-b, c-d)$.

Also, with the definition for the reciprocal of some complex number, we can define division.
$/ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. $\frac{z}{w} := z \cdot \frac{1}{w}$.

We can calculate what that would be. The easiest way to do that would be to express z as $a+bi$, and w as $c+di$. Then, we can perform some basic algebra.

$$\frac{z}{w} := \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac - adi + bci - bdi^2}{c^2 - (di)^2}$$ difference of two squares

$$= \frac{ac+bd+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

$$\therefore \frac{z}{w} := \left[\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right], \text{ where } z \neq (0,0)$$

We should note, that so far, we have abused notation in order to deal with real and complex numbers together in the same algebraic expression. To make our previous abuse of notation a bit more formal, we can define an embedding, $\mathbb{R} \to \mathbb{C}$, such that

$x \mapsto (x, 0)$. By doing this, it is not really abuse of notation to write $a+bi$. The arithmetic operations between the real and complex numbers are defined, as the reals are seen complex numbers.

8. We can define the conjugate of a complex number. Let $z \in \mathbb{C}$, where $z := (a, b)$.
$f: \mathbb{C} \to \mathbb{C}; f(z) := (a, -b)$.
$\bar{z} := f(z)$ we will not use $f(z)$ to refer to $\bar{z}$

We have rigorously defined $\mathbb{C}$. We know that $i^2 = -1$, though that is simply a property that arises from our definitions. We cannot current go from that to $\sqrt{-1} = i$, because the square root function is defined over nonnegative real numbers. We could make the definition $\sqrt{-1} := i$. This is better than $i := \sqrt{-1}$, as i already exists. However, we must consider whether this useful to us, appears natural, and satifies some properties we want.
For $y \in \mathbb{R}$, where $y \geq 0$, $\sqrt{y} := |x|$ s.t. $x^2 = y$.
Here, $|x|$ represents the absolute value function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The reason $|x|$ is used is because there are two values of x for every value of y, except $y=0$, that satisfy $x^2 = y$. For example, $(2)^2 = (-2)^2 = 4$, though $\sqrt{4} \neq -2$. The reason we take the positive number out of the two, is because it makes sense to us. It is a sensible, natural convention to have. Additionally, the identity $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ holds true for all $x, y \geq 0$.

Extending the square root function to $\mathbb{C}$ would present the same issue. $(1+i)^2 = (-1-i)^2 = 2i$. It would make sense to state that $\sqrt{2i} = 1+i$, instead of $\sqrt{2i} = -1-i$. If we split the complex plane along the real axis, we would get two halves. Every complex number $z \in \mathbb{C} \setminus \mathbb{R}$ would lie in one of these halves. There are exactly two values of w, where $w \in \mathbb{R}$, that satisfy $w^2 = z$. One of these lies in the upper half of the complex plane, whilst the other would lie in the bottom half. It makes sense for $\sqrt{z}$ to lie on the same half as z. We can make this more formal by defining two new functions.

Definition (real and imaginary parts):
Let $z \in \mathbb{C}$. $z := (a, b)$.
$\mathrm{Re}: \mathbb{C} \to \mathbb{R}; \mathrm{Re}(z) := a$. $\mathrm{Im}: \mathbb{C} \to \mathbb{R}; \mathrm{Im}(z) := b$.

We have a function that gives us the real part of a function, and one that gives us the imaginary part. This means that $z = (\mathrm{Re}(z), \mathrm{Im}(z)) = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$. Now, there are exactly two values of w that satisfy $w^2 = z$, where $w \in \mathbb{C}$, and $z \in \mathbb{C} \setminus \mathbb{R}$. One of these values is w; the other value is $-w$. If $\mathrm{Im}(w) > 0$ (upper half of complex plane), then $\mathrm{Im}(-w) < 0$ (bottom half). With this, we can define $\sqrt{z}$.

$w^2 = z$. $\sqrt{z} := \begin{cases} w & \text{if } \operatorname{sgn}(\operatorname{Im}(z)) = \operatorname{sgn}(\operatorname{Im}(w)) \\ -w & \text{if } \operatorname{sgn}(\operatorname{Im}(z)) \neq \operatorname{sgn}(\operatorname{Im}(w)) \end{cases}$, where $\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

This definition for $\sqrt{z}$ is not very useful for the purposes of calculating square roots. A better definition may be derived using the polar forms of complex numbers. That is beyond the scope of this introduction.