Problem 73. 2013.03.02

In this question, you may ignore questions of convergence.

Let $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{dy}{dx} - xy - 1 = 0\,,$$

and prove that, for any positive integer n,

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+3)x\frac{d^{n+1}y}{dx^{n+1}} - (n+1)^2\frac{d^n y}{dx^n} = 0\,.$$

Hence obtain the Maclaurin series for $\frac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and even powers of x.

Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2 \times 3^2}{7!} + \ldots + \frac{2^2 \times 3^2 \times \ldots \times n^2}{(2n+1)!} + \ldots$$

Prerequisites.

Firstly, you should know that $\arcsin x$ is an alternative way of writing $\sin^{-1} x$.

To begin with, this question requires you to be totally accurate in applying standard rules for differentiation and be fully confident about constructing a proof by induction.

To obtain the Maclaurin series for $\frac{\arcsin x}{\sqrt{1-x^2}}$ you must of course know the standard result

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

When using this to find the expansion for

$$f(x) = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

you will require some persistence to find a sufficient number of terms to be able to see some sort of pattern emerging in the coefficients. You will need to write separate expressions for the general terms expressing odd and even powers of x. You can then go on to obtain the required expansion but **do not** fully evaluate the coefficients. Instead, leave them as products so that you may more easily recognise the pattern of development of these coefficients that you will need to find, in order to evaluate the infinite sum given at the end of the question. The critical part of