2025-03-01 STEP Practice: Problem 73 (2013.03.02)

Let $y = \frac{\arcsin x}{\sqrt{1-x^2}}$.

$$y' = \frac{\frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} - -\frac{x}{\sqrt{1-x^2}} \cdot \arcsin x}{1-x^2} = \frac{1}{1-x^2} + \frac{x \arcsin x}{[1-x^2]\sqrt{1-x^2}}$$

(*) $[1-x^2]\frac{dy}{dx} - xy - 1 = [1-x^2]y' - xy - 1 = 1 + \frac{x \arcsin x}{\sqrt{1-x^2}} - \frac{x \arcsin x}{\sqrt{1-x^2}} - 1 = 0$

Statement: $[1-x^2]\frac{d^{n+2}y}{dx^{n+2}} - [2n+3]x\frac{d^{n+1}y}{dx^{n+1}} - [n+1]^2\frac{d^n y}{dx^n} = 0$

(**) rewritten as: $[1-x^2]y^{(n+2)} - [2n+3]xy^{(n+1)} - [n+1]^2 y^{(n)} = 0$

For $n=1$: $[1-x^2]y''' - 5xy'' - 4y' = 0$

I just find $y^{(n)}$ easier to deal with than $\frac{d^n y}{dx^n}$

Differentiating (*):

$\frac{d}{dx}[[1-x^2]y' - xy - 1] = \frac{d}{dx}[0]$

We can differentiate LHS and RHS because the equation holds for all x.

$-2xy' + [1-x^2]y'' - y - xy' = 0$

$[1-x^2]y'' - 3xy' - y = 0$

$\frac{d}{dx}[[1-x^2]y'' - 3xy' - y] = \frac{d}{dx}[0]$

$-2xy'' + [1-x^2]y''' - 3y' - 3xy'' - y' = 0$

$[1-x^2]y''' - 5xy'' - 4y' = 0$

$\therefore$ statement is true for $n=1$.

Suppose the statement is true for $n=k$, where $k \in \mathbb{Z}^+$.

Then, $[1-x^2]y^{(k+2)} - [2k+3]xy^{(k+1)} - [k+1]^2 y^{(k)} = 0$

$\frac{d}{dx}[LHS] = \frac{d}{dx}[RHS]$

$-2xy^{(k+2)} + [1+x^2]y^{(k+3)} - [2k+3]y^{(k+1)} - [2k+3]xy^{(k+2)} - [k+1]^2 y^{(k+1)} = 0$

$$[x+1]^2 y^{(k+3)} - [2k+5]xy^{(k+2)} - [[k+1]^2+[2k+3]]y^{(k+1)} = 0$$

$$[x+1]^2 y^{(k+1+2)} - [2[k+1]+3]x^{(k+1+1)} - [k+1+1]^2 y^{(\ +1)} = 0$$

$\therefore$ The statement is true for $n=k+1$, if it is true for $n=k$.
The statement is true for $n=1$.
$\therefore$ By mathematical induction, the statement is true for all $n \in \mathbb{Z}^+$.

Let $f(x)=y$. Then, the Maclaurin series for $f(x)$ is expressed as the following.

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

Replacing $y^{(n)}$ with $f^{(n)}(x)$ in (**)

$f^{(n)}(x)$ satisfies (**). For $x=0$:

$$f^{(n+2)}(0) - [n+1]^2 f^{(n)}(0) \quad 0$$

Middle term disappears

(***) $$f^{(n+2)}(0) = [n+1]^2 f^{(n)}(0)$$

$$f(0) = \frac{\arcsin(0)}{\sqrt{1-0}} = 0$$

$$f'(0) = \frac{1}{1-0} + 0 = 1$$

$f''(x)$ satisfies $[1-x^2]f''(x) - 3xf'(x) - f(x) = 0$
For $x=0$: $f''(0) - 0 - 0 = 0$
$f''(0) = 0 \rightsquigarrow f^{(4)} = f^{(6)} = f^{(8)} = \ldots = 0$ via (***)

It can be seen that $f^{(2n)} = 0$ for all $n \in \mathbb{Z}^+$.

via (***)

$$f'(0) = 1,\ f'''(0) = [1+1]^2 \cdot 1,\ f^{(5)}(0) = [3+1]^2 \cdot [1+1]^2 \cdot 1,\ f^{(7)} = [5+1]^2 \cdot [3+1]^2 + [1+1]^2 \cdot 1, \ldots$$

$$f^{(2n+1)}(0) = \prod_{i=1}^{n} [2i]^2 \text{ for all } n \in \mathbb{Z}^+.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \cancelto{0}{\sum_{n=1}^{\infty} \frac{f^{(2n)}(0)}{[2n]!} x^{2n}} + \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{[2n+1]!} x^{2n+1}$$

$$= x + \sum_{n=1}^{\infty} \frac{f^{(2n+1)}(0)}{[2n+1]!} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{[2n+1]!} \prod_{i=1}^{n} [2i]^2$$

$$f(x) = x + \sum_{n=1}^{\infty} x^{2n+1} \prod_{i=1}^{2n+1}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n}[2i]^2$$

$$n! = \prod_{i=0}^{n} i$$

$$= x + \sum_{n=1}^{\infty} x^{2n+1} \prod_{i=n+1}^{2n+1}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n}[2i]^2$$

$$\prod_{i=1}^{n} a_i \cdot \prod_{i=1}^{n} b_i = \prod_{i=1}^{n} a_i b_i$$

$$= x + \sum_{n=1}^{\infty} x^{2n+1} \prod_{i=n+1}^{2n+1}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n} \frac{[2i]^2}{i}$$

$$= x + \sum_{n=1}^{\infty} x^{2n+1} \prod_{i=n+1}^{2n+1}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n} 4i$$

$$\prod_{i=1}^{n} 4i = \prod_{i=1}^{n} 4 \cdot \prod_{i=1}^{n} i = 4^n i!$$

$$= x + \sum_{n=1}^{\infty} x^{2n+1} 4^n i! \prod_{i=n+1}^{2n+1}\left[\tfrac{1}{i}\right]$$

$$= x + \sum_{n=1}^{\infty} x^{2n+1} 4^n n! \prod_{i=1}^{2n+1}\left[\tfrac{1}{i}\right] \prod_{i=1}^{n} i$$

Both forms of this series verified using Stack Exchange

$$= x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{[2n+1]!} 4^n [n!]^2$$

$$= \sum_{n=0}^{\infty} \frac{4^n [n!]^2}{[2n+1]!} x^{2n+1}$$

Alternatively:

$$f(x) = x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{[2n+1]!} \prod_{i=1}^{n} [2i]^2$$

$$= x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{[2n+1]!} \cdot [[2n]!!]^2$$

n!! is a double factorial:

$$n!! := \prod_{i=1}^{n/2} 2i$$

Like regular factorial but only uses factors of the same parity as n.
e.g. $10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$,
$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$

$$= x + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{[2n+1]!!\,[2n]!!} [[2n]!!]^2$$

$$= x + \sum_{n=1}^{\infty} \frac{[2n]!!}{[2n+1]!!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{[2n]!!}{[2n+1]!!} x^{2n+1}$$

The general term for even powers of x is 0.
The general term for odd powers of x is $\frac{4^n[n!]^2}{[2n+1]!}x^{2n+1}$ or $\frac{[2n]!!}{[2n+1]!!}x^{2n+1}$

We are asked to evaluate $\sum_{n=0}^{\infty}\frac{[n!]^2}{[2n+1]!}$.

Note that this looks very similar to the first form of our Maclaurin series.
If we choose $x=\frac{1}{2}$, then

$$f(\tfrac{1}{2}) = \sum_{n=0}^{\infty}\frac{4^n[n!]^2}{[2n+1]!}\left[\tfrac{1}{2}\right]^{2n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n[n!]^2}{[2n+1]!}\left[\tfrac{1}{2}\right]^{2n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n[n!]^2}{[2n+1]!}\cdot 2^{-2n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{4^n[n!]^2}{[2n+1]!}\cdot 4^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{[n!]^2}{[2n+1]!}$$

To make the two summands the same

Finding value of x.
Choose x s.t. $x^{2n+1} = 4^{-n}$
2^{-2n}
very close $\left[\frac{1}{2}\right]^{2n}$
We can then try $x=\frac{1}{2}$ and see what happens

not the biggest fan of "clearly", but it is pretty clear...

Clearly, the target sum is $2f(\frac{1}{2})$

$$\sum_{n=0}^{\infty}\frac{[n!]^2}{[2n+1]!} = 2f(\tfrac{1}{2}) = \frac{2\arcsin(\frac{1}{2})}{\sqrt{1-(\frac{1}{2})^2}} = \frac{2\cdot\frac{\pi}{6}}{\sqrt{\frac{3}{4}}} = \frac{\pi}{3}\cdot\frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

Notes

All in all, this is not a very difficult question. The first part is straightforward differentiation, though I did make a few careless mistakes at first. The proof by induction is not the standard type that you see in your A-levels. Figuring out that you have to differentiate both sides took some time, and I almost fell into the trap of finding higher-order derivatives of $f(x)$. With the proven formula, finding the Maclaurin series is easy. I definitely spent far too much time finding a 'nice' form for it. Fortunately, it made the last part of the question easier. If I did not spend as much time on the Maclaurin series, then I would have to consider the given sum more closely to figure out a way to evaluate it with what I have.