2024-09-21 # STEP Practice: Problem 3 (2004.1.1)

Lemma 3.1

The square of an even number is even.

pf: Let $k \in \mathbb{Z}^+$

$\therefore 2k$ is even

$(2k)^2 = 4k^2$

$= 2(2k^2)$

$\Rightarrow (2k)^2$ is an even integer <u>Q.E.D</u>

Lemma 3.2

The square of an odd number is odd.

pf: Let $k \in \mathbb{Z}^+$

$\therefore 2k-1$ is odd

$(2k-1)^2 = 4k^2 - 4k + 1$

$= 2(2k^2 - 2k) + 1$

$\Rightarrow (2k-1)^2$ is an odd integer <u>Q.E.D</u>

$\sqrt{2}$ cannot be written as a rational number ($\sqrt{2} \neq \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0$)

pf: Suppose that $\sqrt{2} = \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0, \gcd(p,q) = 1$

$\sqrt{2} = \frac{p}{q}$

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

$\therefore p^2$ is even

$\Rightarrow p$ is even (Lemma 3.1)

$\therefore p = 2k, k \in \mathbb{Z}$

$p^2 = (2k)^2 = 4k^2$

$\therefore \cancel{2}q^2 = \cancel{4}^{2}k^2$

$q^2 = 2k^2$

$\therefore q^2$ is even

$\Rightarrow q$ is even (Lemma 3.1)

p is even $\wedge$ q is even
$\therefore \gcd(p, q) \neq 1$

There is a contradiction
$\therefore$ supposition is false
$\therefore$ proposition is true. Q.E.D

Note:
A rational number cannot equal an irrational number. That is evident via the definition of a rational number x, where $x \in \mathbb{R} \setminus \mathbb{Q}$

i) $(3+2\sqrt{5})^3 \equiv a+b\sqrt{5} \mid a,b \in \mathbb{Z}$

$27+120+60 = 27+180 = 207$
$18+36+40 = 54+40 = 94$

$(3+2\sqrt{5})^3 = (3+2\sqrt{5})^2(3+2\sqrt{5})$
$= (9+12\sqrt{5}+20)(3+2\sqrt{5})$
$= 27+18\sqrt{5}+36\sqrt{5}+120+60+40\sqrt{5}$
$= 207+94\sqrt{5}$

$\therefore 207+94\sqrt{5} \equiv a+b\sqrt{5}$
$207 = a$
$94\sqrt{5} = b\sqrt{5} \implies 94 = b$

$\boxed{a = 207,\ b = 94}$

ii) $\sqrt[3]{99-70\sqrt{2}} \equiv c-d\sqrt{2}$, where $c,d \in \mathbb{Z}^+$.

$99-70\sqrt{2} \equiv (c-d\sqrt{2})^3$
$\equiv (c-d\sqrt{2})^2(c-d\sqrt{2})$
$\equiv (c^2-2cd\sqrt{2}+2d^2)(c-d\sqrt{2})$
$\equiv c^3-c^2d\sqrt{2}-2c^2d\sqrt{2}+4cd^2+2cd^2-2d^3\sqrt{2}$
$\equiv c^3-3c^2d\sqrt{2}+6cd^2-2d^3\sqrt{2} \equiv 99-70\sqrt{2}$

① $\therefore c^3+6cd^2 = 99$
② & $-3c^2d-2d^3 = -70 \implies 3c^2d+2d^3 = 70$
Since $c,d \in \mathbb{Z}^+$, $c \leq 4$ and $d \leq 4$

If $c=1$, $98 = 6d^3$ from ①
$\therefore c \neq 1 \because 6 \nmid 98$

Similarly, $c \neq 2 \because 6 \nmid 91$
and $c \neq 4 \because c \nmid 35$

$\therefore c = 3, d = 2$

iii) $x^6 - 198x^3 + 1 = 0$ (Quadratic Equation in x^3)

$$x^3 = \frac{198 \pm \sqrt{198^2 - 4}}{2} = \frac{198 \pm \sqrt{(198+2)(198-2)}}{2} = \frac{198 \pm \sqrt{200 \times 196}}{2} = \frac{198 \pm 140\sqrt{2}}{2}$$
$$= 99 \pm 70\sqrt{2}$$

$\therefore x = 3 \pm 2\sqrt{2}$

Notes

Being able to recognise the significance of the identity is crucial for solving this problem. Since the structure of the LHS and the RHS were the same, we were able to equate terms. Trial and error was the method used to complete (ii). It was effective here because the information we had greatly limited the number of values we had to try. The ability to extract all the relevent information from questions like these is vital to solving them. One other thing to note is that (iii) was incredibly straightforward and simple to complete only because we had the result from the previous part. This highlights the way the various parts of a STEP question work with each other.