18-01-2025 STEP Practice : Problem 14 (2000.01.01)

$\log 2 = 0.301029996$ $\quad \log 3 = 0.477121255$

i) $\log 5 = \log\left(\frac{10}{2}\right) = \log 10 - \log 2 = 1 - \log 2 = 0.699$ 3d.p.
$\log 6 = \log(2\times 3) = \log 2 + \log 3 = 0.778$ 3d.p.

$\log(5\times 10^{47}) = \log 5 + 47 = 47.699$ 3d.p.
$\log(3^{100}) = 100\log 3 = 47.712$ 3d.p.
$\log(6\times 10^{47}) = \log 6 + 47 = 47.778$ 3d.p.
$\therefore \log(5\times 10^{47}) < \log(3^{100}) < \log(6\times 10^{47})$
$\therefore 5\times 10^{47} < 3^{100} < 6\times 10^{47}$ $\quad \because \log x$ is strictly increasing
$\therefore$ First digit of 3^{100} is 5.

ii) $\log(2^{1000}) = 1000\log 2 = 301.030$ 3d.p.
$\log 9 = 2\log 3 = 0.954$ 3d.p.
$\log(1\times 3^{301}) < \log(2^{1000}) < \log(2\times 10^{301})$
$\therefore$ First digit of 2^{1000} is 1.

$\log(2^{10000}) = 10000\log 2 = 3010.300$ 3d.p.
$\log(1\times 10^{3010}) < 2^{10000} < \log(2\times 10^{3010})$
$\therefore$ First digit of 2^{10000} is 1.

$\log(2^{100000}) = 100000\log 2 = 30102.9996$ 4d.p.
$\log(9\times 10^{30102}) < \log(2^{100000}) < \log(1\times 10^{30103})$
$\therefore$ First digit of 2^{100000} is 9.

We can do this because the function $\log x$ is strictly increasing. It allows us to take the logarithm of an inequality and preserve the order. That, alongside some properties of logarithms, allowed for me to solve this STEP problem pretty easily.