

02-11-2024 STEP Practice: Polynomial Graphs and Triangles

A polynomial is a finite expression of the form:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n \equiv \sum_{i=0}^n a_i x^i =: P\{A\}(x); A := \{a_0, a_1, \dots, a_n\}$$

Let $k \in \mathbb{Z}^+$.

$$P'(x) = \frac{d}{dx} \sum_{i=0}^n a_i x^i = \sum_{i=0}^n \frac{d}{dx} [a_i x^i] = \sum_{i=0}^n i a_i x^{i-1}$$

$$\lim_{x \rightarrow \infty} x^{2k} = +\infty$$

$$\lim_{x \rightarrow -\infty} x^{2k} = +\infty$$

$$\lim_{x \rightarrow \infty} x^{2k+1} = +\infty$$

$$\lim_{x \rightarrow -\infty} x^{2k+1} = -\infty$$

$$\lim_{x \rightarrow \infty} x^2 = \infty \Leftrightarrow [\forall M > 0 \exists N > 0 \text{ s.t. } x > N \Rightarrow x^2 > M]$$

$$x^2 \geq 0 \quad \forall x \in \mathbb{R}$$

$$x^2 > x \quad \forall x > 1$$

$$M = \frac{1}{2}: N = \max(1, \frac{1}{2}) = 1 \quad 3 > 1 \Rightarrow 3^2 > 1$$

$$M = 2: N = \max(1, 2) = 2 \quad 3 > 2 \Rightarrow 3^2 > 2$$

pf: Given $M > 0$,
let $N = \max(1, M)$.

If $x > N$,

Then:

$$x^2 > \max(1, M)^2 \geq M^2 > \underline{M} \quad \text{Q.E.D.}$$

$$\lim_{x \rightarrow -\infty} x^2 = \infty \Leftrightarrow [\forall M > 0 \exists N < 0 \text{ s.t. } x < N \Rightarrow x^2 > M]$$

$$x^2 > M \Rightarrow |x| > \sqrt{M} \Rightarrow x < -\sqrt{M} \quad \therefore x < N < 0$$

pf: Given $M > 0$,
let $N = -\sqrt{M}$.

If $x < N$,

Then:

$$x < -\sqrt{M} \Rightarrow -x > \sqrt{M} \Rightarrow |x| > \sqrt{M} \Rightarrow x^2 > \underline{M} \quad \text{Q.E.D.}$$

$$\sin \theta \equiv \tan \theta \quad \cot^2 \theta + \tan^2 \theta \equiv 1$$

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\Rightarrow \sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$$

$$\text{Let } P := A+B$$

$$\text{Let } Q := A-B$$

$$P+Q = 2A$$

$$A = \frac{P+Q}{2}$$

$$\Rightarrow \sin P + \sin Q = 2 \sin \left[\frac{P+Q}{2} \right] \cos \left[\frac{P-Q}{2} \right]$$

$$P-Q = 2B$$

$$B = \frac{P-Q}{2}$$

$$\Rightarrow \sin P - \sin Q = 2 \cos \left[\frac{P+Q}{2} \right] \sin \left[\frac{P-Q}{2} \right]$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

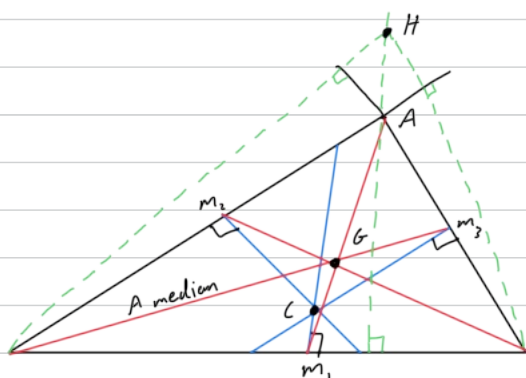
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\Rightarrow \cos(A+B) + \cos(A-B) = 2 \cos A \cos B$$

$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$$

$$\Rightarrow \cos P + \cos Q = 2 \cos \left[\frac{P+Q}{2} \right] \cos \left[\frac{P-Q}{2} \right]$$

$$\Rightarrow \cos P - \cos Q = -2 \sin \left[\frac{P+Q}{2} \right] \sin \left[\frac{P-Q}{2} \right]$$



$$AG:GM \\ 2:1$$

The 3 perpendicular bisectors are concurrent to the circumcentre.

A median is the line joining a vertex joining to the midpoint of the opposite side.

The medians are concurrent to the centroid of the triangle.

The altitudes of a triangle are concurrent at the orthocentre.

↑
Centre of gravity of a uniform triangular lamina

O, G, and H lie on a straight line called the Euler line.