STEP Notes (2011.02.07)

The Melon Man

2024-11-28

I am typesetting my notes for this STEP problem, because I it rather interesting. Additionally, it is one of the first STEP problems that I have attempted and completed myself – with almost no aid. It is a geometric progressions' problem, and a fairly difficult one. However, I found it simpler than most of the ones that I have seen far. As I attempt more problems, I will get better at learning how to approach them. Time-Management skills are not as important now, but they will be developed within due time.

Firstly, it is useful to express the formula for the sum of the first n terms of a geometric progression.

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^n = \frac{a(1-r^n)}{1-r} \tag{1}$$

We are given some information, all of which is useful. Two sequences, $a_0, a_1, a_2, \ldots$, and $b_0, b_1, b_2, \ldots$ have the general terms,

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n,$$

where

$$\lambda = 1 + \sqrt{2} \quad \text{and} \quad \mu = 1 - \sqrt{2}.$$

Part (i) provides us with a result which we must show is true.

$$\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}} a_{n+1} \tag{2}$$

Additionally, it asks us to give a corresponding result for:

$$\sum_{r=0}^{n} a_r$$

Given the information we have, we can write the sum of b_r in terms of λ and μ.

$$\sum_{r=0}^{n} b_r = \sum_{r=0}^{n} \lambda^r - \mu^r = \sum_{r=0}^{n} \lambda^r - \sum_{r=0}^{n} \mu^r$$

Now we have the difference between two geometric series; both of them can be evaluated using (1). The initial term for both sums is 1 ($\lambda^0 = \mu^0 = 1$), and the common ratios are λ and μ respectively.

$$\sum_{i=0}^{n} \lambda^i = \sum_{i=1}^{n} \lambda^{i-1} = \frac{1-\lambda^n}{1-\lambda} + \lambda^n \tag{3}$$

$$\sum_{i=0}^{n} \mu^i = \sum_{i=1}^{n} \mu^{i-1} = \frac{1-\mu^n}{1-\mu} + \mu^n \tag{4}$$

With these, we can find the sum of b_r. It is better not write λ and μ as the expressions they were given as for the sake of keeping our work simple. Those definitions will be utilised to simplify when possible.

$$\begin{aligned}
\sum_{r=0}^{n} b_r &= \frac{1-\lambda^n}{1-\lambda} - \frac{1-\mu^n}{1-\mu} + \lambda^n - \mu^n \\
&= \frac{(1-\lambda^n)(1-\mu) - (1-\mu^n)(1-\lambda)}{(1-\lambda)(1-\mu)} + \lambda^n - \mu^n \\
&= \frac{(1-\lambda^n)(1-1+\sqrt{2}) - (1-\mu^n)(1-1-\sqrt{2})}{(1-1-\sqrt{2})(1-1+\sqrt{2})} + \lambda^n - \mu^n \\
&= \frac{\sqrt{2}(1-\lambda^n) + \sqrt{2}(1-\mu^n)}{-2} + \lambda^n - \mu^n \\
&= -\frac{\sqrt{2}}{2}\left(1-\lambda^n+1-\mu^n\right) + \lambda^n - \mu^n \\
&= -\frac{1}{\sqrt{2}}\left(2-\lambda^n-\mu^n\right) + \lambda^n - \mu^n
\end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{2} + \frac{1}{\sqrt{2}}\left(\lambda^n + \mu^n\right) + \lambda^n - \mu^n \\
&= -\sqrt{2} + \frac{1}{\sqrt{2}}\left(\lambda^n + \mu^n + \sqrt{2}\lambda^n - \sqrt{2}\mu^n\right) \\
&= -\sqrt{2} + \frac{1}{\sqrt{2}}\left(\left(1+\sqrt{2}\right)\lambda^n + \left(1-\sqrt{2}\right)\mu^n\right) \\
&= -\sqrt{2} + \frac{1}{\sqrt{2}}\left(\lambda\lambda^n + \mu\mu^n\right) \\
&= -\sqrt{2} + \frac{1}{\sqrt{2}}\left(\lambda^{n+1} + \mu^{n+1}\right) \\
&= -\sqrt{2} + \frac{1}{\sqrt{2}}a_{n+1} \qquad \square
\end{aligned}
$$

Now, we may do the same for the sum of a_r.

$$
\begin{aligned}
\sum_{r=0}^{n} a_r &= \sum_{r=0}^{n} \lambda^r + \sum_{r=0}^{n} \mu^r \\
&= \frac{1-\lambda^n}{1-\lambda} + \frac{1-\mu^n}{1-\mu} + \lambda^n + \mu^n \\
&= \frac{(1-\lambda^n)(1-\mu) + (1-\mu^n)(1-\lambda)}{(1-\lambda)(1-\mu)} + \lambda^n + \mu^n \\
&= \frac{\sqrt{2}(1-\lambda^n) - \sqrt{2}(1-\mu^n)}{-2} + \lambda^n + \mu^n
\end{aligned}
$$