Python Function of Runge Kutta Method

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1 Four Order Runge Kutta Method

Runge–Kutta method is an effective and widely used iterative method for solving the initial-value problems of ordinary differential equations $\frac{dr(t)}{dt} = f(r,t)$. Runge–Kutta methods perform several evaluation of function $f()$ around the point $(r(t_n), t_n)$ and then $r(t_{n+1})$ is computed using a weighted average of those $f()'s$ values. The Runge-Kutta of four order is defined using the following recursion formula,

$$r_{n+1} = r_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4), \tag{1}$$

where the $k's$ are slopes values at the different points given by the function $f()$ evaluated as follow

$$k_1 = f(r_0, t_0), \tag{2a}$$

$$k_2 = f(r_0 + k_1\frac{h}{2}, t_0 + \frac{h}{2}), \tag{2b}$$

$$k_3 = f(r_0 + k_2\frac{h}{2}, t_0 + \frac{h}{2}), \tag{2c}$$

$$k_4 = f(r_0 + k_3 h, t_0 + h), \tag{2d}$$

where k_1 is the beginnig slope, k_2, k_3 two midpoint slopes and the endpoint slope k_4, as shown in the figure 1

The generalization of to vectorial functions is straightfoward as well as for second order differential equations.

The code solve the vectorial equation which can be written in a unique function, but because of a matter of code clean up taste the first and second order differential equations method applications are explicitly given.

1.1 Vectorial Differential Equation

To clarify the usage of the method and so the functions delivered in this work, the generalization can be explicitly written as follows. Consider the the differential equation as,

$$\frac{d\vec{r}(t)}{dt} = \vec{F}(\vec{r}, t, parameters). \tag{3}$$

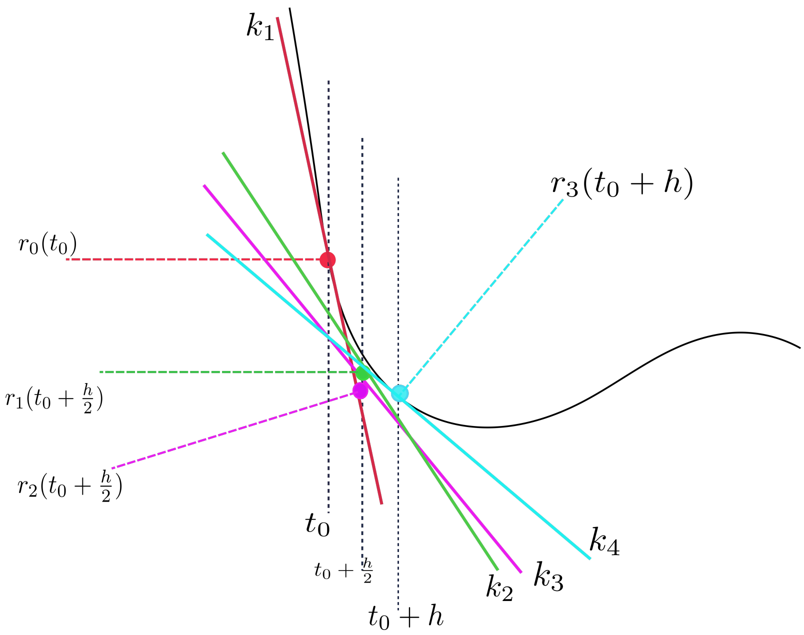

Figure 1: Evaluation of the function $f()$ around the point $(r(t_n), t_n)$ to obtain the slopes and perform the approximation.

where the function $\vec{F}()$ contains the for example the partial derivatives of $\vec{r}$. The slopes of the method are vectorial since are the function itself evaluated. Lorentz oscillator is the chosen example to ilustrate this vectorial solution, shown at the end of the file.

2 First Order Differential Equation Function

Numerical integration within an interval of integration, suppose $t = (t_0, t_f)$ to get the solution of the differential equation,

$$\frac{d\vec{r}(t)}{dt} = f(\vec{r}, t, parameters). \tag{4}$$

where parameters are parameters of the function. As theory of differential equations tell us all the time, to find the solution of a differential equation we need the initial conditions, $(t_0, \vec{r}_0)$. The applying the RK method one finds the slopes and the approximated solution as stated in the equations ((2a)-(16)) and (1).

3 Second Order Differential Equation Function

The ordinary second order differential equations have the following form

$$\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r}, \dot{\vec{r}}, t, parameters), \tag{5}$$

and are similarly solved as the first order differential equations but solving both the first derivative and the the function desired, at once. Because of the number of terms in the second order differential equation case grows, I avoid writting the vector form of the equations.

The approximated first derivative is recursively obtained as

$$\dot{r}_{n+1} = \dot{r}_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4), \tag{6}$$

and is used to obtain the approximated solution of the function as in (1). In (6) the $k's$ are the second derivatives, the slopes of the first derivative function curve at different points around one given by the function $\vec{F}()$ evaluated as,

$$k_1 = F(r_0, \dot{r}_0, t_0), \tag{7}$$

used to obtain the first derivative and solution stepping halfway through the time step,

$$\dot{r}1 = \dot{r}_0 + k_1 * h/2, \tag{8}$$

$$r1 = r_0 + h * \frac{(\dot{r}_0 + \dot{r}1)}{2}, \tag{9}$$

and similarly the other slopes and approximations,

$$k_2 = F(r1, \dot{r}1, t_0 + \frac{h}{2}), \tag{10}$$

$$\dot{r}2 = \dot{r}_0 + k_2 * h/2, \tag{11}$$

$$r2 = r_0 + h * \frac{(\dot{r}_0 + \dot{r}2)}{2}, \tag{12}$$

$$k_3 = F(r2, \dot{r}2, t_0 + \frac{h}{2}), \tag{13}$$

$$\dot{r}3 = \dot{r}_0 + k_3 * h/2, \tag{14}$$

$$r3 = r_0 + h * \frac{(\dot{r}_0 + \dot{r}3)}{2}, \tag{15}$$

$$k_4 = F(r3, \dot{r}3, t_0 + h), \tag{16}$$

so, the approximated first derivative and solution are given by the average of their corresponding slopes,

$$\dot{r}_{n+1} = \dot{r}_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4), \tag{6}$$

and

$$r_{n+1} = r_n + \frac{1}{6}(\dot{r}1 + \dot{r}2 + \dot{r}3 + \dot{r}_{n+1}). \tag{17}$$