Tracking control of the orbitally flat kinematic car with a new time-scaling input

Bálint Kiss and Emese Szádeczky-Kardoss

Abstract— **The paper defines a time-scaling scheme such that the scaling function depends on the state variables of the system and on an external variable referred to as the time-scaling input. The system evolving according to the scaled time has thus one more input variable compared to the original system. This time-scaling scheme is applied to the kinematic car with one input, namely the steering angle. The longitudinal velocity of the car is a measurable external signal and cannot be influenced by the controller. Using an on-line time-scaling, which is driven by the longitudinal velocity of the car, by a time-scaling output of the tracking controller, and by their time derivatives up to the second order, one can achieve exponential tracking of any sufficiently smooth reference trajectory with non-vanishing velocity, similar to the two-input differentially flat model of the kinematic car. The time derivatives of the longitudinal velocity and the time-scaling output are no longer necessary if one designs the tracking controller for the linearized error dynamics around the reference path.**

I. INTRODUCTION

Time-scaling is a widely used concept in control theory. The motivation to introduce time-scaling is often to gain useful properties for the scaled system, which evolves according to the modified time, compared to the original system. It is shown in [1] that such a property to gain with time-scaling may be feedback linearizability.

The notion of orbital flatness introduced by Fliess et al. [2], [3] involves time-scaling to define an equivalence between a class of nonlinear systems and finite chains of integrators. The problem of checking the orbital flatness property of single input systems is addressed in [4], [5], together with the well known example of the kinematic car with constant longitudinal velocity which is shown to be orbitally flat. Other orbitally flat systems are also reported such as the crystallization process model in [6].

The time-scaling introduced by these concepts involves the state variables to express the relation between the time scales, hence the time-scaling does not involve any new input or variable external to the system. This paper contributes a time-scaling scheme which is not driven only by the state variables of the system but also by a new input, referred to as the time-scaling input.

Another concept related to time-scaling is the use of the tracking error in closed loop to modify the time distribution along the reference path [7], [8]. These methods change the traveling time of the reference path according to the actual tracking error by decelerating if the motion is not accurate enough and by accelerating if the errors are small or vanish.

In the scheme suggested in this paper the new time-scaling input variable, which is not an input of the original physical system but can be produced by a tracking controller, is used to drive the time-scaling of the reference. The usefulness of our approach is demonstrated for the nonholonomic model of the kinematic car with one input and can be in particular applied to the problem related to the automatic parking of a passenger car such that the longitudinal velocity is generated by the driver during the maneuver. Notice that solutions to the motion planning and tracking algorithms are reported to the kinematic car model with two inputs and to its extension with n trailers exploiting its differentially flat property [9], [10] or using other methods [11], [12].

Using the time-scaling scheme and an appropriate feedback law, we show that the kinematic car with one input can track any smooth trajectory with non-vanishing longitudinal velocity such that the tracking error is reduced exponentially along the path. The feedback law for the model with scaled time is based on its differential flatness property.

The remaining part of the paper is organized as follows. The next section defines our time-scaling concept by introducing an additional, time-scaling input. The kinematic model of the car used in our work is described in Section III. Section IV presents two time-scaling based feedback laws to achieve exponential tracking. Simulation results and a real measurement on a passenger car are shown in Section V to demonstrate the usefulness of our approach. A short summary concludes the paper.

II. TIME-SCALING

Consider the system given by the equation

$$\dot{\xi} = f(\xi, u), \quad \xi \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{1}$$

such that the notation $\dot{\xi}$ stands for $\frac{d\xi}{dt}$ where t is the time. Introduce a modified time-scale according to a new time variable τ and define the relation between the time t and τ as

$$\frac{dt}{d\tau} = g(\xi, u_s), \tag{2}$$

where u_s is referred to as the time-scaling input. One may obtain the resulting system evolving according to the time τ as

$$\frac{d\xi}{d\tau} = \frac{d\xi}{dt}\frac{dt}{d\tau} = g(\xi, u_s) f(\xi, u), \tag{3}$$

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which has $m+1$ inputs. Notice that one gets back the usual form of time-scaling as defined in [1] if the function g does not depend on the new scaling input u_s. The time variables t and τ will be used throughout the paper such that $\xi' = \frac{d\xi}{d\tau}$ and $\dot{t} = \tau' = 1$.

III. KINEMATIC CAR MODEL WITH ONE INPUT

Consider the kinematic car depicted in Fig. 1. The Ackermann steering geometry is supposed to be respected, hence all wheels turn around the same point, denoted by P, which is on the line collinear with the rear axle of the car. The position of the car is given by the coordinates (x,y) of the rear axle midpoint and its orientation angle is denoted by θ. Under these assumptions, the kinematics of the car can be fully described if one considers the kinematics of the bicycle fitted on the longitudinal symmetry axis of the car (see Fig. 1). The angle of the front wheel of this bicycle w.r.t. its longitudinal axis is denoted by φ, which is the single input of the model.

Suppose that the longitudinal velocity v_{car} is generated by the driver hence it cannot be influenced by any feedback controller but it can be measured. The kinematic model is given by the equations

$$\dot{x} = v_{car}\cos\theta, \tag{4}$$
$$\dot{y} = v_{car}\sin\theta, \tag{5}$$
$$\dot{\theta} = \frac{v_{car}}{l}\tan\varphi, \tag{6}$$

where $l > 0$ denotes the distance between the front and the rear axels. For $v_{car} = 1$ (or for any other constant velocity that differs from zero), this model is orbitally (but not differentially) flat [2], [5]. One would obtain a differentially flat model if v_{car} were a second input variable.

Consider now the time-scaling defined as

$$\frac{dt}{d\tau} = \frac{u_s}{v_{car}}, \tag{7}$$

which results the scaled dynamics

$$x' = u_s\cos\theta, \tag{8}$$
$$y' = u_s\sin\theta, \tag{9}$$
$$\theta' = \frac{u_s}{l}tan\varphi. \tag{10}$$

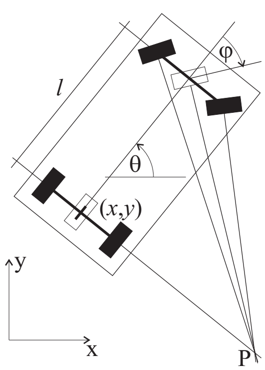

Fig. 1. Kinematic car in the horizontal plane

with two independent inputs, namely φ and u_s. This is the differentially flat version of the kinematic car [9]. Observe that the time-scaling does not rewind the time if both v_{car} and u_s are positive (respectively negative) but it is singular if $v_{car} = 0$.

Recall also that the time-scaling introduced by (7) does not require the knowledge of $v_{car}(t)$ for all values of t but only its instantaneous value, which is supposed to be measured.

IV. TRACKING CONTROL USING TIME-SCALING

The control objective is to track any reference trajectory with non-vanishing velocity generated by the driver. Note that this problem is solved for the kinematic car model with two inputs but not yet addressed in the presented context. The idea is to use the differentially flat time-scaled model (8)-(10) to solve the motion planning problem with the time τ. Then one would use a tracking feedback controller designed again for the differentially flat model (8)-(10), which produces u_s and φ. The signal u_s produced by the controller is used to drive the time-scaling $\tau \mapsto t$ of the reference trajectory designed for the time τ.

The control loop is illustrated in Fig. 2, where the controller provides φ to the single input physical system modeled by the equations (4)-(6).

A. Motion Planning

The motion planning is done for the model (8)-(10) exploiting its differential flatness property. Since the flat output variables are x and y, the motion planning realizes the mappings

$$\tau \rightarrow \{x_{\tau,ref}, x'_{\tau,ref}, x''_{\tau,ref}, x'''_{\tau,ref}\}, \tag{11}$$
$$\tau \rightarrow \{y_{\tau,ref}, y'_{\tau,ref}, y''_{\tau,ref}, y'''_{\tau,ref}\} \tag{12}$$

for $\tau \in [0,T]$ where T is the desired traveling time along the path. Since the velocity of the car is determined by an external signal v_{car}, the real traveling time in closed loop will be obtained as $t(T)$.

Several motion planning schemes [13] can be used to realize (11) and (12). For the sake of simplicity, seventh degree polynomial trajectories are considered here such that

$$x_{\tau,ref} = \sum_{i=0}^{7} a_{x,i}\tau^i, \qquad y_{\tau,ref} = \sum_{i=0}^{7} a_{y,i}\tau^i. \tag{13}$$

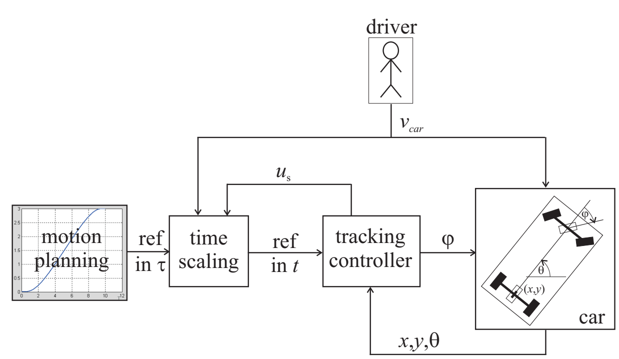

Fig. 2. Control loop including time-scaling.

The coefficients ($a_{x,i}$, $a_{y,i}$) are obtained as solutions of a set of linear algebraic equations determined by the constraints which the polynomials and their three successive derivatives must satisfy at $\tau = 0$ and $\tau = T$. Notice that the non-zero constraints are no longer respected according to the scaled time t for the derivatives of the references, unless $\dot{\tau} \equiv 1$. It follows in particular that the constraints imposed on the traveling velocities along the trajectory at $\tau = 0$ (respectively at $\tau = T$) will be scaled by $\dot{\tau}(0)$ (respectively by $\dot{\tau}(t(T))$ which depends on the velocity generated by the driver and on time-scaling input.

The motion planning can be done prior to the motion and the time-scaling does not need the redesign the geometry of the reference. It follows that more involved methods can be also applied to determine the geometry of the path.

B. Time-Scaling of the Reference Trajectory

Suppose that the references are obtained for (8)-(10) and one disposed the time functions $x_{\tau,ref}$, $x'_{\tau,ref}$, $x''_{\tau,ref}$, $x'''_{\tau,ref}$, $y_{\tau,ref}$, $y'_{\tau,ref}$, $y''_{\tau,ref}$, and $y'''_{\tau,ref}$. The time-scaling is defined by the mappings

$$\{x_{\tau,ref},\ldots,x'''_{\tau,ref}\} \;\rightarrow\; \{x_{ref}(t),\ldots,x^{(3)}_{ref}(t)\}, \qquad (14)$$

$$\{y_{\tau,ref},\ldots,y'''_{\tau,ref}\} \;\rightarrow\; \{y_{ref}(t),\ldots,y^{(3)}_{ref}(t)\}. \qquad (15)$$

First one uses (7) in order to determine the relations between τ, v_{car}, u_s, and their derivatives

$$\begin{aligned}
\tau(t) &= \int_0^t \frac{v_{car}}{u_s} d\vartheta, \qquad \tau(0) = t(0) = 0, && (16)\\
v_{car} &= \dot{\tau} u_s, && (17)\\
\dot{v}_{car} &= \ddot{\tau} u_s + \dot{\tau}^2 u'_s, && (18)\\
\ddot{v}_{car} &= \tau^{(3)} u_s + 3\dot{\tau}\ddot{\tau} u'_s + \dot{\tau}^3 u''_s, && (19)
\end{aligned}$$

which allow to express $\tau(t)$, $\dot{\tau}$, $\ddot{\tau}$, and $\tau^{(3)}$. Observe that the resulting expressions do not require the knowledge of future values of v_{car}, hence can be calculated in real-time. Notice also that one needs the current values of $\dot{v}_{car}$ and $\ddot{v}_{car}$ to determine $\ddot{\tau}$ and $\tau^{(3)}$ which is a drawback since these derivatives may be difficult to measure or estimate.

Than the scaling of the reference is defined by

$$\begin{aligned}
x_{ref}(t) &= x_{\tau,ref}(\tau(t)), && (20)\\
\dot{x}_{ref}(t) &= x'_{\tau,ref}(\tau(t))\dot{\tau}, && (21)\\
\ddot{x}_{ref}(t) &= x''_{\tau,ref}(\tau(t))\dot{\tau}^2 + x'_{\tau,ref}(\tau(t))\ddot{\tau}, && (22)\\
x^{(3)}_{ref}(t) &= x'''_{\tau,ref}(\tau(t))\dot{\tau}^3 + 3x''_{\tau,ref}(\tau(t))\dot{\tau}\ddot{\tau} + \\
&\quad + x'_{\tau,ref}(\tau(t))\tau^{(3)}, && (23)
\end{aligned}$$

and one obtains similar expressions for the mapping (15).

C. Tracking Feedback

The tracking feedback is designed again using the flatness property of the time-scaled model (8)-(10). By virtue of this property, one can apply a linearizing feedback such that the resulting system is two chains of integrators

$$x''' = \omega_x, \qquad y''' = \omega_y. \qquad (24)$$

Suppose moreover that one specifies the tracking behavior in terms of the tracking errors $e_x = x - x_{\tau,ref}$ and $e_y = y - y_{\tau,ref}$ such that the differential equations

$$\begin{aligned}
e'''_x + k_{x,2}e''_x + k_{x,1}e'_x + k_{x,0}e_x &= 0, && (25)\\
e'''_y + k_{y,2}e''_y + k_{y,1}e'_y + k_{y,0}e_y &= 0 && (26)
\end{aligned}$$

hold true. The coefficients $k_{a,i}$ ($a \in \{x,y\}$, $i = 0,1,2$) are design parameters and have to be chosen such that the corresponding characteristic polynomials have all their roots in the left half of the complex plane.

Define first the dynamics of the feedback as

$$\begin{aligned}
\zeta'_1 &= \zeta_2 = u'_s, & \zeta'_3 &= v_2, && (27)\\
\zeta'_2 &= v_1 = u''_s, & \varphi &= \zeta_3, && (28)\\
u_s &= \zeta_1, & & && (29)
\end{aligned}$$

where ζ_1, ζ_2, and ζ_3 are the inner states of the feedback. Observe that ζ_2 and v_1 give precisely the derivatives of u_s which need to realize the time-scaling in (16)-(19), hence no numerical differentiation is needed.

The inputs v_1 and v_2 of the feedback dynamics must be determined such that the tracking errors e_x and e_y satisfy (25) and (26), respectively.

For, one needs to determine first x, x', x'', x''', y, y', y'', and y''' as functions of v_1, v_2, and x, y, θ, ζ_1, ζ_2, ζ_3, which are the states of the closed loop system including the measured states of the kinematic car model, and the states of the feedback (27)-(29). After some cumbersome but elementary differentiations one obtains

$$\begin{aligned}
x' &= \zeta_1\cos\theta, && (30)\\
x'' &= \zeta_2\cos\theta - \frac{\zeta_1^2\sin\theta\tan\zeta_3}{l}, && (31)\\
x''' &= v_1\cos\theta - \frac{3\zeta_1\zeta_2\sin\theta\sin\zeta_3}{l\cos\zeta_3} - \frac{\zeta_1^3\cos\theta}{l^2\cos^2\zeta_3}\\
&\quad + \frac{\zeta_1^3\cos\theta}{l^2} - \frac{v_2\zeta_1^2\sin\theta}{l\cos^2\zeta_3}, && (32)\\
y' &= \zeta_1\sin\theta, && (33)\\
y'' &= \zeta_2\sin\theta + \frac{\zeta_1^2\cos\theta\tan\zeta_3}{l}, && (34)\\
y''' &= v_1\sin\theta + \frac{3\zeta_1\zeta_2\cos\theta\sin\zeta_3}{l\cos\zeta_3} - \frac{\zeta_1^3\sin\theta}{l^2\cos^2\zeta_3}\\
&\quad + \frac{\zeta_1^3\sin\theta}{l^2} + \frac{v_2\zeta_1^2\cos\theta}{l\cos^2\zeta_3}. && (35)
\end{aligned}$$

These expressions allow to calculate e_x, e'_x, e''_x, e_y, e'_y, and e''_y using the reference trajectory and the states of the closed loop system. Plugging in these expressions into (25) and (26), and using (24) one gets

$$\begin{bmatrix} \cos\theta & -\frac{\zeta_1^2\sin\theta}{l\cos^2\zeta_3} \\ \sin\theta & \frac{\zeta_1^2\cos\theta}{l\cos^2\zeta_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \omega_x - A \\ \omega_y - B \end{bmatrix}, \qquad (36)$$

with

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\frac{3\zeta_1\zeta_2\sin\theta\sin\zeta_3}{l\cos\zeta_3} - \frac{\zeta_1^3\cos\theta}{l^2\cos^2\zeta_3} + \frac{\zeta_1^3\cos\theta}{l^2} \\ \frac{3\zeta_1\zeta_2\cos\theta\sin\zeta_3}{l\cos\zeta_3} - \frac{\zeta_1^3\sin\theta}{l^2\cos^2\zeta_3} + \frac{\zeta_1^3\sin\theta}{l^2} \end{bmatrix}, \qquad (37)$$

where the inverse of the coefficient matrix can be calculated symbolically. One obtains

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \frac{-l\sin\theta\cos^2\zeta_3}{\zeta_1^2} & \frac{l\cos\theta\cos^2\zeta_3}{\zeta_1^2} \end{bmatrix} \begin{bmatrix} \omega_x - A \\ \omega_y - B \end{bmatrix}. \tag{38}$$

The tracking feedback law is defined by (27)-(29) and by (38). A singularity occurs if $\zeta_1^2 = u_s^2 = 0$ which corresponds to zero longitudinal velocity. Another singular situation corresponds to $\zeta_3 = \varphi = \pm\pi/2$ which may occur if the steered wheels are perpendicular to the longitudinal axis of the car. Singularities imply the loss of controllability of the kinematic car model.

D. Time-Scaling Feedback for the Linearized Error Dynamics

The above method needed the time derivatives of the velocity to carry out the time-scaling, which may be difficult to measure or estimate in real applications. The method presented in this section overcomes this difficulty. It uses a transformation of the tracking error expressed in the configuration variables to obtain the error dynamics along the reference path [11]. This non-linear error dynamics is then linearized around the zero error and controlled by a state feedback. The missing input, which is the longitudinal velocity of the car is again replaced by the time-scaling input, which drives the time-scaling of the reference trajectory.

Suppose that the desired behavior of the car is given by the time functions $x_{\tau,ref}$, $y_{\tau,ref}$, $\theta_{\tau,ref}$ such that these functions identically satisfy (8)-(10) for the corresponding reference input signals $u_{s,ref}$ and $\varphi_{\tau,ref}$.

We suggest to scale this reference trajectory according to the time t. The scaled reference trajectory is given by $x_{ref}(t) = x_{\tau,ref}(\tau)$, $y_{ref}(t) = y_{\tau,ref}(\tau)$, and $\theta_{ref}(t) = \theta_{\tau,ref}(\tau)$ and similarly to (8)-(10)

$$x'_{\tau,ref} = u_{s,ref}\cos\theta_{\tau,ref}, \tag{39}$$

$$y'_{\tau,ref} = u_{s,ref}\sin\theta_{\tau,ref}, \tag{40}$$

$$\theta'_{\tau,ref} = \frac{u_{s,ref}}{l}\tan\varphi_{\tau,ref}. \tag{41}$$

The tracking errors are defined for the configuration variables as $e_x = x - x_{ref}(t)$, $e_y = y - y_{ref}(t)$, and $e_\theta = \theta - \theta_{ref}(t)$. Let us now consider the transformation

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} \tag{42}$$

of the error vector (e_x, e_y, e_θ) to a frame fixed to the car such that the longitudinal axis of the car coincides the transformed x axis. Differentiating this equation w.r.t. time t and using the general rule $\dot{a}(\tau) = \frac{da(\tau)}{dt} = \frac{\partial a}{\partial \tau}\frac{\partial \tau}{\partial t} = a'\frac{v_{car}}{u_s}$, we get the differential equation

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin e_3 \\ 0 \end{bmatrix} u_{s,ref}\frac{v_{car}}{u_s} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{43}$$

which describes the evolution of the errors with respect to the reference path. The inputs w_1 and w_2 are

$$w_1 = v_{car} - \frac{v_{car}}{u_s}u_{s,ref}\cos e_3, \tag{44}$$

$$w_2 = \frac{1}{l}\left(v_{car}\tan\varphi - \frac{v_{car}}{u_s}u_{s,ref}\tan\varphi_{\tau,ref}\right). \tag{45}$$

Notice that from the expression of w_1 and w_2 the time-scaling input u_s, the first derivative of the time-scaling ($\dot{\tau}$) and the real input φ can be calculated as

$$u_s = \frac{v_{car}u_{s,ref}\cos e_3}{v_{car} - w_1}, \tag{46}$$

$$\dot{\tau} = \frac{v_{car}}{u_s} = \frac{v_{car} - w_1}{u_{s,ref}\cos e_3}, \tag{47}$$

$$\varphi = \arctan\left(\frac{l w_2 + \dot{\tau}u_{s,ref}\tan\varphi_{\tau,ref}}{v_{car}}\right), \tag{48}$$

if the reference value for the time-scaling input $u_{s,ref} \neq 0$, the error of the orientation $e_3 \neq \pm\pi/2$, and the longitudinal velocity of the car $v_{car} \neq 0$ and $v_{car} \neq w_1$.

This system can be linearized along the reference trajectory with zero input signals, (i.e. for $[\ e_1 \quad e_2 \quad e_3\ ]^T = 0$ and $[\ w_1 \quad w_2\ ]^T = 0$). The linearized system is controllable if the reference value for the time-scaling input ($u_{s,ref}$) and the longitudinal velocity of the car (v_{car}) are non-zero. The linearized system obtained from (43) can be locally stabilized by a state feedback of the form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -K \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{49}$$

such that the gain matrix K puts the eigenvalues of the closed loop system in the left half of the complex plane.

The way of calculations is as follows. One supposes that the tracking errors of the configuration variables (e_x, e_y, e_θ) are measured, hence the the error (e_1, e_2, e_3) can be determined using (42). Then the state feedback (49) allows to calculate w_1 and w_2. From the actual value of w_1 one can determine u_s and $\dot{\tau}$ using the current value of v_{car}, e_3, and the reference time-scaling input $u_{s,ref}$. The input φ is calculated according to (48) using w_2. The function $\tau(t)$ is obtained by the on-line integration of $\dot{\tau}$ determined by (47) using the initial condition $\tau(0) = 0$. The time distribution of the reference trajectory is finally modified according to τ and $\dot{\tau}$.

Using (47) we should get a non-rewinding time function, i.e. $\dot{\tau} > 0$ is expected. Unfortunately this condition is not always satisfied, since for large tracking errors $w_1 \geq v_{car}$ can be achieved, which results $\dot{\tau} \leq 0$ for $u_{s,ref} > 0$ and $\cos e_3 > 0$, which is not allowed since time cannot rewind.

Since a linearized model was used for the controller design, only local stability is guaranteed.

V. TRACKING PERFORMANCE RESULTS

The usefulness of our approach is demonstrated to the problem related to the automatic parking of a passenger car without automatic gear. In this case one wants to realize a parking assist system such that the human driver generates

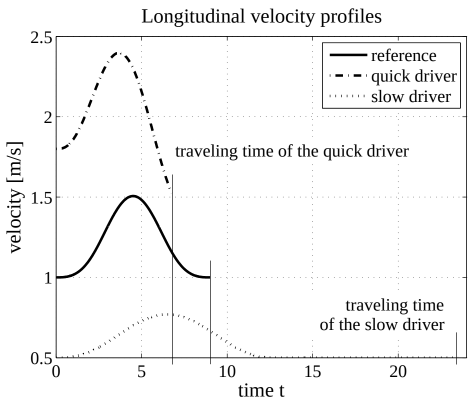

Fig. 3. Velocity profiles. The traveling times are obtained in closed loop.

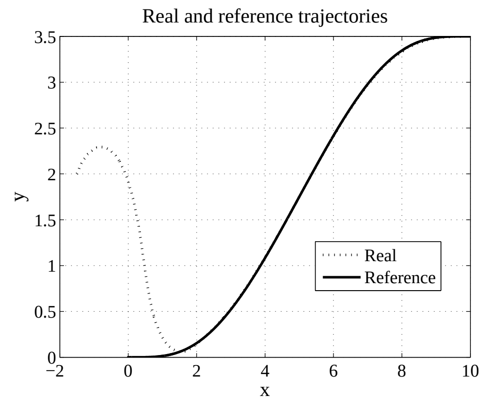

Fig. 4. Trajectories in the horizontal plane for both drivers.

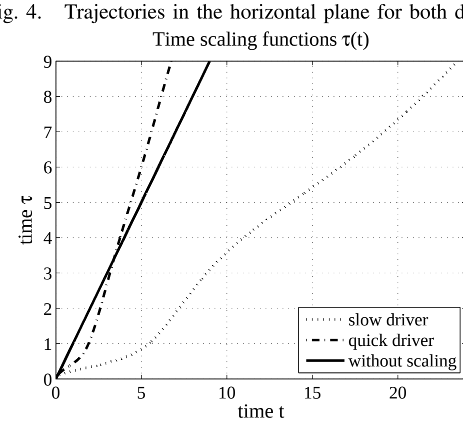

Fig. 5. The time-scaling functions $\tau(t)$.

the velocity of the car with an appropriate management of the gas, clutch, and break pedals while the controller may only influence the angle of the steered wheels. The parking maneuvers start and finish in idle positions, where the feedback is singular. Switching on the feedback only when the velocity of the car exceeds a minimal value results an initial error in position and orientation for the closed loop behavior, but exponential decay of this error is ensured while the feedback is functioning. The very end of the parking maneuver may again be carried out in open loop. According to measurements with passenger car drivers while parking (in a parking row), the low velocity interval (0-0.8 km/h) occurs only when starting and stopping the car. The distance traveled while in low velocity range is 2-7 cm.

Simulation results and a measurement are shown to demonstrate the functioning of the time-scaling based tracking controller for the orbitally flat kinematic car. (For the simulations we chose $l = 1$.)

A. Simulation Results

First we use the feedback described in Subsection IV-C, such that the new time-scaling input u_s drives the time-scaling given in Section III together with the measured v_{car} and its two successive time derivatives. The reference trajectory starts from the point $(x_{\tau,ref}(0) = 0, y_{\tau,ref}(0) = 0, \theta_{\tau,ref}(0) = 0)$ and arrives to the point $(x_{\tau,ref}(T) = 10, y_{\tau,ref}(T) = 3.5, \theta_{\tau,ref}(T) = 0)$, all distances are given in meters and the orientation is given in radians. The traveling time of the reference trajectory is $T = 9$ seconds.

The real initial configuration of the car differs from the one used for motion planning, since $x(0) = -1.5$, $y(0) = 2$, and $\theta(0) = \pi/4$.

Two cases are presented such that the geometry of the reference trajectory and the reference velocity profile obtained are the same, but the driver's behavior is different. In the first case, referred to as the *slow driver* case, the driver imposes considerably slower velocities than those obtained by the motion planning. In the second case, referred to as the *quick driver* case, the driver generates higher velocities than the reference velocity profile. All velocity profiles are given in Fig. 3.

The geometries of the reference trajectories and the real trajectories in the horizontal plane are depicted in Fig. 4 both for the slow and the quick driver, since the time scaling does not modify the geometry of the path itself. Exponential tracking of the reference trajectory is achieved for each scenario with similar geometry of the real path.

Fig. 5 and Fig. 6 show the effects of on-line time-scaling. If the car was driven by a slow driver it needed more than 23 seconds according to time t to achieve the traveling time $T = 9$ sec which was given for the reference trajectory according to the time τ. The reference in τ was decelerated all along the trajectory ($\dot{\tau} < 1$). The deceleration was also accentuated at low values of t which corresponds to the large tracking errors.

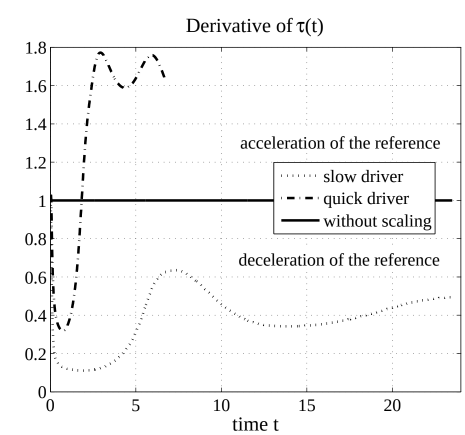

Fig. 6. The derivative of the time-scaling functions $\dot{\tau}$.

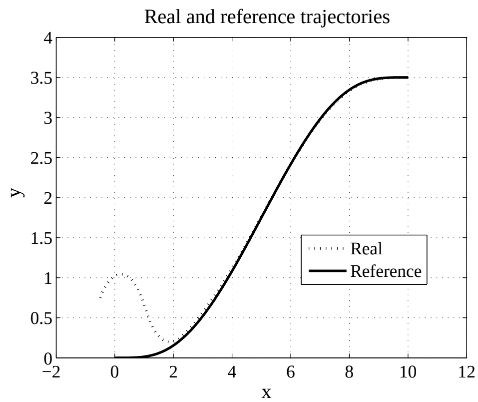

Fig. 7. Trajectories in the horizontal plane for simulation 2.

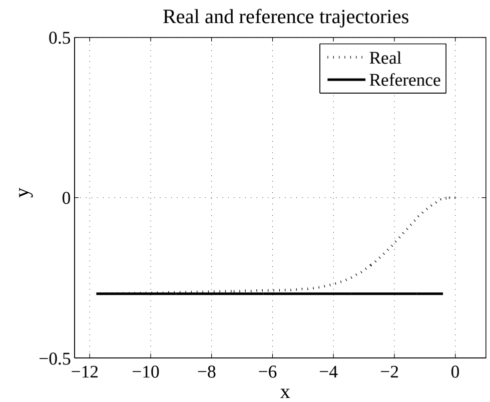

Fig. 8. Trajectories in the horizontal plane in the real measurement.

The time-scaling is completely different for the quick driver who reached the end of the trajectory faster according to the time t than according to the time τ which means that the reference was accelerated except a short section at the beginning where the tracking error elimination slowed down the time-scaling despite the driver's efforts.

Next we apply the feedback law described in Subsection IV-D such that the same reference trajectory is used as in the previous simulations. In this case the difference between the real and the reference initial configuration is smaller, since $x(0) = -0.5$, $y(0) = 0.75$, and $\theta(0) = \pi/4$. The reference trajectory and the real path are shown in Fig. 7. We achieved exponential tracking. If the difference between the real and reference initial configurations is larger, the linearized model is no longer valid and the time-scaling may rewind (i.e. $\dot{\tau} \leq 0$).

B. Measurements on a Real Passenger Car

We implemented the flatness-based tracking controller presented in Section IV-C on a Ford Focus type passenger car using an electronic power assist steering system to realize the steering angle for the front wheels. The position and orientation of the car were calculated from the wheel velocities measured by standard ABS sensors mounted on all wheels of the car. The results of the real measurements correspond to the simulation results. Fig. 8 shows an example such that the car moves backwards. The reference trajectory starts from the point $(x_{\tau,ref}(0) = -0.4, y_{\tau,ref}(0) = -0.3, \theta_{\tau,ref}(0) = 0)$ and arrives to the point $(x_{\tau,ref}(T) = -12, y_{\tau,ref}(T) = -0.3, \theta_{\tau,ref}(T) = 0)$. The real initial configuration of the car differs from the one used for motion planning, since $x(0) = 0$, $y(0) = 0$, and $\theta(0) = 0$. The exponential tracking is achieved.

VI. CONCLUSION

The paper presented two time-scaling based tracking control methods for the orbitally flat kinematic car such that the longitudinal velocity of the vehicle is generated externally and cannot be considered as a control input. A new input, referred to as the time-scaling input was used to drive the time-scaling with the car velocity. For the first control method, which is based on the differentially flat property of the model with the time-scaling input, the derivatives of the car velocity need to be measured or estimated for the feedback. For the second tracking controller, designed for the linearized error dynamics, the time derivatives of the velocity are not needed, but only local stability can be achieved. The exponential decay of the initial error along the trajectory can be ensured in both cases. The results can be extended for the n-trailer case.

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