COMP 598 Summer (May) 2020
Assignment 1
Due Date: 11^{th} May 2020

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Please do as many questions as you can without looking it up on the web. If we are suspicious about whether your solution is really your own, we reserve the right to examine you orally and assign points based on how well you can explain what you have written. If you have a perfect answer and then tell us, "I can't remember anymore how I came up with it or how it works" we will give you zero for that question. By all means collaborate with your friends but you must be able to explain what you turn in.

Question 1[10 points] The collection of strings Σ^* with the operation of concatenation forms an algebraic structure called a *monoid*. A monoid is a set with a binary associative operation and with an identity element (necessarily unique) for the operation. A monoid *homomorphism* is a map between monoids that preserves the identity and the binary operation. Let Σ be any finite set and let M be *any* monoid. Show that *any* function $f : \Sigma \to M$ can be extended in a unique way to a monoid homomorphism from $\Sigma^* \to M$. This is an example of what is called a *universal property*.

Question2[10 points] We defined the notion of acceptance of a language L by a DFA in class. We also defined the notion of acceptance of a language by a finite monoid. Show that these two notions are equivalent.

Question 3[10 points] Recall that a *well-ordered* set is a set equipped with an order that is well-founded as well as linear (total). For any poset $(S, \leq)$ and monotone function $f : S \to S$, we say f is *strictly monotone* if $x < y$ implies that $f(x) < f(y)$; recall that $x < y$ means $x \leq y$ and $x \neq y$. A function $f : S \to S$ is said to be *inflationary* if for every $x \in S$ we have $x \leq f(x)$. Suppose that W is a well-ordered set and that $h : W \to W$ is strictly monotone. *Prove* that h must be inflationary.

Question 4[10 points] Give deterministic finite automata accepting the following languages over the alphabet $\{0, 1\}$.

1. The set of all words ending in 00. [2 points]
2. The set of all words ending in 00 *or* 11. [3 points]
3. The set of all words such that the *second* last element is a 1. By "second last" I mean the second element counting backwards from the end[1]. Thus, 0001101 is not accepted but

[1]The proper English word is "penultimate."

10101010 is accepted. [5 points]

Question 5[10 points] Suppose that L is a language accepted by a DFA (i.e. a regular language) show that the following language is also regular:

$$\text{middle}(L) := \{v \in \Sigma^* | \exists u, w \in \Sigma^* \text{ such that } uvw \in L \text{ and } |u| = |v| = |w|\}.$$

Question 6[10 points]

1. Give a deterministic finite automaton accepting the following language over the alphabet $\{0, 1\}$: The set of all words containing 100 or 110. [1 point]
2. Show that *any* DFA for recognizing this language must have at least 5 states. [9 points]

Question 7[10 points] Suppose that L is a language accepted by a DFA (i.e. a regular language) show that the following language is also regular:

$$\text{LOG}(L) := \{x | \exists y \in \Sigma^* \text{ such that } xy \in L \text{ and } |y| = 2^{|x|}\}.$$

This is very tedious to write out in detail so I am happy with a description of the idea. Just because it is an informal description does not mean that it is vague or unclear.

Question 8[10 points] Show that the following languages are not regular by using the pumping lemma.

1. $\{a^{2n}b^n\}$.
2. $\{x \in \{a, b, c\}^* || x| \text{ is a square.}\}$ Here $|x|$ means the length of x.

Question 9[10 points] Suppose that L is a language accepted by a DFA (i.e. a regular language) show that the following language is not necessarily regular:

$$\text{outer}(L) := \{uw \in \Sigma^* | \exists v \in \Sigma^* \text{ such that } uvw \in L \text{ and } |u| = |v| = |w|\}.$$

You have to give me a specific L which is regular and *prove* that outer(L) is not regular.

Question 10[10 points] Design an **NFA** K with n states, over a one-letter alphabet, such that K rejects some strings, but the *shortest* string that it rejects has length *strictly* greater than n.