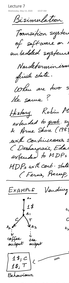
Lecture 7
Jan 25, 2010

Bisimulation

Transition systems as models of software or hardware, or labelled systems.

Nondeterminism, need not be finite state.

Often we have systems obviously the same.



Sequences are an inadequate description of the behaviour. What is a good description of behavioural equivalence?

Bisimulation

Def $s \sim t$ iff

$S \to$ set of states, partition $\mathcal{A}$ into $\to$ set of actions, finite $\to$ transition $\subseteq S \times \mathcal{A} \times S$

$s \xrightarrow{a} s'$

From state s of action a to successor state s'.

Def of Bisimulation (same relation) Use $\sim$ as the bisimulation

(1) $s \sim t$ implies $s \xrightarrow{a} s' \Rightarrow \exists t' \; t \xrightarrow{a} t' \wedge s' \sim t'$

(2) $s \sim t$ implies $t \xrightarrow{a} t' \Rightarrow \exists s' \; s \xrightarrow{a} s' \wedge s' \sim t'$

A good bisimulation

Now we say $s \sim t$

(HIGHER ORDER)

$s \sim t$, but not $s \xrightarrow{a} s'$ then $\forall t \; t \xrightarrow{a} t' \Rightarrow s' \not\sim t'$

(2) $s \sim t$ implies $t \xrightarrow{a} t' \Rightarrow \exists s' \; s \xrightarrow{a} s' \wedge s' \sim t'$

(3) $s \sim t$ implies $s \xrightarrow{a} s' \Rightarrow \forall t' \; t \xrightarrow{a} t' \Rightarrow s' \not\sim t'$

DAVID PARK'S PHRASEOLOGY

(3) We say R is a bisimulation relation if (1) $s \sim t$ implies $s \xrightarrow{a} s' \Rightarrow \exists t' \; t \xrightarrow{a} t' \wedge s' R t'$

(2) $s \sim t$ implies $t \xrightarrow{a} t' \Rightarrow \exists s' \; s \xrightarrow{a} s' \wedge s' R t'$

Note $\sim$ is not concrete. Big given, bisimulation is very many. Can be infinite.

Now we say $s \sim t$ iff $\exists R$ a bisimulation relation $s R t$.

Bisimulation is well defined. Bisimulation relations are closed under union.

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DICKERSON & GATICH THESIS & PIERCE

I assume you know what a poset is, or partially ordered set.

$(S, \leq)$ We say x is the least upper bound of $X \subseteq S$ if

$\forall x \in X \quad x \leq y \quad \text{UPPER BOUND}$

$\forall y \; (\forall x \in X \; x \leq y) \Rightarrow x \leq y$

(This is often called the supremum)

$\sqcup X$

If $X \subseteq S$ we say x is a greatest lower bound if $\forall x \in X \; x \geq y$.

If x is called a greatest lower bound (infimum) if $\forall y \; (\forall x \in X \; x \geq y) \Rightarrow x \geq y$

$\sqcap X$

Def $\bot$ is a bottom element if $\bot$ is a lub of the empty set of elements. This are necessarily unique.

So the figure on the left shows a bottom. $(R, \leq)$ is a lattice if

$x \sqcap y$

$x \sqcup y$

Def $(L, \leq)$ is a lattice if every finite (including $\emptyset$) we call it a complete lattice.

FACT If follows that any set has a glb (complete lattice)

Let $\mathcal{M}$ of finite set $\mathcal{M}$ is a finite element of the lattice.

Proof of short fact

Let $X \subseteq L$, L is complete lattice.

Let $Y = \{x \in L \mid \forall x \in X \; x \leq y\}$

Let z be the least bound of X

Let $y = \text{glb}\, Y$ (exists Y)

Then $z \leq y$ for all $x \in X$

Note $\forall x \in X \; \forall x \in Y \Rightarrow x \leq y$

$\Rightarrow x \leq y$ $\Rightarrow$ y is an upper bound of X

so y is in Y $\Rightarrow$ $y \leq z$ $\Rightarrow$ $y = z$

Since y was arbitrary $\forall x \in X$ upper bound of X $\Rightarrow$ z is the greatest lower bound of X.

Def $f: L \to L$ is monotone if $\forall x, y \in L \; x \leq y \Rightarrow f(x) \leq f(y)$

FACT If $f: L \to L$ is monotone L is complete lattice then the least fixpoint of f, i.e. $\mu f = \sqcap \{x \in L \mid f(x) = x\}$ is a complete lattice. So you know there is a greatest fixed point and a least fixed point.

Proof (FACT will only show the existence of a greatest fixpoint.)

We say x is a prefixpoint if $f(x) \leq x$.

Let $D = \{x \in L \mid f(x) \leq x\}$

Let $y = \text{glb}\, M$ where $M = \{x \in L \mid f(x) \leq x\}$

Claim y is a fixed point of f

Proof y is a prefixpoint $\forall x \in D \; f(x) \leq x$ $\Rightarrow$ $\forall x \in D \; f(f(x)) \leq f(x)$ $\Rightarrow$ $f(y) \in D$ $\Rightarrow$ $f(y) \geq y$ $\Rightarrow$ $f(y) = y$ because y is an lower bound of D.

Now $f(y) = y$ $\Rightarrow$ y is a prefixpoint $\Rightarrow$ $y \in D$ $\Rightarrow$ $y \leq y$ $\Rightarrow$ $f(y) \leq y$ $\Rightarrow$ $f(f(y)) \leq f(y)$ $\Rightarrow$ $f(y) \in M$.

But $y = \text{glb}\, M$ $\Rightarrow$ $y \leq f(y) \in M$ $\Rightarrow$ $y \leq f(y)$ $\Rightarrow$ $y = f(y)$

Hence y is a fixed point and M is a lower bound of D.

Note to be a bisimulation relation means $R \in \mathcal{P}(S \times S)$

Complete lattice

If $R \to R$ is monotone

Def of $f(R)$ Bisimulation

$= U \cap \{R \mid f(R) \subseteq R\}$

(1) $f(S)$ is really the union.

(2) $f(R)$ is a bisimulation relation. Use the bisimulation equation to show special property.

(3) $f(R) \subseteq f(f(R))$ is a fixed image. Functions.

If f is a monotonic function then $f(R)$ is a fixpoint.

The definition of bisimulation is an example of a co-inductive def.

LOGICAL CHARACTERIZATION OF BISIMULATION

Given a LTS $(S, \to, \mathcal{A})$

Define a modal logic called Hennessy-Milner logic $\mathcal{L}$ and $\Phi ::= T \mid \neg\Phi \mid \Phi \wedge \Phi \mid \langle a \rangle \Phi$

$s \models T$ $\quad$ always true

$s \models \neg\Phi$ iff $s \not\models \Phi$

$s \models \Phi \wedge \Psi$ iff $s \models \Phi$ and $s \models \Psi$

$s \models \langle a \rangle \Phi$ iff $\exists s' \; s \xrightarrow{a} s' \wedge s' \models \Phi$ (and)

Example $s \models \langle a \rangle \neg \langle b \rangle T$

$s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$ (NOT)

Example $s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$

Great Power $\models T$

$s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$

$s \models \langle a \rangle \neg \langle b \rangle T$

$s \models \langle a \rangle T$

$s \models T$

$s \models T$

$s \models \langle a \rangle \neg \langle b \rangle T$ $\quad$ $s \models \langle a \rangle T$

$s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$

Fact $s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$ $\quad$ $s \models \langle a \rangle \neg \langle b \rangle T$

Fact $s \models \langle a \rangle \neg \langle b \rangle \neg \langle a \rangle T$ $\quad$ $s \models \langle a