# COMP 598 Summer (May) 2020
# Assignment 2
# **Due Date:** $18^{th}$ May 2020

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**Question 1**[10 points] Show that the language

$$F = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$$

is not regular. Show, however, that it satisfies the statement of the pumping lemma as I proved it in class, i.e. there is a $p$ such that all three conditions for the pumping lemma are met. Explain why this does not contradict the pumping lemma.

**Question 2**[10 points] Are the following statements true or false? Prove your answer in each case. We have some fixed alphabet $\Sigma$ with at least two letters. In the following $A$ and $B$ stand for languages, *i.e.* subsets of $\Sigma^*$.

- If $A$ is regular and $A \subseteq B$ then $B$ must be regular.
- If $A$ and $AB$ are both regular then $B$ must be regular.
- If $\{A_i | i \in \mathbb{N}\}$ is an infinite family of regular sets then $\bigcup_{i=1}^{\infty} A_i$ is regular.
- If $A$ is not regular it cannot have a regular subset.

**Question 3**[10 points] If $L$ is a language over an alphabet with *strictly more than one letter* we define $CYC(L) = \{uv | u, v \in \Sigma^*, vu \in L\}$. Show that if $L$ is regular then $CYC(L)$ is also regular;[7]. Give an example of a *non-regular* language such that $CYC(L)$ is regular. [3]

**Question 4**[10 points] Last week I *incorrectly* stated that the relation

$$x \equiv_L y \iff \forall z \in \Sigma^*,\ xz \in L \iff yz \in L,$$

defined for any language $L$ is a congruence relation. Recall that a congruence relation $\sim$ is one for which

$$x \sim y, u \sim v \Rightarrow xu \sim yv.$$

However, Rahul pointed out that this is false. Give a counter-example showing that $\equiv_L$ is not a congruence. The correct definition that I should have given is the following

$$x \approx_L y \iff \forall u, v \in \Sigma^*,\ uxv \in L \iff uyv \in L.$$

Show that $\approx_L$ is indeed a congruence relation. The quotient of $\Sigma^*$ by $\approx_L$ is called the *syntactic monoid* of $L$. The relation $\equiv_L$ is correctly defined for the Myhill-Nerode theorem; it is only the claim that it is a congruence relation that is false.

**Question 5**[10 points] Which of the following pairs of formulas are equivalent? If they are not equivalent, give a transition system which shows that the formulas have different interpretations. If they are equivalent give a proof of the equivalence based on the semantics of the temporal operators.

1. $\Box\Diamond\phi \Rightarrow \Box\Diamond\psi$ and $\Box(\phi \Rightarrow \Diamond\psi)$.
2. $\Diamond(\phi \wedge \psi)$ and $\Diamond\phi \wedge \Diamond\psi$.
3. $\bigcirc\Diamond\phi$ and $\Diamond \bigcirc \phi$.

In the next two questions we will compare the expressive power of LTL and a version of fixed-point logic adapted to paths.
Consider the following syntax for LTL:

$$\phi ::== \mathcal{P}|\phi_1 \wedge \phi_2|\neg\phi| \bigcirc \phi|\Diamond\phi|\Box\phi|\phi_1 U \phi_2$$

where $\mathcal{P}$ is a set of atomic propositions. The semantics is – as usual – defined in terms of infinite execution paths and the temporal operators have their usual meanings. We can enrich the language by introducing fixed point operators to get the logic $\mu$-LTL below

$$\phi ::== \mathcal{P}|X|\phi_1 \wedge \phi_2|\neg\phi| \bigcirc \phi|\mu X.\phi(X)|\nu X.\phi(X)$$

where $X$ stands for a variable that ranges over formulas and $\mu$ and $\nu$ are least and greatest fixed-point operators respectively. The semantics of $\mu$ and $\nu$ are given in terms of fixed points as we defined in class.

With the fixed point operators present we do not need any of the LTL operators except $\bigcirc$. For example, we can write $\Box\phi \equiv \nu X.\phi \wedge \bigcirc X$. We can then use negation to get $\Diamond$.

**Question 6**[10 points]

1. Show how to write a formula in $\mu$-LTL that allows one to say that a path has the property that $p$ (which is some fixed proposition) holds in the first state and every alternate state after that; we call this formula $odd(p)$. Note, I am **not** saying that $p$ does **not** hold in any of the other states. Thus, your formula must be satisfied by paths like: $(pq)^\infty$ as well as paths like $pqpqppppqpqpp\ldots$ or $p^\infty$. Here we are overloading the symbol $p$ to stand for the state where only $p$ (the proposition) is true.
2. What would happen if you used the other fixed-point operator in your formula?
3. Explain why the formula $p \wedge \Box(p \Rightarrow \bigcirc\neg p) \wedge \Box(\neg p \Rightarrow \bigcirc p)$ does not express $odd(p)$.

**Question 7**[10 points]

Let $\sigma_i = p^i(\neg p)p^\infty$ stand for a path where $p$ is true for the first $i$ steps then it is false for one state then $p$ is true forever. Prove: given a proposition $p$ and *any* LTL (not $\mu$-LTL) formula $\phi$ containing $n$ next operators, the formula $\phi$ has the same truth value on every $\sigma_i$ with $i > n$.

Now show that $odd(p)$ cannot be expressed in LTL.

**Question 8**[10 points] Which of the following identities hold for $\omega$-regular expressions?

1. $(E_1 + E_2) \cdot F^\omega \equiv E_1 \cdot F^\omega + E_2 \cdot F^\omega$
2. $E \cdot (F_1 + F_2)^\omega \equiv E \cdot F_1^\omega + E \cdot F_2^\omega$
3. $(E^* \cdot F)^\omega \equiv E^* \cdot F^\omega$.

Either prove they are equal or give a counterexample.

**Question 9**[10 points] Consider the processes

$$\begin{tikzcd}
 & s_0 \arrow[dl, "a"'] \arrow[d, "a"] & & t_0 \arrow[d, "a"] \\
s_3 & s_1 \arrow[d, "b"] & & t_1 \arrow[d, "b"] \\
 & s_2 & & t_2
\end{tikzcd}$$

Write a formula of Hennessy-Milner logic that distinguishes the states $t_0$ and $s_0$.

The negation-free fragment of Hennessy-Milner logic is

$$\phi ::== \mathsf{true} | \phi_1 \wedge \phi_2 | \phi_1 \vee \phi_2 | \langle a \rangle \phi$$

where $a$ is an action. Show that the states $t_0$ and $s_0$ agree on *all* the formulas of the negation-free fragment. This should be done by induction on the structure of formulas.

**Question 10**[10 points]

Let $\Sigma = \{A, B\}$. Construct an NBA that accepts the set of infinite words $\sigma$ over $\Sigma$ that start with $A$ and such that $A$ occurs infinitely often in $\sigma$ and between any two consecutive $A$'s there are an *odd* number of $B$'s.