Lecture 5

Basic propositional logic

PROP = {P, P→P, ¬P, P∧P, ...}

P, P→P, ¬P, P∧P, ... are formulas and propositional formulas.

Formulas are defined inductively

(1) P is a formula

(2) If P is a formula then ¬P is a formula

(3) If P and Q are formulas so are P→Q, P∧Q, P∨Q

PROOF THEORY: How to use & manipulate formulas in deduction.

JUDGEMENT $\Gamma \vdash P$

Γ — set of formulas (assumptions)

P — formula (conclusion)

RULES OF INFERENCE

Premises / Conclusion

$\frac{\Gamma \vdash P \quad \Delta \vdash Q}{\Gamma, \Delta \vdash P \wedge Q}$

AXIOM

$\frac{}{\Gamma, P \vdash P}$

RULES

Weakening $\frac{\Gamma \vdash P}{\Gamma, Q \vdash P}$

$\frac{\Gamma, P, P \vdash Q}{\Gamma, P \vdash Q}$ not in CPL

INTRO ELIM

$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}$ $\frac{\Gamma \vdash P \to Q \quad \Delta \vdash P}{\Gamma, \Delta \vdash Q}$

$\frac{\Gamma \vdash P \quad \Delta \vdash Q}{\Gamma, \Delta \vdash P \wedge Q}$ $\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P}$ $\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}$

$\frac{\Gamma \vdash P}{\Gamma, \Delta \vdash P \vee Q}$ $\frac{\Gamma \vdash P \vee Q \quad \Delta, P \vdash R \quad \Delta, Q \vdash R}{\Gamma, \Delta \vdash R}$ CASE ANALYSIS

$\frac{\Gamma, \neg P \vdash \bot}{\Gamma \vdash P}$ PROOF BY CONTRADICTION

$\frac{\Gamma \vdash \bot}{\Gamma \vdash P}$ EXPLOSION

$\frac{\Gamma \vdash P \quad \Delta \vdash \neg P}{\Gamma, \Delta \vdash \bot}$

$\neg P \to \bot$ is a weaker form of ¬-I.

$\frac{\Gamma \vdash P}{\Gamma \vdash \neg\neg P}$ $\frac{\Gamma \vdash \neg\neg P}{\Gamma \vdash P}$ DOUBLE NEGATION

INTUITIONISTIC LOGIC REFUTES THIS!

A proof is an act where lemma and theorem, the media are visible evidences. In the media is a single proposition P.

If the media has the form P then we say it is a theorem.

$\frac{\Gamma \vdash P}{\Gamma \vdash P}$

TYPE-P: Theorem

SEMANTICS: Interpret logical formulas as elements of a single mathematical structure.

$\mathbb{T}$ — 2 element Boolean algebra

operations $\wedge, \vee, \neg$

defined by truth tables

A valuation $\langle v, \text{PROP} \to \mathbb{T} \rangle$

Extend v to a unique formula by structural induction

$v(P \to Q) = v(P) \to v(Q)$

$v(P \wedge Q) = v(P) \wedge v(Q)$

$v(P \vee Q) = v(P) \vee v(Q)$

$v(\neg P) = \neg v(P)$

Tautology $\models P$ if $v(P) = \mathbb{T}$ for all valuations v

$\models P \to P$ is a tautology

$\models P \to Q$ is not a tautology

Thm (CORRECTNESS) If $\vdash P$ then $\models P$

or formally: $\vdash P \Rightarrow \models P$

Thm (COMPLETENESS) If $\models P$ then $\vdash P$

COMPLETENESS FAILS FOR PROOFS FRAGMENTS CONTAINING ARITHMETIC

Suppose Γ has no axioms. $\Gamma \vdash P$ iff $\models P$ iff P is a tautology.

Thm (COMPLETENESS) If $\models P$ then $\vdash P$ is provable.

Thm (COMPLETENESS) If $\Gamma \models P$ then $\Gamma \vdash P$ is provable.

$\Gamma \models P$ iff $v(P) = \mathbb{T}$ for all valuations v satisfying Γ.

$\Gamma \models P$ — Γ entails P

$\models P$ — P is a tautology

$\models P \to Q$ — P implies Q

$\models P \to Q$ iff $v(P) \leq v(Q)$

$\models P \to Q$ iff $v(P) = \mathbb{F}$ or $v(Q) = \mathbb{T}$

$\models P \to Q$ iff $\models \neg P \vee Q$

$\models P \to Q$ iff $\models \neg(P \wedge \neg Q)$

$\models P \to Q$ iff $\models \neg Q \to \neg P$

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