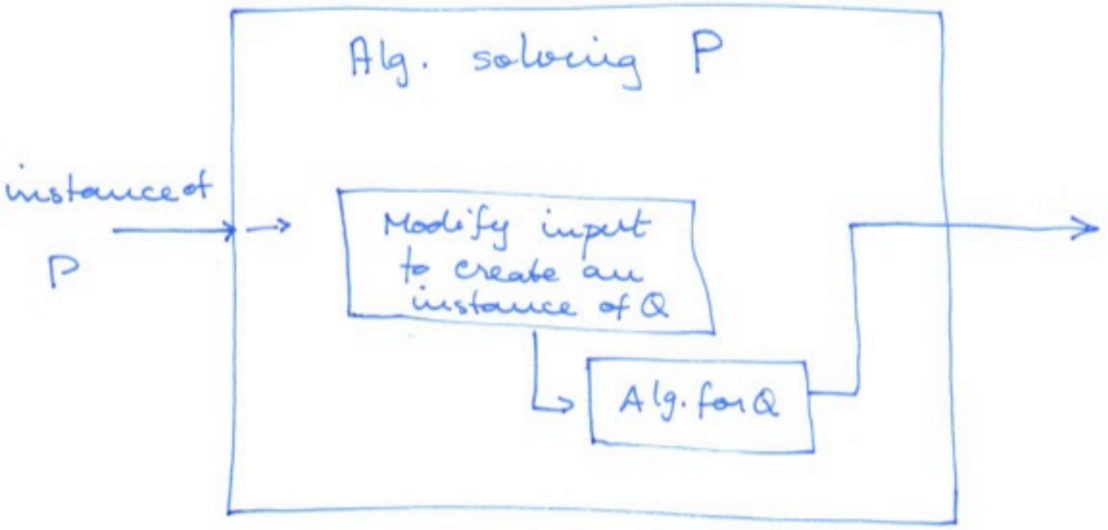
P reduces to Q

≡ If I can solve Q I can solve P

≡ Q is "harder than" P [P not harder than Q]

$$P \leq Q$$



If P is undecidable then Q must be undecidable.

If Q is decidable then P is decidable.

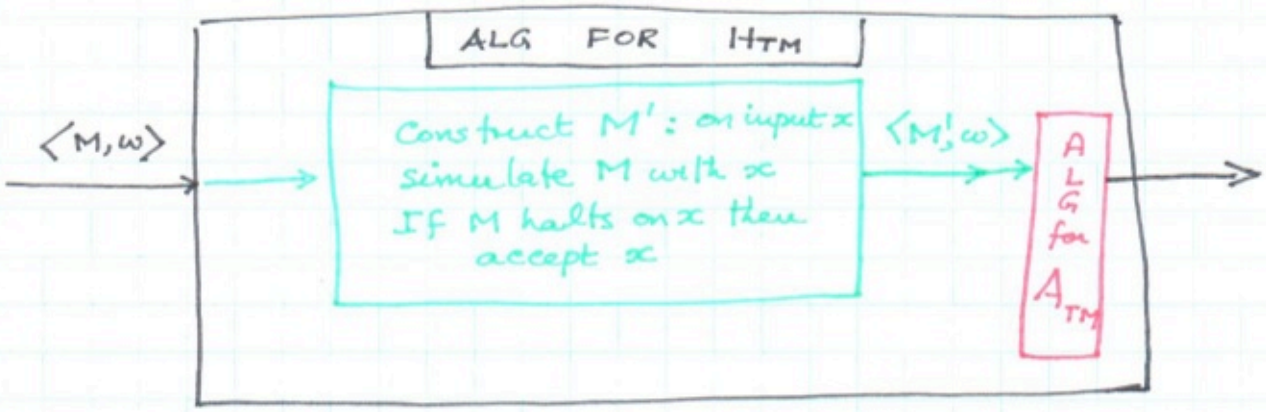
When proving $P \leq Q$ I do not have to explain how to solve Q because it is a conditional statement.

$$H_{TM} = \{ \langle M, \omega \rangle \mid M \text{ halts on } \omega \}$$

$$A_{TM} = \{ \langle M, \omega \rangle \mid M \text{ accepts } \omega \}$$

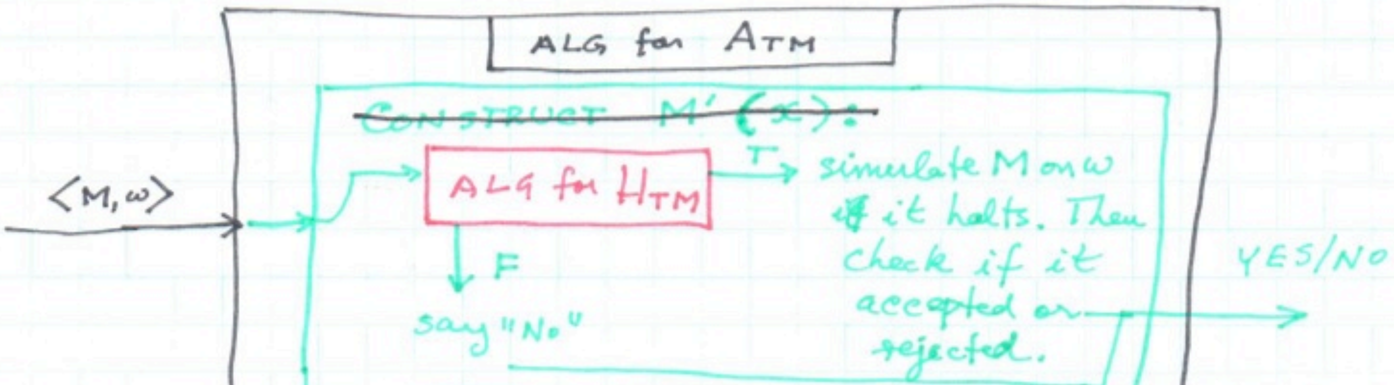
We showed H_{TM} is undecidable

$$H_{TM} \leq A_{TM}$$



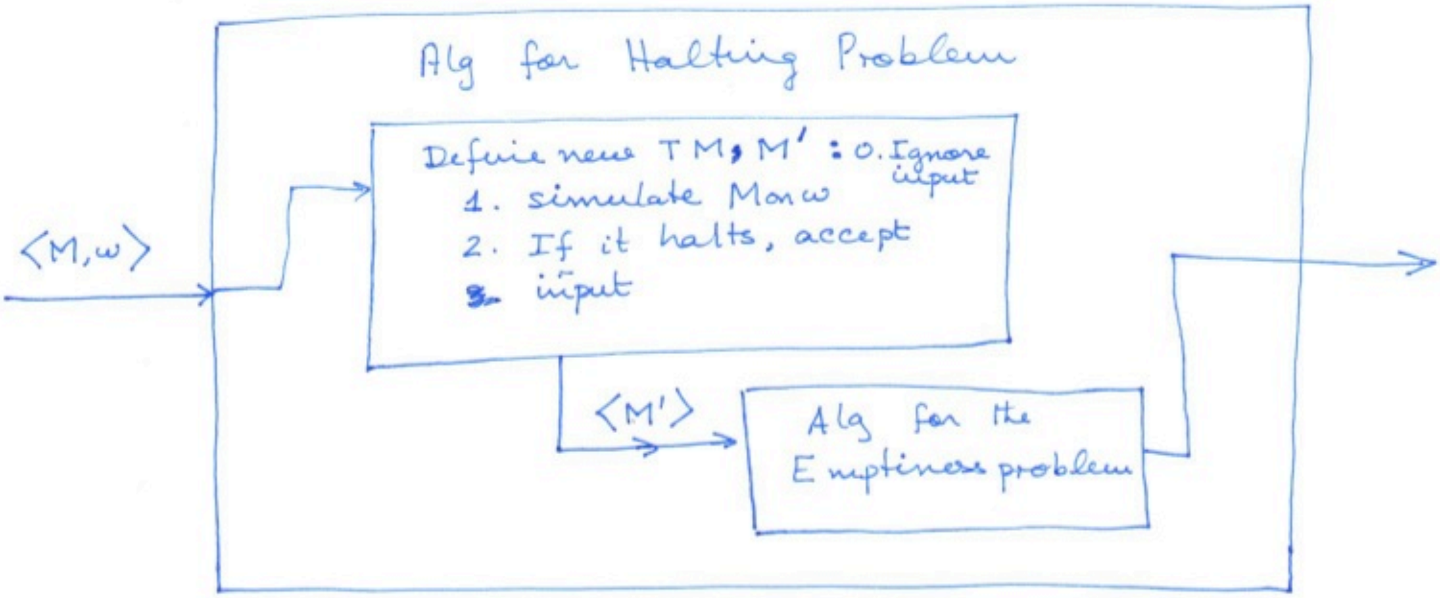
We could have proved A_{TM} is undecidable directly.

$$A_{TM} \leq H_{TM}$$



Halting Problem $\leq$ Emptiness Problem

$$E_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}$$



$L(M') = \emptyset$ if $M(\omega)\uparrow$

$L(M') \neq \emptyset \; (= \Sigma^*)$ if $M(\omega)\downarrow$

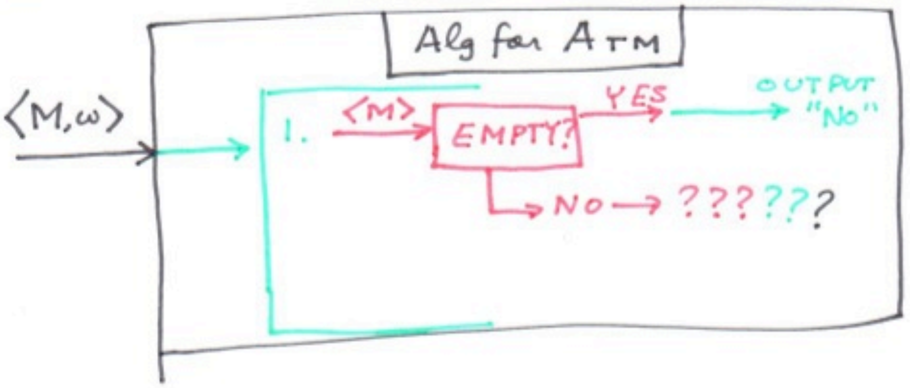
Halting problem is undecidable

Hence Emptiness problem is undecidable

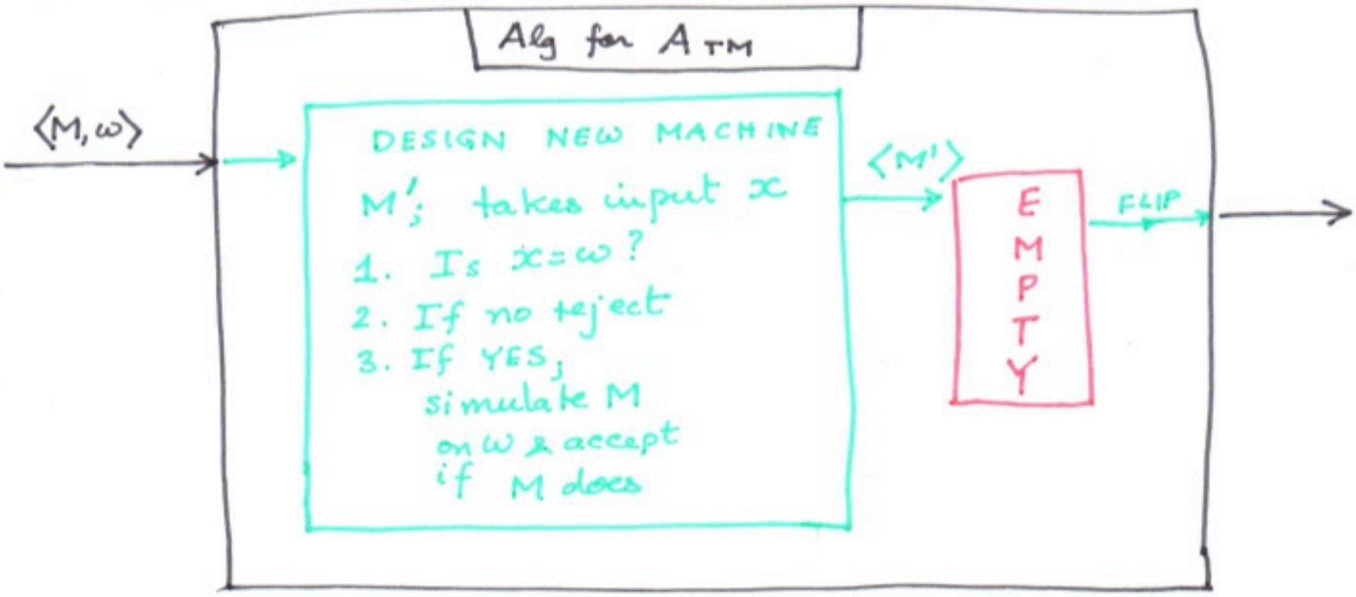
$$EMPTY_{TM} = \{\langle M \rangle \mid L(M) = \phi\}$$ ④

$$A_{TM} \leqslant EMPTY_{TM}$$

First TRY



2^{nd} TRY



If M accept ω $L(M') = \{\omega\}$

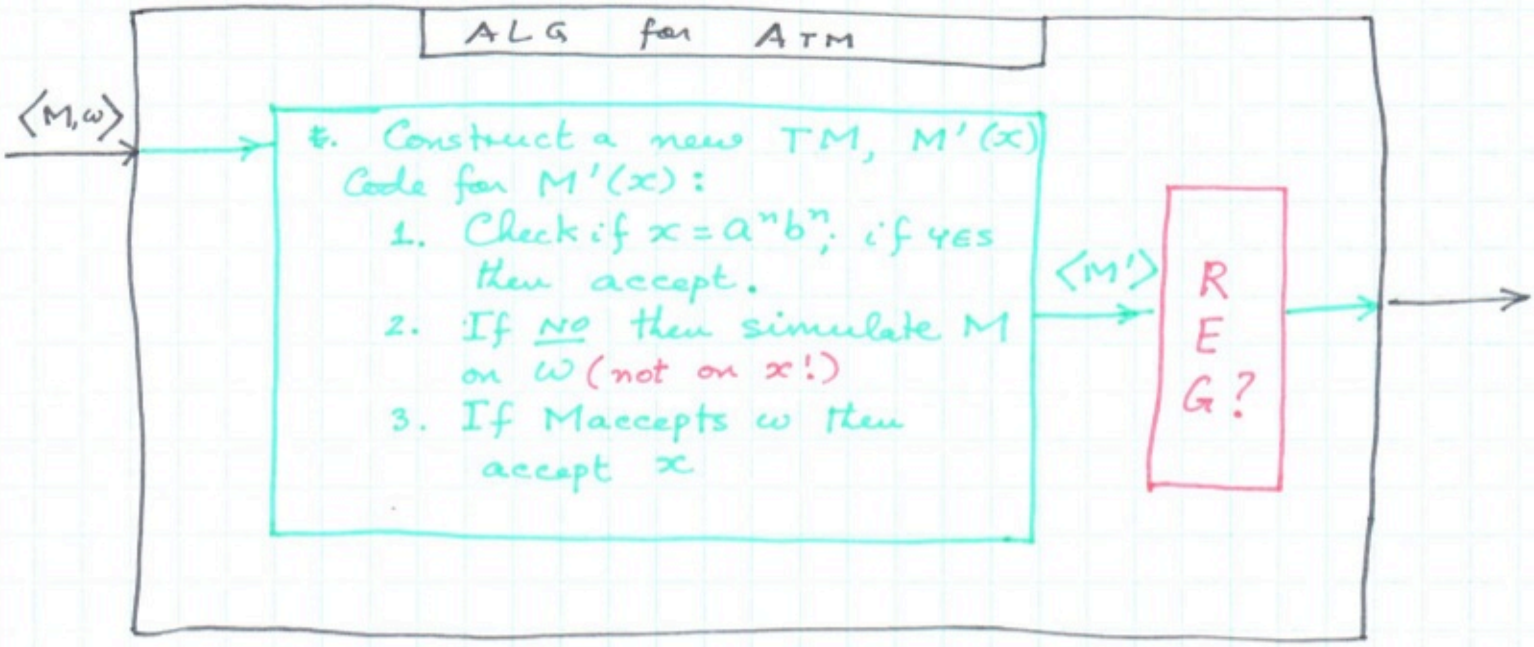
If M does not accept ω then $L(M') = \phi$

Is $L(M)$ a regular language?

UNDECIDABLE!

REG?

$A_{TM} \leq REG$



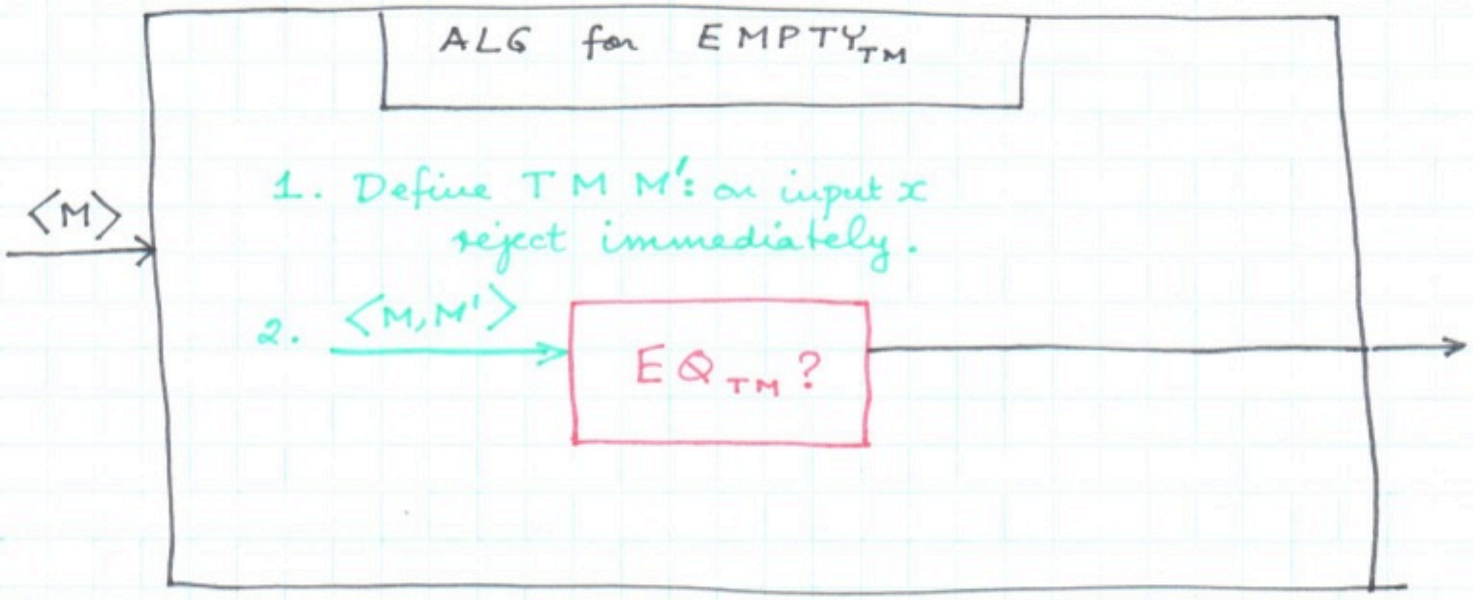
$L(M') = \Sigma^*$ if M accepts ω

$L(M') = \{a^n b^n \mid n \geq 0\}$ if M does not accept ω

$$L(M_1) = L(M_2)\ ?$$

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$$

$$EMPTY_{TM} \leq EQ_{TM}$$



EXERCISES

1. $L(M) = L(M')$ where M' always halts
2. $L(M)$ is context-free
3. $|L(M)| < \infty$
4. $L(M) = \Sigma^*$

Sharper notion of reduction

MAPPING REDUCTION

Suppose $L_1, L_2 \subseteq \Sigma^*$

$$L_1 \leq_m L_2$$

if there is a <u>TOTAL</u> <u>COMPUTABLE</u> function

$$f : \Sigma^* \longrightarrow \Sigma^*$$

such that $\forall \omega \in \Sigma^*$

$\omega \in L_1$ iff $f(\omega) \in L_2$

f is called the mapping reduction

Note (1) $L_1 \leq_m L_2$

implies $\overline{L_1} \leq_m \overline{L_2}$

(2) $\leq_m$ has a DIRECTION

it is not the same as ${}_m\geq$

$L_1 \leq_m L_2$ does <u>NOT</u> mean

$L_2 \leq_m L_1$

1. If $P \leq_m Q$ & P is undecidable then Q is undecidable

2. If $P \leq_m Q$ & Q is decidable then P is decidable

3. If $P \leq_m Q$ & Q is CE then P is CE

4. If $P \leq_m Q$ & P is not CE then Q cannot be CE

5. If $P \leq_m Q$ & P is not coCE then Q cannot be coCE

H_{TM}, A_{TM} are both CE but not coCE.

The reductions we gave were mapping reductions.

$\overline{A_{TM}}$ is co ~~RE~~ CE but not CE.

$$A_{TM} \leqslant EMPTY_{TM}$$

BUT

$$\cancel{A_{TM} \leqslant_m EMPTY_{TM}}$$

if there were then

$$\overline{A_{TM}} \leqslant_m \overline{EMPTY_{TM}}$$

But $\overline{A_{TM}}$ is not CE
& $\overline{EMPTY_{TM}}$ is CE.

<u>Thm</u> EQ_{TM} is not CE nor co CE

<u>PROOF</u> (1) We show $\overline{A_{TM}} \leq_m \overline{EQ_{TM}}$

Thus $\overline{EQ_{TM}}$ is not CE so
EQ_{TM} is not co CE.

(2) We show $A_{TM} \leq_m \overline{EQ_{TM}}$

Thus $\overline{EQ_{TM}}$ is not co CE

so EQ_{TM} is not CE.

Now for the reductions:

(1) same as $A_{TM} \leq_m EQ_{TM}$

Input $\langle M, \omega \rangle$

Construct (a) $M_1(x)$: ignore input & accept

(b) $M_2(x)$: ignore input, run $M(\omega)$ & if it accepts, accept x.

$L(M_1) = \Sigma^*$ $\quad L(M_2) = \begin{cases} \Sigma^* & \text{if } M \text{ accepts } \omega \\ \phi & \text{otherwise} \end{cases}$

$L(M_1) = L(M_2) \iff M \text{ accepts } \omega$

(2) Input $\langle M, \omega \rangle$

M_1: ignore input & reject

M_2: just as above

$L(M_1) = \phi$ so $L(M_1) = L(M_2) \iff A$ does not accept ω

$|L(M)| = \infty$

HOW HARD IS THIS?

Claim: $INF = \{\langle M \rangle \mid L(M) \text{ is infinite}\}$

is not even CE

$\overline{H_{TM}} \leq_m INF$

Given $\langle M, \omega \rangle$ we define a TM

M' with input x

M' works as follows:

1. Simulate M on ω for $|x|$ steps.
   (a) if M halts before the end of the simulation reject x
2. else accept x.

(This is OK because we are doing a controlled simulation)

If M does <u>not</u> halt on ω then

$L(M')$ is infinite else finite so

$\langle M, \omega \rangle \in \overline{H_{TM}} \iff \langle M' \rangle \in INF$