Homework 1
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Question 1

We have $f : \Sigma \to M$ and want $f^* : \Sigma^* \to M$ to be a homomorphism.

First we define $\Sigma^* = \Sigma_0 \cup \Sigma_1 \cup ...$ where Σ_k is the set containing words of length k

We define f^* on Σ_0 be the map to identity element. $f^*(\epsilon) = \epsilon$, on Σ_1 as equal to the element in Σ such that $\forall e \in \Sigma : f^*(e) = f(e)$, and on Σ_{i+1} given $mn \in \Sigma_{i+1}$ where $m \in \Sigma_i$ and $n \in \Sigma$ we define $f^*(mn) = f^*(m)f(n)$

In order to prove f^* is a monoid homomorphism, we show that the identity element is preserved $f^*(\epsilon) = \epsilon$ by definition. We also need to prove the binary operations is preserved. Let $\star$ be the binary operation on M and $\circ$ be the binary operation on Σ^*.Let $ai, bi \in \Sigma$

$f^*(x) \star f^*(y) = (f(a_1) \star f(a_2) \star f(a_3) \star \cdots \star f(a_n)) \star (f(b_1) \star f(b_2) \star f(b_3) \star \cdots \star f(b_n))$

$= f(a_1) \star f(a_2) \star f(a_3) \star \cdots \star f(a_n) \star (f(b_1) \star f(b_2) \star f(b_3) \star \cdots \star f(b_n)$ by associativity

$= f^*(a_1 \circ a_2 \circ a_3 \circ \cdots b_n)$

$= f^*(x \circ y)$

In order to prove uniqueness, we can do it with induction. Any other g which agree with f^* on Σ_i will agree on Σ_{i+1} since $g(xy) = g(x)f(y) = f^*(x)f(y) = f^*(xy)$

Question 2

($\to$)Let $A = (Q, A, i, \delta, T)$ be a DFA such that it recognizes L and let $T(A)$ be the transition monoid of A. Let $f : A^* \to T(A)$ the function associating the string with the effect of the string on the set Q. Let $I = f(L)$ be the image of the language in the transition monoid. Clearly, $L \subseteq f^{-}1(I)$ Let $x \in f^{-}1(I)$, Then by definition there exist an $y \in L$ such that $f(x) = f(y)$. But $y \in L(A)$ and so $x \in L(A)$ hence $x \in L$

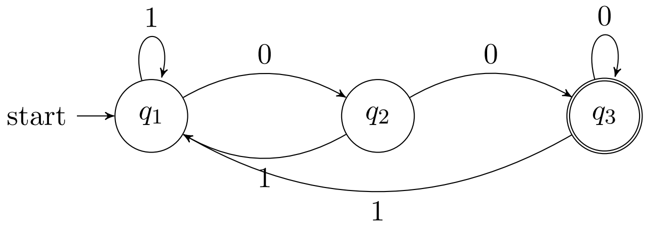
($\leftarrow$)Proving the converse: Suppose that L is recognized by a finite monoid. There is a monoid homomorphism $f : A^* \to M$ and a subset $S \subseteq M$ such that $L = f^{-1}(S)$. Define $A = \{M, A, i, \delta, S\}$ where i is the identity of M and $\delta(p, q) = p \circ f(q)$ It is clear that A is a finite automaton. If q is a string that is in A^* then $\delta^*(p, q) = p \circ f(q)$ We have that $x \in L(A)$ if and only if $i \circ f(x) \in S$ if and only if $x \in f^{-1}(S)$ which is L. So L(A) = L

Question 3

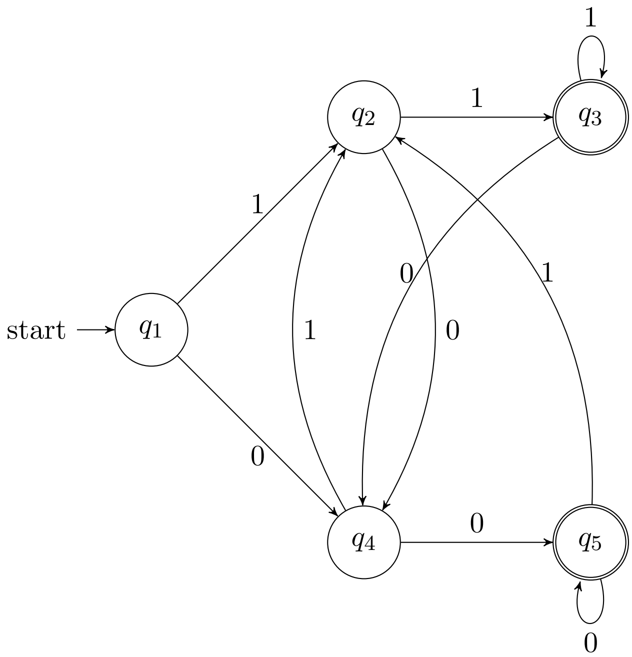
Suppose h is not inflationary, i.e. $V = \{x | x > h(x)\}$ and V not emptyset. W is well-founded so that V has minimal element such that $v_0 > h(v_0)$. Since $\forall y < v_0,\ y \le h(y)$, we have $h(v_0) \le h(h(v_0))$. However, since h is strictly monotone, we can deduce $h(v_0) \ge h(h(v_0))$ from $v_0 \ge h(v_0)$ Thus an contradiction is found.

Question 4

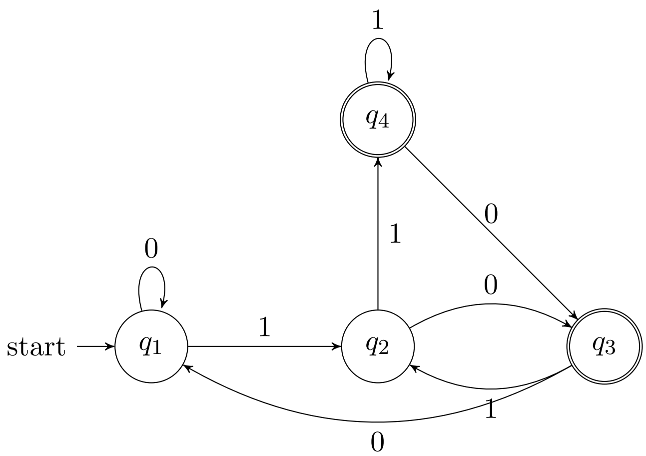
4.1



4.2



4.3



Question 5

We assume (S, s_0, F, δ) is a DFA that recognizes L. We now construct a NFA (Q, Q_0, F, Δ) for middle(L).

$$Q = \overbrace{S}^{\text{track } v} \times \underbrace{S}_{\substack{\text{guess where DFA will} \\ \text{be when } u \text{ ends}}} \times \overbrace{S}^{\substack{\text{guess where DFA will} \\ \text{be when } v \text{ ends}}} \times \underbrace{2^S}_{\text{track } u} \times \underbrace{2^S}_{\text{track } w}$$

$Q_0 = \{(s, s, t, \{s_0\}, \{t\})\}$

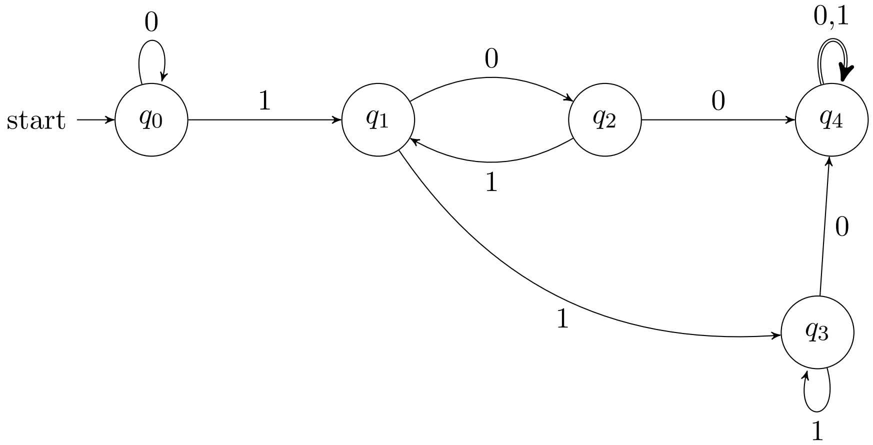
$F' = \{(t, s, t, X, Y) | Y' \cup F \neq \emptyset, s \in X\}$

$\Delta((s, s_c, t, X, Y), a) = (s', s_c, t, X', Y')$ where $\delta(s, a) = s'$, X' and Y' are the sets of states the DFA can reach from X and Y respectively reading any symbol

Since this is clearly a NFA that recognizes middle(L), this is a regular language.

Question 6

6.1



6.2

Suppose that for contradiction we can find a correct DFA that has 4 states.

By the Pigeonhole Principle, if we choose 5 strings over Σ, then at least two of those strings must end at the same state q.

Let wi and wj be these 2 strings that satisfies the above. For any string x, if wi and wj end at q, then wix and wjx must also end at the same state r and hence must both be accepted or rejected by the DFA. If we can pick for 5 strings such that, for any pair of them, for some x, that wix is NOT in L (rejected) and wjx is in L (accepted), then we have a contradiction, and what must assumed (we can find a correct DFA with 4 states) must be FALSE.

We pick the following four strings

$W0 = \epsilon$

$W1 = 110$

$W2 = 010$

$W3 = 111$

$W4 = 101$

For one of the pairs of strings, the supposed 4-state DFA is forced into the same state for both strings (because of Pigeonhole principle), and $W_i x$ and $W_j x$ must be both accepted or rejected, for any string x. We will now show that for each pair this is NOT true.

Pair 1 Choose $x = \epsilon$
$W_0 x = \epsilon$ reject $W_1 x = 110$ accept
Pair 2 Choose $x = 0$
$W_0 x = 0$ reject $W_2 x = 0100$ accept
Pair 3 Choose $x = 10$
$W_0 x = 10$ reject $W_3 x = 11110$ accept
Pair 4 Choose $x = 10$
$W_0 x = 10$ reject $W_4 x = 10110$ accept
Pair 5 Choose $x = \epsilon$
$W_1 x = 110$ accept $W_2 x = 010$ reject
Pair 6 Choose $x = \epsilon$
$W_1 x = 110$ accept $W_3 x = 111$ reject
Pair 7 Choose $x = \epsilon$
$W_1 x = 110$ accept $W_4 x = 101$ reject
Pair 8 Choose $x = 10$
$W_2 x = 01010$ reject $W_3 x = 11110$ accept
Pair 9 Choose $x = 10$
$W_2 x = 01010$ reject $W_4 x = 10110$ accept
Pair 10 Choose $x = 0$
$W_3 x = 1110$ accept $W_4 x = 1010$ reject
A contradiction is found.

Question 7

Assume (S, s_0, F, δ) is a DFA that recognizes L. We know that a boolean matrix is finite and can be used in a NFA. Let M_{n*n} be a boolean matrix where M_{ij} stores if NFA can get from s_i to s_j in $2^{|w|}$ steps where $|w|$ is the number of characters that has been read. So from M we can know if there exists a y such that $xy \in L$ and $|y| = 2^{|x|}$

Lets now construct an NFA (Q, Q_0, F, Δ)

$$Q = \underbrace{S}_{\text{track the state of the machine reading } x} \times \underbrace{S}_{\text{track where the machine will end up}} \times M$$

$Q_0 = \{(s_0, M)\}$
$F' = \{(s_i, M) | M_{ij} = 1 \text{ for } s_j \in F\}$
$\Delta((s, M), a) = \{(\delta(s, a), M^2)\}$
This is clearly regular

Question 8

8.1

Demon pick $p > 0$ We choose $w = a^{2p}b^p$ such that $|w| > p$ always hold. Demon pick $y = a^k$, where $1 \leq k \leq p$, I pick $i = 0$, then $xy^0z = xz = a^{2p-k}b^p$ If $xy^0z \in L$, then $2p - k = 2p$ which means $k = 0$, but $k \leq 1$ so we conclude that it is not regular.

8.2

Demon pick $p > 0$. We choose $w = a^{(p+1)^2}$. In order to satisfy pumping lemma $|xy| \leq p$ and $|y| > 0$, y need to consist exclusively with a. So $y = a^k$ for $0 \leq k < p$. We then pick $i = 2$, then new string $= xy^2z = a^{(p+1)^2+k}$ Since $(p+1)^2 + k > (p+1)^2$ and $k \leq p$ we can then conclude that $(p+1)^2 + k \leq (p+1)^2 + p < (p+2)^2$ and obviously $xy^2z \notin L$ So by pumping lemma it is not regular.

Question 9

Proof by counterexample. Let $\Sigma = \{x, y, z\}$, $L = \{x^*zy^*\}$ Since L and x^*y^* can be written as a regular expression, they are both regular. Now we need to prove outer(L) is not regular by showing $outer(L) \cap \{x^*y^*\}$ is not regular. Since there is no z in $\{x^*y^*\}$, for $uw \in outer(L) \cap \{x^*y^*\}$ there is no z in u or w, so z can only be in v Since $L = x^*zy^*$ and that z can only be in v, $u = x^*$ and $w = y^*$ and $v = x^*zy^*$ with $|u| = |w| = |v|$ So $u = x^n$, $w = y^n$ with some $n \geq 1$ So $outer(L) \cap \{x^*y^*\} = \{x^*y^*, n \geq 1\}$ Then, using pumping lemma to prove this is not regular. Demon pick $p > 0$, I choose $w = x^p y^p$. Demon is forced to pick $Y = x^k, k \geq 1$ in order to satisfy $|XY| < p$ I choose i=2. Then $XY^iZ = x^{p+k}y^p \notin \{x^n y^n | n \geq 1\}$ Since $outer(L) \cap \{x^*y^*\}$ is not regular, $outer(L)$ cannot be regular.

Question 10

$\Sigma = \{1\}$ Build an NFA with nine states arranged in loops of length two and seven, Assign start and final state so that the shortest rejected string is of length $14 = 2 * 7$ and it is strictly greater than nine.