A set that is neither CE nor co-CE

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Proposition 1. If $Q \leq_m P$ and if P is CE then so is Q. If P is co-CE so is Q.

Proof . Suppose that f is the required mapping reduction: then $w \in Q$ if and only if $f(w) \in P$. Now assume that P is CE. If I want to know whether $w \in Q$ I will ask the algorithm for P if $f(w) \in P$, if $(w) \in P$ I will eventually find this out but if it is not, the algorithm may loop forever. If P is co-CE I have an algorithm B with the property that for any word **not** in P the algorithm will eventually tell me that it is not in P; for words that are in P the algorithm may loop forever. Now I want to know if $w \in Q$, I ask B if $f(w) \in P$, if it is **not** then I will find out eventually but if it is in P B may loop forever. Thus Q is also co-CE. ∎

Corollary 2. If $Q \leq_m P$ and if Q is not CE then neither is P.

Proof . This is just the contrapositive of the proposition. ∎

Now for the main point of this note. Consider the set:

$$EQ_{TM} = \{\langle M_1,\ M_2 \rangle \mid L(M_1) = L(M_2)\}.$$

I claim that this set is neither CE nor co-CE. Recall that the set A_{TM} is not decidable but it is CE. Recall also that a set is decidable if it is both CE and co-CE, so A_{TM} is not co-CE. Its complement $\overline{A_{TM}}$ is co-CE but not CE. First, I will show that $\overline{A_{TM}} \leq_m EQ_{TM}$, which will establish that EQ_{TM} is not CE. Then I will show that $A_{TM} \leq_m EQ_{TM}$, which will show that EQ_{TM} is not co-CE.

To show the first claim we proceed as follows. Suppose I want to know whether $w \notin L(M)$ for some Turing machine M. Define two Turing machines M_1 and M_2 as follows. M_1 rejects everything, thus $L(M_1) = \emptyset$. M_2 checks if the input word is w, if it is not M_2 will reject it. If the input word is w then M_2 simulates M on w: if M accepts w then M_2 will as well. Thus $\langle M,\ w \rangle \in \overline{A_{TM}}$ if and only if $L(M_1) = L(M_2)$. Thus if EQ_{TM} is CE then so is $\overline{A_{TM}}$; but we know that the latter is not.

To show the second claim we proceed as follows. This time I want to know if $w \in L(M)$. I define M_1 to accept everything and M_2 as follows. M_2 checks its input x: if $x \neq w$ then M_2 accepts it, if $x = w$ then M_2 simulates M on w and accepts w if M does. In this case $\langle M, w \rangle \in A_{TM}$ iff $L(M_1) = L(M_2)$. Thus if EQ_{TM} is co-CE so is A_{TM} but we know that A_{TM} is not co-CE.