Σ^* - infinite. Σ^{ω} - finite words

Σ^{∞} - the set of infinite words over Σ

if $\Sigma = \{a, b\}$ Σ^{∞} is uncountable

$\Sigma^{\infty} = \Sigma^{\omega} \cup \Sigma^{\omega}$

a string in Σ^{∞} is countable, in particular, a string can be ordered by the Σ and Σ^{ω} Σ Σ^{ω} function.

$\leq$ on Σ^{∞}

$u \leq v$ if u is a prefix of v

$aab \leq aabaa$

$aab \leq aabaabaa$

Additional model

We can define a notion $\leq$ on Σ^{∞} by saying $u \leq v$ if u is the final position where $u[i] = v[i]$ or just this is an alternative.

Σ^{∞} is the family consisting of Σ^{ω} and the words u infinitely many $u \in \Sigma^{\omega}$ $u \leq v$ $\{u \mid u \leq v\}$ $\{u \mid u \leq v\}$

All finitely are countable, all ones below cannot

ω-regular expressions

E - regular expression. $A \in E(\Sigma)$

E^{ω} - is an ω-regular expression

$L(E^{\omega}) = \{u_1 u_2 \ldots \mid u_i \in L(E)\}$

$L(E^{\omega})$ - is a language defined by the ω-reg expression.

$(a^*bb)^{\omega}$

a^{ω}

let E be regular expression

E, E be regular expressions

$E(E)^* + E(E)^{\omega} \rightarrow E + E(E)^{\omega}$ is general ω-reg expression

$(a^*bb)^{\omega}$

$(a^*bb)^{\omega}$

E^{ω} $E \cdot E^{\omega} = E^{\omega}$ is also an ω-regular expression

Concatenating ω-strings

$u_1 u_2 \in \Sigma^{\omega}$ $u_1 \cdot u_2 = u$

$u \in \Sigma^{\omega}$ $v \in \Sigma^{\omega}$ $u \cdot v = u$

$u \in \Sigma^{\omega}$ $v \in \Sigma^{\omega}$ Then $u \cdot v$ has the meaning you would expect.

Def. If α - language $(\Sigma^* \Sigma^{\omega})$ is called ω-regular if it is defined by some ω-regular expression.

Example. This ω-language may not be unique

- concatenation
- union
- $(\ )^{\omega}$ infinite repetition

ω-automata specifically (Buchi automata) (NBA)

$A = (Q, \Sigma, \delta, Q_0, F)$ finite

$\delta: Q \times \Sigma \rightarrow 2^Q$ $Q_0 \subseteq F$

$S \subseteq Q$ $S \rightarrow 2^Q$

$S \in S(A, a) = A \rightarrow 2^{\Sigma}$

Given a word $\sigma \in \Sigma^{\omega}$ a run of σ on A is an infinite sequence of states

$\rho = q_0, q_1, q_2, \ldots$ s.t.

$q_0 \in Q_0$, $\forall i$ $q_{i+1} \in \delta(q_i, \sigma_i)$

where $\sigma[i]$ is the ith position of σ.

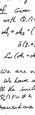
$\inf(\rho) = \{q \mid \exists i \text{ infinitely many } q = q_i\}$

A run is accepting if it hits an accept state infinitely often.

$\inf(\rho) \cap F \neq \emptyset$

We say a word is accepted by A if that word has some accepting run.

Example $\Sigma = \{a, b\}$



σ^{ω} is not accepted

ab^{ω} is accepted $\quad b^{\omega}$ a^{ω}

$(ab)^{\omega}$ is accepted

$(a^*ab)^{\omega}$ ✗

$(a^*ab)^{\omega}$ ✗

$L_{\omega}(A) = (a^*ab)^{\omega} \cup (a^*ab)^{\omega}$

The NBA is exactly the ω-regular languages.

Proof Given $A = (Q, \Sigma, \delta, Q_0, F)$ we will $R(i, j, k)$

$A_i = (Q, \Sigma, \delta, \{i\}, Q_0, F)$

$S(i, a) = S_i(i, a) = j \in Q$

$L(A_i \cdot a_i) = L_i(A) \cdot L_i(a_i)$

$L(A_i \cdot a_i) = L_i(A) \cup L_i(a_i)$

Let L_i - set corresponding to L_i for an NFA for F_i, we can with the initial state an non accepting $Q_0 \setminus F_i$ & then we can accepting transitions to any state in F_i.

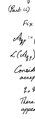
Given $A = (Q, \Sigma, \delta, Q_0, F)$ an NFA for L with $Q_0 \setminus F$ with no incoming transitions to Q_0.

NBA $A' = (Q, \Sigma, \delta', Q_0, F)$

$L_{\omega}(A') = (L(A))^{\omega}$

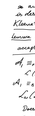
$\delta'(q, a) = \begin{cases} \delta(q, a) \text{ if } \delta(q, a) \cap F = \emptyset \\ \delta(q, a) \cup Q_0 \text{ otherwise} \end{cases}$

$Q_0' = F'$



NBA $(a^*b)^{\omega}$

(3) $E \cdot (F)^{\omega}$



NFA $\quad$ NBA

$L(E) \quad L(F)^{\omega}$

All the claim that reg language defined by an ω-reg can be accepted by an NBA

Build (3) Suppose $A = (Q, \Sigma, \delta, Q_0, F)$ NFA

For any S state, any $q, q' \in Q$

$A_{q'}^{q} = (Q, \Sigma, \delta, \{q\}, \{q'\})$ NFA

$L(A_{q'}^{q}) = L_{q'}^{q} = \{u \mid u \in \Sigma^* \mid \delta^*(q, u) = q'\}$

Consider an accepted word σ of $L(A)$ accepting run

$q_0 q_1 q_2 \ldots$

Then there is the same $q \in F$ that appears infinitely often.

$\sigma = \underset{L_{q_0}^{q_0}}{\underbrace{\sigma_0 \ldots \sigma_1}} \underset{L_{q_1}^{q_1}}{\underbrace{\sigma_1 \ldots \sigma_2}} \underset{L_{q_2}^{q_2}}{\underbrace{\sigma_2 \ldots \sigma_3}} \ldots$

ω - we can copy

Any of the σ that appears infinitely. We can an accepting run.

$L_{\omega}(A) = \underset{L_{q_0}^{q_0}}{L_{q_0}^{q_0}} \cdot \underset{L_{q_1}^{q_1}}{L_{q_1}^{q_1}} \cdot \underset{L_{q_2}^{q_2}}{L_{q_2}^{q_2}} \ldots \quad (L_{q_i}^{q_i} \cdot L_{q_i}^{q_i})^{\omega}$

ω - anything accepted by an NBA is accepted by an ω-reg exp. (Buchi) Thus the ω-languages.

Lemma $L_{\omega}(A) \neq \emptyset$ iff $\exists$ a reachable accept state that belongs to a cycle.

If $\exists$ σ_0, σ_1 the same in NFA. $L(A_i) = L(A_i)$

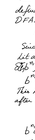
If, is σ_0 σ_1 the same in NBA

$L_{\omega}(A_i) = L_{\omega}(A_i)$.

Then $\sigma_0 \rightarrow \sigma_1$ NFA

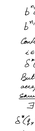
Then $\sigma_0 \rightarrow \sigma_1$ NBA

(1) $\sigma_0 \rightarrow \sigma_1$ for $\sigma_0 \rightarrow \sigma_1$



$a^* \rightarrow a^{\omega}$ $\quad a^* \rightarrow a^{\omega}$

(2)



$L(A) = a^{\omega}$ $\quad L(A) = \emptyset$

$L(A) = (aa)^{\omega}$ $\quad L(A) = a(aa)^{\omega}$

Deterministic BA - same as DFA with finite acceptance

NBA

$L_{\omega}(A) = (a+b)^{\omega}$

Then there can't BA that can recognize the above language.

Proof Suppose we have a DBA for this language. Can we define S as small as the minimal DFA. Consider the word

$b^{\omega} = (a+b)^{\omega}$

Since b^{ω} is accepted we must let an accept state at some point $S(q_0^{\omega}) = q$

b^{ω} $b^{\omega} \in L$

Then there is let an accept state again after reading n bs

$b^n a$ b^n b^n $b^n