A model predictive control approach to attitude control of an A-4D fighter aircraft

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***Abstract*—This work carries out a study of a model predictive control (MPC) approach applied to control the attitude of an A-4D fighter aircraft. The dynamics model used in the MPC controller is a linearization of the rotational dynamics of an aircraft. Two types of MPC control are considered, namely state-based reference tracking MPC and output-based reference tracking MPC with disturbance rejection. A stability analysis is carried out to prove asymptotic closed-loop stability of the tracking problem. It was discovered that the linearized attitude dynamics of the A-4D fighter aircraft are open-loop stable, leading to stability irrespective of the prediction horizon length. Furthermore, it was found that the A-4D fighter aircraft is able to reject disturbances when tracking a certain pitch angle.**

I. Introduction

An aircraft is described by complicated dynamics, making the design of a control algorithm for this system difficult. On top of this, fighter jets in particular are designed to be inherently unstable since this allows them to be more agile and enable them to perform aggressive maneuvers. For this reason, advanced control techniques are required to ensure stability and control of the aircraft. A lot of the current research effort in aircraft, as well as spacecraft, attitude control is focused on incremental nonlinear dynamic inversion. This technique consists on canceling the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form, allowing conventional linear controllers to be applied [1]. The inversion of these dynamics is implicit and the dynamics are only of a linear form once the control loop is closed. For this reason, a linear model predictive control (MPC) approach can not be used since this relies on an explicit model of the system dynamics. However, as mentioned earlier, being able to fly a fighter jet at its operational limits is especially beneficial, making the ability to set state and control input constraints especially attractive. Therefore, this work involves designing an MPC controller for linearized pitch dynamics of an A-4D aircraft. Specifically, this study explores the pitch attitude hold capabilities and the disturbance rejection.

The paper is structures as follows: section II gives the relevant system dynamics, finally arriving at the linearized longitudinal dynamics in state-space form. Subsequently, section III formulates the MPC problem in detail. The asymptotic stability of the controller is covered in section IV. Finally, the results of numerical simulations are discussed in section V.

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II. A-4D Attitude Dynamics

This section describes the rotational dynamics obtained from Euler's rotation equations for a rigid body. From these equations, the longitudinal dynamics are extracted and subsequently linearized. Lastly, the constraints under which the system operates are presented.

A. Rotational Dynamics

The system's rotational dynamics are nonlinear, described by Euler's rotation equations using a rotating reference frame with its axes fixed to the body and parallel to the body's principal axes of inertia [2, sec. 3.2], given in Equation 1.

$$\vec{M} = \mathbf{I}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{I}\vec{\omega} \tag{1}$$

where $\vec{\omega}$ and $\mathbf{I}$ are the angular velocity vector and inertia tensor respectively, and $\vec{M}$ the moments acting at the center of gravity of the system. Extending this vector equation gives the following three scalar equations:

$$I_{xx}\dot{p} - I_{xz}\dot{r} - I_{xz}pq + (I_{zz} - I_{yy})rq = L \tag{2}$$

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) = M \tag{3}$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} - I_{xz}qr + (I_{yy} - I_{xx})pq = N \tag{4}$$

where p, q and r are the rolling, pitching and yawing rates respectively and L, M and N are the rolling, pitching and yawing moments respectively. These moments are a result of the aerodynamic force which acts through the center of pressure of the aircraft. It is thus assumed that the thrust line passes through the center of gravity, hence will not generate a moment about the center of gravity.

B. Linearized System Dynamics

The longitudinal attitude dynamics are represented by Equation 3. These are linearized using the small disturbance theory [2, sec. 3.5] in which the equations are linearized about the operating point, usually trim point, by replacing a variable with a reference value and a perturbation. The operating point is taken to be at steady, level cruise conditions. Taking these conditions leads to $p_0 = 0$, $q_0 = 0$ and $r_0 = 0$. Linearizing the pitching moment gives the following result:

$$\begin{aligned} \Delta M = {} & \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial \dot{w}}\Delta \dot{w} \\ & + \frac{\partial M}{\partial q}\Delta q + \frac{\partial M}{\partial \delta_e}\Delta \delta_e + \frac{\partial M}{\partial \delta_T}\Delta \delta_T \end{aligned} \tag{5}$$

The full linearized small-disturbance longitudinal rigid body equation of motion can then be represented in state-space form as follows [3]:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T}\\ Z_{\delta_e} & X_{\delta_T}\\ M_{\delta_e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_e\\ \Delta\delta_T\end{bmatrix} \tag{6}$$

Assuming that θ does not affect velocity, the row corresponding to the axial velocity can be removed. Furthermore constant thrust is assumed and the effect of change in thrust is ignored. Moreover the vertical velocity w has been replaced with the angle of attack in the model ($\Delta\alpha = \frac{\Delta w}{u_0}$). This yields following simplified state-space model:

$$\begin{bmatrix}\Delta\dot{\alpha}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 & 0\\ M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} Z_{\delta_e}\\ M_{\delta_e} + M_{\dot{w}}Z_{\delta_e}\\ 0\end{bmatrix}\Delta\delta_e \tag{7}$$

where $\vec{x} = [\Delta\alpha \quad \Delta q \quad \Delta\theta]^T$ are the system states variables and $u = \Delta\delta_e$ is the control input. The equations to compute the relevant longitudinal derivatives are given in Table I, obtained from [2, pg.123]:

TABLE I
LONGITUDINAL DERIVATIVES

$M_w = C_{m_\alpha}\frac{QS\bar{c}}{u_0 I_y}$	$Z_w = \frac{-(C_{L_\alpha}+C_{D_0})QS}{mu_0}$
$M_{\delta_e} = C_{m_{\delta_e}}\frac{QS\bar{c}}{I_y}$	$Z_{\delta_e} = -C_{Z_{\delta_e}}\frac{QS}{m}$
$M_q = C_{m_q}\frac{\bar{c}}{2u_0}\frac{QS\bar{c}}{I_y}$	$Z_\alpha = \frac{-(C_{L_u}+2C_{L_0})QS}{m}$
$M_{\dot{w}} = C_{m_{\dot{\alpha}}}\frac{\bar{c}}{2u_0}\frac{QS\bar{c}}{u_0 I_y}$	

Dimensionless coefficients are characteristic constants of any aircraft geometry at a specific flight condition. This allows the state-space model for any particular aircraft to be generated if the dimensionless coefficients are available since the general form of the state-space model for an aircraft is known.

The geometric, mass and aerodynamic characteristic data of the A-4D Fighter Airplane model are given in Table II. This data is taken from [2, pg.404].

TABLE II
CHARACTERISTIC DATA FOR THE A-4D FIGHTER AIRCRAFT

Geometric Characteristics	
Reference Area S $[m^2]$	24.2
Reference Area b $[m]$	8.33
Mean Aerodynamic Chord $\bar{c}$ $[m^2]$	24.2
Mass characteristics	
Mass m $[kg]$	7973.25
Center of Gravity Location	$0.25\bar{c}$
Moment of Inertia about y-axis I_y $[kgm^2]$	35115.7
Aerodynamic characteristics	
C_{L_0} = 0.30	C_{D_0} = 0.038
C_{L_α} = 4.0	C_{m_α} = -0.38
$C_{m_{\dot{\alpha}}}$ = -1.65	C_{m_q} = -4.30
$C_{Z_{\delta_e}}$ = -0.4	$C_{m_{\delta_e}}$ = -0.60

Using the data presented in Table II, the longitudinal derivatives can be computed. Furthermore, the aircraft is taken to be flying at a velocity u_0 of $237.2\ m/s$. This yields the continuous state-space model of the longitudinal dynamics of the A-4D fighter aircraft:

$$\begin{bmatrix}\Delta\dot{\alpha}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} -0.5507 & 1.0 & 0.0\\ -9.7621 & -0.9983 & 0.0\\ 0.0 & 1.0 & 0.0\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}-0.0545\\ -14.494\\ 0.0\end{bmatrix}\Delta\delta_e \tag{8}$$

In order to use this state-space model in a model predictive control formulation, discretization was performed using a time step of $\Delta t = 0.1\ s$. Additionally, taking the equilibrium state about which the system has been linearized to be when $\alpha = 0$, $q = 0$, $\theta = 0$ and $\delta_e = 0$, the following discrete state-space model is obtained:

$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix} 0.9013 & 0.0911 & 0.0\\ -0.8889 & -0.8605 & 0.0\\ -0.0460 & 0.0937 & 1.0\end{bmatrix}\begin{bmatrix}\alpha\\ q\\ \theta\end{bmatrix} + \begin{bmatrix}-0.0735\\ -1.3549\\ -0.0695\end{bmatrix}\delta_e \tag{9}$$

C. Physical System Constraints

Aircrafts have physical operational boundaries up till which they can function without significant risk of failure. This is reflected in the aircraft's flight envelope; a diagram which reflects the structural, and stall limitations of the aircraft. An absolute limit for fighter jets is a maximum load factor of $8g$ because of structural limitations and pilot's comfort [4].

Aircraft have a flight manual in which its operating limits are reported. For the A-4D aircraft, this document is published as 'NAVWEPS 01-40AVA-1'. Unfortunately, section V covering operating limitations is confidential. Therefore, the same values as reported in [3] are taken as constraints. Hence, an angle of attack less than 35^o is required and maximum control surface deflections of $\pm 25^o$.

III. Model predictive control design

This section describes the construction of the optimal control problem for tracking of a reference attitude. First, the general MPC law is described. Subsequently, the implementation of this law using state feedback is presented, after which an implementation using output feedback is given.

A. Model Predictive Control Law

A model predictive controller is designed to control the pitch angle of the aircraft. MPC is a control scheme which optimizes a given cost function over a finite prediction horizon subject to specified constraints. This yields a sequence of optimal control inputs of which the first one is injected into the system. At the next time step, this procedure is repeated, creating a moving prediction horizon. This is the receding horizon principle.

The optimal control problem requires a cost function and constraints subject to which the cost function should be minimized. The states of the discretized system (9) must satisfy the system dynamics $x^+ = Ax + Bu$, with $x(0) = x_0$.

As mentioned in subsection II-C, the aircraft also has physical limitations on its states, meaning that while solving the optimization problem, the state should be bounded to ensure proper functioning of the aircraft. Hence, it should remain within a given set $\mathbb{X}$. Loose constraints are also set on the pitch rate and pitch angle for them to remain bounded, also taking into account validity of the linearized model. The states shall then be constrained to the following set $\mathbb{X}$:

$$
\begin{aligned}
x_k \in \mathbb{X} =& \{(\alpha, q, \theta)^T | -15^o < \alpha < 35^o, \\
& -100^o/s < q < 100^o/s, -15^o < \theta < 30^o\}
\end{aligned}
$$

The elevator control surface angle deflection is constraint to the set $\mathbb{U}$:

$$
\delta_{e,k} \in \mathbb{U} = \{\delta_e | -25 < \alpha < 25\}
$$

The state constraints are related to input constraints by the state solution function $x_k = \phi(k, x_0, \mathbf{u}_k) = Px_0 + S\mathbf{u}_k$. This yields following linear constraint set on the control input:

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [Px_0 + S\mathbf{u}_k] \leq \begin{bmatrix} 35 \\ 100 \\ 30 \\ -15 \\ -100 \\ -15 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 \\ -1 \end{bmatrix} \delta_{e,k} \leq \begin{bmatrix} 25 \\ 25 \end{bmatrix}
$$

Lastly, a terminal set $\mathbb{X}_f$ is chosen appropriately, such that asymptotic stability of the closed-loop linear system can be proven. This is covered in detail in section IV.

The cost function of the optimal control problem has the general form given in 10.

$$
V_N(x_0, \mathbf{u}_N) = \sum_{k=0}^{N-1} l(x_k, u_k) + V_f(x_N) \tag{10}
$$

where $x_k = \phi(k, x_0, \mathbf{u}_k)$. The function l represents the stage cost which takes into account how well the state converges to the attractor set and V_f represents the terminal cost which relates to the stability of the closed-loop system.

The full optimal control problem is then formulated as:

$$
\mathbb{P}_N(x_0, t) = \begin{cases} min_{\mathbf{u}_N} & V_N(x_0, \mathbf{u}_N) \\ s.t. & x(0) = x_0 \\ & x_{k+1} = Ax_k + Bu_k, \quad \forall k \\ & x_k \in \mathbb{X}, \quad \forall k \\ & u_k \in \mathbb{U}, \quad \forall k \\ & x_N \in \mathbb{X}_f \end{cases} \tag{11}
$$

B. Reference Tracking using State Feedback

In this case, full-state knowledge is assumed. Therefore, the output of the system consists of the full state, hence the output matrix C is identity. The goal is to have the aircraft perform a pitch attitude hold maneuver, meaning that it should track a certain constant pitch angle as reference. The reference tracking MPC problem to be solved at each iteration is given in 11 with the stage and terminal cost defined as:

$$
\begin{aligned}
l(x_k, u_k) &= (x_k - x_{ref})^T Q (x_k - x_{ref}) \\
&\quad + (u_k - u_{ref})^T R (u_k - u_{ref}) \\
V_f(x_N) &= (x_N - x_{ref})^T P (x_N - x_{ref})
\end{aligned} \tag{12}
$$

The cost weighing matrices were chosen to be $Q = diag(10, 10, 50)$ and $R = I$. The higher cost corresponding to the pitch angle is chosen to ensure faster convergence of the state to the desired reference pitch. The cost matrix P is the solution to the discrete-time algebraic Riccati equation used to solve the infinite horizon, unconstrained optimal control problem. The reason for selecting this terminal cost function concerns the closed-loop stability as discussed in section IV.

For the purpose of reference tracking, an optimal target selection (OTS) needs to be solved to determine the required x_{ref} and u_{ref} such that steady-state tracking of the reference is ensured. The OTS to be solved offline is as described in 13.

$$
(x_{ref}, u_{ref})(y_{ref}) \in \begin{cases} \underset{x_{ref}, u_{ref}}{argmin} J(x_{ref}, y_{ref}) \\ s.t. \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_{ref} \\ u_{ref} \end{bmatrix} = \begin{bmatrix} 0 \\ y_{ref} \end{bmatrix} \\ (x_{ref}, u_{ref}) \in \mathbb{X} \times \mathbb{U} = \mathbb{Z} \end{cases} \tag{13}
$$

With the reference state and input known, the reference tracking MPC problem described in 11 can be solved at each time step. For this, the prediction horizon is chosen to be 5, which with a sampling time of 0.1 s works out to a horizon of 0.5 s. The control scheme for this reference tracking problem is schematically represented in Figure 1.

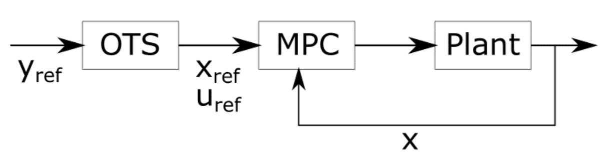

Fig. 1. Control Scheme for Reference Tracking using State Feedback

C. *Offset-free Reference Tracking using Output Feedback*

Output-based feedback is used in case direct access to the state can not be obtained. Knowledge of the state is obtained through measurements of the output using sensors and is estimated using these measurements and a model of the system. The A-4D is equipped with an electrically operated attitude gyro [5, pg.36] which gives pitch angle θ and pitch rate q measurements. Hence the system has two outputs, namely pitch angle θ and pitch rate q.

In order to handle model mismatch with the real dynamics in steady-state, also an estimate of a constant disturbance is made to guarantee offset-free tracking of the reference. The state and disturbance estimates are made by a Luenberger observer. This requires a disturbance model augmented with the auxiliary state $d^+ = d$, giving the augmented dynamics as given in Equation 14.

$$\begin{aligned}\begin{bmatrix} x^+ \\ d^+ \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}\begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} w \\ w_d \end{bmatrix} \\ y &= \begin{bmatrix} C & C_d \end{bmatrix}\begin{bmatrix} x \\ d \end{bmatrix} + v\end{aligned} \tag{14}$$

where d is a constant, unknown disturbance, w is a state disturbance with $\mathbb{E}[w(t)] = 0$ and v a measurement noise with $\mathbb{E}[v(t)] = 0$. The augmented system 14 is observable if (C, A) is observable and

$$\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{15}$$

Since the relation of the output to the state changes in real time, optimal target selection (OTS) has to be performed online at each step of the iteration to obtain x_{ref} and y_{ref}. The OTS problem is as described in Equation 16.

$$(x_r, u_r)(\hat{d}, y_r) \in \begin{cases} \underset{x_r, u_r}{argmin} J(x_r, y_r) \\ s.t. \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} x_r \\ u_r \end{bmatrix} = \begin{bmatrix} B_d \hat{d} \\ y_r - C_d \hat{d} \end{bmatrix} \\ \quad (x_r, u_r) \in \mathbb{X} \times \mathbb{U} = \mathbb{Z} \\ \quad C x_r + \hat{d} \in \mathbb{Y} \end{cases} \tag{16}$$

The observer model is derived from Equation 14 to include additional terms such that the model's estimate of the augmented state converges to that of the plant based on measured inputs and outputs of the plant. The observer model used in this case is the Luenberger observer which includes the output of the observer subtracted from the plant output multiplied by a gain matrix L, as shown in Equation 17.

$$\begin{bmatrix} \hat{x}^+ \\ \hat{d}^+ \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_x \\ L_d \end{bmatrix}(y - \hat{y}) \tag{17}$$

in which $\hat{x} \in \mathbb{R}^n$ and $\hat{d} \in \mathbb{R}^{n_d}$ are the current state and disturbance estimates, $u \in \mathbb{R}^m$ is the current control action, $\hat{x}^+$ and $\hat{d}^+$ are the next state estimates, $\hat{y}$ is the current output estimate and $L_x \in \mathbb{R}^{n \times p}$, $L_d \in \mathbb{R}^{n_d \times p}$.

The error of the Luenberger observer satisfies the equation $e_{k+1} = (\tilde{A} - \tilde{L}\tilde{C})e_k$. Therefore, the observer is asymptotically stable; the state estimation error $e_k = \hat{x}_k - x_k$ converges to zero for $k \to \infty$, when the matrix $\tilde{A} - \tilde{L}\tilde{C}$ has eigenvalues all within the unit circle, with $\tilde{A}, \tilde{L}$ and $\tilde{C}$ as given in 18.

$$\tilde{A} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \quad \tilde{L} = \begin{bmatrix} L_x \\ L_d \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} C & C_d \end{bmatrix} \tag{18}$$

The formulation of the offset-free tracking using a disturbance model and observer given previously is a general formulation. A particular case is obtained by choosing the observer gain $\tilde{L}$ and the disturbance matrices B_d and C_d as given in 19, where the gain matrix K is chosen such that $(A - KCA)$ has eigenvalues within the unit circle [6].

$$B_d = K \quad C_d = I - CK \quad L_x = K \quad L_d = I \tag{19}$$

According to *proposition 10* as stated in [6], choosing the matrices as specified in 19 leads to the observability condition of the augmented system given in 15 holding true. Furthermore, the author noted that the distrubance $\hat{d}$ does not alter the closed-loop dynamics of the state estimate error being governed by $(A - KCA)$. Besides, this approach also reduces the tuning process to tuning just one parameters, K.

Using the state and disturbance estimates, the optimal target selection described in 16 can be solved at each step producing the reference state and input. These are then used to solve the reference tracking MPC problem given in 11, using the stage and terminal cost as described in 12. For this, the prediction horizon is again chosen to be 5, which with a sampling time of 0.1 s works out to a horizon of 0.5 s. The control scheme for this reference tracking problem is schematically represented in Figure 2.

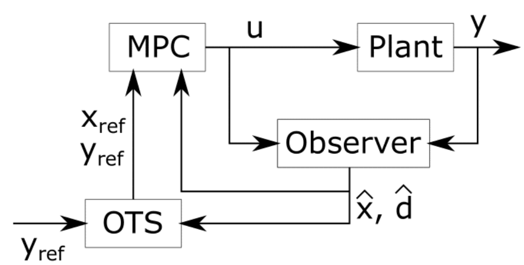

Fig. 2. Control Scheme for Reference Tracking using Output Feedback

IV. Asymptotic stability

This section considers the asymptotic closed-loop stability of the system controlled by the MPC controller designed in section III. For the purpose of proving asymptotic stability of the reference tracking problem, it is first proven that the regulation problem is asymptotically stable. This implicates that the origin can be translated to the reference state, proving that the system state will asymptotically stabilize around the reference state x_{ref}, with a similarly translated terminal constraint set: $\mathbb{X}_f = \{x_{ref}\} \oplus \mathbb{X}'_f$.

Proving asymptotic stability of the regulation problem is done through verifying the assumptions of Theorem 2.19 as posed in the book *'Model Predictive Control: Theory and Design'* by *J. Rawlings and D.Mayne* [7, pg.119].

The system to be controlled has the form $x^+ = Ax + Bu$ where the control u is subject to the constraint $u \in \mathbb{U}$ and the state x is subject to the constraint $x \in \mathbb{X}$. Both sets $\mathbb{U}$ and $\mathbb{X}$ are chosen such that they are closed, compact and contain the origin in its interior. In case of the tracking problem, the origin is translated to x_{ref} and the set $\mathbb{X}$ contains this reference state in its interior.

The stage cost of the regulation problem was chosen to be the quadratic function $l(x,u) = 1/2(x^T x + u^T Ru)$ where Q and R were both selected such that they are positive definite.

The terminal cost of the regulation problem was chosen to be the cost function V_∞^{uc} for the infinite horizon unconstrained optimal problem $\mathbb{P}_\infty^{uc}(x)$ with optimal control $u = Kx$. Thus, $V_f(x) = V_\infty^{uc} = 1/2(x^T Px)$ where P is the solution to the discrete algebraic Riccati equation. The terminal cost function then satisfies:

$$V_f(A_k x) = (1/2)x^T Q_K x - V_f(x) \leq 0 \quad \forall x \in \mathbb{R}^n$$

where $Q_K := Q + K^T RK$.

The terminal constraint set $\mathbb{X}_f$ is chosen as the largest set W satisfying:

(a) $W \subseteq \{x \in \mathbb{X} | Kx \in \mathbb{U}\}$
(b) $x \in W \implies x_k = A_K^k \in W \quad \forall k \geq 0$

These two conditions imply that if the initial state of the system is in the terminal constraint set $\mathbb{X}_f$, then the controller $u = Kx$ maintains the state in $\mathbb{X}_f$ and satisfies the state and control constraints for all future time. The terminal constraint set is then said to be control invariant for $x^+ = Ax + Bu, \quad u \in \mathbb{U}$.

Assumption 2.2: Continuity of system and cost
The function $f(x,u) = Ax + Bu$ is linear, hence it is a continuous function with an equilibrium point at $(x,y) = (0,0)$. Furthermore, both stage cost $l(x,u)$ and terminal cost $V_f(x)$ are positive definite, continuous functions, as they are quadratic functions with Q, R, $P \succ 0$. These functions also have that $l(0,0) = 0$ and $V_f(0) = 0$. Hence, *assumption 2.2* as posed in [7, pg. 97] is satisfied.

Assumption 2.3: Properties of constraint sets
This assumption requires the set $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ to be a closed set. This is the case since the constraints on the state $x \in \mathbb{X}$ and on the control input $u \in \mathbb{U}$ are closed and compact. Furthermore, since the set $\mathbb{X}_f$ is chosen such that $\mathbb{X}_f \subseteq \mathbb{X}$, also $\mathbb{X}_f$ is a compact set. All of these sets also contain the origin in their interior, as has been stated earlier. Hence, *assumption 2.3* as posed in [7, pg. 98] is satisfied.

Assumption 2.14: Basic stability assumption
Having chosen the terminal constraint set $\mathbb{X}_f$ to be control invariant, the terminal cost $V_f(\cdot)$ to be the cost function for the infinite horizon unconstrained optimal problem and the stage cost $l(\cdot)$ to be quadratic, both (a) and (b) of *assumption 2.14* as posed in [7, pg. 114] are satisfied.

Theorem 2.19: Asymptotic stability of the origin
Theorem 2.19 of [7, pg.119] states that with assumptions *2.2*, *2.3* and *2.14* satisfied, *(a)* there exists $\mathcal{K}_\infty$ functions α_1 and α_2 such that:

$$\alpha_1(|X|) \leq V_N^0(x) \leq \alpha_2(|X|)$$
$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\alpha(|x|)$$

for all $x \in \mathcal{X}_N$ and *(b)* that the origin is asymptotically stable in $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$. The set $\mathcal{X}_N$ is the region of attraction which contains all controllable states which have guaranteed asymptotic convergence towards the origin.

V. Numerical simulations

In this section, several numerical simulations are produced in which the A-4D fighter aircraft is simulated to perform a pitch attitude hold maneuver of 20^o. The performance of the controller in bringing the aircraft to this attitude is compared for different choices of prediction horizon and cost matrices. Varying these parameters helps in getting the desired system response to receiving a reference signal.

Furthermore, in some scenarios, the output cannot be accurately measured. Therefore, the presence of a random measurement disturbance is assumed, testing the disturbance rejection of the offset-free MPC controller.

Source files for the simulations can be found on github [8].

A. *Effect of prediction horizon*

From Figure 3, it can be seen that changing the prediction horizon for this particular maneuver does not change the stability of the tracking problem. This is due to the fact that the system is open loop stable, meaning that all eigenvalues of the state matrix A lie on the unit disk.

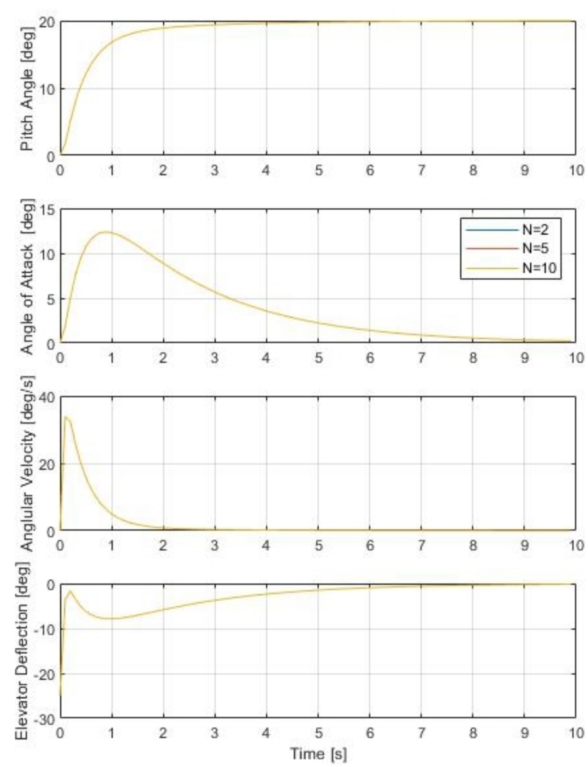

Fig. 3. State trajectories for different prediction horizons N

B. Effect of cost matrices

The effect of changing the cost matrices on the trajectory tracking is demonstrated on the MPC controller which utilizes state feedback. By varying the matrices, either more importance is put on tracking the reference state or tracking the reference input.

Figure 4, 5 and 6 show the pitch hold maneuver for various Q and R matrices. These cost matrices are used as the stage cost, as specified in Equation 12. The terminal cost matrix P in these simulations remains specified as the solution to the discrete-time algebraic Riccati equation.

From Figure 4, it is clear that a higher cost on the state tracking results in a faster rise time of the state towards the reference state. This on the other hand translates in more control effort, as can be seen on the graph showing the elevator deflection. In case the cost matrix Q is too low, a very slow convergence of the state towards the reference takes place. This is due to the fact that the reference input $u_{ref} = 0$ for this tracking problem, hence the cost of putting in more control effort is higher than the cost of not tracking the reference state.

From Figure 5, it can be seen that there is a relation between increasing the cost matrix Q and decreasing the cost matrix R, namely the same state trajectory is obtained for $Q = r\mathbf{I}_3$ and $R = \mathbf{I}_1$ as for $R = \frac{1}{r}\mathbf{I}_1$ and $Q = \mathbf{I}_3$, $r \in \mathbb{R}$.

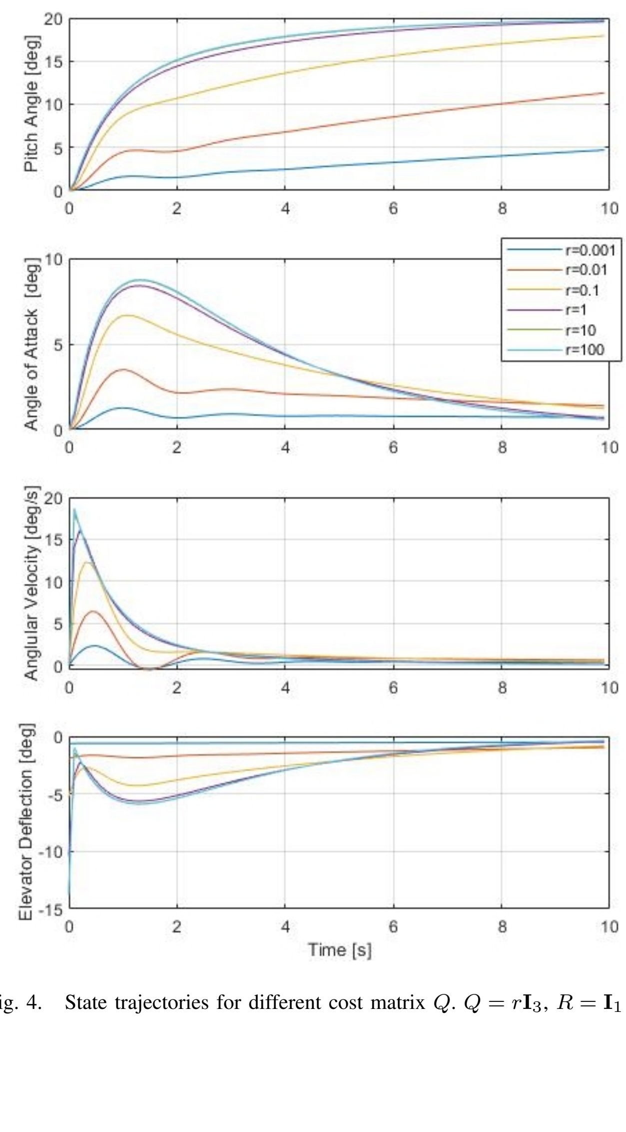

Fig. 4. State trajectories for different cost matrix Q. $Q = r\mathbf{I}_3$, $R = \mathbf{I}_1$

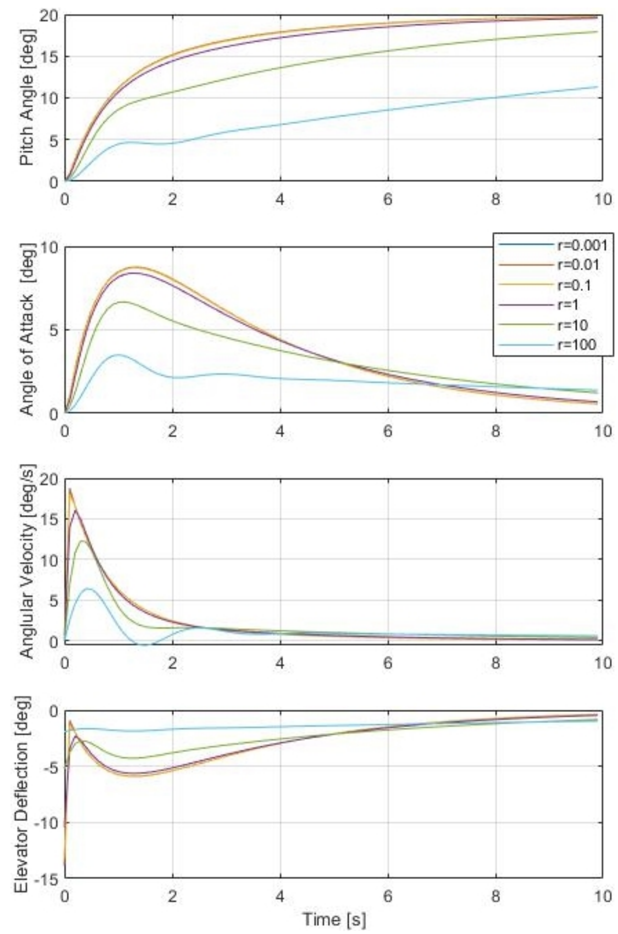

Fig. 5. State trajectories for different cost matrix R. $R = r\mathbf{I}_1$, $Q = \mathbf{I}_3$

In case more desirable tracking properties are required, each individual state should be weighted differently. Since the system tracks a reference on the pitch angle, more weight is put on this state. Furthermore, there is no downside to using more control effort as long as $u \in \mathbb{U}$, therefore a lower cost is put on the control input. Figure 6 shows a faster rise time of the aircraft pitch angle to the reference angle.

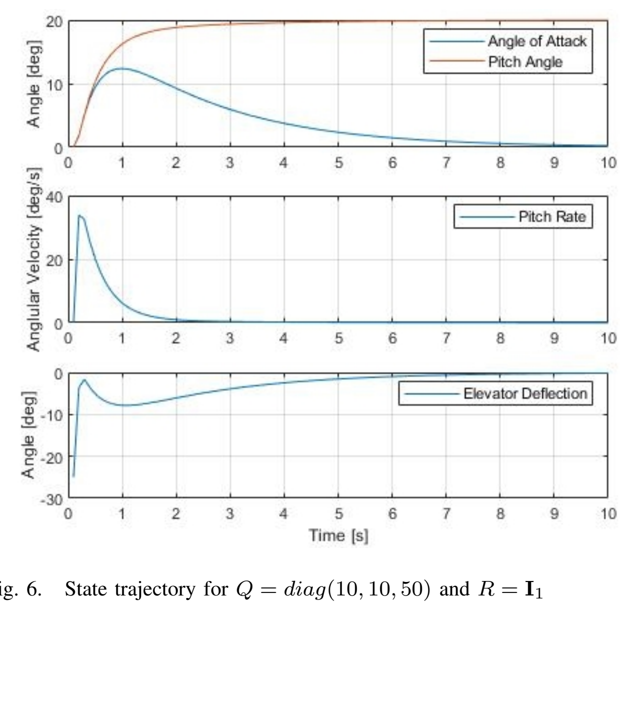

Fig. 6. State trajectory for $Q = diag(10, 10, 50)$ and $R = \mathbf{I}_1$

C. Disturbance rejection

In some circumstances the aircraft might be flying in strong wind conditions or the gyroscopes might be suffering from inaccurate tuning causing them to drift. In these cases, proper disturbance rejection properties of the controller are required. For this, the offset-free reference tracking problem as stated in subsection III-C is used.

The tuning of the general offset-free reference tracking formulation is not straightforward since it requires the choice of the disturbance model matrices (B_d, C_d) and the observer gain matrices (L_x, L_d). However, as stated in [6], any choice of (B_d, C_d) satisfying 15 is equivalent since there exist observer gain matrices (L_x, L_d) which make the augmented system 14 algebraically equivalent to any augmented system based on the same model matrices (A, B, C) with different disturbance matrices (B_d, C_d).

A simpler approach, proposed in [6], for tuning (B_d, C_d, L_x, L_d) is by choosing the gain matrix K such that the non-augmented system characteristic matrix $(A - KCA)$ has its eigenvalues within the unit circle. Then, the matrices (B_d, C_d, L_x, L_d) are calculated as given in 19. For the following simulation, the gain matrix K is tuned by placing the poles of $(A - KCA)$ at $(0.85, 0.90, 0.55)$, for which the observer error converges to zero. Furthermore, in order to have a closer tracking of the pitch angle to the reference, its cost is multiplied by a factor of 10, making the stage cost matrix $Q = diag(10, 10, 500)$.

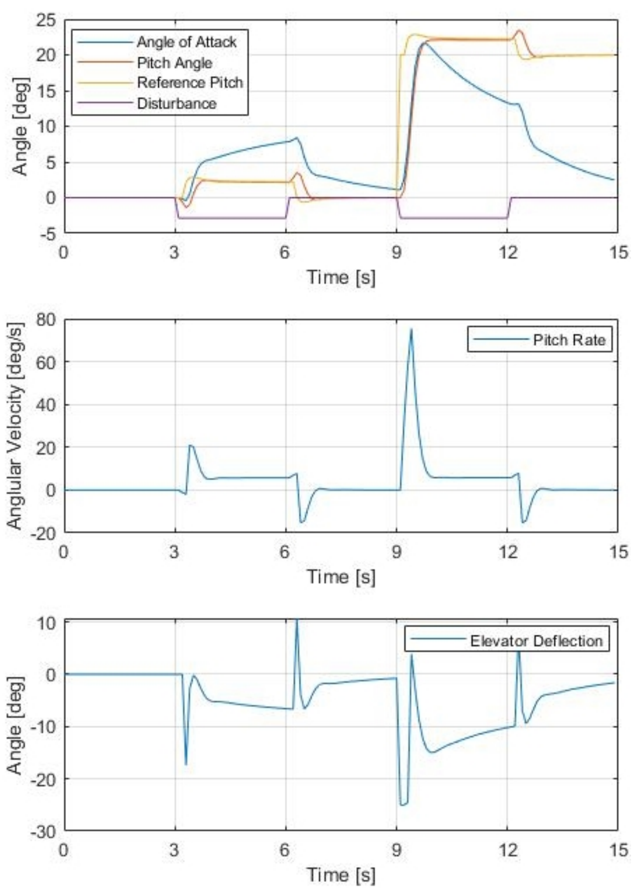

Fig. 7. State trajectories under influence of constant disturbance signals

The system output is simulated to firstly, track a 0^o pitch angle with a disturbance acting from $3s$ till $6s$ and secondly, after $9s$ to track a 20^o pitch angle with a disturbance acting from $9s$ till $12s$. As can be seen from Figure 7, the pitch angle is able to track the reference signal, however with a small delay of $0.4s$ due to the disturbance being introduced.

Figure 8 shows that the measured states; the output of the system, is able to track the reference signal accurately, even in the presence of disturbances. At $3s$, $6s$ and $12s$ an error is observed due to disturbance signals being introduced or removed. These tracking error are about 3.5^o in magnitude and span $0.6s$. Furthermore, at $9s$ there is a delay of $0.3s$ in following the change in reference angle due to a disturbance also being introduce at $9s$.

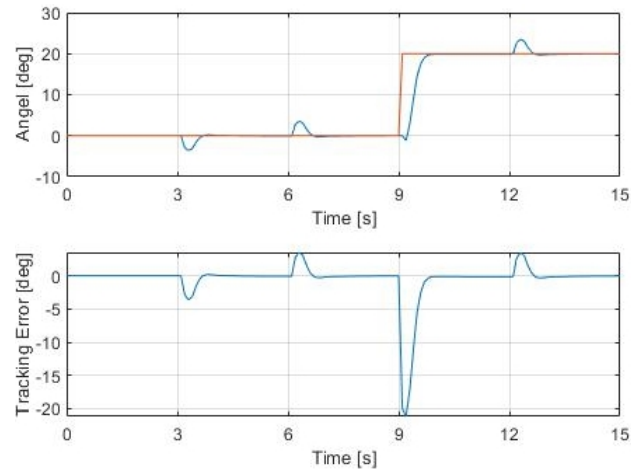

Fig. 8. Measured pitch angle and reference tracking error

Figure 9 shows the error between the predicted state and the true system state. At each point where a disturbance is either introduced or removed, an increase in the prediction error can be observed. After each of these points however, the error quickly converges to zero with minimum amount of oscillation. This is achieved through placing the poles of $(\tilde{A} - \tilde{L}\tilde{C})$ such that the convergence of the observer error has desirable properties.

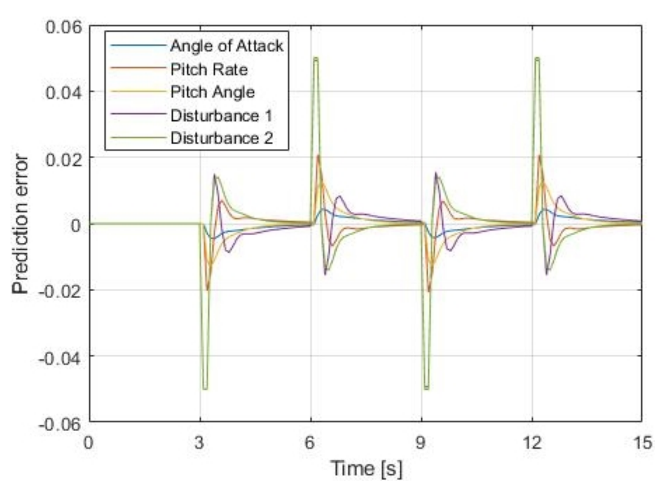

Fig. 9. State observer prediction error

In order to investigate the convergence of the disturbance prediction, the rate at which the error goes towards zero is considered. Let's take the disturbance prediction error to be $\epsilon_k := |\hat{d}_k - \hat{d}_{k-1}|$. Then, for the sum of this error to converge to zero as $k \to \infty$, it is required that $\epsilon_k < \frac{1}{k}$, thus $log(\epsilon_k) < -log(k)$. The curves corresponding to $log(\epsilon_k)$ and $-log(k)$ are plotted in Figure 10. This shows the disturbance error evolution from 3s till 6s. As can be seen, the logarithm of the error does indeed remain below the blue curve. This means that the error converges to zero which, together with *Theorem 2.19* from IV, implies stability of the offset-free tracking problem.

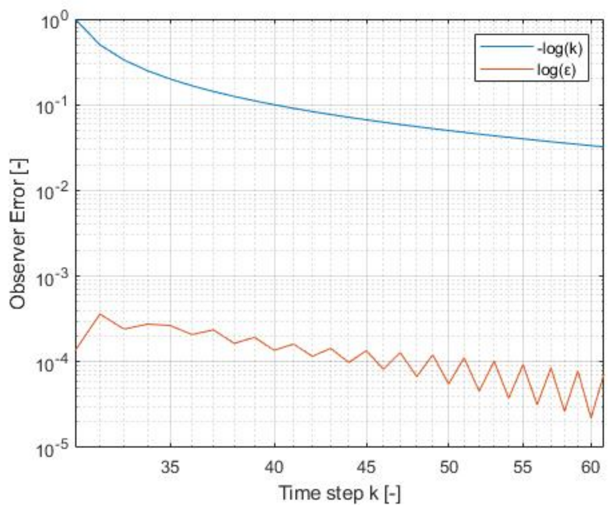

Fig. 10. Convergence of the observer error

VI. Conclusion

Designing a controller for the A-4D fighter aircraft has given a lot of insight in how aircraft dynamics work and how dimensional and dimensionless coefficients allow to quickly linearize any aircraft for which a linear control strategy can be employed. For this work, additional assumptions were made during the linearization, namely that the pitch angle does not affect velocity and that thrust is constant. However, these assumptions are believed to be too restrictive and deviating from a realistic situation. Therefore, it is recommended to extend this work to design a controller for the system with four states $[u, \alpha, q, \theta]$ and two inputs $[\delta_e, \delta_T]$ to closer resemble reality.

However, by reducing the states and inputs to just three states and two inputs allowed for focussing more on controller design. Using this model, it was shown that the system is stable irrespective of the prediction horizon, due to the fact that this simplified linear model of the attitude dynamics is open-loop stable. Furthermore, the system was shown to have desirable disturbance rejection properties through using a Luenberger state observer, predicting the augmented state of the system.

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