Lecture Notes

Lecture 1 - 30 Aug 22

Office hrs → Wed 1000-1200 Room 106
↳ Any TA or prof. Rabinoff hours too

admin

Discussion will have quiz over earlier week's notes

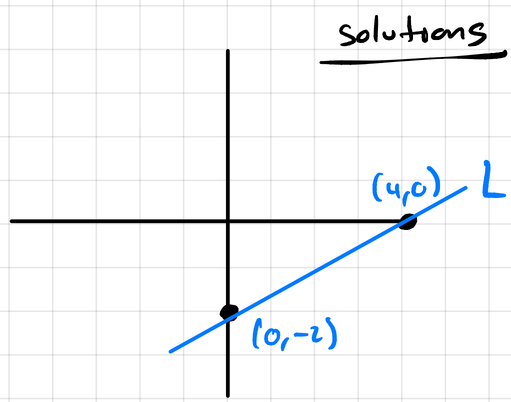
Submit homework on Gradescope every Wed @ 2359

- Lin alg studies linear equations
- Lin eq's for variables $x_1, \dots, x_n$ are $a_1x_1, \dots, a_nx_n$

  where $= a_1x_1 + \dots + a_nx_n = b$

<u>Ex</u> lin eq

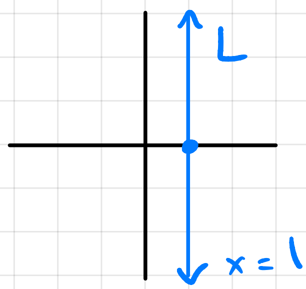
$x - 2y = 4$



any point in L solves lin eq

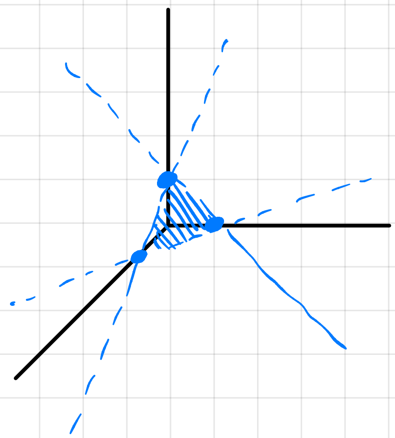
Ex

$x + 0y = 1$



Ex (3-vars)

$x + y + z = 1$



Sol is plane passing through points

Ex (4-var)

$x_1 + x_2 - x_3 - x_4 = 2$

Sol's

(2, 0, 0, 0)

(0, 2, 0, 0)

(0, 0, -2, 0)

(0, 0, 0, -2)

⋮

Usually

- lin eq in 2-var is line in cartesian plane $(\mathbb{R}^2)$
- " " " 3-var is plane in space $(\mathbb{R}^3)$
- " " " 4-var is hyperplane $(\mathbb{R}^4)$

Except

- $0x + 0y = 0 \rightarrow$ all of $\mathbb{R}^2$, not line
- $0x + 0y = 1 \rightarrow$ no sols ("solution set is empty")

Systems of lin eq's

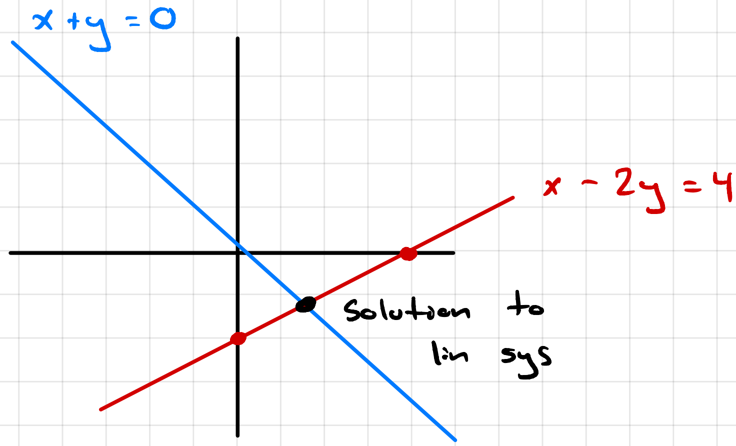
↳ sys of lin eq's in vars x_1 through x_n is a list of lin eq's

↳
$$a_1 x_1 + \cdots + a_{1n} x_{1n}$$
$$a_2 x_2 + \cdots + a_{2n} x_{2n}$$
$$\vdots$$
$$a_m x_m + \cdots + a_{mn} x_{mn}$$

Solution to lin sys is simultaneous sols to all equations

Ex

$$x + y = 0$$
$$x - 2y = 4$$



Calculate sol

$$x + y = 0$$
$$x - 2y = 4$$

$\hookrightarrow$

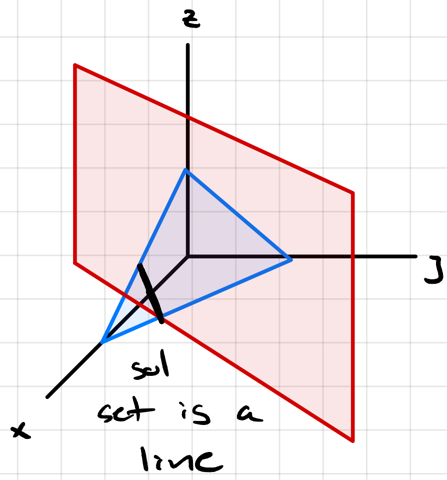
$$x + y = 0$$
$$0x - 3y = 4$$

$\hookrightarrow$

$$0x - 3y = 4$$
$$y = -4/3$$
$$\therefore\ x = 4/3$$

Ex (3-var)

$$x + y + z = 1$$
$$x - y \qquad = 2$$



Usually

- 2 lin eq, 2 vars give point
- " " ", 3 vars gives line
- 3 " ", 3 vars gives point

Except

- 2 lin eq's are parallel → no solutions
- 2 lin eq's are same line → infinitely many sols

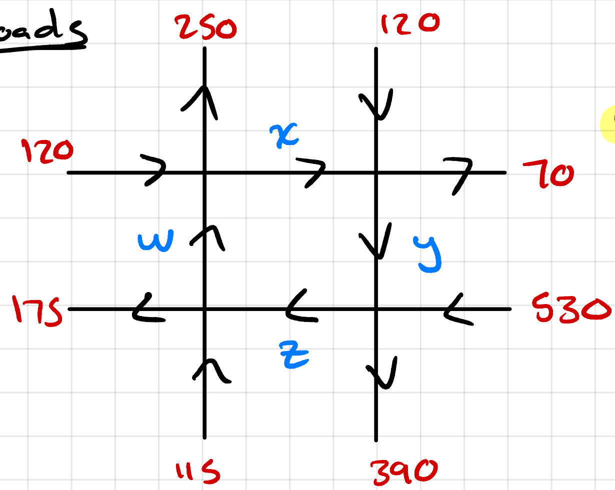
How many sols?

lin sys is inconsistent if there are no sols
↳ consistent w/ one or more sols

if consistent:
↳ 1 sol
↳ line of sols
↳ more
⋮

Ex (networks, pipes, flows)

Roads



red = car traffic in / out

Q: How many cars on inner roads?

4 intersections (in = out)

$\therefore$ $w + 120 = x + 250$

$x + 120 = y + 70$

$y + 530 = z + 390$

$\underbrace{z + 115}_{\text{in}} = \underbrace{w + 175}_{\text{out}}$

$w + 120 = x + 250$

$x + 120 = y + 70$

$y + 530 = z + 390$

$z + 115 = w + 175$

$\hookrightarrow$ $x - w = -130$

$y - x = 50$

$z - y = 140$

$w - z = -60$

$$\hookrightarrow \left.\begin{aligned} x + 0y + 0z - w &= -130 \\ -x + y + 0z + 0w &= 50 \\ 0x - y + z + 0w &= 140 \end{aligned}\right\} \text{(sum) } z - w = 60$$

$$0x + 0y - z + w = -60$$

which makes eq 4 redundant, meaning 3 eq's and 4 vars

$\therefore$

A: ∞ many sols (cars circle in center)

Q: Can we change nums to make "no solutions"?

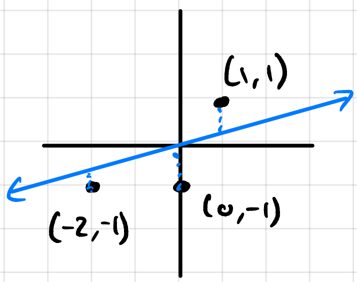
A: yes, as long as 4 nums don't cancel

Note:

↳ Fewer eq's than vars = ∞ many sols

Approximate solution:

Ex (data fitting):



Q: how to fit a line through points?

↳ "least squares": find line $y = ax + b$ that minimizes vertical errors

TBD

fit line $\Rightarrow$ $y = Ax + B$

System

$$y_1 \quad 1 = \overset{x_1}{1}A + B$$

$$y_2 \quad -1 = \overset{x_2}{0}A + B$$

$$y_3 \quad -1 = \overset{x_3}{-2}A + B$$

Note: 3 eq, 2 vars is inconsistent since no line passes thru all three points

Recurrence Relations : algos that produce sequence of nums

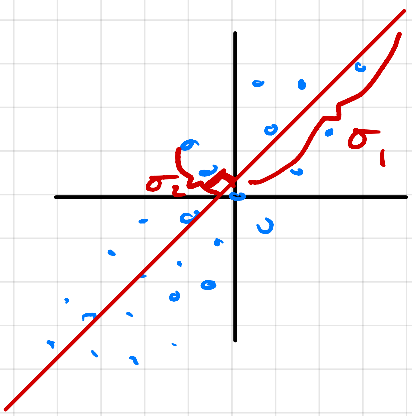
↳ Fibonacci Sequence: 1, 1, 2, 3, 5, 8, ...

↳ $x_n = x_{n-1} + x_{n-2}$

Q: how fast is seq growing?

A: $x_n \approx \left(\frac{1+\sqrt{5}}{2}\right)^n$ TBD

Principal Component Analysis TBD



spread along best fit line and perpendicularly

Online Lecture

Scalars are real numbers

$$c \in \mathbb{R}$$

↑ subset of

Vectors are finite ordered lists of nums

Vectors' **size** is the length of the list

Nums in list are **coordinates**

$$v \in \mathbb{R}^n \leftarrow \text{"set of all vectors of size } n\text{"}$$

Ex

$$v = \begin{bmatrix} 2 \\ -\pi \\ e^{\sqrt{3}} \end{bmatrix} \in \mathbb{R}^3$$

$$w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Note: we write vectors as columns

↳ aka **column vectors**

NB: often decorated

↳ $\vec{v}$ $\mathbf{v}$

<u>Ex</u> (important)

Unit coord vecs in $\mathbb{R}^n$ are:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

<u>Ex</u>

in $\mathbb{R}^3$, unit coord vecs:

$$e_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

NB: zero vector

↳ $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$

Vectors are equal if they have same size and coords

$$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vector Algebra

Scaler Multi (returns vector)

$$c \in \mathbb{R} \qquad v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$\hookrightarrow \quad c \cdot v = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} \in \mathbb{R}^n$$

Ex

$$2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Vector Add / Sub (returns vector)

$$v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad w = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

$$\hookrightarrow \quad v \pm w = \begin{bmatrix} x_1 \pm y_1 \\ \vdots \\ x_n \pm y_n \end{bmatrix} \in \mathbb{R}^n$$

NB can only add/sub same-size vecs

Ex

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 + \pi \\ 2 + e \\ 3 + \sqrt{2} \end{bmatrix}$$

Dot Product (aka inner product)

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

$$\hookrightarrow \quad u \cdot v = x_1 y_1 + \cdots + x_n y_n \in \mathbb{R}$$

NB: Dot prods are (vec)(vec) = scalar

Ex (dot prod)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 12$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14$$

NB if $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ then $v \cdot v \geq 0$

this is $0 \iff v = 0$

"if and only if"

Rules for vec algebra

1. $c(u \pm v) = cu \pm cv$

2. $u \cdot (cv) = c(u \cdot v) = cu \cdot v$

3. $u \cdot v = v \cdot u$

4. $u \cdot (v \pm w) = u \cdot v \pm u \cdot w$

Ex

$$(u+v)\cdot(w+x) = u\cdot(w+x) + v\cdot(w+x)$$

$$= u\cdot w + u\cdot x + v\cdot w + v\cdot x$$

A linear combination (LC) of vecs $v_1 \cdots v_n \in \mathbb{R}^n$ w/ weights $x_1, \cdots, x_n \in \mathbb{R}$ is the vec $x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^n$

Ex

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

The coordinates of v are the weights when I express v as a LC of unit coord vecs

Matrix Algebra

Matrix is a box holding 2D grid of nums

Size of matrix is (# rows) x (# cols)

↳ usually (not always) **m = # rows**
n = # cols

∴ A is an $m \times n$ matrix

The (i,j) entry is i^{th} row, j^{th} column

Ex

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Ex

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

size = 3×2

$(3,2)$ entry $= 6$

the **diagonal entries** of A are the (i,j) entries where $i = j$

Mat is diagonal if all diag entries are the only non-0's

Mat is square if $m=n$ (same # rows + columns)

<u>Ex</u> ($n\times n$ identity matrix)

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix}$$

square, diagonal

<u>Ex</u> ($m\times n$ zero matrix)

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

diagonal

Matrix Algebra

Scaler multi, add, sub:

↳ component-wise like vectors

$$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

NB
Can only add / sub mats of same size

NB
Vec of size n is $n \times 1$ matrix

A row vector is a matrix w/ one row

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Mat x Vec = Vec

① By columns

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$Ax = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^n$$

NB

coords of x are weights of cols of A in a LC

↳ $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is LC of the cols of A w/ weights $x_1, \dots, x_n$

② **By rows**

$$A = \begin{pmatrix} — w_1 — \\ \vdots \\ — w_m — \end{pmatrix} \text{ then}$$

$$Ax = \begin{pmatrix} — w_1 — \\ \vdots \\ — w_m — \end{pmatrix} = \begin{pmatrix} w_1 \cdot x_1 \\ \vdots \\ w_m \cdot x_m \end{pmatrix}$$

↳ the i^{th} coord of Ax is $(\text{row } i) \cdot x$

Ex

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \overset{①}{=} 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \overset{②}{=} \begin{pmatrix} 1(2) + 4(-1) \\ 2(2) + 5(-1) \\ 3(2) + 6(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

NB only works if size of x = # cols of A

- A is $m \times n \Rightarrow x \in \mathbb{R}^n \quad Ax \in \mathbb{R}^m$

- $\underset{(m\times n)}{A} \cdot \underset{(n\times 1)}{x} \Rightarrow \underset{(m\times 1)}{Ax}$

Ex

$$Ae_i = 0\cdot(\text{col } 1) + \dots + 1\cdot(\text{col } i) + \dots + 0(\text{col } n)$$

unit coord vector

$\therefore$ Ae_i = the i^{th} col of A

Ex

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 0 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

e_2

Ex

$$I_n x = \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + \cdots + x_n e_n = x$$

$$\boxed{I_n x = x}$$

Matrix x Matrix = Matrix Col form

A is $m \times n$

B is $n \times p$ w/ cols $u_1, \dots, u_p \in \mathbb{R}^n$

Product AB is $m \times p$ matrix

w/ cols $Au_1, \dots, Au_p \in \mathbb{R}^m$

$$AB = A \begin{pmatrix} | & & | \\ u_1 & \cdots & u_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Au_1 & \cdots & Au_p \\ | & & | \end{pmatrix}$$

NB only works if # cols A = # rows B

$$\therefore \underset{m \times n}{A} \cdot \underset{n \times p}{B} \Rightarrow \underset{m \times p}{AB}$$

Mat x Mat row wise

$$\begin{pmatrix} — w_1 — \\ \vdots \\ — w_m — \end{pmatrix} \begin{pmatrix} | & & | \\ u_1 & \cdots & u_p \\ | & & | \end{pmatrix} = \begin{pmatrix} w_1 \cdot u_1 & \cdots & w_1 \cdot u_p \\ w_2 \cdot u_1 & \cdots & w_2 \cdot u_p \\ \vdots & \ddots & \vdots \\ w_m \cdot u_1 & \cdots & w_m \cdot u_p \end{pmatrix}$$

So, the (i,j) entry of AB

= (row i of A) $\cdot$ (row j of B)

Ex !

compute $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$

$2 \times 3 \qquad 3 \times 2 \longrightarrow 2 \times 2$

Col-wise

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$$

$$\hookrightarrow \left[1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ -4 \end{pmatrix} \right]$$

$$\hookrightarrow \boxed{\begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}}$$

Row wise

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$$

* each row's dot poduct w/ each col of second matrix, mapped to each row of final matrix *

$$\hookrightarrow \begin{pmatrix} 1(1) + 2(2) + 3(4) & 1(3) + 2(1) + 3(-1) \\ -1(1) + 2(2) + -4(4) & -1(3) + 2(1) + -4(-1) \end{pmatrix}$$

$$\hookrightarrow \boxed{\begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}}$$

Ex

$$AI_n = A\begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ae_1 & \cdots & Ae_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ (\text{col } 1) & \cdots & (\text{col } n) \\ | & & | \end{pmatrix} = A$$

$$\therefore \quad I_m A = A = A I_n$$

An outer product is a row vector times a column vector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \end{pmatrix}$$

Ex

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 4 & 5 \end{pmatrix} = \left[3\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad 4\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{pmatrix}$$

Mat × Mat : outer product form

$A : m \times n$ w/ cols $v_1, \dots, v_n \in \mathbb{R}^m$

$B : n \times p$ w/ rows $w_1, \dots, w_n \in \mathbb{R}^p$

$AB : m \times p$ mat

$$\hookrightarrow AB = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} — & w_1 & — \\ & \vdots & \\ — & w_n & — \end{pmatrix}$$

$$= v_1 w_1^T + \cdots + v_n w_n^T$$

(T means transposed to row vector)

Ex

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \begin{pmatrix} 4 & -1 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 12 & -3 \\ -16 & 4 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}$$

Transpose

↳ A is $m \times n$ matrix, it's transpose is the $n \times m$ matrix A^T whose rows are the colums of A and vice versa

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Ex

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad \therefore \quad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

NB if $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ then

$$v^T w = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = v \cdot w = \text{scalar}$$

$\therefore$ $v^T w = v \cdot w$ for $v, w \in \mathbb{R}^n$

Compare

$v^T w$ = inner product (scalar)

$v w^T$ = outer product (matrix)

A mat S is **symmetric** if $S = S^T$

↳ always square

Ex

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \leftarrow \text{square}$$

Ex A is $m \times n$ w/ cols $v_1, \dots, v_n \in \mathbb{R}^n$

A^T is $n \times m$

$\therefore$ $A^T A$ is $(n \times m) \cdot (m \times n) = n \times n$

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad A^T = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{pmatrix}$$

$$A^T A = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ v_2 \cdot v_1 & \cdots & v_2 \cdot v_n \\ \vdots & \ddots & \vdots \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix} = \text{symmetric}$$

$\therefore$ the (i,j) entry of A^TA is $v_i \cdot v_j$

and A^TA is the mat of column dot products, and is symmetric

Rules for mat algebra

A, B, C: matrices. Assume compatible sizes.

① $A(BC) = (AB)C$

↳ ABC makes sense

↳ If $C = v$ is a vector, $A(Bv) = (AB)v$

② $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

③ $I_m A = A = A I_n$

④ $(A^T)^T = A$

⑤ $(A \pm B)^T = A^T \pm B^T$

⑥ $(AB)^T \neq A^T B^T$

$(AB)^T = B^T A^T$

Ex

$$(A^TA)^T = A^T(A^T)^T = A^TA \leftarrow \text{symmetrical}$$

Ex

if A is square, AA makes sense, as does AAA, $AAAA$, etc.

If A is square, then its n^{th} power is

$$A^n = AA \cdots A$$

Caveats

① No communtativity $\therefore AB \neq BA$

② No cancellation $\therefore A \neq O$

Then $AB = AC$

$\hookrightarrow B \neq C$

Lecture 2 - 01SEP22

Solving systems using elimination

Ex $\mathbb{R}^3$

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 6 \\
2x_1 - 3x_2 + 2x_3 &= 14 \\
3x_1 + x_2 - x_3 &= -2
\end{aligned}
$$

opt a

- solve one for x_1
- sub into 2^{nd} & 3^{rd} eq

elimination

opt b

- add a multiple of an eq to another

$R_2 -= 2R_1$

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 6 \\
0x_1 - 7x_2 - 4x_3 &= 2 \\
3x_1 + x_2 - x_3 &= -2
\end{aligned}
$$

$R_3 -= 3R_1$

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 6 \\
0x_1 - 7x_2 - 4x_3 &= 2 \\
0x_1 - 5x_2 - 10x_3 &= -20
\end{aligned}
$$

$R_3 -= \frac{5}{7} R_2$

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 6 \\
0x_1 - 7x_2 - 4x_3 &= 2 \\
0x_1 + 0x_2 - \frac{50}{7}x_3 &= -\frac{150}{7}
\end{aligned}
$$

cont'd

Elimination stage complete

$$-\frac{50}{7}x_3 = -\frac{150}{7} \qquad \therefore \quad x_3 = 3$$

sub

$$x_1 + 2x_2 + 3(3) = 6$$

$$0x_1 - 7x_2 - 4(3) = 2 \quad \dashrightarrow \quad -7x_2 - 12 = 2$$

$$-7x_2 = 14$$

$$\therefore \quad x_2 = -2$$

sub

$$x_1 + 2(-2) + 3(3) = 6$$

$$x_1 + 5 = 6$$

$$x_1 = 1$$

$$\therefore \quad \boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}}$$

Ex

$$4x_2 + 3x_3 = 2$$
$$x_1 + x_2 - x_3 = 3$$
$$2x_1 - 3x_2 - 6x_3 = -3$$

NB rows can be swapped to any order

$R_3 -= 2R_1$ and $R_2 \leftrightarrow R_1$

$$x_1 + x_2 - x_3 = 3$$
$$4x_2 + 3x_3 = 2$$
$$0x_1 - 5x_2 - 4x_3 = -9$$

$R_3 += \frac{5}{4} R_2$

$$x_1 + x_2 - x_3 = 3$$
$$4x_2 + 3x_3 = 2$$
$$0x_2 - \frac{1}{4}x_3 = -\frac{13}{2}$$

$\therefore x_3 = 26$

$$4x_2 + 3(26) = 2$$
$$\therefore x_2 = -19$$

$$x_1 - 19 - 26 = 3$$
$$x_1 = 48$$

$\therefore$
$$x_1 = 48$$
$$x_2 = -19$$
$$x_3 = 26$$

Elementary Row Operations

↳ ways of changing lin sys to get a new one

① row addition / subtraction

$$R_n \mathrel{+}= cR_m$$

② row swapping

$$R_n \longleftrightarrow R_m$$

③ row scaling / multiplication (cannot scale by 0)

$$10x_1 + 5x_2 = 10$$

$$\times \tfrac{1}{5} \hookrightarrow \; 2x_1 + x_2 = 2$$

NB

$$\begin{bmatrix} \text{lin} \\ \text{sys} \end{bmatrix} \xrightarrow[\text{operation}]{\text{elem row}} \begin{bmatrix} \text{new} \\ \text{lin} \\ \text{sys} \end{bmatrix}$$

any solution to original sys is also a solution to new sys, and vice-versa.

Q: why are there no new solu's to new sys?

A: All elem row ops are reversible

∴ elem row ops do not change solution sets

Q: why are row ops reversible?

A:

① row addition

$$[\text{old}] \xrightarrow{R_1 += 2R_2} [\text{new}] \xrightarrow{R_1 -= 2R_2} [\text{old}]$$

② row swap

$$[\text{old}] \xrightarrow{R_1 \leftrightarrow R_2} [\text{new}] \xrightarrow{R_2 \leftrightarrow R_1} [\text{old}]$$

③ row scaling

FILL Here!

Elimination uses elem row ops to simplify systems to ones w/ same solution set

Augmented Matrices

Ex

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 2 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 2 \end{array}\right]$$

Aug mat

$R_2 -= 2R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -4 \end{array}\right]$$

Matrix form of lin sys

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \longrightarrow \begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 2 \end{aligned}$$

matrix form

General Case

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$ matrix form

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1m} & \cdots & a_{mn} & b_m \end{array}\right)$$ augmented form

! **Ex**

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{array}\right]$$

$R_2 -= 4R_1$

$R_3 -= 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & -6 & -12 & -8 \end{array}\right]$$

cont'd

$R_3 -= 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Done w/ elim

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0x_3 &= 0 \end{aligned}$$

can set x_3 to anything and solve using back-sub

$$\therefore \boxed{(x_1, x_2, x_3) = \left(\tfrac{8}{3} - 7x_3,\ \tfrac{4}{3} - 2x_3,\ x_3\right)}$$

Ex (slightly modified)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right) \xrightarrow[\text{then } R_3 -= 2R_2]{R_2 -= 4R_1,\ R_3 -= 7R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$0 = 1$

$\therefore$ **no solution**

Row Echelon Form (REF)

↳ how to tell when done w/ elimination

↳ look thus:

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$*$ = non-zero

REF rules

① any zero-rows are below all rows w/ non-zero values

② circle first non-0 entries in each row
↳ these are pivots

③ pivots go down and to the right

Ex (are REF?)

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2 & 0 \end{array}\right)$$ yes

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{array}\right)$$ no

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ yes

Aug mats in REF

↳ see how many solutions lin eq's have

Ex

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

inconsistent because there is a pivot in augmentation column

Ex (consistent)

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & \boxed{*} & * \\ 0 & 0 & 0 & 0 \end{array}\right)$$

set x_3 to any value

NB

↳ number of non-aug columns w/o a pivot is the number of free variables

↳ number of non-aug columns w/ pivots is the rank of the aug matrix

Lecture 3 - 06SEP22

Recap: elem. row ops help solve lin sys until arrival at REF

Elim algorithm (Gaussian Elimination)

↳ input = any mat. A

output = row-equivalent mat. in REF

Gaussian Elim:

① look in C_1 of A, circle pivot

or

do a row swap to move 1st pivot to R_1

↳ Ex

$$\begin{pmatrix} 0 & 4 & 3 & 3 \\ 1 & 1 & -1 & 3 \\ 5 & -3 & -6 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 5 & -3 & -6 & -6 \end{pmatrix}$$

② use row add. to elim all entries below 1st pivot

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 5 & -3 & -6 & -6 \end{pmatrix} \xrightarrow{R_3 -= 5R_1} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{pmatrix}$$

③ Repeat steps 1 and 2 on submatrix until REF.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 3 & 3 \\ -8 & -1 & -21 \end{pmatrix}$$

$$\xrightarrow{R_2 += 2R_1} \begin{pmatrix} 4 & 3 & 3 \\ 0 & 5 & -15 \end{pmatrix}$$

NB a mat. A can have many differen REF's. This algo gives one.

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

NB when solving lin. sys, we do back-substitution

<u>Back-sub w/ aug mats</u>

$$\begin{aligned} x + 2y &= 3 \\ 2y &= 4 \end{aligned} \iff \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2 & 4 \end{array}\right) \xrightarrow{\text{row scale}} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

$$\Downarrow R_1 -= 2R_2$$

row-reduced echelon form (RREF) $\boxed{\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)}$

RREF

① mat. is in REF
② all pivots = 1
③ all entries above pivots = 0

NB when lin sys is in RREF, there is no more work to be done

Ex

$$\underset{\text{REF}}{\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)} \xrightarrow{R_1 -= 2R_2} \underset{\text{RREF}}{\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)}$$

* ∞ many sols by setting y to anything

Algo (Jordan Substitution)

↳ input = mat in REF
output = mat in RREF

① scale all rows to make pivots = 1
② use lowest row to eliminate rows above it
③ repeat 1 and 2 w/ next lowest rows

NB Given a matrix A, there is exactly <u>one</u> RREF mat. row-equivalent to A

$$A \Rightarrow \text{RREF}(A)$$

Computational Complexity of Algos

↳ measured by counting number of floating point operations (flops) to complete

Comp. Complex. of Gaussian Elim

$$n \underset{n}{\begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}} \rightarrow$$

takes $n-1$ multiplications and $n-1$ additions to do row op to second row

$$\therefore 2(n-1)^2 \text{ total flops}$$

↳ then the same for second submat

$$\therefore 2(n-2)^2 \text{ flops}$$

↳ continuing...

$$\text{total} = \underbrace{2(n-1)^2}_{\text{base of pyramid}} + \underbrace{2(n-2)^2}_{\text{next layer}} + \cdots + \underbrace{2(1)^2}_{\text{top}}$$

NB this is a pyramidal number which is $\approx \underbrace{\frac{2n^3}{3}}_{\text{area of pyramid}}$ flops

Comp. Complex. of Jordan Sub.

C_n

$n-1$ multiplications
$n-1$ adds/subtracts $\quad \therefore 2(n-1)$ flops

Total

$$2(n-1) + 2(n-2) + \cdots + 2 = \frac{2[n(n-1)]}{2}$$

$$\approx \boxed{n^2 \text{ flops}}$$

Therefore: Gaussian Elim is <u>much slower</u> than Jordan Sub

Row Operation Matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 -= R_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Goal: find matrix E where multiplying EA gives result of row op

A E

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

E is a row op mat. correlating to $R_1 -= R_2$

Row scaling row op mat:

$$\underset{E}{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \text{scale } R_1 \text{ by } 2$$

Row swapping row op mat:

$$\underset{E}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Q: which row op mat corresponds to nothing?

A: nxn identity matrix (I_n) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

NB

Elem row ops are reversible

$\therefore$ For any row op mat. E_1, we can find another mat E_2 that will undo E_1.

↳ $\boxed{E_2 E_1 = I_n}$

Invertible matrix

↳ An nxn matrix A is invertible if another nxn matrix B exists such that $BA = I_n$

Facts about invertible mats:

"there exists"

① If $\exists$ B such that $BA = I_n$, then $\exists$ C such that $AC = I_n$

② There is exactly one B such that $BA = I_n$

↳ call that A^{-1}, or "A inverse"

③ If A^{-1} inverts A to the left, it also does on the right

↳ $A^{-1}A = I_n$ and $AA^{-1} = I_n$

$\therefore$ If A is invertible, $A^{-1}A = AA^{-1} = I_n$

NB we only talk about invertible matrices for square matrices

Inverse of 1x1 matrix

$$(a)^{-1} = (a^{-1})$$

Inverse of 2x2 matrix   ! memorize

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\therefore$ a 2x2 matrix is invertible only when $ad-bc \neq 0$

Ex

ⓐ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ not invertible; $ad-bc=0$

ⓑ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ invertible; $\underset{A}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\underset{A^{-1}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Facts

① if A and B are invertible, then AB is invertible

↳ $B^{-1}A^{-1}(AB) = I_n$

$\therefore$ why $(AB)^{-1} = B^{-1}A^{-1}$

Lecture 4 – 08SEP22

Inverse Mats

↳ a square mat A is invertible when A^{-1} exists

such that $\boxed{\text{st}}$ $AA^{-1} = A^{-1}A = I_n$

Larger than 2x2 test for invertibility

$\boxed{\text{TFAE}}$

the following are equivalent
(if one true, all true; if one false, all false)

① An $n \times n$ mat is invertible

② REF of A has a pivot in every col

③ $\text{RREF}(A) = I_n$

Computing the inverse

① Augment w/ I_n:

$$n\left(\underset{n}{A} \,\middle|\, I_n\right)$$

② Compute $\text{RREF}(A \mid I_n)$

$$\text{rref}(A \mid I_n) = \left(\underbrace{\square}_{I_n} \,\middle|\, \underbrace{\square}_{A^{-1}}\right)$$

Ex

Compute A^{-1} $\quad A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1}$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 -= \frac{1}{2}R_1} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right) \xrightarrow{2R_2} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 -= 3R_2} \left(\begin{array}{cc|cc} 2 & 0 & 4 & -6 \\ 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\overset{I_n}{\boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}} \middle| \overset{A^{-1}}{\boxed{\begin{array}{cc} 2 & -3 \\ -1 & 2 \end{array}}} \right)$$

(answer)

Inv. mats + lin. sys.

↳ when $n \times n$ mat A is inv'ble, then $Ax = B$ always has exactly one solution for any B.

NB

- Suppose v solves $Ax = B$;
- Multiply both sides by A^{-1} on left:

$$A^{-1}Av = A^{-1}B$$
$$\therefore v = A^{-1}B$$

- so, $A^{-1}B$ is unique solution to $Ax = B$

Ex

Solve:

$$\begin{aligned} 2x + 3y &= b_1 \\ x + 2y &= b_2 \end{aligned} \implies \underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{B}$$

$$A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \qquad \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

computed in prev. example

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2b_1 - 3b_2 \\ -b_1 + 2b_2 \end{pmatrix}$$

Warning: Do not calculate A^{-1} to solve for many B's if:

- ↳ Difficult to find A^{-1}
- ↳ if A is "sparse" (many 0 entries)

NB

Each column of A^{-1} is the solution to $Ax = \begin{pmatrix} \text{corresponding} \\ \text{col of } I_n \end{pmatrix}$

Q: why do tests for inv'ability and algo for inversion true?

Test: $\text{rref}(A) = I_n$

A: to compute rref, we use elem row ops mats

$$A \to E_1 A \to E_2(E_1 A) \implies \underbrace{E_n \cdot E_{n-1} \cdot \ldots \cdot E_1 A}_{\text{rref}(A)} = I_n$$

and if rref(A) is many E's $\cdot A$ and $A^{-1}A$, then all E's multiplied together is A^{-1}

Test (algo)

$$(A \mid I_n) \xRightarrow[]{\text{same } E\text{'s}} (\underbrace{E_k \cdots E_1 A}_{I_n} \mid E_k \cdots E_1)$$

A: $\therefore$ since $E_k \cdots E_1 A = I_n$ and $A^{-1}A = I_n$, then $E_k \cdots E_1$ is A^{-1}, which is what algo claims

Exercise

If ref of A has a pivot in each col, then $\text{rref}(A) = I_n$

Prove

Upper and Lower Triangular Mats (only for square mats)

(U_n) Upper triangular if every entry below the diagonal is 0

$$\begin{pmatrix} * & & \\ 0 & \ddots & \\ \vdots & \ddots & \\ 0 & \cdots\ 0 & * \end{pmatrix}$$

(L_n) Lower triangular if every entry above the diagonal is 0

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ * & \cdots & & * \end{pmatrix}$$

Properties of tri. mats.

↳ $L_1 L_2$ is still lower tri
↳ $U_1 U_2$ is still upper tri

↳ any U is invertible if all entries on the diagonal is non-zero

↳ any L is invertible if L^T also is (diags all non-zero)

NB

$$(A^T)^{-1} = (A^{-1})^T$$

LU factorization

↳ if $m \times n$ mat. A can be put into REF by Gaussian elim w/o row swaps, then $A = LU$ where L is lower tri. w/ 1's on diag and U is REF

$L = m \times m$, $U = m \times n$

Ex

given

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{pmatrix} = A \qquad \text{solve } Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} = L \qquad \therefore LUx = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix} = U$$

First $Lc = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ where $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

Second

$$Ux = c \implies \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$x = \begin{pmatrix} 0 \\ 1 \\ -3/4 \end{pmatrix}$$

$$\therefore \quad Ax = LUx = B$$

$$\begin{aligned} Ux &= c \\ Lc &= B \end{aligned} \implies L(Ux) = B \leftarrow \text{proof}$$

NB

- LU factorization is fast but calculating L and U is slow
- for solving for many B's, finding L and U is still faster than Gauss

Algo for computing $A = LU$

- Do Gauss, but track all row ops done
  - Gauss outputs U

Gauss

- $U = E_k \cdots E_1 A$
- we assumed no row swaps, so each E_i corresponding to row-additions

$$R_i \mathrel{+}= cR_j \quad \text{where} \quad i > j$$

$\therefore$ Each E_i is low-tri w/ 1's on the diag

$$\therefore \quad U = \underbrace{(E_k \cdots E_1)}_{\text{lower-tri}} A \quad \text{and} \quad L_1 L_2 \text{ is lower tri}$$

- $A = LU$ since L^{-1} is always lower-tri

2-column method to find $A = LU$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

2 columns

NB these steps must be subtraction

L	U	
	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	$R_2 -= \boxed{4}R_1$ $R_3 -= \boxed{7}R_1$
$\begin{pmatrix} 1 & & \\ \boxed{4} & & \\ \boxed{7} & & \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$	$R_3 -= \boxed{2}R_2$
$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & \boxed{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$	

Partial Pivoting

↳ only row interchanges

↳ full pivoting is switching rows and columns

Maximal Partial Pivoting

① set up 3-column table; P | L | U

② change P at every step to make the upper pivot be the largest possible absolute value

③ The final resulting matrices will be those where $PA = LU$

Ex

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{pmatrix}.$$

	P	L	U
Start	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} (?) & (?) & (?) \\ (?) & (?) & (?) \\ (?) & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{pmatrix}$
Choose pivot for C_1 ($R_1 \longleftrightarrow R_2$)	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} (?) & (?) & (?) \\ (?) & (?) & (?) \\ (?) & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} -10 & -20 & -30 \\ 1 & 1 & 1 \\ 5 & 15 & 10 \end{pmatrix}$
Eliminate in C_1	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & (?) & (?) \\ -0.1 & (?) & (?) \\ -0.5 & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} -10 & -20 & -30 \\ 0 & -1 & -2 \\ 0 & 5 & -5 \end{pmatrix}$
Choose pivot for C_2 ($R_2 \longleftrightarrow R_3$)	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & (?) & (?) \\ -0.5 & (?) & (?) \\ -0.1 & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & -1 & -2 \end{pmatrix}$
Eliminate in C_2	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & (?) \\ -0.5 & 1 & (?) \\ -0.1 & -0.2 & (?) \end{pmatrix}$	$\begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{pmatrix}$
Eliminate in C_3	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.1 & -0.2 & 1 \end{pmatrix}$	$\begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{pmatrix}$

$-\frac{1}{2}$

P L U

Lecture 5 - 13SEP22

Parametric Form + lin. sys. w/ ∞ many sol's

Ex

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

RREF

$$\begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned} \Leftarrow \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

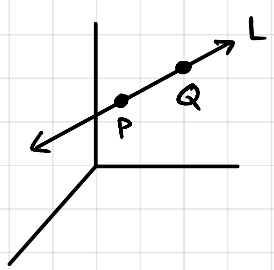
z is a free variable

$$\therefore \begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \end{aligned}$$

$\Rightarrow$ when you plug a z, you get solutions for x and y

z is the parameter here

Parametric Eq's of lines



$\vec{v} = \overrightarrow{PQ}$

any point on L is obtained by starting at P and moving some amount in $\vec{v}$ direction

Vector form of Param Eq

From prev Ex.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix} \xrightarrow{\text{rewrite}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}}_{\text{direction vector}} z + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\text{particular point on a line}}$$

General Method for Parametric Form

Ex

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

no pivot = free variable

Ex

$$x + y + z = 1 \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right]$$

y + z are free vars

Param form

$$x = 1 - y - z$$
$$y = y$$
$$z = z$$

vector form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

← parametric plane

P			L			U		
0	1	0	1	0	0	-10	-20	-30
0	0	1	1/10	1	0	0	5	-5
1	0	0	-1/2	3/5	1	0	3	4

Param Eq Algo

input: $A\vec{x} = \vec{b}$

method:

1. aug mat form
2. RREF of aug mat
3. find free variables w/ pivots
4. convert back to eq's
5. move free vars to right side of eq
6. add tautological eq $x_i = x_i$ for free vars
7. group eq's into vector form
8. write sol'n as

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underset{\text{free}}{x_i} \begin{pmatrix} \vdots \end{pmatrix} + \underset{\text{other free}}{x_j} \begin{pmatrix} \vdots \end{pmatrix} + \cdots + \begin{pmatrix} \vdots \end{pmatrix}$$

constants

def. The dimension of the solution set of consistent lin sys is the number of free variables

↳ 1 var = line
↳ 2 var = plane

∇ Ex

$$\begin{aligned} x + 2y + 2z + w &= 1 \\ 2x + 4y + z - w &= -1 \end{aligned} \Longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 1 & -1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & -3 & -3 & -3 \end{array}\right)$$

free

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

RREF

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \longrightarrow \begin{aligned} x + 2y \quad - w &= -1 \\ z + w &= 1 \end{aligned}$$

param eq's

$$\begin{aligned} x &= -2y + w - 1 \\ z &= -w + 1 \end{aligned}$$

$$\begin{aligned} x &= -2y + w - 1 \\ y &= y \\ z &= -w + 1 \\ w &= w \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

final parametric vector form

Q: what is the dimension of solution set?

A: 2-dimension

Geometry of vectors

- vector in $\mathbb{R}^n$ is a list of n numbers $x_1 \dots x_n$
  - in $\mathbb{R}^2$, $\vec{v}$ is (x, y)
    - (x,y) is a point
    - " " an arrow from origin to (x,y)
  - difference, or "displacement" of two points is the arrow between them

Vector Addition

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

$\vec{v} + \vec{w}$ represents displacement by moving along v and then by $\vec{w}$

Parallelogram Law

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

Scalar Multiplication

$$c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$$

Vector Subtraction

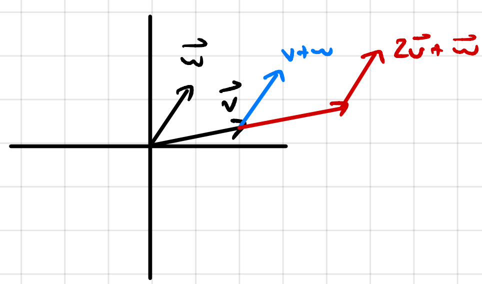
$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \end{pmatrix}$$

$$\therefore \vec{v} = \vec{w} + (\vec{v} - \vec{w})$$

Linear Combinations

$$c_1\vec{v}_1 + c_2\vec{w}_2$$

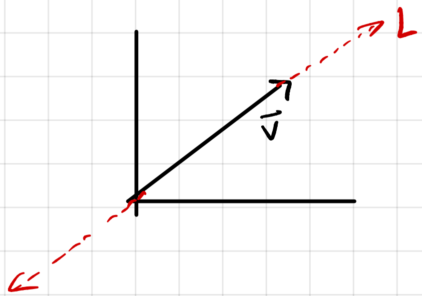


def The **span** of a list of vectors $\vec{v}_1 \cdots \vec{v}_n$ is the collection of all linear combination of these vectors

Notation

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \{x_1\vec{v}_1 + \cdots + x_n\vec{v}_n \mid x_1, \cdots, x_n \in \mathbb{R}\}$$

EX what is span of non-zero vector $\vec{v} \in \mathbb{R}^2$



$L = \text{span}\{\vec{v}\}$ line

Ex

in 3D, what is span of non-colinear vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$

$\text{span}\{\vec{v}, \vec{w}\}$ is a plane

Ex

find span (non-coplanar) $\vec{u}, \vec{v}, w \in \mathbb{R}^3$

$\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

NB
- $\vec{0} \in$ any span
- lines pass through origin

↳ **Notation** $\text{span}\{\} = \{\vec{0}\}$ ← span of no vectors is the zero vector

Lecture 6 – 15SEP22

Recall

$$\text{span}\{\vec{v}_1, \dots \vec{v}_n\} = \{x_1\vec{v}_1 + \dots + x_n\vec{v}_n\}$$

$\vec{0}$ in all spans

Ex of spans:

- line through origin
- plane " "

Ex find $\text{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}4\\3\\4\end{pmatrix} \right\}$

$$\begin{pmatrix}4\\3\\4\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$\therefore$ $\begin{pmatrix}4\\3\\4\end{pmatrix}$ is in span of first two, thereby coplanar

$\therefore$ span is a plane

Spans and Solution Sets

Ex $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right) \Rightarrow \begin{aligned} x \quad + 2z &= 1 \\ y + 3z &= 2 \end{aligned}$

$x = 1 - 2z$
$y = 2 - 3z$
$z = z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 - 2z\\2 - 3z\\z\end{pmatrix}$$

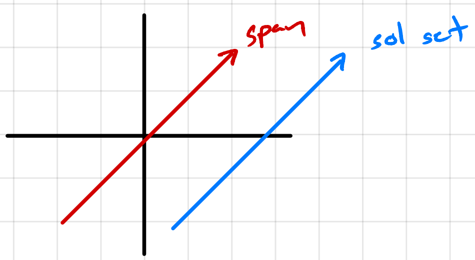
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = z\begin{pmatrix}-2\\-3\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} \quad \text{PVF}$$

sol set

$\therefore$ solutions to lin. sys. are vectors

$$\begin{pmatrix}1\\2\\0\end{pmatrix} + \text{span}\left\{\begin{pmatrix}-2\\-3\\1\end{pmatrix}\right\}$$

$\therefore$ solution sets are translates of span



Ex (cont'd)

$$z=0 \longrightarrow \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\3\\-1\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}-2\\-3\\1\end{pmatrix}\right\}$$

$$z=1 \longrightarrow \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

Ex

describe sol'n set of $x+2y+z=4$ as the translate of a span

$$\begin{aligned} x &= -2y-z+4 \\ y &= y \\ z &= z \end{aligned} \quad \therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = y\begin{pmatrix}-2\\1\\0\end{pmatrix} + z\begin{pmatrix}-1\\0\\1\end{pmatrix} + \begin{pmatrix}4\\0\\0\end{pmatrix}$$

$\therefore$ sol'n set

$$\hookrightarrow \begin{pmatrix}4\\0\\0\end{pmatrix} + \operatorname{span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

translate of a span

In general

The sol'n set of $A\vec{x}=\vec{b}$ is always the translate of a span.

Another way spans help w/ lin. sys.

Consistency Set

↳ the con. set of mat. A is the collection of all vectors b s.t. $Ax = b$ is consistent

Nb

If $Ax = b$ is consistent,

$$\vec{b} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + \cdots + x_n v_n$$

to be consistent, all x's must be numbers, not variables.

$\therefore$ $Ax = b \iff b$ is linear combo of $v_1 \cdots v_n$ $\iff b \in \text{span}\{\text{col. vecs. of } A\}$

$\therefore$ consistency set = span of cols of A

Ex

is $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in $\text{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$? no

same as ↳ is $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in consistency set of $\begin{pmatrix}1 & 1\\2 & 0\\3 & 1\end{pmatrix}$? no

same as ↳ is $\left(\begin{array}{cc|c}1 & 1 & 1\\2 & 0 & 1\\3 & 1 & 1\end{array}\right)$ consistent ? no

$$\left(\begin{array}{cc|c}1 & 1 & 1\\2 & 0 & 1\\3 & 1 & 1\end{array}\right) \xRightarrow{\text{RREF}} \left(\begin{array}{cc|c}1 & 1 & 1\\0 & -2 & 1\\0 & 0 & 1\end{array}\right)$$

Ex

are $\vec{u}$, $\vec{v}$, and $\vec{w}$ coplanar?

same as ↳ is $\vec{u}$ in $\text{span}\{\vec{v}, \vec{w}\}$ or $\vec{v}$ in $\text{span}\{\vec{u}, \vec{w}\}$, etc.

↳ find answer using last example technique

Homogeneous and Inhomogeneous Equations

↳ eq's of form $Ax = 0$ → homogeneous

↳ " " " $Ax = b$, $b \neq 0$ → inhomogeneous

Ex

inhomo

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

homo

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right)$$

same row ops

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 0 & 3/2 & 3 & -3/2 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 0 & 3/2 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 10 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 2 & 0 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

RREF

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

PVF

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

PVF

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

soln set

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \operatorname{span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$$

soln set

$$\operatorname{span}\left\{\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}\right\}$$

Facts

↳ $Ax = b$ sol'n set $\vec{p} + \text{span}\{(\vdots)\}$

↳ $Ax = \emptyset$ sol'n set $\vec{\emptyset} + \text{span}\{(\vdots)\} = \text{span}\{(\vdots)\}$

warning

↳ this only works if $Ax = b$ is consistent

↳ if inconsistent, sol'n set empty by definition

Why?

↳ same steps for RREF

↳ why is $Ax = \emptyset$ always consistent and why always a span, not a translate of a span?

↳ $\vec{x} = \emptyset$ solves any homo. sys.

↳ Why diff of 2 sol'ns give sol'n to $Ax = 0$?

↳ $\vec{v}$ and $\vec{w}$ are sol'ns to $Ax = b$

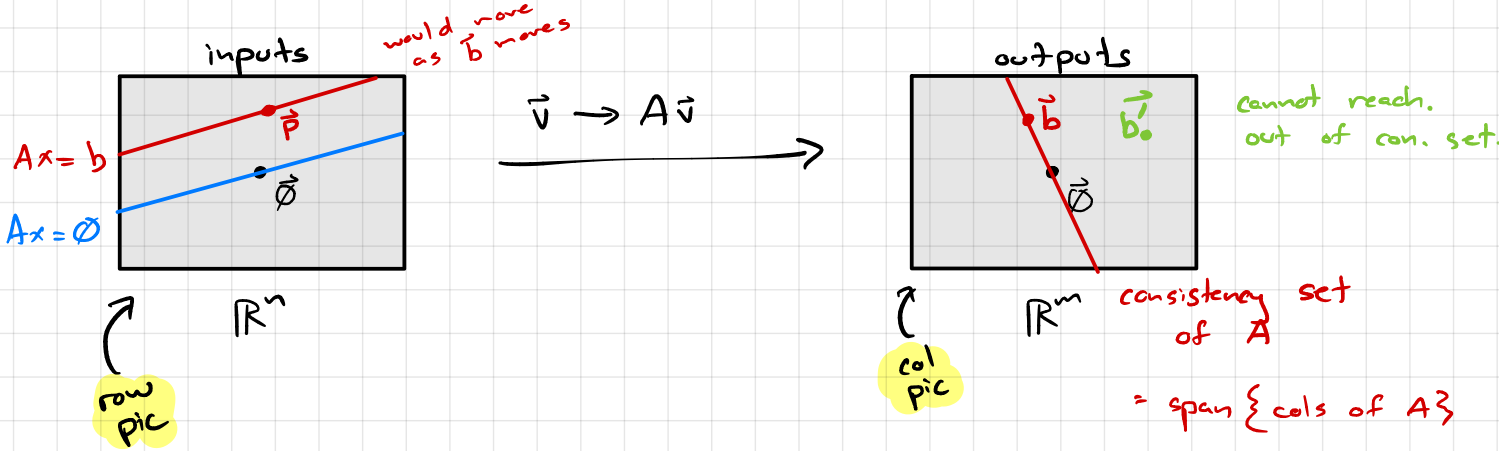
$$\begin{matrix} Av = b \\ Aw = b \end{matrix} \implies \boxed{A(v - w) = Av - Aw = b - b = 0}$$

↳ To get any sol'n to $Ax = b$, pick any sol'n $\vec{p}$ and add any sol'n to $Ax = 0$ (call it $\vec{v}$)

↳ $\boxed{A(\vec{p} + \vec{v}) = Ap + Av = b + 0 = b}$

Row and Col Pictures

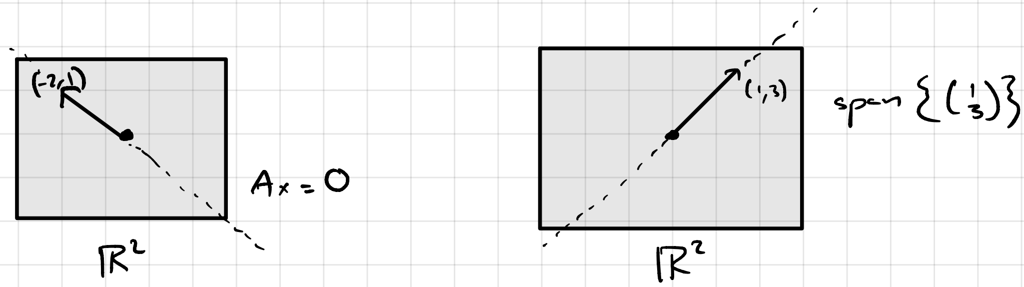
Given $m \times n$ mat A,



No vectors in soln set to $A\vec{x} = \vec{b}'$ since $\vec{b}'$ is not in the consistency set.

Ex

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

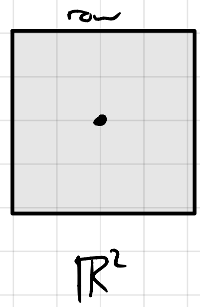


$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

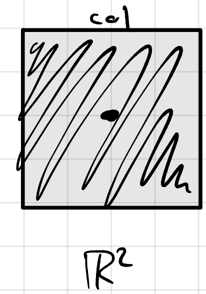
$$Ax = 0 \longrightarrow x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \therefore \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ is one solution}$$

Ex

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



sol'n set to $A x = \vec{0}$ is $\{\vec{0}\}$



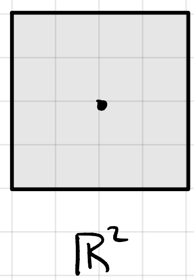
con. set $= \mathbb{R}^2$

$$A x = \vec{0}$$

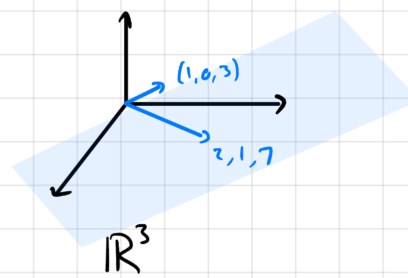
RREF $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \longrightarrow x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Ex

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$$



sol'n set $= \{\vec{0}\}$



con set is plane

$$A\vec{x} = \vec{0}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 7 & 0 \end{array}\right) \overset{\text{RREF}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Lecture 7 - 20SEP22

Subspaces

↳ a subset V of $\mathbb{R}^n$ is a collection of points in $\mathbb{R}^n$

↳ notation: "is contained in"

$$V \subseteq \mathbb{R}^n$$

↳ Ex: lines, shapes in $\mathbb{R}^n$

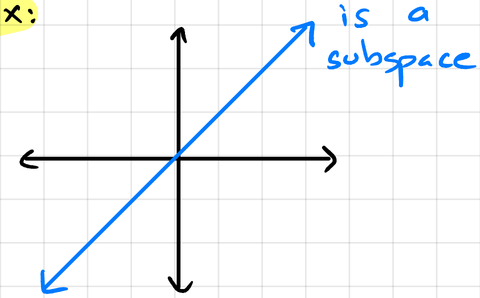
↳ a subspace V of $\mathbb{R}^n$ is a subset $V \subseteq \mathbb{R}^n$:

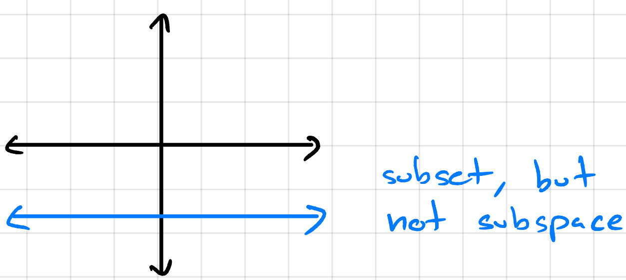
"contains zero" → ① $\emptyset \in V$ "is in"

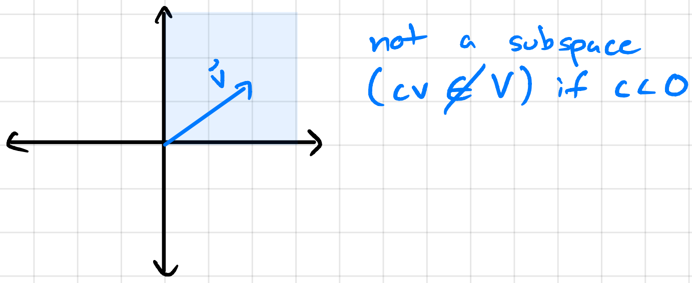
"closed under addition" → ② if $u, v \in V$, then $u+v \in V$

"closed under scaler multiplication" → ③ if $v \in V$ and $c \in \mathbb{R}$, then $cv \in V$

Ex:







Alternitive def. of subspace

↳ $V \subseteq \mathbb{R}^n$ is a subspace if:

① V is nonempty

② if $\vec{v}_1, \dots, \vec{v}_n \in V$ then any linear combo $x_1 v_1 + \dots + x_n v_n \in V$

Ex – prove that $\text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ is a subspace

① zero is in span ✓

② $\vec{w} = c_1\begin{pmatrix}1\\2\end{pmatrix}$ $\quad \therefore \quad (\vec{w} + \vec{v}) = c_1 c_2 \begin{pmatrix}1\\2\end{pmatrix}$ which is a linear combo of $\begin{pmatrix}1\\2\end{pmatrix}$ ✓

$\vec{v} = c_2\begin{pmatrix}1\\2\end{pmatrix}$

③ $\vec{v} \in \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}, \ c \in \mathbb{R}$

$\therefore \ c\vec{v} \in \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ because if $\vec{v} = d\begin{pmatrix}1\\2\end{pmatrix}$

$c\vec{v} = cd\begin{pmatrix}1\\2\end{pmatrix} \in \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ ✓

Ex – prove solution set of $x + y - z = 0$ is a subspace of $\mathbb{R}^3$

① $(0,0,0)$ is in solution set. ✓

② start w/ $\vec{u}, \vec{v}$ in sol'n set we know,

$$u_1 + u_2 - u_3 = 0$$
$$v_1 + v_2 - v_3 = 0$$

$$\therefore \ \underbrace{(u_1 + v_1) + (u_2 + v_2) - (u_3 + v_3)}_{\vec{u} + \vec{v}} = 0$$

$\vec{u} + \vec{v} = 0$ ✓

③ start w/ $\vec{v}$ in sol'n set.

$$v_1 + v_2 - v_3 = 0$$

find if $c\vec{v}$ in sol'n set:

$$cv_1 + cv_2 - cv_3 = 0$$
$$c(v_1 + v_2 - v_3) = 0$$
$$\therefore \ c\vec{v} = 0 \ ✓$$

NB Any subspace $V \subseteq \mathbb{R}^n$ can be described as either:

① solution set of homogenius eq $Ax = 0$

② span of a list of vectors

The sol'n set of $Ax=0$ is a subspace

① $\emptyset$ is in sol'n set ✓

② if $\vec{u}$ and $\vec{v}$ are in sol'n set, is $\vec{u}+\vec{v}$?

$A\vec{v} = 0 \qquad A\vec{u} + A\vec{v} = 0$

$A\vec{u} = 0 \qquad A(\vec{u}+\vec{v}) = 0$ ✓

③ if $\vec{v}$ a sol'n and c is scaler, is $c\vec{v}$ a sol'n?

$A\vec{v} = 0$

$c(A\vec{v}) = 0$

$A(c\vec{v}) = 0$ ✓

NB inhomogeneous lin sys sol'n are never subspaces since $\emptyset$ is not in sol'n set

Any span is a subspace

$$V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$$

① is $\emptyset \in V$?

$$\emptyset v_1 + \cdots + \emptyset v_n = \emptyset \checkmark$$

② if $\vec{u}$ and $\vec{w}$ in span, is $\vec{u}+\vec{w}$?

$$\vec{u} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$
$$\vec{w} = y_1\vec{v}_1 + \cdots + y_n\vec{v}_n$$

$$\vec{u}+\vec{w} = (x_1+y_1)\vec{v}_1 + \cdots + (x_n+y_n)\vec{v}_n$$

$$\therefore \vec{u}+\vec{w} \in V \checkmark$$

③ if $c \in \mathbb{R}$ and $\vec{u} \in V$, is $c\vec{u} \in V$?

$$\vec{u} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

$$c\vec{u} = cx_1v_1 + \cdots + cx_nv_n = c(x_1\vec{v}_1 + \cdots + x_nv_n)$$

$$\therefore c\vec{u} \in V \checkmark$$

Ex The sol'n set of $x+y-z=0$ is a subspace of V

① Find $\vec{u}, \vec{v}$ st $V = \text{span}\{\vec{u}, \vec{v}\}$

Method (PVF)

$$x = -y + z$$
$$y = y$$
$$z = z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \boxed{V = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

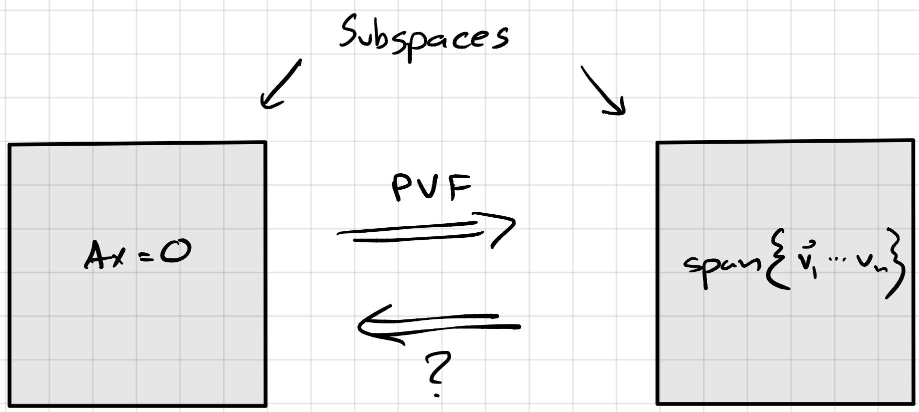
Ex Write sol'n set of $\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as a span

RREF

↳ $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

↳ $x = -2y + w$
$y = y$
$z = -w$
$w = w$

sol'n set is $\boxed{\text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}\right\}}$



Four fundamental subspaces of a matrix

Consider an $m \times n$ matrix A.

① **null space**: subspace denoted $N(A) \subseteq \mathbb{R}^n$

↳ $N(A) = \{\text{the sol'n set of } Ax = 0\}$

$= \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = 0\}$

② **column space**: denoted $C(A) \subseteq \mathbb{R}^m$

↳ $C(A) = \{\text{the consistency set of } A\}$

$= \{\vec{b} \in \mathbb{R}^m \mid Ax = b \text{ has a sol'n}\}$

$= \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{v} \text{ for some } \vec{v} \in \mathbb{R}^n\}$

$= \text{span}\{\text{columns of } A\}$

③ left null space (null space of transpose):

↳ $N(A^T) \subseteq \mathbb{R}^m$

↳ $N(A^T) := \{ \vec{v} \in \mathbb{R}^m \mid A^T\vec{v} = 0 \}$

$:= \{ \vec{v} \in \mathbb{R}^m \mid \vec{v}^T A = 0 \}$ ↙ multiplying on left of A, hence the name

④ row space

↳ $R(A) \subseteq \mathbb{R}^n$

↳ $R(A) = C(A^T) = \text{span}\{\text{rows of matrix } A\}$

Subspaces

$$\boxed{N(A)} \xRightarrow{\text{PVF}} \boxed{C(B)}$$

Ex

$$N\begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 1 & -1 \end{pmatrix} = C\begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$$

NB

These descriptions are not unique:

$$N\begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 4 & -1 \end{pmatrix} = N\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{pmatrix} = \dots \text{ for all steps to RREF}$$

$$C\begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = C\begin{pmatrix} -2 & -1 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \dots$$

NB If you have a $\text{span}\{v_1 \cdots v_n\}$, how many vectors do you actually need?

Ex

$$\text{span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$$

The subspace is (surprisingly) a plane.

Why?

$$\begin{pmatrix}-1\\5\\12\end{pmatrix} = \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

three vectors are coplanar / redundant

so $\begin{pmatrix}-1\\5\\12\end{pmatrix}$ is in $\text{span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$

$$\therefore \text{span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$$

NB in this ex, any vector is in the span of the other two, but that is not always the case

Ex

$$v_1 = \begin{pmatrix}1\\1\end{pmatrix} \quad v_2 = \begin{pmatrix}-2\\-2\end{pmatrix} \quad v_3 = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$V = \text{span}\{v_1, v_2, v_3\}$$

$$v_1 = -\frac{1}{2}v_2 + 0v_3$$

$$v_2 = -2v_1 + 0v_3$$

v_3 is not a linear combo of v_1 and v_2

$$\therefore V = \text{span}\{v_1, v_3\} = \text{span}\{v_2, v_3\}$$

w/o v_3, you'd get a smaller subspace

Linear dependence

A list of vectors $v_1 \cdots v_k$ is linearly dependent if one of the vectors is in the span of the remaining vectors.

ie: $\vec{v}_i = c_1 v_1 + \cdots c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_k v_k$

prev. Ex cont'd

$$\begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ are linearly dependent}$$

NB vectors are linearly dependent (LD) if you can add them to get $\emptyset$ when at least one coefficient is non-zero

↳ $v_1 \cdots v_k$ are LD

if $x_1 v_1 + \cdots + x_k v_k = 0$

where $(x_1, \cdots, x_k) \neq 0$

Lecture 8 - 22SEP22

Linear dependence

↳ list of vectors $v_1 \cdots v_k$ is LD if there exists a way to add non-zero multiples of those vectors arrives at the zero vector

↳ we call those coefficients $x_1 \cdots x_k$ a linear dependents

Suppose that, for example, $x_1 \neq 0$

Then, $v_1 = -\frac{1}{x_1}\left(x_2 v_2 + \cdots + x_k v_k\right)$

$\Rightarrow v_1 \in \text{span}\{v_2, \cdots, v_k\}$

$\therefore v_1$ is redundant in $\text{span}\{v_1 \cdots v_k\}$

Linearly independent lists of vectors $v_1 \cdots v_k$ if they are not LD.

ie: if the only way to make them sum to zero is w/ the trivial sol'n

lin indep.

↳ LI: no vector is in the span of the other vectors

How to test LI/LD?

Ex $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$ LI or LD?

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Does this have free variables? If so, $\exists$ non trivial sol'n

REF $\hookrightarrow \begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix} \implies$ one free variable means vectors are LD.

$\hookrightarrow$ $x = z$
$y = -2z$
$z = z$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = z\begin{pmatrix}1\\-2\\1\end{pmatrix}$

plug in non-zero free var

$\hookrightarrow$ $\underline{z=1}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$ non-trivial sol'n to $Ax=0$

Ex $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$

↳ $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix} \xRightarrow{RREF} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$\therefore$ no free vars → LI

What's good about LI vectors?

① If $\vec{b}$ is in span{LI vectors},

then $b = x_1v_1 + \cdots + x_kv_k$ and the coefficients x_i are unique

↳ why?

↳ $b = x_1v_1 + \cdots x_kv_k$

$b = x_1'v_1 + \cdots + x_k'v_k$

↳ subtract $= b = 0 = (x_1 - x_1')v_1 + \cdots + (x_k - x_k')v_k$ which is false

Dimension of span of LI vectors

① $\text{span}\{\} = \{\vec{0}\}$ = zero-dimensional

② $\text{span}\{\vec{v}\}$ = one-dimensional line

↳ to say that $\{\vec{v}\}$ is LI is to say that the only way for $cv = 0$ is for $c = 0$

③ $\text{span}\{\vec{v}, \vec{w}\}$ = two-dimensional plane

↳ If $\vec{v}, \vec{w}$ were LD, $x_1v + x_2w = 0$ w/ either x_1 or $x_2 \neq 0$

↳ either $v = cw$ or $w = cv$

↳ not colinear $\therefore$ plane

NB dimension of span{LI} is # of free vars

Bases and Dimensional

consider subspace $V \subseteq \mathbb{R}^n$

↳ we say that V is a **spanning set** of V if $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$

If a spanning set of V is LI, we say these vectors form a **basis** of V.

In other words, a basis of a subspace is a spanning set w/o redundancies.

In particular, and vector $\vec{b} \in V$ can be written as a unique combination of basis vectors

↳ $b = x_1 v_1 + \cdots + x_k v_k$

∴ The dimension of a subspace V can be defined to be the number of vectors in any basis of V

Q: If given subspace and asked dimension:

A:

not necessarily a basis

$V = \text{span}\{v_1, \dots, v_k\}$

1. repeatedly use the method of finding dependencies to throw out redundant vectors from this list
2. when no more dependencies, you're left w/ basis
3. # of vectors in final list is **dim(V)**

Ex What is a basis of:

$$V = \text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \right\}$$

↳ we prev. found a dependency $(1, -2, 1)$

drop any vector w/ non-zero, which is all in this case

dropped $\begin{pmatrix}7\\8\\9\end{pmatrix}$

↳ $\text{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix} \right\}$ is LI

$\therefore \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}$ is a basis of V

Q: What is $\dim(\mathbb{R}^n)$?

A:

① write spanning set w/ unit vectors $e_1 \cdots e_n$

② any vector in $(x_1 \cdots x_n) \in \mathbb{R}^n = \overbrace{x_1 e_1 + \cdots + x_n e_n}^{\text{unique}}$

↳ These are LI since $\begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} = I_n$ has no free vars

③ $\therefore \dim(\mathbb{R}^n) = n$

Ex

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ is a basis of $\mathbb{R}^3$

$e_1, e_1+e_2, e_1+e_2+e_3$ is a different basis

Fact

↳ if you have list of vectors $\vec{v}_1, \dots, \vec{v}_{\dim(V)} \in V$, they are a basis if either:

① they are LI or
② they are a spanning set

↳ come together given correct size list

Bases for fundemental subspaces

- We'll find algos which produce bases for $N(A)$, $C(A)$, $N(A^T)$, $Row(A)$
- we won't first find spanning set then throw out redundancies, we'll just find a basis

Basis for C(A)

theorem: the pivot cols of A form a basis of $C(A)$

↳ ① find pivot columns using REF
② use original cols where pivots are

Warning: doing row ops during REF calculations changes the column space

$$C(A) \neq C(\mathrm{rref}(A))$$

NB row operations don't change the "space of dependencies."

ie: if $(x_1, \cdots, x_n)$ gave a dependance on the columns of A, they do the same for columns of $\mathrm{rref}(A)$.

Why do row ops not change its dependencies?

- If $(x_1, \ldots x_n)$ are a dependence on cols of A, then

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

- Let's do a row op to A:

$$EA\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = E\left(A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = E(0) = 0$$

$\Longrightarrow$ $(x_1, \cdots, x_n)$ gives dep. on cols of EA

- Same in reverse:
  - any dep. on cols of EA gives dep. on cols of A.

$\Longrightarrow$ deps on cols of A = deps on cols of $\mathrm{rref}(A)$

Ex

$$A = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{pmatrix} = 4\times 5 \text{ matrix}$$

$$\text{RREF (given)} = \begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots

$v_3 = 3v_1 + 2v_1$
$v_5 = 4v_1 + 6v_2 - v_4$

Thm. $\Rightarrow$ $C(A)$ has basis (v_1, v_2, v_4) w/o free vars

$\Rightarrow$ v_3 and v_5 are

A **spanning set** of a subspace is a list of vectors spanning that subspace, and if none are redundant, it is a **basis**.

NB

$$\dim(C(A)) = \underbrace{\#\text{ pivot cols}}_{\text{"rank"}} = \#\text{ cols} - \#\text{ free vars}$$

Find basis of N(A)

Theorem: the vectors you get when you find PVF of $Ax = 0$ are a basis of $N(A)$.

Fact:

$$\dim(N(A)) = \#\text{ free variables} = \#\text{ cols} - \dim(C(A)) = \#\text{ cols} - \text{rank}(A)$$

Lecture 9 - 27SEP22

Goal: find bases for $N(A)$, $C(A)$, $N(A^T)$, $R(A)$

TLDR

For $N(A)$:
↳ vectors occuring in PVF of $Ax=0$

For $C(A)$:
↳ made using pivot cols of A

NB these same two methods can be used w/ A^T to find $N(A^T)$ and $R(A)$, but there are other methods that are easier

Instead: we will find bases for $N(A^T)$ and $R(A)$ using things related to finding rref(A) instead of rref(A^T)

Row space $R(A)$

Ex

$$R\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$$

! NB row ops don't change the row space

↳ if E is elem matrix, then $R(EA) = R(A)$

↳ row ops do change the col space

Why don't ops effect $R(A)$?

$R_1 \mathrel{{*}{=}} 3: \quad \begin{pmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ ✓

$R_2 \mathrel{+}= 2R_1: \quad \begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ ✓

row swaps just change order in the span ✓

Thm. The pivot rows of $\text{ref}(A)$ form a basis for $R(A)$

Ex

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix} \overset{\text{REF}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

← pivot rows

$\therefore R(A)$ basis $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}$

NB bases are not spans

Why thm. works

↳ row ops don't change $R(A)$

$$R(A) = R\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↳ two rows are LD b/c R_2 has zero where R_1 has a 1.

Dimension of R(A)

! ↳ $\dim(R(A)) = \underline{\#\text{ pivot rows} = \text{rank}(A)}$ ← def: Rank

↳ $\dim(C(A)) = \#\text{ pivots} = \dim(R(A))$

↳ $\dim(C(A)) = \dim(R(A))$

TLDR NB: To find bases of $N(A)$, $C(A)$, $R(A)$:

① RREF

② PVF → $N(A)$

Pivot cols of A → $C(A)$

pivot rows of A → $R(A)$

Find basis of Left Null Space $N(A^T)$

↳ A is $m \times n$ matrix

↳ $N(A^T) = \{ \vec{v} \in \mathbb{R}^m \mid A^T\vec{v} = 0 \} = \{ \vec{v} \in \mathbb{R}^m \mid v^T A = 0 \}$

- Suppose E is invertible s.t.

$$EA = U \quad \text{where} \quad U = \text{ref}(A)$$

↓

Thm Suppose $\text{ref}(A)$ has k zero-rows. Then the final k rows of E form a basis of $N(A^T)$

How to keep track of E.

↳ compute ref of $[A \mid I]$

↳ $E[A \mid I] = [U \mid E]$

! ∴ to compute basis of $N(A^T)$:

① compute $\text{ref}[A \mid I]$

② take rows on right side where there are zero-rows on the left

Ex - $N(A^T)$

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix} \Longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{REF}} \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

$$\therefore\ N(A^T) = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

To check

↳ $(1\ \ -1\ \ 1)\, A = \vec{0}$ ✓

Dimension of $N(A^T)$

↳ $\dim(N(A^T))$ = # zero-rows in ref(A) = # rows − # pivots
$= m - \text{rank}(A)$

Summary (4 fundamental subspaces)

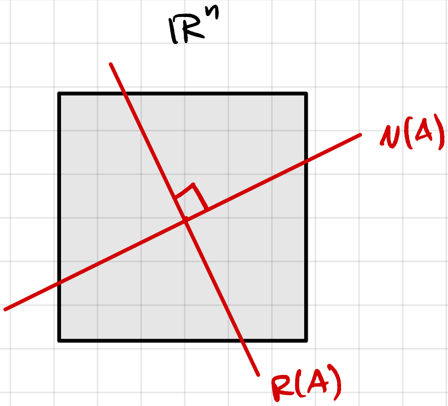
↳ A is $m \times n$ matrix

↳ each of 4 subspaces is in either $\mathbb{R}^m$ or $\mathbb{R}^n$

! O

Name	Inside	Dim	Basis
$C(A)$	$\mathbb{R}^m$	rank(A)	pivot cols of A
$N(A)$	$\mathbb{R}^n$	n − rank(A)	vectors found in PVF of $Ax = 0$
$R(A)$	$\mathbb{R}^n$	rank(A)	pivot rows in ref(A)
$N(A^T)$	$\mathbb{R}^m$	m − rank(A)	the last m − rank(A) rows of E where $EA = U$ and U is one REF

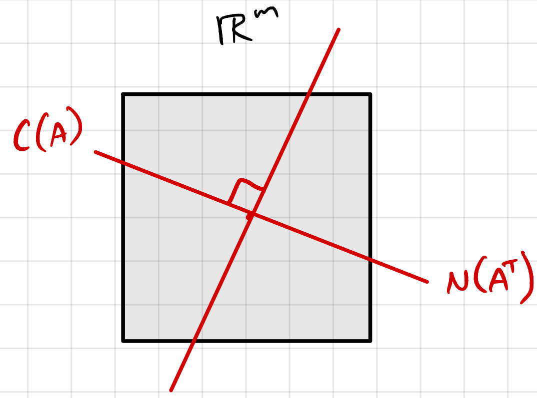
Row picture



$N(A) \perp R(A)$

$\dim(R(A)) + \dim(N(A)) = n$

Col picture



$C(A) \perp N(A^T)$

$\dim(C(A)) + \dim(N(A^T)) = m$

NB $\dim(R(A)) = \dim(C(A)) = \text{rank}(A)$

NB # pivots in ref(A) = # pivots in ref(A^T)

NB you can get a basis of $C(A)$, $N(A)$, $R(A)$ from ref(A)

↳ try to avoid calculating $N(A^T)$ by checking if $\dim(N(A^T)) = 0$ or if $\dim(N(A^T)) = m$ using ideas above

Full rank matrices

- ↳ A "random" matrix typically the maximum possible number of pivots
  - ↳ This is $\min\{m,n\}$
- ↳ A matrix has full column rank if $r=n$ where notation $r:=\text{rank}(A)$
  - ↳ Ex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
- ↳ A matrix has full row rank if $r=m$
  - ↳ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
- ↳ A matrix has full rank if it has full row or col rank

Notes on mat w/ full column rank

the following are equivalent (if one, all) TFAE:

① A has full col rank
  - ↳ pivot in all cols
  - ↳ no free cols

② $N(A) = \{0\}$
  - ↳ the only sol'n of $Ax=0$ is only trivial ← $\vec{0}$
  - ↳ for every $b \in \mathbb{R}^m$, $Ax=b$ has zero or one sol'n

③ The cols of A are LI

④ $\dim(C(A)) = \#\text{cols} = n$

⑤ $\dim(R(A)) = n$

⑥ $R(A) = \mathbb{R}^n$

Notes on mats w/ full row rank

TFAE:

1. A has full row rank
   ↳ pivot in all rows
   ↳ any ref(A) has no zero-rows

2. $Ax=b$ is consistent for <u>every</u> b in $\mathbb{R}^m$!
   ↳ $C(A) = \mathbb{R}^m$
   ↳ $\dim(C(A)) = m$

3. The cols of A span all of $\mathbb{R}^m$

4. $\dim(R(A)) = \dim(C(A)) = m$

5. $N(A^T) = \{0\}$