UNIT III

Definition 1.1: Let F be a non empty set (we require F to have at least two elements) on which there are two binary operations $'+'$ and $'.'$ called addition and multiplication, respectively. Then the set F together with these operations is said to be a **field**, if the following axioms are satisfied:

i. *Addition is closed*: For all $a, b \in F,\ a + b \in F$
ii. *Addition is commutative*: For all $a, b \in F,\ a + b = b + a$
iii. *Addition is associative*: For all $a, b, c \in F,\ a + (b + c) = (a + b) + c$
iv. *Additive identity element*: There exists an element $0 \in F$, such that $a + 0 = a$ for all $a \in F$.
v. *Additive inverse exists*: For all $a \in F$, there exists an element $-a \in F$ such that $a + (-a) = 0$.
vi. *Multiplication is closed*: For all $a, b \in F, a.b \in F$
vii. *Multiplication is commutative*: For all $a, b \in F, a.b = b.a$
viii. *Multiplication is associative*: For all $a, b, c \in F, a(b.c) = (a.b)c$
ix. *Multiplicative identity element*: There exists an element of $1 \in F$, such that $1.a = a$ for all $a \in F$.
x. *Reciprocals exists*: For all $a \neq 0 \in F$, there exists an element $a^{-1} \in F$ such that $a.(a^{-1}) = 1$.
xi. *Distributivity*: For all $a, b, c \in F,\ a(b + c) = a.b + a.c$ and $(a + b)c = a.c + b.c$

Examples of Field: $\mathbb{R}, \mathbb{C}, \mathbb{Q}$, etc.

Definition 1.2*:* A **vector space** over a field F is a set V on which two operations $'+'$ and $'.'$ are defined, called vector addition and scalar multiplication. The operation $+$ (vector addition) must satisfy the following conditions:

Closure: If u and v are any vectors in V, then the sum $u + v \in V$.

i. *Commutative law*: For all $\boldsymbol{u}, \boldsymbol{v} \in V,\ \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
ii. *Associative law*: For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V,\ \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
iii. *Additive identity*: There exists an additive identity element $\mathbf{0} \in V$, such that for any vector $\boldsymbol{v}$ in V, $\ \boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$

iv. *Additive inverse*: For each vector $\boldsymbol{v} \in V$, the equations $\boldsymbol{v} + \boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x} + \boldsymbol{v} = \boldsymbol{0}$ have a solution x in V, called an additive inverse of $\boldsymbol{v}$, and denoted by $-\boldsymbol{v}$.

The operation . (scalar multiplication) is defined between real numbers and vectors, and must satisfy the following conditions:

Closure: If $\boldsymbol{v}$ is any vector in V, and c is any real number, then the product $c.\boldsymbol{v}$ belongs to V.

v. *Distributive law (a)*: For all real numbers c and all vectors $\boldsymbol{u}, \boldsymbol{v}$ in V, $c.(\boldsymbol{u} + \boldsymbol{v}) = c.\boldsymbol{u} + c.\boldsymbol{v}$

vi. *Distributive law (b)*: For all real numbers c, d and all vectors $\boldsymbol{v}$ in V, $(c + d).\boldsymbol{v} = c.\boldsymbol{v} + d.\boldsymbol{v}$

vii. *Associative law*: For all real numbers c, d and all vectors $\boldsymbol{v}$ in V, $c.(d.\boldsymbol{v}) = (cd).\boldsymbol{v}$

viii. *Unitary law*: For all vectors v in V, $1.\boldsymbol{v} = \boldsymbol{v}$

Definition 1.3: Let V be a vector space, and let W be a subset of V. If W is a vector space with respect to the operations in V, then W is called a **subspace** of V.

Theorem 1.1: Let V be a vector space, with operations $+$ and ., and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

i. *W is nonempty*: The zero vector belongs to W.

ii. *Closure under* $+$: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any vectors in W, then $\boldsymbol{u} + \boldsymbol{v} \in W$.

iii. *Closure under* .: If $\boldsymbol{v}$ is any vector in W, and c is any real number, then $c.\boldsymbol{v} \in W$

e.g. (a) $W = \left\{\begin{bmatrix} a \\ 0 \end{bmatrix}; a \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$, while the set $W = \left\{\begin{bmatrix} a \\ 1 \end{bmatrix}; a \in \mathbb{R}\right\}$ is not.

(b) For any vector space V the subset $W = \{\boldsymbol{0}\}$, consisting of only the zero vector, is a subspace of V, called the trivial subspace.

Ex 1.1: Let $W = \left\{\begin{bmatrix} a \\ a+1 \end{bmatrix}; a \in \mathbb{R}\right\}$ be subset of the vector space $V = \mathbb{R}^2$. Determine whether W is a subspace of V.

Ex 1.2: Let $W = \left\{\begin{bmatrix} 3t \\ 0 \\ -2t \end{bmatrix}; t \in \mathbb{R}\right\}$ be subset of the vector space $V = \mathbb{R}^3$. Determine whether W is a subspace of V.

Ex 1.3: Given $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ in a vector space V, Let $H = span\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$. Show that H is a subspace of V. ***(Attempt this question after studying spanning sets).***

Ex.1.4: Let V be a set in $\mathbb{R}^2$ with usual vector addition, but with scalar multiplication defined by $\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha y \\ \alpha x \end{bmatrix}$.

Determine whether or not V is a vector space with these operations.

Ex. 1.5: Let W_1 and W_2 be the subspaces of $\mathbb{R}^2$ with the standard operations given by

$W_1 = \left\{\begin{bmatrix} x \\ 0 \end{bmatrix}; x \in \mathbb{R}\right\}$ and $W_2 = \left\{\begin{bmatrix} 0 \\ y \end{bmatrix}; y \in \mathbb{R}\right\}$

Show that $W_1 \cup W_2$ is not a subspace.

Definition 1.4: Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots \boldsymbol{v_k}\}$ be a set of vectors in $\mathbb{R}^n$, and let $c_1, c_2, \cdots, c_k$ be scalars. An expression of the form

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \sum_{i=1}^{k} c_i\boldsymbol{v_i}$$

is called a **linear combination** of the vectors of S. Any vector $\boldsymbol{v}$ that can be written in this form is also called a **linear combination** of the vectors of S.

Every vector in $\mathbb{R}^3$ can be obtained from the three coordinate vectors $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\boldsymbol{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

for example, the vector $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The vector $\boldsymbol{v}$ is obtained by adding scalar multiples of the coordinate vectors. The vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, and\ \boldsymbol{e_3}$ are not unique in this respect. For example, the vector $\boldsymbol{v}$ can also be written as the combination of the vectors

$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, that is $3\boldsymbol{v_1} - \boldsymbol{v_2} + \boldsymbol{v_3} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \boldsymbol{v}$.

Thus the vector $\boldsymbol{v}$ is the linear combination of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$, and $\boldsymbol{v_3}$.

Ex. 1.6: Determine whether the vector $\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \\ 10 \end{bmatrix}$ is the linear combination of the vectors $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$ and $\boldsymbol{v_3} = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$.

Ex. 1.7: Determine whether the vector $\boldsymbol{v} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is the linear combination of the vectors $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$ and $\boldsymbol{v_3} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Ex. 1.8: Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the linear combination of the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(Attempt examples 1.6, 1.7 and 1.8 after studying solution of simultaneous linear equations)

Definition 1.5: The set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots \boldsymbol{v_m}\}$ in $\mathbb{R}^n$ is **linearly independent** provided that the only solution to the equation

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_m\boldsymbol{v_m} = \boldsymbol{0}$$

is the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. If the above linear combination has a nontrivial solution, then the set S is called **linearly dependent.**

For example, the set of coordinate vectors $S = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \cdots, \boldsymbol{e_n}\}$ in $\mathbb{R}^n$ is linearly independent.

Ex. 1.9: Check whether the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are linearly dependent or independent.

Ex. 1.10: Check whether the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly dependent or independent.

Theorem 1.2: If a set of vectors S is linearly independent, then any subset of S is also a linearly independent set of vectors.

Proof: Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}, \boldsymbol{v_{k+1}}, \cdots, \boldsymbol{v_m}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ be a subset of S. Consider the equation

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{0}$$

Next let $c_{k+1} = c_{k+2} = \cdots = c_m = 0$, and consider the linear combination

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} + 0\boldsymbol{v_{k+1}} + \cdots + 0\boldsymbol{v_m} = \boldsymbol{0}$$

Since S is linearly independent, $c_1 = c_2 = \cdots = c_k = 0$, and hence T is linearly independent.

Theorem 1.3: If a set of vectors T is linearly dependent and S is a set of vectors that contains T , then S is also a linearly dependent set of vectors.

Proof: Let $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ and suppose that $T \subset S$. Label the vectors of S that are not in T as $\boldsymbol{v_{k+1}}, \cdots, \boldsymbol{v_m}$. Since T is linearly dependent, there are scalars $c_1, c_2, \cdots, c_k$ not all zero, such that

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{0}$$

Then $c_1, c_2, \cdots, c_k, c_{k+1} = c_{k+2} = \cdots = c_m = 0$ is a collection of m scalars, not all 0, with

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} + 0\boldsymbol{v_{k+1}} + \cdots + 0\boldsymbol{v_m} = \boldsymbol{0}$$

Consequently, S is linearly dependent.

Theorem 1.4: A set of nonzero vectors is linearly dependent if and only if at least one of the vectors is a linear combination of other vectors in the set.

Proof: Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ be a set of nonzero vectors that is linearly dependent. Then there are scalars $c_1, c_2, \cdots, c_n$, not all zero, with

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} = \boldsymbol{0}$$

Suppose that $c_k \neq 0$, for some index k. Then solving the previous equation for the vector v_k, we have

$$\boldsymbol{v_k} = -\frac{c_1}{c_k}\boldsymbol{v_1} - \cdots - \frac{c_{k-1}}{c_k}\boldsymbol{v_{k-1}} - \frac{c_{k+1}}{c_k}\boldsymbol{v_{k+1}} - \cdots - \frac{c_n}{c_k}\boldsymbol{v_n}$$

Conversely, let $\boldsymbol{v_k}$ be such that

$$\boldsymbol{v_k} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_{k-1}\boldsymbol{v_{k-1}} + c_{k+1}\boldsymbol{v_{k+1}} + \cdots + c_n\boldsymbol{v_n}$$

Then

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_{k-1}\boldsymbol{v_{k-1}} + (-1)\boldsymbol{v_k} + c_{k+1}\boldsymbol{v_{k+1}} + \cdots + c_n\boldsymbol{v_n} = \boldsymbol{0}$$

Since the coefficient of $\boldsymbol{v_k}$ is -1, the linear system has a nontrivial solution. Hence, the set S is linearly dependent.

Theorem 1.5 [1]**:** $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ be a linearly independent set. Suppose that there are scalars $c_1, c_2, \cdots, c_n$ such that

$$\boldsymbol{v} = \sum_{k=1}^{n} c_k\boldsymbol{v_k}$$

Then the scalars are unique.

Proof: Let $\boldsymbol{v}$ be written as

$\boldsymbol{v} = \sum_{k=1}^{n} c_k\boldsymbol{v_k}$ and as $\boldsymbol{v} = \sum_{k=1}^{n} d_k\boldsymbol{v_k}$

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[1] *Given a set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ and an arbitrary vector $\boldsymbol{v}$ not in S, it may or may not be possible to write $\boldsymbol{v}$ as a linear combination of S. Also sometimes when the set S is LD , $\boldsymbol{v}$ can be written as a linear combination of the vectors of S in infinitely many ways [e.g. S={(1,1), (3,3)} and $\boldsymbol{v}$ =(2,2)]. This cannot happen for a linearly independent set as is shown in Theorem 1.5.*

Then

$$\mathbf{0} = \boldsymbol{v} - \boldsymbol{v} = \sum_{k=1}^{n} c_k \boldsymbol{v_k} - \sum_{k=1}^{n} d_k \boldsymbol{v_k}$$

$$= \sum_{k=1}^{n} (c_k - d_k) \boldsymbol{v_k}$$

Since the set of vectors S is linearly independent, the only solution to this last equation is the trivial one. that is,

$c_1 - d_1 = 0, c_2 - d_2 = 0, \cdots, c_n - d_n = 0$

or $c_1 = d_1, c_2 = d_2, \cdots, c_n = d_n$

Theorem 1.6: If a set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ contains the zero vector, then S is linearly dependent.

Proof: *Do it yourself....*

Theorem 1.7: A set of mutually orthogonal vectors is linearly independent.

Proof: Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ is a set of mutually orthogonal vectors.

Suppose $\sum_{i=1}^{k} c_i \boldsymbol{v_i} = \mathbf{0}$, with $c_k \neq 0$ (i.e. assume that S is a set of linearly dependent vectors)

Taking the scalar product of the above equation with $\boldsymbol{v_k}$ we obtain

$$\boldsymbol{v_k} \sum_{i=1}^{k} c_i \boldsymbol{v_i} = \boldsymbol{v_k}.\mathbf{0}$$

Using the orthogonal property $\boldsymbol{v_1}.\boldsymbol{v_2} = 0$, the above equation reduces to

$$c_k \boldsymbol{v_k}.\boldsymbol{v_k} = 0$$

Thus $c_k = 0$, as $\boldsymbol{v_k}$ is not a null vector (because set of linearly independent vectors cannot have a null vector).

This contradicts our initial assumption of linear dependence of S.

Hence S is a set of linearly independent vectors.

Theorem 1.8 [2]**:** Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ be a set of n nonzero vectors in $\mathbb{R}^m$. If $n > m$, then the set S is linearly dependent.

Proof: Let A be the $m \times n$ matrix with column vectors the vectors of S so that

$A_i = \boldsymbol{v_i}$ for $i = 1,2,\cdots,n$

In this way

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} = \boldsymbol{0}$$

in matrix form, is the homogeneous linear system

$$A\boldsymbol{c} = \boldsymbol{0} \text{ where } \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

As A is not square with $n > m$, there is at least one free variable. Thus, the solution is not unique and $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ is linearly dependent.

Definition 1.6: A set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_r}\}$ from $\mathbb{R}^n$ is said to be a **spanning set** for $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors in S.

Definition 1.7 [3]**:** Let V be a vector space and let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ be a (finite) set of vectors in V. The **span** of S, denoted by $\boldsymbol{span}(S)$, is the set

$$\boldsymbol{span}(S) = \{c_1v_1 + c_2v_2 + \cdots c_nv_n : c_1, c_2, \cdots c_n \in \mathbb{R}\}$$

Proposition 1.1 notes: If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ is a set of vectors in a vector space V, then $\boldsymbol{span}(S)$, is a subspace.

[2] *From theorem 1.8, any set of three or more vectors in $\mathbb{R}^2$, four or more vectors in $\mathbb{R}^3$, five or more vectors in $\mathbb{R}^4$, and so on, is linearly dependent. this theorem does not address the case for which $n \le m$. In this case, a set of n vectors in $\mathbb{R}^m$ may be either linearly independent or linearly dependent.*

[3] *Consider the set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$. This is the **spanning set** for $\mathbb{R}^3$. On the other hand, span of this set S, i.e.* $\boldsymbol{span}(s) = \boldsymbol{span}\{(1,0,0), (0,1,0), (0,0,1)\} = \mathbb{R}^3$

Proof: *Do it yourself... (hint: let* $\boldsymbol{u}$ *and* $\boldsymbol{w}$ *be vectors in* $\boldsymbol{span}(S)$ *and c a scalar)*

Ex. 1.11: Let S be the subset of the vector space $\mathbb{R}^3$ defined by

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Show that

$$\boldsymbol{v} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$$

is in $\boldsymbol{span}(S)$.

Definition 1.8: A subset B of a vector space V is a **basis for $\boldsymbol{V}$** provided that

1. B is a linearly independent set of vectors in V.
2. $\boldsymbol{span}(B) = V$ (i.e each vector in V can be written as the linear combination of vectors in B).

Definition 1.9: The dimension of the vector space V, denoted by dim (V), is the number of vectors in any basis of V.

For example, since the standard basis for $\mathbb{R}^n$ is $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \cdots \boldsymbol{e_n}\}$, we have $\dim(\mathbb{R}^n) = n$

We call a vector space V **finite dimensional** if there exists a basis for V with a finite number of vectors. If such a basis does not exist, then V is called **infinite dimensional.**

Note[4]: To determine whether a set of $\boldsymbol{n}$ vectors from a vector space of dimension $\boldsymbol{n}$ is or not a basis, it is sufficient to verify either that the set spans the vector space or that the set is linearly independent.

[4] *This also comes from 'The Basis Theorem' which states "Let* V *be a* $n-dimensional$ *vector space. Any linearly independent set of* ***exactly*** $\boldsymbol{n}$ *vectors in* V *is automatically a basis for* V*."*

Ex. 1.12: Determine whether

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Solution: Since $(dim\mathbb{R}^3) = 3$, the set B is a basis if it is linearly independent.

Now show that the given set is linearly independent.........*(complete it yourself)*

Theorem 1.9: The Representation of any vector in terms of the given basis is unique.

Proof: *Same as theorem 1.5*

Theorem 1.10: Given a set of basis vectors $S = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_r}\}$ for $\mathbb{R}^n$ and any other vector $\boldsymbol{b} \neq \boldsymbol{0}$ from $\mathbb{R}^n$. Then, if in the expression of $\boldsymbol{b}$ as a linear combination of the vectors in S,

$$\boldsymbol{b} = \sum_{i=1}^{r} \alpha_i \boldsymbol{a_i}$$

any vector $\boldsymbol{a_i}$ for which $\alpha_i \neq 0$ is removed from the set $S = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_r}\}$ and $\boldsymbol{b}$

is added to the set, the new collection of r vectors is also a basis for $\mathbb{R}^n$.

Proof: Given

$$\boldsymbol{b} = \sum_{i=1}^{r} \alpha_i \boldsymbol{a_i} = \alpha_1 \boldsymbol{a_1} + \alpha_2 \boldsymbol{a_2} + \cdots + \alpha_r \boldsymbol{a_r} \qquad \cdots\cdots (1)$$

Without loss of generality we can assume that $\alpha_r \neq 0$.

Replacing $\boldsymbol{a_r}$ by $\boldsymbol{b}$ we get the new set as

$$S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \boldsymbol{b}\} \qquad \cdots\cdots (2)$$

To show that (2) is also a basis of $\mathbb{R}^n$ we have to show that the set of vectors (2) is linearly independent.

Let (2) is linearly dependent, that is

$$\beta_1 \boldsymbol{a_1} + \beta_2 \boldsymbol{a_2} + \cdots + \beta_{r-1} \boldsymbol{a_{r-1}} + \beta_r \boldsymbol{b} = \boldsymbol{0} \qquad \cdots\cdots (3)$$

where all $\beta_i'^s$ are not zero.

As any subset of linearly independent set is linearly independent, thus subset of S, i.e. $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \}$ is linearly independent. Substituting the value of $\boldsymbol{b}$ from (1) in (3), we get

$$\beta_1 \boldsymbol{a_1} + \beta_2 \boldsymbol{a_2} + \cdots + \beta_{r-1} \boldsymbol{a_{r-1}} + \beta_r (\alpha_1 \boldsymbol{a_1} + \alpha_2 \boldsymbol{a_2} + \cdots + \alpha_r \boldsymbol{a_r}) = \boldsymbol{0}$$

$$(\beta_1 + \beta_r \alpha_1)\boldsymbol{a_1} + (\beta_2 + \beta_r \alpha_2)\boldsymbol{a_2} + \cdots + (\beta_{r-1} + \beta_r \alpha_{r-1})\boldsymbol{a_{r-1}} + \beta_r \alpha_r \boldsymbol{a_r} = \boldsymbol{0}$$

or $\delta_1 \boldsymbol{a_1} + \delta_2 \boldsymbol{a_2} + \cdots + \delta_{r-1} \boldsymbol{a_{r-1}} + \delta_r \boldsymbol{a_r} = \boldsymbol{0}$

where $\delta_i = (\beta_i + \beta_r \alpha_i), \ \ i = 1,2, \cdots (r-1)$

$\delta_r = \beta_r \alpha_r$

$S = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_r}\}$ is linearly independent, therefore

$\delta_i = 0, \ \ i = 1,2, \cdots (r-1)$ and $\delta_r = 0$.

As $\delta_r = \beta_r \alpha_r = 0$,

but $\alpha_r \neq 0$ (assumed initially)

$$\Rightarrow \beta_r = 0$$

Therefore, (3) $\Rightarrow$ $S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \boldsymbol{b}\}$ is **linearly independent.**

Now to show that $S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \boldsymbol{b}\}$ forms a basis of $\mathbb{R}^n$, we have to show that any vector $\boldsymbol{x} \in \mathbb{R}^n$ can be expressed as a linear combination of these vectors.

Vector $\boldsymbol{x} \in \mathbb{R}^n$ can be expressed as a linear combination of the given basis $S = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_r}\}$ as

$$\boldsymbol{x} = \gamma_1 \boldsymbol{a_1} + \gamma_2 \boldsymbol{a_2} + \cdots + \gamma_r \boldsymbol{a_r} \qquad \cdots\cdots (4)$$

In (1) $\alpha_r \neq 0$, we can write

$$\frac{\boldsymbol{b}}{\alpha_r} = \frac{\alpha_1}{\alpha_r}\boldsymbol{a_1} + \frac{\alpha_2}{\alpha_r}\boldsymbol{a_2} + \cdots + \boldsymbol{a_r}$$

or

$$\boldsymbol{a_r} = \frac{\boldsymbol{b}}{\alpha_r} - \sum_{i=1}^{r-1}\frac{\alpha_i}{\alpha_r}\boldsymbol{a_i} \qquad \cdots\cdots (5)$$

Substituting this value of $\boldsymbol{a_r}$ in (4) we get

$$\boldsymbol{x} = \gamma_1\boldsymbol{a_1} + \gamma_2\boldsymbol{a_2} + \cdots + \gamma_{r-1}\boldsymbol{a_{r-1}} + \gamma_r\left(\frac{\boldsymbol{b}}{\alpha_r} - \sum_{i=1}^{r-1}\frac{\alpha_i}{\alpha_r}\boldsymbol{a_i}\right)$$

$$\boldsymbol{x} = \gamma_1\boldsymbol{a_1} + \gamma_2\boldsymbol{a_2} + \cdots + \gamma_{r-1}\boldsymbol{a_{r-1}} + \frac{\gamma_r}{\alpha_r}\boldsymbol{b} - \sum_{i=1}^{r-1}\frac{\alpha_i}{\alpha_r}\gamma_r\boldsymbol{a_i}$$

$$\boldsymbol{x} = \sum_{i=1}^{r-1}\left(\gamma_i - \frac{\alpha_i}{\alpha_r}\gamma_r\right)\boldsymbol{a_i} + \frac{\gamma_r}{\alpha_r}\boldsymbol{b} \qquad \cdots\cdots (6)$$

$$= \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{r-1}a_{r-1} + \lambda_r a_r \qquad \cdots\cdots (7)$$

where $\lambda_i = \gamma_i - \frac{\alpha_i}{\alpha_r}\gamma_r,\ i = 1,2,\cdots,r-1$

and $\lambda_r = \frac{\gamma_r}{\alpha_r}$

Equation (7) is a linear combination of any $x \epsilon \mathbb{R}^n$ in terms of the new set

$S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \boldsymbol{b}\}$

$\Rightarrow\ S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{r-1}}, \boldsymbol{b}\}$ forms a basis of $\mathbb{R}^n$.

Theorem 1.11: Every basis of $\mathbb{R}^n$ has exactly n vectors.

Proof: We will first show that every basis of $\mathbb{R}^n$ has same number of vectors.

Let $S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_k}\}$ and $S_2 = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_l}\}$ be two bases of $\mathbb{R}^n$ having different number of vectors.

Expressing $\boldsymbol{b_l}$ in terms of the vectors of basis S_1, we get

$$\boldsymbol{b_l} = \sum_{i=1}^{k} \alpha_i \boldsymbol{a_i} = \alpha_1 \boldsymbol{a_1} + \alpha_2 \boldsymbol{a_2} + \cdots + \alpha_k \boldsymbol{a_k}$$

with at least one $\alpha_i \neq 0$ (since $\boldsymbol{b_l}$ is one of the vectors in the basis S_2, and being linearly independent set S_2 cannot have a null vector [5]).

Let $\alpha_k \neq 0$, then $S_3 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots \boldsymbol{a_{k-1}}, \boldsymbol{b_l}\}$ will form a new basis [6] of $\mathbb{R}^n$.

Now we can express $\boldsymbol{b_{l-1}}$ as a linear combination of the vectors in new basis S_3 as

$$\boldsymbol{b_{l-1}} = \beta_1 \boldsymbol{a_1} + \beta_2 \boldsymbol{a_2} + \cdots + \beta_{k-1} \boldsymbol{a_{k-1}} + \beta_k \boldsymbol{b_l}$$

with at least one $\beta_i \neq 0$, because $\boldsymbol{b_l} \neq \boldsymbol{0}$.

Let $\beta_{k-1} \neq 0$

then $S_4 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_{k-2}}, \boldsymbol{b_{l-1}}, \boldsymbol{b_l}\}$ will form a new basis of $\mathbb{R}^n$.

This process will continue until either we have $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_{k-l}}, \boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_l}\}$ as a new basis $(if\ k > l)$ or $\{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_l}\}$ as a new basis $(if\ k = l)$.

There must be at least as many $\boldsymbol{a_i'^s}$ as there are $\boldsymbol{b_j'^s}$, otherwise we get a basis of the form $\{\boldsymbol{b_{l-k+1}}, \cdots, \boldsymbol{b_l}\}$ and the remaining $\boldsymbol{b_j'^s}$ can be expressed as a linear combination of this basis which contradicts that $S_2 = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_l}\}$ is a basis of $\mathbb{R}^n$.

Thus,

$$k \geq l \quad \cdots\cdots (1)$$

Similarly if we start with $S_2 = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_l}\}$ and insert $a_i'^s$ one by one to form new bases, we will conclude that

$$l \geq k \quad \cdots\cdots (2)$$

Therefore, (1) and (2) $\Longrightarrow l = k$

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[5] *See theorem 1.6*

[6] *See theorem 1.10*

Thus, every basis of $\mathbb{R}^n$ has same number of vectors.

Now, as the set of unit vectors $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \cdots, \boldsymbol{e_n}\}$ forms a basis of $\mathbb{R}^n$, it immediately follows that every basis of $\mathbb{R}^n$ has exactly n vectors.

Theorem 1.12: If a set of nonzero vectors $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ from $\mathbb{R}^n$ are mutually orthogonal then this set of vectors forms a basis for $\mathbb{R}^n$.

Proof: The proof follows immediately if we can show that the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ is linearly independent (since any set of n linearly independent vectors from $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$)[7].

Now, the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ can be proved to be a set of linearly independent vectors in a same way as done in theorem 1.7. ***(now proceed yourself....)***

Definition 1.10: A set of n mutually orthogonal vectors of unit length from $\mathbb{R}^n$ forms, what is called an **orthonormal basis** for $\mathbb{R}^n$.

[7] *See footnote 4*