Row Space, Column Space, and Nullspace

Linear Algebra

MATH 2010

- **Terminology:** Let A be the 2x4 matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 4 & 5 & 6 & 2 \end{bmatrix}$$

The *row vectors of A* are

$$\begin{bmatrix} 2 & 3 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 5 & 6 & 2 \end{bmatrix}$$

(the rows of A) in $\mathbb{R}^4$.

The *column vectors of A* are

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

(the columns of A) in $\mathbb{R}^2$

- **Definition:** Let A be a *mxn* matrix (recall m is the number of rows and n is the number of columns), then
  - The *row space* of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A
  - The *column space* of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.
- **Theorem:** If a *mxn* matrix A is row-equivalent to a *mxn* matrix B, then the row space of A is equal to the row space of B. (NOT true for the column space)
- **Theorem:** If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.
- **Example - Finding a Basis for Row Space** Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

Find a basis for the row space of A.

We must reduce A:

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$w_1 = [1, 1, 4, 1, 2] \quad w_2 = [0, 1, 2, 1, 1] \quad w_3 = [0, 0, 0, 1, 2]$$

form a basis for the row space of A.

- **Example- Finding a basis spanned by a set S:** Let $S = \{v_1, v_2, v_3, v_4\}$ where

$$\begin{aligned} v_1 &= [1, -2, 0, 3, -4] \\ v_2 &= [3, 2, 8, 1, 4] \\ v_3 &= [2, 3, 7, 2, 3] \\ v_4 &= [-1, 2, 0, 4, -3] \end{aligned}$$

Find a basis for the subspace of $\Re^5$ spanned by S.

If we look the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$$

then a basis for the row space of A gives a basis for the subspace of $\Re^5$ spanned by S.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 8 & 8 & -8 & 16 \\ 0 & 7 & 7 & -4 & 11 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 7 & 7 & -4 & 11 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 7 & -7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} w_1 &= [1, -2, 0, 3, -4] \\ w_2 &= [0, 1, 1, -1, 2] \\ w_3 &= [0, 0, 0, 1, -1] \end{aligned}$$

form a basis for the subspace spanned by S. The dimension of the row space is 3.

- **Example - Finding a basis for the column space of A:** There are two ways to find a basis for the column space:

1. Find the row-echelon form of A:

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns from the *original matrix* which have leading ones when reduced form a basis for the column space of A. In the above example, columns 1, 2, and 4 have leading ones. Therefore columns 1, 2, and 4 of the original matrix form a basis for the column space of A.So,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the column space of A. The dimension of the column space of A is 3.

2. The second way to find a basis for the column space of A is to recognize that the column space of A is equal to the row space of A^T. Finding a basis for the row space of A^T is the same as finding a basis for the column space of A.

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix}$$

Reducing A^T,we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & -4 & -2 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 0 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} w_1 &= (1,0,0,1,2) \\ w_2 &= (0,1,0,-2,-1) \\ w_3 &= (0,0,1,1,1) \end{aligned}$$

form a basis for the column space of A.

- **Example:** Let $S = \{v_1, v_2, v_3, v_4\}$ from above where

$$\begin{aligned} v_1 &= [1,-2,0,3,-4] \\ v_2 &= [3,2,8,1,4] \\ v_3 &= [2,3,7,2,3] \\ v_4 &= [-1,2,0,4,-3] \end{aligned}$$

Find a basis for the subspace of $\Re^5$ spanned by S *that is a subset of the vectors in* S. To do this, we set the columns of a matrix A as the vectors v_1, v_2, v_3 and v_4:

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix}$$

Then find the column space of A:

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 7/8 & 0 \\ 0 & 0 & 1 & 7/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are leading ones in columns 1,2, and 3, so columns 1,2, and 3 from the original matrix form a basis for the column space of A. This corresponds to the vectors

$$\begin{aligned} v_1 &= [1, -2, 0, 3, -4] \\ v_2 &= [3, 2, 8, 1, 4] \\ v_3 &= [2, 3, 7, 2, 3] \end{aligned}$$

- **Theorem:** If A is an mxn matrix, then the row space and column space of A have the *same* dimension.
- **Definition:** The dimension of the row (or column) space of a matrix A is called the **rank** of A, denoted rank(A).
- **Example:** Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\text{rank}(A) = 2$ because there are 2 leading ones.

- **Definition:** If A is a mxn matrix, then the set of all solutions of the homogeneous system of linear equations

$$Ax = 0$$

is a subspace of $\mathbb{R}^n$ called the *nullspace* of A and denoted $N(A)$.

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

The dimension of $N(A)$ is called the *nullity* of A.

- **Example:** **Finding a basis for the nullspace of A:** Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

We need to solve the system $Ax = 0$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 4 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $x_3 = s$ and $x_5 = t$ are free parameters. The solution to the system is given by

$$x = \begin{bmatrix} -2s - t \\ -2s + t \\ s \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} t$$

Therefore, the basis for $N(A)$ is given by

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

We have $dim(N(A)) = 2$, i.e. nullity$(A) = 2$. Note that dim(rowspace(A)) = 3 and nullity$(A) = 2$. 3+2 = 5 = n (the number of columns of A).

- **Theorem:** If A is a mxn matrix of rank A (r), then the dimension of the solution space of $Ax = 0$ is $n - r$, i.e.

$$n = rank(A) + nullity(A)$$

- **Example:** Let

$$A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$$

Find

1. basis for row space of A
2. basis for column space of A that is a subset of the column vectors of A
3. basis for nullspace of A
4. rank(A)
5. nullity(A)

Answers:

1. basis for row space of A: $\{[1, 2, -1, -1], [0, 0, 1, 4]\}$
2. basis for column space of A that is a subset of the column vectors of A:

$$\left\{ \begin{bmatrix} 2 \\ 7 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 1 \\ -2 \end{bmatrix} \right\}$$

3. basis for nullspace of A:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

4. rank(A): 2
5. nullity(A): 2

- **Solutions of Systems of Linear Equations:** The solution x to

$$Ax = b$$

can be written as

$$x = x_p + x_h$$

where x_p is called a particular solution of $Ax = b$ and x_h is called the homogeneous solution of $Ax = 0$.

- **Example:** Consider the system

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\ 3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\ x_1 & + & 2x_2 & & & - & 5x_4 & = & -9 \end{array}$$

Solving the system we have

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 2 & 2 & -6 & -14 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, we get the solution

$$x = \begin{bmatrix} 5+2s-t \\ -7-s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

The particular solution is given by

$$x_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

and the homogeneous solution is given by

$$x_h = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

- **Theorem:** A system of linear equations $Ax = b$ is consistent (has a solution) if and only if b is in the column space of A (i.e., b can be written as a linear combination of the columns of A).

- **Equivalent Statements:** If A is an nxn matrix, then the following are equivalent:

  (a) A is invertible
  (b) $Ax = 0$ has only the trivial solution
  (c) The reduced row-echelon form of A is I_n
  (d) A is expressible as a product of elementary matrices
  (e) $Ax = b$ is consistent for every nx1 matrix b
  (f) $Ax = b$ has exactly one solution for every nx1 matrix b
  (g) $|A| \neq 0$
  (h) $\lambda = 0$ is not an eigenvalue of A
  (i) rank(A)=n
  (j) n row vectors of A are linearly independent
  (k) n column vectors of A are linearly independent

LINEAR TRANSFORMATION

When a matrix A multiplies a vector $\underline{v}$, it "transforms" $\underline{v}$ into another vector $A\underline{v}$. This transformation follows the same idea as a function.

Defn: A transformation T assigns an output $T(\underline{v})$ to each input vector $\underline{v}$. The transformation is linear if it meets the following requirements for all $\underline{v}$ and $\underline{w}$:

(a) $T(\underline{v}+\underline{w}) = T(\underline{v})+T(\underline{w})$

(b) $T(c\underline{v}) = cT(\underline{v})$ for all scalar c.

(a) & (b) can be combined into one:

$$T(c\underline{v}+d\underline{w}) = cT(\underline{v}) + dT(\underline{w})$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a+b \\ a-b \\ 3c \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}$$

Ex.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $T(\underline{x}) = A\underline{x}$

Also $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\underline{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(a) Find the linear transformation $T(\underline{u})$

(b) Find an $\underline{x}$ whose image under T is $\underline{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

Soln.

(a) $T(\underline{u}) = A\underline{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

(b) we want to find $T(x)$ which produces

$b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

i.e. $A\underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

Solving

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]$$

Ex. Let A be a matrix $A = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$

Let $T(v) = A.\underline{v}$, where $v \in \mathbb{R}^3$

Check if the transformation is linear

Soln. Let $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

Then $T(\underline{v}+\underline{w}) = \cancel{\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}}\ A.(\underline{v}+\underline{w})$

$$= \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \left\{ \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix} \right\}$$

$$T(\underline{v}+\underline{w}) = v_1 + w_1 + 3v_2 + 3w_2 + 4v_3 + 4w_3 \quad \text{——— } (1)$$

$$T(\underline{v}) = A.\underline{v} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= v_1 + 3v_2 + 4v_3 \quad \text{——— } (2)$$

$$T(\underline{w}) = A.\underline{w} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= w_1 + 3w_2 + 4w_3 \quad \text{——— } (3)$$

From ①, ② & ③, we see that

$T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$

Also, $T(c.\underline{v}) = A(c\underline{v})$ ——————— (i)

$$= \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}$$

$$= cv_1 + 3cv_2 + 3cv_3 \quad \text{———— ④}$$

$cT(\underline{v}) = c(A\underline{v})$

$$= c\left\{ \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\} = c\{v_1 + 3v_2 + 4v_3\}$$

$$= cv_1 + 3cv_2 + 4cv_3 \quad \text{——— ⑤}$$

From ④ & ⑤

$T(cv) = cT(v)$ ——————— (ii)

Thus from (i) & (ii) we see that the given transformation is linear.

ex: Determine whether the following transformations are linear or not.

a) $\mathbb{R}^2$ maps to $\mathbb{R}^3$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \longrightarrow \begin{bmatrix}x-y\\x+y\\2x\end{bmatrix}$$

b) $\mathbb{R}^2$ maps to $\mathbb{R}^2$

$$W\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \longrightarrow \begin{bmatrix}x+y\\y+2\end{bmatrix}$$

Soln:

First verify $T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$

Let $\underline{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\underline{w} = \begin{bmatrix}w_1\\w_2\end{bmatrix}$

LHS

$$T(\underline{v}+w) = T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}+\begin{bmatrix}w_1\\w_2\end{bmatrix}\right) = T\begin{pmatrix}v_1+w_1\\v_2+w_2\end{pmatrix}$$

$$= \begin{bmatrix}v_1+w_1-v_2-w_2\\v_1+w_1+v_2+w_2\\2v_1+2w_1\end{bmatrix} \longrightarrow (1)$$

RHS

$$T(v)+T(\underline{w}) = T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)+T\left(\begin{bmatrix}w_1\\w_2\end{bmatrix}\right)$$

$$= \begin{bmatrix}v_1-v_2\\v_1+v_2\\2v_1\end{bmatrix} + \begin{bmatrix}w_1-w_2\\w_1+w_2\\2w_1\end{bmatrix}$$

$$= \begin{bmatrix}v_1-v_2+w_1-w_2\\v_1+v_2+w_1+w_2\\2v_1+2w_1\end{bmatrix} \longrightarrow (2)$$

from ① & ②

$T(v+w) = T(v) + T(w)$ ——— (i)

Now to verify $T(c\underline{v}) = cT(\underline{v})$

LHS

$T(cv) = T\left(\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}\right)$

$= \begin{bmatrix} cv_1 - cv_2 \\ cv_1 + cv_2 \\ 2cv_1 \end{bmatrix}$ ——— (3)

RHS

$cT\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} cv_1 + cv_2 \\ cv_1 - cv_2 \\ 2c_1 \end{bmatrix}$ ——— (4)

from ③ & ④

$T(cv) = cT(v)$. ——— (ii)

Thus, from (i) & (ii) we see that the given transformation is linear.

(b) $W\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \longrightarrow \begin{bmatrix} x+y \\ y+2 \end{bmatrix}$

Soln: First verify $T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$

LHS

$$T(v+w) = T\left(\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} v_1+w_1+v_2+w_2 \\ v_2+w_2+2 \end{bmatrix} \text{———— (1)}$$

RHS

$$T(\underline{v}) + T(\underline{w}) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} v_1+v_2 \\ v_2+2 \end{bmatrix} + \begin{bmatrix} w_1+w_2 \\ w_2+2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1+v_2+w_1+w_2 \\ v_2+w_2+4 \end{bmatrix} \text{———— (2)}$$

Since from (1) & (2)

$$T(\underline{v}+\underline{w}) \neq T(\underline{v}) + T(\underline{w})$$

Therefore, the transformation is not linear

Matrix Representation of Linear Transformation

Let $T: V \to W$

Let V is a n dimensional vector space

W is a m dimensional vector space.

Consider a basis for V as $e_V = \{\underline{v}_1, \underline{v}_2, --- \underline{v}_n\}$

Also consider " " " W " $e_W = \{\underline{w}_1, \underline{w}_2, ---, \underline{w}_m\}$

Since $T: V \to W$

$\therefore$ $T(\underline{v}_1) \in W$, $T(v_2) \in W$, - — $T(v_n) \in W$

i.e. $T(\underline{v}_1), T(v_2), ---, T(v_n)$ are vectors of W.

$\{\underline{w}_1, \underline{w}_2, ——, w_m\}$ is a basis for W.

Therefore every vector $T(\underline{v}_1), --, T(\underline{v}_n)$ can be written as linear combination of $\{\underline{w}_1, \underline{w}_2, -- w_m\}$

$\therefore$ $T(\underline{v}_1) = a_{11}\underline{w}_1 + a_{12}w_2 + --- + a_{1m}\underline{w}_m$

$T(\underline{v}_2) = a_{21}\underline{w}_1 + a_{22}\underline{w}_2 + —— + a_{2m}\underline{w}_m$

$\vdots$

$T(\underline{v}_n) = a_{n1}\underline{w}_1 + a_{n2}\underline{w}_2 + ——— + a_{nm}\underline{w}_m$.

Now, the coefficient matrix of above system of equations is

$$\begin{bmatrix} a_{11} & a_{12} & —— & a_{1m} \\ a_{21} & a_{22} & —— & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & —— & a_{nm} \end{bmatrix}$$

The transpose of above matrix is the matrix of linear transformation T with respect to the basis e_v and e_w, written as $[T]_{e_v}^{e_w}$.

Thus $[T]_{e_v}^{e_w} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & & a_{nm} \end{bmatrix}$

Note:

It will be the matrix of order $m \times n$

If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

then $[T]_{e_2}^{e_3}$ will be of order 3×2.

Ex: If $T: \mathbb{R}^3 \to \mathbb{R}^2$

and $T(x, y, z) = (x+y, y+z)$; $x, y, z \in \mathbb{R}$

Find the matrix representation for the linear transformation T with respect to standard basis.

Soln: Here $e_3 = \{\underset{u_1}{(1,0,0)}, \underset{u_2}{(0,1,0)}, \underset{u_3}{(0,0,1)}\}$

$e_2 = \{(1,0), (0,1)\}$

Since $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(1,0,0) = (1+0, 0+0) = (1, 0)$

$T(0,1,0) = (0+1, 1+0) = (1,1)$

$T(0,0,1) = (0+0, 0+1) = (0,1)$

Now

$(1,0) = 1(1,0) + 0(0,1)$

$(1,1) = 1(1,0) + 1(0,1)$

$(0,1) = 0(1,0) + 1(0,1)$

$\therefore$ Matrix of T, related to e_2 & e_3 is

$$[T]_{e_2}^{e_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Remark: If we now change the basis e_2 to e_2'

where $e_2' = \{(1,0), (1,1)\}$

then again

$T(1,0,0) = (1,0)$

$T(0,1,0) = (1,1)$

$T(0,0,1) = (0,1)$

Now

$(1,0) = c_1(1,0) + c_2(1,1) \Rightarrow \left.\begin{matrix} c_1 + c_2 = 1 \\ c_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix}$

$\therefore (1,0) = 1(1,0) + 0(1,1)$

Similarly

$(1,1) = 0(1,0) + 1(1,1)$

and $(0,1) = -1(1,0) + 1(1,1)$

Thus $[T]_{e_2'}^{e_3} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

Ex. Find the matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$ represented by $T(a, b, c) = (2b+c, a-4b, 3a)$ with respect to the basis

1) $B = \{(1,0,0)\ (0,1,0), (0,0,1)\}$

2) $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$

Change of Basis (Transition Matrix)

Consider in $\mathbb{R}^2$, two basis

$$B_1 = \left\{ \underset{\underline{u}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{\underline{u}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\} \text{ and } B_2 = \left\{ \underset{\underline{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \underset{\underline{v}_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} \right\}$$

Since v_1 & $v_2 \in \mathbb{R}^2$

$\therefore$ we can write

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \underline{u}_1 + 1 \cdot \underline{u}_2$$

$$\underline{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \underline{u}_1 + 0 \underline{u}_2$$

$\underline{v}_1$ & $\underline{v}_2$ are written as the linear combination of $\underline{u}_1$ & $\underline{u}_2$

Again writing it in matrix form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

↑

matrix of coefficient

Taking transpose of coefficient matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \longrightarrow \text{Change of Basis matrix or Transition matrix}$$

Transition matrix is always <u>invertible</u>

Thus

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

<u>Defn.</u> Let $B_1 \{ \underline{u}_1, \dots u_n \}$ be a basis of vector space V and $B_2 = \{ \underline{v}_1, \dots \underline{v}_n \}$ be an another basis of V. Each element in B_2 can be expressed expressed as linear combination of vectors in B_1 as

$$\underline{v}_1 = a_{11} \underline{u}_1 + a_{12} \underline{u}_2 + \dots + a_{1n} u_n$$

$$\underline{v}_2 = a_{21} \underline{u}_1 + a_{22} u_2 + \dots + a_{2n} u_n$$

$$\vdots$$

$$\underline{v}_n = a_{n1} \underline{u}_1 + a_{n2} \underline{u}_2 + \dots + a_{nn} \underline{u}_n$$

Then $\begin{bmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_n \end{bmatrix}$

Then P is the change of basis matrix (transition matrix)

where $P = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ a_{12} & & a_{n2} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix}$

Similar Matrices

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and take any invertible matrix which can be multiplied by A.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 \\ 1 & 5 \end{bmatrix} = B$$

$$P^{-1}AP = \begin{bmatrix} -1 & -8 \\ 1 & 5 \end{bmatrix} = B$$

Now consider matrix A & B (they are similar matrix)

$|A| = 3 \quad |B| = -5+8 = 3$ —— (1)

Also eigen values of A & B are same

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = \underline{\lambda^2 - 4\lambda + 3}$$

$$|B - \lambda I| = \begin{vmatrix} -1-\lambda & -8 \\ 1 & 5-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda) + 8 = -5 + \lambda - 5\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 4\lambda + 3$$

Also rank of A & B are same.

Thus A & B are similar matrices

(A can have several similar matrices)

<u>Defn.</u> A matrix B is said to be similar to A, if $\exists$ a non-singular matrix P s.t.

$$B = P^{-1}AP$$

Also

$$PB = P(P^{-1}AP)$$

$$\boxed{PB = AP}$$

~~Also $B = P^{-1}AP$~~

Also $PBP^{-1} = APP^{-1}$

or $\boxed{A = PBP^{-1}}$

Prove that similar matrices have same eigen values.

Let A & B are similar matrices

then $B = P^{-1}AP$.

Let λ be the eigenvalue of B i.e. $B\underline{x} = \lambda \underline{x}$

$(P^{-1}AP)\underline{x} = \lambda \underline{x}$

Premultiplying both sides by P we get

$P(P^{-1}AP)\underline{x} = P\lambda \underline{x}$

or $(AP)\underline{x} = \lambda P\underline{x}$

or $A(P\underline{x}) = \lambda(P\underline{x})$

i.e. A multiplied by a vector is equal to λ times same vector.

$\therefore$ λ is the eigen value of A.

5 Show that similar matrices have same characteristic eqn.

$$B = P^{-1}AP$$

$$|B-\lambda I| = |P^{-1}AP-\lambda I|$$

$$= |P^{-1}AP - \lambda P^{-1}P|$$

$$= |P^{-1}(AP-\lambda P)|$$

$$= |P^{-1}(A-\lambda I)P|$$

$$= |P^{-1}|\,|(A-\lambda I)|\,|P|$$

$$= |P^{-1}|\,|P|\,|A-\lambda I|$$

$$= |P^{-1}P|\,|A-\lambda I)$$

$$= 1 \cdot |A-\lambda I|$$

$$\therefore |B-\lambda I| = |A-\lambda I|$$