CUSP UCSL Summer 2016:
Introduction to Linear Algebra

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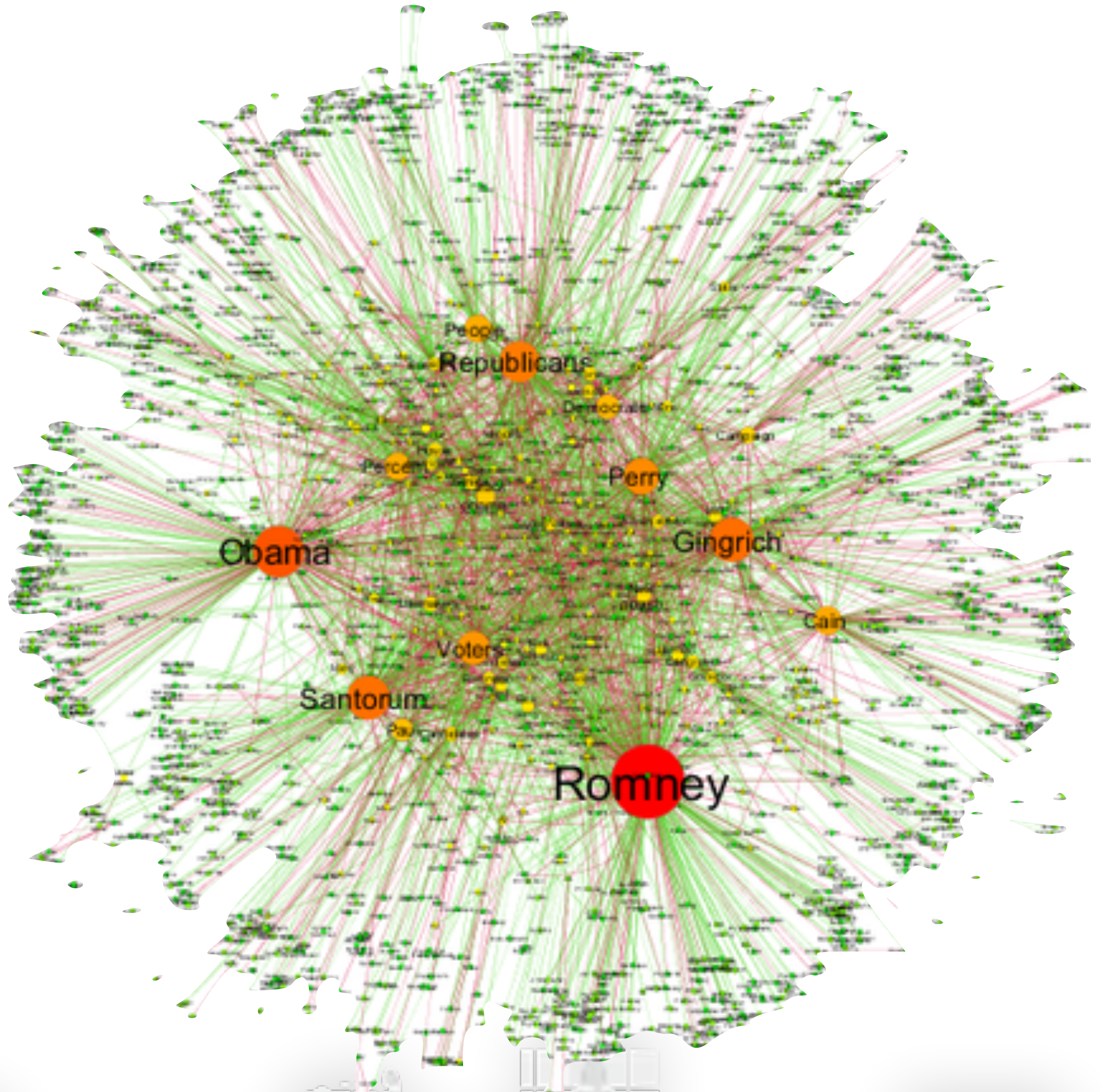
Material related to this lecture can be found at
https://github.com/fedhere/UInotebooks/tree/master/UCSL2016

why do we bother with linear algebra?

LINEAR ALGEBRA ALLOWS US TO SOLVE COMPLEX SYSTEMS OF LINEAR EQUATIONS EFFICIENTLY. Most Data Science and Computational problems can be described by sets of linear equations.

- Finding similarity patterns in data:e.g. network analysis
- Processing an image: e.g. rotation
- Regression



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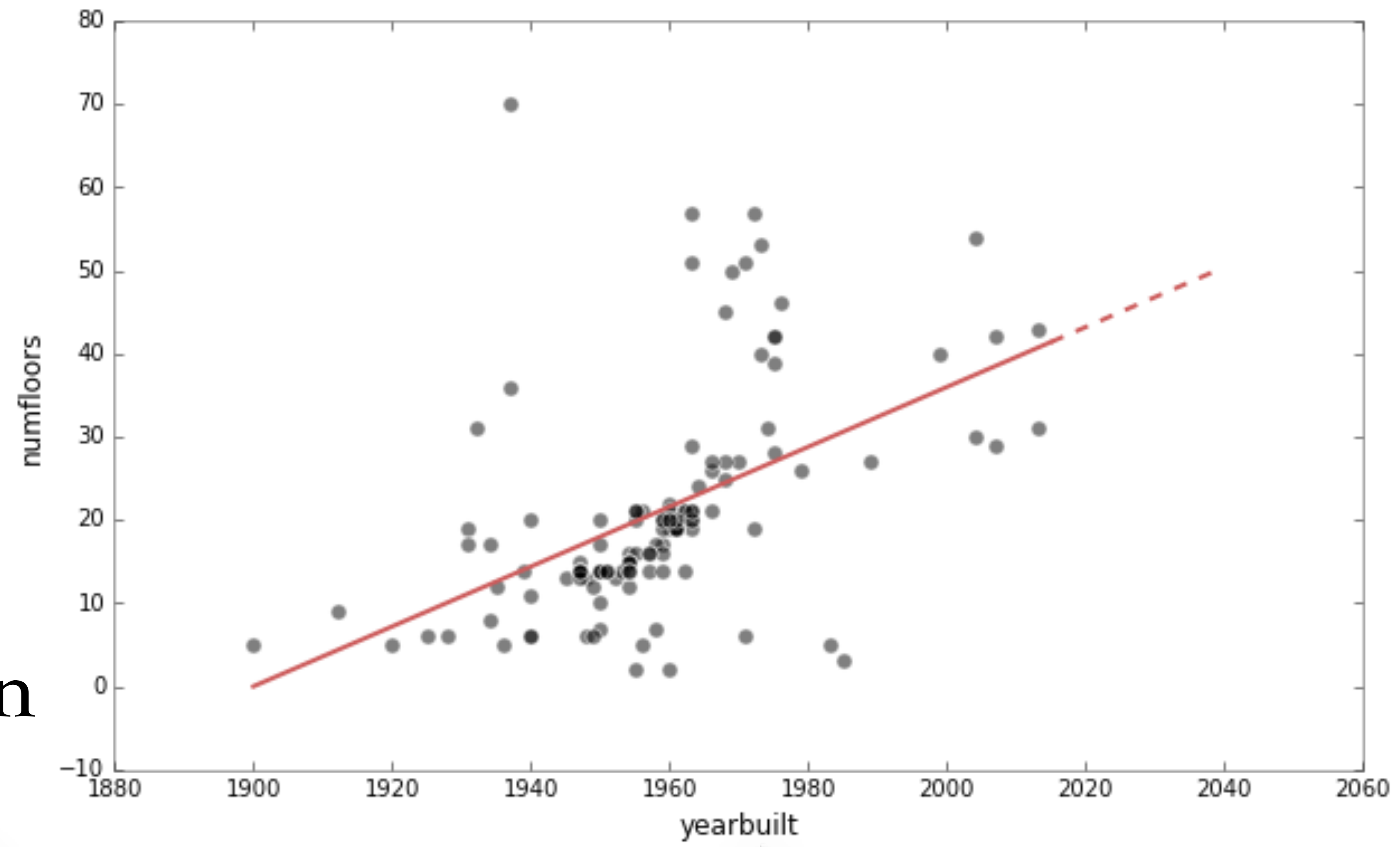
- Finding similarity patterns in data:e.g. network analysis
- Processing an image: e.g. rotation
- Regression: predicting trends, understanding correlation



outline:

- What is a vector
- What is a matrix
- vector & matrix transformations

Vectors

- What is a vector

$$v = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \end{bmatrix}$$

a collection of N numbers corresponding to coordinates in some N-dimensional space

- What is a vector

$$v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

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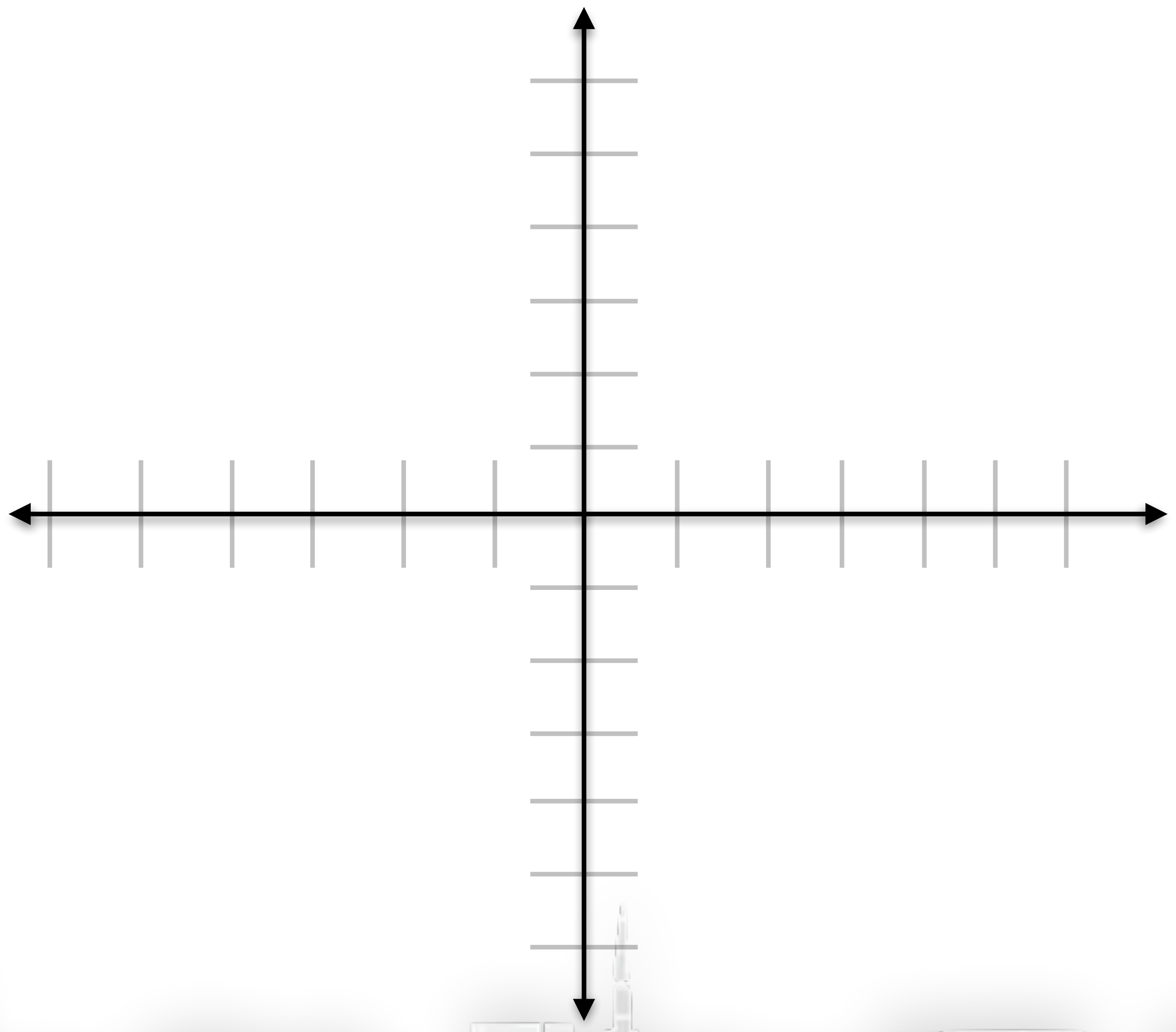
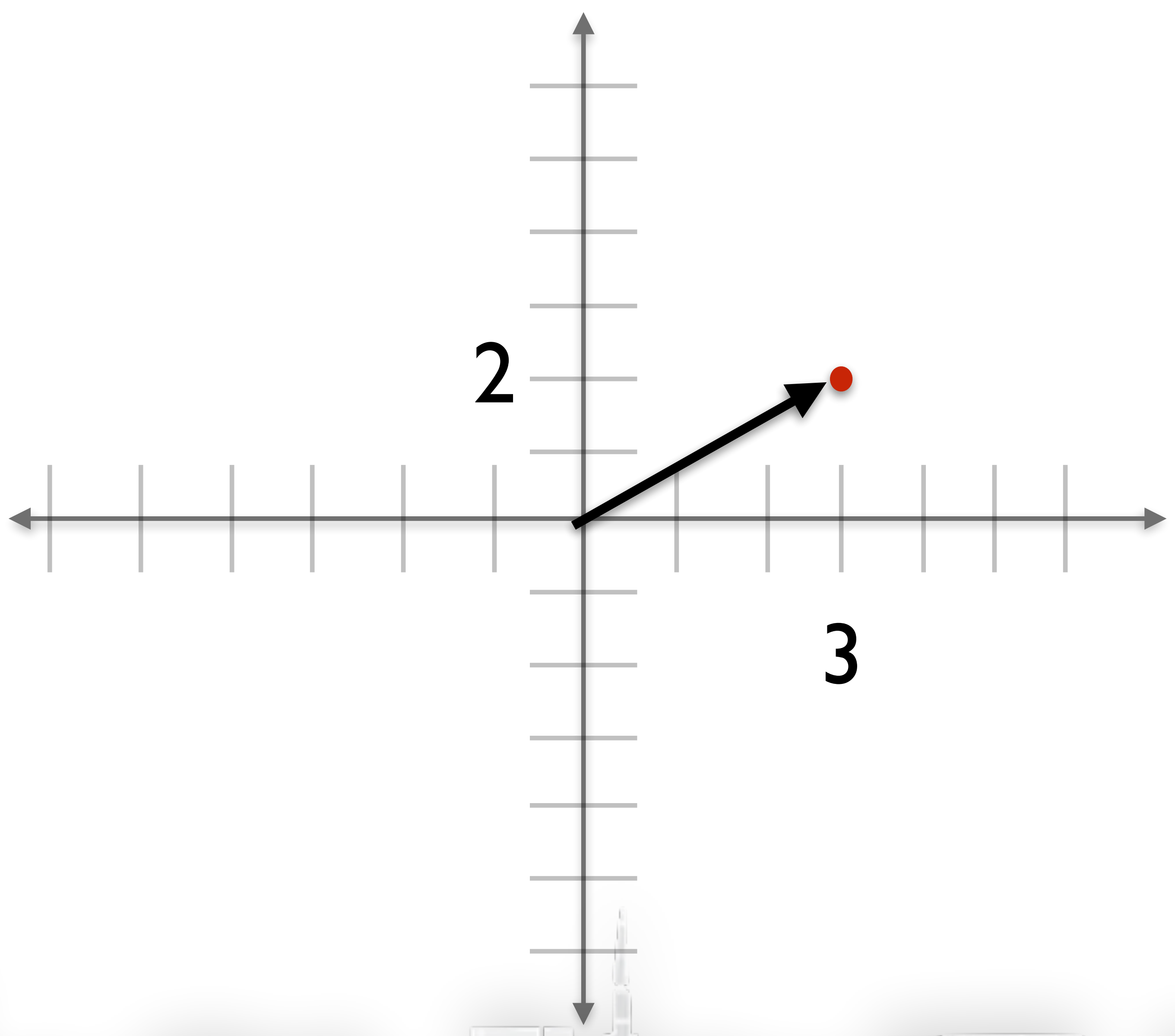


Properties of vectors:

magnitude:

- What is a vector



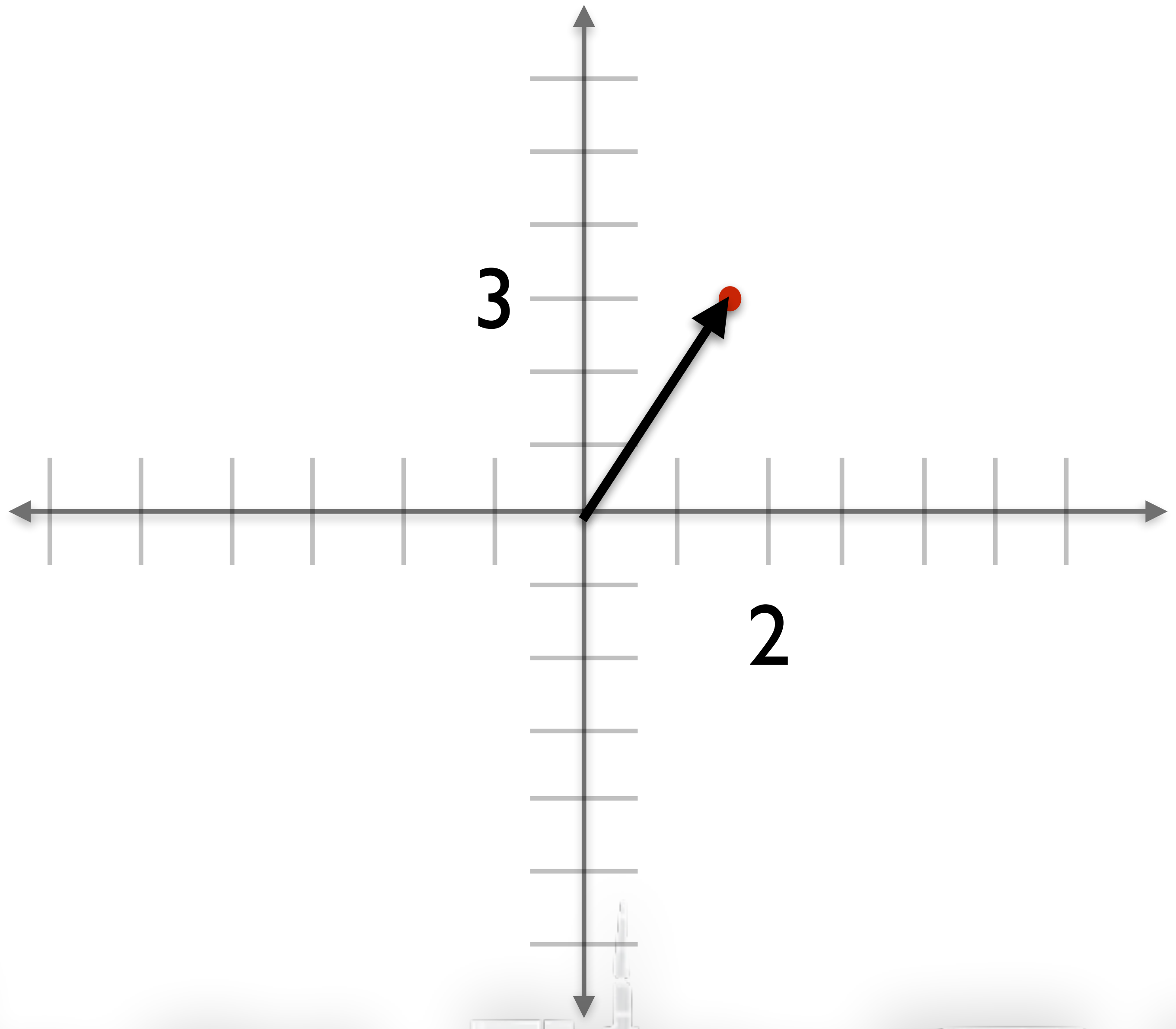
$$v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

Properties of vectors:

magnitude: the length of the vector

- What is a vector



$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

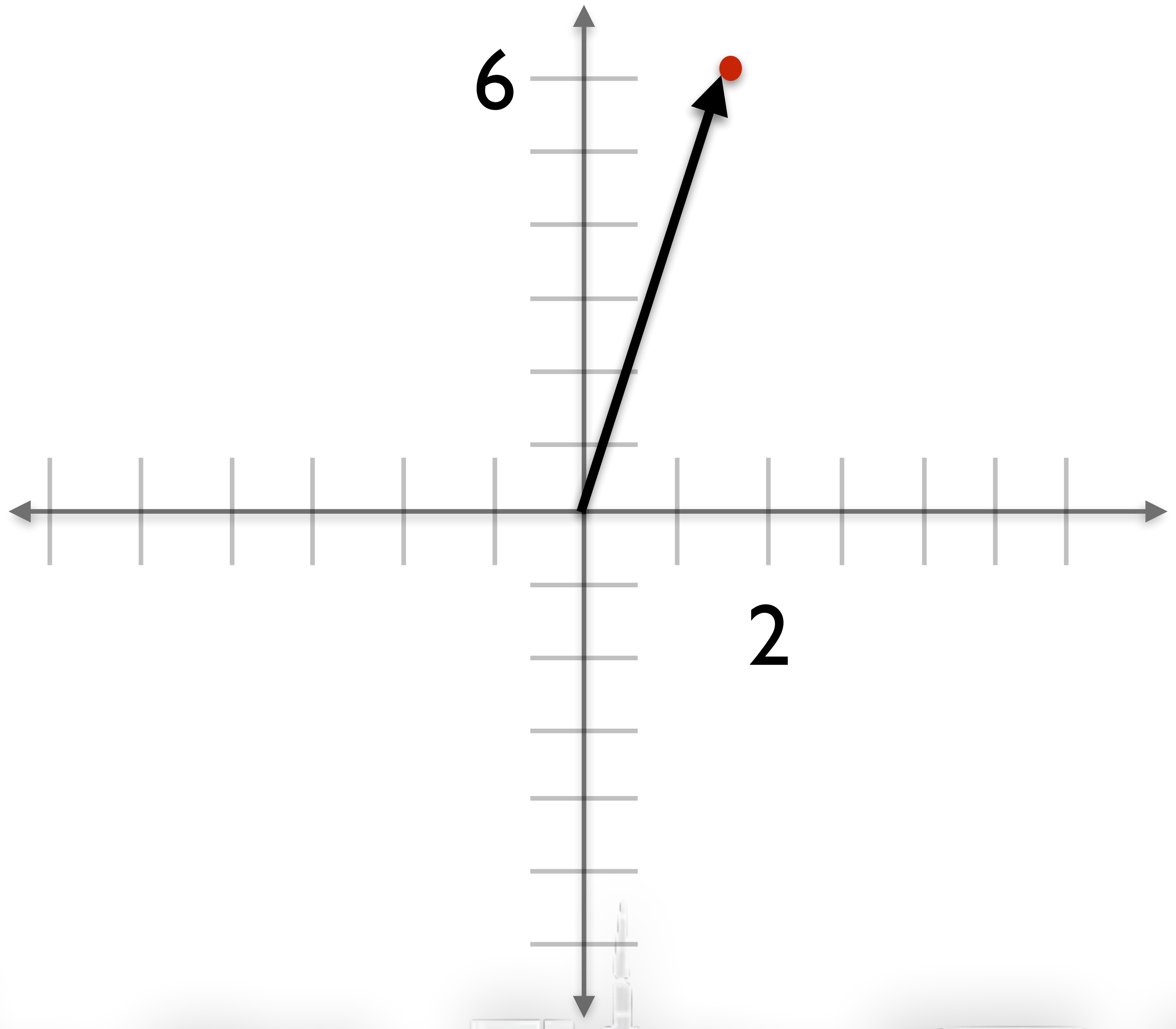
Properties of vectors:

magnitude: the length of the vector

- What is a vector

$$v = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

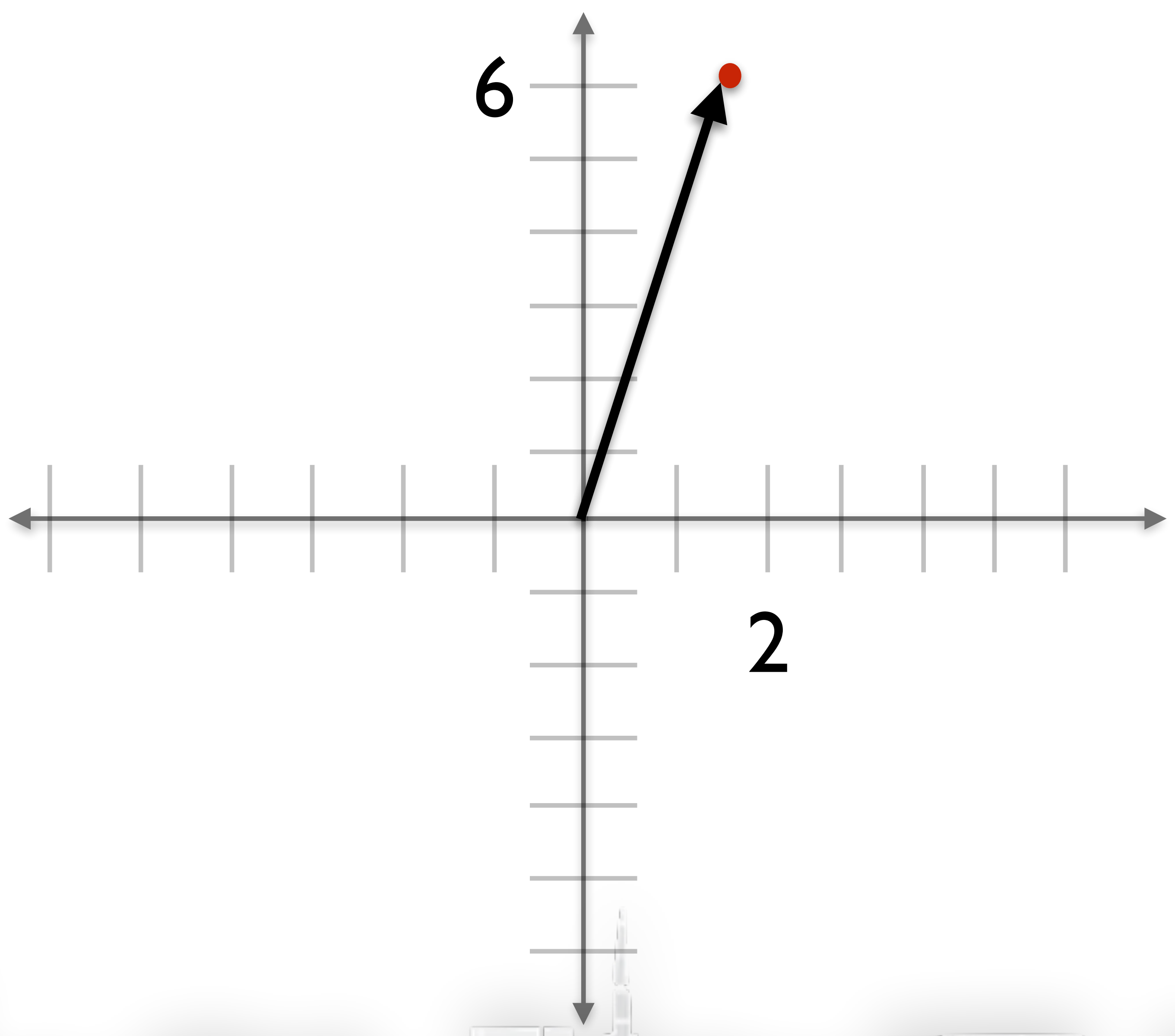
a collection of 2 numbers corresponding to coordinates in some 2-dimensional space



Properties of vectors:

magnitude: the length of the vector

- What is a vector



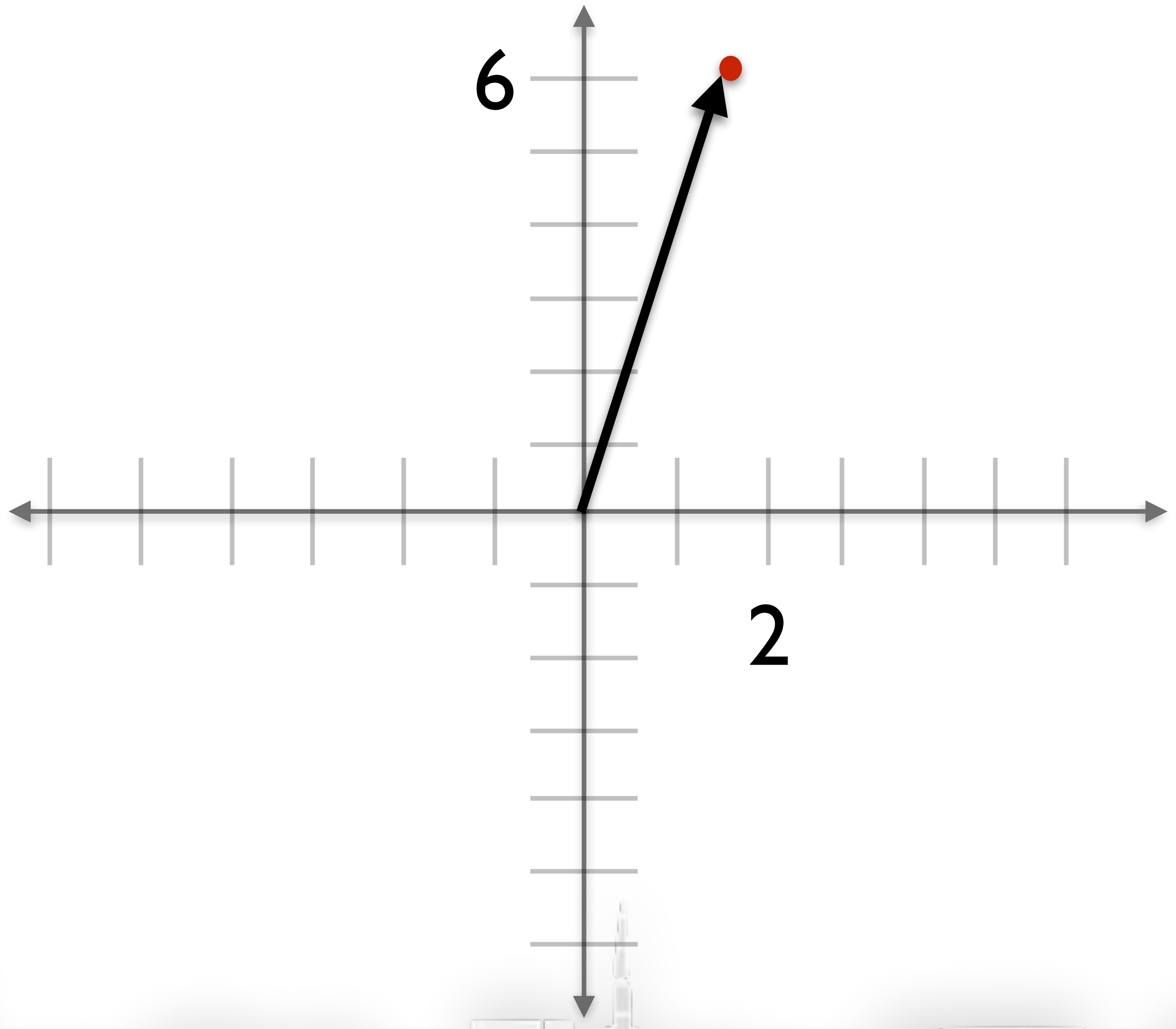
$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

Properties of vectors:

magnitude: $\sqrt{x^2 + y^2}$

- What is a vector



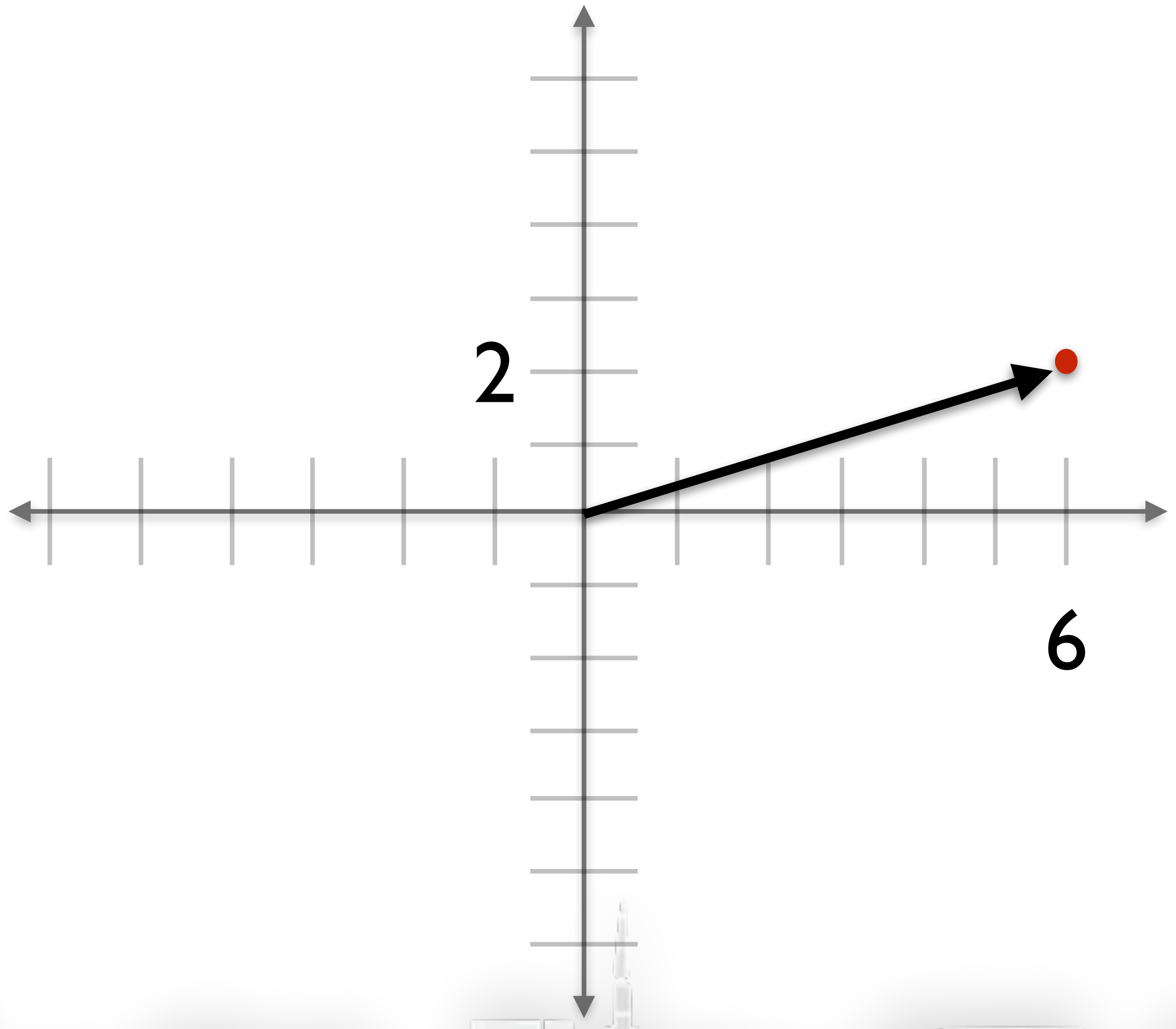
$$v = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

Properties of vectors:

magnitude: $\sqrt{2^2 + 6^2} = 6.3$

- What is a vector



$$v = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

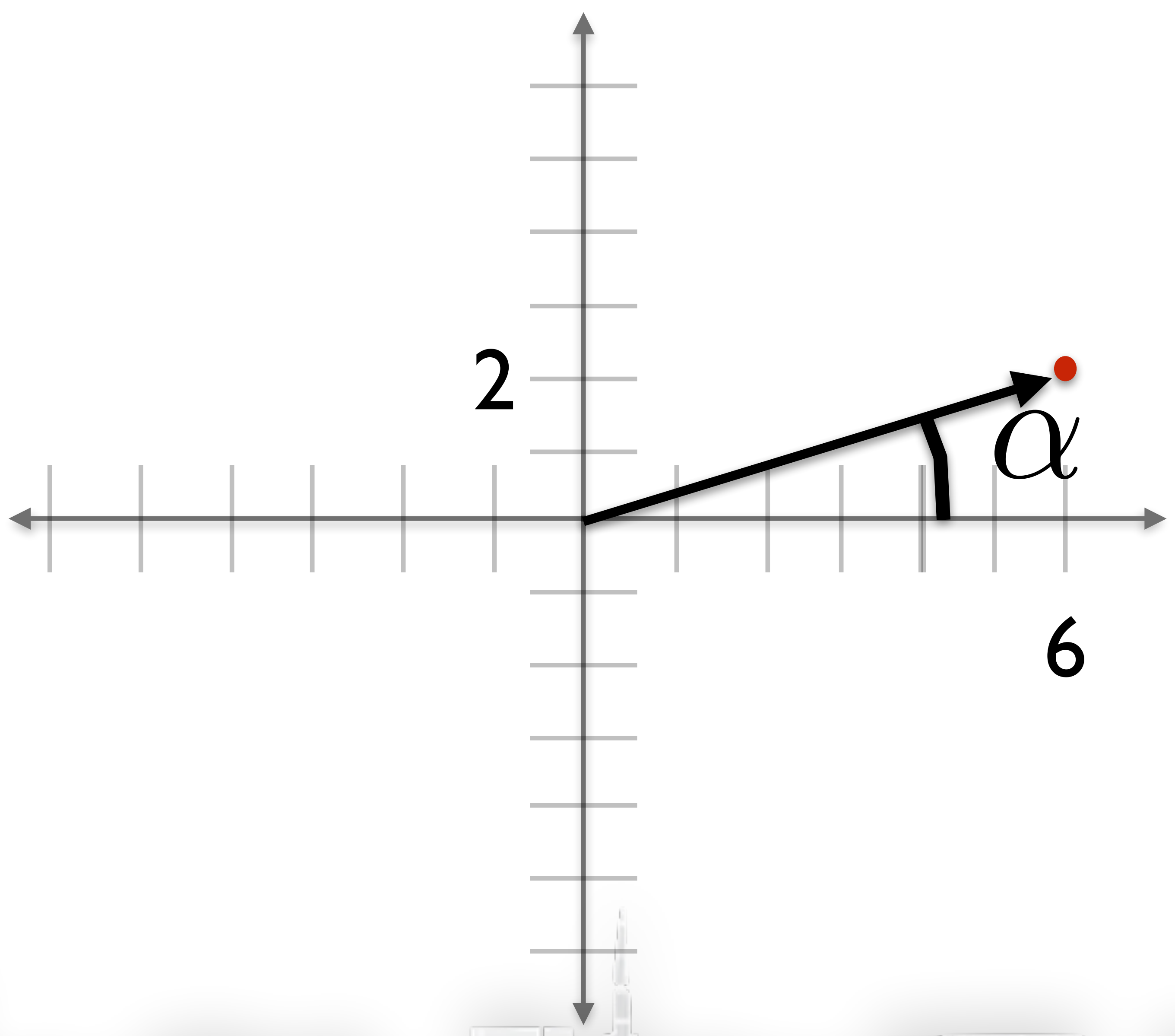
Properties of vectors:

magnitude: $\sqrt{2^2 + 6^2} = 6.3$

- What is a vector

$$v = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

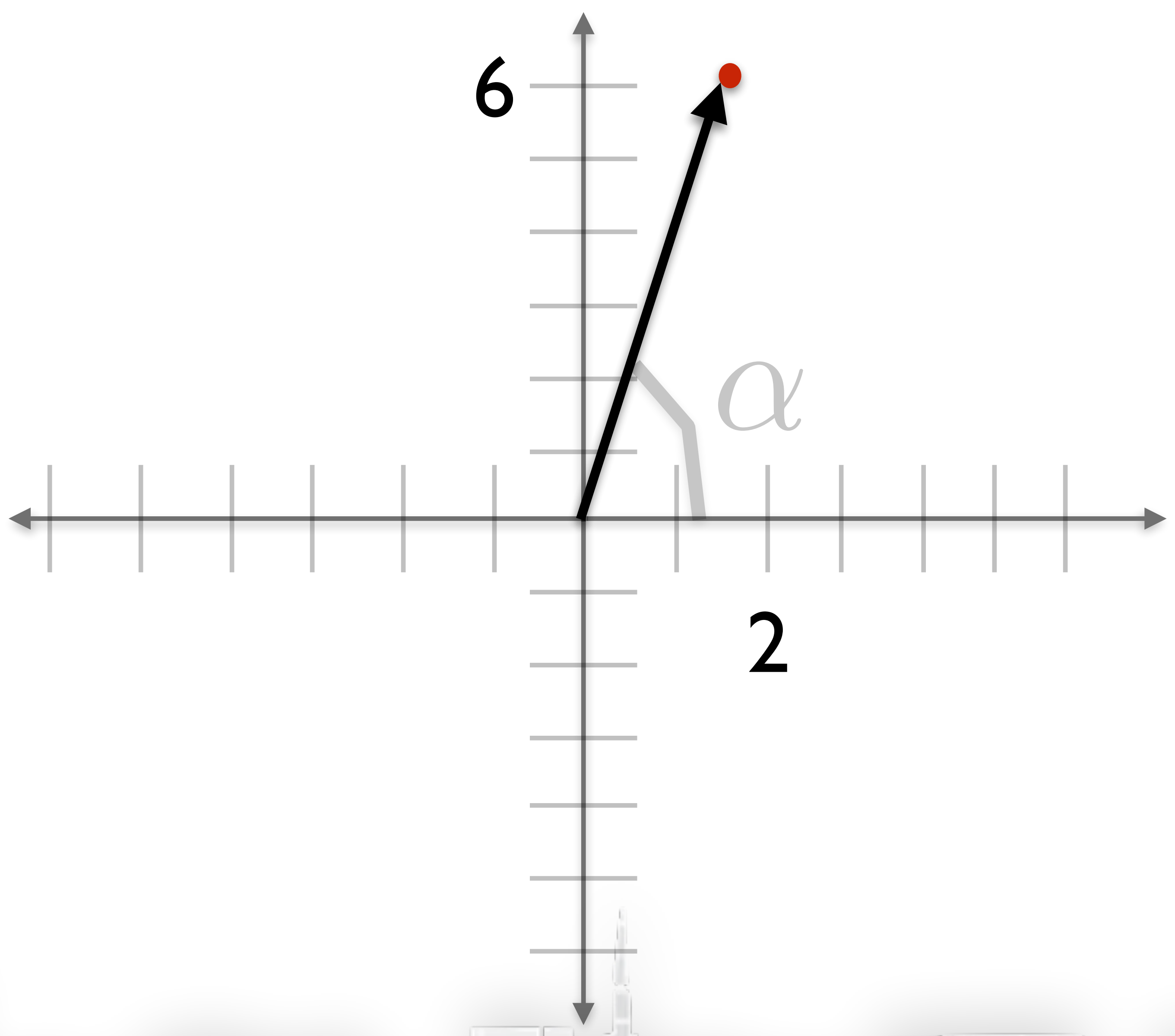


Properties of vectors:

magnitude: $\sqrt{2^2 + 6^2} = 6.3$

direction: $\alpha = 18.5^o = tan^{-1}\frac{x}{y}$

- What is a vector



$$v = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space

Properties of vectors:

magnitude: $\sqrt{2^2 + 6^2} = 6.3$

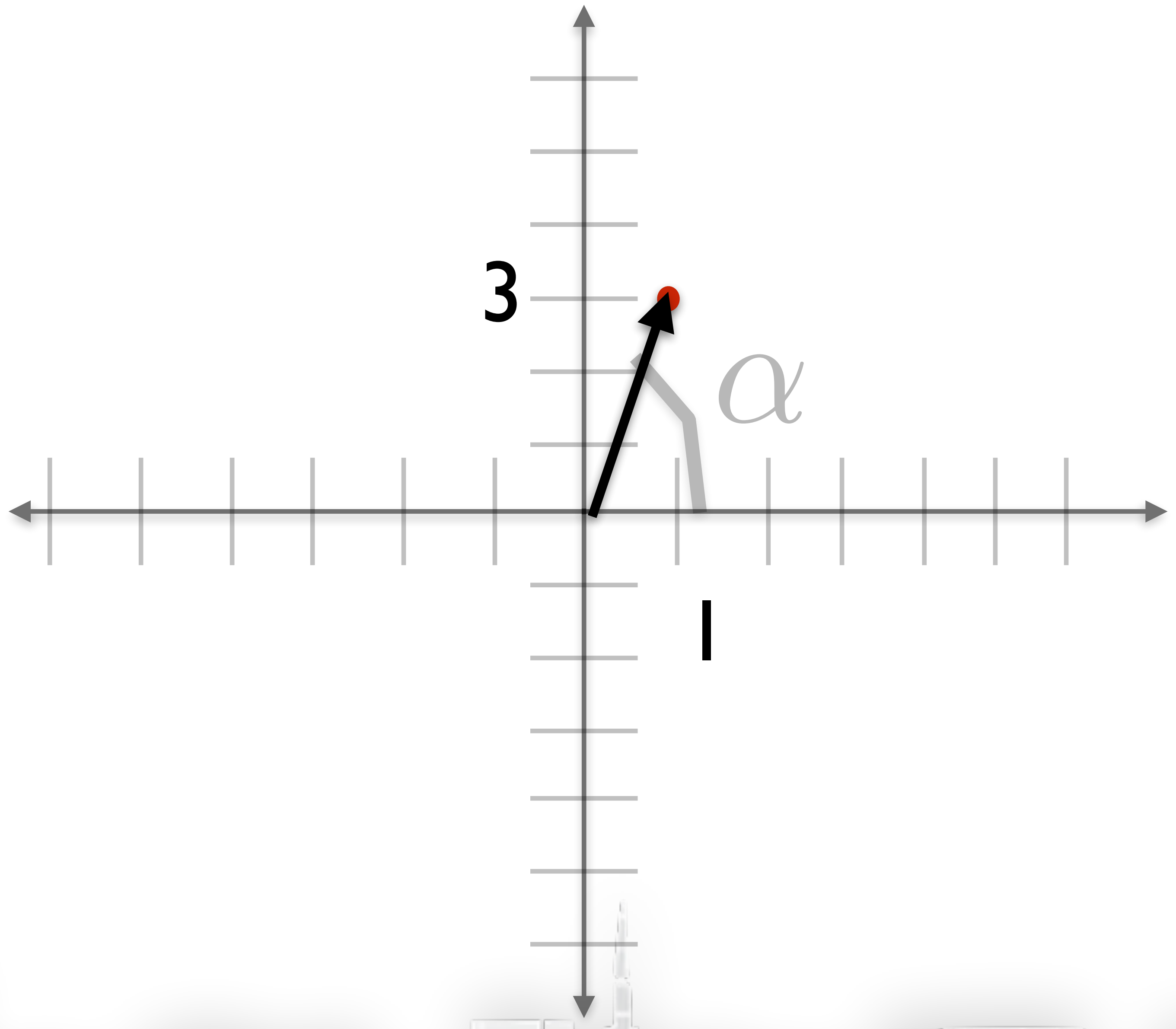
direction: $\alpha = 71.5^o$

Scalar multiplication

- What is a vector

$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

a collection of 2 numbers corresponding to coordinates in some 2-dimensional space



Properties of vectors:

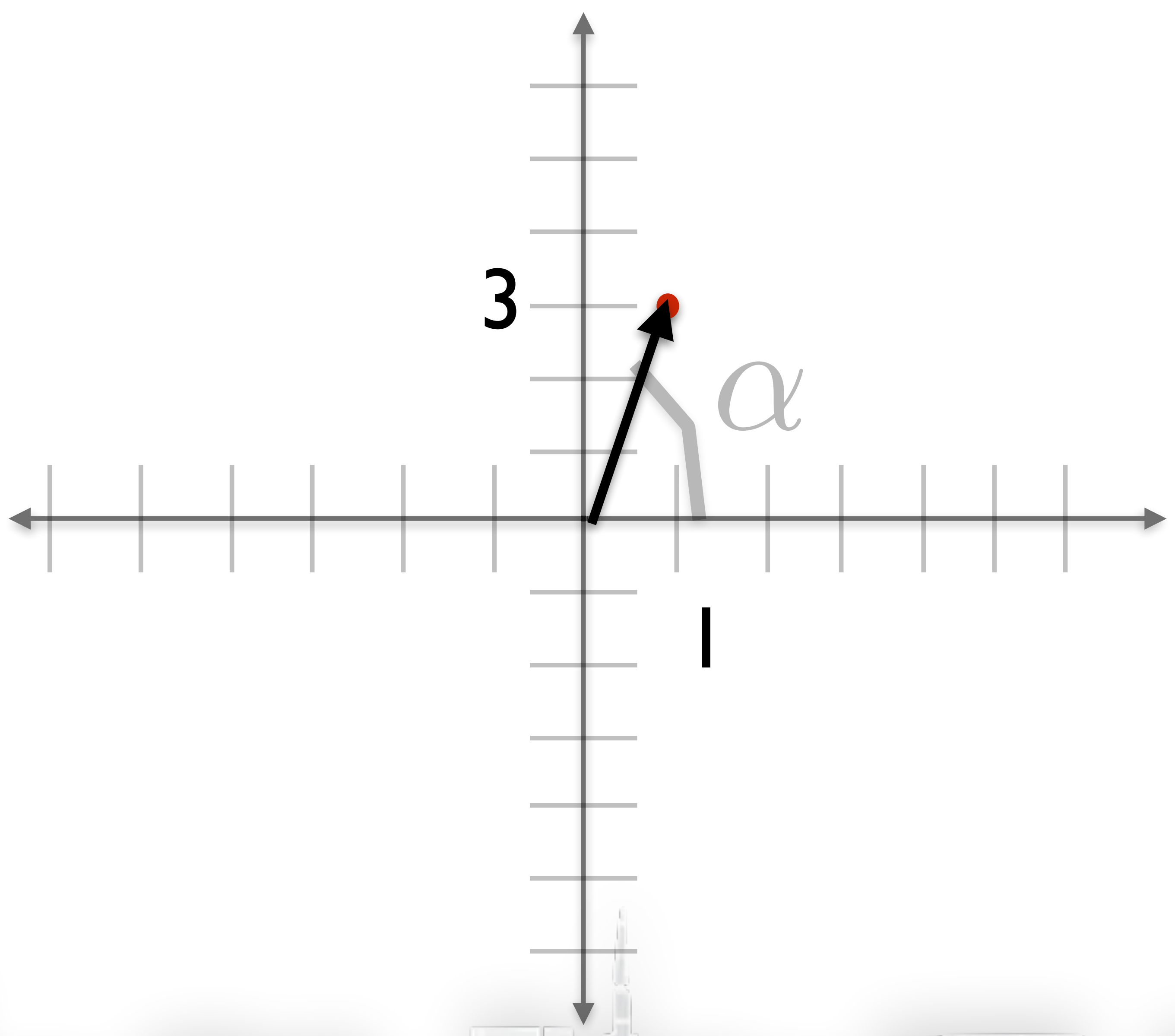
magnitude: $\sqrt{1^2 + 3^2} = 3.15$

direction: $\alpha = 71.5^o$

Scalar multiplication

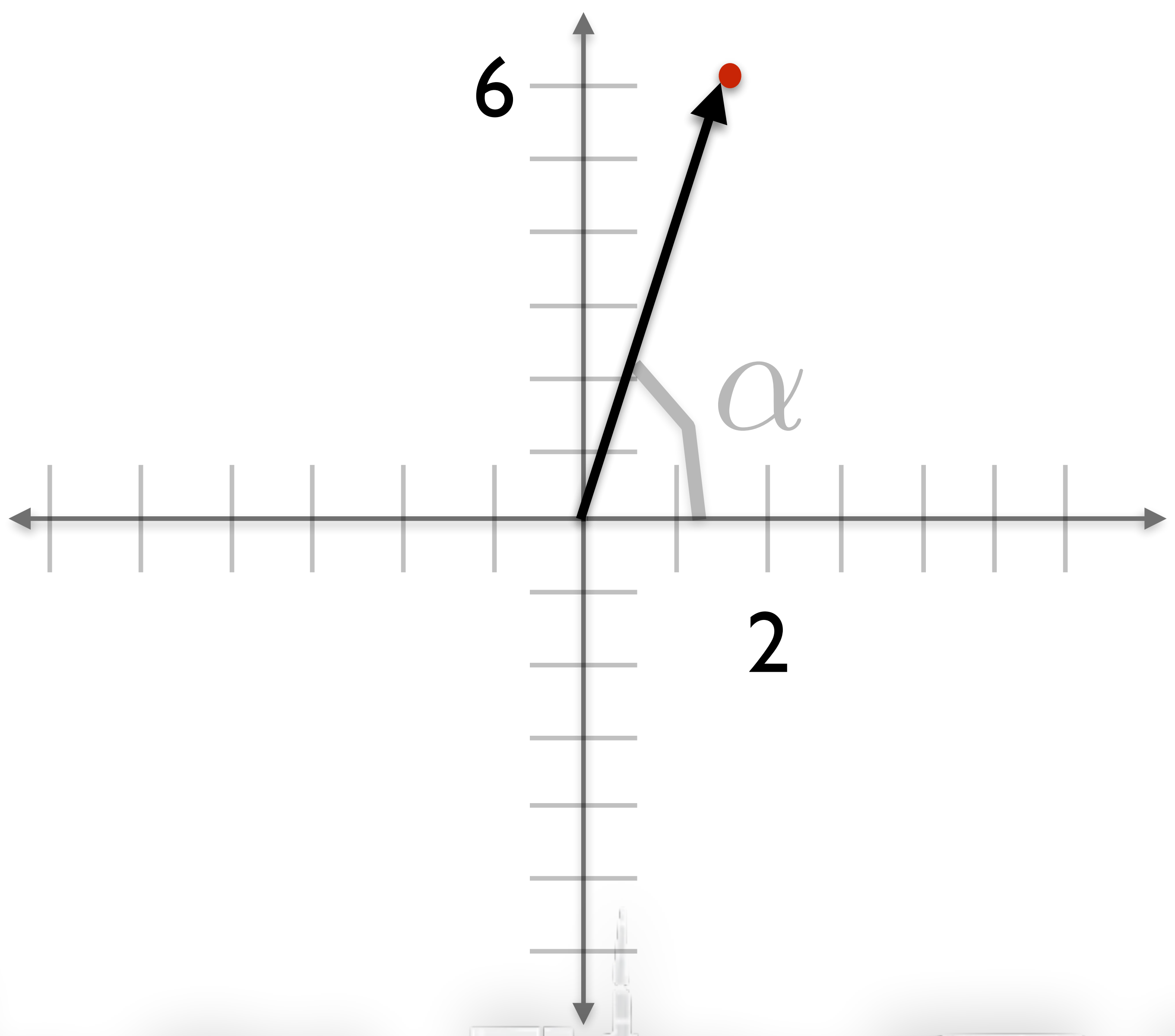
- What is a vector

$$a \cdot v = a \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \cdot 1 \\ a \cdot 3 \end{bmatrix}$$



Scalar multiplication

- What is a vector

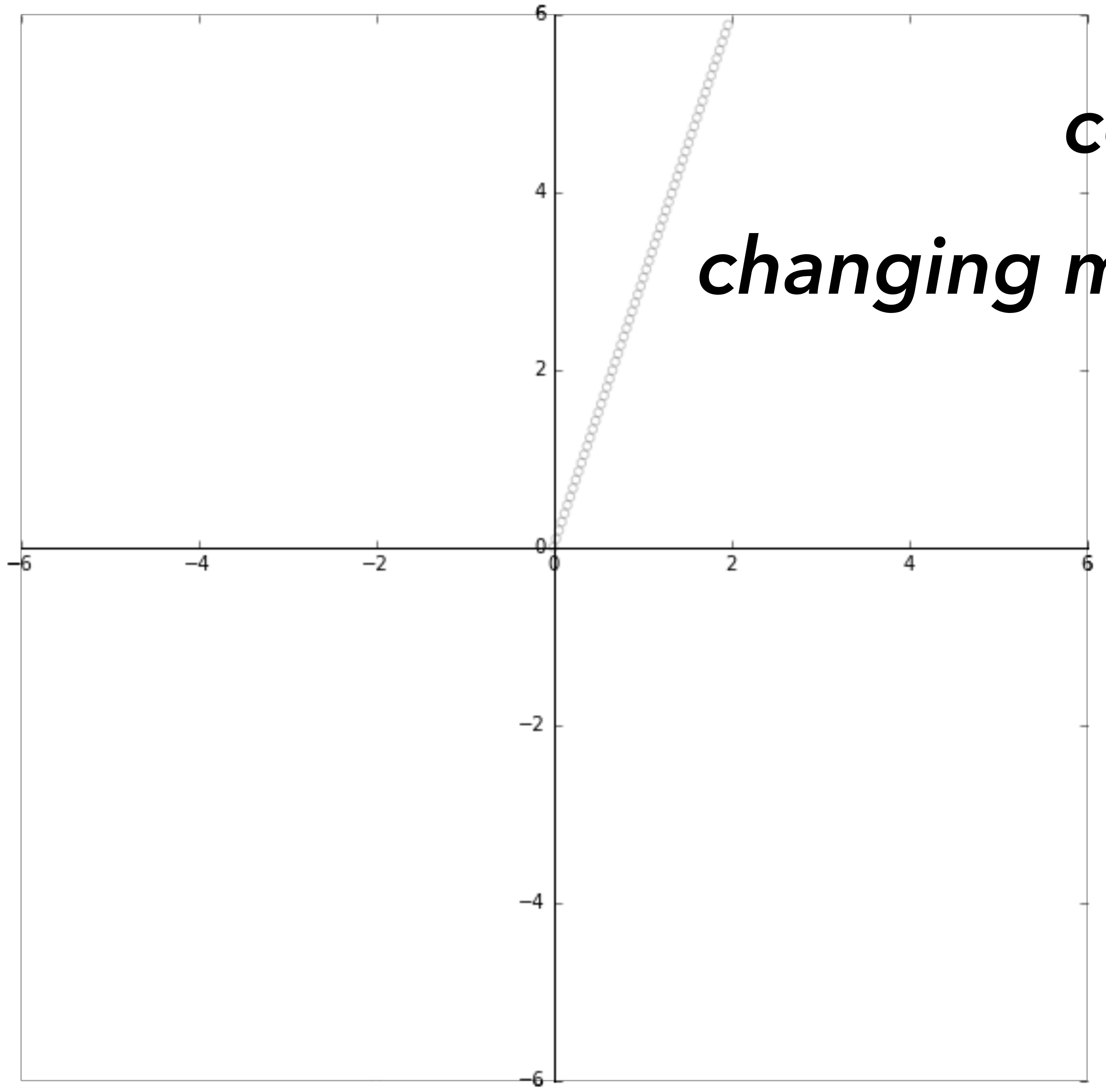


$$a \cdot v = a \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \cdot 1 \\ a \cdot 3 \end{bmatrix}$$

$$2v = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Scalar multiplication

- What is a vector

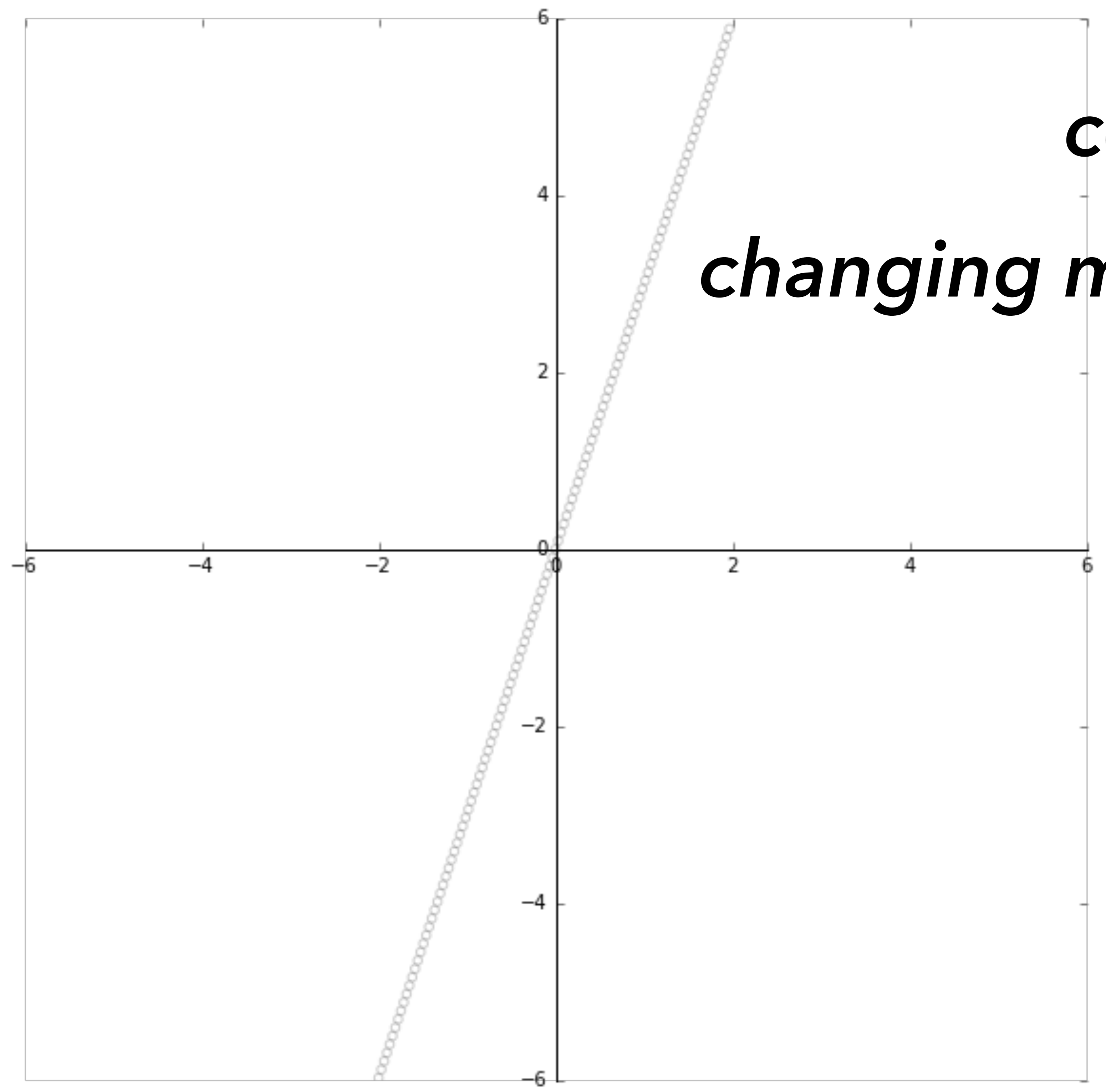


constant direction

changing magnitude 0 to 6

Scalar multiplication

- What is a vector

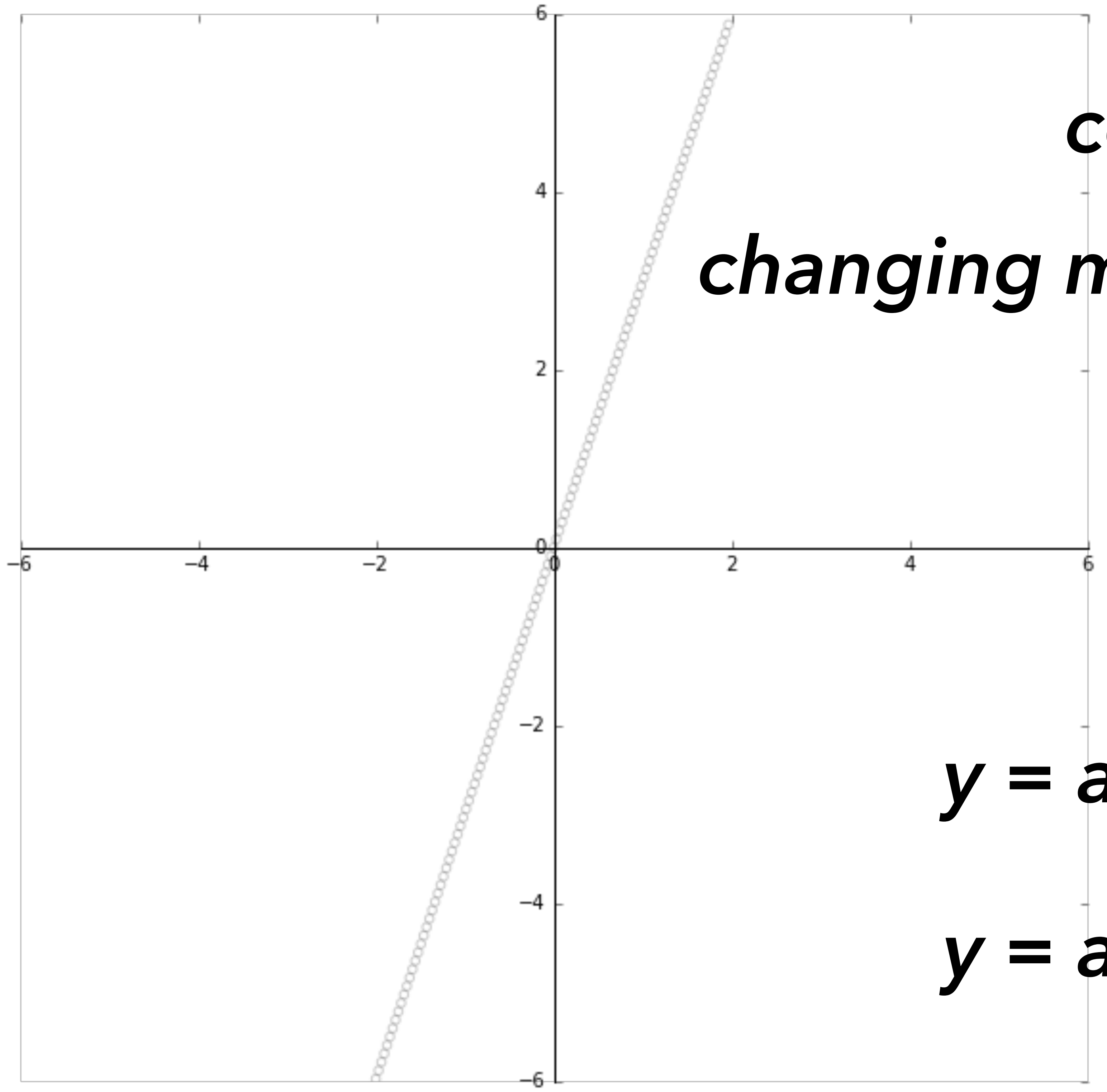


constant direction

changing magnitude -6 to 6

Scalar multiplication

- What is a vector



constant direction

changing magnitude -6 to 6

y = ax + b

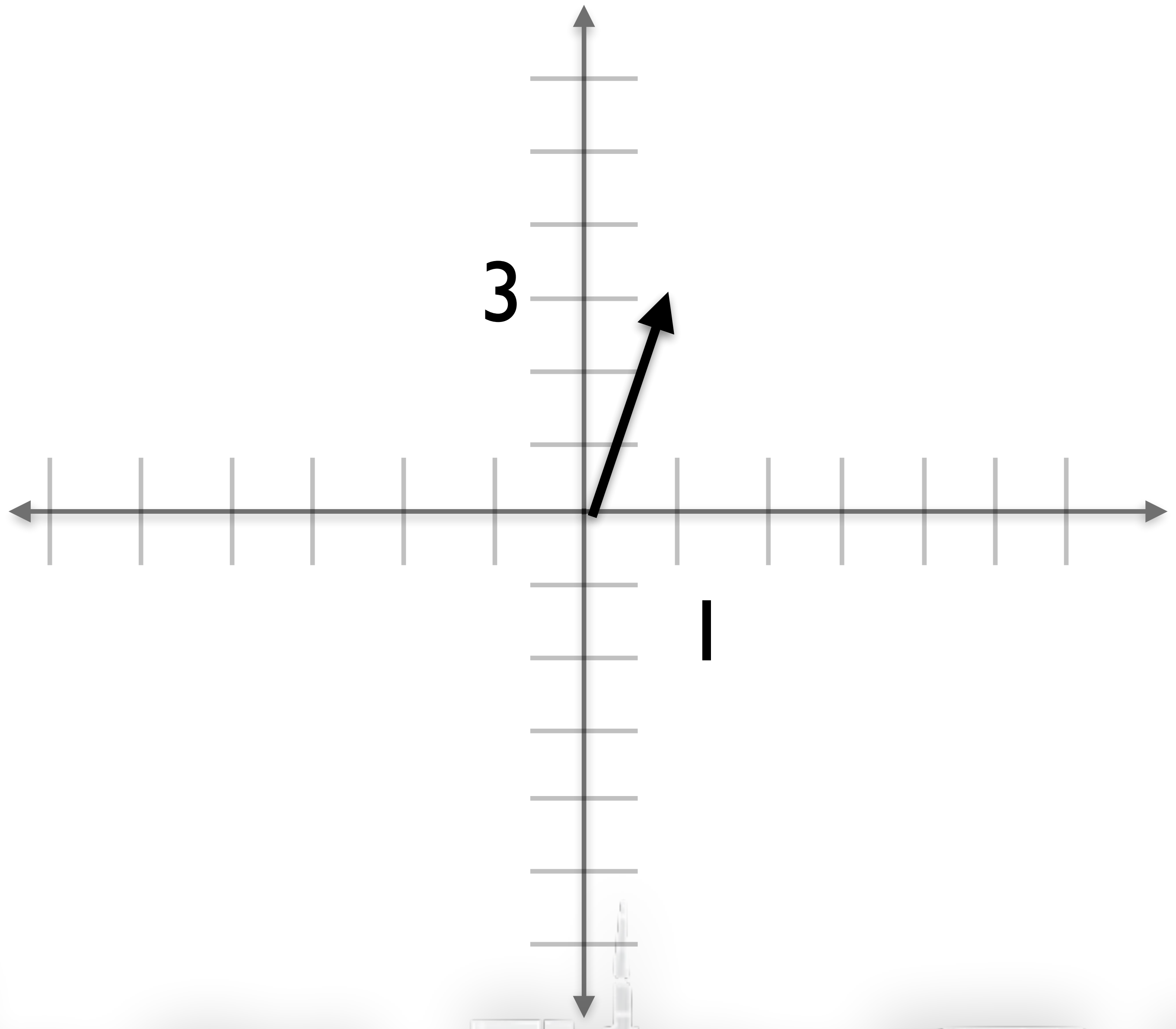
y = ax

a = x/y = 1/3

Vector Sum

- What is a vector

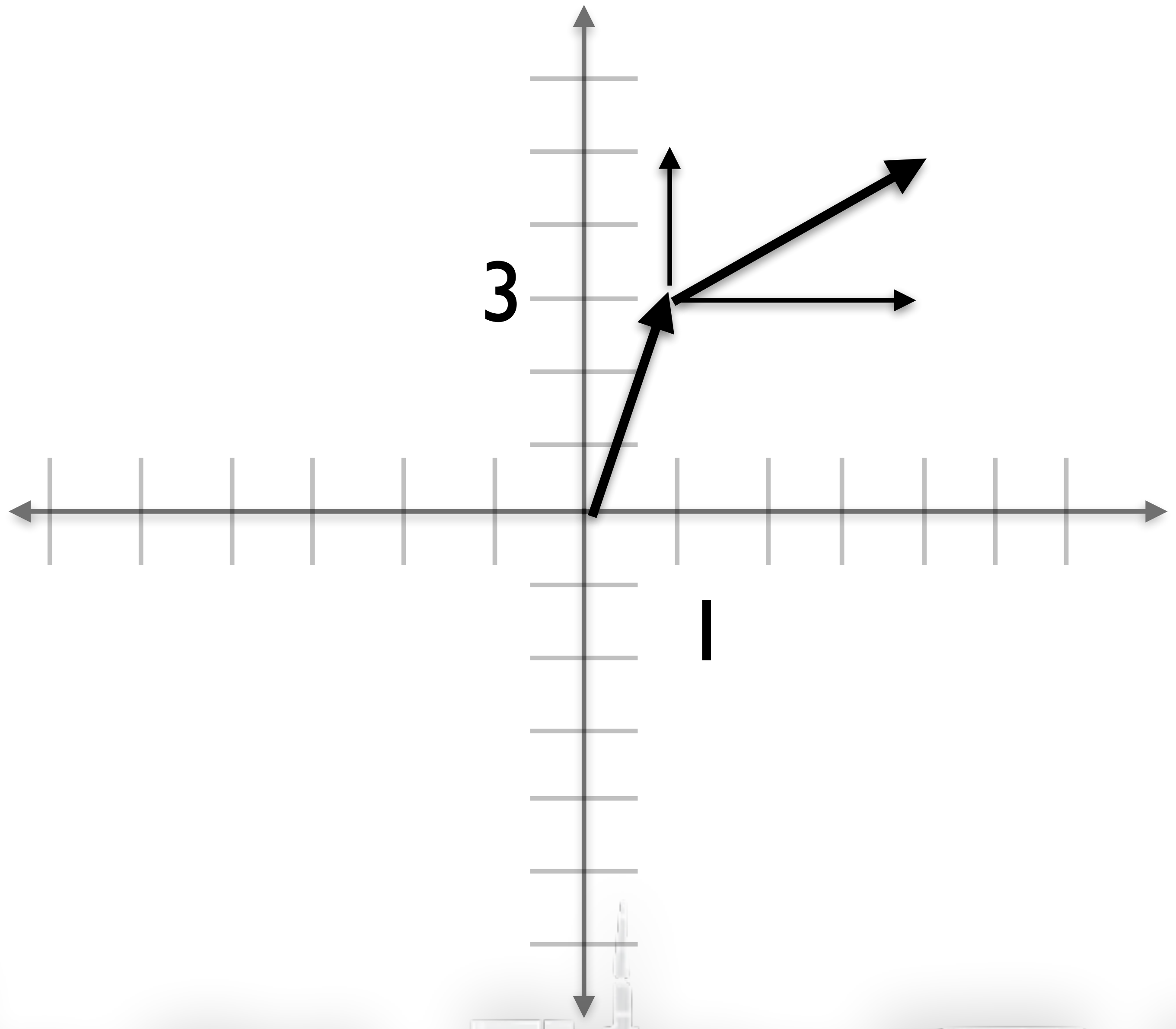
$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Vector Sum

- What is a vector

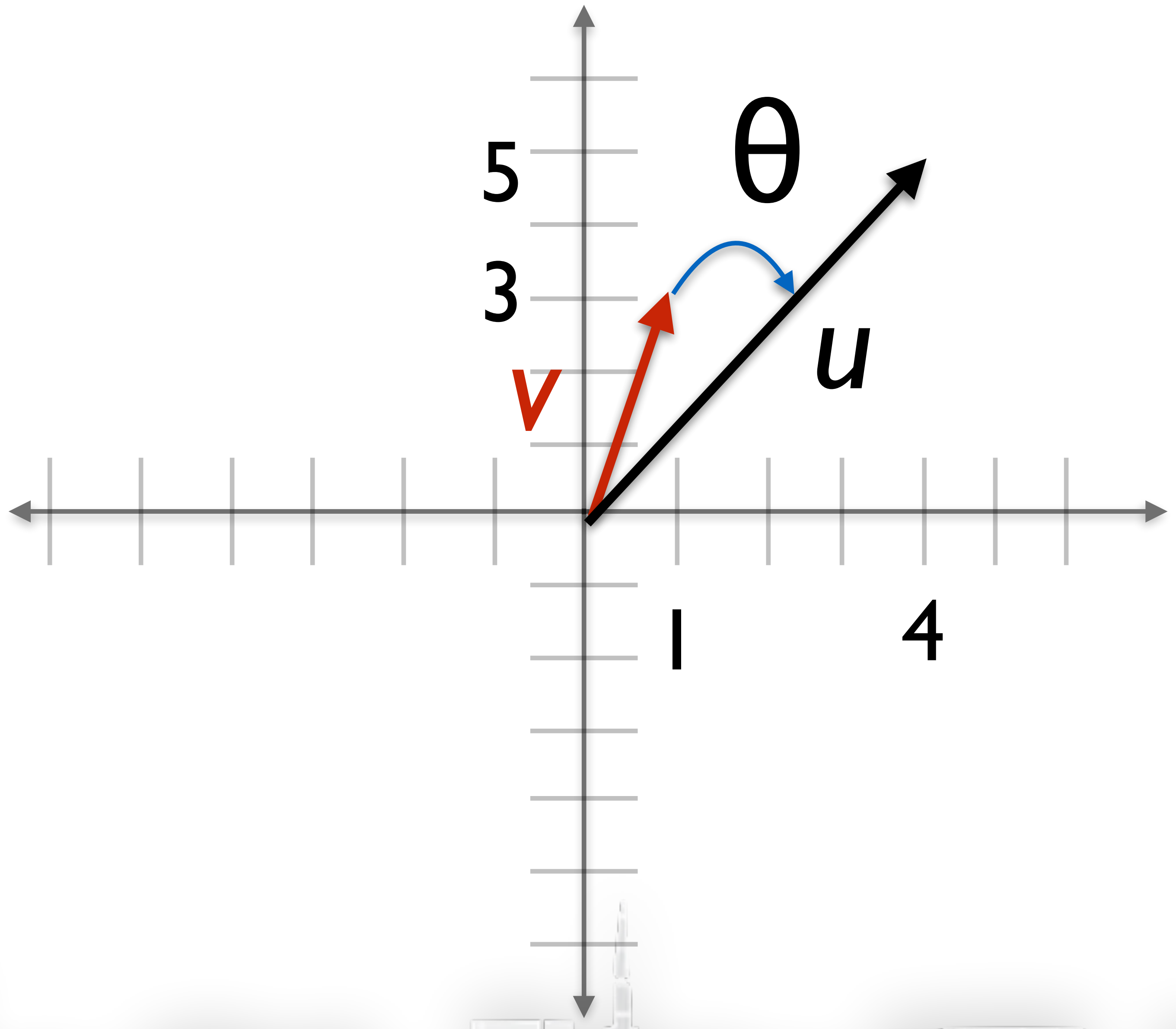
$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$



Dot Product

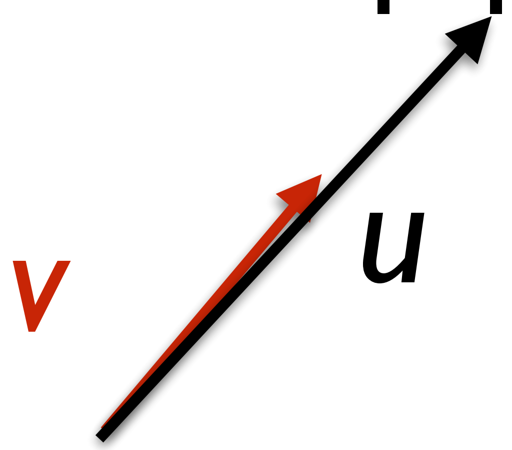
$$u \cdot v = |u||v| \cos\theta$$

Its a measure of similarity between vectors



$\cos(0) = 1$

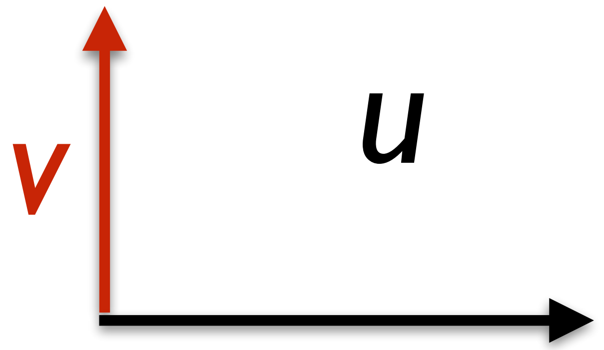
$u \cdot v = |u||v|$



MAX

$\cos(90^o) = 0$

$u \cdot v = 0$



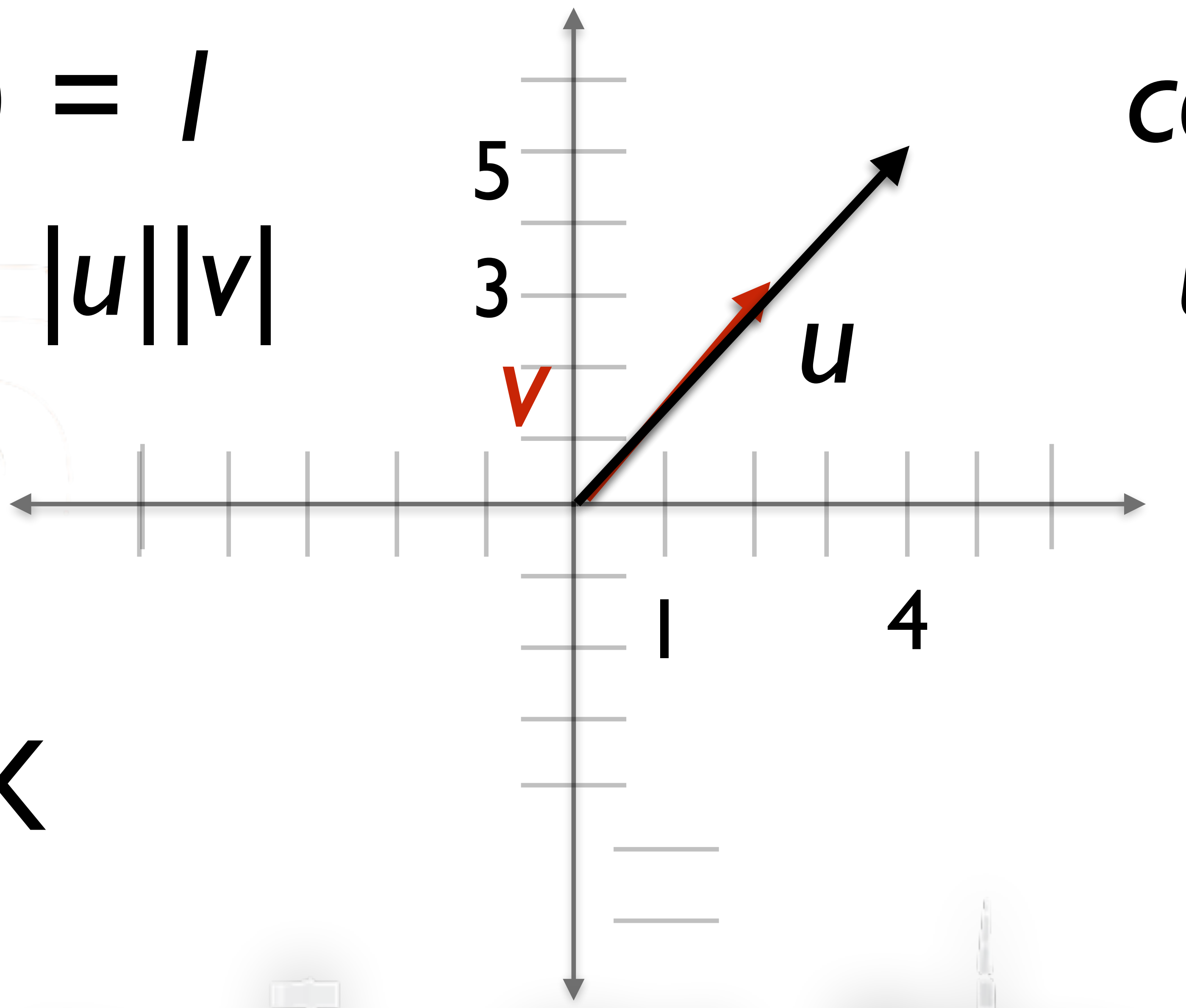
Dot Product

$$u \cdot v = |u||v| \cos\theta$$

Its a measure of similarity between vectors

$\cos(0) = 1$

$u \cdot v = |u||v|$



MAX

$\cos(90^o) = 0$

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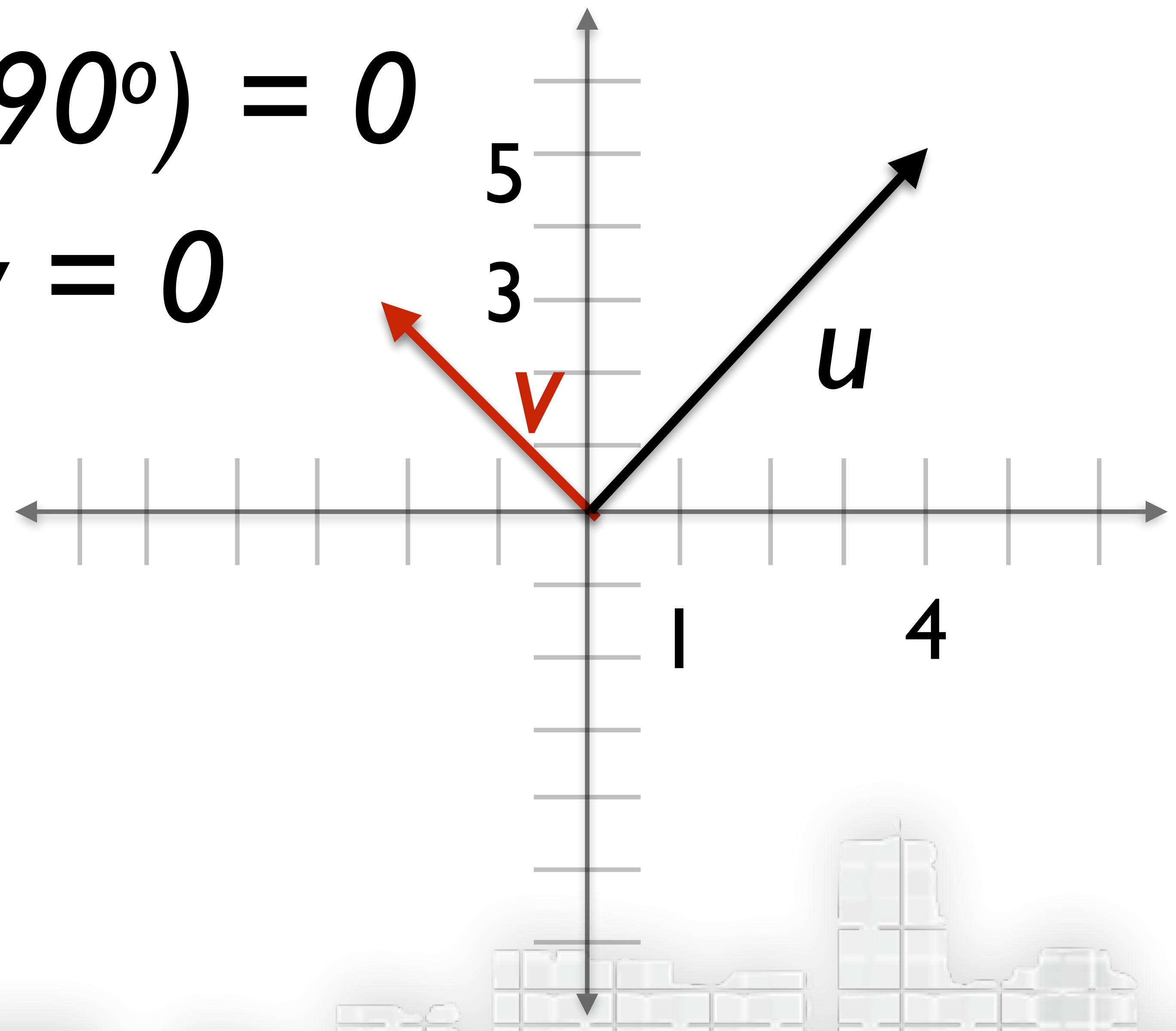
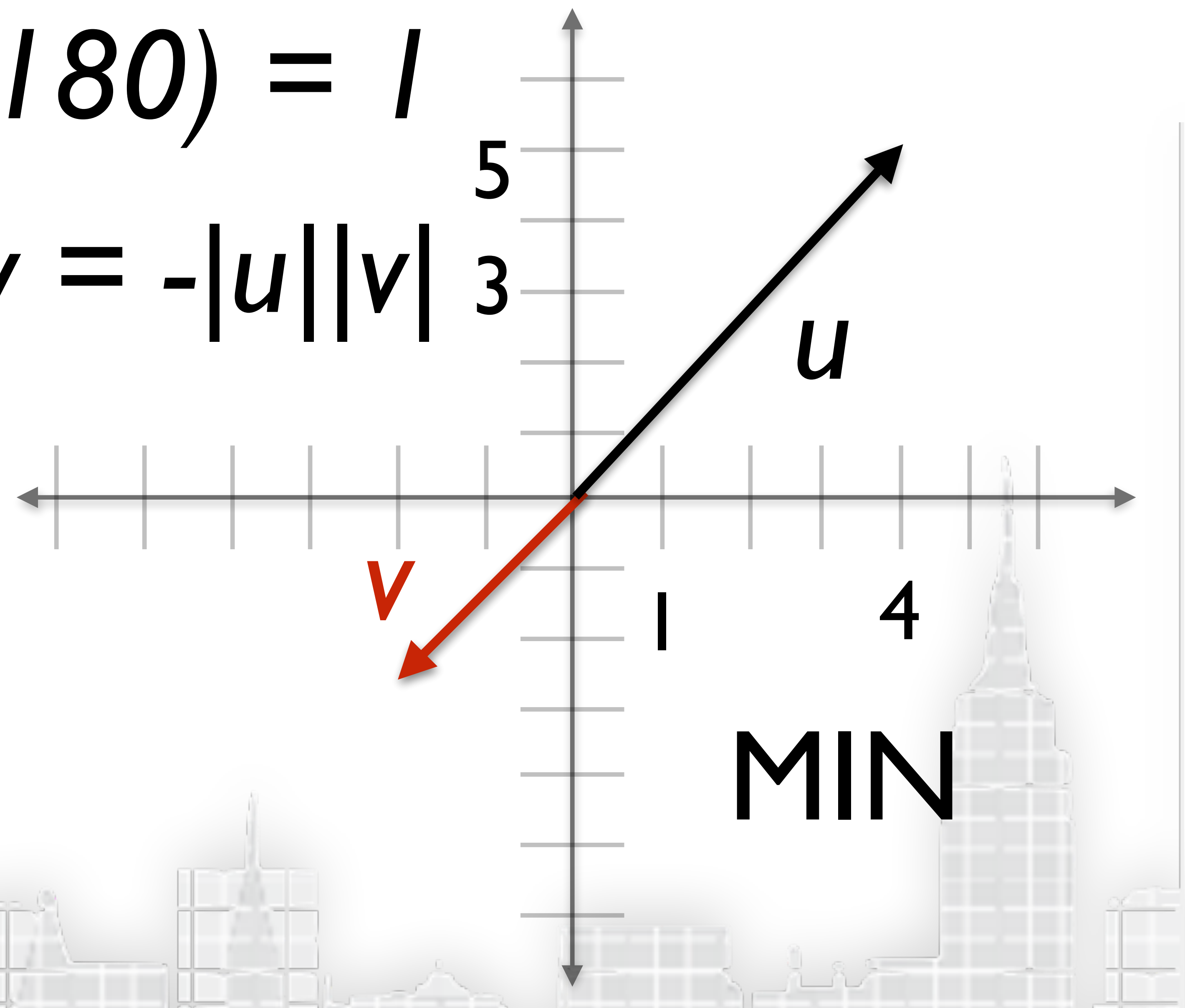
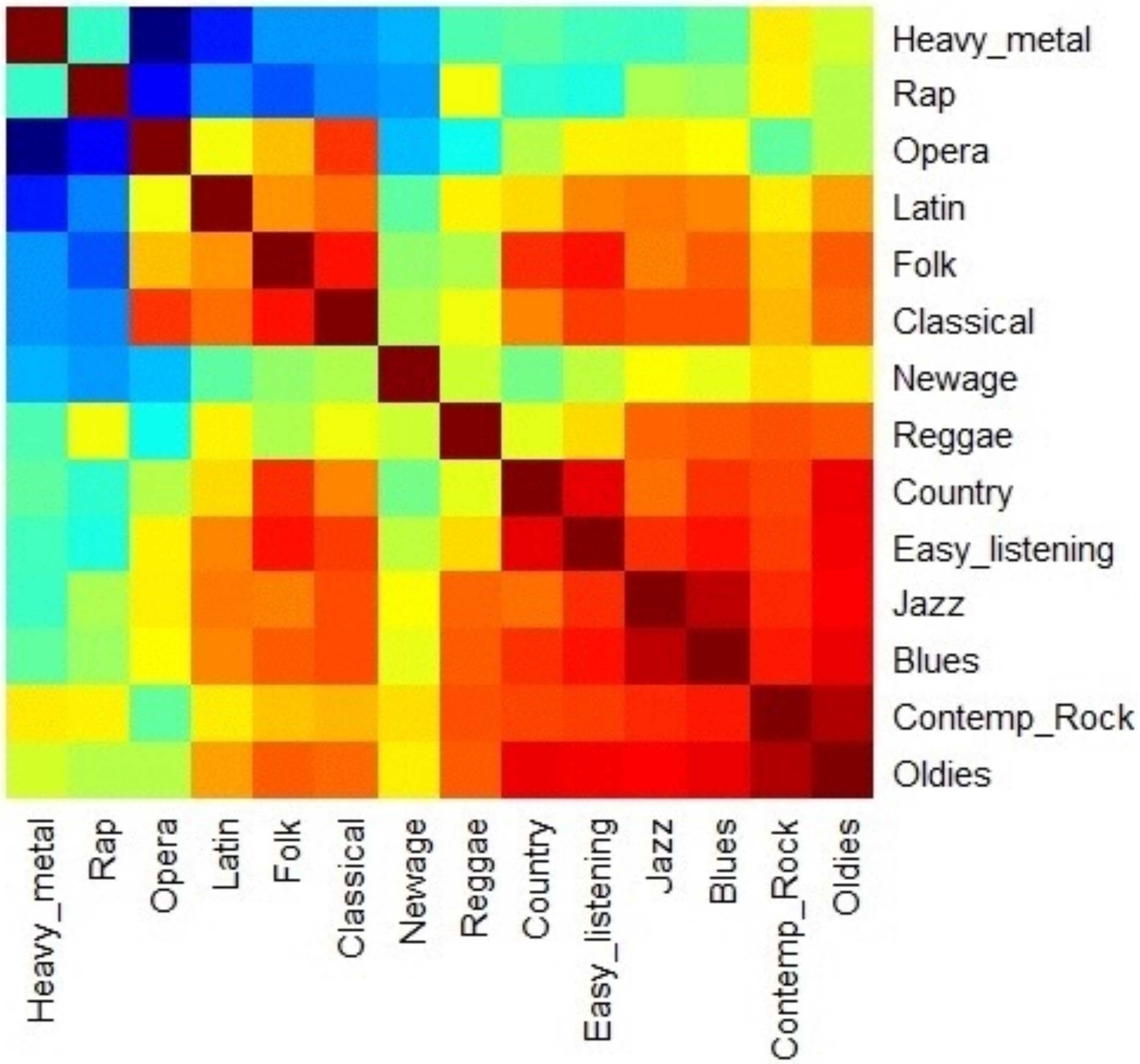


$\cos(180) = 1$

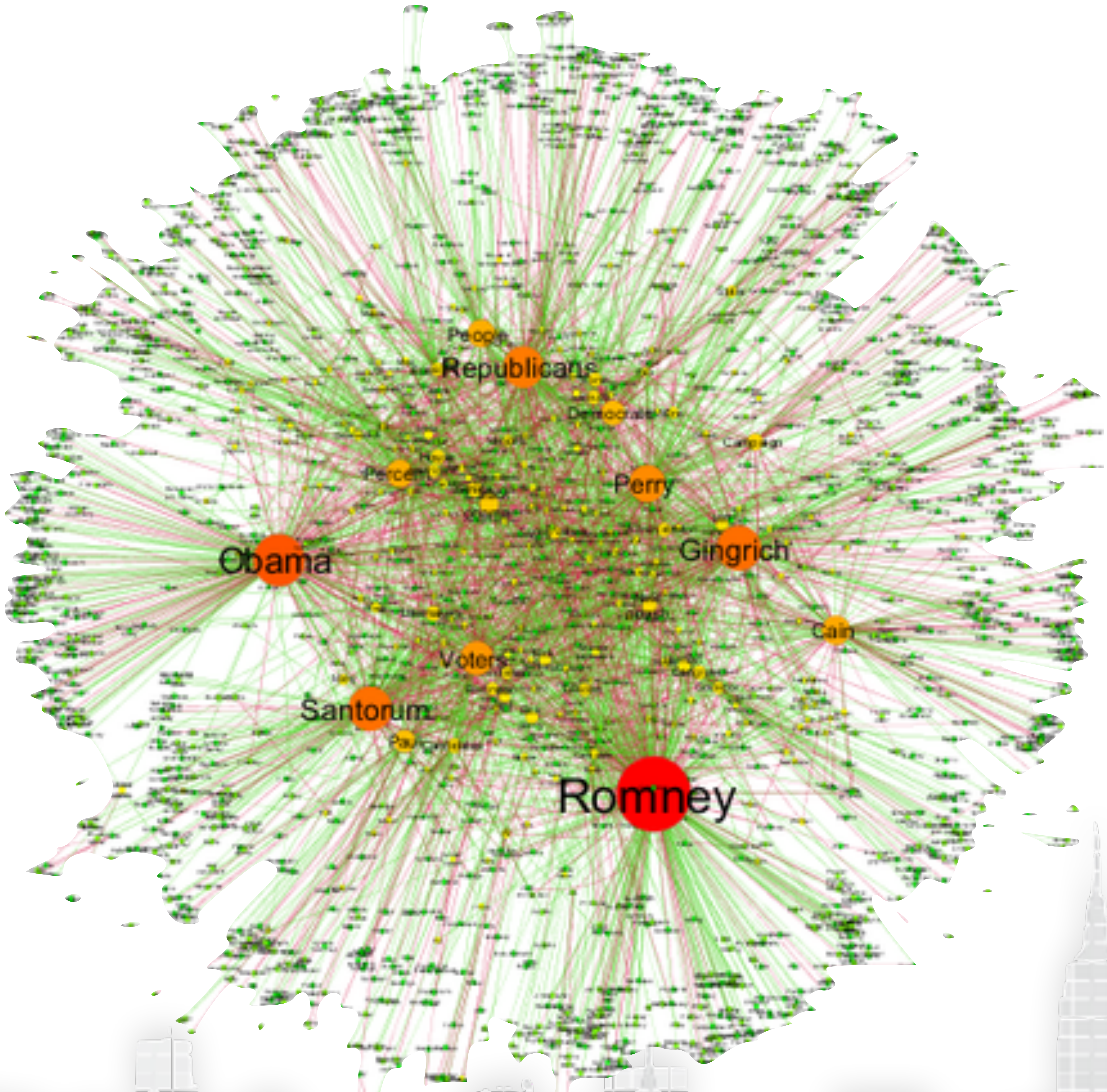
$u \cdot v = -|u||v|$



MIN



http://faculty.sites.uci.edu/skoppman/2015/07/09/different-like-me/heatmap-2/



Sudhahar, Giuseppe A Veltri, Nello Cristianini, 2015
http://bds.sagepub.com/content/2/1/2053951715572916

Matrices

- What is a vector
- What is a matrix

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

A is a 2x3 matrix

2 rows

3 columns

- What is a vector
- What is a matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

A is a 2x3 matrix

2 rows

3 columns

Scalar multiplication

- What is a vector
- What is a matrix

$$2A = \begin{bmatrix} 2a_{00} & 2a_{01} & 2a_{02} \\ 2a_{10} & 2a_{11} & 2a_{12} \end{bmatrix}$$

Scalar multiplication

- What is a vector
- What is a matrix

$$2A = 2\begin{bmatrix}4 & 2 & 0\\ 5 & 3 & 1\end{bmatrix} = \begin{bmatrix}8 & 4 & 0\\ 10 & 6 & 2\end{bmatrix}$$

Sum of matrices

- What is a vector
- What is a matrix

$$A+B = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{00}+b_{00} & a_{01}+b_{01} & a_{02}+b_{02} \\ a_{10}+b_{10} & a_{11}+b_{11} & a_{12}+b_{12} \end{bmatrix}$$

A and B must have the same dimensions

A+B also has the same dimensions

Transpose

- What is a vector
- What is a matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \\ a_{02} & a_{12} \end{bmatrix}$$

is A is nxm A^T mxn

$A[n,m] = A^T[m,n]$

Matrix multiplication

- What is a vector
- What is a matrix

$$A \times C = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \times \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ c_{20} & c_{21} \end{bmatrix}$$

$$\begin{bmatrix} a_{00}c_{00} + a_{01}c_{10} + a_{02}c_{20} & a_{00}c_{01} + a_{01}c_{11} + a_{02}c_{21} \\ a_{10}c_{00} + a_{11}c_{10} + a_{12}c_{20} & a_{10}c_{01} + a_{11}c_{11} + a_{12}c_{21} \end{bmatrix}$$

Matrix multiplication

- What is a vector
- What is a matrix

$$A \times C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times \begin{bmatrix} o & p & q & r \\ s & t & u & v \\ x & y & w & z \end{bmatrix} =$$

$$= \begin{bmatrix} (ao + bs + cx) & (ap + bt + cy) & (aq + bu + cw) & (ar + bv + cz) \\ (do + es + fx) & (dp + et + fy) & (dq + eu + fw) & (dr + ev + fz) \end{bmatrix}$$

A and C must

have the same INNER dimension : 2x3 3x4 -> 2x4

Matrix multiplication

- What is a vector
- What is a matrix

$$A \times C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times \begin{bmatrix} o & p & q & r \\ s & t & u & v \\ x & y & w & z \end{bmatrix} =$$

$$= \begin{bmatrix} (ao + bs + cx) & (ap + bt + cy) & (aq + bu + cw) & (ar + bv + cz) \\ (do + es + fx) & (dp + et + fy) & (dq + eu + fw) & (dr + ev + fz) \end{bmatrix}$$

A and *C* must

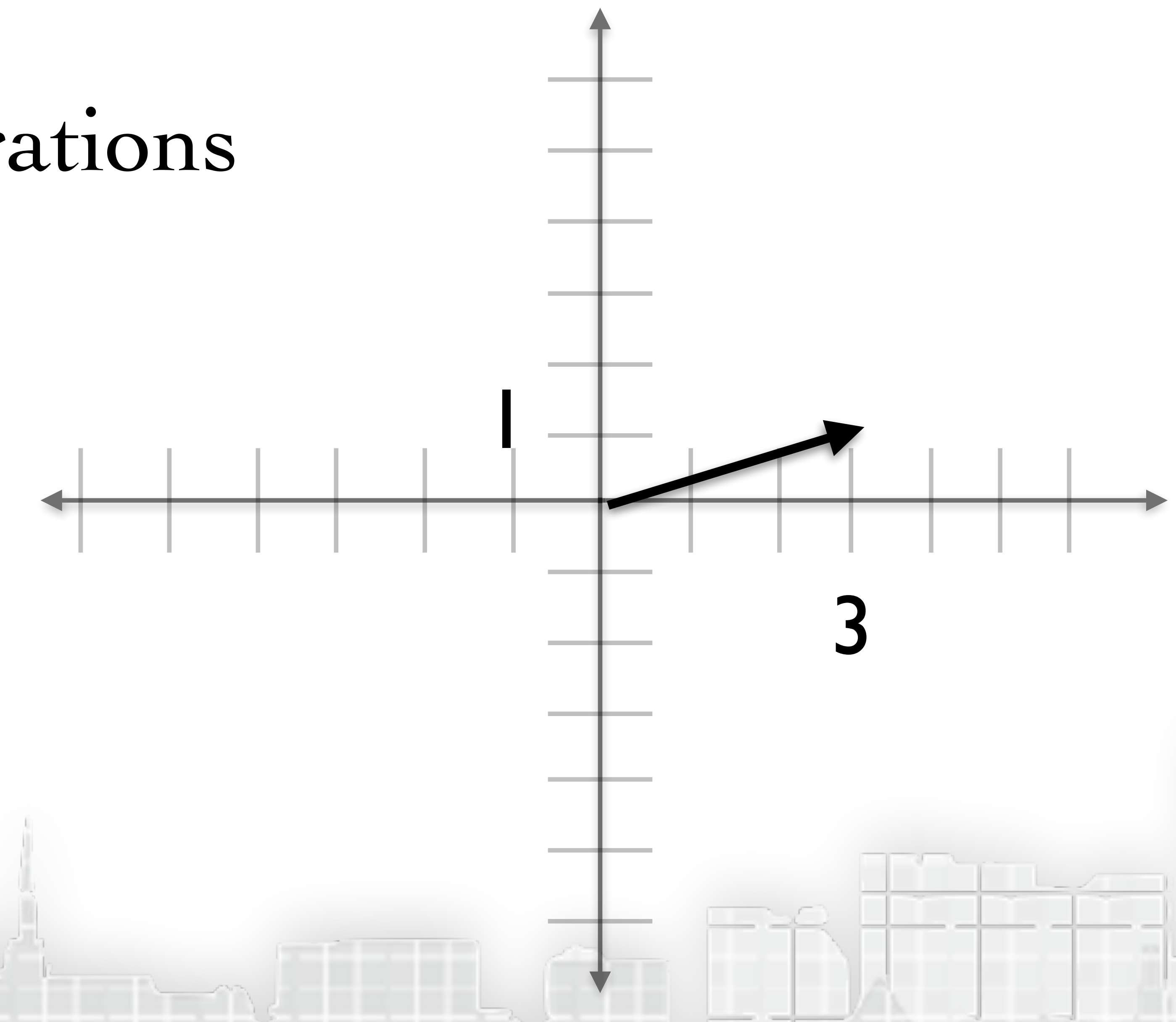
have the same INNER dimension : 2x3 3x4 -> 2x4

*A*x*B* has the outer dimensions

Matrix multiplication

$$Av = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

- What is a vector
- What is a matrix
- vector and matrix operations

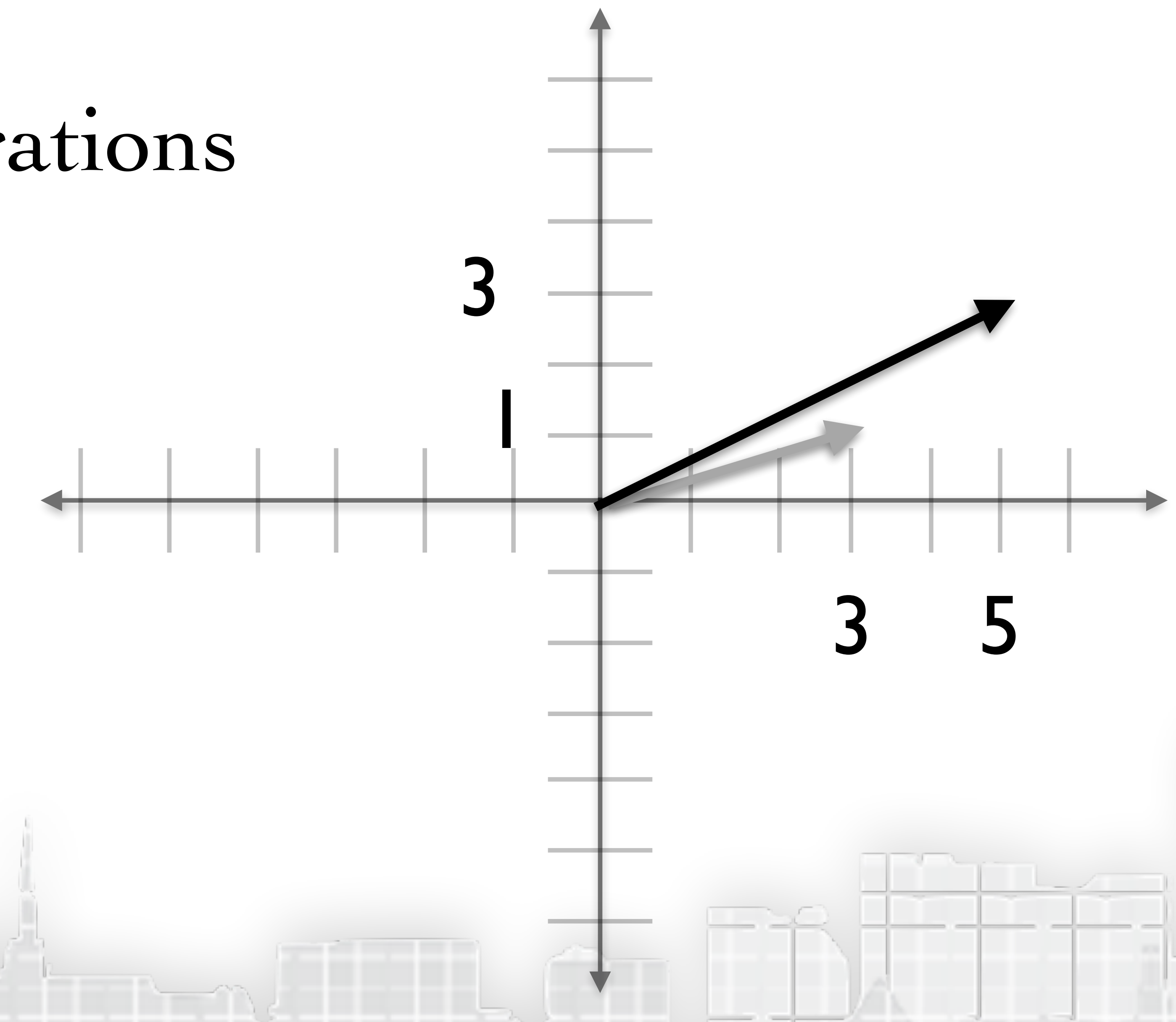


Matrix multiplication

$$Av = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \times 3 + 2 \times 1 & 0 \times 3 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations



Matrix multiplication

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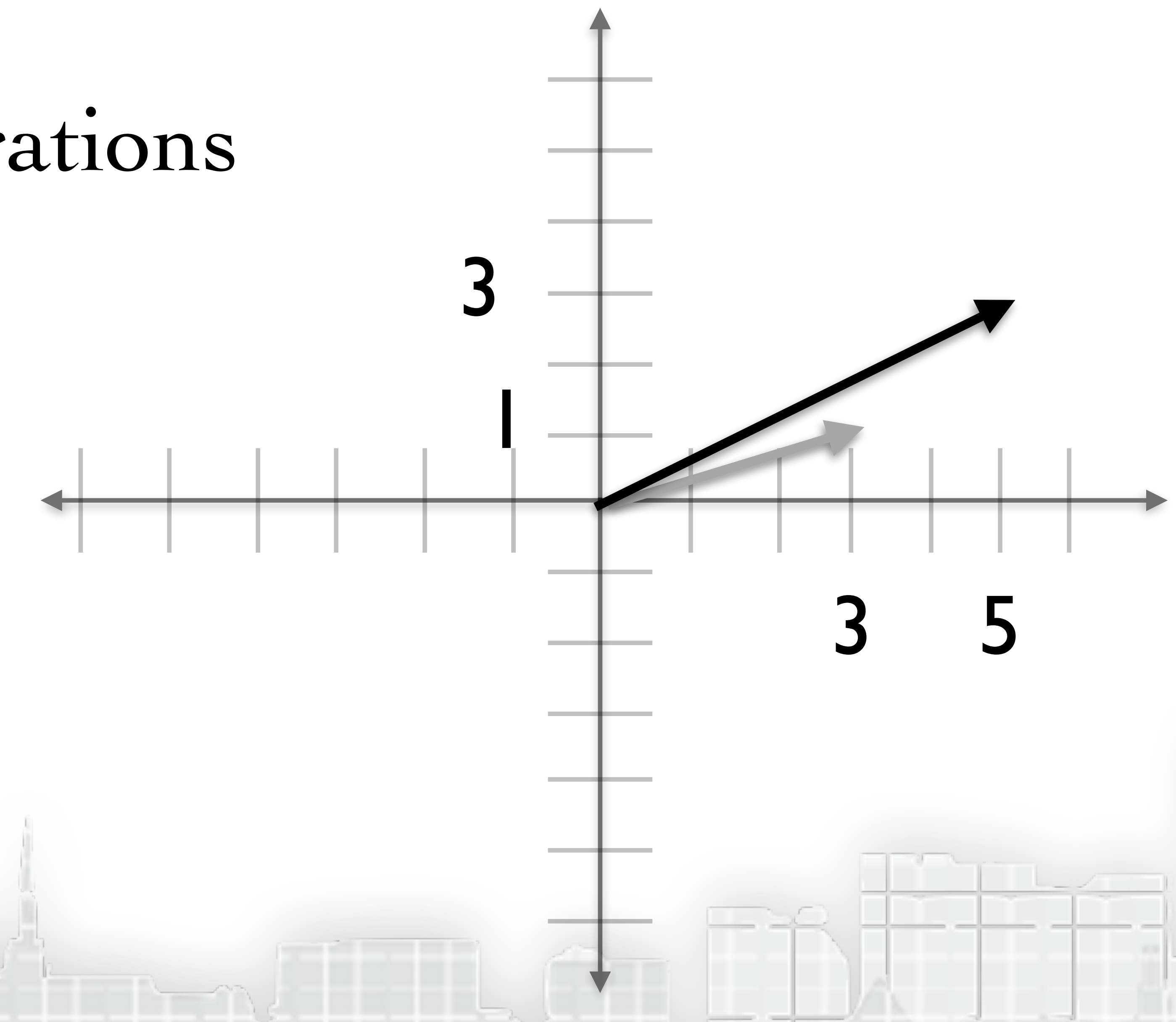


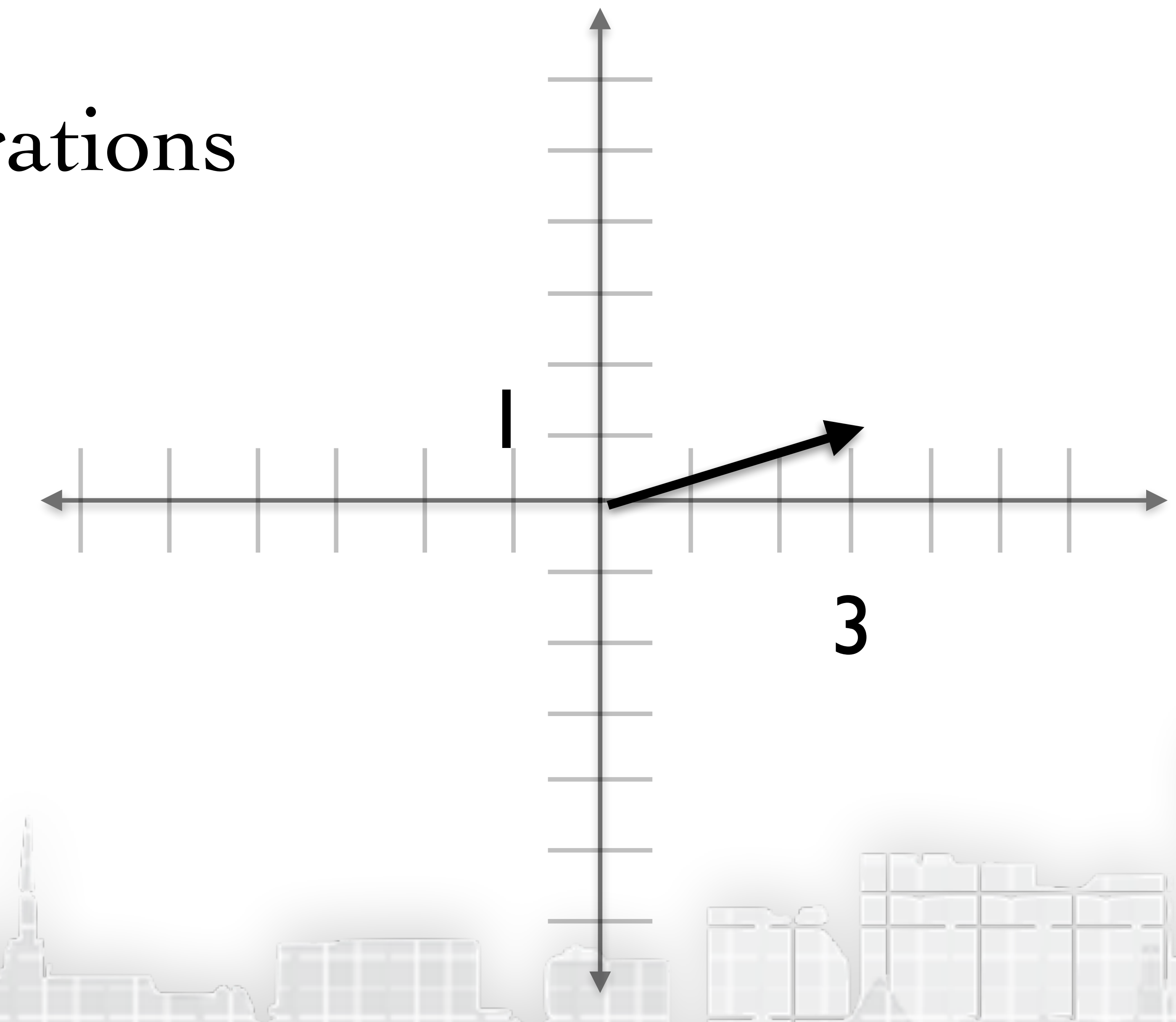
MATRIXES ARE TRANSFORMATIONS THAT MAP ELEMENTS OF A SPACE (vectors) INTO OTHER ELEMENTS OF THE SAME SPACE

Identity Matrix

$$Iv = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \times 3 + 0 \times 1 & 0 \times 3 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations

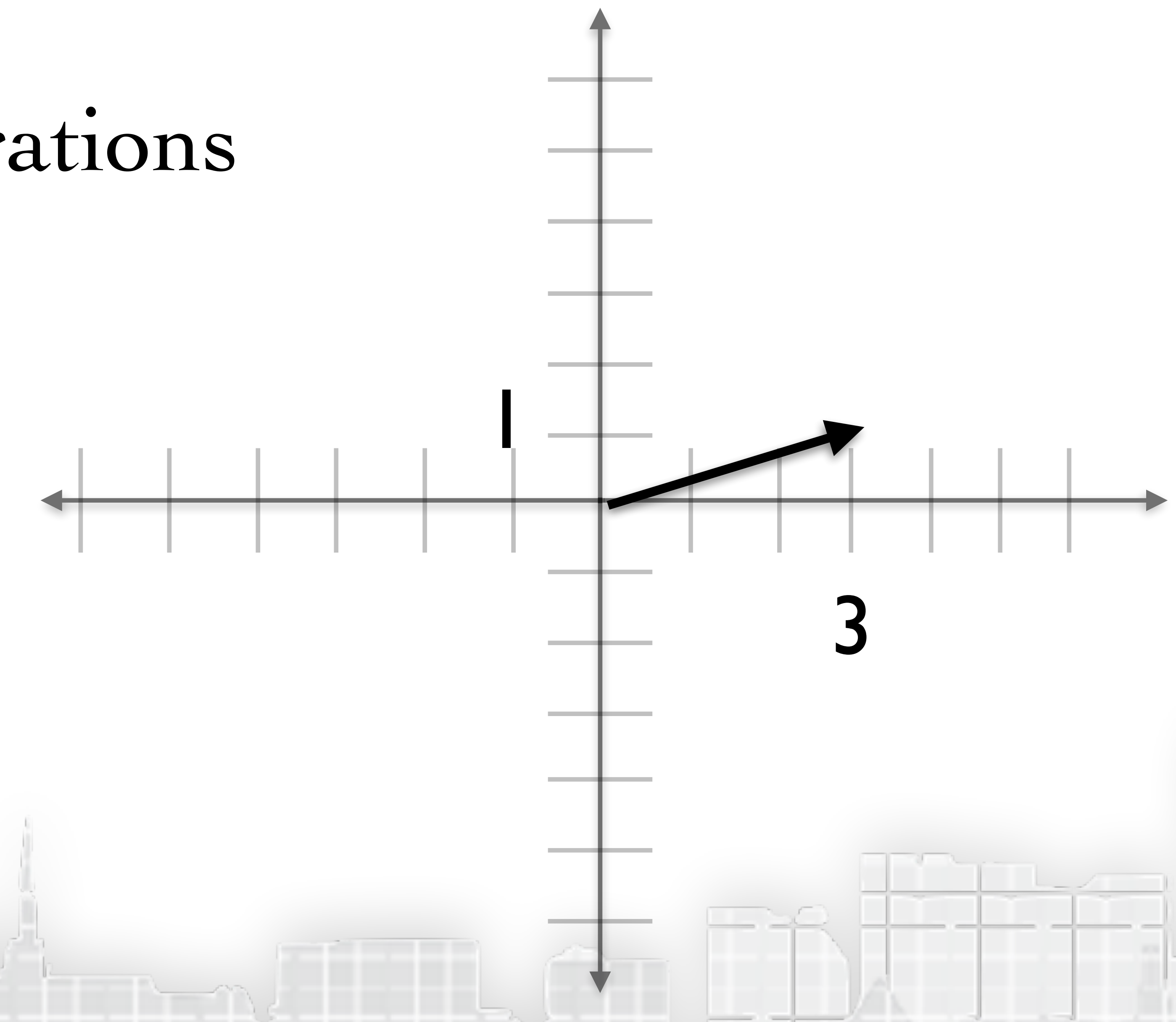


THE IDENTITY MATRIX *I* MAPS AN OBJECT ONTO ITSELF

Inverse of a Matrix

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations

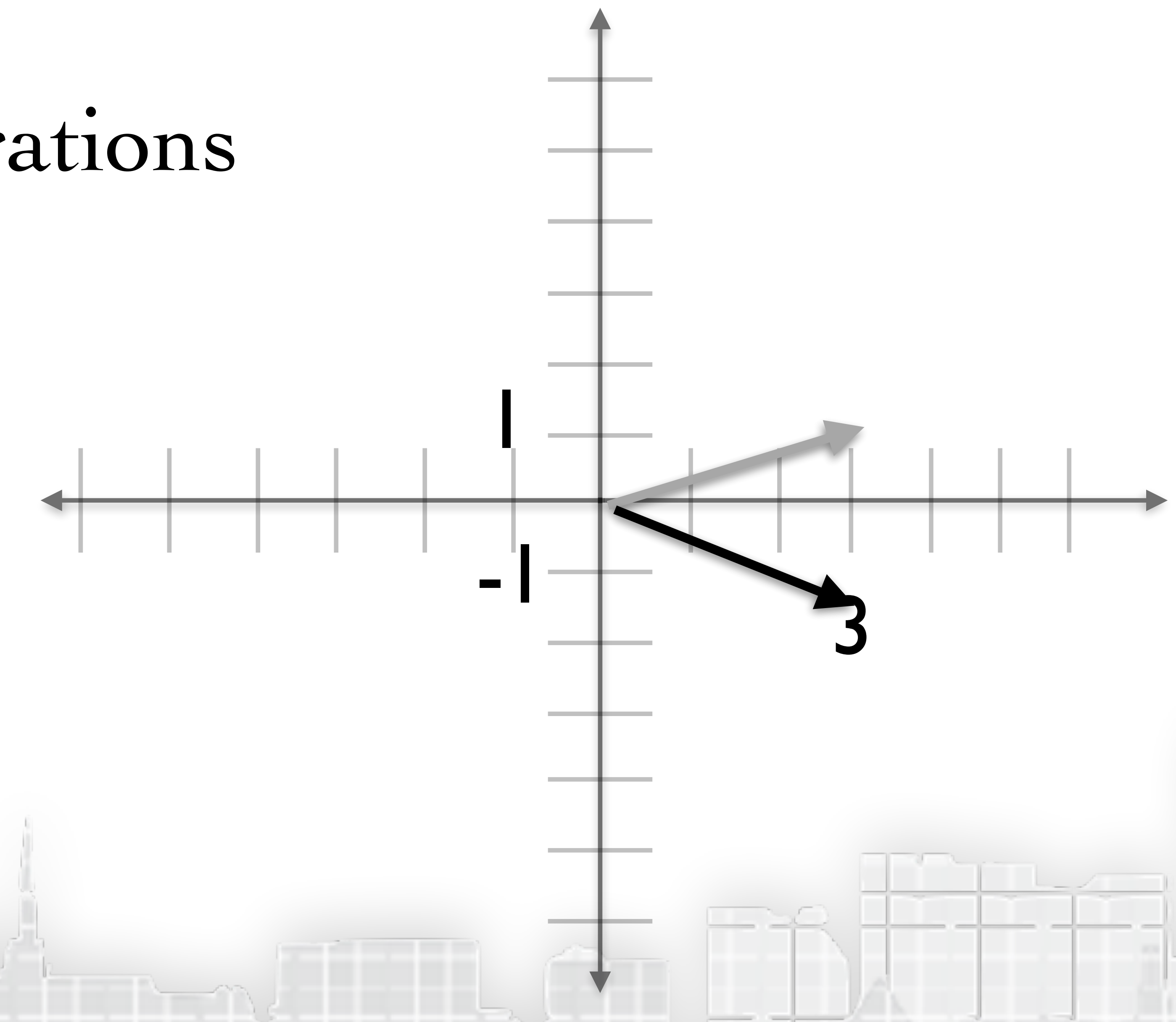


A MATRIX *A* MULTIPLIED BY ITS INVERSE A^{-1} GIVES THE IDENTITY MATRIX ***I***

REFLECTION

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations



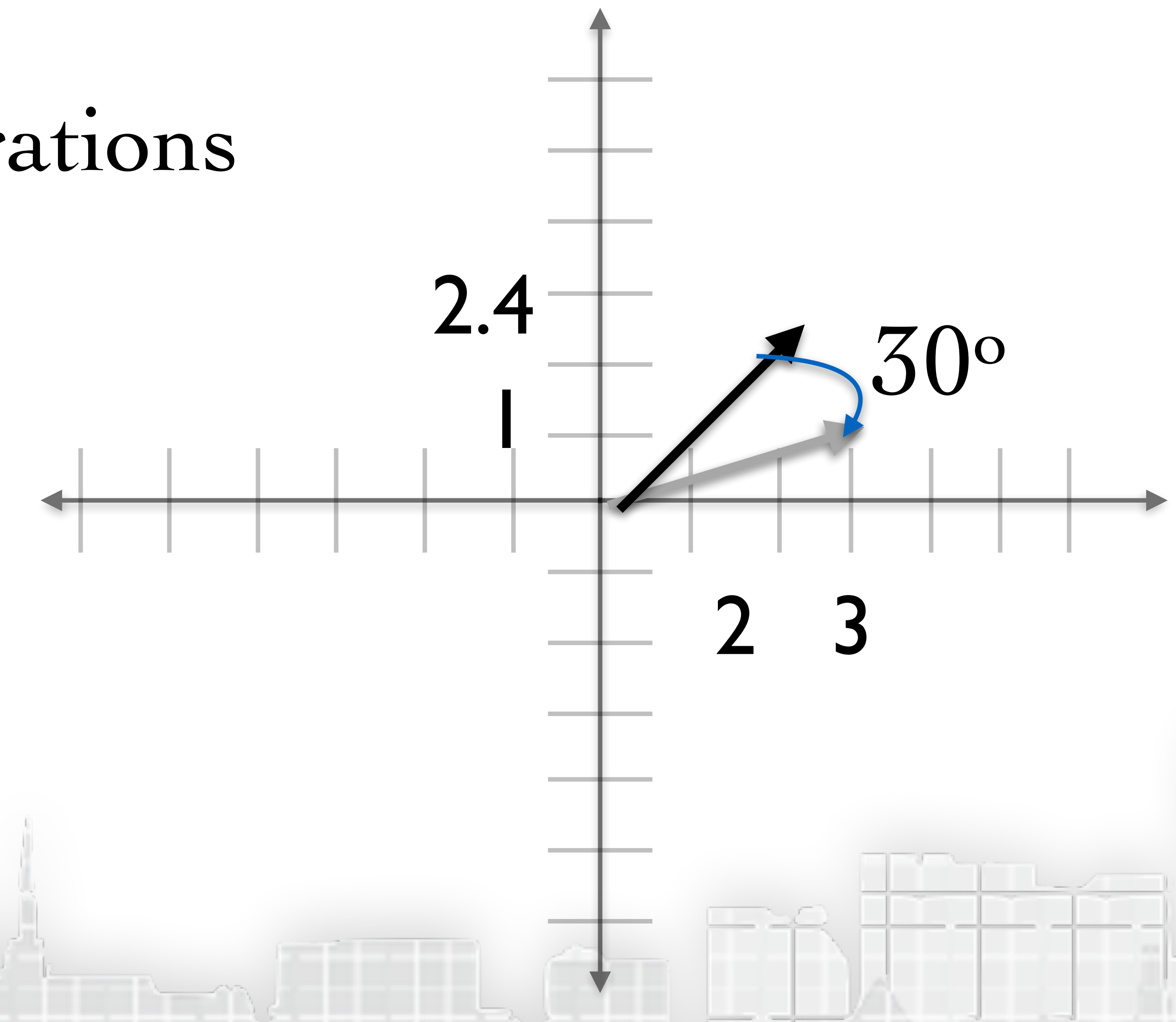
CHANGING THE SIGN ON ELEMENTS OF THE IDENTITY MATRIX ***I***

REFLECTS AN OBJECT WITH RESPECT TO THAT AXIS

ROTATION

$$Rv = \begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 2.4 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations



MATRICES OF cos AND sin ALLOW US TO

ROTATE AN OBJECT WITH RESPECT TO THE ORIGIN

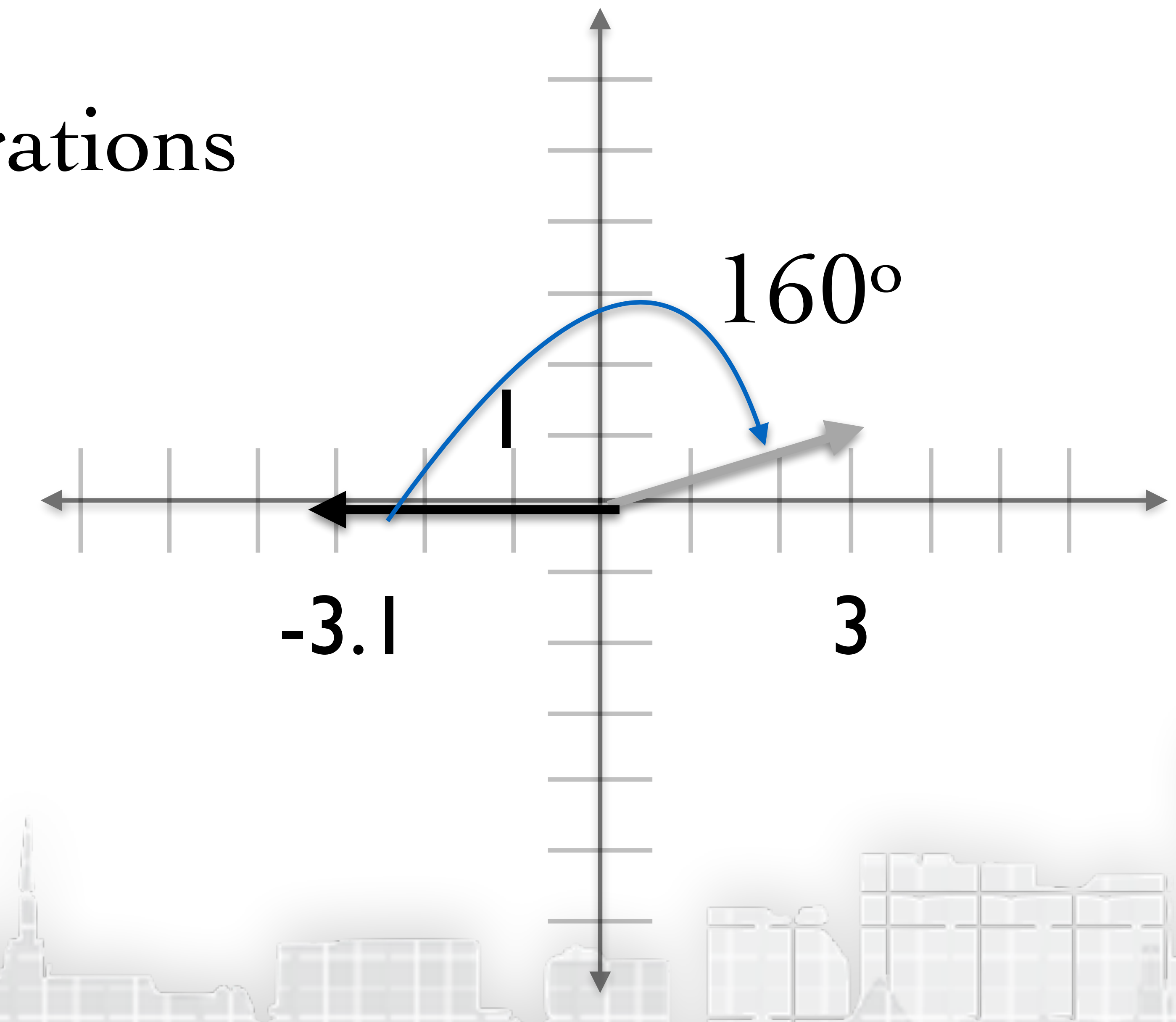
WITHOUT CHANGING THE MAGNITUDE

$$3^2 + 1^2 = 10$$
$$2.1^2 + 2.4^2 = 10$$

ROTATION

$$Rv = \begin{bmatrix} \cos(160) & -\sin(160) \\ \sin(160) & \cos(160) \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2 \\ 0.1 \end{bmatrix}$$

- What is a vector
- What is a matrix
- vector and matrix operations

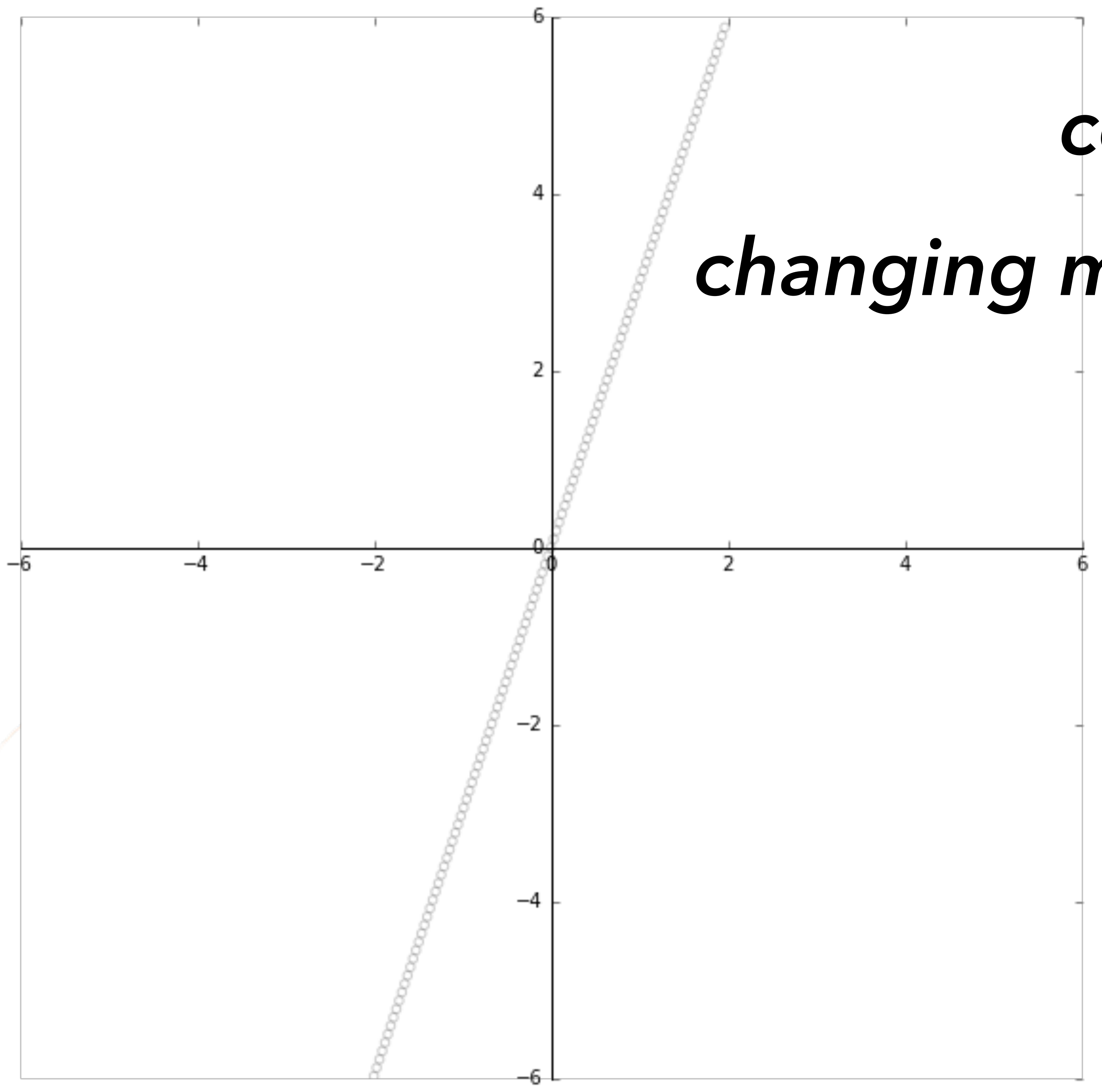


MATRICES OF cos AND sin ALLOW US TO

ROTATE AN OBJECT WITH RESPECT TO THE ORIGIN

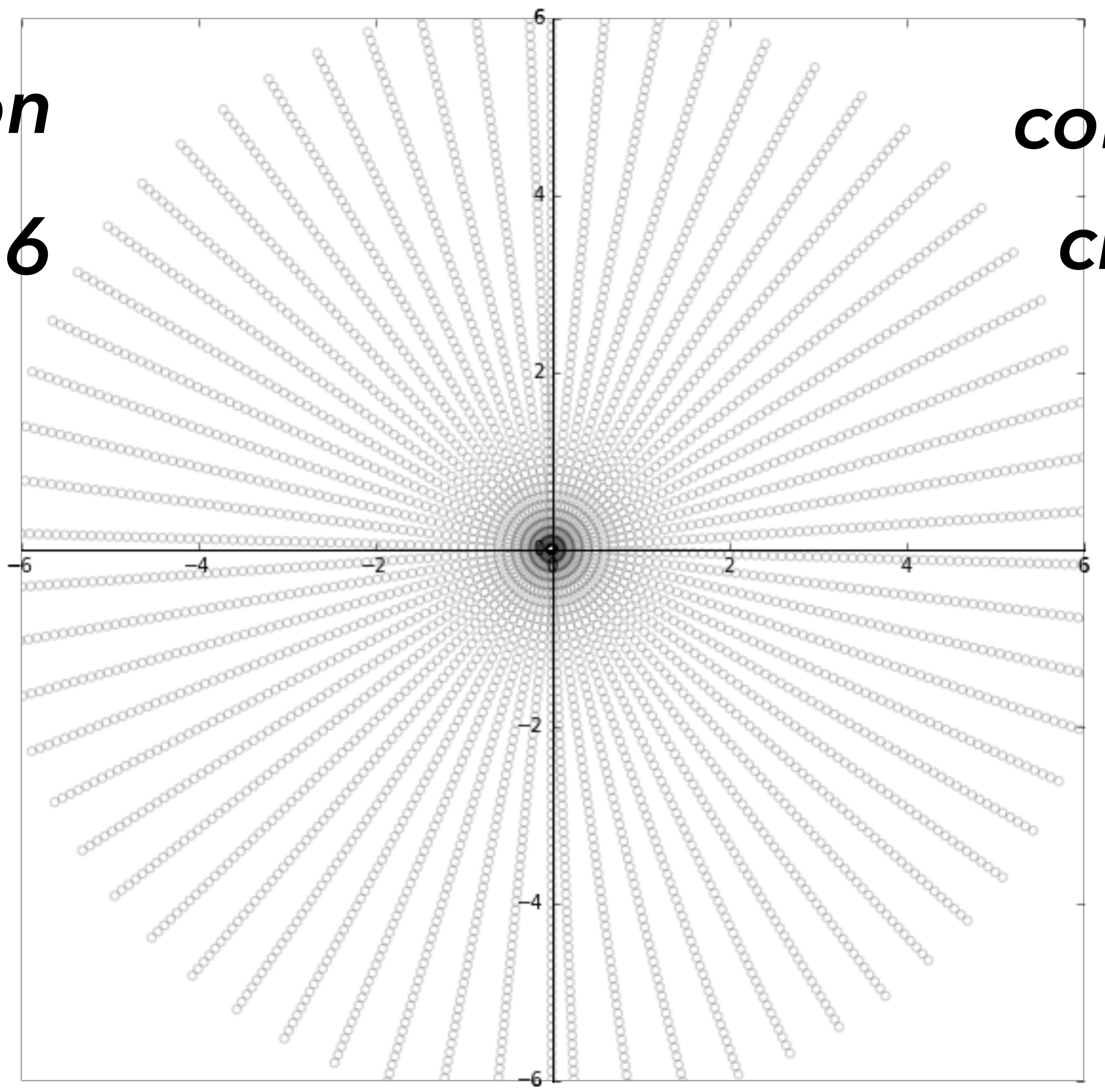
WITHOUT CHANGING THE MAGNITUDE

ROTATION



constant direction

changing magnitude -6 to 6

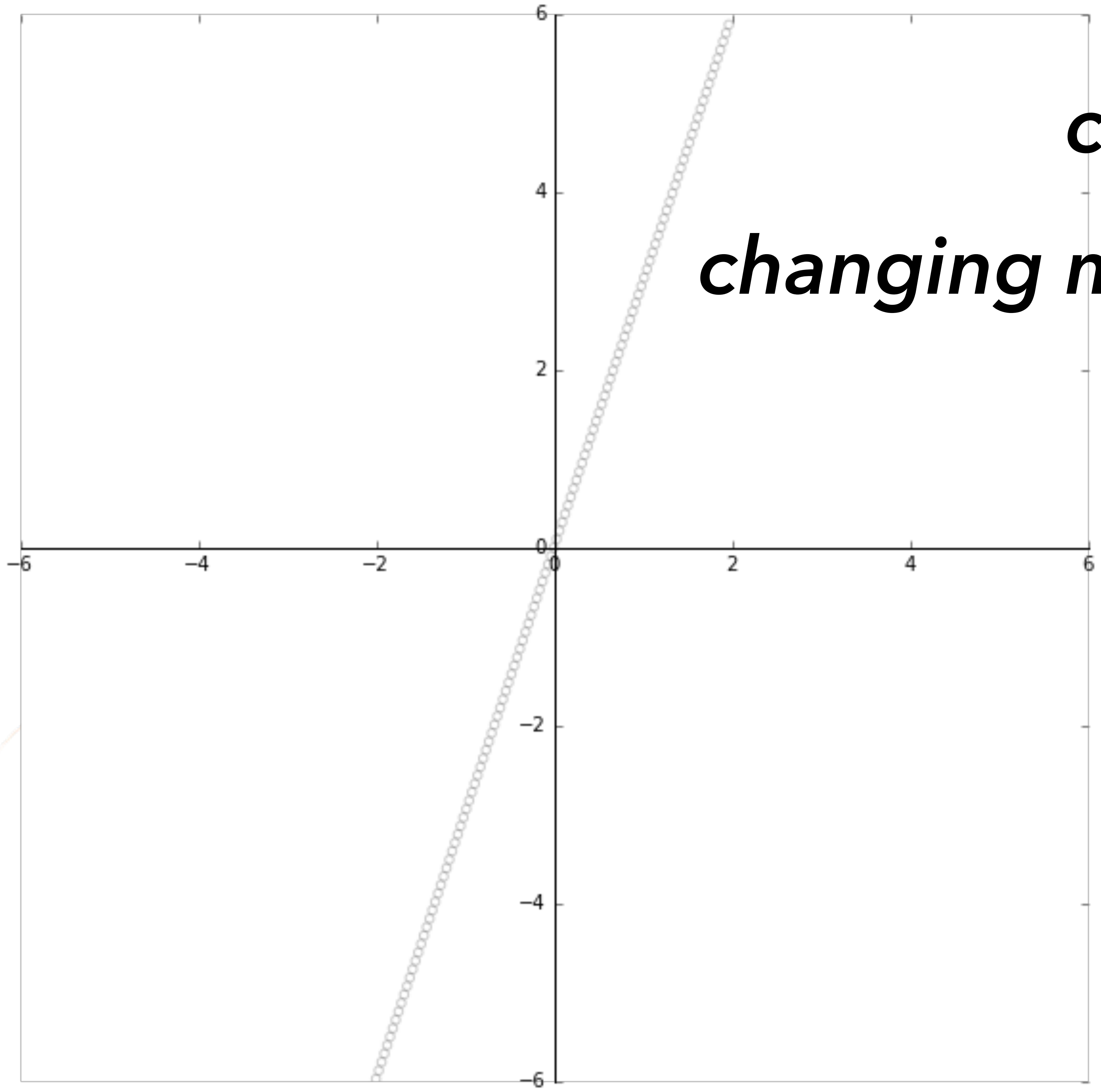


constant magnitude

changing direction

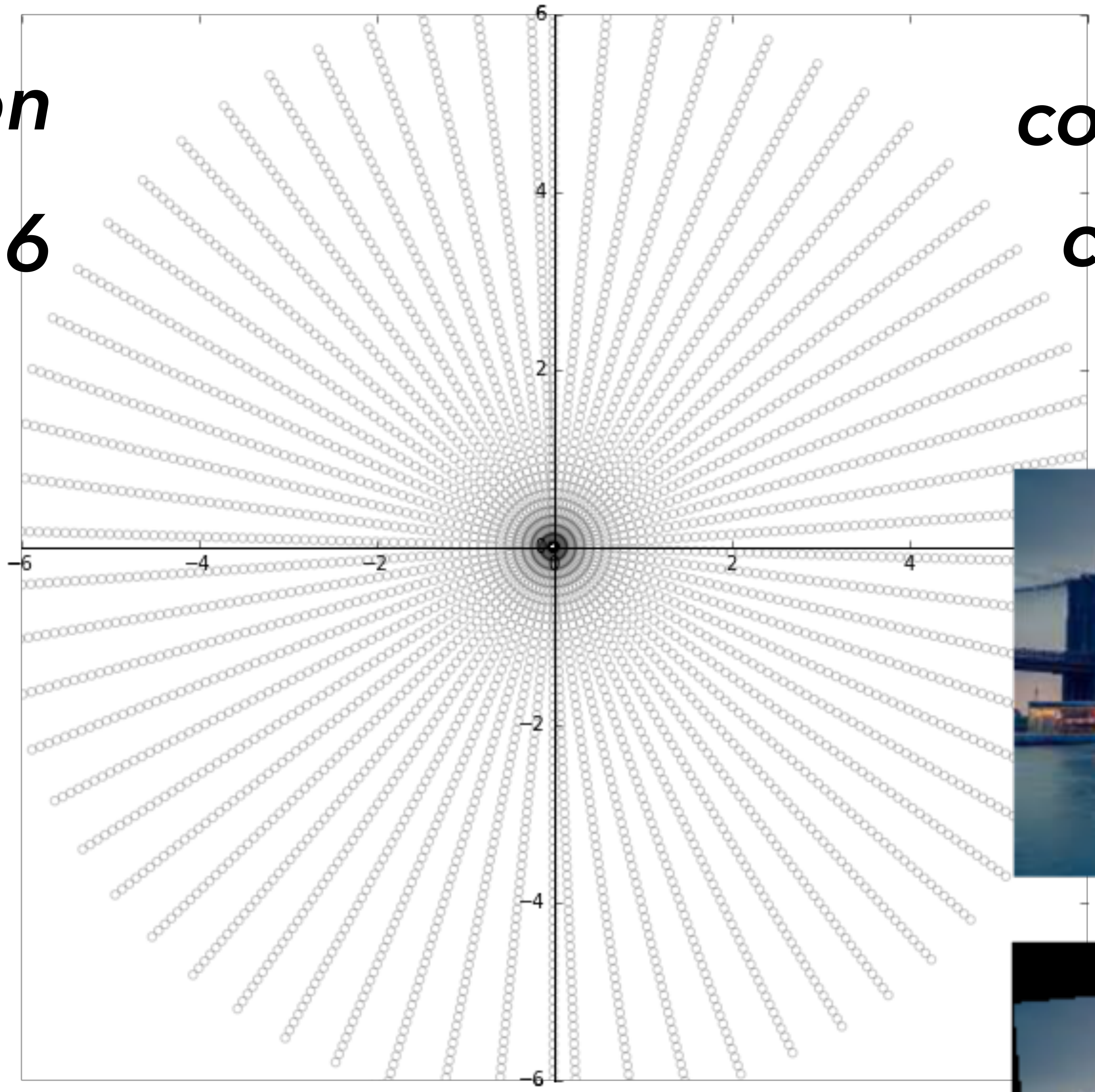
0 to 2π

ROTATION



constant direction

changing magnitude -6 to 6



constant magnitude

changing direction

0 to 2π

- Vectors are series of numbers that indicate locations in a multidimensional space. A vector can indicate a point in the city for example. But also an image is a vector in the space of all (similarly sized) possible images (and the elements of the vectors are the values in each pixel).
- Matrices are transformations that map elements of a space (vectors) into other elements of the same space
- Special Matrices include the roation matric, the identity matrix I
- Matrices and vectors can be operated on, and interact with each other. The dimensions of the matrix and vectors need to be compatible to operate.