A New Foundation of Attack Trees in Linear Logic

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Abstract. In this paper I provide an update on an ongoing research project investigating founding attack trees in the resource conscious theory called linear logic. I introduce a new linear logic called the Attack Tree Linear Logic (ATLL) which models attack trees as formulas and uses implication to reason about attack trees. I then introduce a new generalization of Bucciarelli and Ehrhards indexed linear logic to obtain Indexed ATLL which adds costs to attack trees. The most interesting aspect of Indexed ATLL is that implication is endowed with a relation for taking into consideration the costs while reasoning about attack trees.

1 Introduction

What is a mathematical model of attack trees? There have been numerous proposed answers to this question. Some examples are propositional logic, multisets, directed acyclic graphs, source sink graphs (or parallel-series pomsets), Petri nets, and Markov processes. Is there a unifying foundation in common to each of these proposed models? Furthermore, can this unifying foundation be used to further the field of attack trees and build new tools for conducting threat analysis?

The answer to the first question is positive, but the answer to the second question is open. Each of the proposed models listed above have something in common. They can all be modeled in some form of a symmetric monoidal category [17,2,4,5]. That is all well and good, but what can we gain from monoidal categories?

Monoidal categories are a mathematical model of linear logic as observed through the beautiful Curry-Howard-Lambek correspondence [11]. In linear logic every hypothesis must be used exactly once, and hence, if we view a hypothesis as a resource, then this property can be stated as every resource must be consumed. This linearity property is achieved by restricting the logic so that hypothesis are never duplicated or removed.

Multisets and Petri nets both capture the idea that the nodes of an attack tree consist of the attack action and the state – the resource – of the system being analyzed. As it turns out, linear logic has been shown to be a logical foundation for multisets [17] and Petri Nets [2]. Thus, linear logic has the ability to model the state as well as attack actions of the goals of an attack tree.

In this paper[1] I introduce a new logical foundation of attack trees – with and without costs – in linear logic; the type of attack trees considered in this paper are attack trees with sequential conjunction due to Jhawar et al. [7]. This new foundation is based on a new logic called the Attack Tree Linear Logic (ATLL).

In ATLL (Section 3) attack trees are modeled as linear formulas where base attacks are atomic formulas, and each branching node corresponds to a binary logical connective. Consider the attack tree for an ATM attack from Figure 1. We can model this attack tree as a formula in ATLL as follows:

$B_1 :=$ "steal card"
$B_2 :=$ "social engineer"
$B_3 :=$ "find a post-it"
$B_4 :=$ "withdrawal money"
$T_1 := (B_1 \odot (B_2 \sqcup B_3)) \rhd B_4$

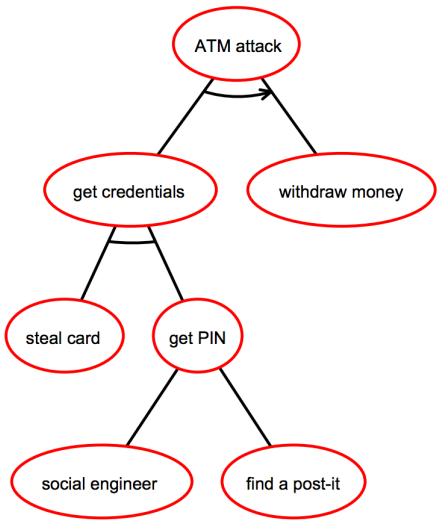


Fig. 1. Attack Tree for an ATM attack from Figure 1 on page 2 of Kordy et al. [8]

Each B_i is an atomic formula, parallel conjunction of attack trees is denoted by $T_1 \odot T_2$, choice between attacks by $T_1 \sqcup T_2$, and sequential conjunction of attacks by $T_1 \rhd T_2$. Parallel conjunction and choice are both symmetric, but sequential conjunction is not. Now that we can model attack trees as formulas we should be able to use the logic to reason about attack trees.

Reasoning about attack trees corresponds to proving implications between them. In fact, every equation from Jhawar et al.'s work on attack trees with sequential conjunction [7] can be proven as an implication in ATLL. Consider a second attack tree from Figure 2. We can model this attack tree as a formula in ATLL as well (the base attacks are the same):

$$T_2 := ((B_1 \odot B_2) \rhd B_4) \sqcup ((B_1 \odot B_3) \rhd B_4)$$

The attack trees T_1 and T_2 represents the same attack, and this can be proven in ATLL by showing that $T_1 \multimap\!\!\circ\!\!- T_2$ where $\multimap\!\!\circ\!\!-$ denotes bi-implication. The proof holds by using the distributive rules for choice.

In addition, implication can be used to prove subattack tree relationships between attack trees. For example, supposed we wanted to show that the tree $B_1 \odot B_2$ is a subattack tree of T_2, then this is the case if $T_2 \multimap (B_1 \odot B_2)$.

[1] This material is based upon work supported by the National Science Foundation CRII CISE Research Initiation grant, "CRII:SHF: A New Foundation for Attack Trees Based on Monoidal Categories", under Grant No. 1565557.

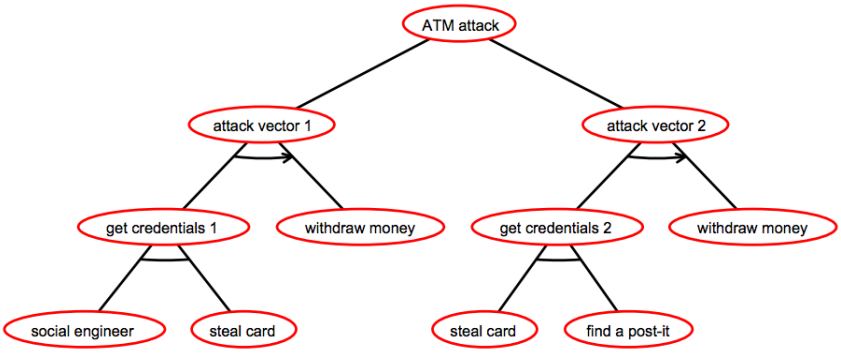


Fig. 2. Canonical Attack Tree for an ATM attack from Figure 2 on page 3 of Kordy et al. [8]

This requires some additional rules be added to ATLL and this is discussed in Section 3.

Using a generalization of Bucciarelli and Ehrhards [3] indexed linear logic I extend ATLL with costs. I call this extension Indexed ATLL (Section 4). In this system every base attack is annotated with a cost, and every branching node is associated with a binary operation used for computing the cost at that node. Then implication is associated with a binary relation on costs. This new binary relation is used to insure that the costs of two trees are related when proving implications between attack trees. Thus, one can prove facts about attack trees in Indexed ATLL that require one to reason about both the tree structure as well the quantitative data in the tree simultaneously.

Modeling attack trees in linear logic has a number of benefits. First, it connects attack trees back to logic, but in elegant and simple way. Kordy et al. [9,10] proposed that attack trees be modeled in propositional logic, but in this model attacks can be freely duplicated and contracted which goes against the process/resources nature of an attack tree. However, linear logic restores this natural interpretation without loosing the process interpretation of attack trees and without having to resort to complicated notation unlike models similar to the situation calculus [16]. By connecting attack trees to logic we can also tap into the long standing research and development of automation, for example, SAT, SMT, proof search, etc. Finally, by connecting to logic we also connect to the theory of statically typed functional programming through the Curry-Howard-Lambek correspondence which will allow for the development of a programming language that can be used to certify the correctness of attack trees and the analysis performed using them.

Contributions. This paper offers the following contributions:

– The Attack Tree Linear Logic (ATLL): a new linear logic for the specification and analysis of attack trees.
– The Indexed Attack Tree Linear Logic (Indexed ATLL): an extension of ATLL with costs and the ability to reason about them while reasoning about attack trees.
– A few new and interesting open problems.

2 A Brief Introduction to Ordered Linear Logic

In order to aid the reader unfamiliar with linear logic and how it and other logics can be used as a modeling framework I first give a brief introduction to (ordered) linear logic and how it can be used to model various structures in computer science. I assume very little of the reader, and start with the basics of the specification of logics. Ordered linear logic (OLL) [13] will be the ongoing example throughout this section, because it is the logic ATLL is based on. If the reader is familiar with linear logic, then they can safely skip this section.

Every logic presented in this paper is in the form of sequent-style natural deduction. This type of formalization begins with the specifying the syntax of formulas and sequents. The latter is defined by a set of inference rules. The syntax for OLL is as follows:

$$\begin{array}{rl} \text{(formulas)} & A, B, C ::= b \mid \top \mid A \odot B \mid A \rhd B \mid A \times B \mid A \multimap B \\ \text{(ordered contexts)} & \Delta ::= \cdot \mid A \mid \Delta_1, \Delta_2 \\ \text{(unordered contexts)} & \Gamma ::= \cdot \mid A \mid \Gamma_1, \Gamma_2 \end{array}$$

Formulas of OLL consist of atomic formulas denoted by b, a symmetric tensor product $A \odot B$, its unit $\top$, a non-symmetric tensor product $A \rhd B$, non-linear conjunction $A \times B$, and linear implication $A \multimap B$. Note that the symmetric tensor product is to linear implication as conjunction is to implication in classical logic. The other forms of conjunction are additional. Contexts are lists of hypothesis where Δ is a list of ordered hypothesis, and Γ is a list of unordered hypothesis. The former are associated with the non-symmetric tensor product, and the latter with the symmetric tensor product. We denote the empty context by $\cdot$, and appending of contexts Γ_1 and Γ_2 by Γ_1, Γ_2.

Linear logic has a resource interpretation, and the locus of this interpretation is linear implication. In linear logic hypotheses cannot be freely duplicated or contracted. That is, $A \multimap (A \odot A)$ does not hold, nor does $(A \odot A) \multimap A$. In addition, hypotheses cannot be freely created, that is, $A \multimap \top$ does not hold in general. Each of these examples either uses an hypotheses more than once, or does not use an hypotheses at all. This implies that every hypotheses must be used exactly once, and this property is called the linearity property.

The linearity property produces a resource interpretation of formulas. Linear implication $A \multimap B$ can be interpreted as, consume the resource A, and then produce B. Under this interpretation the linearity property can be read as, every resource must be consumed exactly once. The symmetric tensor product $A \odot B$ can be read as, consume both the resources A and B. Similarly, the

non-symmetric tensor product $A \rhd B$ can be read as, consume the resource A, and then consume the resource B. Now non-linear conjunction $A \times B$ is ordinary conjunction from classical and intuitionistic logic, and hence, formulas like $(A \times A) \multimap A$ and $A \multimap (A \times A)$ both hold. Similarly, $(A \times B) \multimap A$ and $(A \times B) \multimap B$ hold. Thus, one should interpret this as a way of pairing resources up. I give an example of how to use the resources interpretation to model a simple state-based system at the end of this section.

Sequents of OLL are denoted by $\Delta; \Gamma \vdash A$. We say that the sequent $\Delta; \Gamma \vdash A$ holds if and only if given the hypothesis in Δ and Γ one can construct a proof of the formula A using the following inference rules:

$$\frac{}{\cdot; A \vdash A}\,\mathrm{id}_{\odot} \qquad \frac{}{A; \cdot \vdash A}\,\mathrm{id}_{\rhd} \qquad \frac{}{\cdot; \cdot \vdash \top}\,\top \qquad \frac{\Delta_1; \Gamma \vdash A \quad \Delta_2; \Gamma \vdash B}{\Delta_1, \Delta_2; \Gamma \vdash A \times B}\,\times_I$$

$$\frac{\Delta; \Gamma \vdash A \times B}{\Delta; \Gamma \vdash A}\,\times_{E_1} \qquad \frac{\Delta; \Gamma \vdash A \times B}{\Delta; \Gamma \vdash A}\,\times_{E_2} \qquad \frac{\Delta_1; \Gamma_1 \vdash A \quad \Delta_2; \Gamma_2 \vdash B}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash A \odot B}\,\odot_I$$

$$\frac{\Delta_1; \Gamma_2 \vdash A \odot B \quad \Delta_2; \Gamma_1, A, B, \Gamma_3 \vdash C}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\odot_E \qquad \frac{\Delta_1; \Gamma_1 \vdash A \quad \Delta_2; \Gamma_2 \vdash B}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash A \rhd B}\,\rhd_I$$

$$\frac{\Delta_2; \Gamma_1 \vdash A \rhd B \quad \Delta_1, A, B, \Delta_3; \Gamma_2 \vdash C}{\Delta_1, \Delta_2, \Delta_3; \Gamma_1, \Gamma_2 \vdash C}\,\rhd_E \qquad \frac{\Delta; \Gamma_1, A, B, \Gamma_2 \vdash C}{\Delta; \Gamma_1, B, A, \Gamma_2 \vdash C}\,\mathrm{sym}_{\odot}$$

$$\frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \multimap B}\,\multimap_I \qquad \frac{\Delta_1; \Gamma_1 \vdash A \multimap B \quad \Delta_2; \Gamma_2 \vdash A}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash B}\,\multimap_E$$

An inference rule should be read from top to bottom as an implication. The sequents on top of the line are the premises, and the sequent below the line is the conclusion. For example, one should read the rule $\odot_I$ as if $\Delta_1; \Gamma_1 \vdash A$ and $\Delta_2; \Gamma_2 \vdash B$ both hold, then $\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash A \odot B$ holds. An inference rule with no premises is called an axiom. For example, the rules $\mathrm{id}_{\odot}$ and $\mathrm{id}_{\rhd}$ are both axioms.

Inference rules are read from top to bottom, but applied from bottom to top. A sequent, $\Delta; \Gamma \vdash A$, holds if and only if there is a series of inference rules that can be composed into a derivation. Every derivation is a tree where the root is the sequent we wish to prove, and then rules are composed to form a tree. A rule can be composed with another if the formers conclusion matches a premise of the latter. If every branch of this tree reaches an axiom, then it is a valid proof, but if any branch gets stuck, then the sequent does not hold. The following is a proof that the symmetric tensor is indeed symmetric:

$$\dfrac{\dfrac{\dfrac{}{\cdot; A \odot B \vdash A \odot B}\,\mathrm{ID}_{\odot} \qquad \dfrac{\dfrac{\dfrac{}{\cdot; B \vdash B}\,\mathrm{ID}_{\odot} \qquad \dfrac{}{\cdot; A \vdash A}\,\mathrm{ID}_{\odot}}{\cdot; B, A \vdash B \odot A}\,\odot_I}{\cdot; A, B \vdash B \odot A}\,\mathrm{SYM}_{\odot}}{\cdot; A \odot B \vdash B \odot A}\,\odot_E}{\cdot; \cdot \vdash (A \odot B) \multimap (B \odot A)}\,\multimap_I$$

We call the root of the tree the goal of the proof, and as we construct a derivation this goal is refined into potentially several new subgoals. This is called goal

directed proof. As we can see the previous derivation is a valid proof of the goal $\cdot;\cdot \vdash (A \odot B) \multimap (B \odot A)$.

The following derivation is an invalid proof that the non-symmetric tensor product is symmetric:

$$\dfrac{\dfrac{\dfrac{}{\cdot; A \rhd B \vdash A \rhd B}\,\mathrm{ID}_\odot \qquad A, B; \cdot \vdash B \rhd A}{\cdot; A \rhd B \vdash B \rhd A}\,\rhd_E}{\cdot;\cdot \vdash (A \rhd B) \multimap (B \rhd A)}\,\multimap_I$$

At this point we have refined our goal to the subgoal $A, B; \cdot \vdash B \rhd A$, but this is impossible to prove, because the exchange rule $\mathrm{sym}_\odot$ cannot be be applied to the ordered context to commute A and B.

Building proofs from the inference rules given above can often be tedious and long. To make this process easier it is often necessary to introduce new rules that can be proven to be valid in terms of the inference rules already given. Rules introduced in this way are called derivable inference rules. One derivable rule we will make use of is the following:

$$\dfrac{\Delta_2; \Gamma_2 \vdash A \multimap B \quad \Delta_1; \Gamma_1 \vdash B \multimap C}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash A \multimap C}\,\mathit{comp}$$

Proving this rule is derivable amounts to creating a derivation whose branches terminate at either an axiom or the premises $\Delta_2; \Gamma_2 \vdash A \multimap B$ or $\Delta_1; \Gamma_1 \vdash B \multimap C$. The proof is as follows:

$$\dfrac{\dfrac{\Delta_1; \Gamma_1 \vdash B \multimap C \qquad \dfrac{\Delta_2; \Gamma_2 \vdash A \multimap B \qquad \dfrac{}{\cdot; A \vdash A}\,\mathrm{ID}_\odot}{\Delta_2; \Gamma_2, A \vdash B}\,\multimap_E}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2, A \vdash C}\,\multimap_E}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash A \multimap C}\,\multimap_I$$

At this point I have introduced the basics of sequent-style natural deduction, but nothing I have said up to now has been specific to ordered linear logic. We call the this logic ordered, because it contains the non-symmetric tensor product. Its non-symmetry is enforced by the separation of hypotheses. What makes a logic linear?

Linear logic was first introduced by Girard [6] in the late eighties. A logic is linear if every hypothesis is used exactly once. Thus, no hypothesis can be duplicated or removed at will, that is, the structural rules for contraction (duplication) and weakening (removal) must be banned from the logic. This property makes linear logic particularly suited for reasoning about state-based systems.

We enforce this property by restricting the inference rules in two ways. First, consider the identity axioms for OLL:

$$\dfrac{}{\cdot; A \vdash A}\,\mathrm{id}_\odot \qquad\qquad \dfrac{}{A; \cdot \vdash A}\,\mathrm{id}_\rhd$$

The contexts are restricted to containing at most the hypothesis the axiom is proving holds. The other restriction is when applying the inference rules during

the construction of derivations one is not allowed to copy or introduce new hypothesis across premises. This is why we take great care at separating the contexts in the definition of the inference rules.

Modeling structures in logic corresponds to interpreting the structures as formulas, and then reasoning about the structures corresponds to proving implications between the interpretations. As we will see, attack trees will be modeled as formulas of linear logic, and then we will prove several properties of attack trees by proving implications between them, but before moving on to ATLL I give an example showing how to model a simple state-based system in linear logic.

There is a common example of problem solving used in artificial intelligence called the block world scenario; I learned of this example from Power and Webster's work on embedding linear logic in Coq [14]. Suppose there is a finite set of blocks and a robot arm that can be used to move blocks around. For example, the arm might break a single stack of blocks into two stacks. This setup can be modeled in linear logic using a few predicates on blocks. The state of the system can be modeled using the following predicates:

$\mathsf{on}\ x\ y$	: block x is on top of block y
$\mathsf{table}\ x$	: block x is on the table (i.e. no block underneath of x)
$\mathsf{clear}\ x$	: there is no block on top of x
$\mathsf{holds}\ x$	: the robot arm is holding block x
empty	: an atomic formula indicating the robot arm is not holding a block.

Next we must characterize the pre- and post-states of the valid actions one can perform during the game. These are defined as a set of linear logic axioms, where the hypotheses are the pre-state and the conclusion is the post-state. In this setting linear implication, $A \multimap B$, is being interpreted as, if before the action executes, A holds, then after the action executes, B holds. The following set of inference rules axiomatize the actions:

$$\frac{}{\cdot;\ \mathsf{empty},\ \mathsf{clear}\ x \vdash (\mathsf{holds}\ x) \odot (((\mathsf{table}\ x) \multimap \top) \times ((\mathsf{on}\ x\ y) \multimap (\mathsf{clear}\ y)))}\ get$$

$$\frac{}{\cdot;\ \mathsf{holds}\ x \vdash \mathsf{empty} \odot (\mathsf{clear}\ x) \odot ((\mathsf{table}\ x) \times ((\mathsf{clear}\ y) \multimap (\mathsf{on}\ x\ y)))}\ put$$

The *get* axiom states that if the robot arm was empty and there was no block on top of x, then after the action executes the robot arm will be holding the block x, and either the block x had been on the table, or it was on another block y which we now declare has no block on top of it. The *put* axiom states that if the robot arm was holding the block x, then after the action executes the robot arm is empty, the block x has no block on top of it, and either the block x is sitting on the table, or it is sitting on the block y which was clear, and now we declare that x is on top of y.

We can see that the definition of the two axioms use the tensor product to indicate when two processes run in parallel. For example, in *get* the robot arm is holding the block x in parallel to checking the configuration of the table and other

blocks. Then we use non-linear conjunction to model choices. Furthermore, the identity to the tensor product can be used to model processes that may consume a resource, but does not produce any new resources. For example, the formula $(\mathsf{table}\, x) \multimap \top$ consumes the resource corresponding to the block x being on the table, but then does not update the state.

At this point we can use our logic to prove properties about the block world scenario, and we can even derive new inference rules. For example, we could derive a new inference rule that swaps the two blocks on top of a stack of at least two, or that performs a get when there is a block underneath the block the arm will pick up. We will use these insights throughout the remainder of this paper to model attack trees.

3 Attack Trees without Costs

In this section I introduce attack trees with sequential conjunction – sometimes referred to as SAND attack trees – without cost annotations, and then the Attack Tree Linear Logic (ATLL). This formulation of attack trees was first proposed by Jhawar et al. [7].

Definition 1. *Suppose* B *is a set of base attacks whose elements are denoted by* b*. Then an* ***attack tree*** *is defined by the following grammar:*

$$T ::= b \mid T_1 \odot T_2 \mid T_1 \sqcup T_2 \mid T_1 \rhd T_2$$

I denote unsynchronized parallel conjunction of attacks by $T_1 \odot T_2$*, choice between attacks by* $T_1 \sqcup T_2$*, and sequential conjunction of attacks by* $T_1 \rhd T_2$*.*

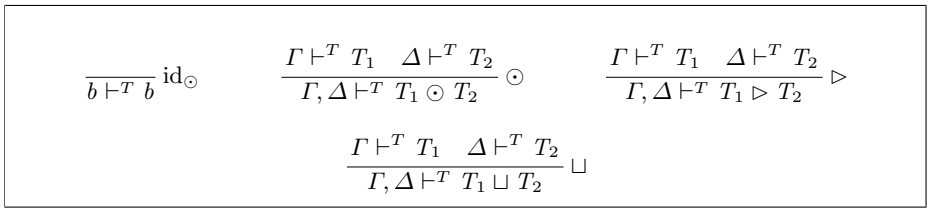
Next I define ATLL and discuss how it can be used for reasoning about attack trees.

In ATLL attack trees are modeled as formulas, and thus, reasoning about attack trees should correspond to proving implications between them. ATLL is an extension of OLL (Section 2) with a new operator for modeling choice between attack trees. The ATLL syntax is defined by the following definition.

Definition 2. *Suppose* B *is a set of base attacks whose elements are denoted by* b*. The syntax of ATLL formulas and contexts are defined as follows:*

$$\begin{array}{rl} \textit{(formulas)} & E ::= b \mid E_1 \odot E_1 \mid E_1 \sqcup E_2 \mid E_1 \rhd E_2 \mid E_1 \multimap E_2 \\ \textit{(base contexts)} & \Gamma, \Delta ::= \cdot \mid b \mid \Gamma, \Delta \\ \textit{(general contexts)} & \Theta, \Phi, \Psi ::= \cdot \mid E \mid \Theta, \Psi \end{array}$$

One can also view ATLL formulas as being defined as an extension of the syntax for attack trees with linear implication denoted by $A \multimap B$. Suppose E is an ATLL formula such that every base attack in E is in the base context Γ. Then E is an attack tree if and only if $\Gamma \vdash^{T} E$ holds with respect to the rules in Figure 3. These inference rules allow one to restrict themselves to attack trees when using ATLL.



$$\frac{}{b \vdash^T b}\,\mathrm{id}_{\odot} \qquad \frac{\Gamma \vdash^T T_1 \quad \Delta \vdash^T T_2}{\Gamma, \Delta \vdash^T T_1 \odot T_2}\,\odot \qquad \frac{\Gamma \vdash^T T_1 \quad \Delta \vdash^T T_2}{\Gamma, \Delta \vdash^T T_1 \rhd T_2}\,\rhd$$

$$\frac{\Gamma \vdash^T T_1 \quad \Delta \vdash^T T_2}{\Gamma, \Delta \vdash^T T_1 \sqcup T_2}\,\sqcup$$

Fig. 3. Well Formed Attack Trees in ATLL

The inference rules in Figure 3 enforce several properties. First, they enforce that every base attack is used exactly once, and that an attack tree does not mention linear implication. Furthermore, they allow us to restrict our reasoning to attack trees in ATLL which I enforce in the hope that it will result in our reasoning being more amenable to automation, but this is left to future work.

The ATLL logical inference rules are defined in Figure 4. The sequent of this system is denoted by $\Theta; \Phi; \Psi \vdash E$ and consists of the general context Θ associated with sequential conjunction, Φ associated with choice, and Ψ associated with parallel conjunction. This is similar to OLL; see Section 2. It is required for hypotheses used with choice to be separated, because the attack tree equations for choice require contraction and a weak form of weakening; see the rules $\mathrm{dup}_{\sqcup}$ and $\mathrm{cont}_{\sqcup}$.

These rules make up a basic linear logic for attack trees. The reader should note that the rule $\mathrm{comp}_{\multimap}$ is very useful when proving properties of attack trees, but is actually a derivable rule. Furthermore, one can see quite clearly that these rules are an extension of the rules for OLL, but instead of non-linear conjunction it has non-linear choice. In ATLL one can prove that choice, sequential conjunction, and parallel conjunction are associative, and choice and parallel conjunction are symmetric. However, one cannot prove the usual distributive laws associated with attack trees with sequential conjunction [7].

To overcome this limitation we add the attack tree axioms in Figure 5 to ATLL. The logical connective $\multimap\!\!\multimap$ is linear biimplication which is defined in the usual way as classical implication. The reader should note that these axioms are restricted to attack trees only and not applicable to full ATLL formulas. In addition, the rule $\mathrm{dis}_{\odot_2}$ is derivable, because parallel conjunction is symmetric.

ATLL allows one to prove every equivalence for attack trees as linear implications that one might expect. The equations provable in ATLL are give in Table 1. These are all of the equations given by Jhawar et al. [7]. I conjecture that it should be possible to define a proof search algorithm for ATLL to automate equational reasoning on attack trees, but this is future work. Using implication as a means to reason about attack trees opens the door for different types of reasoning.

Suppose T_1 is a larger attack tree, and we wish to know if the attack tree T_2 is contained within T_1, that is, T_2 is a subattack tree of T_1. Currently, ATLL will not allow us to prove this, but it can if we add the following rules:

$$\frac{}{\cdot;\cdot;E \vdash E}\,\mathrm{id}_{\odot} \qquad \frac{}{\cdot;E;\cdot \vdash E}\,\mathrm{id}_{\sqcup} \qquad \frac{}{E;\cdot;\cdot \vdash E}\,\mathrm{id}_{\triangleright}$$

$$\frac{\Theta_1;\Phi_1;\Psi_1 \vdash E_1 \quad \Theta_2;\Phi_2;\Psi_2 \vdash E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_1 \odot E_2}\,\odot_I$$

$$\frac{\Theta_1;\Phi_1;\Psi_2 \vdash E_1 \odot E_2 \quad \Theta_2;\Phi_2;\Psi_1,E_1,E_2,\Psi_3 \vdash E_3}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2,\Psi_3 \vdash E_3}\,\odot_E$$

$$\frac{\Theta_1;\Phi_1;\Psi_1 \vdash E_1 \quad \Theta_2;\Phi_2;\Psi_2 \vdash E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_1 \sqcup E_2}\,\sqcup_I$$

$$\frac{\Theta_1;\Phi_2;\Psi_1 \vdash E_1 \sqcup E_2 \quad \Theta_2;\Phi_1,E_1,E_2,\Phi_3;\Psi_2 \vdash E_3}{\Theta_1,\Theta_2;\Phi_1,\Phi_2,\Phi_3;\Psi_1,\Psi_2 \vdash E_3}\,\sqcup_E$$

$$\frac{\Theta_1;\Phi_1;\Psi_1 \vdash E_1 \quad \Theta_2;\Phi_2;\Psi_2 \vdash E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_1 \triangleright E_2}\,\triangleright_I$$

$$\frac{\Theta_2;\Phi_1;\Psi_1 \vdash E_1 \triangleright E_2 \quad \Theta_1,E_1,E_2,\Theta_3;\Phi_2;\Psi_2 \vdash E_3}{\Theta_1,\Theta_2,\Theta_3;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_3}\,\triangleright_E$$

$$\frac{\Theta;\Phi;\Psi_1,E_1,E_2,\Psi_2 \vdash E}{\Theta;\Phi;\Psi_1,E_2,E_1,\Psi_2 \vdash E}\,\mathrm{sym}_{\odot} \qquad \frac{\Theta;\Phi_1,E_1,E_2,\Phi_2;\Psi \vdash E}{\Theta;\Phi_1,E_2,E_1,\Phi_2;\Psi \vdash E}\,\mathrm{sym}_{\sqcup}$$

$$\frac{\Theta;\Phi_1,E_1,\Phi_2;\Psi \vdash E_2}{\Theta;\Phi_1,E_1,E_1,\Phi_2;\Psi \vdash E_2}\,\mathrm{dup}_{\sqcup} \qquad \frac{\Theta;\Phi_1,E_1,E_1,\Phi_2;\Psi \vdash E_2}{\Theta;\Phi_1,E_1,\Phi_2;\Psi \vdash E_2}\,\mathrm{cont}_{\sqcup}$$

$$\frac{\Theta;\Phi;\Psi,E_1 \vdash E_2}{\Theta;\Phi;\Psi \vdash E_1 \multimap E_2}\,\multimap_I \qquad \frac{\Theta_1;\Phi_1;\Psi_1 \vdash E_1 \multimap E_2 \quad \Theta_2;\Phi_2;\Psi_2 \vdash E_1}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_2}\,\multimap_E$$

$$\frac{\Theta_2;\Phi_1;\Psi_1 \vdash E_1 \multimap E_2 \quad \Theta_1;\Phi_2;\Psi_2 \vdash E_2 \multimap E_3}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2 \vdash E_1 \multimap E_3}\,\mathrm{comp}_{\multimap}$$

Fig. 4. ATLL Logical Inference Rules

$$\frac{\Theta, \Phi, \Psi \vdash^T T_1 \odot (T_2 \rhd T_3)}{\Theta; \Phi; \Psi \vdash (T_1 \odot (T_2 \sqcup T_3)) \multimap ((T_1 \odot T_2) \sqcup (T_1 \odot T_3))}\ \mathrm{dis}_{\odot_1}$$

$$\frac{\Theta \vdash^T (T_2 \rhd T_3) \odot T_1}{\Theta; \Phi; \Psi \vdash ((T_2 \sqcup T_3) \odot T_1) \multimap ((T_2 \odot T_1) \sqcup (T_3 \odot T_1))}\ \mathrm{dis}_{\odot_2}$$

$$\frac{\Theta, \Phi, \Psi \vdash^T T_1 \rhd (T_2 \sqcup T_3)}{\Theta; \Phi; \Psi \vdash (T_1 \rhd (T_2 \sqcup T_3)) \multimap ((T_1 \rhd T_2) \sqcup (T_1 \rhd T_3))}\ \mathrm{dis}_{\rhd_1}$$

$$\frac{\Theta, \Phi, \Psi \vdash^T (T_2 \sqcup T_3) \rhd T_1}{\Theta; \Phi; \Psi \vdash ((T_2 \sqcup T_3) \rhd T_1) \multimap ((T_2 \rhd T_1) \sqcup (T_2 \rhd T_1))}\ \mathrm{dis}_{\rhd_2}$$

Fig. 5. ATLL Attack Tree Axioms

$T \sqcup T \multimap T$	$(T_1 \rhd T_2) \rhd T_3 \multimap T_1 \rhd (T_2 \rhd T_3)$
$T_1 \odot T_2 \multimap T_2 \odot T_1$	$T_1 \odot (T_2 \sqcup T_3) \multimap (T_1 \odot T_2) \sqcup (T_1 \odot T_3)$
$T_1 \sqcup T_2 \multimap T_2 \sqcup T_1$	$(T_2 \sqcup T_3) \odot T_1 \multimap (T_2 \odot T_1) \sqcup (T_3 \odot T_1)$
$(T_1 \sqcup T_2) \sqcup T_3 \multimap T_1 \sqcup (T_2 \sqcup T_3)$	$T_1 \rhd (T_2 \sqcup T_3) \multimap (T_1 \rhd T_2) \sqcup (T_1 \rhd T_3)$
$(T_1 \odot T_2) \odot T_3 \multimap T_1 \odot (T_2 \odot T_3)$	$(T_2 \sqcup T_3) \rhd T_1 \multimap (T_2 \rhd T_1) \sqcup (T_2 \rhd T_1)$

Table 1. The Attack Tree Equivalences

$$\frac{\Theta_1, \Theta_2; \Phi; \Psi \vdash E \quad \Delta \vdash^T T}{\Theta_1,\, T, \Theta_2; \Phi; \Psi \vdash E}\ \mathrm{wk}_{\rhd} \qquad \frac{\Theta; \Phi_1, \Phi_2; \Psi \vdash E \quad \Delta \vdash^T T}{\Theta; \Phi_1,\, T, \Phi_2; \Psi \vdash E}\ \mathrm{wk}_{\sqcup}$$

$$\frac{\Theta; \Phi; \Psi_1, \Psi_2 \vdash E \quad \Delta \vdash^T T}{\Theta; \Phi; \Psi_1,\, T, \Psi_2 \vdash E}\ \mathrm{wk}_{\odot}$$

These rules allow us to discard parts of a larger attack tree while constructing proof derivations, but this only works for attack trees not general ATLL formulas. Thus, with these rules ATLL becomes slightly affine. We can now use these rules to determine if $T_1 \multimap T_2$ holds, and if it does, then T_2 is a subattack tree of T_1.

I now give an example to illustrate using the system to prove properties of attack trees. Consider the two example trees from the introduction:

$$T_1 := (B_1 \odot (B_2 \sqcup B_3)) \rhd B_4$$
$$T_2 := ((B_1 \odot B_2) \rhd B_4) \sqcup ((B_1 \odot B_3) \rhd B_4)$$

The attack trees T_1 and T_2 are equivalent if and only if $T_1 \multimap T_2$ holds. The ATLL attack tree axioms are biimplications, and hence, we may treat these axioms as equivalences. Thus, using the ATLL attack tree axioms we can prove

that T_1 is equivalent to T_2 informally as follows:

$$\begin{array}{lll} T_1 := & (B_1 \odot (B_2 \sqcup B_3)) \rhd B_4 & \text{(definition)} \\ \multimap & ((B_1 \odot B_2) \sqcup (B_1 \odot B_3)) \rhd B_4 & (\text{rule } \mathrm{dis}_{\odot_1}) \\ \multimap & ((B_1 \odot B_2) \rhd B_4) \sqcup ((B_1 \odot B_3) \rhd B_4) & (\text{rule } \mathrm{dis}_{\rhd_2}) \\ := & T_2 & \text{(definition)} \end{array}$$

The corresponding formal derivation is quite large, but is easily constructible using the previous reasoning.

4 Attack Trees with Costs

Modeling attack trees without costs is only half of the picture. So how do we add costs? Furthermore, can the quantitative layer impact the logical layer (process tree) and vice versa? I answer both of these questions in this section.

The answer to the first question lies in a generalization of Bucciarelli and Ehrhards indexed linear logic [3]. Hence, I extend ATLL into Indexed ATLL which has the ability to model and reason about attack trees with costs. The formalization of Indexed ATLL I present in this paper is a simplification of the fully general system. This simplified version will make the main ideas easier to see by the reader without those ideas being drowned out by technicalities. Throughout this section I will explain how Indexed ATLL can be generalized to the full system.

Attack trees are generalized to include costs by the following definition.

Definition 3. *Suppose* B *is a set of base attacks whose elements are denoted by* b*, and* C *is a set of costs whose elements are denoted by* c*. Additionally, let* $(\mathsf{C}, \mathsf{op}_\odot)$ *and* $(\mathsf{C}, \mathsf{op}_\sqcup)$ *be symmetric monoids on* C*, and* $(\mathsf{C}, \mathsf{op}_\rhd)$ *be a monoid on* C*, where* $\mathsf{op}_\odot$ *and* $\mathsf{op}_\rhd$ *distribute over* $\mathsf{op}_\sqcup$*. Then an* ***attack tree with costs*** *is defined by the following grammar:*

$$T ::= (b, c) \mid T_1 \odot T_2 \mid T_1 \sqcup T_2 \mid T_1 \rhd T_2$$

$$\frac{}{(b,c) \vdash^T_c b}\,\mathrm{id}_\odot \qquad \frac{\Delta_1 \vdash^T_{c_1} T_1 \quad \Delta_2 \vdash^T_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^T_{\mathsf{op}_\odot(c_1,c_2)} T_1 \odot T_2}\,\odot$$

$$\frac{\Delta_1 \vdash^T_{c_1} T_1 \quad \Delta_2 \vdash^T_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^T_{\mathsf{op}_\rhd(c_1,c_2)} T_1 \rhd T_2}\,\rhd \qquad \frac{\Delta_1 \vdash^T_{c_1} T_1 \quad \Delta_2 \vdash^T_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^T_{\mathsf{op}_\sqcup(c_1,c_2)} T_1 \sqcup T_2}\,\sqcup$$

Fig. 6. Well Formed Attack Trees in Indexed ATLL

Each base attack is annotated with a cost, and every branching node is associated with a binary operation for computing the cost at that node using the costs of the

left subtree and the right subtree. The requirement that each binary operation is monoidal insures that each of the attack tree equations (Table 1) lift to the quantitative layer. As one can see from the definition of attack trees there is a single operation that will compute the cost across each type of branching operator, for example, computing the cost of each sequential branching node will use the exact same operation. This is generalized in the full system by indexing each of the branching operators with a monoidal operation of the same type given here. This will allow each branching node to use a different operator if desired.

I now extend the syntax and rules of ATLL into Indexed ATLL, but due to space I do not give every rule here, but only the most interesting ones, because they are similar to ATLL; the entire system is defined in Appendix A.

Definition 4. *Suppose* B *is a set of base attacks whose elements are denoted by* b*, and* C *is a set of costs whose elements are denoted by* c*. Additionally, let* $(\mathsf{C}, \mathsf{op}_{\odot})$ *and* $(\mathsf{C}, \mathsf{op}_{\sqcup})$ *be symmetric monoids on* C*, and* $(\mathsf{C}, \mathsf{op}_{\triangleright})$ *be a monoid on* C*, where* $\mathsf{op}_{\odot}$ *and* $\mathsf{op}_{\triangleright}$ *distribute over* $\mathsf{op}_{\sqcup}$*. Furthermore, let* $(\mathsf{C}, \mathsf{rel}_{\multimap})$ *be a preorder on* C *where* $\mathsf{rel}_{\multimap}(\mathsf{op}_{\sqcup}(c, c), c)$ *and* $\mathsf{rel}_{\multimap}(c, \mathsf{op}_{\sqcup}(c, c))$ *hold. Then the syntax of Indexed ATLL formulas and contexts are defined as follows:*

$$\begin{array}{rl} \textit{(formulas)} & E ::= (b, c) \mid E_1 \odot E_1 \mid E_1 \sqcup E_2 \mid E_1 \rhd E_2 \mid (E_1, c) \multimap E_2 \\ \textit{(base contexts)} & \Gamma, \Delta ::= \cdot \mid (b, c) \mid \Gamma, \Delta \\ \textit{(general contexts)} & \Theta, \Phi, \Psi ::= \cdot \mid (E, c) \mid \Theta, \Psi \end{array}$$

As we can see the only major changes to the ATLL syntax is that all hypotheses are annotated with costs, and there is a new preorder, $\mathsf{rel}_{\multimap}$, on costs such that $\mathsf{rel}_{\multimap}(\mathsf{op}_{\sqcup}(c, c), c)$ and $\mathsf{rel}_{\multimap}(c, \mathsf{op}_{\sqcup}(c, c))$ both hold. This insures that the contraction equation $T \sqcup T \multimap\!\!\circ\, T$ holds in Indexed ATLL. This implies that the type of monoids one can instantiate $(\mathsf{C}, \mathsf{op}_{\sqcup})$ to must satisfy the previous relationships, but this includes the maximum and minimum monoids which are the most used operations for choice. The hypothesis in implications, now denoted by $(E_1, c) \multimap E_2$, are also annotated with a cost. This is where we begin to link the quantitative layer with the logical layer. In the inference rule, $\multimap_I$, for linear implication the preorder on costs will be used to influence when an implication will be provable.

The inference rules for well-formed attack trees in Indexed ATLL are given in Figure 6. The sequent is now denoted by $\Delta \vdash_c^T T$ where c is the fully computed cost of the root of the attack tree T. Notice that this cost is computed during the construction of a proof derivation.

The main sequent for Indexed ATLL is denoted by $\Theta; \Phi; \Psi \vdash_c E$. The cost, c, is the cost of the root of the syntax tree of E, and if E is an attack tree, then it is the cost of the root of E. This cost is computed during the construction of the derivation proving E holds in Indexed ATLL. Thus, if E is an attack tree, then this will certify it is constructed correctly and the costs are all computed using the given operations. Furthermore, proving implications between attack trees will be able to take this cost into consideration.

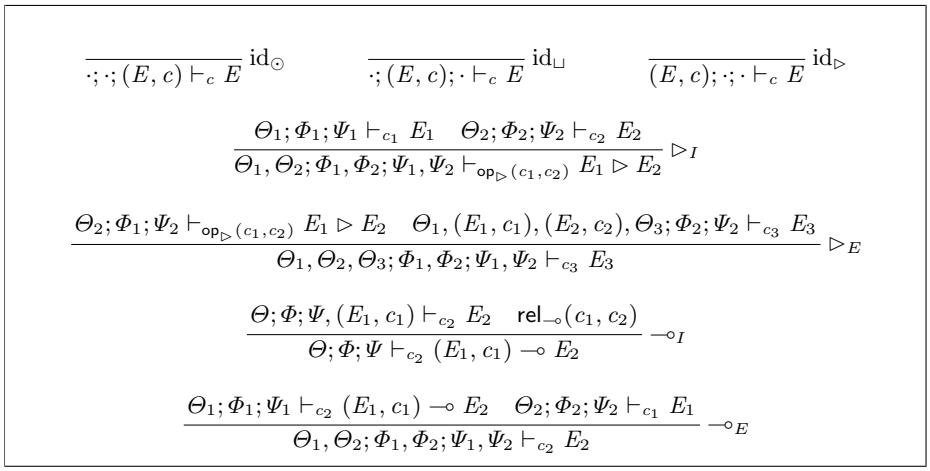


Fig. 7. Select Logical Inference Rules for Indexed ATLL

The logical inference rules for Indexed ATLL are given in Figure 7. As I mentioned above I only give a select few of the rules here, but the complete system can be found in Appendix A. The most interesting rule is the introduction rule for implication, $\multimap_I$, because the second premise is $\mathsf{rel}_{\multimap}(c_1, c_2)$.

Suppose (T_1, c_1) and (T_2, c_2) are two attack trees with their respective costs, and the relation $\mathsf{rel}_{\multimap}(c_1, c_2)$ holds if and only if $c_1 \leq c_2$. If we wanted to show that the attack tree T_1 is equivalent to the attack tree T_2, then we would have to show $T_1 \multimap\!\!\circ\!\!\multimap T_2$ where $\mathsf{rel}_{\multimap\circ\multimap}(c_1, c_2)$ holds if and only if $\mathsf{rel}_{\multimap}(c_1, c_2)$ and $\mathsf{rel}_{\multimap}(c_2, c_1)$ hold. This would require one to prove $\mathsf{rel}_{\multimap\circ\multimap}(c_1, c_2)$ which would require $c_1 = c_2$. Thus, in this example, attack trees would be equivalent if and only if they have the same logical structure and the same overall cost. Thus, several different types of reasoning on attack trees can be modeled in Index ATLL depending on the relation chosen for implication.

The attack tree axioms, see Appendix A, do not differ from ATLL, but this is because of the additional structure imposed on the cost operations. For example, the following is one of the distributive laws:

$$\frac{\Theta,\Phi,\Psi\vdash^T_c T_1\rhd(T_2\sqcup T_3)}{\Theta;\Phi;\Psi\vdash_c (T_1\rhd(T_2\sqcup T_3))\multimap\!\!\circ\!\!\multimap((T_1\rhd T_2)\sqcup(T_1\rhd T_3))}\,\mathrm{dis}_{\rhd_1}$$

Suppose (T_1, c_1), (T_2, c_2), and (T_3, c_3) are all attack trees. Then the attack tree $T_1 \rhd (T_2 \sqcup T_3)$ has cost $\mathsf{op}_{\rhd}(c_1, \mathsf{op}_{\sqcup}(c_2, c_3))$ and the attack $(T_1 \rhd T_2) \sqcup (T_1 \rhd T_3)$ has cost $\mathsf{op}_{\sqcup}(\mathsf{op}_{\rhd}(c_1, c_2), \mathsf{op}_{\rhd}(c_1, c_3))$, and hence, it must be the case that the following hold:

$$\mathsf{rel}_{\multimap}(\mathsf{op}_{\rhd}(c_1, \mathsf{op}_{\sqcup}(c_2, c_3)), \mathsf{op}_{\sqcup}(\mathsf{op}_{\rhd}(c_1, c_2), \mathsf{op}_{\rhd}(c_1, c_3)))$$
$$\mathsf{rel}_{\multimap}(\mathsf{op}_{\sqcup}(\mathsf{op}_{\rhd}(c_1, c_2), \mathsf{op}_{\rhd}(c_1, c_3)), \mathsf{op}_{\rhd}(c_1, \mathsf{op}_{\sqcup}(c_2, c_3)))$$

However, in the definition of Indexed ATLL

$$\mathsf{op}_{\triangleright}(c_1, \mathsf{op}_{\sqcup}(c_2, c_3)) = \mathsf{op}_{\sqcup}(\mathsf{op}_{\triangleright}(c_1, c_2), \mathsf{op}_{\triangleright}(c_1, c_3)),$$

and thus, in the rule above I simply state the cost is some c. Therefore, the additional structure imposed on the binary operations for branching nodes are required to insure the attack tree equivalences from Table 1 hold.

5 Related and Future Work

The following summarizes related work and proposes a few open problems:

- Attack trees with sequential conjunction were first proposed by Jhawar et al. [7]. They found their attack trees in sets of series-parallel graphs. The foundation given here offers a new perspective by founding attack trees in linear logic.
- Kordy et al. [9,10] proposed that attack trees be modeled in propositional logic. Their work provided many insights into how to use logic to model attack trees, but propositional logic does not enforce the concurrency aspect of attack trees. This is regained in ATLL by using linear logic. There is a rich amount of research that has gone into giving linear logic a game semantics [1]. I propose an open problem, in the spirit of Kordy et al. [9] **can we use a game semantics of linear logic to model attack trees?** This would provide the same benefits as their work, but with the resource semantics left intact.
- Samarji et al. [16] propose that graph based attack-defense models can be automatically constructed using and extension of John McCarthy's situation calculus. Originally the situation calculus did not contain the ability to model concurrent attacks, but has since been updated to allow the modeling of concurrency [15,12]. An additional sort had to be added to the situation calculus called concurrent whose inhabitants are all concurrent actions. This is in contrast to linear logic where concurrency and state change are innate properties of the logic. Thus, linear logic is more fundamental than the situation calculus. This shows up already in ATLL which correctly models attack trees in a simpler system – propositional linear logic – than the situation calculus – first-order logic. Furthermore, this opens up an interesting questions, **can one model the situation calculus in first-order linear logic?** This needs to be explored.

Future work. My students and I are working on designing and implementing a term-assignment to the fully general version of Indexed ATLL. Under the Curry-Howard-Lambek correspondence this will produce a statically-typed functional programming language for specifying and reasoning about attack trees. We hope to then study the feasibility of constructing a means of automatically constructing and reasoning about attack trees in a proof producing manner. Finally, we are looking into if it is possible to use current graphical models of linear logic in string diagrams and proof nets to reason about attack trees graphically.

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Appendix

A The Full Specification of Indexed ATLL

Definition 4. *Suppose* B *is a set of base attacks whose elements are denoted by* b*, and* C *is a set of costs whose elements are denoted by* c*. Additionally, let* $(\mathsf{C}, \mathsf{op}_{\odot})$ *and* $(\mathsf{C}, \mathsf{op}_{\sqcup})$ *be symmetric monoids on* C*, and* $(\mathsf{C}, \mathsf{op}_{\rhd})$ *be a monoid on* C*, where* $\mathsf{op}_{\odot}$ *and* $\mathsf{op}_{\rhd}$ *distribute over* $\mathsf{op}_{\sqcup}$*. Furthermore, let* $(\mathsf{C}, \mathsf{rel}_{\multimap})$ *be a preorder on* C *where* $\mathsf{rel}_{\multimap}(\mathsf{op}_{\sqcup}(c, c), c)$ *and* $\mathsf{rel}_{\multimap}(c, \mathsf{op}_{\sqcup}(c, c))$ *hold. Then the syntax of Indexed ATLL formulas and contexts are defined as follows:*

$$
\begin{array}{rl}
\textit{(formulas)} & E ::= (b, c) \mid E_1 \odot E_1 \mid E_1 \sqcup E_2 \mid E_1 \rhd E_2 \mid (E_1, c) \multimap E_2 \\
\textit{(base contexts)} & \Gamma, \Delta ::= \cdot \mid (b, c) \mid \Gamma, \Delta \\
\textit{(general contexts)} & \Theta, \Phi, \Psi ::= \cdot \mid (E, c) \mid \Theta, \Psi
\end{array}
$$

$$
\frac{}{(b, c) \vdash^{T}_{c} b}\,\mathrm{id}_{\odot} \qquad \frac{\Delta_1 \vdash^{T}_{c_1} T_1 \quad \Delta_2 \vdash^{T}_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^{T}_{\mathsf{op}_{\odot}(c_1, c_2)} T_1 \odot T_2}\,\odot
$$

$$
\frac{\Delta_1 \vdash^{T}_{c_1} T_1 \quad \Delta_2 \vdash^{T}_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^{T}_{\mathsf{op}_{\rhd}(c_1, c_2)} T_1 \rhd T_2}\,\rhd \qquad \frac{\Delta_1 \vdash^{T}_{c_1} T_1 \quad \Delta_2 \vdash^{T}_{c_2} T_2}{\Delta_1, \Delta_2 \vdash^{T}_{\mathsf{op}_{\sqcup}(c_1, c_2)} T_1 \sqcup T_2}\,\sqcup
$$

Table 2. Well Formed Attack Trees in Indexed ATLL

$$\frac{}{\cdot;\cdot;(E,c)\vdash_c E}\,\mathrm{id}_{\odot} \qquad \frac{}{\cdot;(E,c);\cdot\vdash_c E}\,\mathrm{id}_{\sqcup} \qquad \frac{}{(E,c);\cdot;\cdot\vdash_c E}\,\mathrm{id}_{\triangleright}$$

$$\frac{\Theta_1;\Phi_1;\Psi_1\vdash_{c_1} E_1 \quad \Theta_2;\Phi_2;\Psi_2\vdash_{c_2} E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2\vdash_{\mathsf{op}_{\odot}(c_1,c_2)} E_1\odot E_2}\,\odot_I$$

$$\frac{\Theta_1;\Phi_1;\Psi_2\vdash_{\mathsf{op}_{\odot}(c_1,c_2)} E_1\odot E_2 \quad \Theta_2;\Phi_2;\Psi_1,(E_1,c_1),(E_2,c_2),\Psi_3\vdash_{c_3} E_3}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2,\Psi_3\vdash_{c_3} E_3}\,\odot_E$$

$$\frac{\Theta_1;\Phi_1;\Psi_1\vdash_{c_1} E_1 \quad \Theta_2;\Phi_2;\Psi_2\vdash_{c_2} E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2\vdash_{\mathsf{op}_{\sqcup}(c_1,c_2)} E_1\sqcup E_2}\,\sqcup_I$$

$$\frac{\Theta_1;\Phi_2;\Psi_1\vdash_{\mathsf{op}_{\sqcup}(c_1,c_2)} E_1\sqcup E_2 \quad \Theta_2;\Phi_1,(E_1,c_1),(E_2,c_2),\Phi_3;\Psi_2\vdash_{c_3} E_3}{\Theta_1,\Theta_2;\Phi_1,\Phi_2,\Phi_3;\Psi_1,\Psi_2\vdash_{c_3} E_3}\,\sqcup_E$$

$$\frac{\Theta_1;\Phi_1;\Psi_1\vdash_{c_1} E_1 \quad \Theta_2;\Phi_2;\Psi_2\vdash_{c_2} E_2}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2\vdash_{\mathsf{op}_{\triangleright}(c_1,c_2)} E_1\triangleright E_2}\,\triangleright_I$$

$$\frac{\Theta_2;\Phi_1;\Psi_2\vdash_{\mathsf{op}_{\triangleright}(c_1,c_2)} E_1\triangleright E_2 \quad \Theta_1,(E_1,c_1),(E_2,c_2),\Theta_3;\Phi_2;\Psi_2\vdash_{c_3} E_3}{\Theta_1,\Theta_2,\Theta_3;\Phi_1,\Phi_2;\Psi_1,\Psi_2\vdash_{c_3} E_3}\,\triangleright_E$$

$$\frac{\Theta;\Phi;\Psi_1,(E_1,c_1),(E_2,c_2),\Psi_2\vdash_c E}{\Theta;\Phi;\Psi_1,(E_2,c_2),(E_1,c_1),\Psi_2\vdash_c E}\,\mathrm{sym}_{\odot}$$

$$\frac{\Theta;\Phi_1,(E_1,c_1),(E_2,c_2),\Phi_2;\Psi\vdash_c E}{\Theta;\Phi_1,(E_2,c_2),(E_1,c_1),\Phi_2;\Psi\vdash_c E}\,\mathrm{sym}_{\sqcup}$$

$$\frac{\Theta;\Phi_1,(E_1,c_1),\Phi_2;\Psi\vdash_{c_2} E_2}{\Theta;\Phi_1,(E_1,c_1),(E_1,c_1),\Phi_2;\Psi\vdash_{c_2} E_2}\,\mathrm{dup}_{\sqcup}$$

$$\frac{\Theta;\Phi_1,(E_1,c_1),(E_1,c_1),\Phi_2;\Psi\vdash_{c_2} E_2}{\Theta;\Phi_1,(E_1,c_1),\Phi_2;\Psi\vdash_{c_2} E_2}\,\mathrm{cont}_{\sqcup}$$

$$\frac{\Theta;\Phi;\Psi,(E_1,c_1)\vdash_{c_2} E_2 \quad \mathsf{rel}_{\multimap}(c_1,c_2)}{\Theta;\Phi;\Psi\vdash_{c_2}(E_1,c_1)\multimap E_2}\,\multimap_I$$

$$\frac{\Theta_1;\Phi_1;\Psi_1\vdash_{c_2}(E_1,c_1)\multimap E_2 \quad \Theta_2;\Phi_2;\Psi_2\vdash_{c_1} E_1}{\Theta_1,\Theta_2;\Phi_1,\Phi_2;\Psi_1,\Psi_2\vdash_{c_2} E_2}\,\multimap_E$$

Table 3. Logical Inference Rules for Indexed ATLL

$$\frac{\Theta, \Phi, \Psi \vdash_c^T T_1 \odot (T_2 \rhd T_3)}{\Theta; \Phi; \Psi \vdash_c (T_1 \odot (T_2 \sqcup T_3)) \multimap ((T_1 \odot T_2) \sqcup (T_1 \odot T_3))} \text{dis}_{\odot_1}$$

$$\frac{\Theta, \Phi, \Psi \vdash_c^T (T_2 \rhd T_3) \odot T_1}{\Theta; \Phi; \Psi \vdash_c ((T_2 \sqcup T_3) \odot T_1) \multimap ((T_2 \odot T_1) \sqcup (T_3 \odot T_1))} \text{dis}_{\odot_2}$$

$$\frac{\Theta, \Phi, \Psi \vdash_c^T T_1 \rhd (T_2 \sqcup T_3)}{\Theta; \Phi; \Psi \vdash_c (T_1 \rhd (T_2 \sqcup T_3)) \multimap ((T_1 \rhd T_2) \sqcup (T_1 \rhd T_3))} \text{dis}_{\rhd_1}$$

$$\frac{\Theta, \Phi, \Psi \vdash_c^T (T_2 \sqcup T_3) \rhd T_1}{\Theta; \Phi; \Psi \vdash_c ((T_2 \sqcup T_3) \rhd T_1) \multimap ((T_2 \rhd T_1) \sqcup (T_2 \rhd T_1))} \text{dis}_{\rhd_2}$$

Table 4. Indexed ATLL Attack Tree Axioms