$$\dfrac{}{\Gamma, A \vdash A}\,ax \qquad\qquad \dfrac{}{\Gamma \vdash 1}\,1_i$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B}\,\times_i \qquad\qquad \dfrac{\Gamma \vdash A \times B}{\Gamma \vdash A}\,\times_{e_1} \qquad\qquad \dfrac{\Gamma \vdash A \times B}{\Gamma \vdash B}\,\times_{e_2}$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\to_i \qquad\qquad \dfrac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\,\to_e$$

$$\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{A \vdash 1}\,1_i}{A \vdash A \times 1}\,\times_i \qquad\qquad \dfrac{\dfrac{}{A \times 1 \vdash A \times 1}\,ax}{A \times 1 \vdash A}\,\times_i$$

$$\dfrac{\dfrac{\dfrac{}{A \times B \vdash A \times B}\,ax}{A \times B \vdash B}\,\times_{e_2} \qquad \dfrac{\dfrac{}{A \times B \vdash A \times B}\,ax}{A \times B \vdash A}\,\times_{e_1}}{A \times B \vdash B \times A}\,\times_i$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \times B) \to C \vdash (A \times B) \to C}\,ax \qquad \dfrac{\dfrac{}{A, B \vdash A}\,ax \qquad \dfrac{}{A, B \vdash B}\,ax}{A, B \vdash A \times B}\,\times_i}{(A \times B) \to C, A, B \vdash C}\,\to_e}{(A \times B) \to C, A \vdash B \to C}\,\to_i}{(A \times B) \to C \vdash A \to (B \to C)}\,\to_i$$

$$t ::= x \mid \mathsf{unit} \mid (t_1, t_2) \mid \lambda x.t \mid t_1\, t_2$$

$$\dfrac{}{\Gamma, x : A \vdash x : A}\,ax \qquad\qquad \dfrac{}{\Gamma \vdash \mathsf{unit} : 1}\,1_i$$

$$\dfrac{\Gamma \vdash t_1 : A \qquad \Gamma \vdash t_2 : B}{\Gamma \vdash (t_1, t_2) : A \times B}\,\times_i \qquad\qquad \dfrac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{fst}\, t : A}\,\times_{e_1} \qquad\qquad \dfrac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{snd}\, t : B}\,\times_{e_2}$$

$$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\,\to_i \qquad\qquad \dfrac{\Gamma \vdash t_1 : A \to B \qquad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1\, t_2 : B}\,\to_e$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : A \times B \vdash z : A \times B}\,ax}{z : A \times B \vdash \mathsf{snd}\, z : B}\,\times_{e_2} \qquad \dfrac{\dfrac{}{w : A \times B \vdash w : A \times B}\,ax}{w : A \times B \vdash \mathsf{fst}\, w : A}\,\times_{e_1}}{w : A \times B \vdash (\mathsf{snd}\, w, \mathsf{fst}\, w) : B \times A}\,\times_i}{\cdot \vdash \lambda w.(\mathsf{snd}\, w, \mathsf{fst}\, w) : (A \times B) \to (B \times A)}\,\to_i$$

$$\dfrac{\dfrac{\dfrac{}{A \times B \vdash A \times B}\ ax}{A \times B \vdash B}\ \times_{e_2} \qquad \dfrac{\dfrac{}{A \times B \vdash A \times B}\ ax}{A \times B \vdash A}\ \times_{e_1}}{\dfrac{A \times B \vdash B \times A}{\cdot \vdash (A \times B) \to (B \times A)}\ \to_i}\ \times_i$$

A **category**, $\mathcal{C}$, consists of a collection of objects $A, \dots, Z \in \mathsf{Obj}(\mathcal{C})$ and a collection of morphisms $f, \dots, z \in \mathsf{Mor}(\mathcal{C})$ such that the following hold:

i. There are two functions $\mathsf{src} : \mathsf{Mor} \to \mathsf{Obj}$ and $\mathsf{tar} : \mathsf{Mor} \to \mathsf{Obj}$, such that, we write $f : A \longrightarrow B$ if $\mathsf{src}(f) = A$ and $\mathsf{tar}(f) = B$

ii. For any $A \in \mathsf{Obj}(\mathcal{C})$, there is a morphism $\mathsf{id}_A : A \longrightarrow A$.

iii. Given any two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ there must be a morphism $f; g : A \longrightarrow C$. In addition,

   a. $f; \mathsf{id}_B = \mathsf{id}_A; f = f : A \longrightarrow B$

   b. $f; (g; h) = (f; g); h$ for $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$

A **cartesian closed category** is a category $\mathcal{C}$ with the following additional data:

i. There is a special object $1 \in \mathsf{Obj}(\mathcal{C})$ such that for any $A \in \mathsf{Obj}(\mathcal{C})$, there is a unique morphism $i : A \longrightarrow 1$.

ii. For any $A, B \in \mathsf{Obj}(\mathcal{C})$, there is an object $A \times B \in \mathsf{Obj}(\mathcal{C})$. In addition,

   a. $\pi_1 : A \times B \longrightarrow A$

   b. $\pi_2 : A \times B \longrightarrow B$

   c. $\langle f, g \rangle : A \longrightarrow B \times C$ for any $f : A \longrightarrow B$ and $g : A \longrightarrow C$

iii. For any $A, B \in \mathsf{Obj}(\mathcal{C})$, there is an object $A \to B \in \mathsf{Obj}(\mathcal{C})$. In addition,

   a. $f^* : G \longrightarrow A \to B$ for any $f : G \times A \longrightarrow B$

   b. $\mathsf{app}_{A,B} : (A \to B) \times A \longrightarrow B$

$$A ::= 1 \mid A \times B \mid A \to B$$

$$\lambda x.x : A \to A$$

$$\lambda x.\lambda y.y : A \to B \to B$$

$$\lambda x.(x, x) : A \to (A \times A)$$

$$\lambda f.\lambda g.\lambda x.g\,(f\,x) : (A \to B) \to (B \to C) \to (A \to C)$$

$$\dfrac{}{(\lambda x.t_2)\, t_1 \rightsquigarrow [t_1/x]t_2}\ \beta \qquad \dfrac{}{\mathsf{fst}\,(t_1, t_2) \rightsquigarrow t_1}\ fst$$

$$\dfrac{}{\mathsf{snd}\,(t_1, t_2) \rightsquigarrow t_2}\ snd$$

A formula A is valid iff there exists a term t and a context Γ, such that, $\Gamma \vdash t : A$.

$$\frac{}{\Gamma, A \vdash A}\ ax \qquad\qquad \Gamma \times A \xrightarrow{\pi_2} A$$

$$\frac{}{\Gamma \vdash 1}\ 1_i \qquad\qquad \Gamma \xrightarrow{i} 1$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B}\ \times_i \qquad\qquad \frac{\Gamma \xrightarrow{f} A \qquad \Gamma \xrightarrow{g} B}{\Gamma \xrightarrow{\langle f, g \rangle} A \times B}$$

$$\frac{\Gamma \vdash A \times B}{\Gamma \vdash A}\ \times_{e_1} \qquad\qquad \frac{\Gamma \xrightarrow{f} A \times B}{\Gamma \xrightarrow{f} A \times B \xrightarrow{\pi_1} A}$$

$$\frac{\Gamma \vdash A \times B}{\Gamma \vdash B}\ \times_{e_1} \qquad\qquad \frac{\Gamma \xrightarrow{f} A \times B}{\Gamma \xrightarrow{f} A \times B \xrightarrow{\pi_2} B}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_i \qquad\qquad \frac{\Gamma \times A \xrightarrow{f} B}{\Gamma \xrightarrow{f^*} A \to B}$$

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to_e \qquad\qquad \frac{\Gamma \xrightarrow{f} A \to B \qquad \Gamma \xrightarrow{g} A}{\Gamma \xrightarrow{\langle f, g \rangle} (A \to B) \times A \xrightarrow{\mathsf{app}_{A,B}} B}$$