A Non-commutative Extension of Multiplicative Exponential Linear Logic

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January 21, 2004

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Abstract *We extend multiplicative exponential linear logic (*MELL*) by a non-commutative, self-dual logical operator. The extended system, called* NEL*, is defined in the formalism of the calculus of structures, which is a generalisation of the sequent calculus and provides a more refined analysis of proofs. We are then able to extend the expressiveness of* MELL *by modelling a broad notion of sequentiality. We show some proof theoretical results: decomposition and cut elimination. The new operator represents a significant challenge: to get our results we use here for the first time some novel techniques, which constitute a uniform and modular approach to cut elimination, contrary to what is possible in the sequent calculus.*

Keywords: *Proof theory, linear logic, non-commutativity, cut elimination.*

Table of Contents

1 Introduction

Non-commutative logical operators have a long tradition [16, 29, 2, 17, 20, 3], and their proof theoretical properties have been studied in the sequent calculus [10] and in proof nets [11]. Recent research has shown that the sequent calculus is not adequate to deal with very simple forms of non-commutativity [12, 13, 28]. On the other hand, proof nets are not ideal for dealing with exponentials and additives, which are desirable for getting good computational power.

In this paper we show a logical system that joins a simple form of non-commutativity with commutative multiplicatives and exponentials. This is done in the formalism of the calculus of structures [12, 13], which overcomes the difficulties encountered in the sequent calculus and in proof nets. Structures are expressions intermediate between formulae and sequents, and in fact they unify those two latter entities into a single one, thereby allowing more control over mutual dependencies of logical relations.

We perform a proof theoretical analysis for cut elimination, with new tools, and we explore some further important properties, which are not available in more traditional settings and which we can collectively regard as 'modularity'. Despite the complexities of the proof theoretical investigation, the system obtained is very simple. This paper contributes the following new results:

1. We define a propositional logical system, called NEL (non-commutative exponential linear logic), which extends MELL (multiplicative exponential linear logic [11]) by a non-commutative, self-dual logical operator called *seq*. This system, which was first imagined in [13], is conservative over MELL augmented by the mix and nullary mix rules [1, 9, 18]. System NEL can be immediately understood by anybody acquainted with the sequent calculus, and is aimed at the same range of applications as MELL. In nearly all computer science languages, sequential composition plays a fundamental role, and it is therefore important to address it in a direct way, in logical representations of those languages. Perhaps surprisingly, parallel composition has been much easier to deal with, due to its commutative nature, which is more similar to the typical nature of traditional logics. The addition of seq opens new syntactic possibilities, for example in dealing with process algebras. It has been used already, in a purely multiplicative setting, to model CCS's prefixing [8]. Furthermore, we show a class of equivalent extensions of NEL, which all enjoy the subformula property. This, together with the finer detail in derivations achieved by the calculus of structures, provides much greater flexibility, as witnessed by the proof theoretical properties mentioned below.

2. We prove for NEL a property called *decomposition* (first pioneered in [13, 26]): we can transform every derivation into an equivalent one, composed of seven derivations carried into seven *disjoint* subsystems of NEL. We can study small subsystems of NEL in isolation and then compose them together with considerable more freedom than in the sequent calculus, where, for example, contraction can not be isolated in a derivation. Decomposition is made available in the calculus of structures by exploiting a new top-down symmetry of derivations.

Since it is a basic compositional result, we expect applications to be very broad in range; we are especially excited about the possibilities in the semantics of derivations.

3. We prove cut elimination for NEL by use of decomposition and a new technique that we call *splitting*. In the calculus of structures the traditional methods for proving cut elimination fail, due to the more general applicability of inference rules. The deep reason for this is in how the calculus deals with associativity. Splitting theorems are a uniform means of recovering control over the way logical operators associate; they allow us to manage the complex inductions required. The cut elimination argument becomes modular, because we can reduce the cut rule to several more primitive inference rules, each of which is separately shown admissible by way of splitting. Only one of these rules (an atomic form of cut) is infinitary, all the others enjoy the subformula property and can be used to extend the system without affecting provability. This result should be handy for software analysis and verification. It is worth noting that this result about splitting holds also in the restriction of MELL (without mix and nullary mix), and is thus an alternative proof of cut elimination for that fragment of linear logic.

The points above correspond, respectively, to Sections 2, 3 and 4. Readers who are not interested in the proof theory of system NEL can just read Section 2.

Other systems extending linear logic with non-commutative operators are studied in [3, 22]. These are more traditional systems in the sequent calculus, for which a more limited proof theory can be developed. The calculus of structures allows us to design a much simpler logic, as witnessed by the fact that we have just one self-dual non-commutative operator instead of two dual ones.

It is worth noting that every system that can be expressed in the one-sided sequent calculus can be trivially expressed in the calculus of structures, but the vice versa is not true. The results in this paper help us to establish the calculus of structures as a natural choice for logical systems aimed at computer science. We showed in [13] that the sequent calculus suffers from excessive restrictions, which are not apparent in the traditional systems of classical and intuitionistic logics, but which start to appear in linear logic and are more and more evident when issues such as non-commutativity, locality of inference rules, and various forms of modularity are taken into account. The calculus of structures was in fact conceived, in [12], as a way to overcome the limitations of the sequent calculus in dealing with non-commutativity. Our calculus has later been used successfully in [26] for defining pure MELL and showing decomposition and cut elimination for it. In [7, 5] a completely local definition of classical logic is shown: in that system, not only the cut rule, but also contraction is atomic. By using atomicity, Brünnler presents in [4] the simplest known syntactic proof of cut elimination for classical logic. In [6] Brünnler and Guglielmi use atomicity to design a finitary system without having to resort to cut elimination.

The calculus of structures essentially introduces two new ideas: 1) it makes derivations top-down symmetric and 2) it allows inference rules to be applied anywhere deep inside structures. We are showing, in this and other papers, that it is possible to produce a rich proof theory in our calculus. This formalism is less dependent than the sequent calculus or natural deduction on the original idiosyncrasies of classical (and intuitionistic) logic, and it is actually designed with notions of locality, atomicity and modularity in mind. For these reasons we promote the calculus of structures as a worthy tool for syntactic investigations related to computer science languages.

In the following sections some proof theory is developed for system NEL. We stress the fact that the methods used are general. As stated above, many techniques in this paper are new, but we tested them privately on the systems that have already been studied, namely BV, MELL and classical logic, and in some systems that we are currently investigating, like full linear logic, also in its entirely atomic presentation [24].

The main results of this paper have already been presented in [14].

2 The System

We call *calculus* a formalism, like natural deduction or the sequent calculus, for specifying logical systems. We say (*formal*) *system* to indicate a collection of inference rules in a given calculus.

A system in our calculus requires a language of *structures*, which are intermediate expressions between formulae and sequents. We now define the language for system NEL and its variants. Intuitively, $[S_1, \dots, S_h]$ corresponds to a sequent $\vdash S_1, \dots, S_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity (and associativity). The structure $(S_1, \dots, S_h)$ corresponds to the associative and commutative times connection of $S_1, \dots, S_h$. The structure $\langle S_1; \dots; S_h \rangle$ is associative and *non-commutative*: this corresponds to the new logical operator, called *seq*, that we add to those of MELL.

For reasons explained in [12, 13], dealing with seq involves adding the rules mix and its nullary version mix0 (see [9, 18, 1]):

$$\mathsf{mix}\,\frac{\vdash \Phi \quad \vdash \Psi}{\vdash \Phi, \Psi} \qquad \text{and} \qquad \mathsf{mix0}\,\frac{}{\vdash} \quad .$$

This has the effect of collapsing the multiplicative units 1 and $\bot$: we will only have one unit $\circ$ common to par, times and seq. Please notice that mix and mix0 are not an artefact of the calculus of structures. But, as shown by Retoré in [18], they are required when using a self-dual non-commutative connective.

2.1 Definition There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by a, b, ... *Structures* are denoted by S, P, Q, R, T, U, V, W, X and Z. The structures of the *language* NEL are generated by

$$S ::= a \mid \circ \mid [\underbrace{S, \dots, S}_{>0}] \mid (\underbrace{S, \dots, S}_{>0}) \mid \langle \underbrace{S; \dots; S}_{>0} \rangle \mid ?S \mid !S \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not an atom; $[S_1, \dots, S_h]$ is a *par structure*, $(S_1, \dots, S_h)$ is a *times structure*, $\langle S_1; \dots; S_h \rangle$ is a *seq structure*, $?S$ is a *why-not structure* and $!S$ is an

Associativity

$$
\begin{array}{rcl}
[\vec{R},[\vec{T}],\vec{U}] & = & [\vec{R},\vec{T},\vec{U}] \\
(\vec{R},(\vec{T}),\vec{U}) & = & (\vec{R},\vec{T},\vec{U}) \\
\langle\vec{R};\langle\vec{T}\rangle;\vec{U}\rangle & = & \langle\vec{R};\vec{T};\vec{U}\rangle
\end{array}
$$

Commutativity

$$
\begin{array}{rcl}
[\vec{R},\vec{T}] & = & [\vec{T},\vec{R}] \\
(\vec{R},\vec{T}) & = & (\vec{T},\vec{R})
\end{array}
$$

Unit

$$
\begin{array}{rcl}
[\circ,\vec{R}] & = & [\vec{R}] \\
(\circ,\vec{R}) & = & (\vec{R}) \\
\langle\circ;\vec{R}\rangle & = & \langle\vec{R}\rangle \\
\langle\vec{R};\circ\rangle & = & \langle\vec{R}\rangle
\end{array}
$$

Singleton

$$[R] = (R) = \langle R\rangle = R$$

Exponentials

$$
\begin{array}{rcl}
?\circ & = & \circ \\
!\circ & = & \circ \\
??R & = & ?R \\
!!R & = & !R
\end{array}
$$

Negation

$$
\begin{array}{rcl}
\bar{\circ} & = & \circ \\
\overline{[R_1,\dots,R_h]} & = & (\bar{R}_1,\dots,\bar{R}_h) \\
\overline{(R_1,\dots,R_h)} & = & [\bar{R}_1,\dots,\bar{R}_h] \\
\overline{\langle R_1;\dots;R_h\rangle} & = & \langle\bar{R}_1;\dots;\bar{R}_h\rangle \\
\overline{?R} & = & !\bar{R} \\
\overline{!R} & = & ?\bar{R} \\
\bar{\bar{R}} & = & R
\end{array}
$$

Contextual Closure

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

Figure 1: *Basic equations for the syntactic equivalence =*

of-course structure; $\bar{S}$ is the *negation* of the structure S. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R,T]$ stands for $S\{[R,T]\}$.

Structures come with equational theories establishing some basic, decidable algebraic laws by which structures are indistinguishable. These are analogous to the laws of associativity, commutativity, idempotency, and so on, usually imposed on sequents. The difference is that we merge the notions of formula and sequent, and we extend the equations to formulae. The structures of the language NEL are equivalent modulo the relation =, defined in Figure 1. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (sequences may contain ',' or ';' separators as appropriate in the context).

2.2 Definition An (*inference*) *rule* is any scheme $\rho\,\dfrac{T}{R}$, where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*; R or T, but not both, may be missing. Rule names are denoted by ρ. A (*formal*) *system*, denoted by $\mathscr{S}$, is a set of rules. A *derivation* in a system $\mathscr{S}$ is a finite chain of instances of rules of $\mathscr{S}$, and is denoted by Δ; a derivation can consist of just one structure. The topmost structure in a derivation is called its *premise*; the bottommost structure is called *conclusion*. A derivation Δ whose premise is T, conclusion is R, and whose rules are in $\mathscr{S}$ is denoted by $\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}$.

The typical inference rules are of the kind $\rho \dfrac{S\{T\}}{S\{R\}}$. This rule scheme ρ specifies that if a structure matches R, in a context $S\{\ \}$, it can be rewritten as specified by T, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). A rule corresponds to implementing in the formal system *any axiom* $T \Rightarrow R$, where $\Rightarrow$ stands for the implication we model in the system, in our case linear implication. The case where the context is empty corresponds to the sequent calculus. For example, the linear logic sequent calculus rule

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$$

could be simulated easily in the calculus of structures by the rule

$$\otimes' \frac{(\Gamma, [A, \Phi], [B, \Psi])}{(\Gamma, [(A, B), \Phi, \Psi])} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. The structure Γ stands for the times structure of the other hypotheses in the derivation tree. More precisely, any sequent calculus derivation

$$\begin{array}{c} \underbrace{\vdash \Gamma_1 \quad \cdots \quad \vdash \Gamma_{i-1} \quad \otimes \dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad \vdash \Gamma_{i+1} \quad \cdots \quad \vdash \Gamma_h}_{\Delta} \\ \vdash \Sigma \end{array}$$

containing the $\otimes$ rule can by simulated by

$$\begin{array}{c} \otimes' \dfrac{(\Gamma'_1, \dots, \Gamma'_{i-1}, [A', \Phi'], [B', \Psi'], \Gamma'_{i+1}, \dots, \Gamma'_h)}{(\Gamma'_1, \dots, \Gamma'_{i-1}, [(A', B'), \Phi', \Psi'], \Gamma'_{i+1}, \dots, \Gamma'_h)} \\ \Delta' \Big\| \\ \Sigma' \end{array} \quad ,$$

in the calculus of structures, where Γ'_j, A', B', Φ', Ψ', Δ' and Σ' are obtained from their counterparts in the sequent calculus by the obvious translation. This means that by this method every system in the one-sided sequent calculus can be ported trivially to the calculus of structures.

Of course, in the calculus of structures, rules could be used as axioms of a generic Hilbert system, where there is no special, structural relation between T and R: then all the good proof theoretical properties of sequent systems would be lost. We will be careful to design rules in a way that is conservative enough to allow us to prove cut elimination, and such that they possess the subformula property.

In our systems, rules come in pairs, $\rho\!\downarrow \dfrac{S\{T\}}{S\{R\}}$ (down version) and $\rho\!\uparrow \dfrac{S\{\bar{R}\}}{S\{\bar{T}\}}$ (up version). Sometimes rules are self-dual, i.e., the up and down version are identical, in which case we omit the arrows. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$. We will be able to get rid of the up rules without affecting provability—after all, $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$ are equivalent statements in many logics. Remarkably, the cut rule reduces into several up rules, and this makes for a modular decomposition of the cut elimination argument because we can eliminate up rules one independently from the other.

Let us now define system NEL by starting from a top-down symmetric variation, that we call SNEL. It is made by two sub-systems that we will call conventionally *interaction* and *structure*. The interaction fragment deals with negation, i.e., duality. It corresponds to identity and cut in the sequent calculus. In our calculus these rules become mutually top-down symmetric and both can be reduced to their atomic counterparts.

The structure fragment corresponds to logical and structural rules in the sequent calculus; it defines the logical operators. Differently from the sequent calculus, the operators need not be defined in isolation, rather complex contexts can be taken into consideration. In the following system we consider *pairs* of logical relations, one inside the other.

2.3 Definition In Figure 2, *system* SNEL is shown (*symmetric non-commutative exponential linear logic*). The rules $\mathsf{ai}\!\downarrow$, $\mathsf{ai}\!\uparrow$, s, $\mathsf{q}\!\downarrow$, $\mathsf{q}\!\uparrow$, $\mathsf{p}\!\downarrow$, $\mathsf{p}\!\uparrow$, $\mathsf{w}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$, are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *weakening*, *coweakening*, *absorption* and *coabsorption*. The *down fragment* of SNEL is $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$, the *up fragment* is $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{w}\!\uparrow, \mathsf{b}\!\uparrow\}$.

There is a straightforward two-way correspondence between structures not involving seq and formulae of MELL: for example

$$![(?a, b), \bar{c}, !\bar{d}] \quad \text{corresponds to} \quad !((?a \otimes b) \mathbin{⅋} c^{\perp} \mathbin{⅋} !d^{\perp}) \quad ,$$

and vice versa. Units are mapped into $\circ$, since $1 \equiv \perp$, when mix and $\mathsf{mix0}$ are added to MELL. System SNEL is just the merging of systems SBV and SELS shown in [12, 13, 26]; there one can find details on the correspondence between our systems and linear logic. The reader can check that the equations in Figure 1 correspond to logical equivalences in MELL, disregarding seq. In particular, $!A \multimap !!A$ and $!!A \multimap !A$ for every MELL formula A, and dually for ?. The rules s, $\mathsf{q}\!\downarrow$ and $\mathsf{q}\!\uparrow$ are the same as in pomset logic viewed as a calculus of cographs [21].

All equations are typical of a sequent calculus presentation, save those for units, exponentials and contextual closure. Contextual closure just corresponds to equivalence being a congruence, it is a necessary ingredient of the calculus of structures. All other equations can be removed and replaced by rules, as in the sequent calculus. This might prove necessary for certain applications. For our purposes, this setting makes for a much more compact presentation, at a more effective abstraction level.

$$
\mathsf{ai}{\downarrow}\,\frac{S\{\circ\}}{S[a,\bar a]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar a)}{S\{\circ\}}
$$

$$
\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{q}{\downarrow}\,\frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \mathsf{q}{\uparrow}\,\frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle (R,T);(U,V)\rangle}
$$

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}}
$$

$$
\mathsf{w}{\downarrow}\,\frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{\circ\}}
$$

$$
\mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \qquad \mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R,R)}
$$

Figure 2: *System* SNEL

Negation is involutive and can be pushed to atoms; it is convenient always to imagine it directly over atoms. Please note that negation does not swap arguments of seq, as happens in the systems of Lambek and Abrusci-Ruet. The unit $\circ$ is self-dual and common to par, times and seq. One may think of it as a convenient way of expressing the empty sequence. Rules become very flexible in the presence of the unit. For example, the following notable derivation is valid:

$$
\mathsf{q}{\downarrow}\,\frac{\mathsf{q}{\uparrow}\,\frac{(a,b)}{\langle a;b\rangle}}{[a,b]} \quad = \quad \mathsf{q}{\downarrow}\,\frac{\mathsf{q}{\uparrow}\,\frac{(\langle a;\circ\rangle,\langle\circ;b\rangle)}{\langle [a,\circ];[\circ,b]\rangle}}{[\langle a;\circ\rangle,\langle\circ;b\rangle]} \quad = \quad \mathsf{q}{\downarrow}\,\frac{\mathsf{q}{\uparrow}\,\frac{(\langle a;\circ\rangle,\langle\circ;b\rangle)}{\langle (a,\circ);(\circ,b)\rangle}}{[\langle a;\circ\rangle,\langle\circ;b\rangle]} \quad .
$$

Each inference rule in Figure 2 corresponds to a linear implication that is sound in MELL plus mix and mix0. For example, promotion corresponds to the implication $!(R \parr T) \multimap (!R \parr ?T)$. Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

2.4 Definition The following rules are called *interaction* and *cut*:

$$
\mathsf{i}{\downarrow}\,\frac{S\{\circ\}}{S[R,\bar R]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar R)}{S\{\circ\}} \quad ,
$$

where R and $\bar R$ are called *principal structures*.

The sequent calculus rule $\bowtie \dfrac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$ is realised as

$$\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{([A,\Phi],[\bar{A},\Psi])}{[([A,\Phi],\bar{A}),\Psi]}}{[(A,\bar{A}),\Phi,\Psi]}}{[\Phi,\Psi]} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. Notice how the tree shape of derivations in the sequent calculus is realised by making use of times structures: in the derivation above, the premise corresponds to the two branches of the cut rule. For this reason, in the calculus of structures rules are allowed to access structures deeply nested into contexts.

The cut rule in the calculus of structures can mimic the classical cut rule in the sequent calculus in its realisation of transitivity, but it is much more general. We believe a good way of understanding it is in thinking of the rule as being about lemmas *in context*. The sequent calculus cut rule generates a lemma valid in the most general context; the new cut rule does the same, but the lemma only affects the limited portion of structure that can interact with it.

We easily get the next two propositions, which say: 1) The interaction and cut rules can be reduced into their atomic forms—note that in the sequent calculus it is possible to reduce interaction to atomic form, but not cut. 2) The cut rule is as powerful as the whole up fragment of the system, and vice versa.

2.5 Definition A rule ρ is *derivable* in the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every instance $\rho\,\dfrac{T}{R}$ there exists a derivation $\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}$. The systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}$ there is a derivation $\begin{array}{c} T \\ \Delta' \| \mathscr{S}' \\ R \end{array}$, and vice versa.

2.6 Proposition *The rule* $\mathsf{i}{\downarrow}$ *is derivable in* $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}\}$*, and, dually, the rule* $\mathsf{i}{\uparrow}$ *is derivable in the system* $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}, \mathsf{p}{\uparrow}\}$*.*

Proof: Induction on principal structures. We show the inductive cases for $\mathsf{i}{\uparrow}$:

$$\mathsf{i}{\uparrow}\,\dfrac{\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(P,Q,[\bar{P},\bar{Q}])}{S(Q,[(P,\bar{P}),\bar{Q}])}}{S[(P,\bar{P}),(Q,\bar{Q})]}}{S(Q,\bar{Q})}}{S\{\circ\}} \quad , \qquad \mathsf{i}{\uparrow}\,\dfrac{\mathsf{i}{\uparrow}\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{S(\langle P;Q\rangle,\langle\bar{P};\bar{Q}\rangle)}{S\langle(P,\bar{P});(Q,\bar{Q})\rangle}}{S(Q,\bar{Q})}}{S\{\circ\}} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\dfrac{\mathsf{p}{\uparrow}\,\dfrac{S(?P,!\bar{P})}{S\{?(P,\bar{P})\}}}{S\{\circ\}} \quad .$$

The cases for $\mathsf{i}{\downarrow}$ are dual. □

2.7 Proposition *Each rule* $\rho{\uparrow}$ *in* SNEL *is derivable in* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\downarrow}\}$*, and, dually, each rule* $\rho{\downarrow}$ *in* SNEL *is derivable in the system* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\uparrow}\}$*.*

$$\circ\!\downarrow \frac{}{\circ} \qquad \mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{q}\!\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}}$$

Figure 3: *System* NEL

Proof: Each instance $\rho\!\uparrow \dfrac{S\{T\}}{S\{R\}}$ can be replaced by $\mathsf{i}\!\uparrow \dfrac{\rho\!\downarrow \dfrac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow \dfrac{S\{T\}}{S(T,[R,\bar{R}])}}{S[R,(T,\bar{R})]}}{S[R,(T,\bar{T})]}}{S\{R\}}$. □

In the calculus of structures, we call *core* the set of rules, other than atomic interaction and cut, used to reduce interaction and cut to atomic form. Rules, other than interaction and cut, that are not in the core are called *non-core*.

2.8 Definition The *core* of SNEL is $\{\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{q}\!\uparrow,\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\}$, denoted by SNELc.

System SNEL is top-down symmetric, and the properties we saw are also symmetric. Provability is an asymmetric notion: we want to observe the possible conclusions that we can obtain from a unit premise. We now break the top-down symmetry by adding an inference rule with no premise, and we join this logical axiom to the down fragment of SNEL.

2.9 Definition The following rule is called *unit*: $\circ\!\downarrow \dfrac{}{\circ}$. *System* NEL is shown in Figure 3.

As an immediate consequence of Propositions 2.6 and 2.7 we get:

2.10 Theorem *The systems* $\mathsf{NEL}\cup\{\mathsf{i}\!\uparrow\}$ *and* $\mathsf{SNEL}\cup\{\circ\!\downarrow\}$ *are strongly equivalent.*

2.11 Definition A derivation with no premise is called a *proof*, denoted by Π. A system $\mathscr{S}$ *proves* R if there is in the system $\mathscr{S}$ a proof Π whose conclusion is R, written $\begin{array}{c}\Pi\,\Vert\,\mathscr{S}\\ R\end{array}$. We say that a rule ρ is *admissible* for the system $\mathscr{S}$ if $\rho\notin\mathscr{S}$ and for every proof $\begin{array}{c}\Pi\,\Vert\,\mathscr{S}\cup\{\rho\}\\ R\end{array}$ there is a proof $\begin{array}{c}\Pi'\,\Vert\,\mathscr{S}\\ R\end{array}$. Two systems are *equivalent* if they prove the same structures.

Except for cut and coweakening, systems SNEL and NEL enjoy a subformula property (which we treat as an asymmetric property, by going from conclusion to premise): premises are made of substructures of the conclusions.

To get cut elimination, so as to have a system whose rules all enjoy the subformula property, we could just get rid of $\mathsf{ai}\!\uparrow$ and $\mathsf{w}\!\uparrow$, by proving their admissibility for the other rules. But we can do more than that: the whole up fragment of SNEL, except for s (which also belongs to the down fragment), is admissible. This entails a *modular* scheme for proving cut elimination. In Sections 3 and 4 we will give the proof of the cut elimination theorem:

2.12 Theorem *System* NEL *is equivalent to every subsystem of* $\mathsf{SNEL} \cup \{\circ\downarrow\}$ *which contains* NEL*.*

2.13 Corollary *The rule* $\mathsf{i}\uparrow$ *is admissible for system* NEL*.*

Proof: Immediate from Theorems 2.10 and 2.12. □

Any implication $T \multimap R$, i.e. $[\bar{T}, R]$, is connected to derivability by:

2.14 Corollary *For any two structures* T *and* R*, we have*

$$\begin{array}{c} T \\ \Vert \mathsf{SNEL} \\ R \end{array} \qquad \text{if and only if} \qquad \begin{array}{c} \\ \Vert \mathsf{NEL} \\ [\bar{T}, R] \end{array} \quad .$$

Proof: For the first direction, perform the following transformations:

$$\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \\ R \end{array} \quad \overset{1}{\leadsto} \quad \begin{array}{c} [\bar{T}, T] \\ \Delta \Vert \mathsf{SNEL} \\ [\bar{T}, R] \end{array} \quad \overset{2}{\leadsto} \quad \begin{array}{c} \mathsf{i}\downarrow \dfrac{\circ\downarrow \dfrac{}{\circ}}{[\bar{T}, T]} \\ \Delta \Vert \mathsf{SNEL} \\ [\bar{T}, R] \end{array} \quad \overset{3}{\leadsto} \quad \begin{array}{c} \\ \Pi \Vert \mathsf{NEL} \\ [\bar{T}, R] \end{array} \quad .$$

In the first step we replace each structure S occurring inside Δ by $[\bar{T}, S]$, which is then transformed into a proof by adding an instance of $\mathsf{i}\downarrow$ and $\circ\downarrow$. Then we apply Proposition 2.6 and cut elimination (Theorem 2.12) to obtain a proof in system NEL. For the other direction, let $\begin{array}{c} \Pi \Vert \mathsf{NEL} \\ [\bar{T}, R] \end{array}$ be given. Then there is a derivation $\begin{array}{c} \circ \\ \Delta \Vert \mathsf{NEL} \setminus \{\circ\downarrow\} \\ [\bar{T}, R] \end{array}$, from which we can construct the derivation

$$\mathsf{i}\uparrow \dfrac{\mathsf{s} \dfrac{\begin{array}{c} T \\ \Delta \Vert \mathsf{NEL} \setminus \{\circ\downarrow\} \\ (T, [\bar{T}, R]) \end{array}}{[(T, \bar{T}), R]}}{R} \quad ,$$

to which we apply Proposition 2.6 to get a derivation $\begin{array}{c} T \\ \Vert \mathsf{SNEL} \\ R \end{array}$. □

Consistency follows as usual and can be proved by way of the same technique used in [13]. It is also easy to prove that system NEL is a conservative extension of MELL plus mix and mix0 (see [12, 25]). The locality properties shown in [13, 26] still hold in this system, of course. In particular, the promotion rule is local, as opposed to the same rule in the sequent calculus.

3 Decomposition

The new top-down symmetry of derivations in the calculus of structures allows to study properties that are not observable in the sequent calculus. The most remarkable results so far are decomposition theorems. In general, a decomposition theorem says that a given system $\mathscr{S}$ can be divided into n pairwise disjoint subsystems $\mathscr{S}_1, \ldots, \mathscr{S}_n$ such that every derivation Δ in system $\mathscr{S}$ can be rearranged as composition of n derivations $\Delta_1, \ldots, \Delta_n$, where Δ_i uses only rules of $\mathscr{S}_i$, for every $1 \leqslant i \leqslant n$.

For system SNEL, we have two such results, which both state a decomposition of every derivation into seven subsystems.

3.1 Theorem (First Decomposition) *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \\ R \end{array}$ *there are derivations* $\Delta_1, \ldots, \Delta_7$*, such that*

$$\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\downarrow\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{ai}\downarrow\} \\ T_3 \\ \Delta_4 \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow\} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{ai}\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\downarrow\} \\ R \end{array}$$

for some structures T_1, T_2, T_3, R_1, R_2 *and* R_3.

Apart from this decomposition into seven subsystems, the first decomposition theorem can also be read as a decomposition into three subsystems that could be called *creation*, *merging* and *destruction*. In the creation subsystem, each rule increases the size of the structure; in the merging system, each rule does some rearranging of substructures, without changing the size of the structures; and in the destruction system, each rule decreases the size of the structure. In a decomposed derivation, the merging part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom:

$$
\begin{array}{rcl}
 & T & \\
\text{destruction} & \Uparrow\ \Downarrow & \text{creation} \\
 & T' & \\
 & \Updownarrow & \text{merging} \\
 & R' & \\
\text{creation} & \Uparrow\ \Downarrow & \text{destruction} \\
 & R &
\end{array}
\quad .
$$

In system SNEL the merging part contains the rules s, $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$, which coincides with the core. In the top-down reading of a derivation, the creation part contains the rules $\mathsf{b}{\uparrow}$, $\mathsf{w}{\downarrow}$ and $\mathsf{ai}{\downarrow}$, and the destruction part consists of $\mathsf{b}{\downarrow}$, $\mathsf{w}{\uparrow}$ and $\mathsf{ai}{\uparrow}$. In the bottom-up reading, creation and destruction are exchanged.

Such a decomposition is not restricted to system SNEL. It also holds for other systems in the calculus of structures, including systems SBV and SELS [13, 26], classical logic [7] and full propositional linear logic [24].

The second decomposition is almost the same statement, with the only difference that the rules $\mathsf{w}{\downarrow}$ and $\mathsf{w}{\uparrow}$ are exchanged.

3.2 Theorem (Second Decomposition) *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \\ R \end{array}$ *there are derivations* $\Delta_1, \dots, \Delta_7$*, such that*

$$
\begin{array}{c}
T \\
\Delta_1 \Vert \{\mathsf{b}{\uparrow}\} \\
T_1 \\
\Delta_2 \Vert \{\mathsf{w}{\uparrow}\} \\
T_2 \\
\Delta_3 \Vert \{\mathsf{ai}{\downarrow}\} \\
T_3 \\
\Delta_4 \Vert \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\} \\
R_3 \\
\Delta_5 \Vert \{\mathsf{ai}{\uparrow}\} \\
R_2 \\
\Delta_6 \Vert \{\mathsf{w}{\downarrow}\} \\
R_1 \\
\Delta_7 \Vert \{\mathsf{b}{\downarrow}\} \\
R
\end{array}
$$

for some structures T_1, T_2, T_3, R_1, R_2 *and* R_3.

The second decomposition theorem allows a separation between core and noncore of the system, such that the up fragment and the down fragment of the noncore are not merged, as it is the case in the first decomposition. More precisely, we can separate the following five subsystems:

$$
\begin{array}{l}
T \\
\Vert \text{noncore (up)} \\
T' \\
\Vert \text{interaction (down)} \\
T'' \\
\Vert \text{core (up and down)} \\
R'' \\
\Vert \text{interaction (up)} \\
R' \\
\Vert \text{noncore (down)} \\
R
\end{array} \quad .
$$

This decomposition will also considerably simplify the cut elimination argument because we no longer have to deal with the noncore rules (up and down) which usually make the cut elimination proof hard.

We prove the two decomposition statements similarly. In the following, we will give only a sketch of the proofs of the decomposition theorems, and then the detailed proofs will follow in Sections 3.1–3.4.

The first decomposition is obtained in three steps as follows:

$$
\begin{array}{c}
T \\
\Delta \Vert \mathsf{SNEL} \\
R
\end{array}
\quad \overset{1}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\Delta_1 \Vert \{\mathsf{b}\uparrow\} \\
T_1 \\
\Delta' \Vert \mathsf{SNEL} \setminus \{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\
R_1 \\
\Delta_7 \Vert \{\mathsf{b}\downarrow\} \\
R
\end{array}
\quad \overset{2}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\Delta_1 \Vert \{\mathsf{b}\uparrow\} \\
T_1 \\
\Delta_2 \Vert \{\mathsf{w}\downarrow\} \\
T_2 \\
\Delta'' \Vert \{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\} \cup \mathsf{SNELc} \\
R_2 \\
\Delta_6 \Vert \{\mathsf{w}\uparrow\} \\
R_1 \\
\Delta_7 \Vert \{\mathsf{b}\downarrow\} \\
R
\end{array}
\quad \overset{3}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\Delta_1 \Vert \{\mathsf{b}\uparrow\} \\
T_1 \\
\Delta_2 \Vert \{\mathsf{w}\downarrow\} \\
T_2 \\
\Delta_3 \Vert \{\mathsf{ai}\downarrow\} \\
T_3 \\
\Delta_4 \Vert \mathsf{SNELc} \\
R_3 \\
\Delta_5 \Vert \{\mathsf{ai}\uparrow\} \\
R_2 \\
\Delta_6 \Vert \{\mathsf{w}\uparrow\} \\
R_1 \\
\Delta_7 \Vert \{\mathsf{b}\downarrow\} \\
R
\end{array} \quad .
$$

This is very similar to the proof of the same statement for system SELS [26]. The second and third step are realized by permutation of rules: In Section 3.1, we will show that the rules $\mathsf{w}\downarrow$ and $\mathsf{ai}\downarrow$ can be permuted over all rules in $\mathsf{SNEL} \setminus \{\mathsf{b}\downarrow, \mathsf{b}\uparrow\}$, and dually, the rules $\mathsf{w}\uparrow$ and $\mathsf{ai}\uparrow$ can be permuted down.

The first step is more difficult. The basic idea is to permute the instances of $\mathsf{b}\uparrow$

$$
\begin{array}{c} T \\ \Vert\,\mathsf{SNEL} \\ R \end{array}
\rightsquigarrow
\begin{array}{c} T \\ \Vert\,\{\mathsf{b}\uparrow\} \\ T_1 \\ \Vert\,\mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ R \end{array}
\rightsquigarrow
\begin{array}{c} T \\ \Vert\,\{\mathsf{b}\uparrow\} \\ T_1 \\ \Vert\,\mathsf{SNEL}\setminus\{\mathsf{b}\downarrow\} \\ R_1 \\ \Vert\,\{\mathsf{b}\downarrow\} \\ R \end{array}
\rightsquigarrow
\begin{array}{c} T \\ \Vert\,\{\mathsf{b}\uparrow\} \\ T_2 \\ \Vert\,\mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ R_1 \\ \Vert\,\{\mathsf{b}\downarrow\} \\ R \end{array}
\rightsquigarrow
\begin{array}{c} T \\ \Vert\,\{\mathsf{b}\uparrow\} \\ T_2 \\ \Vert\,\mathsf{SNEL}\setminus\{\mathsf{b}\downarrow\} \\ R_2 \\ \Vert\,\{\mathsf{b}\downarrow\} \\ R \end{array}
\rightsquigarrow \cdots \rightsquigarrow
\begin{array}{c} T \\ \Vert\,\{\mathsf{b}\uparrow\} \\ T_k \\ \Vert\,\mathsf{SNEL}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R_h \\ \Vert\,\{\mathsf{b}\downarrow\} \\ R \end{array}
$$

Figure 4: *Permuting* $\mathsf{b}\uparrow$ *up and* $\mathsf{b}\downarrow$ *down*

over all the other rules. The most problematic case is when

$$
\mathsf{b}\uparrow \frac{\mathsf{p}\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}}{S[(!R,R),?T]}
\qquad \text{is replaced by} \qquad
\mathsf{b}\downarrow \frac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{p}\downarrow\dfrac{\mathsf{b}\uparrow\dfrac{S\{![R,T]\}}{S(![R,T],[R,T])}}{S([!R,?T],[R,T])}}{S[([!R,?T],R),T]}}{S[(!R,R),?T,T]}}{S[(!R,R),?T]} \quad ,
$$

because a new instance of $\mathsf{b}\downarrow$ is introduced. After all $\mathsf{b}\uparrow$ have reached the top of the derivation, the instances of $\mathsf{b}\downarrow$ are permuted down by the dual procedure, where new instances of $\mathsf{b}\uparrow$ might be introduced; and so on.

The problem is to show that this process, visualized in Figure 4, does terminate eventually. This is done as follows: In Section 3.3, we will introduce the notion of a certain type of cycle inside a derivation. This is very similar to the case of pure MELL in [26]. From a derivation that contains such a cycle, we will extract a derivation

$$
\begin{array}{c}
([!R_1,?T_1],[!R_2,?T_2],\ldots[!R_n,?T_n]) \\
\Delta\Vert\,\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
[(!R_2,?T_1),(!R_3,?T_2),\ldots(!R_1,?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and structures $R_1, \ldots, R_n, T_1, \ldots, T_n$. In Section 3.2, we will show that such a derivation cannot exist. Due to the presence of the seq, this step is more difficult than in the case of MELL. From the non-existence of a cycle, we will, in Section 3.4, be able to deduce the termination.

The proof of the second decomposition is also done in three steps:

$$
\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ R \end{array}
\quad \overset{1}{\leadsto} \quad
\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}{\uparrow}\} \\ T_1 \\ \Delta' \Vert \mathsf{SNEL} \setminus \{\mathsf{b}{\downarrow},\mathsf{b}{\uparrow}\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array}
\quad \overset{2}{\leadsto} \quad
\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}{\uparrow}\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}{\uparrow}\} \\ T_2 \\ \Delta'' \Vert \{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\} \cup \mathsf{SNELc} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}{\downarrow}\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array}
\quad \overset{3}{\leadsto} \quad
\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}{\uparrow}\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}{\uparrow}\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{ai}{\downarrow}\} \\ T_3 \\ \Delta_4 \Vert \mathsf{SNELc} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{ai}{\uparrow}\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}{\downarrow}\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array}
\quad .
$$

The first and the last step are exactly the same as for the first decomposition. In the second step, we have to permute the rule $\mathsf{w}{\uparrow}$ up and the rule $\mathsf{w}{\downarrow}$ down. When permuting $\mathsf{w}{\uparrow}$ up, the problematic case is when

$$
\mathsf{w}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S\{![R,T]\}}{S[!R,?T]}}{S\{?T\}}
\qquad \text{is replaced by} \qquad
\mathsf{w}{\downarrow}\,\dfrac{\mathsf{w}{\uparrow}\,\dfrac{S\{![R,T]\}}{S\{\circ\}}}{S\{?T\}} \quad ,
$$

because a new instance of $\mathsf{w}{\downarrow}$ is introduced. After all $\mathsf{w}{\uparrow}$ have reached the top of the derivation, the instances of $\mathsf{w}{\downarrow}$ are permuted down by the dual procedure, where new instances of $\mathsf{w}{\uparrow}$ might be introduced; and so on. This means that there is the same termination problem as for the separation of absorption in the first step. However, since the weakening rule deletes matter instead of duplicating it, termination will be much easier to show than for absorption.

3.1 Permutation of Rules

In this section we will show the technical details for the second and third step of the proof for the first decomposition theorem.

3.3 Definition A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if for every derivation $\rho\,\dfrac{\pi\,\dfrac{Q}{U}}{P}$ there is a derivation $\pi\,\dfrac{\rho\,\dfrac{Q}{V}}{P}$ for some structure V. A rule ρ *permutes over* a rule π *by a system* $\mathscr{S}$, if for every derivation $\pi \dfrac{\rho \dfrac{Q}{U}}{P}$ there is a derivation

$$\pi \dfrac{\rho \dfrac{Q}{Q'}}{\begin{array}{c} P' \\ \| \mathscr{S} \\ P \end{array}} \quad .$$

Dually, a rule π *permutes under* a rule ρ *by a system* $\mathscr{S}$,

if for every derivation $\pi \dfrac{\rho \dfrac{Q}{U}}{P}$ there is a derivation

$$\begin{array}{c} Q \\ \| \mathscr{S} \\ \rho \dfrac{Q'}{P'} \\ \pi \dfrac{}{P} \end{array} \quad .$$

In order to study the permutation properties of rules, some more definitions are needed. The inference rules of SNEL, as it is presented in Figure 2, are all of the shape $\rho \dfrac{S\{W\}}{S\{Z\}}$: the structure Z is called the *redex* and W the *contractum* of the rule's instance. A substructure that occurs both in the redex and in the contractum of a rule without changing is called *passive*, and all the substructures of redexes and contracta, that are not passive, (i.e. that change, dissappear or are duplicated) are called *active*. Consider for example the rules

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{and} \qquad \mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \quad .$$

In $\mathsf{p}\!\downarrow$, the redex is $[!R,?T]$ and the contractum is $![R,T]$; the structures R and T are passive; the structures $[!R,?T]$, $!R$ and $?T$ are active in the redex; and the structures $![R,T]$ and $[R,T]$ are active in the contractum. In $\mathsf{b}\!\downarrow$ there are no passive structures; in the redex the structures $?R$ and R are active and in the contractum $[?R,R], ?R, R$ and R are active (i.e. both occurrences of the structure R are active).

3.4 Definition An application of a rule $\rho \dfrac{T}{R}$ will be called *trivial* if $R = T$.

3.5 Case Analysis In order to find out whether a rule ρ permutes over a rule π, we have to consider all possibilities of interference of the redex of π and the contractum

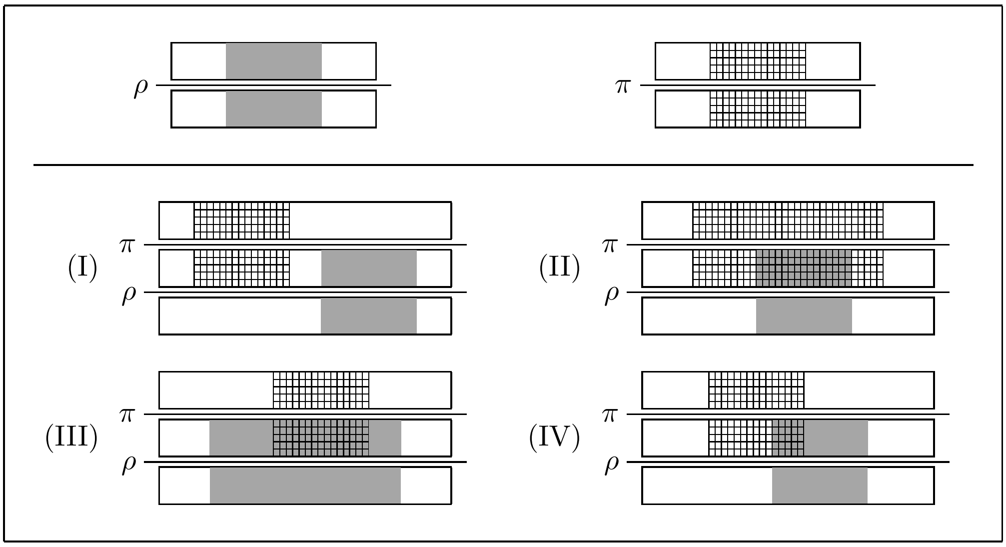


Figure 5: *Possible interferences of redex and contractum of two consecutive rules*

of ρ in a situation

$$\pi \, \dfrac{Q}{\rho \, \dfrac{U}{P}} \quad .$$

It can happen that one is inside the other, that they overlap or that they are independent. The four possible interferences are shown Figure 5. Although the situation is symmetric with respect to ρ and π, in most cases the situation to be considered will be of the shape

$$\pi \, \dfrac{Q}{\rho \, \dfrac{S\{W\}}{S\{Z\}}} \quad ,$$

where the redex Z and the contractum W of ρ are known and we have to make a case analysis for the position of the redex of π inside the structure $S\{W\}$. There are altogether six cases, which are listed below. Figure 6 shows an example for each case. In that figure, we marked the redex with a gray background. (The same kind of highlighting will be employed later on in order to ease the reading. It is therefore advisible to read derivations bottom-up instead of top-down, although they are, in principle, top-down symmetric objects.) In Figure 5 only four cases occur because the there is no distinction between active and passive structures.

(1) The redex of π is inside the context $S\{\ \}$ of ρ. This corresponds to case (I) in Figure 5.

(2) The contractum W of ρ is inside a passive structure of the redex of π. This is a subcase of (II) in Figure 5.

$$(1)\quad \mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{(\circ,d,[a,c],b)}{([b,\bar{b}],d,[a,c],b)}}{([b,\bar{b}],d,[(a,b),c])} \qquad\qquad (2)\quad \mathsf{ai}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{(!(a,\circ,c),[\bar{a},d])}{[\bar{a},(!(a,\circ,c),d)]}}{[\bar{a},(!(a,[b,\bar{b}],c),d)]}$$

$$(3)\quad \mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{([a,\circ,c],b)}{([(a,[b,\bar{b}]),c],b)}}{[(a,[b,\bar{b}],b),c]} \qquad\qquad (4)\quad \mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{(a,![b,(c,d)])}{(a,[!b,?(c,d)])}}{[(a,!b),?(c,d)]}$$

$$(5)\quad \mathsf{ai}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{[a,b,\circ]}{[a,b,?[(c,\bar{c}),\bar{a}]]}}{[a,b,?[\circ,\bar{a}]]} \qquad\qquad (6)\quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{[?[a,b],a,([b,c],d)]}{[?[a,b],a,b,(c,d)]}}{[?[a,b],(c,d)]}$$

Figure 6: *Examples for interferences between two consecutive rules*

(3) The redex of π is inside a passive structure of the contractum W of ρ. This is a subcase of (III) in Figure 5.

(4) The redex of π is inside an active structure of the contractum W of ρ but not inside a passive one. This is a subcase of (III) in Figure 5.

(5) The contractum W of ρ is inside an active structure of the redex of π but not inside a passive one. This is a subcase of (II) in Figure 5.

(6) The contractum W of ρ and the redex of π (properly) overlap. This corresponds to case (IV) in Figure 5. Observe that this case can happen because of associativity.

In the first two cases, we have that $Q = S'\{W\}$ for some context $S'\{\ \}$. This means that the derivation above is of the shape

$$\rho\,\dfrac{\pi\,\dfrac{S'\{W\}}{S\{W\}}}{S\{Z\}}\quad ,$$

where we can permute ρ over π as follows

$$\pi\,\dfrac{\rho\,\dfrac{S'\{W\}}{S'\{Z\}}}{S\{Z\}}\quad .$$

In the third case, we have that $Z = Z'\{R\}$ and $W = W'\{R\}$ for some contexts $Z'\{\ \}$ and $W'\{\ \}$ and some structure R, and $Q = S\{W'\{R'\}\}$ for some structure R'. This means the derivation is

$$\pi\,\cfrac{S\{W'\{R'\}\}}{\rho\,\cfrac{S\{W'\{R\}\}}{S\{Z'\{R\}\}}} \quad ,$$

where R is passive for ρ, and we can permute ρ over π as follows

$$\rho\,\cfrac{S\{W'\{R'\}\}}{\pi\,\cfrac{S\{Z'\{R'\}\}}{S\{Z'\{R\}\}}} \quad .$$

This means that in a proof of a permutation result the cases (1) to (3) are always trivial, whereas for the remaining cases (4) to (6), more elaboration will be necessary. In every proof concerning a permutation result, we will follow this scheme.

In the following, we will show the permutation results concerning weakening and atomic interaction.

In Section 3.3, we will need the following additional rules:

$$\hat{\mathsf{p}}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,U]} \quad , \quad \hat{\mathsf{p}}{\uparrow}\,\frac{S(?R,U)}{S\{?(R,T)\}} \quad , \quad \hat{\mathsf{w}}{\downarrow}\,\frac{S\{\circ\}}{S\{R\}} \quad , \quad \hat{\mathsf{w}}{\uparrow}\,\frac{S\{R\}}{S\{\circ\}} \quad .$$

The reader should not be worried by the fact that they are unsound. They will be introduced and eliminated within a rather technical proof. They are included here in order to avoid a repetition of the same permutation argument later on.

3.6 Lemma *Every rule* $\rho \in \{\mathsf{ai}{\downarrow}, \mathsf{w}{\downarrow}, \hat{\mathsf{w}}{\downarrow}\}$ *permutes over every rule* $\pi \in (\mathsf{SNEL} \cup \{\hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}\}) \setminus \{\mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ *by the system* $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}$*.*

Proof: Consider a derivation

$$\pi\,\cfrac{Q}{\rho\,\cfrac{S\{\circ\}}{S\{Z\}}} \quad ,$$

where $\rho \in \{\mathsf{ai}{\downarrow}, \mathsf{w}{\downarrow}, \hat{\mathsf{w}}{\downarrow}\}$ and $\pi \in (\mathsf{SNEL} \cup \{\hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}\}) \setminus \{\mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ is nontrivial. According to 3.5, there are the following cases to consider:

(4) The redex of π is inside an active structure of the contractum $\circ$ of ρ. This case is impossible because no structure can be inside $\circ$.

(5) The contractum $\circ$ of ρ is inside an active structure of the redex of π but not inside a passive one. Then, the following subcases are possible.

   (i) $\pi = \mathsf{w}{\downarrow}$. Then $S\{\circ\} = S'\{?R\{\circ\}\}$ for some contexts $S'\{\ \}$ and $R\{\ \}$. We have

$$\mathsf{w}{\downarrow}\,\cfrac{S'\{\circ\}}{\rho\,\cfrac{S'\{?R\{\circ\}\}}{S'\{?R\{Z\}\}}} \qquad \text{yields} \qquad \mathsf{w}{\downarrow}\,\frac{S'\{\circ\}}{S'\{?R\{Z\}\}} \quad .$$

   (ii) $\pi = \hat{\mathsf{w}}{\downarrow}$. Similar to (i).

(iii) $\pi = \mathsf{q}{\uparrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'\langle(R,[(R',T),\circ],T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Then we have

$$\rho\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'\langle(R,R',T,T');(U,V)\rangle}}{S'\langle(R,[(R',T),Z],T');(U,V)\rangle}\quad ,$$

which yields

$$\mathsf{s}\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{\rho\,\dfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'(\langle(R,[R',Z]);U\rangle,\langle(T,T');V\rangle)}}{S'\langle(R,[R',Z],T,T');(U,V)\rangle}}{S'\langle(R,[(R',T),Z],T');(U,V)\rangle}\quad .$$

(b) $S\{\circ\} = S'\langle(R,T);(U,[(U',V),\circ],V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (a).

(c) $S\{\circ\} = S'\langle(R,\langle(R',T);\circ\rangle,T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Then we have

$$\rho\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'\langle(R,R',T,T');(U,V)\rangle}}{S'\langle(R,\langle(R',T);Z\rangle,T');(U,V)\rangle}\quad ,$$

which yields

$$\mathsf{q}{\uparrow}\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{\rho\,\dfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'(\langle(R,\langle R';Z\rangle);U\rangle,\langle(T,T');V\rangle)}}{S'\langle(R,\langle R';Z\rangle,T,T');(U,V)\rangle}}{S'\langle(R,\langle(R',T);Z\rangle,T');(U,V)\rangle}\quad .$$

(d) $S\{\circ\} = S'\langle(R,\langle\circ;(R',T)\rangle,T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Similar to (c).

(e) $S\{\circ\} = S'\langle(R,T);(U,\langle(U',V);\circ\rangle,V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (c).

(f) $S\{\circ\} = S'\langle(R,T);(U,\langle\circ;(U',V)\rangle,V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (c).

(g) $S\{\circ\} = S'\langle(R,T);\circ;(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,T,U and V. Then

$$\rho\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{S'(\langle R;U\rangle,\langle T;V\rangle)}{S'\langle(R,T);(U,V)\rangle}}{S'\langle(R,T);Z;(U,V)\rangle}\qquad \text{yields}\qquad \mathsf{q}{\uparrow}\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{\pi\,\dfrac{S'(\langle R;U\rangle,\langle T;V\rangle)}{S'(\langle R;Z;U\rangle,\langle T;V\rangle)}}{S'\langle(\langle R;Z\rangle,T);(U,V)\rangle}}{S'\langle(R,T);Z;(U,V)\rangle}\quad .$$

(iv) $\pi = \mathsf{s}$. Then we have the following subcases:

(a) $S\{\circ\} = S'[(R,[(R',T),\circ],T'),U]$ for some context $S'\{\ \}$ and structures R, R', T, T' and U. This is similar to case (iii.a): We have

$$\rho\,\dfrac{\mathsf{s}\,\dfrac{S'([(R,R'),U],T,T')}{S'[(R,R',T,T'),U]}}{S'[(R,[(R',T),Z],T'),U]}\quad,$$

which yields

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\rho\,\dfrac{S'([(R,R'),U],T,T')}{S'([(R,[R',Z]),U],T,T')}}{S'\langle(R,[R',Z],T,T'),U\rangle}}{S'[(R,[(R',T),Z],T'),U]}\quad.$$

(b) $S\{\circ\} = S'[(R,\langle(R',T);\circ\rangle,T'),U]$ for some context $S'\{\ \}$ and structures R, R', T, T' and U. Similar to case (iii.c).

(c) $S\{\circ\} = S'[(R,\langle\circ;(R',T)\rangle,T'),U]$ for some context $S'\{\ \}$ and structures R, R', T, T' and U. Similar to case (iii.d).

(d) $S\{\circ\} = S'[([(R,T),U],\circ),U']$ for some context $S'\{\ \}$ and structures R, T, U and U'. Then we have

$$\rho\,\dfrac{\mathsf{s}\,\dfrac{S'([R,U,U'],T)}{S'[(R,T),U,U']}}{S'[([(R,T),U],Z),U']}\qquad\text{yields}\qquad\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\rho\,\dfrac{S'([R,U,U'],T)}{S'([R,U,U'],T,Z)}}{S'([(R,T),U,U'],Z)}}{S'[([(R,T),U],Z),U']}\quad.$$

(e) $S\{\circ\} = S'[\langle[(R,T),U];\circ\rangle,U']$ for some context $S'\{\ \}$ and structures R, T, U and U'. Then we have

$$\rho\,\dfrac{\mathsf{s}\,\dfrac{S'([R,U,U'],T)}{S'[(R,T),U,U']}}{S'[\langle[(R,T),U];Z\rangle,U']}\qquad\text{yields}\qquad\mathsf{q}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\rho\,\dfrac{S'([R,U,U'],T)}{S'\langle([R,U,U'],T);Z\rangle}}{S'\langle[(R,T),U,U'];Z\rangle}}{S'[\langle[(R,T),U];Z\rangle,U']}\quad.$$

(f) $S\{\circ\} = S'[(\langle\circ;(R,T),U\rangle),U']$ for some context $S'\{\ \}$ and structures R, T, U and U'. Similar to (e).

(v) $\pi = \mathsf{p}{\uparrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'\{?(R,[(R',T),\circ],T')\}$. for some context $S'\{\ \}$ and structures R, R', T and T'. Then we have

$$\rho\,\dfrac{\mathsf{p}{\uparrow}\,\dfrac{S'(?(R,R'),!(T,T'))}{S'\{?(R,R',T,T')\}}}{S'\{?(R,[(R',T),Z],T')\}}\quad,$$

which is is similar to (iii.a):

$$\mathsf{s}\,\dfrac{\mathsf{p}{\uparrow}\,\dfrac{\rho\,\dfrac{S'(?(R,R'),!(T,T'))}{S'(?(R,[R',Z]),!(T,T'))}}{S'\{?(R,[R',Z],T,T')\}}}{S'\{?(R,[(R',T),Z],T')\}}\quad .$$

(b) $S\{\circ\} = S'\{?(R,\langle(R',T);\circ\rangle,T')\}$. for some context $S'\{\ \}$ and structures R, R', T and T'. Similar to (iii.c).

(c) $S\{\circ\} = S'\{?(R,\langle\circ;(R',T)\rangle,T')\}$. for some context $S'\{\ \}$ and structures R, R', T and T'. Similar to (iii.d).

(vi) $\pi = \hat{\mathsf{p}}{\uparrow}$. Similar to (v).

(vii) $\pi = \mathsf{p}{\downarrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'[(!R,\circ),?T]$ for some $S'\{\ \}$ and R and T. Then the situation is similar as in (iv.d):

$$\rho\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,Z),?T]}\qquad \text{yields}\qquad \mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\rho\,\dfrac{S'\{![R,T]\}}{S'(![R,T],Z)}}{S'([!R,?T],Z)}}{S'[(!R,Z),?T]}\quad .$$

(b) $S\{\circ\} = S'[!R,(?T,\circ)]$. Similar to (a).

(c) $S\{\circ\} = S'[\langle !R;\circ\rangle,?T]$ for some $S'\{\ \}$ and R and T. Then the situation is similar as in (iv.e):

$$\rho\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[\langle !R;Z\rangle,?T]}\qquad \text{yields}\qquad \mathsf{q}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\rho\,\dfrac{S'\{![R,T]\}}{S'\langle ![R,T];Z\rangle}}{S'\langle [!R,?T];Z\rangle}}{S'[\langle !R;Z\rangle,?T]}\quad .$$

(d) $S\{\circ\} = S'[\langle\circ;!R\rangle,?T]$. Similar to (c).

(e) $S\{\circ\} = S'[!R,\langle ?T;\circ\rangle]$. Similar to (c).

(f) $S\{\circ\} = S'[!R,\langle\circ;?T\rangle]$. Similar to (c).

(viii) $\pi = \hat{\mathsf{p}}{\downarrow}$. Similar to (vi).

(ix) $\pi = \mathsf{ai}{\downarrow}$. Similar to (vi).

(x) $\pi = \mathsf{q}{\downarrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'[\langle R;[\langle R';T\rangle,\circ];T'\rangle,\langle U;V\rangle]$ for some context $S'\{\ \}$ and structures R, R', T, T', U and V. Then we have

$$\rho\,\dfrac{\mathsf{q}{\downarrow}\,\dfrac{S'\langle[\langle R;R'\rangle,U];[\langle T;T'\rangle,V]\rangle}{S'[\langle R;R';T;T'\rangle,\langle U;V\rangle]}}{S'[\langle R;[\langle R';T\rangle,Z];T'\rangle,\langle U;V\rangle]}\quad ,$$

which yields

$$
\dfrac{\dfrac{\dfrac{S'\langle[\langle R;R'\rangle,U];[\langle T;T'\rangle,V]\rangle}{S'\langle[\langle R;[R',Z]\rangle,U];[\langle T;T'\rangle,V]\rangle}\rho}{S'[\langle R;[R',Z];T;T'\rangle,\langle U;V\rangle]}\mathsf{q}\!\downarrow}{S'[\langle R;[\langle R';T\rangle,Z];T'\rangle,\langle U;V\rangle]}\mathsf{q}\!\downarrow \quad .
$$

(b) $S\{\circ\} = S'[\langle R;T\rangle,\langle U;[\langle U';V\rangle,\circ];V'\rangle]$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (a).

(c) $S\{\circ\} = S'[\langle R;(\langle R';T\rangle,\circ);T'\rangle,\langle U;V\rangle]$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Then we have

$$
\dfrac{\dfrac{S'\langle[\langle R;R'\rangle,U];[\langle T;T'\rangle,V]\rangle}{S'[\langle R;R';T;T'\rangle,\langle U;V\rangle]}\mathsf{q}\!\downarrow}{S'[\langle R;(\langle R';T\rangle,Z);T'\rangle,\langle U;V\rangle]}\rho \quad .
$$

In this case we need to employ $\mathsf{q}\!\uparrow$ together wit s:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S'\langle[\langle R;R'\rangle,U];[\langle T;T'\rangle,V]\rangle}{S'(\langle[\langle R;R'\rangle,U];[\langle T;T'\rangle,V]\rangle,Z)}\rho}{S'([\langle R;R';T;T'\rangle,\langle U;V\rangle],Z)}\mathsf{q}\!\downarrow}{S'[(\langle R;R';T;T'\rangle,Z),\langle U;V\rangle]}\mathsf{s}}{S'[\langle R;(\langle R';T;T'\rangle,Z)\rangle,\langle U;V\rangle]}\mathsf{q}\!\uparrow}{S'[\langle R;(\langle R';T\rangle,Z);T'\rangle,\langle U;V\rangle]}\mathsf{q}\!\uparrow \quad .
$$

(d) $S\{\circ\} = S'[\langle R;T\rangle,\langle U;(\langle U';V\rangle,\circ);V'\rangle]$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (c).

(xi) If $\pi \in \{\mathsf{ai}\!\downarrow,\mathsf{w}\!\uparrow,\hat{\mathsf{w}}\!\uparrow\}$, then there is no problem because the redex of π is $\circ$.

(6) The contractum $\circ$ of ρ and the redex of π (properly) overlap. This is impossible because $\circ$ cannot overlap with any other structure. ☐

3.7 Lemma *Every rule* $\rho \in \{\mathsf{ai}\!\uparrow,\mathsf{w}\!\uparrow,\hat{\mathsf{w}}\!\uparrow\}$ *permutes under every rule* $\pi \in (\mathsf{SNEL} \cup \{\hat{\mathsf{p}}\!\downarrow,\hat{\mathsf{p}}\!\uparrow,\hat{\mathsf{w}}\!\downarrow,\hat{\mathsf{w}}\!\uparrow\}) \setminus \{\mathsf{b}\!\downarrow,\mathsf{b}\!\uparrow\}$ *by the system* $\{\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{q}\!\uparrow\}$.

Proof: Dual to Lemma 3.6. ☐

Observe that Lemma 3.6 and Lemma 3.7 are sufficient for showing the second and third step of the first decomposition and the third step of the second decomposition.

3.2 A Property of Derivations

In this section, we will show that there is no derivation

$$
\begin{array}{c}
([!R_1,?T_1],[!R_2,?T_2],\dots[!R_n,?T_n]) \\
\Delta\Big\|\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
[(!R_2,?T_1),(!R_3,?T_2),\dots(!R_1,?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and structures $R_1, \dots, R_n, T_1, \dots, T_n$. For this, we will first show a stronger result about the system $\{\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{q}\!\uparrow\}$.

3.8 Lemma *Let $n > 0$ and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ different atoms. Further, let $W_1, \ldots, W_n, Z_1, \ldots, Z_n$ be structures, such that*

- $W_i = [a_i, b_i]$ *or* $W_i = \langle a_i; b_i \rangle$, *for every* $i = 1, \ldots, n$,
- $Z_j = (a_{j+1}, b_j)$ *or* $Z_j = \langle a_{j+1}; b_j \rangle$, *for every* $j = 1, \ldots, n-1$, *and*
- $Z_n = (a_1, b_n)$ *or* $Z_n = \langle a_1; b_n \rangle$.

Then there is no derivation

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ [Z_1, Z_2, \ldots, Z_n] \end{array} \quad .$$

Proof: (In this proof, we will use the convention that $j+1 = 1$ in the case that $j = n$.) Let m be the number of structures Z_j (for $j = 1, \ldots, n$) which are seq-structures, i.e.

$$m \quad = \quad \left| \{ \; j \mid Z_j = \langle a_{j+1}; b_j \rangle \; \} \right| \quad .$$

Proceed by induction on the pair $\langle n, m \rangle$ by using lexicographic ordering, i.e.

$$\langle n', m' \rangle < \langle n, m \rangle \iff n' < n \text{ or } (n' = n \text{ and } m' < m) \quad .$$

Base Case: If $n = 1$, then obviously, there is no derivation

$$\begin{array}{c} [a_1, b_1] \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ (a_1, b_1) \end{array} \quad \text{or} \quad \begin{array}{c} [a_1, b_1] \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ \langle a_1; b_1 \rangle \end{array} \quad \text{or} \quad \begin{array}{c} \langle a_1; b_1 \rangle \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ (a_1, b_1) \end{array} \quad .$$

Inductive Case: Suppose there is no derivation Δ for all $\langle n', m' \rangle < \langle n, m \rangle$, and, by way of contradiction assume there is a derivation

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ [Z_1, Z_2, \ldots, Z_n] \end{array} \quad ,$$

where $Z_j = (a_{j+1}, b_j)$ or $Z_j = \langle a_{j+1}; b_j \rangle$ for every $j = 1, \ldots, n$ and $W_i = [a_i, b_i]$ or $W_i = \langle a_i; b_i \rangle$, for every $i = 1, \ldots, n$. Now consider the bottommost rule instance ρ of Δ. Without loss of generality, we can assume that ρ is not trivial.

(1) $\rho = \mathsf{q}\uparrow$. There are two possibilities to apply $\mathsf{q}\uparrow \dfrac{S(\langle R; U \rangle, \langle T; V \rangle)}{S\langle (R, T); (U, V) \rangle}$.

(i) $R = a_{j+1}$, $T = \circ$, $U = \circ$ and $V = b_j$ for some $j = 1, \ldots, n$. Then there must be some Δ', such that

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta' \Vert \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\ \mathsf{q}\uparrow \dfrac{[Z_1, \ldots, Z_{j-1}, (a_{j+1}, b_j), Z_{j+1}, \ldots, Z_n]}{[Z_1, \ldots, Z_{j-1}, \langle a_{j+1}; b_j \rangle, Z_{j+1}, \ldots, Z_n]} \end{array} \quad ,$$

which is a contradiction to the induction hypothesis.

(ii) $R = \circ, T = a_{j+1}, U = b_j, V = \circ$ for some $j = 1, \ldots, n$. Similar to (i).

(2) $\rho = \mathsf{q}\!\downarrow$. There are the following possibilities to apply $\mathsf{q}\!\downarrow \dfrac{S\langle [R, U]; [T, V]\rangle}{S[\langle R; T\rangle, \langle U; V\rangle]}$.

(i) $R = a_{j+1}, T = b_j$ for some $j = 1, \ldots, n$. Then, without loss of generality, we can assume that $j = 1$. We have the following subcases:

(a) $U = a_{i+1}$ and $V = b_i$ for some $i = 2, \ldots, n$. Then we have a derivation Δ' such that

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta' \Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ \mathsf{q}\!\downarrow \dfrac{[\langle [a_2, a_{i+1}], [b_1, b_i]\rangle, Z_2, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n]}{[\langle a_2; b_1\rangle, Z_2, \ldots, Z_{i-1}, \langle a_{i+1}; b_i\rangle, Z_{i+1}, \ldots, Z_n]} \end{array} .$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $2 \leqslant m \leqslant i$. Then we get

$$\begin{array}{c} (W_1, W_{i+1}, \ldots, W_n) \\ \Delta'' \Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ [\langle a_{i+1}, b_1\rangle, Z_{i+1}, \ldots, Z_n] \end{array} ,$$

which is a contradiction to the induction hypothesis.

(b) $U = \circ$ and $V = [Z_{k_1}, \ldots, Z_{k_v}]$ for some $v > 0$ and $k_1, \ldots, k_v \in \{2, \ldots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_v$. Let $\{2, \ldots, n\} \setminus \{k_1, \ldots, k_v\} = \{h_1, \ldots, h_s\}$, where $s = n - v - 1$. Then there is a derivation Δ' such that

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta' \Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ \mathsf{q}\!\downarrow \dfrac{[\langle a_2; [b_1, Z_{k_1}, \ldots, Z_{k_v}]\rangle, Z_{h_1}, \ldots, Z_{h_s}]}{[\langle a_2; b_1\rangle, Z_2, \ldots, Z_n]} \end{array} .$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $m > k_1$ and for $m = 1$. Then we get

$$\begin{array}{c} (W_2, \ldots, W_{k_1}) \\ \Delta'' \Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ [\langle a_2, b_{k_1}\rangle, Z_2, \ldots, Z_{k_1 - 1}] \end{array}$$

which is a contradiction to the induction hypothesis.

(c) $U = [Z_{k_1}, \ldots, Z_{k_u}]$ and $V = \circ$ for some $u > 0$ and $k_1, \ldots, k_u \in \{2, \ldots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_u$. Let $\{2, \ldots, n\} \setminus \{k_1, \ldots, k_u\} = \{h_1, \ldots, h_s\}$, where $s = n - u - 1$. Then there is a derivation Δ' such that

$$\begin{array}{c} (W_1, W_2, \ldots, W_n) \\ \Delta' \Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ \mathsf{q}\!\downarrow \dfrac{[\langle [a_2, Z_{k_1}, \ldots, Z_{k_u}]; b_1\rangle, Z_{h_1}, \ldots, Z_{h_s}]}{[\langle a_2; b_1\rangle, Z_2, \ldots, Z_n]} \end{array} .$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $1 < m \leqslant k_u$. Then we get

$$\begin{array}{c}(W_1, W_{k_u+1}, \dots, W_n)\\ \Delta''\Big\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ [\langle a_{k_u+1}, b_1\rangle, Z_{k_u+1}, \dots, Z_n]\end{array} ,$$

which is a contradiction to the induction hypothesis.

(ii) $U = a_{j+1}$ and $V = b_j$ for some $j = 1, \dots, n$. Similar to (i).

(iii) $R = [Z_{k_1}, \dots, Z_{k_r}]$, $T = \circ$, $U = \circ$ and $V = [Z_{l_1}, \dots, Z_{l_v}]$ for some $r, v > 0$ and $k_1, \dots, k_r, l_1, \dots, l_v \in \{1, \dots, n\}$. Then there is a derivation Δ' such that

$$\begin{array}{c}(W_1, W_2, \dots, W_n)\\ \Delta'\Big\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ \mathsf{q}\downarrow\dfrac{[\langle [Z_{k_1}, \dots, Z_{k_r}]; [Z_{l_1}, \dots, Z_{l_v}]\rangle, Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]}\end{array} ,$$

where $s = n-r-v$ and $\{h_1, \dots, h_s\} = \{1, \dots, n\} \setminus \{k_1, \dots, k_r, l_1, \dots, l_v\}$. Without loss of generality, we can assume that $r = v = 1$. Otherwise we could replace

$$\mathsf{q}\downarrow\frac{[\langle [Z_{k_1}, \dots, Z_{k_r}]; [Z_{l_1}, \dots, Z_{l_v}]\rangle, Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]} \qquad \text{by}$$

$$\mathsf{q}\downarrow\dfrac{\mathsf{q}\downarrow\dfrac{\mathsf{q}\downarrow\dfrac{[\langle [Z_{k_1}, \dots, Z_{k_r}]; [Z_{l_1}, \dots, Z_{l_v}]\rangle, Z_{h_1}, \dots, Z_{h_s}]}{[\langle [Z_{k_1}, \dots, Z_{k_r}]; Z_{l_1}\rangle, Z_{l_2}, \dots, Z_{l_v}, Z_{h_1}, \dots, Z_{h_s}]}}{[\langle Z_{k_1}; Z_{l_1}\rangle, Z_{k_2}, \dots, Z_{k_r}, Z_{l_2}, \dots, Z_{l_v}, Z_{h_1}, \dots, Z_{h_s}]}}{[Z_1, Z_2, \dots, Z_n]} .$$

Now let $k = k_1$ and $l = l_1$. Then we have

$$\begin{array}{c}(W_1, W_2, \dots, W_n)\\ \Delta'\Big\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ \mathsf{q}\downarrow\dfrac{[\langle Z_k; Z_l\rangle, Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]}\end{array} ,$$

where $s = n - 2$. There are two subcases.

(a) $k < l$. Then replace inside Δ' all atoms a_m and b_m by $\circ$ for every m with $m \leqslant k$ or $l < m$. The derivation Δ' then becomes

$$\begin{array}{c}(W_{k+1}, \dots, W_l)\\ \Delta''\Big\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ [\langle a_{k+1}; b_l\rangle, Z_{k+1}, \dots, Z_{l-1}]\end{array}$$

which is a contradiction to the induction hypothesis.

(b) $k > l$. Then replace inside Δ' all atoms a_m and b_m by $\circ$ for every m with $k \geqslant m > l$. The derivation Δ' then becomes

$$\begin{array}{c}(W_1, \dots, W_l, W_{k+1}, \dots, W_n)\\ \Delta''\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ [Z_1, \dots, Z_{l-1}, \langle a_{k+1}; b_l\rangle, Z_{k+1}, \dots, Z_n]\end{array} \quad ,$$

which is a contradiction to the induction hypothesis.

(iv) $R = \circ$, $T = [Z_{k_1}, \dots, Z_{k_t}]$, $U = [Z_{l_1}, \dots, Z_{l_u}]$ and $V = \circ$ for some $t, u > 0$ and $k_1, \dots, k_t, l_1, \dots, l_u \in \{1, \dots, n\}$. Similar to (iii).

(3) $\rho = s$. There are three possibilities to apply $\mathsf{s}\dfrac{S([R,U],T)}{S[(R,T),U]}$.

(i) $R = a_{j+1}, T = b_j$ for some $j = 1, \dots, n$. Then, without loss of generality, we can assume that $j = 1$. Further, $U = [Z_{k_1}, \dots, Z_{k_u}]$ for some $u > 0$ and $k_1, \dots, k_u \in \{2, \dots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \dots < k_u$. Let $\{2, \dots, n\} \setminus \{k_1, \dots, k_u\} = \{h_1, \dots, h_s\}$, where $s = n - u - 1$. Then there is a derivation Δ' such that

$$\begin{array}{c}(W_1, W_2, \dots, W_n)\\ \Delta'\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ \mathsf{s}\dfrac{[([a_2, Z_{k_1}, \dots, Z_{k_u}], b_1), Z_{h_1}, \dots, Z_{h_s}]}{[(a_2, b_1), Z_2, \dots, Z_n]}\end{array} \quad .$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $1 < m \leqslant k_u$. Then we get

$$\begin{array}{c}(W_1, W_{k_u+1}, \dots, W_n)\\ \Delta''\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ [(a_{k_u+1}, b_1), Z_{k_u+1}, \dots, Z_n]\end{array} \quad ,$$

which is a contradiction to the induction hypothesis.

(ii) $R = b_j, T = a_{j+1}$ for some $j = 1, \dots, n$. Similar to (i).

(iii) $R = \circ$, $T = [Z_{k_1}, \dots, Z_{k_t}]$ and $U = [Z_{l_1}, \dots, Z_{l_u}]$ for some $t, u > 0$ and $k_1, \dots, k_t, l_1, \dots, l_u \in \{1, \dots, n\}$. Similar to (2.iii). □

Now we can easily get:

3.9 Lemma *Let $n \geqslant 1$ and $!R_1, \dots, !R_n, ?T_1, \dots, ?T_n$ be any* NEL *structures. Then there is no derivation*

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \dots [!R_n, ?T_n])\\ \Delta\Vert\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\\ [(!R_2, ?T_1), (!R_3, ?T_2), \dots (!R_1, ?T_n)]\end{array} \quad .$$

Proof: By way of contradiction, assume there is such a derivation Δ. The rules s, $\mathsf{q}\downarrow$ and $\mathsf{q}\uparrow$ have no access to the exponentials. Hence, we can replace inside Δ, the substructures $!R_1, \dots, !R_n, ?T_1, \dots, ?T_n$ by new atoms $a_1, \dots, a_n, b_1, \dots, b_n$, respectively.

This gives us a valid derivation

$$\begin{array}{c} ([a_1, b_1], [a_2, b_2], \ldots [a_n, b_n]) \\ \Delta' \big\Vert \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \\ [(a_2, b_1), (a_3, b_2), \ldots (a_1, b_n)] \end{array} ,$$

which is a contradiction to Lemma 3.8. □

3.3 Chains and Cycles in Derivations

The purpose of this section is to introduce the notion of chains in derivations, which represent the "paths" that are taken by the instances of $\mathsf{b}\!\uparrow$ and $\mathsf{b}\!\downarrow$ when they are permuted up and down, respectively, to obtain the decomposition theorems. If such a "path" loops, then the corresponding chain is called a cycle. The main result of this section is to show that a certain form of cycle cannot exist.

The material of this section has already been presented in [26] (see also [25]), where a similar result is shown for pure MELL. We repeat the development here for the following reasons:

- We want to make the paper as self-contained as possible.
- The collapse of the units and the presence of the seq cause some modifications that have to be taken into account.
- We introduce several improvements of the presentation that will help the reader to understand the constructions.

In Section 2, we introduced the concept of a context as a structure with a hole, and then used $S\{R\}$ for denoting a structure with a substructure R. In this section, we also need to denote structures with more than one distinguished substructure. For example, we will write $S\{R_1\}\{R_2\}$ to denote a structure with two independent substructures R_1 and R_2. More generally, $S\{R_1\}\ldots\{R_n\}$ has $R_1, \ldots, R_n$ as substructures. Removing those substructures and replacing them by $\{\ \}$, yields an n-ary context $S\{\ \}\ldots\{\ \}$. For example, $[!\{\ \}, (a, \{\ \}, b)]$ is a binary context.

3.10 Definition Let Δ be a derivation. A $!$-*link* in Δ is any $!$-structure $!R$ that occurs as substructure of a structure S inside a derivation.

The purpose of the definition of a !-link is to define !-chains, which consist of !-links. In general, in a given derivation Δ, most of the !-links in Δ do not belong to a !-chain. Later, we will mark those !-links that belong to a !-chain or that are for other reason under discussion with a $!^{\blacktriangle}$.

3.11 Example The derivation

$$\begin{array}{rc} & (!^{\blacktriangle}[(b, !a), \bar{a}], !c) \\ \mathsf{p}\!\downarrow & \overline{} \\ & ([!^{\blacktriangle}(b, !a), ?\bar{a}], !c) \\ \mathsf{s} & \overline{} \\ & [!(b, !^{\blacktriangle}a), (?\bar{a}, !^{\blacktriangle}c)] \\ \mathsf{p}\!\uparrow & \overline{} \\ & [!(b, !a), ?(\bar{a}, c)] \end{array}$$

(1)	$\rho\,\dfrac{S\{\boldsymbol{!^{\blacktriangle}R}\}\{W\}}{S\{\boldsymbol{!^{\blacktriangle}R}\}\{Z\}}$	$\rho\,\dfrac{S\{\boldsymbol{?^{\blacktriangledown}T}\}\{W\}}{S\{\boldsymbol{?^{\blacktriangledown}T}\}\{Z\}}$
(2)	$\rho\,\dfrac{S\{W\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S\{Z\{\boldsymbol{!^{\blacktriangle}R}\}\}}$	$\rho\,\dfrac{S\{W\{\boldsymbol{?^{\blacktriangledown}T}\}\}}{S\{Z\{\boldsymbol{?^{\blacktriangledown}T}\}\}}$
(3)	$\rho\,\dfrac{S\{\boldsymbol{!^{\blacktriangle}R\{W\}}\}}{S\{\boldsymbol{!^{\blacktriangle}R\{Z\}}\}}$	$\rho\,\dfrac{S\{\boldsymbol{?^{\blacktriangledown}T\{W\}}\}}{S\{\boldsymbol{?^{\blacktriangledown}T\{Z\}}\}}$
(4.i)	$\mathsf{p}{\downarrow}\,\dfrac{S\{\boldsymbol{!^{\blacktriangle}[R,T]}\}}{S[\boldsymbol{!^{\blacktriangle}R},?T]}$	$\mathsf{p}{\uparrow}\,\dfrac{S(!R,\boldsymbol{?^{\blacktriangledown}T})}{S\{\boldsymbol{?^{\blacktriangledown}(R,T)}\}}$
(4.ii)	$\mathsf{b}{\uparrow}\,\dfrac{S\{\boldsymbol{!^{\blacktriangle}R}\}}{S(\boldsymbol{!^{\blacktriangle}R},R)}$	$\mathsf{b}{\uparrow}\,\dfrac{S(\boldsymbol{?^{\blacktriangledown}T},T)}{S\{\boldsymbol{?^{\blacktriangledown}T}\}}$
(4.iii)	$\mathsf{b}{\uparrow}\,\dfrac{S\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S(!V\{\boldsymbol{!^{\blacktriangle}R}\},V\{!R\})}$	$\mathsf{b}{\downarrow}\,\dfrac{S[?U\{\boldsymbol{?^{\blacktriangledown}T}\},U\{?T\}]}{S\{?U\{\boldsymbol{?^{\blacktriangledown}T}\}\}}$
(4.iv)	$\mathsf{b}{\uparrow}\,\dfrac{S\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S(!V\{!R\},V\{\boldsymbol{!^{\blacktriangle}R}\})}$	$\mathsf{b}{\downarrow}\,\dfrac{S[?U\{?T\},U\{\boldsymbol{?^{\blacktriangledown}T}\}]}{S\{?U\{\boldsymbol{?^{\blacktriangledown}T}\}\}}$
(4.v)	$\mathsf{b}{\downarrow}\,\dfrac{S[?U\{\boldsymbol{!^{\blacktriangle}R}\},U\{!R\}]}{S\{?U\{\boldsymbol{!^{\blacktriangle}R}\}\}}$	$\mathsf{b}{\uparrow}\,\dfrac{S\{!V\{\boldsymbol{?^{\blacktriangledown}T}\}\}}{S(!V\{\boldsymbol{?^{\blacktriangledown}T}\},V\{?T\})}$
(4.vi)	$\mathsf{b}{\downarrow}\,\dfrac{S[?U\{!R\},U\{\boldsymbol{!^{\blacktriangle}R}\}]}{S\{?U\{\boldsymbol{!^{\blacktriangle}R}\}\}}$	$\mathsf{b}{\uparrow}\,\dfrac{S\{!V\{\boldsymbol{?^{\blacktriangledown}T}\}\}}{S(!V\{?T\},V\{\boldsymbol{?^{\blacktriangledown}T}\})}$

Figure 7: *Connection of* !*-links and* ?*-links*

contains many !-links, but only four of them are marked. (So far, there is no relation between markings and chains.)

3.12 Definition Two !-links $\boldsymbol{!^{\blacktriangle}R}$ and $\boldsymbol{!^{\blacktriangle}R'}$ inside a derivation Δ are *connected* if they occur in two consecutive structures, i.e. Δ is of the shape

$$\begin{array}{c} P \\ \Vert \\ \rho\,\dfrac{S'\{\boldsymbol{!^{\blacktriangle}R'}\}}{S\{\boldsymbol{!^{\blacktriangle}R}\}} \\ \Vert \\ Q \end{array} \quad ,$$

and one of the following cases holds:

(1) The link $\boldsymbol{!^{\blacktriangle}R}$ is inside the context of ρ, i.e. $R = R'$ and $S\{\boldsymbol{!^{\blacktriangle}R}\} = S''\{\boldsymbol{!^{\blacktriangle}R}\}\{Z\}$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S''\{\boldsymbol{!^{\blacktriangle}R}\}\{W\}$ for some context $S''\{\ \}\{\ \}$, where Z and W are redex and contractum of ρ:

$$\rho\,\frac{S''\{\boldsymbol{!^{\blacktriangle}R}\}\{W\}}{S''\{\boldsymbol{!^{\blacktriangle}R}\}\{Z\}} \quad .$$

(2) The link $\boldsymbol{!^{\blacktriangle}R}$ is inside a passive structure of the redex of ρ, i.e. $R = R'$ and there are contexts $S''\{\ \}$, $Z'\{\ \}$ and $W'\{\ \}$ such that $S\{\boldsymbol{!^{\blacktriangle}R}\} = S''\{Z\{\boldsymbol{!^{\blacktriangle}R}\}\}$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S''\{W\{\boldsymbol{!^{\blacktriangle}R}\}\}$, where $Z\{\boldsymbol{!^{\blacktriangle}R}\}$ and $W\{\boldsymbol{!^{\blacktriangle}R}\}$ are redex and contractum of ρ:

$$\rho\,\frac{S''\{W\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S''\{Z\{\boldsymbol{!^{\blacktriangle}R}\}\}} \quad .$$

(3) The redex of ρ is inside R, i.e. $S\{\ \} = S'\{\ \}$ and there is a context $R''\{\ \}$ such that $S\{\boldsymbol{!^{\blacktriangle}R}\} = S\{\boldsymbol{!^{\blacktriangle}R''\{Z\}}\}$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S\{\boldsymbol{!^{\blacktriangle}R''\{W\}}\}$, where Z and W are redex and contractum of ρ:

$$\rho\,\frac{S\{\boldsymbol{!^{\blacktriangle}R''\{W\}}\}}{S\{\boldsymbol{!^{\blacktriangle}R''\{Z\}}\}} \quad .$$

(4) The link $\boldsymbol{!^{\blacktriangle}R}$ is inside an active structure of the redex of ρ, but not inside a passive one. Then six subcases are possible:

(i) $\rho = \mathsf{p}{\downarrow}$ and there is a structure T such that we have $S\{\boldsymbol{!^{\blacktriangle}R}\} = S'[\boldsymbol{!^{\blacktriangle}R}, ?T]$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S'\{\boldsymbol{!^{\blacktriangle}[R,T]}\}$, i.e. $R' = [R, T]$:

$$\mathsf{p}{\downarrow}\,\frac{S'\{\boldsymbol{!^{\blacktriangle}[R,T]}\}}{S'[\boldsymbol{!^{\blacktriangle}R}, ?T]} \quad .$$

(ii) $\rho = \mathsf{b}{\uparrow}$, $R = R'$, $S\{\boldsymbol{!^{\blacktriangle}R}\} = S'(\boldsymbol{!^{\blacktriangle}R}, R)$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S'\{\boldsymbol{!^{\blacktriangle}R}\}$:

$$\mathsf{b}{\uparrow}\,\frac{S'\{\boldsymbol{!^{\blacktriangle}R}\}}{S'(\boldsymbol{!^{\blacktriangle}R}, R)} \quad .$$

(iii) $\rho = \mathsf{b}{\uparrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{\boldsymbol{!^{\blacktriangle}R}\} = S''(!V\{\boldsymbol{!^{\blacktriangle}R}\}, V\{!R\})$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S''\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}$:

$$\mathsf{b}{\uparrow}\,\frac{S''\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S''(!V\{\boldsymbol{!^{\blacktriangle}R}\}, V\{!R\})} \quad .$$

(iv) $\rho = \mathsf{b}{\uparrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{\boldsymbol{!^{\blacktriangle}R}\} = S''(!V\{!R\}, V\{\boldsymbol{!^{\blacktriangle}R}\})$ and $S'\{\boldsymbol{!^{\blacktriangle}R'}\} = S''\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}$:

$$\mathsf{b}{\uparrow}\,\frac{S''\{!V\{\boldsymbol{!^{\blacktriangle}R}\}\}}{S''(!V\{!R\}, V\{\boldsymbol{!^{\blacktriangle}R}\})} \quad .$$

(v) $\rho = \mathsf{b}{\downarrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that $S\{\boldsymbol{!^\blacktriangle R}\} = S''\{?U\{\boldsymbol{!^\blacktriangle R}\}\}$ and $S'\{\boldsymbol{!^\blacktriangle R'}\} = S''[?U\{\boldsymbol{!^\blacktriangle R}\}, U\{!R\}]$:

$$\mathsf{b}{\downarrow}\,\frac{S''[?U\{\boldsymbol{!^\blacktriangle R}\}, U\{!R\}]}{S''\{?U\{\boldsymbol{!^\blacktriangle R}\}\}} \quad .$$

(vi) $\rho = \mathsf{b}{\downarrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that $S\{\boldsymbol{!^\blacktriangle R}\} = S''\{?U\{\boldsymbol{!^\blacktriangle R}\}\}$ and $S'\{\boldsymbol{!^\blacktriangle R'}\} = S''[?U\{!R\}, U\{\boldsymbol{!^\blacktriangle R}\}]$:

$$\mathsf{b}{\downarrow}\,\frac{S''[?U\{!R\}, U\{\boldsymbol{!^\blacktriangle R}\}]}{S''\{?U\{\boldsymbol{!^\blacktriangle R}\}\}} \quad .$$

On the left-hand side of Figure 7, we have listed all possibilities, how !-links can be connected.

3.13 Example In the derivation shown in Example 3.11, the two !-links $\boldsymbol{!^\blacktriangle[(b, !a), \bar{a}]}$ and $\boldsymbol{!^\blacktriangle(b, !a)}$ are connected (by case (4.i)), whereas the !-link $\boldsymbol{!^\blacktriangle a}$ is neither connected to $\boldsymbol{!^\blacktriangle(b, !a)}$ nor to $\boldsymbol{!^\blacktriangle c}$.

3.14 Definition A *!-chain* χ inside a derivation Δ is a sequence of connected !-links. The bottommost !-link of χ is called its *tail* and the topmost !-link of χ is called its *head*.

Throughout this work, we will visualise !-chains by giving the derivation and marking all !-links of the chain by $\boldsymbol{!^\blacktriangle}$.

3.15 Example The following derivation shows a !-chain with tail $\boldsymbol{!^\blacktriangle(b, ?a)}$ and head $\boldsymbol{!^\blacktriangle b}$:

$$
\begin{array}{rcl}
 & (\boldsymbol{!^\blacktriangle b}, !c) & \\ \hline
\mathsf{b}{\uparrow} & (\boldsymbol{!^\blacktriangle b}, b, !c) & (4.\mathrm{ii}) \\ \hline
\mathsf{ai}{\downarrow} & (\boldsymbol{!^\blacktriangle(b, [a, \bar{a}])}, !c, b) & (3) \\ \hline
\mathsf{w}{\downarrow} & (\boldsymbol{!^\blacktriangle(b, [?a, a, \bar{a}])}, !c, b) & (3) \\ \hline
\mathsf{s} & (\boldsymbol{!^\blacktriangle[(b, [?a, a]), \bar{a}]}, !c, b) & (3) \\ \hline
\mathsf{b}{\downarrow} & (\boldsymbol{!^\blacktriangle[(b, ?a), \bar{a}]}, !c, b) & (3) \\ \hline
\mathsf{p}{\downarrow} & ([\boldsymbol{!^\blacktriangle(b, ?a)}, ?\bar{a}], !c, b) & (4.\mathrm{i}) \\ \hline
\mathsf{s} & ([\boldsymbol{!^\blacktriangle(b, ?a)}, (?\bar{a}, !c)], b) & (2) \\ \hline
\mathsf{p}{\uparrow} & ([\boldsymbol{!^\blacktriangle(b, ?a)}, ?(\bar{a}, c)], b) & (1)
\end{array}
\quad .
$$

On the right-hand side of the derivation we indicated by which case (of Figure 7) the links are connected. (Observe that every *subchain* is also a !-chain. For example the bottommost four !-links are also a !-chain.)

3.16 Definition The notion of *?-link* is defined in the same way as the one of !-link. The notion of *?-chain* is defined dually to !-chain, in particular, the *tail* of a ?-chain is its topmost ?-link and its *head* is its bottommost ?-link.

Similar to !-links, we will mark ?-links that are under discussion with $\boldsymbol{?^\blacktriangledown}$. In particular, ?-chains are marked by marking every link of the ?-chain by $\boldsymbol{?^\blacktriangledown}$. For convenience, we have listed on the right-hand side of Figure 7 all possibilities how ?-links can be connected inside a ?-chain.

3.17 Example The following derivation shows an example for a ?-chain with tail $\boldsymbol{?^{\blacktriangledown}a}$ and head $\boldsymbol{?^{\blacktriangledown}(a,c)}$:

$$
\begin{array}{rcl}
 & ![!c,(\boldsymbol{?^{\blacktriangledown}a},!c),\circ] & \\
\mathsf{p}\!\downarrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (2)\\
 & [?!c,![(\boldsymbol{?^{\blacktriangledown}a},!c),\circ]] & \\
\mathsf{w}\!\downarrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (1)\\
 & [?!([?a,\circ],!c),![(\boldsymbol{?^{\blacktriangledown}a},!c),\circ]] & \\
\mathsf{s} & \overline{\qquad\qquad\qquad\qquad\qquad} & (1)\\
 & [?![(?a,!c),\circ],![(\boldsymbol{?^{\blacktriangledown}a},!c),\circ]] & \\
\mathsf{b}\!\downarrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (4.\mathrm{iv})\\
 & ?![(\boldsymbol{?^{\blacktriangledown}a},!c),\circ] & \\
\mathsf{ai}\!\downarrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (1)\\
 & ?![(\boldsymbol{?^{\blacktriangledown}a},!c),b,\bar{b}] & \\
\mathsf{p}\!\downarrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (2)\\
 & ?[![(\boldsymbol{?^{\blacktriangledown}a},!c),b],?\bar{b}] & \\
\mathsf{b}\!\uparrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (4.\mathrm{v})\\
 & ?[(![(\boldsymbol{?^{\blacktriangledown}a},!c),b],[(?a,!c),b]),?\bar{b}] & \\
\mathsf{p}\!\uparrow & \overline{\qquad\qquad\qquad\qquad\qquad} & (4.\mathrm{i})\\
 & ?[(![\boldsymbol{?^{\blacktriangledown}(a,c)},b],[(?a,!c),b]),?\bar{b}] & .
\end{array}
$$

Again, we indicated on the right-hand side of the derivation the cases by which the links are connected.

3.18 Definition Let Δ be a derivation. An *upper link* in Δ is any structure $[!R,?T]$ that occurs as substructure of a structure S inside Δ. Dually, a *lower link* is any structure $(?T,!R)$ that occurs as substructure of a structure S inside Δ.

Similar to the marking of !-links and ?-links, we will mark upper links as $\boldsymbol{[}\boldsymbol{!^{\blacktriangle}R},\boldsymbol{?^{\blacktriangledown}T}\boldsymbol{]}$ and lower links as $\boldsymbol{(}\boldsymbol{?^{\blacktriangledown}T},\boldsymbol{!^{\blacktriangle}R}\boldsymbol{)}$.

3.19 Definition Let Δ be a derivation. The set $X(\Delta)$ of *chains* in Δ is defined inductively as follows:

(1) For every !-chain χ in Δ, we have $\chi \in X(\Delta)$.

(2) For every ?-chain χ in Δ, we have $\chi \in X(\Delta)$.

(3) If Δ contains two chains χ_1 and χ_2 and an upper link $\boldsymbol{[}\boldsymbol{!^{\blacktriangle}R},\boldsymbol{?^{\blacktriangledown}T}\boldsymbol{]}$ such that $\boldsymbol{!^{\blacktriangle}R}$ is the head of χ_1 and $\boldsymbol{?^{\blacktriangledown}T}$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(4) If Δ contains two chains χ_1 and χ_2 and a lower link $\boldsymbol{(}\boldsymbol{?^{\blacktriangledown}T},\boldsymbol{!^{\blacktriangle}R}\boldsymbol{)}$ such that $\boldsymbol{?^{\blacktriangledown}T}$ is the head of χ_1 and $\boldsymbol{!^{\blacktriangle}R}$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(5) There are no other chains in $X(\Delta)$.

3.20 Definition The *length* of a chain χ is the number of !-chains and ?-chains it is composed of.

$$
\mathsf{w}{\uparrow}\dfrac{\mathsf{p}{\uparrow}\dfrac{\mathsf{b}{\downarrow}\dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{w}{\downarrow}\dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{p}{\downarrow}\dfrac{(![c,d],!a,?b)}{(\boldsymbol{[}!^{\blacktriangle}\boldsymbol{c},?^{\blacktriangledown}\boldsymbol{d}\boldsymbol{]},!a,?b)}}{([!^{\blacktriangle}\boldsymbol{c},(!a,?^{\blacktriangledown}\boldsymbol{d})],?b)}}{[(!^{\blacktriangle}\boldsymbol{c},?b),(!a,?^{\blacktriangledown}\boldsymbol{d})]}}{[?(!c,\boldsymbol{[}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{b}\boldsymbol{]},?d),(!^{\blacktriangle}\boldsymbol{c},?b),(!a,?^{\blacktriangledown}\boldsymbol{d})]}}{[?(!c,[(!^{\blacktriangle}\boldsymbol{a},?d),?^{\blacktriangledown}\boldsymbol{b}]),(!^{\blacktriangle}\boldsymbol{c},?b),(!a,?^{\blacktriangledown}\boldsymbol{d})]}}{[?[(!^{\blacktriangle}\boldsymbol{a},?d),(!c,?^{\blacktriangledown}\boldsymbol{b})],(!^{\blacktriangle}\boldsymbol{c},?b),(!a,?^{\blacktriangledown}\boldsymbol{d})]}}{?[\boldsymbol{(}?^{\blacktriangledown}\boldsymbol{b},!^{\blacktriangle}\boldsymbol{c}\boldsymbol{)},(!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{d})]}}{?[?(b,c),\boldsymbol{(}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{d}\boldsymbol{)}]}}{?[?(b,c),?d]}
$$

Figure 8: *A cycle χ with $n(\chi)=2$*

3.21 Example Here are two examples of chains in derivations:

$$
\mathsf{ai}{\downarrow}\dfrac{\mathsf{p}{\uparrow}\dfrac{\mathsf{s}\dfrac{\mathsf{ai}{\uparrow}\dfrac{\mathsf{p}{\downarrow}\dfrac{(![a,(c,\bar{c}),b],!^{\blacktriangle}\boldsymbol{c})}{([![a,(c,\bar{c})],?b],!^{\blacktriangle}\boldsymbol{c})}}{(\boldsymbol{[}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{b}\boldsymbol{]},!^{\blacktriangle}\boldsymbol{c})}}{[!^{\blacktriangle}\boldsymbol{a},\boldsymbol{(}?^{\blacktriangledown}\boldsymbol{b},!^{\blacktriangle}\boldsymbol{c}\boldsymbol{)}]}}{[!^{\blacktriangle}\boldsymbol{a},?(b,c)]}}{[!^{\blacktriangle}\boldsymbol{(a,[d,\bar{d}])},?(b,c)]}
\qquad\text{and}\qquad
\mathsf{b}{\downarrow}\dfrac{\mathsf{w}{\uparrow}\dfrac{\mathsf{s}\dfrac{\mathsf{w}{\downarrow}\dfrac{\mathsf{s}\dfrac{\mathsf{p}{\downarrow}\dfrac{(a,![c,d])}{(a,\boldsymbol{[}!^{\blacktriangle}\boldsymbol{c},?^{\blacktriangledown}\boldsymbol{d}\boldsymbol{]})}}{[(a,!^{\blacktriangle}\boldsymbol{c}),?^{\blacktriangledown}\boldsymbol{d}]}}{[([a,?^{\blacktriangledown}\boldsymbol{b}],!^{\blacktriangle}\boldsymbol{c}),?^{\blacktriangledown}\boldsymbol{d}]}}{[a,\boldsymbol{(}?^{\blacktriangledown}\boldsymbol{b},!^{\blacktriangle}\boldsymbol{c}\boldsymbol{)},?^{\blacktriangledown}\boldsymbol{d}]}}{[a,?b,?^{\blacktriangledown}\boldsymbol{d}]}}{[?b,?^{\blacktriangledown}\boldsymbol{d}]}\quad.
$$

In the first chain, the tail is $!^{\blacktriangle}\boldsymbol{(a,[d,\bar{d}])}$ and the head is $!^{\blacktriangle}\boldsymbol{c}$. In the second example the tail is $?^{\blacktriangledown}\boldsymbol{b}$ and the head is $?^{\blacktriangledown}\boldsymbol{d}$. Both have length $l=3$.

3.22 Definition Let Δ be a derivation. A chain $\chi \in X(\Delta)$ is called a *cycle* if Δ contains an upper link $\boldsymbol{[}!^{\blacktriangle}\boldsymbol{R},?^{\blacktriangledown}\boldsymbol{T}\boldsymbol{]}$ such that $!^{\blacktriangle}\boldsymbol{R}$ is the head of χ and $?^{\blacktriangledown}\boldsymbol{T}$ is the tail of χ, or Δ contains a lower link $\boldsymbol{(}?^{\blacktriangledown}\boldsymbol{T},!^{\blacktriangle}\boldsymbol{R}\boldsymbol{)}$ such that $?^{\blacktriangledown}\boldsymbol{T}$ is the head of χ and $!^{\blacktriangle}\boldsymbol{R}$ its tail.

In other words, a cycle can be seen as a chain without head or tail. Figure 8 shows an example for a cycle. Observe that for every cycle χ there is a number $n=n(\chi)\geqslant 1$ such that χ consists of n !-chains, n ?-chains, n upper links and n lower links. we will call this $n(\chi)$ the *characteristic number* of χ. For the example in Figure 8, we have $n=2$.

3.23 Definition A cycle χ is called a *promotion cycle* if every upper link of χ is redex of a $\mathsf{p}{\downarrow}$-rule (called *link promotion*) and every lower link of χ is contractum of a $\mathsf{p}{\uparrow}$-rule (called *link copromotion*).

The example in Figure 8 is not a promotion cycle because the upper link $\boldsymbol{[}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{b}\boldsymbol{]}$ is not redex of a $\mathsf{p}{\downarrow}$-rule and the lower link $\boldsymbol{(}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{d}\boldsymbol{)}$ is not contractum of a $\mathsf{p}{\uparrow}$-rule.

$$
\begin{array}{rc}
 & !(![a,b],![c,d],![e,f]) \\ \cline{2-2}
\mathsf{p}\downarrow & !(\mathbf{[}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{b}\mathbf{]},![c,d],![f,e]) \\ \cline{2-2}
\mathsf{p}\downarrow & !([!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{b}],![c,d],\mathbf{[}?^{\blacktriangledown}\boldsymbol{f},!^{\blacktriangle}\boldsymbol{e}\mathbf{]}) \\ \cline{2-2}
\mathsf{b}\uparrow & (!([!^{\blacktriangle}\boldsymbol{a},?b],![c,d],[!e,?^{\blacktriangledown}\boldsymbol{f}]),[!a,?^{\blacktriangledown}\boldsymbol{b}],![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{w}\uparrow & (!([!^{\blacktriangle}\boldsymbol{a},?b],[!e,?^{\blacktriangledown}\boldsymbol{f}]),[!a,?^{\blacktriangledown}\boldsymbol{b}],![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{s} & (![(!^{\blacktriangle}\boldsymbol{a},[!e,?^{\blacktriangledown}\boldsymbol{f}]),?b],[!a,?^{\blacktriangledown}\boldsymbol{b}],![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{s} & (![\mathbf{(}!^{\blacktriangle}\boldsymbol{a},?^{\blacktriangledown}\boldsymbol{f}\mathbf{)},!e,?b],[!a,?^{\blacktriangledown}\boldsymbol{b}],![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{p}\uparrow & (![?(a,f),!e,?b],[!a,?^{\blacktriangledown}\boldsymbol{b}],![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{w}\uparrow & ([!a,?^{\blacktriangledown}\boldsymbol{b}]![c,d],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{p}\downarrow & ([!a,?^{\blacktriangledown}\boldsymbol{b}],\mathbf{[}!^{\blacktriangle}\boldsymbol{c},?^{\blacktriangledown}\boldsymbol{d}\mathbf{]},[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{s} & ([([!a,?^{\blacktriangledown}\boldsymbol{b}],!^{\blacktriangle}\boldsymbol{c}),?^{\blacktriangledown}\boldsymbol{d}],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{s} & ([\mathbf{(}?^{\blacktriangledown}\boldsymbol{b},!^{\blacktriangle}\boldsymbol{c}\mathbf{)},!a,?^{\blacktriangledown}\boldsymbol{d}],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{p}\uparrow & ([?(b,c),!a,?^{\blacktriangledown}\boldsymbol{d}],[!^{\blacktriangle}\boldsymbol{e},?f]) \\ \cline{2-2}
\mathsf{s} & [?(b,c),!a,(?^{\blacktriangledown}\boldsymbol{d},[!^{\blacktriangle}\boldsymbol{e},?f])] \\ \cline{2-2}
\mathsf{s} & [?(b,c),!a,\mathbf{(}?^{\blacktriangledown}\boldsymbol{d},!^{\blacktriangle}\boldsymbol{e}\mathbf{)},?f] \\ \cline{2-2}
\mathsf{p}\uparrow & [?(b,c),!a,?(d,e),?f]
\end{array}
$$

Figure 9: *A promotion cycle χ with $n(\chi) = 3$*

Figure 9 shows an example for a promotion cycle. Observe that it is not necessarily the case that all upper links are above all lower links in the derivation.

3.24 Definition Let χ be a cycle inside a derivation Δ, and let all !-links and ?-links of χ be marked with $\mathbf{!}^{\blacktriangle}$ or $\mathbf{?}^{\blacktriangledown}$, respectively. Then, χ is called *forked* if one of the following holds:

(i) There is an instance of $\mathsf{b}\downarrow \dfrac{S[?T,T]}{S\{?T\}}$ inside Δ, such that both substructures $?T$ and T of the contractum contain at least one substructure marked by $\mathbf{!}^{\blacktriangle}$ or $\mathbf{?}^{\blacktriangledown}$.

(ii) There is an instance of $\mathsf{b}\uparrow \dfrac{S\{!R\}}{S(!R,R)}$ inside Δ, such that both substructures $?R$ and R of the redex contain at least one substructure marked by $\mathbf{!}^{\blacktriangle}$ or $\mathbf{?}^{\blacktriangledown}$.

A cycle is called *non-forked* if it is not forked.

Both examples for cycles, that we have shown, are forked cycles. In the remainder of this section, we will show that there are no non-forked cycles.

3.25 Definition A context where the hole does not occur inside an !- or ?-structure, is called a *basic context*.

3.26 Example The contexts $[a, b, (\bar{a}, [c, d, \bar{b}, \{\ \}, a], ?c)]$ and $([!(b, ?a), \{\ \}], b)$ are basic, whereas $([!(\{\ \}, ?a), ?(\bar{a}, c)], b)$ is not basic.

3.27 Lemma *Let $S\{\ \}$ be a basic context and R and T be any structures. Then there is a derivation*

$$\begin{array}{c} S[R,T] \\ \Delta \Vert \{\mathsf{s}, \mathsf{q}{\downarrow}\} \\ [S\{R\}, T] \end{array} .$$

Proof: By structural induction on $S\{\ \}$.

- $S = \{\ \}$. Trivial because $S[R,T] = [R,T] = [S\{R\}, T]$.
- $S = [S', S''\{\ \}]$ for some structure S' and context $S''\{\ \}$. Then by induction hypothesis we have

$$\begin{array}{c} [S', S''[R,T]] \\ \Delta \Vert \{\mathsf{s}\} \\ [S', S''\{R\}, T] \end{array} .$$

- $S = (S', S''\{\ \})$ for some structure S' and context $S''\{\ \}$. Then let Δ be

$$\begin{array}{c} (S', S''[R,T]) \\ \Delta' \Vert \{\mathsf{s}\} \\ \mathsf{s}\dfrac{(S', [S''\{R\}, T])}{[(S', S''\{R\}), T]} \end{array} ,$$

  where Δ' exists by induction hypothesis.

- $S = \langle S'; S''\{\ \}; S''' \rangle$ for some context $S''\{\ \}$ and structures S' and S'''. Then let Δ be

$$\begin{array}{c} \langle S'; S''[R,T]; S''' \rangle \\ \Delta' \Vert \{\mathsf{s}, \mathsf{q}{\downarrow}\} \\ \mathsf{q}{\downarrow}\dfrac{\langle S'; [S''\{R\}, T]; S''' \rangle}{\mathsf{q}{\downarrow}\dfrac{\langle S'; [\langle S''\{R\}; S''' \rangle, T] \rangle}{[\langle S'; S''\{R\}; S''' \rangle, T]}} \end{array} ,$$

  where Δ' exists by induction hypothesis. □

3.28 Definition A cycle χ is called *pure* if

(i) for each !-chain and each ?-chain contained in χ, head and tail are equal, and

(ii) all upper links occur in the same structure and all lower links occur in the same structure.

For example, the two cycles in Figures 8 and 9 are not pure. Although in both cases condition (i) is fulfilled, condition (ii) is not. Figure 10 shows an example for a pure cycle.

$$
\begin{array}{rc}
 & [?(!c,[!a,?b],?d),(![c,d],!a,?b)] \\ \hline
\mathsf{p}\!\downarrow & [?(!c,[\mathbf{!^{\blacktriangle}a},\mathbf{?^{\blacktriangledown}b}],?d),([\mathbf{!^{\blacktriangle}c},\mathbf{?^{\blacktriangledown}d}],!a,?b)] \\ \hline
\mathsf{s} & [?(!c,[\mathbf{!^{\blacktriangle}a},\mathbf{?^{\blacktriangledown}b}],?d),([\mathbf{!^{\blacktriangle}c},(!a,\mathbf{?^{\blacktriangledown}d})],?b)] \\ \hline
\mathsf{s} & [?(!c,[\mathbf{!^{\blacktriangle}a},\mathbf{?^{\blacktriangledown}b}],?d),(\mathbf{!^{\blacktriangle}c},?b),(!a,\mathbf{?^{\blacktriangledown}d})] \\ \hline
\mathsf{s} & [?(!c,[(\mathbf{!^{\blacktriangle}a},?d),\mathbf{?^{\blacktriangledown}b}]),(\mathbf{!^{\blacktriangle}c},?b),(!a,\mathbf{?^{\blacktriangledown}d})] \\ \hline
\mathsf{s} & [?[(?d,\mathbf{!^{\blacktriangle}a}),(\mathbf{?^{\blacktriangledown}b},!c)],(\mathbf{!^{\blacktriangle}c},?b),(!a,\mathbf{?^{\blacktriangledown}d})] \\ \hline
\mathsf{b}\!\downarrow & ?[(\mathbf{?^{\blacktriangledown}b},\mathbf{!^{\blacktriangle}c}),(\mathbf{!^{\blacktriangle}a},\mathbf{?^{\blacktriangledown}d})] \\ \hline
\mathsf{p}\!\uparrow & ?[?(c,b),(!a,?d)]
\end{array}
$$

Figure 10: *A pure cycle χ with $n(\chi) = 2$*

If a derivation

$$
\begin{array}{c} P \\ \Delta \Vert \mathsf{SNEL} \\ Q \end{array}
$$

contains a pure cycle then there are structures $R_1, \dots, R_n$ and $T_1, \dots, T_n$ (for some $n \geqslant 1$) and two n-ary contexts $S\{\ \}\dots\{\ \}$ and $S'\{\ \}\dots\{\ \}$, such that Δ is of the shape

$$
\begin{array}{c}
P \\
\Delta_1 \Vert \mathsf{SNEL} \\
S[\mathbf{!^{\blacktriangle}}R_1, \mathbf{?^{\blacktriangledown}}T_1][\mathbf{!^{\blacktriangle}}R_2, \mathbf{?^{\blacktriangledown}}T_2]\dots[\mathbf{!^{\blacktriangle}}R_n, \mathbf{?^{\blacktriangledown}}T_n] \\
\Delta_2 \Vert \mathsf{SNEL} \\
S'(\mathbf{?^{\blacktriangledown}}T_1, \mathbf{!^{\blacktriangle}}R_2)(\mathbf{?^{\blacktriangledown}}T_2, \mathbf{!^{\blacktriangle}}R_3)\dots(\mathbf{?^{\blacktriangledown}}T_n, \mathbf{!^{\blacktriangle}}R_1) \\
\Delta_3 \Vert \mathsf{SNEL} \\
Q
\end{array}
,
$$

where inside Δ_1 and Δ_3 no structures are marked with $\mathbf{!^{\blacktriangle}}$ or $\mathbf{?^{\blacktriangledown}}$ because the structure

$$
S[\mathbf{!^{\blacktriangle}}R_1, \mathbf{?^{\blacktriangledown}}T_1][\mathbf{!^{\blacktriangle}}R_2, \mathbf{?^{\blacktriangledown}}T_2]\dots[\mathbf{!^{\blacktriangle}}R_n, \mathbf{?^{\blacktriangledown}}T_n]
$$

contains all upper links and

$$
S'(\mathbf{?^{\blacktriangledown}}T_1, \mathbf{!^{\blacktriangle}}R_2)(\mathbf{?^{\blacktriangledown}}T_2, \mathbf{!^{\blacktriangle}}R_3)\dots(\mathbf{?^{\blacktriangledown}}T_n, \mathbf{!^{\blacktriangle}}R_1)
$$

contains all lower links of the pure circle. In other words, the cycle is completely inside Δ_2.

It seems rather intuitive that every cycle can be transformed into a pure cycle by permuting rules up and down until all upper links are in one structure, all lower links are in one structure, and all links belonging to a !-subchain (or ?-subchain) are equal. The following proposition makes this precise for non-forked cycles. More precisely, every non-forked promotion cycle will be transformed into a non-forked pure cycle.

The proof is rather long because of the technical details to be carried out. But the basic idea is quite simple: Whenever there is a situation

$$\rho\,\frac{\pi\,\dfrac{S\{\mathbf{!}^{\blacktriangle}\boldsymbol{R}\}}{S'\{\mathbf{!}^{\blacktriangle}\boldsymbol{R'}\}}}{S''\{\mathbf{!}^{\blacktriangle}\boldsymbol{R''}\}}\quad,$$

where the three links are connected and ρ can be permuted over π by one of the cases (1), (2), or (3) of 3.5 to get

$$\pi\,\frac{\rho\,\dfrac{S\{\mathbf{!}^{\blacktriangle}\boldsymbol{R}\}}{V}}{S''\{\mathbf{!}^{\blacktriangle}\boldsymbol{R''}\}}\quad,$$

for some structure V, then V must contain a link $\mathbf{!}^{\blacktriangle}\boldsymbol{R'''}$ that is connected to $\mathbf{!}^{\blacktriangle}\boldsymbol{R}$ and $\mathbf{!}^{\blacktriangle}\boldsymbol{R''}$. In other words, we have $V = S'''\{\mathbf{!}^{\blacktriangle}\boldsymbol{R'''}\}$ for some context $S'''\{\ \}$:

$$\pi\,\frac{\rho\,\dfrac{S\{\mathbf{!}^{\blacktriangle}\boldsymbol{R}\}}{S'''\{\mathbf{!}^{\blacktriangle}\boldsymbol{R'''}\}}}{S''\{\mathbf{!}^{\blacktriangle}\boldsymbol{R''}\}}\quad.$$

This means that a cycle cannot be destroyed by simple rule permutations.

3.29 Proposition *If there is a derivation* $\begin{array}{c}P\\ \Delta\Big\Vert\mathsf{SNEL}\\ Q\end{array}$ *that contains a non-forked promotion cycle, then there is a derivation* $\begin{array}{c}\tilde{P}\\ \tilde{\Delta}\Big\Vert\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{q}\!\uparrow\}\\ \tilde{Q}\end{array}$ *that contains a pure cycle.*

Proof: Let χ be the non-forked promotion cycle inside Δ and let all !-links and ?-links of χ be marked with $\mathbf{!}^{\blacktriangle}$ and $\mathbf{?}^{\blacktriangledown}$, respectively (see Figure 11, first derivation). Further, let all instances of a link promotion (Definition 3.22) and all instances of a link copromotion be marked as $\mathsf{p}\!\downarrow^{\Cap}$ and $\mathsf{p}\!\uparrow^{\Cup}$, respectively (see Figure 11, second derivation). Now we will stepwise construct $\tilde{\Delta}$ from Δ by adding some further markings and by permuting, adding and removing rules until the cycle is pure. Observe that the transformations will not destroy the cycle but might change premise and conclusion of the derivation.

I. Let n be the characteristic number of χ. For each of the n marked instances of

$$\mathsf{p}\!\downarrow^{\Cap}\frac{S\{![R_i,T_i]\}}{S[\mathbf{!}^{\blacktriangle}\boldsymbol{R_i},\mathbf{?}^{\blacktriangledown}\boldsymbol{T_i}]}$$

proceed as follows: Mark the contractum $![R_i,T_i]$ as $\mathbf{!}^{\blacktriangle}[\boldsymbol{R_i},\boldsymbol{T_i}]$ and continue the marking for all !-links of the (maximal) !-chain that has $\mathbf{!}^{\blacktriangle}[\boldsymbol{R_i},\boldsymbol{T_i}]$ as tail. There is always a unique choice how to continue the marking (see Definition 3.12), except for one case: If the marking reaches a

$$\mathsf{b}\!\downarrow\frac{S[?U,U]}{S\{?U\}}$$

$$
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_1\{![R_1,T_1]\}}{S_1[\boldsymbol{!^{\blacktriangle}R_1},\boldsymbol{?^{\blacktriangledown}T_1}]}\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_2\{![R_2,T_2]\}}{S_2[\boldsymbol{!^{\blacktriangle}R_2},\boldsymbol{?^{\blacktriangledown}T_2}]}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_1(\boldsymbol{!^{\blacktriangle}R'_2},\boldsymbol{?^{\blacktriangledown}T'_1})}{S'_1\{?(R'_2,T'_1)\}}\\
\|\\
\mathsf{p}{\downarrow}\,\dfrac{S_3\{![R_3,T_3]\}}{S_3[\boldsymbol{!^{\blacktriangle}R_3},\boldsymbol{?^{\blacktriangledown}T_3}]}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_3(\boldsymbol{!^{\blacktriangle}R'_1},\boldsymbol{?^{\blacktriangledown}T'_3})}{S'_3\{?(R'_1,T'_3)\}}\\
\|\\
\mathsf{p}{\uparrow}\,\dfrac{S'_2(\boldsymbol{!^{\blacktriangle}R'_3},\boldsymbol{?^{\blacktriangledown}T'_2})}{S'_2\{?(R'_3,T'_2)\}}\\
\|\\
Q
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_1\{![R_1,T_1]\}}{S_1[\boldsymbol{!^{\blacktriangle}R_1},\boldsymbol{?^{\blacktriangledown}T_1}]}\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_2\{![R_2,T_2]\}}{S_2[\boldsymbol{!^{\blacktriangle}R_2},\boldsymbol{?^{\blacktriangledown}T_2}]}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_1(\boldsymbol{!^{\blacktriangle}R'_2},\boldsymbol{?^{\blacktriangledown}T'_1})}{S'_1\{?(R'_2,T'_1)\}}\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_3\{![R_3,T_3]\}}{S_3[\boldsymbol{!^{\blacktriangle}R_3},\boldsymbol{?^{\blacktriangledown}T_3}]}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_3(\boldsymbol{!^{\blacktriangle}R'_1},\boldsymbol{?^{\blacktriangledown}T'_3})}{S'_3\{?(R'_1,T'_3)\}}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_2(\boldsymbol{!^{\blacktriangle}R'_3},\boldsymbol{?^{\blacktriangledown}T'_2})}{S'_2\{?(R'_3,T'_2)\}}\\
\|\\
Q
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}
P\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_1\{\boldsymbol{!^{\blacktriangle}[R_1,T_1]}\}}{S_1[\boldsymbol{!^{\blacktriangle}R_1},\boldsymbol{?^{\blacktriangledown}T_1}]}\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_2\{\boldsymbol{!^{\blacktriangle}[R_2,T_2]}\}}{S_2[\boldsymbol{!^{\blacktriangle}R_2},\boldsymbol{?^{\blacktriangledown}T_2}]}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_1(\boldsymbol{!^{\blacktriangle}R'_2},\boldsymbol{?^{\blacktriangledown}T'_1})}{S'_1\{\boldsymbol{?^{\blacktriangledown}(R'_2,T'_1)}\}}\\
\|\\
\mathsf{p}{\downarrow}^{\Cap}\,\dfrac{S_3\{\boldsymbol{!^{\blacktriangle}[R_3,T_3]}\}}{S_3[\boldsymbol{!^{\blacktriangle}R_3},\boldsymbol{?^{\blacktriangledown}T_3}]}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_3(\boldsymbol{!^{\blacktriangle}R'_1},\boldsymbol{?^{\blacktriangledown}T'_3})}{S'_3\{\boldsymbol{?^{\blacktriangledown}(R'_1,T'_3)}\}}\\
\|\\
\mathsf{p}{\uparrow}^{\Cup}\,\dfrac{S'_2(\boldsymbol{!^{\blacktriangle}R'_3},\boldsymbol{?^{\blacktriangledown}T'_2})}{S'_2\{\boldsymbol{?^{\blacktriangledown}(R'_3,T'_2)}\}}\\
\|\\
Q
\end{array}
$$

Figure 11: *Example (with $n(\chi) = 3$) for the marking inside Δ*

and the last marked $\boldsymbol{!^{\blacktriangle}}$-structure is inside the redex $?U$. Then there are two possibilities: either continue inside $?U$ (case (4.v) of Definition 3.12) or continue inside U (case (4.vi) of Definition 3.12). Choose that side that already contains a marked $\boldsymbol{!^{\blacktriangle}}$- or $\boldsymbol{?^{\blacktriangledown}}$-structure. Since the cycle χ is non-forked, it cannot happen that both sides already contain a marked $\boldsymbol{!^{\blacktriangle}}$- or $\boldsymbol{?^{\blacktriangledown}}$-structure. If there is no marked $\boldsymbol{!^{\blacktriangle}}$- or $\boldsymbol{?^{\blacktriangledown}}$-structure inside the contractum $[?U, U]$ of the $\mathsf{b}{\downarrow}$, then choose either one.

Proceed dually for all marked

$$
\mathsf{p}{\uparrow}^{\Cup}\,\frac{S(\boldsymbol{!^{\blacktriangle}R'_i},\boldsymbol{?^{\blacktriangledown}T_i}')}{S\{?(R'_i,T'_i)\}}\quad ,
$$

i.e. mark the redex $?(R'_i, T'_i)$ as $\boldsymbol{?^{\blacktriangledown}(R'_i, T'_i)}$ and mark also all links of the ?-chain that has $\boldsymbol{?^{\blacktriangledown}(R'_i, T'_i)}$ as tail (see Figure 11, third derivation).

II. Now consider all !-substructures and all ?-substructures that occur somewhere in the derivation Δ. They can be divided into three groups:

(a) those which are marked with ,

(b) those which are a substructure of a marked $\boldsymbol{!^{\blacktriangle}}$- or $\boldsymbol{?^{\blacktriangledown}}$-structure, and

(c) all the others.

In this step replace all substructures $!R$ and $?T$ that fall in group (c) by R and T respectively, i.e. remove the exponential. This rather drastic step will, of course, yield a non-valid derivation because correct rule applications might become incorrect. Observe that all instance of $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, s, $\mathsf{q}{\downarrow}$ and $\mathsf{q}{\uparrow}$ inside Δ do not suffer from this step, i.e. they remain valid. Let us now inspect more closely what could happen to the instances of $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$ and $\mathsf{b}{\uparrow}$.

- An instance of $\mathsf{p}{\downarrow}$ either remains unchanged (because it is inside the scope of a $\mathbf{!}^{\blacktriangle}$ or $\mathbf{?}^{\blacktriangledown}$, or is marked as $\mathsf{p}{\downarrow}^{\text{ⓜ}}$), or becomes trivial, or becomes

$$\hat{\mathsf{p}}{\downarrow}\,\frac{S\{\mathbf{!}^{\blacktriangle}[\boldsymbol{R},\boldsymbol{T}]\}}{S[\mathbf{!}^{\blacktriangle}\boldsymbol{R},T']}\quad,$$

  where T' is obtained from T by removing the exponentials. The new instances of $\hat{\mathsf{p}}{\downarrow}$ will be removed in Step V.

- Dually, for $\mathsf{p}{\uparrow}$, we obtain the rule

$$\hat{\mathsf{p}}{\uparrow}\,\frac{S(R',\mathbf{?}^{\blacktriangledown}\boldsymbol{T})}{S\{\mathbf{?}^{\blacktriangledown}(\boldsymbol{R},\boldsymbol{T})\}}\quad,$$

  which will also be removed in Step V.

- The instances of $\mathsf{w}{\downarrow}$ either remain unchanged, or become

$$\hat{\mathsf{w}}{\downarrow}\,\frac{S\{\circ\}}{S\{T'\}}\quad,$$

  where T' is obtained from T by removing some or all exponentials.

- For the rule $\mathsf{w}{\uparrow}$ the situation is dual and we obtain

$$\hat{\mathsf{w}}{\uparrow}\,\frac{S\{R'\}}{S\{\circ\}}\quad,$$

  where R' is obtained from R by removing the exponentials. The two rules $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$ will be removed in Step IV.

- Similarly, an instance of $\mathsf{b}{\downarrow}$ either remains unchanged, or becomes

$$\hat{\mathsf{b}}{\downarrow}\,\frac{S[T,T']}{S\{T\}}\quad,$$

  where $S\{\ \}$ is a basic context, and T and T' are arbitrary structures.

- Dually, for $\mathsf{b}{\uparrow}$, we obtain

$$\hat{\mathsf{b}}{\uparrow}\,\frac{S\{R\}}{S(R,R')}\quad,$$

  where $S\{\ \}$ is a basic context. The new instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$ will be removed in the next step.

Before, let us summarize what is achieved after this step: The original derivation

$$\begin{array}{c} P \\ \Delta \Vert \mathsf{SNEL} \\ Q \end{array}$$

has been transformed into

$$\begin{array}{c} P \\ \hat{\Delta} \Vert \mathsf{SNEL} \cup \{\mathsf{p}{\downarrow}^{\Cap}, \mathsf{p}{\uparrow}^{\Cup}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}, \hat{\mathsf{b}}{\downarrow}, \hat{\mathsf{b}}{\uparrow}\} \\ Q \end{array} \quad ,$$

where the cycle together with the extentions of its !- and ?-chains is marked. In the following steps, we will remove all rules (including $\hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}, \hat{\mathsf{b}}{\downarrow}, \hat{\mathsf{b}}{\uparrow}$) that prevent the cycle from being pure.

III. First, we will remove all instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$. Consider the bottommost occurrence of $\hat{\mathsf{b}}{\downarrow}\dfrac{S[T,T']}{S\{T\}}$ inside $\hat{\Delta}$. Replace

$$\hat{\Delta} \quad = \quad \begin{array}{c} P \\ \Delta_1 \Vert \\ \hat{\mathsf{b}}{\downarrow}\dfrac{S[T,T']}{S\{T\}} \\ \Delta_2 \Vert \\ Q \end{array} \qquad \text{by} \qquad \begin{array}{c} P \\ \Delta_1 \Vert \\ S[T,T'] \\ \Delta_3 \Vert \{\mathsf{s},\mathsf{q}{\downarrow}\} \\ [S\{T\},T'] \\ \Delta_2 \Vert \\ [Q,T'] \end{array} \quad ,$$

where Δ_2 does not contain any $\hat{\mathsf{b}}{\downarrow}$ and Δ_3 exists by Lemma 3.27. Repeat this until there are no more $\hat{\mathsf{b}}{\downarrow}$ in the derivation. Then proceed dually to remove all $\hat{\mathsf{b}}{\uparrow}$, i.e. start with the topmost $\hat{\mathsf{b}}{\uparrow}$. This gives us a derivation

$$\begin{array}{c} P' \\ \hat{\Delta}' \Vert \mathsf{SNEL} \cup \{\mathsf{p}{\downarrow}^{\Cap}, \mathsf{p}{\uparrow}^{\Cup}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}\} \\ Q' \end{array} \quad .$$

Observe that premise and conclusion of the derivation have changed now, but the cycle is still present.

IV. In this step, we will remove all instances of $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$. For this, observe that it can never happen that the contractum $\circ$ of

$$\hat{\mathsf{w}}{\downarrow}\frac{S\{\circ\}}{S\{T\}}$$

is inside an active structure of the redex of $\mathsf{p}{\uparrow}$, $\hat{\mathsf{p}}{\uparrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{b}{\uparrow}$ or $\mathsf{w}{\downarrow}$ because then the redex T would be inside a marked $\mathbf{!}^{\blacktriangle}$- or $\mathbf{?}^{\blacktriangledown}$-structure, which is not possible by the construction of $\hat{\mathsf{w}}{\downarrow}$ in Step II. Hence, the rule $\hat{\mathsf{w}}{\downarrow}$ permutes (by $\{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}$)

over all other rules in the derivation $\hat{\Delta}'$ (by Lemma 3.6). Dually, $\hat{\mathsf{w}}\!\uparrow$ permutes under all other rules in $\hat{\Delta}'$. This means that $\hat{\Delta}'$ can easily be transformed into

$$\begin{array}{c} P' \\ \Delta_1' \Big\Vert \{\hat{\mathsf{w}}\!\downarrow\} \\ P'' \\ \hat{\Delta}'' \Big\Vert \mathsf{SNEL} \cup \{\mathsf{p}\!\downarrow^{\Cap}, \mathsf{p}\!\uparrow^{\Cup}, \hat{\mathsf{p}}\!\downarrow, \hat{\mathsf{p}}\!\uparrow\} \\ Q'' \\ \Delta_2' \Big\Vert \{\hat{\mathsf{w}}\!\uparrow\} \\ Q' \end{array}$$

by permuting stepwise all $\hat{\mathsf{w}}\!\downarrow$ up and all $\hat{\mathsf{w}}\!\uparrow$ down. Let us now consider only $\begin{array}{c} P'' \\ \hat{\Delta}'' \Vert \mathsf{SNEL} \cup \{\mathsf{p}\!\downarrow^{\Cap}, \mathsf{p}\!\uparrow^{\Cup}, \hat{\mathsf{p}}\!\downarrow, \hat{\mathsf{p}}\!\uparrow\} \\ Q'' \end{array}$, in which the cycle χ is still present.

V. Inside $\hat{\Delta}''$ mark all rules ρ whose redex is inside a marked $\mathbf{!}^{\blacktriangle}$-structure as $\rho^{\vartriangle}$. Additionally, mark all instances of $\hat{\mathsf{p}}\!\downarrow$ as $\hat{\mathsf{p}}\!\downarrow^{\vartriangle}$. Dually, mark all rules $\hat{\mathsf{p}}\!\uparrow$ as well as all rules ρ whose contractum is inside a marked $\mathbf{?}^{\blacktriangledown}$-structure as $\rho^{\triangledown}$. Now mark all remaining, i.e. not yet marked, rules ρ as ρ°. This means, we now have a derivation

$$\begin{array}{c} P'' \\ \hat{\Delta}'' \Big\Vert \{\mathsf{p}\!\downarrow^{\Cap}, \mathsf{p}\!\uparrow^{\Cup}, \rho^{\vartriangle}, \rho^{\triangledown}, \rho^{\circ}\} \\ Q'' \end{array} ,$$

which will in this step be decomposed into

$$\begin{array}{c} P'' \\ \hat{\Delta}_1'' \Big\Vert \{\rho^{\vartriangle}\} \\ P''' \\ \hat{\Delta}_2'' \Big\Vert \{\mathsf{p}\!\downarrow^{\Cap}\} \\ \tilde{P} \\ \tilde{\Delta} \Big\Vert \{\rho^{\circ}\} \\ \tilde{Q} \\ \hat{\Delta}_3'' \Big\Vert \{\mathsf{p}\!\uparrow^{\Cup}\} \\ Q''' \\ \hat{\Delta}_4'' \Big\Vert \{\rho^{\triangledown}\} \\ Q'' \end{array}$$

only by permutation of rules. In order to obtain this decomposition, we need to show that

(a) all rules marked as $\rho^{\vartriangle}$ permute over all other rules,

(b) all rules marked as $\rho^{\triangledown}$ permute under all other rules,

(c) all rules $\mathsf{p}\!\downarrow^{\Cap}$ permute over all rules marked as ρ° or $\mathsf{p}\!\uparrow^{\Cup}$, and

(d) all rules $\mathsf{p}\!\uparrow^{\Cup}$ permute under all rules marked as ρ° or $\mathsf{p}\!\downarrow^{\Cap}$.

We will apply the scheme of 3.5 to show the four statements.

(a) Consider

$$\pi \frac{Q}{\rho^{\vartriangle} \frac{S\{W\}}{S\{Z\}}} \quad ,$$

where π is not marked as $\pi^{\vartriangle}$ and nontrivial. According to 3.5 the cases to consider are:

(4) The redex of π is inside an active structure of the contractum W of $\rho^{\vartriangle}$. This is impossible because

(i) if the redex of $\rho^{\vartriangle}$ is inside a $\mathbf{!}^{\blacktriangle}$-structure, then the contractum of $\rho^{\vartriangle}$ is also inside a $\mathbf{!}^{\blacktriangle}$-structure, and hence, the redex of π is inside a $\mathbf{!}^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$;

(ii) if $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$, then the redex of π is also inside a $\mathbf{!}^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$.

(5) The contractum W of $\rho^{\vartriangle}$ is inside an active structure of the redex of π but not inside a passive one. There are the following subcases:

(i) The redex of $\rho^{\vartriangle}$ is inside a $\mathbf{!}^{\blacktriangle}$-structure. This is impossible because then the contractum of $\rho^{\vartriangle}$ is also inside a $\mathbf{!}^{\blacktriangle}$-structure. Since it is also inside an active structure of the redex of π, we have that either this active structure is a $\mathbf{!}^{\blacktriangle}$-structure and therefore $\pi = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$, or the whole redex of π is inside a $\mathbf{!}^{\blacktriangle}$-structure and therefore π must be marked as $\pi^{\vartriangle}$.

(ii) $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ and $\pi = \hat{\mathsf{p}}{\downarrow}$. This is impossible because then π is marked as $\pi^{\vartriangle}$.

(iii) $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ and $\pi = \mathsf{p}{\downarrow}$. Then $\pi = \mathsf{p}{\downarrow}^{⋒}$ because there are no other $\mathsf{p}{\downarrow}$ that have a marked $\mathbf{!}^{\blacktriangle}$-structure in the redex, and we can replace

$$\mathsf{p}{\downarrow}^{⋒} \frac{S'\{\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T_1}, \boldsymbol{T_2}]\}}{\hat{\mathsf{p}}{\downarrow}^{\vartriangle} \frac{S'[\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T_1}], \mathbf{?}^{\blacktriangledown}\boldsymbol{T_2}]}{S'[\mathbf{!}^{\blacktriangle}\boldsymbol{R}, T_1', \mathbf{?}^{\blacktriangledown}\boldsymbol{T_2}]}} \quad \text{by} \quad \hat{\mathsf{p}}{\downarrow}^{\vartriangle} \frac{S'\{\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T_1}, \boldsymbol{T_2}]\}}{\mathsf{p}{\downarrow}^{⋒} \frac{S'[\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T_2}], T_1']}{S'[[\mathbf{!}^{\blacktriangle}\boldsymbol{R}, \mathbf{?}^{\blacktriangledown}\boldsymbol{T_2}], T_1']}} \quad .$$

(6) The contractum W of ρ° and the redex of π overlap. This is impossible.

(b) Dual to (a).

(c) Consider

$$\pi \frac{Q}{\mathsf{p}{\downarrow}^{⋒} \frac{S\{\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T}]\}}{S[\mathbf{!}^{\blacktriangle}\boldsymbol{R}, \mathbf{?}^{\blacktriangledown}\boldsymbol{T}]}} \quad ,$$

where $\pi \in \{\rho^{\circ}, \mathsf{p}{\uparrow}^{⋓}\}$ is nontrivial.

(4) The redex of π is inside an active structure of the contractum of $\mathsf{p}{\downarrow}^{⋒}$. This is impossible because then the redex of π is inside a $\mathbf{!}^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$.

(5) The contractum $\mathbf{!}^{\blacktriangle}[\boldsymbol{R}, \boldsymbol{T}]$ of $\mathsf{p}{\downarrow}^{⋒}$ is inside an active structure of the redex of π but not inside a passive one. This is impossible because then π were $\hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ or $\mathsf{p}{\downarrow}^{⋒}$.

(6) The contractum of $\mathsf{p}{\downarrow}^{\Cap}$ and the redex of π overlap. As before, this is impossible.

(d) Dual to (c).

Now it only remains to show that the subderivation $\tilde{\Delta}\,\begin{array}{c}\tilde{P}\\ \| \{\rho^{\circ}\}\\ \tilde{Q}\end{array}$ obtained in the last step has indeed the desired properties (i.e. contains a pure cycle and consists only of the rules $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}$). Observe that all rules $\rho \in \{\mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ in Δ

- either have been transformed into $\hat{\rho}$ in Step II and then been removed in the Steps III, IV and V,
- or remained unchanged in Step II (because they occurred inside a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure) and have then been marked as $\rho^{\vartriangle}$ or $\rho^{\triangledown}$ and removed in Step V.

This means that only the rules $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\downarrow}$ and $\mathsf{q}{\uparrow}$ are left inside $\tilde{\Delta}$. Now consider the premise $\tilde{P}$ of $\tilde{\Delta}$. Since it is also the conclusion of $\hat{\Delta}''_2$ which consists only of $\mathsf{p}{\downarrow}^{\Cap}$, it is of the shape

$$S[\,!^{\blacktriangle}\boldsymbol{R_1}\,,\,?^{\blacktriangledown}\boldsymbol{T_1}\,][\,!^{\blacktriangle}\boldsymbol{R_2}\,,\,?^{\blacktriangledown}\boldsymbol{T_2}\,]\ldots[\,!^{\blacktriangle}\boldsymbol{R_n}\,,\,?^{\blacktriangledown}\boldsymbol{T_n}\,]$$

for some structures $R_1,\ldots,R_n,T_1,\ldots,T_n$ and some n-ary context $S\{\ \}\{\ \}\ldots\{\ \}$. Similarly, we have that

$$\tilde{Q} = S'(\,!^{\blacktriangle}\boldsymbol{R'_1}\,,\,?^{\blacktriangledown}\boldsymbol{T'_1}\,)(\,!^{\blacktriangle}\boldsymbol{R'_2}\,,\,?^{\blacktriangledown}\boldsymbol{T'_2}\,)\ldots(\,!^{\blacktriangle}\boldsymbol{R'_n}\,,\,?^{\blacktriangledown}\boldsymbol{T'_n}\,)$$

for some structures $R'_1,\ldots,R'_n,T'_1,\ldots,T'_n$ and some n-ary context $S'\{\ \}\{\ \}\ldots\{\ \}$. Since no transformation in Steps II–V destroyed the cycle, it must still be present in $\tilde{\Delta}$. Since $\tilde{\Delta}$ contains no rule that operates inside a $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, we have that $R'_1 = R_2$, $R'_2 = R_3$, …, $R'_n = R_1$ and $T'_1 = T_1$, $T'_2 = T_2$, …, $T'_n = T_n$. This means that $\tilde{\Delta}$ does indeed contain a pure cycle. □

3.30 Lemma *There is no derivation* $\Delta\,\begin{array}{c}P\\ \| \{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow},\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\\ Q\end{array}$ *that contains a pure cycle.*

Proof: By Lemma 3.6 and Lemma 3.7, the derivation Δ can be decomposed into

$$\begin{array}{c}P\\ \Delta_1 \| \{\mathsf{ai}{\downarrow}\}\\ P'\\ \Delta_2 \| \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\\ Q'\\ \Delta_3 \| \{\mathsf{ai}{\uparrow}\}\\ Q\end{array} \quad .$$

This transformation does not destroy the cycle. Hence, the pure cycle is contained in Δ_2. In other words, Δ_2 has a subderivation

$$\begin{array}{c}S[\,!^{\blacktriangle}R_1\,,\,?^{\blacktriangledown}T_1\,][\,!^{\blacktriangle}R_2\,,\,?^{\blacktriangledown}T_2\,]\ldots[\,!^{\blacktriangle}\boldsymbol{R_n}\,,\,?^{\blacktriangledown}\boldsymbol{T_n}\,]\\ \Delta' \| \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\\ S'(\,!^{\blacktriangle}\boldsymbol{R_2}\,,\,?^{\blacktriangledown}T_1\,)(\,!^{\blacktriangle}\boldsymbol{R_3}\,,\,?^{\blacktriangledown}T_2\,)\ldots(\,!^{\blacktriangle}R_1\,,\,?^{\blacktriangledown}\boldsymbol{T_n}\,)\end{array}$$

for some structures $R_1, \dots, R_n, T_1, \dots, T_n$ and two n-ary contexts $S\{\ \}\dots\{\ \}$ and $S'\{\ \}\dots\{\ \}$. In the premise of Δ', for every $i = 1, \dots, n$, the substructures $\boldsymbol{!^{\blacktriangle}R_i}$ and $\boldsymbol{?^{\blacktriangledown}T_i}$ are in par-relation. No rule in $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}$ is able to transform a par-relation into a times-relation while going down in a derivation. Hence, for every $i = 1, \dots, n$, the substructures $\boldsymbol{!^{\blacktriangle}R_i}$ and $\boldsymbol{?^{\blacktriangledown}T_i}$ are also in par-relation in the conclusion of Δ'. This means that the context $S'\{\ \}\dots\{\ \} = S'_0[S'_1\{\ \}, \dots, S'_n\{\ \}]$ for some contexts $S'_0\{\ \}, S'_1\{\ \}, \dots, S'_n\{\ \}$. Dually, we have that $S\{\ \}\dots\{\ \} = S_0(S_1\{\ \}, \dots, S_n\{\ \})$ for some contexts $S_0\{\ \}, S_1\{\ \}, \dots, S_n\{\ \}$. Hence, the derivation Δ' has the shape

$$\begin{array}{c} S_0(S_1[\boldsymbol{!^{\blacktriangle}R_1}, \boldsymbol{?^{\blacktriangledown}T_1}], S_2[\boldsymbol{!^{\blacktriangle}R_2}, \boldsymbol{?^{\blacktriangledown}T_2}], \dots, S_n[\boldsymbol{!^{\blacktriangle}R_n}, \boldsymbol{?^{\blacktriangledown}T_n}]) \\ \Delta' \Big\Vert \{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\} \\ S'_0[S'_1(\boldsymbol{!^{\blacktriangle}R_2}, \boldsymbol{?^{\blacktriangledown}T_1}), S'_2(\boldsymbol{!^{\blacktriangle}R_3}, \boldsymbol{?^{\blacktriangledown}T_2}), \dots, S'_n(\boldsymbol{!^{\blacktriangle}R_1}, \boldsymbol{?^{\blacktriangledown}T_n})] \end{array} .$$

Observe that the two contexts $S_0(S_1\{\ \}, \dots, S_n\{\ \})$ and $S'_0[S'_1\{\ \}, \dots, S'_n\{\ \}]$ must contain the same atoms because Δ' contains no rules that could create or destroy any atoms. Hence, the derivation Δ' remains valid if those atoms are replaced by $\circ$ (i.e. are removed from the derivation), which gives us the derivation

$$\begin{array}{c} ([\boldsymbol{!^{\blacktriangle}R_1}, \boldsymbol{?^{\blacktriangledown}T_1}], [\boldsymbol{!^{\blacktriangle}R_2}, \boldsymbol{?^{\blacktriangledown}T_2}], \dots [\boldsymbol{!^{\blacktriangle}R_n}, \boldsymbol{?^{\blacktriangledown}T_n}]) \\ \tilde{\Delta} \Big\Vert \{\mathsf{s}\} \\ [(\boldsymbol{!^{\blacktriangle}R_2}, \boldsymbol{?^{\blacktriangledown}T_1}), (\boldsymbol{!^{\blacktriangle}R_3}, \boldsymbol{?^{\blacktriangledown}T_2}), \dots (\boldsymbol{!^{\blacktriangle}R_1}, \boldsymbol{?^{\blacktriangledown}T_n})] \end{array} ,$$

which is a contradiction to Lemma 3.9. □

3.31 Theorem *There exists no derivation containing a non-forked promotion cycle.*

Proof: Immediate from Proposition 3.29 and Lemma 3.30. □

3.32 Corollary *There exists no derivation containing a non-forked cycle.*

Proof: Any (non-forked) cycle can easily be transformed into a (non-forked) promotion cycle by adding instances of $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$. □

3.4 Separation of Absorption and Weakening

In this section we will show the technical details of the first step in the proofs of the decomposition theorems, i.e. we show how absorption can be separated from the other rules. For this, the results of the previous section is used in a crucial way. Then we will also show the second step in the second decomposition, i.e. the separation of weakening.

The process of permuting up all instances of $\mathsf{b}{\uparrow}$ in a given derivation $\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \\ R \end{array}$ is realized by the procedure $\mathsf{b}{\uparrow}\mathsf{up}$:

3.33 Algorithm $\mathsf{b}{\uparrow}\mathsf{up}$ for permuting coabsorption up Consider the topmost occurrence of a subderivation

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{Q}{S\{!R\}}}{S(!R, R)} ,$$

where $\pi \neq \mathsf{b}{\uparrow}$. According to 3.5 there are the following cases (cases (3) and (6) are not possible):

(1) If the redex of π is inside $S\{\ \}$, or

(2) if the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside a passive structure of the redex of π, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S'\{!R\}}{S\{!R\}}}{S(!R,R)} \quad \text{by} \quad \pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!R\}}{S'(!R,R)}}{S(!R,R)} \quad .$$

(4) If the redex of π is inside the contractum $!R$ of $\mathsf{b}{\uparrow}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S\{!R'\}}{S\{!R\}}}{S(!R,R)} \quad \text{by} \quad \pi\,\dfrac{\pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S\{!R'\}}{S(!R',R')}}{S(!R',R)}}{S(!R,R)} \quad .$$

(5) If the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside an active structure of the redex of π but not inside a passive one, then there are three subcases:

(i) If $\pi = \mathsf{p}{\downarrow}$ and $S\{!R\} = S'[!R, ?T]$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,R),?T]} \quad \text{by} \quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{![R,T]\}}{S'(![R,T],[R,T])}}{S'([!R,?T],[R,T])}}{S'[([!R,?T],R),T]}}{S'[(!R,R),?T,T]}}{S'[(!R,R),?T]} \quad .$$

(ii) If $\pi = \mathsf{w}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\circ\}}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \quad \text{by} \quad \mathsf{w}{\downarrow}\,\dfrac{S'\{\circ\}}{S'\{?S''(!R,R)\}} \quad .$$

(iii) If $\pi = \mathsf{b}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \quad \text{by} \quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'[?S''(!R,R),S''\{!R\}]}}{S'[?S''(!R,R),S''(!R,R)]}}{S'\{?S''(!R,R)\}} \quad .$$

Repeat until all instances of $\mathsf{b}\uparrow$ are at the top of the derivation.

It is easy to see that if the Algorithm 3.33 terminates, then the resulting derivation has the shape

$$\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T' \\ \Vert\mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ R \end{array} \quad .$$

However, it is not obvious that this procedure does indeed terminate, because while permuting the rule $\mathsf{b}\uparrow$ up, it might happen that new instances of $\mathsf{b}\uparrow$ as well as new instances of $\mathsf{b}\downarrow$ are introduced.

3.34 Lemma *The* $\mathsf{b}\uparrow\mathsf{up}$ *algorithm terminates for any input derivation* $\begin{array}{c} T \\ \Delta\Vert\mathsf{SNEL} \\ R \end{array}$ *.*

Proof: The proof of this lemma is literally the same as already presented in [26]. For this reason, we will show here only a sketch. The problem of showing termination is that the number of instances of $\mathsf{b}\uparrow$ might increase during the process of permuting up $\mathsf{b}\uparrow$. For this reason we first show the termination for an input derivation

$$\begin{array}{c} T \\ \Delta'\Vert\mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ \mathsf{b}\uparrow\dfrac{S\{!R\}}{S(!R,R)} \end{array} \quad .$$

This is done by marking inside Δ' all !-chains that have the contractum $!R$ of the $\mathsf{b}\uparrow$ instance as tail. Based on this marking a global induction measure is constructed which decreases in each permutation step. □

The dual procedure to $\mathsf{b}\uparrow\mathsf{up}$ is $\mathsf{b}\downarrow\mathsf{down}$, in which all instances of $\mathsf{b}\downarrow$ are moved down in the derivation:

3.35 Algorithm $\mathsf{b}\downarrow\mathsf{down}$ for permuting absorption down Repeat until all instances of $\mathsf{b}\downarrow$ are at the bottom of the derivation: Consider the bottommost occurrence of a subderivation

$$\rho\,\dfrac{\mathsf{b}\downarrow\dfrac{S[?T,T]}{S\{?T\}}}{P} \quad ,$$

where $\rho \neq \mathsf{b}\downarrow$. The possible cases are (dual to 3.5):

(1) The contractum of ρ is inside $S\{\ \}$, or

(2) the redex $?T$ of $\mathsf{b}\downarrow$ is inside a passive structure of the contractum of ρ. Then replace

$$\rho\,\dfrac{\mathsf{b}\downarrow\dfrac{S[?T,T]}{S\{?T\}}}{S'\{?T\}} \quad \text{by} \quad \mathsf{b}\downarrow\dfrac{\rho\,\dfrac{S[?T,T]}{S'[?T,T]}}{S'\{?T\}} \quad .$$

(4) The contractum of ρ is inside the redex $?T$ of $\mathsf{b}{\downarrow}$ Then replace

$$\mathsf{b}{\downarrow}\,\dfrac{\rho\,\dfrac{S[?T,T]}{S\{?T\}}}{S\{?T'\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\dfrac{\rho\,\dfrac{\rho\,\dfrac{S[?T,T]}{S[?T,T']}}{S[?T',T']}}{S\{?T'\}} \quad .$$

(5) The redex $?T$ of $\mathsf{b}{\downarrow}$ is inside an active structure of the contractum of ρ but not inside a passive one. Then there are three cases:

(i) If $\rho = \mathsf{p}{\uparrow}$ and $S\{?T\} = S'(!R, ?T)$, then replace

$$\mathsf{p}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S'(!R,[?T,T])}{S'(!R,?T)}}{S'\{?(R,T)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{p}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'(!R,[?T,T])}{S'(!R,R,[?T,T])}}{S'([(!R,R),?T],T)}}{S'[(!R,?T),(R,T)]}}{S'[?(R,T),(R,T)]}}{S'\{?(R,T)\}} \quad .$$

(ii) If $\rho = \mathsf{w}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{w}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'\{\circ\}} \qquad \text{by} \qquad \mathsf{w}{\uparrow}\,\dfrac{S'\{!S''[?T,T]\}}{S'\{\circ\}} \quad .$$

(iii) If $\rho = \mathsf{b}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'(!S''\{?T\},S''\{?T\})} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!S''[?T,T]\}}{S'(!S''[?T,T],S''[?T,T])}}{S'(!S''[?T,T],S''\{?T\})}}{S'(!S''\{?T\},S''\{?T\})} \quad .$$

Remark: Cases (3) and (6) are not possible.

3.36 Lemma *The* $\mathsf{b}{\downarrow}\mathsf{down}$ *procedure terminates for every input derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \\ R \end{array}$ *and yields a derivation*

$$\begin{array}{c} T \\ \Delta' \Vert \mathsf{SNEL} \setminus \{\mathsf{b}{\downarrow}\} \\ R' \\ \Delta'' \Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array} \quad .$$

Proof: Dual to Lemma 3.34. □

3.37 Algorithm $\mathsf{b{\uparrow}{\downarrow}split}$ **for separating absorption and coabsorption**

I. If there are no subderivations of the shape $\pi \frac{Q}{U}$ / $\mathsf{b}{\uparrow} \frac{U}{P}$, where $\pi \neq \mathsf{b}{\uparrow}$, or of the shape

$$\mathsf{b}{\downarrow}\frac{Q}{V} \quad \rho\frac{V}{P}$$

, where $\rho \neq \mathsf{b}{\downarrow}$, then terminate.

II. Permute all instances of $\mathsf{b}{\uparrow}$ up by applying $\mathsf{b{\uparrow}up}$.

III. Permute all instances of $\mathsf{b}{\downarrow}$ down by applying $\mathsf{b{\downarrow}down}$.

IV. Go to step I.

Lemma 3.34 and Lemma 3.36 ensure that each step of Algorithm 3.37 (depicted in Figure 4, page 15) does terminate. It remains to show that the whole algorithm $\mathsf{b{\uparrow}{\downarrow}split}$ does terminate eventually.

3.38 Lemma *Then the algorithm* $\mathsf{b{\uparrow}{\downarrow}split}$ *does terminate for every input derivation* Δ.

Proof: Agaim, the proof is literally the same as the one already presented in [26] for pure MELL. For this reason, we will, as before, show here only a sketch. The proof is done in two parts. First, we show that the algorithm terminates for a derivation that does not contain a promotion cycle. For this we mark all chains that have the contractum $!R$ of a $\mathsf{b}{\uparrow}$ as tail with $!^{\blacktriangle}$ and $?^{\blacktriangledown}$ as in the previous section, and let l_{max} be the maximal length of those chains (see Definition 3.20). By running $\mathsf{b{\uparrow}up}$ each marked chain reduces its length by one. Hence l_{max} decreases (although the number of chains can be increased). For $\mathsf{b{\downarrow}down}$ the situation is similar. The second step of the proof is to show that after a consecutive run of $\mathsf{b{\uparrow}up}$ and $\mathsf{b{\downarrow}down}$ the derivation cannot contain a promotion cycle. For this, assume there is a promotion cycle χ, which is forked by k_χ different instances of $\mathsf{b}{\uparrow}$. By induction on k_χ a contradiction is obtained, where the base case follows immediately from Theorem 3.31. □

3.39 Proposition *For every derivation* $\begin{array}{c} T \\ \Delta \Big\| \mathsf{SNEL} \\ R \end{array}$ *there is a derivation*

$$\begin{array}{c} T \\ \Delta_1 \Big\| \{\mathsf{b}{\uparrow}\} \\ T' \\ \Delta' \Big\| \mathsf{SNEL} \setminus \{\mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\} \\ R' \\ \Delta_2 \Big\| \{\mathsf{b}{\downarrow}\} \\ R \end{array} .$$

Proof: Apply the algorithm $\mathsf{b{\uparrow}{\downarrow}split}$, which terminates by the previous lemma. □

This completes the proof of the first decomposition theorem. For the second decomposition, we have to separate weakening and coweakening.

3.40 Proposition *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SNEL} \setminus \{\mathsf{b}\downarrow, \mathsf{b}\uparrow\} \\ R \end{array}$ *there is a derivation*

$$\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{w}\uparrow\} \\ T' \\ \Delta' \Vert \mathsf{SNEL} \setminus \{\mathsf{b}\downarrow, \mathsf{b}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow\} \\ R' \\ \Delta_2 \Vert \{\mathsf{w}\downarrow\} \\ R \end{array} \quad .$$

Proof: First, all instances of $\mathsf{w}\uparrow$ inside Δ are permuted up to the top of the derivation. For this, consider the topmost subderivation

$$\mathsf{w}\uparrow \frac{\pi \dfrac{Q}{S\{!R\}}}{S\{\circ\}} \quad ,$$

where $\pi \in \mathsf{SNEL} \setminus \{\mathsf{b}\downarrow, \mathsf{b}\uparrow, \mathsf{w}\uparrow\}$ is not trivial. According to 3.5 there are the following cases to consider:

(4) The redex of π is inside an active structure of the contractum $!R$ of $\mathsf{w}\uparrow$. Then replace

$$\mathsf{w}\uparrow \frac{\pi \dfrac{S\{!R'\}}{S\{!R\}}}{S\{\circ\}} \quad \text{by} \quad \mathsf{w}\uparrow \frac{S\{!R'\}}{S\{\circ\}} \quad .$$

(5) The contractum $!R$ of $\mathsf{w}\uparrow$ is inside an active structure of the redex of π but not inside a passive one. There are two possibilities:

(i) $\pi = \mathsf{p}\downarrow$ and $S\{!R\} = S'[!R, ?T]$ for some context $S'\{\ \}$ and some structure T. Then replace

$$\mathsf{w}\uparrow \frac{\mathsf{p}\downarrow \dfrac{S'\{![R,T]\}}{S'[!R, ?T]}}{S'[\circ, ?T]} \quad \text{by} \quad \mathsf{w}\downarrow \frac{\mathsf{w}\uparrow \dfrac{S'\{![R,T]\}}{S'\{\circ\}}}{S'[\circ, ?T]} \quad .$$

(ii) $\pi = \mathsf{w}\downarrow$ and $S\{!R\} = S'\{?S''\{!R\}\}$ for some contexts $S'\{\ \}$ and $S''\{\ \}$. Then replace

$$\mathsf{w}\uparrow \frac{\mathsf{w}\downarrow \dfrac{S'\{\circ\}}{S'\{?S''\{!R\}\}}}{S'\{?S''\{\circ\}\}} \quad \text{by} \quad \mathsf{w}\downarrow \frac{S'\{\circ\}}{S'\{?S''\{\circ\}\}} \quad .$$

(6) The contractum $!R$ of $\mathsf{w}\uparrow$ and the redex of π (properly) overlap. Not possible.

This obviously terminates because the number of instances of $\mathsf{w}\uparrow$ does not increase and all reach the top eventually. Then, proceed dually, to permute all instances of $\mathsf{w}\downarrow$ down to the bottom of the derivation. Repeat the permuting up of $\mathsf{w}\uparrow$ and the permuting down of $\mathsf{w}\downarrow$ until the derivation has the desired shape. Since during the permuting up of $\mathsf{w}\uparrow$ new instances of $\mathsf{w}\downarrow$ are introduced and during the permuting down of $\mathsf{w}\downarrow$ new instances of $\mathsf{w}\uparrow$ are introduced, it remains to show that this does indeed terminate. The only possibility of introducing a new $\mathsf{w}\downarrow$ while a $\mathsf{w}\uparrow$ is permuted up, is in case (5.i), when the $\mathsf{w}\uparrow$ meets a $\mathsf{p}\downarrow$. But then this $\mathsf{p}\downarrow$ disappears. Since the number of instances of $\mathsf{p}\downarrow$ in the initial derivation is finite and this number is not increased during the process of permuting $\mathsf{w}\uparrow$ up and $\mathsf{w}\downarrow$ down, the process must terminate eventually. □

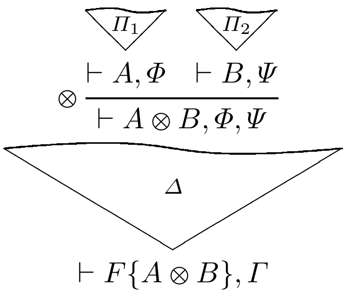
4 Cut Elimination

The classical arguments for proving cut elimination in the sequent calculus rely on the following property: when the principal formulae in a cut are active in both branches, they determine which rules are applied immediately above the cut. This is a consequence of the fact that formulae have a root connective, and logical rules only hinge on that, and nowhere else in the formula.

This property does not necessarily hold in the calculus of structures. Further, since rules can be applied anywhere deep inside structures, everything can happen above a cut. This complicates considerably the task of proving cut elimination. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form, which happens simply and independently of cut elimination. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus: complex interactions with their context are possible. As a consequence, our techniques are largely different than the traditional ones.

Two approaches to cut elimination in the calculus of structures have been explored in previous papers: in [13, 26] we relied on permutations of rules, in [7] Brünnler and Tiu relied on semantics, and in [4] Brünnler presents a simple syntactic method that employs the atomicity of cut together with certain proof theoretical properties of classical logic. In this paper we use a third technique, called *splitting* [12], which has the advantage of being more uniform than the one based on permutations and which yields a much simpler case analysis. It also establishes a deep connection to the sequent calculus, at least for the fragments of systems that allow for a sequent calculus presentation (in this case, the commutative fragment). Since many systems are expressed in the sequent calculus, our method appears to be entirely general; still it is independent of the sequent calculus and of a complete semantics.

Splitting can be best understood by considering a sequent system with no weakening and absorption (or contraction). Consider for example multiplicative linear logic: If we have a proof of the sequent $\vdash F\{A \otimes B\}, \Gamma$, where $F\{A \otimes B\}$ is a formula that contains the subformula $A \otimes B$, we know for sure that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context. We are in the following situation:



corresponds to

$$
\begin{array}{c}
\Pi_1 \Vert \\
[A, \Phi] \\
\Pi_2 \Vert \\
\mathsf{s}\,\dfrac{([A, \Phi], [B, \Psi])}{[([A, \Phi], B), \Psi]} \\
\mathsf{s}\,\dfrac{}{[(A, B), \Phi, \Psi]} \\
\Delta \Vert \\
[F(A, B), \Gamma]
\end{array}
\quad .
$$

We can consider, as shown at the left, the proof for the given sequent as composed of three pieces, Δ, Π_1 and Π_2. In the calculus of structures, many different proofs correspond to the sequent calculus one: they differ for the possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all these proofs, there must also be one that fits the scheme at the right of the figure above. This precisely illustrates the idea behind the splitting technique.

The derivation Δ above implements a *context reduction* and a proper splitting. We can state, in general, these principles as follows:

1. Context reduction: If $S\{R\}$ is provable, then $S\{\ \}$ can be reduced to the structure $[\{\ \}, U]$, such that $[R, U]$ is provable. In the example above, $[F\{\ \}, \Gamma]$ is reduced to $[\{\ \}, \Gamma']$, for some Γ'.

2. Splitting: If $[(R, T), P]$ is provable, then P can be reduced to $[P_1, P_2]$, such that $[R, P_1]$ and $[T, P_2]$ are provable. In the example above Γ' is reduced to $[\Phi, \Psi]$. Splitting holds for all logical operators.

Context reduction is in turn proved by splitting, which is then at the core of the matter. The biggest difficulty resides in proving splitting, and this mainly requires finding the right induction measure.

4.1 Splitting

In this section we will state and prove splitting, as we will need it for cut elimination. Before, let us explain the induction measure that will be used in the proofs. It is based on the size of a structure and the length of a proof.

4.1 Definition The *size* of a NEL structure R, denoted by $\mathsf{size}\,R$, is inductively defined as follows:

$$\begin{array}{rcl} \mathsf{size}\,\circ &=& 0 \quad , \\ \mathsf{size}\,a &=& 1 \quad , \\ \mathsf{size}[R,T] = \mathsf{size}(R,T) = \mathsf{size}\langle R;T\rangle &=& \mathsf{size}\,R + \mathsf{size}\,T \quad , \\ \mathsf{size}\,!R = \mathsf{size}\,?R &=& \mathsf{size}\,R \quad . \end{array}$$

4.2 Example The structure $R = [!(a, \bar{c}, \langle ?\bar{a}; b\rangle), c]$ has $\mathsf{size}\,R = 5$.

4.3 Definition The *size* of a proof $\begin{array}{c}\Pi \| \mathsf{NEL} \\ R\end{array}$ is the pair

$$\mathsf{size}\,\Pi = \langle \mathsf{size}\,R, \mathrm{length}(\Pi)\rangle \quad ,$$

where $\mathrm{length}(\Pi)$ is the number of rule instances applied in Π. Given two proofs $\begin{array}{c}\Pi \| \mathsf{NEL} \\ R\end{array}$ and $\begin{array}{c}\Pi' \| \mathsf{NEL} \\ R'\end{array}$, define

$$\begin{array}{rcll} \mathsf{size}\,\Pi < \mathsf{size}\,\Pi' & \iff & \mathsf{size}\,R < \mathsf{size}\,R' & \text{or} \\ & & \mathsf{size}\,R = \mathsf{size}\,R' & \text{and} \quad \mathrm{length}(\Pi) < \mathrm{length}(\Pi') \quad . \end{array}$$

The lexicographic order $<$ defined on the size of proofs is, of course, well-founded and can therefore be used in an induction proof.

4.4 Definition For notational convenience, we define *system* NELm to be the system obtained from NEL by removing the non-core rules:

$$\mathsf{NELm} = \mathsf{NEL} \setminus \{\mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\} = \{\circ\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{p}\!\downarrow\} \quad .$$

4.5 Lemma (Splitting) *Let R, T, P be any NEL structures.*

(a) *If $[(R,T),P]$ is provable in NELm, then there are structures P_R and P_T, such that*

$$\begin{array}{c}[P_R, P_T] \\ \| \mathsf{NELm} \\ P\end{array} \quad \textit{and} \quad \begin{array}{c} \| \mathsf{NELm} \\ [R, P_R]\end{array} \quad \textit{and} \quad \begin{array}{c} \| \mathsf{NELm} \\ [T, P_T]\end{array} \quad .$$

(b) *If $[\langle R;T\rangle,P]$ is provable in NELm, then there are structures P_R and P_T, such that*

$$\begin{array}{c}\langle P_R; P_T\rangle \\ \| \mathsf{NELm} \\ P\end{array} \quad \textit{and} \quad \begin{array}{c} \| \mathsf{NELm} \\ [R, P_R]\end{array} \quad \textit{and} \quad \begin{array}{c} \| \mathsf{NELm} \\ [T, P_T]\end{array} \quad .$$

Proof: If $R = \circ$ or $T = \circ$, then the both statements are trivially true. Let us now consider the case $R \neq \circ \neq T$. Both statements are proved simultaneously by induction on the size of the proof of $[(R,T),P]$ or $[\langle R;T\rangle,P]$, respectively.

(a) Consider the bottommost rule instance ρ in the proof $\begin{array}{c}\Pi \Vert \mathsf{NELm} \\ [(R,T),P]\end{array}$. We can assume that the application of ρ is not trivial. We can distinguish between three different cases:

(1) The redex of ρ is inside R, T or P.

(2) The substructure (R,T) is inside a passive structure of the redex of ρ.

(3) The substructure (R,T) is inside an active structure of the redex of ρ.

Compared with a cut elimination proof in the sequent calculus, case (2) is similar to a commutative case and case (3) is similar to a key case. There is no counterpart to case (1) in the sequent calculus because there is no possibility of deep inference.

(1) The redex of ρ is inside R, T or P. There are three subcases.

(i) The redex of ρ is inside R. Then the proof Π has the shape

$$\rho \frac{\begin{array}{c}\Pi' \Vert \mathsf{NELm} \\ [(R',T),P]\end{array}}{[(R,T),P]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\begin{array}{c}[P_{R'},P_T] \\ \Delta_P \Vert \mathsf{NELm} \\ P\end{array} \qquad \text{and} \qquad \begin{array}{c}\Pi'_R \Vert \mathsf{NELm} \\ [R',P_{R'}]\end{array} \qquad \text{and} \qquad \begin{array}{c}\Pi_T \Vert \mathsf{NELm} \\ [T,P_T]\end{array} \quad .$$

Let $P_R = P_{R'}$. From Π'_R, we can get

$$\rho \frac{\begin{array}{c}\Pi'_R \Vert \mathsf{NELm} \\ [R',P_{R'}]\end{array}}{[R,P_{R'}]} \quad .$$

(ii) The redex of ρ is inside T. Analogous to the previous case.

(iii) The redex of ρ is inside P. Then the proof Π has the shape

$$\rho \frac{\begin{array}{c}\Pi' \Vert \mathsf{NELm} \\ [(R,T),P']\end{array}}{[(R,T),P]} \quad .$$

By applying the induction hypothesis, we get

$$\begin{array}{c}[P_R,P_T] \\ \Delta'_P \Vert \mathsf{NELm} \\ P'\end{array} \qquad \text{and} \qquad \begin{array}{c}\Pi_R \Vert \mathsf{NELm} \\ [R,P_R]\end{array} \qquad \text{and} \qquad \begin{array}{c}\Pi_T \Vert \mathsf{NELm} \\ [T,P_T]\end{array} \quad .$$

From Δ'_P, we can get

$$\begin{array}{c}[P_R, P_T]\\ \Delta'_P \Big\Vert \mathsf{NELm}\\ \rho\,\dfrac{P'}{P}\end{array} \quad .$$

(2) The substructure (R, T) is inside a passive structure of the redex of ρ. This is only possible if $\rho = \mathsf{s}$ or $\rho = \mathsf{q}{\downarrow}$. There are again three subcases.

(i) $\rho = \mathsf{s}$ and $P = [(P_1, P_2), P_3, P_4]$ and Π is

$$\begin{array}{c}\Pi' \Big\Vert \mathsf{NELm}\\ \mathsf{s}\,\dfrac{[([(R,T), P_1, P_3], P_2), P_4]}{[(R,T), (P_1, P_2), P_3, P_4]}\end{array} \quad .$$

By applying the induction hypothesis to Π', we get

$$\begin{array}{c}[Q_1, Q_2]\\ \Delta_1 \Big\Vert \mathsf{NELm}\\ P_4\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_1 \Big\Vert \mathsf{NELm}\\ [(R,T), P_1, P_3, Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_2 \Big\Vert \mathsf{NELm}\\ [P_2, Q_2]\end{array} \quad .$$

We can apply the induction hypothesis again to Π_1 (because the instance of s is not trivial). We get

$$\begin{array}{c}[P_R, P_T]\\ \Delta_2 \Big\Vert \mathsf{NELm}\\ [P_1, P_3, Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_R \Big\Vert \mathsf{NELm}\\ [R, P_R]\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_T \Big\Vert \mathsf{NELm}\\ [T, P_T]\end{array} \quad .$$

We can now build

$$\begin{array}{c}[P_R, P_T]\\ \Delta_2 \Big\Vert \mathsf{NELm}\\ [P_1, P_3, Q_1]\\ \Pi_2 \Big\Vert \mathsf{NELm}\\ \mathsf{s}\,\dfrac{[(P_1, [P_2, Q_2]), P_3, Q_1]}{[(P_1, P_2), P_3, Q_1, Q_2]}\\ \Delta_1 \Big\Vert \mathsf{NELm}\\ [(P_1, P_2), P_3, P_4]\end{array} \quad .$$

(ii) $\rho = \mathsf{q}{\downarrow}$ and $P = [\langle P_1; P_2\rangle, P_3, P_4]$ and Π is

$$\begin{array}{c}\Pi' \Big\Vert \mathsf{NELm}\\ \mathsf{q}{\downarrow}\,\dfrac{[\langle [(R,T), P_1, P_3]; P_2\rangle, P_4]}{[(R,T), \langle P_1; P_2\rangle, P_3, P_4]}\end{array} \quad .$$

By applying the induction hypothesis to Π', we get

$$\begin{array}{c}\langle Q_1; Q_2\rangle\\ \Delta_1 \Big\Vert \mathsf{NELm}\\ P_4\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_1 \Big\Vert \mathsf{NELm}\\ [(R,T), P_1, P_3, Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_2 \Big\Vert \mathsf{NELm}\\ [P_2, Q_2]\end{array} \quad .$$

We can apply the induction hypothesis again to Π_1 (because the instance of $\mathsf{q}{\downarrow}$ is not trivial). We get

$$
\begin{array}{c}[P_R, P_T]\\ \Delta_2 \Vert \mathsf{NELm}\\ [P_1, P_3, Q_1]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_R \overline{\Vert} \mathsf{NELm}\\ [R, P_R]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_T \overline{\Vert} \mathsf{NELm}\\ [T, P_T]\end{array}
\quad .
$$

We can now build

$$
\begin{array}{c}
[P_R, P_T]\\
\Delta_2 \Vert \mathsf{NELm}\\
[P_1, P_3, Q_1]\\
\Pi_2 \Vert \mathsf{NELm}\\
\mathsf{q}{\downarrow}\dfrac{[\langle [P_1, Q_1]; [P_2, Q_2]\rangle, P_3]}{[\langle P_1; P_2\rangle, P_3, \langle Q_1; Q_2\rangle]}\\
\Delta_1 \Vert \mathsf{NELm}\\
[\langle P_1; P_2\rangle, P_3, P_4]
\end{array}
\quad .
$$

(iii) $\rho = \mathsf{q}{\downarrow}$ and $P = [\langle P_1; P_2\rangle, P_3, P_4]$ and Π is

$$
\begin{array}{c}
\Pi' \overline{\Vert} \mathsf{NELm}\\
\mathsf{q}{\downarrow}\dfrac{[\langle P_1; [(R,T), P_2, P_3]\rangle, P_4]}{[(R,T), \langle P_1; P_2\rangle, P_3, P_4]}
\end{array}
\quad .
$$

Analogous to the previous case.

(3) The substructure (R,T) is inside an active structure of the redex of ρ. In this case we have only one possibility:

(i) $\rho = \mathsf{s}$, $R = (R_1, R_2)$, $T = (T_1, T_2)$, $P = [P_1, P_2]$ and Π is

$$
\begin{array}{c}
\Pi' \overline{\Vert} \mathsf{NELm}\\
\mathsf{s}\dfrac{[([(R_1, T_1), P_1], R_2, T_2), P_2]}{[(R_1, R_2, T_1, T_2), P_1, P_2]}
\end{array}
\quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c}[Q_1, Q_2]\\ \Delta_1 \Vert \mathsf{NELm}\\ P_2\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_1 \overline{\Vert} \mathsf{NELm}\\ [(R_1, T_1), P_1, Q_1]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_2 \overline{\Vert} \mathsf{NELm}\\ [(R_2, T_2), Q_2]\end{array}
\quad .
$$

Since the instance of s is not trivial, we can apply the induction hypothesis to Π_1 and Π_2 and get

$$
\begin{array}{c}[P_{R_1}, P_{T_1}]\\ \Delta_3 \Vert \mathsf{NELm}\\ [P_1, Q_1]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_{R_1} \overline{\Vert} \mathsf{NELm}\\ [R_1, P_{R_1}]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_{T_1} \overline{\Vert} \mathsf{NELm}\\ [T_1, P_{T_1}]\end{array}
\quad\text{and}
$$

$$
\begin{array}{c}[P_{R_2}, P_{T_2}]\\ \Delta_4 \Vert \mathsf{NELm}\\ Q_2\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_{R_2} \overline{\Vert} \mathsf{NELm}\\ [R_2, P_{R_2}]\end{array}
\quad\text{and}\quad
\begin{array}{c} \Pi_{T_2} \overline{\Vert} \mathsf{NELm}\\ [T_2, P_{T_2}]\end{array}
\quad .
$$

Now let $P_R = [P_{R_1}, P_{R_2}]$ and $P_T = [P_{T_1}, P_{T_2}]$. We can build

$$
\begin{array}{c}
[P_{R_1}, P_{R_2}, P_{T_1}, P_{T_2}] \\
\Delta_4 \Vert \mathsf{NELm} \\
[P_{R_1}, P_{T_1}, Q_2] \\
\Delta_3 \Vert \mathsf{NELm} \\
[P_1, Q_1, Q_2] \\
\Delta_1 \Vert \mathsf{NELm} \\
[P_1, P_2]
\end{array}
\qquad \text{and}
$$

$$
\begin{array}{c}
\Pi_{R_1} \Vert \mathsf{NELm} \\
[R_1, P_{R_1}] \\
\Pi_{R_2} \Vert \mathsf{NELm} \\
\mathsf{s}\,\dfrac{[(R_1, [R_2, P_{R_2}]), P_{R_1}]}{[(R_1, R_2), P_{R_1}, P_{R_2}]}
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_{T_1} \Vert \mathsf{NELm} \\
[T_1, P_{T_1}] \\
\Pi_{T_2} \Vert \mathsf{NELm} \\
\mathsf{s}\,\dfrac{[(T_1, [T_2, P_{T_2}]), P_{T_1}]}{[(T_1, T_2), P_{T_1}, P_{T_2}]}
\end{array}
\quad .
$$

(b) This is very similar to (a). Consider the bottommost rule instance ρ in the proof $\begin{array}{c}\Pi \Vert \mathsf{NELm} \\ [\langle R; T\rangle, P]\end{array}$. We can assume that the application of ρ is not trivial. Again, we distinguish between three cases:

(1) The redex of ρ is inside R, T or P. Analogous to (a.1).

(2) The substructure $\langle R; T\rangle$ is inside a passive structure of the redex of ρ. This case is the same as (a.2), with the only difference that the derivation

$$
\begin{array}{c}
[P_R, P_T] \\
\Delta_2 \Vert \mathsf{NELm} \\
[P_1, P_3, Q_1]
\end{array}
\qquad \text{is replaced by} \qquad
\begin{array}{c}
\langle P_R; P_T\rangle \\
\Delta_2 \Vert \mathsf{NELm} \\
[P_1, P_3, Q_1]
\end{array}
\quad .
$$

(3) The substructure $\langle R; T\rangle$ is inside an active structure of the redex of ρ. This case is also similar to (a.3). But here we have that $\rho = \mathsf{q}{\downarrow}$ and there are two possibilities:

(i) $R = \langle R_1; R_2\rangle$, $P = [\langle P_1; P_2\rangle, P_3]$ and Π is

$$
\begin{array}{c}
\Pi' \Vert \mathsf{NELm} \\
\mathsf{q}{\downarrow}\,\dfrac{[\langle [R_1, P_1]; [\langle R_2; T\rangle, P_2]\rangle, P_3]}{[\langle R_1; R_2; T\rangle, \langle P_1; P_2\rangle, P_3]}
\end{array}
\quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c}
\langle Q_1; Q_2\rangle \\
\Delta_1 \Vert \mathsf{NELm} \\
P_3
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_1 \Vert \mathsf{NELm} \\
[R_1, P_1, Q_1]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_2 \Vert \mathsf{NELm} \\
[\langle R_2; T\rangle, P_2, Q_2]
\end{array}
\quad .
$$

Since the instance of $\mathsf{q}{\downarrow}$ is not trivial, we can apply the induction hypothesis to Π_2 and get

$$
\begin{array}{c}
\langle P_{R_2}; P_T\rangle \\
\Delta_2 \Vert \mathsf{NELm} \\
[P_2, Q_2]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_{R_2} \Vert \mathsf{NELm} \\
[R_2, P_{R_2}]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_T \Vert \mathsf{NELm} \\
[T, P_T]
\end{array}
\quad .
$$

Now let $P_R = \langle [P_1, Q_1]; P_{R_2} \rangle$. We can build

$$\begin{array}{c} \langle [P_1,Q_1];P_{R_2};P_T\rangle \\ \Delta_2 \Vert \mathsf{NELm} \\ \mathsf{q}{\downarrow}\,\dfrac{\langle [P_1,Q_1];[P_2,Q_2]\rangle}{[\langle P_1;P_2\rangle,\langle Q_1;Q_2\rangle]} \\ \Delta_1 \Vert \mathsf{NELm} \\ [\langle P_1;P_2\rangle,P_3] \end{array} \quad\text{and}\quad \begin{array}{c} \Pi_1 \overline{\Vert} \mathsf{NELm} \\ [R_1,P_1,Q_1] \\ \Pi_{R_2} \Vert \mathsf{NELm} \\ \mathsf{q}{\downarrow}\,\dfrac{\langle [R_1,P_1,Q_1];[R_2,P_{R_2}]\rangle}{[\langle R_1;R_2\rangle,\langle [P_1,Q_1];P_{R_2}\rangle]} \end{array} \quad .$$

(ii) $T = \langle T_1; T_2 \rangle$, $P = [\langle P_1; P_2 \rangle, P_3]$ and Π is

$$\begin{array}{c} \Pi' \overline{\Vert} \mathsf{NELm} \\ \mathsf{q}{\downarrow}\,\dfrac{[\langle [\langle R;T_1\rangle,P_1];[T_2,P_2]\rangle,P_3]}{[\langle R;T_1;T_2\rangle,\langle P_1;P_2\rangle,P_3]} \end{array} \quad .$$

Similar to (i). □

4.6 Lemma (Splitting for Exponentials) *Let R and P be any* NEL *structures.*

(a) *If $[!R, P]$ is provable in* NELm*, then there are structures $P_1, \dots, P_h$ for some $h \geqslant 0$, such that*

$$\begin{array}{c} [?P_{R1},\dots,?P_{Rh}] \\ \Vert \mathsf{NELm} \\ P \end{array} \quad \textit{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [R,P_{R1},\dots,P_{Rh}] \end{array} \quad .$$

(b) *If $[?R, P]$ is provable in* NELm*, then there is a structure P_R, such that*

$$\begin{array}{c} !P_R \\ \Vert \mathsf{NELm} \\ P \end{array} \quad \textit{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [R,P_R] \end{array} \quad .$$

Proof: In the case of $R = \circ$, both statements are trivially true. Let us now assume that $R \neq \circ$. We will proceed by induction on the size of the proof of $[!R, P]$ (or $[?R, P]$, respectively). In other words, we use the same measure as in the previous proof.

(a) Consider the bottommost rule instance ρ in the proof $\begin{array}{c} \Pi \overline{\Vert} \mathsf{NELm} \\ [!R,P] \end{array}$. We can assume that the application of ρ is not trivial. There are again the same three cases:

(1) The redex of ρ is inside R or P. Analogous to (a.1) in the previous proof.

(2) The substructure $!R$ is inside a passive structure of the redex of ρ. This case is the same as (a.2) in the previous proof, but this time we have

$$\begin{array}{c} [?P_{R1},\dots,?P_{Rh}] \\ \Delta_2 \Vert \mathsf{NELm} \\ [P_1,P_3,Q_1] \end{array} \quad \text{instead of} \quad \begin{array}{c} [P_R,P_T] \\ \Delta_2 \Vert \mathsf{NELm} \\ [P_1,P_3,Q_1] \end{array} \quad .$$

(3) The substructure $!R$ is inside an active structure of the redex of ρ. There is only one possibility

(i) $\rho = \mathsf{p}{\downarrow}$, $P = [?P_1, P_2]$ and Π is

$$\mathsf{p}{\downarrow}\,\frac{\begin{array}{c}\Pi' \big\Vert \mathsf{NELm}\\ [![R,P_1],P_2]\end{array}}{[!R,?P_1,P_2]} \quad .$$

By applying the induction hypothesis to Π', we get structures $?P_{R2}$, $\dots$, $?P_{Rh}$ such that

$$\begin{array}{c}[?P_{R2},\dots,?P_{Rh}]\\ \big\Vert \mathsf{NELm}\\ P_2\end{array} \qquad \text{and} \qquad \begin{array}{c}\big\Vert \mathsf{NELm}\\ [R,P_1,P_{R2},\dots,P_{Rh}]\end{array} \quad .$$

Now let $P_{R1} = P_1$. We immediately get

$$\begin{array}{c}[?P_1,?P_{R2},\dots,?P_{Rh}]\\ \big\Vert \mathsf{NELm}\\ [?P_1,P_2]\end{array} \quad .$$

(b) Consider the bottommost rule instance ρ in the proof $\begin{array}{c}\Pi \big\Vert \mathsf{NELm}\\ [?R,P]\end{array}$. We can assume that the application of ρ is not trivial. There are again the same three cases:

(1) The redex of ρ is inside R or P. Analogous to (a.1) in the previous proof.

(2) The substructure $?R$ is inside a passive structure of the redex of ρ. This case is the same as (a.2) in the previous proof, but this time we have

$$\begin{array}{c}!P_R\\ \Delta_2 \big\Vert \mathsf{NELm}\\ [P_1,P_3,Q_1]\end{array} \qquad \text{instead of} \qquad \begin{array}{c}[P_R,P_T]\\ \Delta_2 \big\Vert \mathsf{NELm}\\ [P_1,P_3,Q_1]\end{array} \quad .$$

(3) The substructure $?R$ is inside an active structure of the redex of ρ. There is only one possibility

(i) $\rho = \mathsf{p}{\downarrow}$, $P = [!P_1, P_2]$ and Π is

$$\mathsf{p}{\downarrow}\,\frac{\begin{array}{c}\Pi' \big\Vert \mathsf{NELm}\\ [![R,P_1],P_2]\end{array}}{[?R,!P_1,P_2]} \quad .$$

By applying part (a), we get

$$\begin{array}{c}[?Q_1,\dots,?Q_h]\\ \big\Vert \mathsf{NELm}\\ P_2\end{array} \qquad \text{and} \qquad \begin{array}{c}\big\Vert \mathsf{NELm}\\ [R,P_1,Q_1,\dots,Q_h]\end{array} \quad .$$

Now let $P_R = [P_1, Q_1, \ldots, Q_h]$. We can build:

$$\begin{array}{c} ![P_1, Q_1, \ldots, Q_h] \\ \| \{\mathsf{p}\!\downarrow\} \\ [!P_1, ?Q_1, \ldots, ?Q_h] \\ \| \mathsf{NELm} \\ [!P_1, P_2] \end{array} .$$

□

4.7 Lemma (Splitting for Atoms) *Let a be any atom and P be any* NEL *structure.*

$$\textit{If there is a proof} \quad \begin{array}{c} \| \mathsf{NELm} \\ [a, P] \end{array} \quad \textit{then there is a derivation} \quad \begin{array}{c} \bar{a} \\ \| \mathsf{NELm} \\ P \end{array} .$$

Proof: This is very similar to the previous two proofs. Consider the bottommost rule instance ρ in the proof $\begin{array}{c} \Pi \| \mathsf{NELm} \\ [a, P] \end{array}$, where the application of ρ is not trivial. There are again the same three cases:

(1) The redex of ρ is inside P. As before.

(2) The atom a is inside a passive structure of the redex of ρ. This case is analogous to the case (2) in the previous two proofs. But this time we have

$$\begin{array}{c} \bar{a} \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array} \quad \text{instead of} \quad \begin{array}{c} [P_R, P_T] \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array} .$$

(3) The atom a is inside an active structure of the redex of ρ. There is only one possibility: $\rho = \mathsf{ai}\!\downarrow$, $P = [\bar{a}, P_1]$ and Π is

$$\mathsf{ai}\!\downarrow \frac{\begin{array}{c} \Pi' \| \mathsf{NELm} \\ P_1 \end{array}}{[a, \bar{a}, P_1]} .$$

Then, we immediately get $\begin{array}{c} \bar{a} \\ \| \mathsf{NELm} \\ [\bar{a}, P_1] \end{array}$. □

4.2 Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \}, P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

4.8 Lemma (Context Reduction) *Let R be a* NEL *structure and $S\{\ \}$ be a context. If $S\{R\}$ is provable in* NELm*, then there is a structure P_R, such that $[R, P_R]$ is provable in* NELm *and such that for every structure X, we have*

$$\begin{array}{c}[X, P_R]\\ \Vert\mathsf{NELm}\\ S\{X\}\end{array} \quad \textit{or} \quad \begin{array}{c}![X, P_R]\\ \Vert\mathsf{NELm}\\ S\{X\}\end{array} \quad .$$

Proof: This proof will be carried out by structural induction on the context $S\{\ \}$.

(1) $S\{\ \} = \{\ \}$. Then the lemma is trivially true for $P_R = \circ$.

(2) $S\{\ \} = [S'\{\ \}, P]$ for some P, such that $S\{R\}$ is not a proper par.

(i) $S'\{\ \} = \{\ \}$. Then the lemma is trivially true for $P_R = P$.

(ii) $S'\{\ \} = (S''\{\ \}, T)$ for some context $S''\{\ \}$ and structure $T \neq \circ$. Then we can apply splitting (Lemma 4.5) to the proof of $[(S''\{R\}, T), P]$ and get:

$$\begin{array}{c}[P_S, P_T]\\ {\scriptstyle\Delta_P}\Vert\mathsf{NELm}\\ P\end{array} \quad \text{and} \quad \begin{array}{c}{\scriptstyle\Pi_S}\Vert\mathsf{NELm}\\ [S''\{R\}, P_S]\end{array} \quad \text{and} \quad \begin{array}{c}{\scriptstyle\Pi_T}\Vert\mathsf{NELm}\\ [T, P_T]\end{array} \quad .$$

By applying the induction hypothesis we get P_R such that $\begin{array}{c}\Vert\mathsf{NELm}\\ [R, P_R]\end{array}$ and for every X

$$\begin{array}{c}[X, P_R]\\ \Vert\mathsf{NELm}\\ [S''\{X\}, P_S]\end{array} \quad \text{or} \quad \begin{array}{c}![X, P_R]\\ \Vert\mathsf{NELm}\\ [S''\{X\}, P_S]\end{array} \quad .$$

From this we can build:

$$\begin{array}{c}[X, P_R]\\ \Vert\mathsf{NELm}\\ [S''\{X\}, P_S]\\ {\scriptstyle\Pi_T}\Vert\mathsf{NELm}\\ \mathsf{s}\,\dfrac{[(S''\{X\}, [T, P_T]), P_S]}{[(S''\{X\}, T), P_S, P_T]}\\ {\scriptstyle\Delta_P}\Vert\mathsf{NELm}\\ [(S''\{X\}, T), P]\end{array} \quad \text{or} \quad \begin{array}{c}![X, P_R]\\ \Vert\mathsf{NELm}\\ [S''\{X\}, P_S]\\ {\scriptstyle\Pi_T}\Vert\mathsf{NELm}\\ \mathsf{s}\,\dfrac{[(S''\{X\}, [T, P_T]), P_S]}{[(S''\{X\}, T), P_S, P_T]}\\ {\scriptstyle\Delta_P}\Vert\mathsf{NELm}\\ [(S''\{X\}, T), P]\end{array} \quad .$$

(iii) $S'\{\ \} = \langle S''\{\ \}; T\rangle$ for some context $S''\{\ \}$ and structure $T \neq \circ$. Then we can apply splitting (Lemma 4.5) to the proof of $[\langle S''\{R\}; T\rangle, P]$ and get:

$$\begin{array}{c}\langle P_S; P_T\rangle\\ {\scriptstyle\Delta_P}\Vert\mathsf{NELm}\\ P\end{array} \quad \text{and} \quad \begin{array}{c}{\scriptstyle\Pi_S}\Vert\mathsf{NELm}\\ [S''\{R\}, P_S]\end{array} \quad \text{and} \quad \begin{array}{c}{\scriptstyle\Pi_T}\Vert\mathsf{NELm}\\ [T, P_T]\end{array} \quad .$$

By applying the induction hypothesis we get P_R such that $\begin{array}{c} \overline{\Vert}\,\mathsf{NELm} \\ [R, P_R] \end{array}$ and for every X

$$\begin{array}{c} [X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} \quad \text{or} \quad \begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} \quad .$$

From this we can build:

$$\begin{array}{c} [X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \\ \Pi_T \Vert\,\mathsf{NELm} \\ \mathsf{q}\!\downarrow \dfrac{\langle [S''\{X\}, P_S]; [T, P_T] \rangle}{[\langle S''\{X\}; T\rangle, \langle P_S; P_T \rangle]} \\ \Delta_P \Vert\,\mathsf{NELm} \\ [\langle S''\{X\}; T\rangle, P] \end{array} \quad \text{or} \quad \begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \\ \Pi_T \Vert\,\mathsf{NELm} \\ \mathsf{q}\!\downarrow \dfrac{\langle [S''\{X\}, P_S]; [T, P_T] \rangle}{[\langle S''\{X\}; T\rangle, \langle P_S; P_T \rangle]} \\ \Delta_P \Vert\,\mathsf{NELm} \\ [\langle S''\{X\}; T\rangle, P] \end{array} \quad .$$

(iv) $S'\{\ \} = !S''\{\ \}$ for some context $S''\{\ \}$. Then we can apply splitting (Lemma 4.6) to the proof of $[!S''\{R\}, P]$ and get:

$$\begin{array}{c} [?P_1, \dots, ?P_h] \\ \Delta_P \Vert\,\mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_S \overline{\Vert}\,\mathsf{NELm} \\ [S''\{R\}, P_1, \dots, P_h] \end{array} \quad .$$

By applying the induction hypothesis to Π_S we get P_R such that $\begin{array}{c} \overline{\Vert}\,\mathsf{NELm} \\ [R, P_R] \end{array}$ and for every X

$$\begin{array}{c} [X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_1, \dots, P_h] \end{array} \quad \text{or} \quad \begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_1, \dots, P_h] \end{array} \quad .$$

From this we can build:

$$\begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ ![S''\{X\}, P_1, \dots, P_h] \\ \Vert\,\{\mathsf{p}\!\downarrow\} \\ [!S''\{X\}, ?P_1, \dots, ?P_h] \\ \Delta_P \Vert\,\mathsf{NELm} \\ [!S''\{X\}, P] \end{array} \quad .$$

(v) $S'\{\ \} = ?S''\{\ \}$ for some context $S''\{\ \}$. Then we can apply splitting (Lemma 4.6) to the proof of $[?S''\{R\}, P]$ and get:

$$\begin{array}{c} !P_S \\ \Delta_P \Vert\,\mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_S \overline{\Vert}\,\mathsf{NELm} \\ [S''\{R\}, P_S] \end{array} \quad .$$

By applying the induction hypothesis to Π_S we get P_R such that $\begin{array}{c} \overline{\Vert}\,\mathsf{NELm} \\ [R, P_R] \end{array}$ and for every X

$$\begin{array}{c} [X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} \quad \text{or} \quad \begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} .$$

From this we can build:

$$\begin{array}{c} ![X, P_R] \\ \Vert\,\mathsf{NELm} \\ \mathsf{p}\!\downarrow \dfrac{[?S''\{X\}, !P_S]}{![S''\{X\}, P_S]} \\ \Delta_P \Vert\,\mathsf{NELm} \\ [?S''\{X\}, P] \end{array} .$$

(3) $S\{\ \} = (S'\{\ \}, T)$ for some context $S'\{\ \}$ and $T \neq \circ$. This is a special case of (2.ii) for $P = \circ$.

(4) $S\{\ \} = \langle S'\{\ \}; T\rangle$ for some context $S'\{\ \}$ and $T \neq \circ$. This is a special case of (2.iii) for $P = \circ$.

(5) $S\{\ \} = !S'\{\ \}$ for some context $S'\{\ \}$. This is a special case of (2.iv) for $P = \circ$.

(6) $S\{\ \} = ?S'\{\ \}$ for some context $S'\{\ \}$. This is a special case of (2.v) for $P = \circ$.

□

4.3 Elimination of the Up Fragment

In this section, we will first show three lemmata, which are all easy consequences of splitting and which say that the core up rules of system SNEL are admissible if they are applied in a shallow context $[\{\ \}, P]$. Then we will show how context reduction is used to extend these lemmata to any context. As a result, we get a proof of cut elimination that can be considered modular, in the sense that the three core up rules $\mathsf{ai}\!\uparrow$, $\mathsf{q}\!\uparrow$, and $\mathsf{p}\!\uparrow$ are shown to be admissible independently from each other.

4.9 Lemma *Let P be a structure and let a be an atom. If $[(a, \bar{a}), P]$ is provable in* NELm*, then P is also provable in* NELm*.*

Proof: Apply splitting to the proof of $[(a, \bar{a}), P]$. This yields:

$$\begin{array}{c} [P_a, P_{\bar{a}}] \\ \Vert\,\mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert}\,\mathsf{NELm} \\ [a, P_a] \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert}\,\mathsf{NELm} \\ [\bar{a}, P_{\bar{a}}] \end{array} .$$

By applying Lemma 4.7, we get

$$\begin{array}{c} \bar{a} \\ \Vert\,\mathsf{NELm} \\ P_a \end{array} \quad \text{and} \quad \begin{array}{c} a \\ \Vert\,\mathsf{NELm} \\ P_{\bar{a}} \end{array} .$$

From this we can build:

$$\begin{array}{c}
\circ\!\downarrow \dfrac{\quad}{\circ} \\
\mathsf{ai}\!\downarrow \dfrac{\circ}{[\bar{a}, a]} \\
\Vert \mathsf{NELm} \\
[P_a, P_{\bar{a}}] \\
\Vert \mathsf{NELm} \\
P
\end{array} \quad .$$

□

4.10 Lemma *Let R, T, U, V and P be any* NEL *structures. If $[(\langle R; U\rangle, \langle T; V\rangle), P]$ is provable in* NELm*, then $[\langle (R,T); (U,V)\rangle, P]$ is also provable in* NELm.

Proof: By applying splitting to the proof of $[(\langle R; U\rangle, \langle T; V\rangle), P]$, we get two structures P_1 and P_2 such that:

$$\begin{array}{c} [P_1, P_2] \\ \Vert \mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [\langle R; U\rangle, P_1] \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [\langle T; V\rangle, P_2] \end{array} \quad .$$

By applying splitting again, we get P_R, P_T, P_U and P_V such that

$$\begin{array}{c} \langle P_R, P_U\rangle \\ \Vert \mathsf{NELm} \\ P_1 \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [R, P_R] \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [U, P_U] \end{array} \quad \text{and}$$

$$\begin{array}{c} \langle P_T, P_V\rangle \\ \Vert \mathsf{NELm} \\ P_2 \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [T, P_T] \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [V, P_V] \end{array} \quad .$$

From this we can build:

$$\begin{array}{c}
\overline{\Vert} \mathsf{NELm} \\
\langle ([R, P_R], [T, P_T]); ([U, P_U], [V, P_V])\rangle \\
\Vert \{\mathsf{s}\} \\
\langle [(R,T), P_R, P_T]; [(U,V), P_U, P_V]\rangle \\
\Vert \{\mathsf{q}\!\downarrow\} \\
[\langle (R,T); (U,V)\rangle, \langle P_R, P_U\rangle, \langle P_T, P_V\rangle] \\
\Vert \mathsf{NELm} \\
[\langle (R,T); (U,V)\rangle, P_1, P_2] \\
\Vert \mathsf{NELm} \\
[\langle (R,T); (U,V)\rangle, P]
\end{array} \quad .$$

□

4.11 Lemma *Let R, T and P be any* NEL *structures. If $[(?R, !T), P]$ is provable in* NELm*, then $[?(R,T), P]$ is also provable in* NELm.

Proof: By applying splitting to the proof of $[(?R, !T), P]$, we get two structures P_1 and P_2 such that:

$$\begin{array}{c} [P_1, P_2] \\ \Vert \mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [?R, P_1] \end{array} \quad \text{and} \quad \begin{array}{c} \overline{\Vert} \mathsf{NELm} \\ [!T, P_2] \end{array} \quad .$$

By applying splitting again, we get P_R, P_{T1},. . . ,P_{Tn}, such that

$$\begin{array}{c} !P_R \\ \Vert \mathsf{NELm} \\ P_1 \end{array} \quad \text{and} \quad \begin{array}{c} \prod \mathsf{NELm} \\ [R, P_R] \end{array} \quad \text{and}$$

$$\begin{array}{c} [?P_{T1}, \dots, ?P_{Tn}] \\ \Vert \mathsf{NELm} \\ P_2 \end{array} \quad \text{and} \quad \begin{array}{c} \prod \mathsf{NELm} \\ [T, P_{T1}, \dots, P_{Tn}] \end{array} \quad .$$

From this we can build:

$$\begin{array}{c} \prod \mathsf{NELm} \\ !([R, P_R], [T, P_{T1}, \dots, P_{Tn}]) \\ \Vert \{\mathsf{s}\} \\ ![(R, T), P_R, P_{T1}, \dots, P_{Tn}] \\ \Vert \{\mathsf{p}\!\downarrow\} \\ [?(R, T), !P_R, ?P_{T1}, \dots, ?P_{Tn}] \\ \Vert \mathsf{NELm} \\ [?(R, T), P_1, P_2] \\ \Vert \mathsf{NELm} \\ [?(R, T), P] \end{array} \quad .$$

□

By the use of context reduction, we can extent the statements of Lemmata 4.9, 4.10, and 4.11 from shallow contexts $[\{\ \}, P]$ to arbitrary contexts $S\{\ \}$.

4.12 Lemma *Let R, T, U and V be any structures, let a be an atom and let $S\{\ \}$ be any context. Then we have the following*

- *If* $\begin{array}{c} \prod \mathsf{NELm} \\ S(a, \bar{a}) \end{array}$, *then* $\begin{array}{c} \prod \mathsf{NELm} \\ S\{\circ\} \end{array}$.
- *If* $\begin{array}{c} \prod \mathsf{NELm} \\ S(\langle R; U\rangle, \langle T; V\rangle) \end{array}$, *then* $\begin{array}{c} \prod \mathsf{NELm} \\ S\langle (R, T); (U, V)\rangle \end{array}$.
- *If* $\begin{array}{c} \prod \mathsf{NELm} \\ S(?R, !T) \end{array}$, *then* $\begin{array}{c} \prod \mathsf{NELm} \\ S\{?(R, T)\} \end{array}$.

Proof: All three statements are proved similarly. We will here show only the third: Let

$$\begin{array}{c} \prod \mathsf{NELm} \\ S(?R, !T) \end{array}$$

be given. Apply context reduction, to get a structure P, such that $\begin{array}{c} \prod \mathsf{NELm} \\ [(?R, !T), P] \end{array}$

and for all structures X, either $\begin{array}{c}[X,P]\\ \Vert\,\mathsf{NELm}\\ S\{X\}\end{array}$ or $\begin{array}{c}![X,P]\\ \Vert\,\mathsf{NELm}\\ S\{X\}\end{array}$. In particular, we have

$$\begin{array}{c}[?(R,T),P]\\ \Vert\,\mathsf{NELm}\\ S\{?(R,T)\}\end{array} \quad\text{or}\quad \begin{array}{c}![?(R,T),P]\\ \Vert\,\mathsf{NELm}\\ S\{?(R,T)\}\end{array} \quad .$$

By Lemma 4.11 there is a proof $\begin{array}{c}\overline{\Vert}\,\mathsf{NELm}\\ [?(R,T),P]\end{array}$, from which we get $\begin{array}{c}\overline{\Vert}\,\mathsf{NELm}\\ S\{?(R,T)\}\end{array}$. □

Now we can very easily give a proof for the cut elimination theorem for system NEL.

Proof of Theorem 2.12: First, we apply to a given proof $\begin{array}{c}\overline{\Vert}\,\mathsf{SNEL}\cup\{\circ\}\\ R\end{array}$ the second decomposition theorem, which yields a proof

$$\begin{array}{c}\circ\!\downarrow\,\overline{}\\ \circ\\ \Vert\,\{\mathsf{ai}\!\downarrow\}\\ R_4\\ \Vert\,\mathsf{SNELc}\\ R_3\\ \Vert\,\{\mathsf{ai}\!\uparrow\}\\ R_2\\ \Vert\,\{\mathsf{w}\!\downarrow\}\\ R_1\\ \Vert\,\{\mathsf{b}\!\downarrow\}\\ R\end{array}$$

for some structures R_1, R_2, R_3 and R_4. The instances of $\mathsf{b}\!\uparrow$ and $\mathsf{w}\!\uparrow$ disappear because their premise is the unit $\circ$. The obtained derivation can also bee seen as

$$\begin{array}{c}\Pi\,\overline{\Vert}\,\mathsf{NELm}\cup\{\mathsf{ai}\!\uparrow,\mathsf{q}\!\uparrow,\mathsf{p}\!\uparrow\}\\ R_2\\ \Vert\,\{\mathsf{w}\!\downarrow\}\\ R_1\\ \Vert\,\{\mathsf{b}\!\downarrow\}\\ R\end{array} \quad .$$

All instances of $\mathsf{ai}\!\uparrow$, $\mathsf{q}\!\uparrow$ and $\mathsf{p}\!\uparrow$ are now eliminated from Π, starting with the topmost instance: Let $\rho \in \{\mathsf{ai}\!\uparrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow\}$ and replace

$$
\begin{array}{c}
\overline{\Vert}\,\mathsf{NELm} \\
\rho\,\dfrac{S\{W\}}{S\{Z\}} \\
\Vert\,\mathsf{NELm}\cup\{\mathsf{ai}\!\uparrow,\mathsf{q}\!\uparrow,\mathsf{p}\!\uparrow\} \\
R_2 \\
\Vert\,\{\mathsf{w}\!\downarrow\} \\
R_1 \\
\Vert\,\{\mathsf{b}\!\downarrow\} \\
R
\end{array}
\qquad \text{by} \qquad
\begin{array}{c}
\overline{\Vert}\,\mathsf{NELm} \\
S\{Z\} \\
\Vert\,\mathsf{NELm}\cup\{\mathsf{ai}\!\uparrow,\mathsf{q}\!\uparrow,\mathsf{p}\!\uparrow\} \\
R_2 \\
\Vert\,\{\mathsf{w}\!\downarrow\} \\
R_1 \\
\Vert\,\{\mathsf{b}\!\downarrow\} \\
R
\end{array}
\quad ,
$$

by applying Lemma 4.12. □

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules than cut. So, our technique is much more general than cut elimination in the sequent calculus, for two reasons:

1. it applies to operators that admit no sequent calculus definition, as seq;
2. it can be used to show admissibility of non-infinitary rules that involve no negation, like $\mathsf{q}\!\uparrow$ and $\mathsf{p}\!\uparrow$.

4.13 Corollary *For every proof* $\begin{array}{c}\overline{\Vert}\,\mathsf{NEL} \\ R\end{array}$ *, there is a proof*

$$
\begin{array}{c}
\Pi\,\overline{\Vert}\,\mathsf{NELm} \\
R_2 \\
\Vert\,\{\mathsf{w}\!\downarrow\} \\
R_1 \\
\Vert\,\{\mathsf{b}\!\downarrow\} \\
R
\end{array}
\quad ,
$$

for some structures R_1 and R_2.

5 Conclusions and Future Work

We have shown a class of logical systems, built around system NEL, that integrate multiplicative commutativity and non-commutativity, together with exponentials. This has been done in the formalism of the calculus of structures, which allows us to obtain very simple systems. In addition, we get properties of locality, atomicity and modularity that do not hold in other known calculi.

System NEL was originally inspired by Retoré's pomset logic [20]. There is research in progress to show that the multiplicative fragments of his logic and ours coincide. In this case, our system and the work [28] would explain why sequentialising pomset logic has been so hard and unfruitful. It should be possible to extend our system NEL to other logical operators, perhaps to full linear logic, and also to the self-dual modality associated to Retoré's non-commutative operator [19]. In this paper we limited ourselves to the bare necessary to include MELL.

In [27], it is shown that NEL is Turing-complete. This result establishes an interesting boundary to MELL, whose decidability is still an open problem. If it turns out, as many believe, that MELL is decidable, then the boundary with undecidability is crossed by our simple extension to seq. This would give a precise technical content to the perceived difficulty of getting Turing-completeness for MELL, namely the trouble in realising the tape of a Turing machine. In this sense, our sequentiality would be even more strongly motivated by a basic computational mechanism.

One of the biggest open problems we have is understanding when and why decomposition theorems work. They seem to have a strong relation to the notion of core system, but we fail to understand the deep reasons for this. For the time being we observe that decomposition theorems hold for all logics we studied so far (classical, linear and several commutative/non-commutative systems).

The calculus of structures generalises the sequent calculus for one-sided sequent systems, which correspond to logics with involutive negation. Preliminary work shows that it is also possible to design intuitionistic systems in the calculus of structures, by way of polarities. We believe that there is a close relation to the theory of structads that has recently been developed by Lamarche [15]. The exploration of this promises to be an active area of research.

In [23], Stewart and Stouppa use the calculus of structures to present a new approach to the proof theory of modal logics, which is more systematic than it is possible in the sequent calculus.

Proving cut elimination is more difficult than in the sequent calculus. However, the methods we used are more general than the traditional ones, and, we believe, unveil some fundamental properties of logical systems that were previously hidden. We make an essential use of a top-down symmetric notion of derivation, which leads to a reduction of the cut rule into constituents which are dual to the common logical rules.

We did not attempt to base our calculus on philosophical grounds. We believe that this can only happen after several systems are thoroughly studied and discussed. For the time being we are still collecting empirical evidence.

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