

Coeffects and Graded Linear Logic (GLL)

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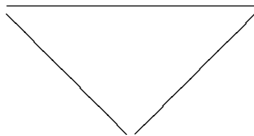
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Logic



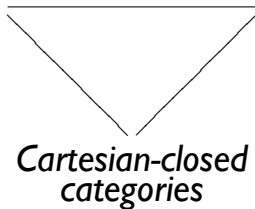
Types

Semantics

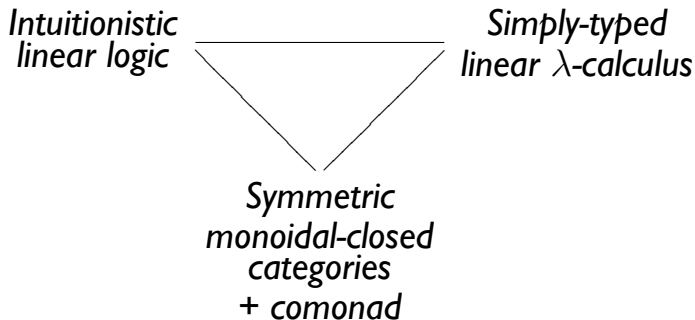
Curry-Howard-Lambek correspondence

*Intuitionistic
propositional
logic*

*Simply-typed
 λ -calculus*



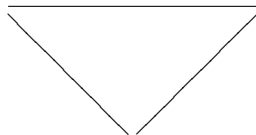
Let's be linear!



! as logical connective, type constructor, and modelled by a comonad

This line of work...

Graded
Linear Logic



Linear λ -calculus
+ *coeffacts*

Symmetric
monoidal-closed
categories
+ *graded exponential
comonad*

$!_r$ as family of logical connectives, type constructors, and modelled
by a *graded comonad*

$!r$

for $r \in \mathcal{R}$, which is a semiring

What are coeffects?

aspects of computation which “consume” the context

As a type-based analysis: variable dependency properties:

- ▶ Live-variable analysis (live/dead)
- ▶ How many times is the variable used? (reuse bounds cf. BLL)
- ▶ Information flow, e.g., security levels (High/Low)
- ▶ Variable “schedules”
- ▶ Probabilities
- ▶ Deconstructors (strictness analysis)

Whole-context dependencies (not the main focus here)

- ▶ Additional hardware resources
- ▶ Type class instances
- ▶ Library versions

Coeffect-and-type systems

- ▶ *Coeffects: Unified static analysis of context-dependence* [Petricek et al., 2013]

$$C^r \Gamma \vdash e : \tau$$

“term e has type τ , free variables Γ , and contextual requirements r ”.

- ▶ Whole context: global resource analysis and approximations to variable use
 - ▶ Dualising “type-and-effect systems”
- ▶ *Bounded Linear Types in a Resource Semiring* [Ghica and Smith, 2014]

$$x : A \cdot r, \dots, y : B \cdot s \vdash e : \tau$$

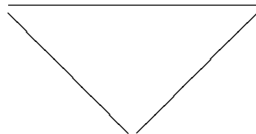
- ▶ Provided as a kind of generalisation to BLL.
 - ▶ Per variable: fine-grained resource analysis for hardware synthesis, e.g. variable schedules.
- ▶ *Coeffects: A calculus of context-dependent computation* [Petricek et al., 2014]
 - ▶ Combines per-variable and whole-context type-based analysis

Coeffect meta-languages: coeffects as first-class types

- ▶ *A Core Quantitative Coeffect Calculus* [Brunel et al., 2014]
 - ▶ Based on linear λ -calculus with indexed exponential constructor $!_r$
 - ▶ Generalises bounded linear exponential (indexed by \mathbb{N}) to arbitrary semiring \mathcal{R}
- ▶ *Modelling Coeffects in the Relational Semantics of Linear Logic*
[Breuvart and Pagani, 2015]
- ▶ *Combining Effects and Coeffects via Grading* [Gaborardi et al., 2016]
 - ▶ Adds graded monadic metalanguage for effects T_e graded by monoid \mathcal{E}
 - ▶ “Matched pair” (distributive law) between effect monoid and coeffect semiring

This line of work...

*Graded
Linear Logic*



*Linear λ -calculus
+ coefficients*

*Symmetric
monoidal-closed
categories
+ graded exponential
comonad*

Graded Linear Logic

Let $(\mathcal{R}, *, 1, +, 0)$ be a semiring of *coeffect* “grades” over the set \mathcal{R} .

Propositions:

$$\phi, \psi ::= \text{atoms} \mid \phi \Rightarrow \psi \mid !_{\textcolor{blue}{r}} \psi$$

Contexts comprise linear and “non-linear” graded hypotheses:

$$\Gamma, \Delta ::= \emptyset \mid \phi, \Gamma \mid [\phi]_{\textcolor{blue}{r}}, \Gamma$$

where $[\phi]_{\textcolor{blue}{r}}$ represent non-linear hypotheses (called *discharged*)

Some shorthand:

- ▶ $[\Gamma]$ means all hypotheses are non-linear with some grade;
- ▶ $[\Gamma]_{\textcolor{blue}{r}}$ means all hypotheses are non-linear with grade $\textcolor{blue}{r}$.

Graded Linear Logic (2), for semiring $(\mathcal{R}, *, 1, +, 0)$

$\Gamma + \Delta$ operation on contexts:

$$\begin{aligned}
 \Gamma + \emptyset &= \Gamma \\
 \emptyset + \Delta &= \Delta \\
 (\psi, \Gamma) + \Delta &= \psi, (\Gamma + \Delta) && \text{if } \psi \notin \Delta \\
 \Gamma + (\psi, \Delta) &= \psi, (\Gamma + \Delta) && \text{if } \psi \notin \Gamma \\
 [\psi]_r, \Gamma + [\psi]_s, \Delta &= [\psi]_{r+s}, (\Gamma + \Delta)
 \end{aligned}$$

Natural deduction rules:

$$\begin{array}{lcl}
 \text{ax} \frac{}{[\Delta]_0, \phi \vdash \phi} & \Rightarrow_I \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi} & \Rightarrow_E \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Delta \vdash \phi}{\Gamma + \Delta \vdash \psi} \\
 \text{der} \frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_1 \vdash \psi} & \text{pr} \frac{[\Gamma] \vdash \psi}{r*[\Gamma] \vdash!_r \psi} & \text{cut!} \frac{\Gamma \vdash!_r \phi \quad \Delta, [\phi]_r \vdash \psi}{\Gamma + \Delta \vdash \psi}
 \end{array}$$

e.g. Bounded Linear Logic [Girard et al., 1992] via $(\mathbb{N}, *, 1, +, 0)$

Example: Bounded Linear Logic

Bounded Linear Logic [Girard et al., 1992] recovered from Graded Linear Logic via $(\mathbb{N}, *, 1, +, 0)$

or, quotiented versions:

- ▶ *none-one-tons*: $(\{0, 1, \infty\}, *, 1, +, 0)$ with ∞ saturating;
- ▶ *liveness*: $(\{\top, \perp\}, \wedge, \top, \vee, \perp)$ where \top means live and \perp means dead;

Example: security levels

- ▶ Two pointed lattice $\mathcal{R} = \{\text{Lo}, \text{Hi}\}$
- ▶ additive monoid $(\mathcal{R}, \vee, \text{Lo})$
 - ▶ weaken with low security, and contract via least-upper bound;
- ▶ multiplicative $(\mathcal{R}, \wedge, \text{Hi})$

$$\text{pr} \frac{[\Gamma] \vdash \psi}{\text{Lo} \wedge [\Gamma] \vdash!_{\text{Lo}} \psi} \quad \text{pr} \frac{[\Gamma] \vdash \psi}{\text{Hi} \wedge [\Gamma] \vdash!_{\text{Hi}} \psi}$$

- ▶ ground truths of varying security level, e.g. $\vdash!_{\text{Hi}} \text{secret}$
- ▶ use *cut* for security-level observing substitution:

$$\text{cut!} \frac{\Gamma \vdash!_r \phi \quad \Delta, [\phi]_r \vdash \psi}{\Gamma + \Delta \vdash \psi}$$

- ▶ There is no way to prove $x : [\phi]_{\text{Hi}} \vdash!_{\text{Lo}} \phi$
- ▶ Generalises to security lattice.

Example: probabilistic logic

Let $\mathcal{R} = [0, 1]$ with multiplicative monoid $([0, 1], *, 1)$.

i.e. linear part has probability 1, by dereliction:

$$\text{der} \frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_1 \vdash \psi}$$

and propagation of probabilities via promotion:

$$\text{pr} \frac{x : [A]_{0.5}, y : [B]_{0.8} \vdash \psi}{x : [A]_{0.25}, y : [B]_{0.4} \vdash !_{0.5} \psi}$$

The additive part is $([0, 1], \vee, 0)$ i.e., weakened variables have probability 0 and contraction takes maximum probability.

Graded necessity \subset GLL

The multiplicative part $(\mathcal{R}, *, 1)$ of GLL yields $!_r$ as *graded necessity*.

In Hilbert-style axiomatisation:

$$(T) \quad !_1 A \rightarrow A$$

$$(4) \quad !_r !_s A \rightarrow !_r !_s A$$

$$(K) \quad !_r (A \rightarrow B) \rightarrow !_r A \rightarrow !_r B$$

$$(Nec) \quad \frac{\vdash A}{\vdash !_r A}$$

Example: LTL “neXt” as graded necessity

Recall the unary connective X from Linear Temporal Logic.
There is a graded necessity

$$!_r A = X^r A$$

with monoid $(\mathbb{N}, +, 0)$ such that:

$$(T) \quad X^0 A \rightarrow A$$

$$(4) \quad X^{r+s} A \rightarrow X^r X^s A$$

$$(K) \quad X^r (A \rightarrow B) \rightarrow X^r A \rightarrow X^r B$$

$$(Nec) \quad \frac{\vdash A}{\vdash X^r A}$$

Example: LTL “neXt” as graded necessity (2)

Back in GLL this gives us:

$$\text{der} \frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_0 \vdash \psi} \quad \text{pr} \frac{[\Gamma] \vdash \psi}{r+[\Gamma] \vdash X_r \psi}$$

e.g.

$$\text{pr} \frac{\text{der} \frac{A \vdash B}{[A]_0 \vdash B}}{[A]_2 \vdash XXB}$$

if A is true in 2 time steps then we can conclude B at 2 time steps.

What about the additive part for the rest of GLL?

- ▶ $(\mathbb{N}, \vee, 0)$ changes the meaning of X to “next and all future states”.
- ▶ **Better:** make $+$ partial: $r + r = r$ otherwise $r + s = \perp$

Semantics: Graded exponential comonad

- ▶ Symmetric monoidal-closed category \mathbb{C}
- ▶ Semiring \mathcal{R} as category (i.e. strict monoidality via bifunctors $+$ and $*$ etc.)
- ▶ and a model for $!$ via the functor D with:

Functor	$D :$	\mathcal{R}	\rightarrow	$[\mathbb{C}, \mathbb{C}]$
0-Monoidality	$m_{r,1} :$	1	\rightarrow	$D r 1$
2-Monoidality	$m_{r,A,B} :$	$D r A \otimes D r B$	\rightarrow	$D r (A \otimes B)$
Weakening	$w_A :$	$D 0 A$	\rightarrow	1
Contraction	$c_{r,s,A} :$	$D(r + s) A$	\rightarrow	$D r A \otimes D s A$
Dereliction	$\varepsilon_A :$	$D 1 A$	\rightarrow	A
Digging	$\delta_{r,s,A} :$	$D(r * s) A$	\rightarrow	$D r (D s A)$

making a number of diagrams commute.

- ▶ When the semiring is trivial gives linear exponential comonad on \mathbb{C} .
- ▶ Similar structure in all coeffect papers; here based on [Gabori et al., 2016]

Conclusions and what's next

- ▶ Coeffects: a relatively new way of understanding (intensional) program behaviour (state of the art [Gaboardi et al., 2016]).
- ▶ Graded semantics provide refined models; aids reasoning.
- ▶ Graded Linear Logic provides parametric quantitative version of LL.
- ▶ Applications to quantitative models, e.g., timed, probabilistic, communicating automata.
- ▶ Get in touch! d.a.orchard@kent.ac.uk

Thanks to my coeffect coauthors: Tomas Petricek, Alan Mycroft, Marco Gaboardi, Shin-ya Katsumata, Flavien Breuvert, Tarmo Uustalu

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