Coeffects and Graded Linear Logic (GLL)

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$$\begin{tikzcd} \textit{Logic} \arrow[rr, dash] \arrow[dr, dash] & & \textit{Types} \arrow[dl, dash] \\ & \textit{Semantics} & \end{tikzcd}$$

Curry-Howard-Lambek correspondence

$$
\begin{tikzcd}
\textit{Intuitionistic propositional logic} \arrow[rr, dash] \arrow[dr, dash] & & \textit{Simply-typed } \lambda\textit{-calculus} \arrow[dl, dash] \\
& \textit{Cartesian-closed categories} &
\end{tikzcd}
$$

Let's be linear!

Intuitionistic linear logic — *Simply-typed linear λ-calculus*

Symmetric monoidal-closed categories + comonad

$!$ as logical connective, type constructor, and modelled by a comonad

This line of work...

Graded Linear Logic

Linear λ-calculus + *coeffects*

Symmetric monoidal-closed categories + *graded exponential comonad*

$!_r$ as family of logical connectives, type constructors, and modelled by a *graded* comonad

$$!_r$$

for $r \in \mathcal{R}$, which is a semiring

What are coeffects?

aspects of computation which "consume" the context

As a type-based analysis: variable dependency properties:

- Live-variable analysis (live/dead)
- How many times is the variable used? (reuse bounds cf. BLL)
- Information flow, e.g., security levels (High/Low)
- Variable "schedules"
- Probabilities
- Deconstructors (strictness analysis)

Whole-context dependencies (not the main focus here)

- Additional hardware resources
- Type class instances
- Library versions

Coeffect-and-type systems

- *Coeffects: Unified static analysis of context-dependence* [Petricek et al., 2013]

$$C^r \Gamma \vdash e : \tau$$

"term e has type τ, free variables Γ, and contextual requirements r".

  - Whole context: global resource analysis and approximations to variable use
  - Dualising "type-and-effect systems"

- *Bounded Linear Types in a Resource Semiring* [Ghica and Smith, 2014]

$$x : A \cdot r, \ldots, y : B \cdot s \vdash e : \tau$$

  - Provided as a kind of generalisation to BLL.
  - Per variable: fine-grained resource analysis for hardware synthesis, e.g. variable schedules.

- *Coeffects: A calculus of context-dependent computation* [Petricek et al., 2014]
  - Combines per-variable and whole-context type-based analysis

Coeffect meta-languages: coeffects as first-class types

- *A Core Quantitative Coeffect Calculus* [Brunel et al., 2014]
  - Based on linear λ-calculus with indexed exponential constructor $!_r$
  - Generalises bounded linear exponential (indexed by $\mathbb{N}$) to arbitrary semiring $\mathcal{R}$
- *Modelling Coeffects in the Relational Semantics of Linear Logic* [Breuvart and Pagani, 2015]
- *Combining Effects and Coeffects via Grading* [Gaboardi et al., 2016]
  - Adds graded monadic metalanguage for effects T_e graded by monoid $\mathcal{E}$
  - "Matched pair" (distributive law) between effect monoid and coeffect semiring

This line of work...

Graded Linear Logic

Linear λ-calculus + *coeffects*

Symmetric monoidal-closed categories + *graded exponential comonad*

Graded Linear Logic

Let $(\mathcal{R}, *, 1, +, 0)$ be a semiring of *coeffect* “grades” over the set $\mathcal{R}$.

Propositions:

$$\phi, \psi ::= \textit{atoms} \mid \phi \Rightarrow \psi \mid !_r \psi$$

Contexts comprise linear and “non-linear” graded hypotheses:

$$\Gamma, \Delta ::= \emptyset \mid \phi, \Gamma \mid [\phi]_r, \Gamma$$

where $[\phi]_r$ represent non-linear hypotheses (called *discharged*)

Some shorthand:

- $[\Gamma]$ means all hypotheses are non-linear with some grade;
- $[\Gamma]_r$ means all hypotheses are non-linear with grade r.

Graded Linear Logic (2), for semiring $(\mathcal{R}, *, 1, +, 0)$

$\Gamma + \Delta$ operation on contexts:

$$
\begin{array}{rcll}
\Gamma + \emptyset & = & \Gamma & \\
\emptyset + \Delta & = & \Delta & \\
(\psi, \Gamma) + \Delta & = & \psi, (\Gamma + \Delta) & \text{if } \psi \notin \Delta \\
\Gamma + (\psi, \Delta) & = & \psi, (\Gamma + \Delta) & \text{if } \psi \notin \Gamma \\
[\psi]_r, \Gamma + [\psi]_s, \Delta & = & [\psi]_{r+s}, (\Gamma + \Delta) &
\end{array}
$$

Natural deduction rules:

$$
\text{ax}\,\frac{}{[\Delta]_0, \phi \vdash \phi}
\qquad
\Rightarrow_I \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}
\qquad
\Rightarrow_E \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Delta \vdash \phi}{\Gamma + \Delta \vdash \psi}
$$

$$
\text{der}\,\frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_1 \vdash \psi}
\qquad
\text{pr}\,\frac{[\Gamma] \vdash \psi}{r * [\Gamma] \vdash !_r \psi}
\qquad
\text{cut!}\,\frac{\Gamma \vdash !_r \phi \qquad \Delta, [\phi]_r \vdash \psi}{\Gamma + \Delta \vdash \psi}
$$

e.g. Bounded Linear Logic [Girard et al., 1992] via $(\mathbb{N}, *, 1, +, 0)$

Example: Bounded Linear Logic

Bounded Linear Logic [Girard et al., 1992] recovered from Graded Linear Logic via $(\mathbb{N}, *, 1, +, 0)$

or, quotiented versions:

- *none-one-tons*: $(\{0, 1, \infty\}, *, 1, +, 0)$ with ∞ saturating;
- *liveness*: $(\{\mathsf{T}, \bot\}, \wedge, \mathsf{T}, \vee, \bot)$ where $\top$ means live and $\bot$ means dead;

Example: security levels

- Two pointed lattice $\mathcal{R} = \{\mathsf{Lo}, \mathsf{Hi}\}$
- additive monoid $(\mathcal{R}, \vee, \mathsf{Lo})$
  - weaken with low security, and contract via least-upper bound;
- multiplicative $(\mathcal{R}, \wedge, \mathsf{Hi})$

$$\text{pr}\frac{[\Gamma] \vdash \psi}{\mathsf{Lo} \wedge [\Gamma] \vdash !_{\mathsf{Lo}} \psi} \qquad \text{pr}\frac{[\Gamma] \vdash \psi}{\mathsf{Hi} \wedge [\Gamma] \vdash !_{\mathsf{Hi}} \psi}$$

- ground truths of varying security level, e.g. $\vdash !_{\mathsf{Hi}}$ *secret*
- use *cut* for security-level observing substitution:

$$\text{cut!}\frac{\Gamma \vdash !_r \phi \qquad \Delta, [\phi]_r \vdash \psi}{\Gamma + \Delta \vdash \psi}$$

- There is no way to prove $x : [\phi]_{\mathsf{Hi}} \vdash !_{\mathsf{Lo}} \phi$
- Generalises to security lattice.

Example: probabilistic logic

Let $\mathcal{R} = [0, 1]$ with multiplicative monoid $([0, 1], *, 1)$.
i.e. linear part has probability 1, by dereliction:

$$\text{der}\,\frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_1 \vdash \psi}$$

and propagation of probabilities via promotion:

$$\text{pr}\,\frac{x : [A]_{0.5}, y : [B]_{0.8} \vdash \psi}{x : [A]_{0.25}, y : [B]_{0.4} \vdash !_{0.5}\psi}$$

The additive part is $([0, 1], \vee, 0)$ i.e., weakened variables have probability 0 and contraction takes maximum probability.

Graded necessity $\subset$ GLL

The multiplicative part $(\mathcal{R}, *, 1)$ of GLL yields $!_r$ as *graded necessity*.

In Hilbert-style axiomatisation:

$$
\begin{aligned}
(T) &\quad !^{1} A \to A \\
(4) &\quad !^{r*s} A \to !^{r} !^{s} A \\
(K) &\quad !^{r} (A \to B) \to !^{r} A \to !^{r} B \\
(Nec) &\quad \frac{\vdash A}{\vdash !^{r} A}
\end{aligned}
$$

Example: LTL "neXt" as graded necessity

Recall the unary connective X from Linear Temporal Logic.
There is a graded necessity

$$!_r A = X^r A$$

with monoid $(\mathbb{N}, +, 0)$ such that:

$$
\begin{array}{rl}
(T) & X^0 A \to A \\
(4) & X^{r+s} A \to X^r X^s A \\
(K) & X^r (A \to B) \to X^r A \to X^r B \\
(Nec) & \dfrac{\vdash A}{\vdash X^r A}
\end{array}
$$

Example: LTL "neXt" as graded necessity (2)

Back in GLL this gives us:

$$\text{der}\frac{\Gamma, \phi \vdash \psi}{\Gamma, [\phi]_0 \vdash \psi} \qquad \text{pr}\frac{[\Gamma] \vdash \psi}{r{+}[\Gamma] \vdash X_r \psi}$$

e.g.

$$\text{pr}\frac{\text{der}\dfrac{A \vdash B}{[A]_0 \vdash B}}{[A]_2 \vdash XXB}$$

if A is true in 2 time steps then we can conclude B at 2 time steps.

What about the additive part for the rest of GLL?

- $(\mathbb{N}, \vee, 0)$ changes the meaning of X to "next and all future states".
- **Better**: make $+$ partial: $r + r = r$ otherwise $r + s = \bot$

Semantics: Graded exponential comonad

- Symmetric monoidal-closed category $\mathbb{C}$
- Semiring $\mathcal{R}$ as category (i.e. strict monoidality via bifunctors $+$ and $*$ etc.)
- and a model for ! via the functor D with:

$$
\begin{array}{llrcl}
\text{Functor} & D : & \mathcal{R} & \to & [\mathbb{C}, \mathbb{C}] \\
\text{0-Monoidality} & m_{r,1} : & 1 & \to & Dr1 \\
\text{2-Monoidality} & m_{r,A,B} : & DrA \otimes DrB & \to & Dr(A \otimes B) \\
\text{Weakening} & w_A : & D0A & \to & 1 \\
\text{Contraction} & c_{r,s,A} : & D(r+s)A & \to & DrA \otimes DsA \\
\text{Dereliction} & \varepsilon_A : & D1A & \to & A \\
\text{Digging} & \delta_{r,s,A} : & D(r*s)A & \to & Dr(DsA)
\end{array}
$$

  making a number of diagrams commute.
- When the semiring is trivial gives linear exponential comonad on $\mathbb{C}$.
- Similar structure in all coeffect papers; here based on [Gaboardi et al., 2016]

Conclusions and what's next

- Coeffects: a relatively new way of understanding (intensional) program behaviour (state of the art [Gaboardi et al., 2016]).
- Graded semantics provide refined models; aids reasoning.
- Graded Linear Logic provides parametric quantitative version of LL.
- Applications to quantitative models, e.g., timed, probabilistic, communicating automata.
- Get in touch! d.a.orchard@kent.ac.uk

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