# Comparison of three call option pricing models

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This report aims at describing three different techniques that are commonly used in order to price American call options. In particular, the Cox, Ross and Rubenstein's binomial model, the Black-Scholes' model and the Leisen and Reimer's model. Through Excel simulations, convergence rates of the three models have been investigated and compared, moreover, in order to get things more "practical", the models have been tested on the call options that were taken into consideration in the past first report.

## BRIEF DESCRIPTION OF THE MODELS

Before going through the details of the methodologies, it is useful to firstly introduce some basic concepts regarding the three models that have been taken into consideration for the option pricing.

### Binomial option pricing model

This technique for pricing an option was introduced by Cox, Ross and Rubenstein in 1979 and involves constructing a binomial tree (in particular, a recombining tree), generalizing the one-step binomial model that was taken into consideration in the first report. Binomial tree is a diagram representing different possible paths that might be followed by the stock price over the life of an option. The underlying assumption is that the stock price follows a random walk. In each time step, it has a certain probability of moving up by a certain percentage amount and a certain probability of moving down by a certain percentage amount and no assumptions are required about the actual real-world probabilities of up and down movements in the stock price.

By only assuming the principles of risk-neutral valuation and the absence of arbitrage opportunities, it is possible to derive the formulas for pricing a call option in the general case of a n-step binomial tree. The call option price is given by:

$$c_B = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} max(S_0 u^j d^{n-j} - K; 0) \quad (1)$$

Where $r$ denotes the interest rate, $T$ the maturity, $n$ the number of steps in the tree, $S_0$ the initial stock price, $K$ the strike price, $\sigma$ the volatility, while $u$, $d$ and $p$ are given by the following equivalences and respectively represent the amounts of an upper or lower movement inside a tree and their probability to occur.

$$u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = e^{-\sigma\sqrt{\frac{T}{n}}} \qquad p = \frac{e^{r\frac{T}{n}} - d}{u - d} \quad (2)$$

### Black-Scholes option pricing model

In the limit of the time steps approaching zero (or equivalently, the number of time steps approaching infinity), the binomial model gives the same prediction as the Black–Scholes model. By taking this limit and manipulating Equations 1 and 2, it can be found that the call option price of the Black-Scholes model is the following.

$$c_{BS} = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2) \quad (3)$$

Where $\mathcal{N}(x)$ represents the cumulative function of a normal randomly distributed variable and $d_1$, $d_2$ are given by:

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad d_2 = \frac{\log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (4)$$

### Leisen-Reimer option pricing model

This model was firstly introduced in 1995 by Leisen and Reimer [2] and displays a greater precision and a faster convergence with a smaller number of steps with respect to the Cox, Ross and Rubenstein's binomial model.

As will be shown, binomial model becomes more and more precise increasing the number of steps. However, this convergence is not smooth, but instead oscillates. Leisen and Reimer provided a method that not only improves the convergence of the binomial model, but also converges smoothly, decreasing computation times in some situations.

Leisen and Reimer's idea was to center the underlying price binomial tree around the option's strike price at expiration, contrarily to what is done in the binomial model, where instead it is centered around the current underlying price. The calculations of the tree nodes and the option price are the same as in the binomial model (Equation 1), what does change is the calculation of up and down move sizes and their respective probabilities.

The probability of an up movement in a Leisen-Reimer tree is given by:

$$p_{LR} = h^{-1}(d_2) \quad (5)$$

While the probability of a down move is simply $1-p_{LR}$. $h^{-1}(x)$ represent the Peizer-Pratt inversion function, which provides discrete binomial estimates for continuous normal cumulative distribution functions for $n$ odd. Explicitly:

$$h^{-1}(x) = \frac{1}{2} + \frac{sign(x)}{2}\sqrt{1 - e^{-\left(\frac{x}{n+\frac{1}{3}\frac{1}{10(n+1)}}\right)^2 (n+\frac{1}{6})}} \tag{6}$$

Defining $p'_{LR} = h^{-1}(d_1)$ is then possible to calculate up and down move sizes as:

$$u = e^{(r-\sigma)\frac{T}{n}}\frac{p'_{LR}}{p_{LR}} \qquad d = e^{(r-\sigma)\frac{T}{n}}\frac{1-p'_{LR}}{1-p_{LR}} \tag{7}$$

## METHODS

All the formulas of the previous section have been implemented through some functions into a VBA script, using Excel. In this way, just by fixing some values for $S_0$, $K$, $T$, $r$, $\sigma$ and the number of step $n$, it is possible to recall the created functions and compute the call options pricing for each considered model. Three sets of values have been considered for the above variables. The first one served just for a first visualization of the performances of the models, while the other two involved the same data that had been collected for the first report and thus, consisted in a real case application. For each set of variables, the step's number $n$ was set as a column of integers spanning from 1 to 100. As will be shown, this range of values provided a good visualization of models' convergence.

## RESULTS

The first set of fixed parameters is shown in Table I. As said in the previous section, the step number $n$ of the binomial tree, spanned integer values between 1 and 100.

| $S_0$ [\$] | $K$ [\$] | $T$ [$months \backslash year$] | $r$ | $\sigma$ |
|---|---|---|---|---|
| 100 | 100 | 1 | 0,01 | 0,2 |

TABLE I. First set of input parameters.

By inserting these values as input parameters of the VBA functions, together with $n$, call option prices estimations have been obtained. All the results are plotted in Figure 1.

The values for the second and the third set of input parameters are the same that were used and computed in the first report for two Alibaba (BABA) call options with maturities of 3 months and 6 months. These data

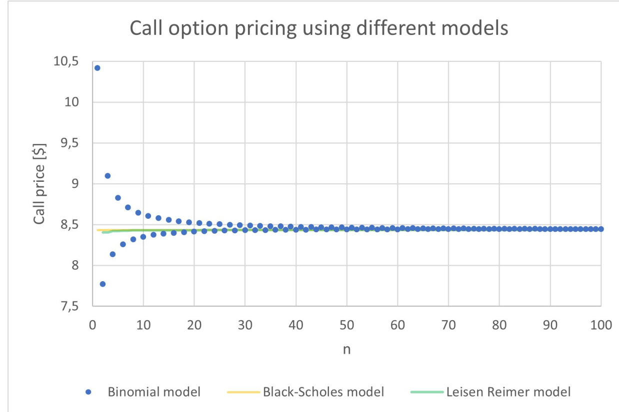


FIG. 1. Call option pricing using first set of parameters.

were collected by checking on site [3] and are resumed in Tables II, III.

| | |
|---|---|
| Contract name | BABA220617C00115000 |
| Last trade date | 2022-03-22 12:37PM EDT |
| $T$ [$months \backslash year$] | 0,25 |
| $S_0$ [\$] | 116,49 |
| $K$ [\$] | 115,00 |
| Bid [\$] | 14,40 |
| Ask [\$] | 14,80 |
| Mid price [\$] | 14,60 |
| $r$ | 0,95371% |
| $\sigma_{year}$ | 0,981 |

TABLE II. 3 months maturity BABA call option.

| | |
|---|---|
| Contract name | BABA220916C00115000 |
| Last trade date | 2022-03-22 12:24PM EDT |
| $T$ [$months \backslash year$] | 0,50 |
| $S_0$ [\$] | 116,49 |
| $K$ [\$] | 115,00 |
| Bid [\$] | 18,50 |
| Ask [\$] | 19,65 |
| Mid price [\$] | 19,08 |
| $r$ | 1,38457% |
| $\sigma_{year}$ | 0,789 |

TABLE III. 6 months maturity BABA call option.

These values, given in input to the functions implemented in VBA gave the results shown in Figure 2.

In particular, recall from first report, that by using a single step binomial tree, obtained prices for the two calls were: $c_3^{1\,step} = 28,69\$$ for the 3 months maturity call option and $c_6^{1\,step} = 32,52\$$ for the 6 months maturity call option. Now instead, considering a higher number of

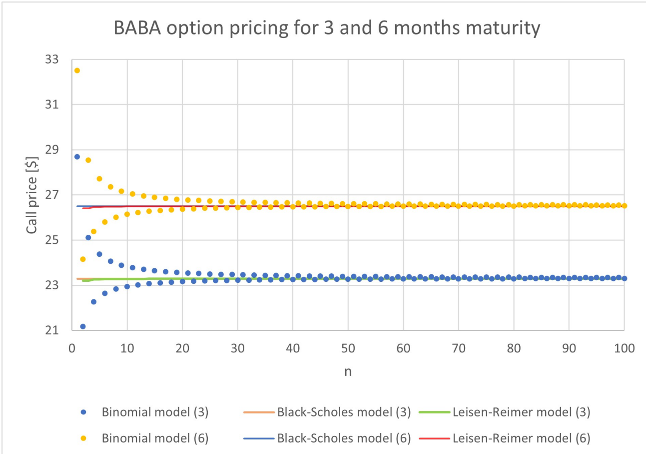


FIG. 2. BABA call options pricing.

steps, call prices asymptotically converge to the respective values $c_3^{BS} = 23,29\$$ and $c_6^{BS} = 26,50\$$ predicted by the Black-Scholes model.

## CONCLUSIONS

It can be observed that both binomial and Leisen-Reimer models converge asymptotically to the pricing solution predicted by the Black-Scholes model. Indeed Leisen and Reimer technique converges very fast to the solution with only few tens steps, while binomial price continues to oscillate for a really long time, even increasing $n$.

Note that however, even if precision is undoubtedly increased, the mid prices of Alibaba options differ again from the respective ones, predicted by the Black-Scholes formula.


[1] Hull, John C. *Options futures and other derivatives*. Pearson Education India, 2003.

[2] Leisen, Dietmar PJ, and Matthias Reimer. "Binomial models for option valuation-examining and improving convergence." *Applied Mathematical Finance* 3.4 (1996): 319-346.

[3] https://finance.yahoo.com/quote/BABA?p=BABA&.tsrc=fin-srch