EECS 461 PROBABILITY & STATISTICS

MULTIPLE RANDOM VARIABLES — HOMEWORK 10 5.6 – 5.9

MORGAN BERGEN — OCT 27 2022

---

5 MULTIPLE RANDOM VARIABLES

5.6 INDEPENDENT RANDOM VARIABLES

5.7 EXPECTED VALUE OF A FUNCTION OF TWO RANDOM VARIABLES

5.8 COVARIANCE, CORRELATION, AND INDEPENDENCE

5.9 BIVARIATE GAUSSIAN RANDOM VARIABLES

1. RECALL THE AIR SUPPORT ROOF EXAMPLE FROM CLASS, IN WHICH G REPRESENTED INSIDE BAROMETRIC PRESSURE & H REPRESENTED OUTSIDE BAROMETRIC PRESSURE. THE JOINT PDF WAS GIVEN AS:

$$f_{G,H}(g,h) = c/g \qquad \text{FOR} \qquad 27 \le h \le g \le 33$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS

FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

A. FIND $E[G]$ & $E[H]$ $\quad c = 1.7185$

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{\infty} \frac{c}{g}\,dh = \frac{c}{g}\int_{h=27}^{h=g} 1\,dh = \frac{c}{g}\left[h \Big|_{27}^{g} \right] = \frac{c}{g}\left(g - 27\right) = \frac{1.7185}{g}\left(g - 27\right) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dg = \int_{-\infty}^{\infty} \frac{c}{g}\,dg = c\int_{g=h}^{g=33} \frac{1}{g}\,dg = c\int_{h}^{33} \frac{1}{g}\,dg = c\left[\mathrm{LOG}(g)\right]\Big|_{h}^{33} = c\left[\mathrm{LOG}(33) - \mathrm{LOG}(h)\right] = -1.7185\left[\mathrm{LOG}\left(\frac{33}{h}\right)\right]$$

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(1.7185 - \dfrac{46.3995}{g}\right), & h \le g \le 33 \\ 0, & \text{OTHERWISE} \end{cases} \qquad f_H(h) = \begin{cases} -1.7185\left[\mathrm{LOG}\left(\dfrac{33}{h}\right)\right], & 27 \le h \le g \\ 0, & \text{OTHERWISE} \end{cases}$$

EECS 461 PROBABILITY & STATISTICS

MULTIPLE RANDOM VARIABLES — HOMEWORK 10 5.6 – 5.9

MORGAN BERGEN — OCT 27 2022

1. RECALL THE AIR SUPPORT ROOF EXAMPLE FROM CLASS, IN WHICH G REPRESENTED INSIDE BAROMETRIC PRESSURE & H REPRESENTED OUTSIDE BAROMETRIC PRESSURE. THE JOINT PDF WAS GIVEN AS:

$$f_{G,H}(g,h) = c/g \quad \text{FOR} \quad 27 \le h \le g \le 33 \qquad f_{G,H}(g,h) = f_G(g)\cdot f_H(g)$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

A. FIND $E[G]$ & $E[H]$ $\quad c = 1.7185$

C. FIND THE VARIANCE OF BOTH G & H

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{\infty} \frac{c}{g}\,dh = \frac{c}{g}\int_{h=27}^{h=g} 1\,dh = \frac{c}{g}\left[h\Big|_{27}^{g}\right] = \frac{c}{g}(g-27) = \frac{1.7185}{g}(g-27) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dg = \int_{-\infty}^{\infty}\frac{c}{g}\,dg = c\int_{g=h}^{g=33}\frac{1}{g}\,dg = c\int_{h}^{33}\frac{1}{g}\,dg = c\left[\log(g)\right]\Big|_{h}^{33} = c\left[\log(33) - \log(h)\right] = 1.7185\left[\log\left(\frac{33}{h}\right)\right]$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF THE EXPECTED VALUE

$$E[G] = \int_{-\infty}^{\infty} g\, f_G(g)\,dg = \int_{g=27}^{g=33} g\left(\frac{c}{g}(27-g)\right)dg = c\int_{27}^{33}(g-27)\,dg = c\int_{27}^{33} g\,dg - 27c\int_{27}^{33} 1\,dg = c\left(\frac{g^2}{2}\right)\Bigg|_{27}^{33} + -27cg\Bigg|_{27}^{33}$$

$$= c\left[\frac{33^2}{2} - \frac{27^2}{2}\right] + \left[(-27\cdot 33c) - (-27\cdot 27c)\right] = 180c - 162c = 18c = 18\cdot 1.7185$$

$$E[G] = 1.7185\,(18) = 30.933 \qquad \boxed{\therefore\ E[G] = 30.933}$$

$$E[G^2] = \int_{-\infty}^{\infty} g^2 f_G(g)\,dg = \int_{g=27}^{g=33} g^2\left(\frac{c}{g}(27-g)\right)dg = c\int_{27}^{33}(27-g)g\,dg =$$

INTEGRATION BY PARTS

$u = 27 - g$ $\quad$ LOWER BOUND $\quad$ $U = 27-27 = 0$

$du = -dv$ $\quad$ UPPER BOUND $\quad$ $U = 27-33 = 6$

$$= \int_0^{-6}(u-27)u\,du = \int_0^{-6} u^2\,du + 27\int_0^{-6} u\,du = 558$$

$$E[G^2] = 1.7185\,(558) = 958.923$$

$$\text{VAR}[G] = E[G^2] - (E[G])^2 = 958\ 923 - (30.933)^2 = 2.072511 \qquad \boxed{\therefore\ \text{VAR}[G] = 2.0725}$$

C. FIND THE VARIANCE OF BOTH G & H

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(\frac{c}{g}(g-27)\right), & 27 \le g \le 33 \\ & c = 1.7185 \\ 0, & \text{OTHERWISE} \end{cases} \qquad f_H(h) = \begin{cases} 1.7185\left[\log\left(\frac{33}{h}\right)\right], & 27 \le h \le 33 \\ 0, & \text{OTHERWISE} \end{cases}$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF H TO THEN FIND THE EXPECTED VALUE

$$E[H] = \int_{-\infty}^{\infty} h f_H(h)\,dh = \int_{27}^{33} h\left(c\left[\log(33) - \log(h)\right]\right)dh = c\int_{27}^{33}\left[h\log(33) - h\log(h)\right]dh = c\left[\left(\log(23)\int_{27}^{33} h\,dh\right) - \int_{27}^{33} h\log(h)\,dh\right] = c\left[\frac{1}{2}h^2\log(23)\Big|_{27}^{33} - \int_{27}^{33} h\log(h)\,dh\right]$$

$$= c\left[\frac{33^2}{2}\log(23) - \frac{27^2}{2}\log(23) - \int_{27}^{33} h\log(h)\,dh\right] = c\left[\frac{360}{2}\log(23) - \int_{27}^{33} h\log(h)\,dh\right] =$$

INTEGRATION BY PARTS $\int v\,du = vu - \int v\,du$ $\Rightarrow$

LET $V = \log(h)$ $\quad dv = h\,dh$

$U = h^2/2$ $\quad du = h^2/2$

$$= c\left[\frac{360}{2}\log(23) - \left(\frac{1}{2}h^2\log(h)\Big|_{27}^{33} - \frac{1}{2}\int_{27}^{33} h\,dh\right)\right] = c\left[180\log(23) - \frac{9}{4}\left(40 + 81\log(729) - 121\log(1089)\right)\right] = c\left[90 - \frac{729}{4}\log\left(\frac{121}{81}\right)\right], \quad c = 1.7185$$

$E[H] \simeq 16.855\,(1.7185)$ $\quad \boxed{\therefore E[H] \simeq 28.9662}$

NEXT WE WILL FIND THE SECOND MOMENT OF H

$$E\left[H^2\right] = \int_{-\infty}^{\infty} h^2 f_H(h)\,dh = \int_{27}^{33} h^2\left(c\left[\log(33) - \log(h)\right]\right)dh = c\left[\log(33)\int_{27}^{33} h^2\,dh - \int_{27}^{33} h^2\log(h)\,dh\right] = c\left[\frac{1}{3}h^3\log(33)\Big|_{27}^{33} - \int_{27}^{33} h^2\log(h)\,dh\right] = c\left[5418\log(33) - \int_{27}^{33} h^2\log(h)\,dh\right]$$

INTEGRATION BY PARTS

$\int u\,dv = uv - \int v\,du$

$u = \log(h)$ $\quad v = h^3/3$

$du = 1/h\,dh$ $\quad dv = h^2\,dh$

$$= c\left[5418\log(33) - \left(\frac{1}{3}h^3\log(h)\Big|_{27}^{33} - \frac{1}{3}\int_{27}^{33} h^2\,dh\right)\right] = c\left[5418\log(33) - \left(-3\left(602 + 2187(27) - 3993\log(33)\right)\right)\right] = 489.4$$

$E\left[H^2\right] \simeq 489.4\,(1.7185)$ $\quad \therefore E\left[H^2\right] \simeq 841.033$

$VAR[H] = E\left[H^2\right] - \left(E[H]\right)^2 = 841.033 - (28.9662)^2 \simeq 1.992$ $\quad \boxed{\therefore VAR[H] \simeq 1.992}$

B. FIND $E[GH]$ AND $\text{COV}[G, H]$

$$E[GH] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} gh\, f_{G,H}(g,h)\, dg\, dh = \int_{27}^{33} \int_{27}^{33} \left(\frac{c}{g}\right)(gh)\, dg\, dh = 6c \log(11/9) = 1080c$$

$$E[GH] = 1080\,(1.7185)$$

$$\boxed{\therefore E[GH] = 1855.98}$$

$$\text{COV}[GH] = E[GH] - E[G]\,E[H]$$
$$= 1855.98 - (30.933 \cdot 28.9662)$$

$$\boxed{\therefore \text{COV}[GH] = 959.9685}$$

D. FIND THE VARIANCE OF $(G + H)$

$$\text{VAR}[G + H] = \text{VAR}[G] + \text{VAR}[H] = 2.0725 + 1.992 = 4.0645$$

$$\boxed{\therefore \text{VAR}[G + H] = 4.0645}$$

E. FIND THE CORRELATION COEFFICIENT $\rho_{G,H}$

$$\rho_{G,H} = \frac{\text{COV}[GH]}{\sqrt{\text{VAR}[G]\,\text{VAR}[H]}} = \frac{959.9685}{\sqrt{(2.0725)(1.992)}} = 472.460$$

$$\boxed{\therefore \rho_{G,H} = 472.460}$$

F. FIND $E[G - H]$ AND GIVE A PRACTICAL PHYSICAL INTERPRETATION FOR THIS EXPECTATION

$E[G - H] \Rightarrow$

EECS 461 PROBABILITY & STATISTICS

5 MULTIPLE RANDOM VARIABLES — HOMEWORK 10 5.6 – 5.9

MORGAN BERGEN

OCT 27 2022

---

2. MANUFACTURE OF A WIDGET REQUIRES WELDING 2 JOINTS & TIGHTENING 3 BOLTS. LET X BE THE NUMBER OF DEFECTIVE WELDS AND Y BE THE NUMBER OF IMPROPERLY TIGHTENED BOLTS. FROM PAST EXPERIENCE, THE JOINT PMF IS GIVEN BELOW

— — —	X = 0	X = 1	X = 2	X = 3
Y = 0	0.840	0.030	0.020	0.010
Y = 1	0.060	0.010	0.008	0.002
Y = 2	0.010	0.005	0.004	0.001

A. FIND THE EXPECTED NUMBER OF DEFECTIVE WELDS, THE EXPECTED NUMBER OF IMPROPERLY TIGHTENED BOLTS, AND THE MEAN TOTAL NUMBER OF MANUFACTURING PROBLEMS

$$E[X] = 0(0.840 + 0.060 + 0.010) + 1(0.03 + 0.01 + 0.005) + 2(0.020 + 0.008 + 0.004) + 3(0.01 + 0.002 + 0.001)$$
$$= 0(0.91) + 1(0.045) + 2(0.032) + 3(0.013)$$
$$\boxed{\therefore E[X] = 0.148}$$

$$E[X^2] = 0(0.91) + 1^2(0.045) + 2^2(0.032) + 3^2(0.013)$$
$$E[X^2] = 0.2837$$

$$E[Y] = 0(0.84 + 0.03 + 0.02 + 0.01) + 1(0.06 + 0.01 + 0.008 + 0.002) + 2(0.010 + 0.005 + 0.004 + 0.001)$$
$$\boxed{\therefore E[Y] = 0.12}$$

$$E[Y^2] = 1^2(0.08) + 2^2(0.02)$$
$$E[Y^2] = 0.16$$

B. FIND THE CORRELATION (NOT CORRELATION COEFFICIENT) AND COVARIANCE OF X & Y

$$E[XY] = \sum_{y=0}^{2} \sum_{x=0}^{3} xy\, P_{XY}(x,y) = (1)(1)(0.01) + (1)(2)(0.008) + (1)(3)(0.002) + (1)(2)(0.005) + (2)(2)(0.004) + (3)(2)(0.001)$$
$$\boxed{\therefore E[XY] = 0.064}$$ CORRELATION

$$\mathrm{COV}[XY] = E[XY] - E[X]\,E[Y]$$
$$= 0.064 - 0.148(0.12)$$
$$\boxed{\therefore \mathrm{COV}[XY] = 0.04624}$$ COVARIANCE

C. FIND THE VARIANCE OF BOTH X AND Y

$$\mathrm{VAR}[X] = E[X^2] - (E[X])^2$$
$$= 0.2837 - (0.148^2)$$
$$\boxed{\therefore \mathrm{VAR}[X] = 0.261796}$$

$$\mathrm{VAR}[Y] = E[Y^2] - (E[Y])^2$$
$$= 0.16 - (0.12^2)$$
$$\boxed{\therefore \mathrm{VAR}[Y] = 0.1456}$$

2. MANUFACTURE OF A WIDGET REQUIRES WELDING 2 JOINTS & TIGHTENING 3 BOLTS. LET X BE THE NUMBER OF DEFECTIVE WELDS AND Y BE THE NUMBER OF IMPROPERLY TIGHTENED BOLTS. FROM PAST EXPERIENCE, THE JOINT PMF IS GIVEN BELOW

– – –	X = 0	X = 1	X = 2	X = 3
y = 0	0.840	0.030	0.020	0.010
y = 1	0.060	0.010	0.008	0.002
y = 2	0.010	0.005	0.004	0.001

$0.84 + 0.06 + 0.01 = 0.91$

$0.84 + 0.03 + 0.02 + 0.01 = .90$

D. FIND THE CORRELATION COEFFICIENT $\rho_{X,Y}$

$$\rho_{X,Y} = \frac{\text{COV}[XY]}{\sqrt{\text{VAR}[X]\ \text{VAR}[Y]}} = \frac{0.04624}{\sqrt{(0.2617)(0.1456)}} =$$

$\therefore \rho_{G,H} =$

E. ARE X AND Y INDEPENDENT? JUSTIFY YOUR ANSWER MATHMATICALLY

NO THE REASON AS TO WHY IS AS FOLLOWS,

$$P_{XY}(x,y) \neq P_X(x)\, P_Y(y) \quad \text{FOR } ! \forall x, y \text{ VALUES}$$

$$P_{XY}(0,0) = 0.840 \neq P_X(0) = 0.91 \cdot P_Y(y) = 0.90$$

$$0.84 \neq (0.91 \cdot 0.90)$$

3. THE LENGTH L AND WIDTH W OF A RECTANGLE HAVE JOINT PDF GIVEN BY.

$$f_{L,W}(l,w) = 2e^{-(l+2w)} \quad \text{FOR } l \geq 0,\ w \geq 0, \text{ AND } 0 \text{ OTHERWISE}$$

A. FIND THE CORRELATION OF L AND W : $E[LW]$, WHICH IS ALSO THE EXPECTED AREA OF THE RECTANGLE

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$E[LW] = \int_0^{\infty}\int_0^{\infty} 2e^{-(l+2w)}\, lw \; dl\,dw = \int_0^{\infty} w^2 e^{-2w}\, dw = \boxed{\frac{1}{4}}$$

B. ARE L AND W INDEPENDENT? JUSTIFY YOUR ANSWER MATHMATICALLY

IN ORDER FOR L & W TO BE INDEPENDENT THE FOLLOWING MUST BE TRUE,

$$f_L(l) = \int_{-\infty}^{\infty} f_{L,W}(l,w)\,dl = \int_0^{\infty} 2e^{-(l+2w)}\,dl = 2e^{-2w}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{L,W}(l,w)\,dw = \int_0^{\infty} 2e^{-(l+2w)}\,dw = e^{-l}$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} == f_L(l)\cdot f_W(w)$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} == \left(f_L(l) = 2e^{-2w}\right)\left(f_W(w) = e^{-l}\right)$$

$$\boxed{2e^{-(l+2w)} = e^{-l}\left(2e^{-2w}\right)} \Longleftarrow \text{TRUE}$$