PROBABILITY & STOCHASTIC PROCESSES
06 PROBABILITY MODELS OF DERIVED RANDOM VARIABLES

MORGAN BERGEN

TUES NOV 08 2022

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TRANSFORMATION OF RANDOM VARIABLES

INTRO

TRANSFORMATION OF DRVS

A SINGLE DRV FROM SINGLE DRV

SINGLE DRV FROM TWO DRV

RESULT $D = m(G, H)$

$$P_D(d) = \sum_{(g,h):\, m(g,h) = d} P_{G,H}(g,h)$$

EXAMPLE

FOR GIVEN HOUSE ON HALOWEEN, LET G BE THE # OF CANDIES GIVEN PER GROUP OF KIDS BY 2 ADULTS (ONE GIVES 4, OTHER GIVES 8) & LET H BE THE # KIDS PER GROUP (ASSUME 2 OR 4)

JOINT PMF

h \ g	4	8
2	.4	.2
4	.3	.1

SUPPOSE WE WANT PMF OF CANDIES PER KIDS

$D = G/H = m(G, H)$

G	H	G/H	$P_{G,H}$
4	2	2	.4
4	4	1	.3
8	2	4	.2
8	4	2	.1

$S_D = \{1, 2, 4\}$

$$P_D(d) = \begin{cases} 0.3, & d = 1 \\ 0.5, & d = 2 \\ 0.2, & d = 4 \end{cases}$$

TRANSFORMATIONS OF CRVs

REVIEW WE DO HAVE RESULTS ALREADY FOR SPECIAL CASE : LINEAR COMBINATION OF GAUSSIAN CRVs IS GAUSSIAN

SINGLE DERIVED CRV FROM SINGLE CRV

MOST GENERAL APPROACH FOR CRV G, TO GET PDF OF $J = m(G)$

(i) DETERMINE CDF OF J : $F_J(j) = P[J \le j]$

(ii) TAKE DERIVATIVE TO GET PDF

$$f_J(j) = \frac{d}{dj} F_J(j)$$

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SIMPLE EXAMPLE

GENERAL THEOREM 6.2 $J = aG$ WITH $a > 0$

THEN $F_J(j) = P[J \le j] = P[J \le j] = P[aG < j] = P[G \le j/a] = F_G(j/a)$

THEN $f_J(j) = \frac{d}{dj} F_J(j) = \frac{d}{dj} F_G(j/a) = f_G(j/a) \cdot \frac{d}{dj}(j/a)$

CHAIN RULE

$f_J(j) = \frac{1}{a} f_G(j/a)$

SPECIFIC G IS SPECIFIC $\exp(\lambda)$ & $J = mG$ $a > 0$

THEN $f_J(j) = \frac{1}{a} f_G\left(\frac{j}{a}\right) = \frac{1}{a}(\lambda)e^{-\lambda\left(\frac{j}{a}\right)}$

$f_J(j) = \frac{\lambda}{a}^{-\left(\frac{\lambda}{a}\right)j}$

SO J IS EXP $\left(\frac{\lambda}{a}\right)$

SEE THM 6.3 FOR OTHER COMMON G RVs

SEE THM 6.4 $J = G + b$, THEN

$F_J(j) = F_G(j-b)$ AND $f_J(j) = f_G(j-b)$

THM 6.5

LET U BE A UNIFORM $(0,1)$ CRV & LET $F(g)$ DENOTE ANY VALID CDF WITH INVERSE $F^{-1}(u)$ DEFINED FOR $0 \le u \le 1$

THEN THE RV $G = F^{-1}(u)$ HAS CDF $F_G(g) = F(g)$

PROOF

$F_G(g) = P[G \le g] = P[F^{-1}(u) \le g] = P[U \le F(g)] = F(g)$

$F_G(g) = F(g)$

TO SHOW $F^{-1}(u)$ IS NON-DECREASING:

LET $u \ge u'$ & $g = F^{-1}(u)$ & $g' = F'(u')$ OF $U(0,1)$

SHOW THAT $g \ge g'$

HERE $u = F(g)$ & $u' = F(g')$ SO $F(g) \ge F(g')$ BUT F IS A CDF HENCE NON-DECREASING

THIS IMPLIES $g \ge g'$

QED

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U IS UNIFORM$(0,1)$. WHAT $m(u)$ WILL MAKE $G = m(u)$ ON EXP(λ) RV??

DESIRED CDF IS $F(g) = 0 \quad g < 0$

$$1 - e^{-\lambda g} \quad g \geq 0$$

WITH $U = 1 - e^{-\lambda g}$, THEN $0 \leq u \leq 1$

SO $F^{-1}(u) = \frac{1}{\lambda} \ln(1-u)$

SO $G = F^{-1}(u) = \frac{-1}{\lambda} \ln(1-U)$ GIVEN

$$F_G(g) = F(g) = \begin{cases} 0 & g < 0 \\ 1 - e^{-\lambda g} & g \geq 0 \end{cases}$$

STRICTLY MONOTONIC FUNCTIONS (TRANSFORMATIONS)

ALMOST AS GENERAL & MORE DIRECT

LET $J = m(G)$ WITH m STRICTLY MONOTONIC, SO THE MAPPING m IS 1 TO 1 & FOR ANY $j \in S_J$ THERE IS A UNIQUE $g \in S_G$, THEN FOR ANY ARBITRARY INTERVAL $[j_1, j_2] \in S_J$

LET $g_1 = m^{-1}(j_1)$ & $g_2 = m^{-1}(j_2)$

THEN $[j_1, j_2]$ CORRESPOND TO $[g_1, g_2]$ IF m IS INCREASING OR $[g_2, g_1]$ IF m IS DECREASING

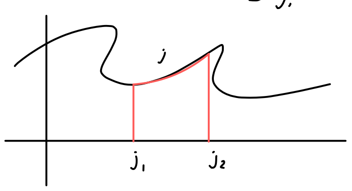
CAN FIND $f_J(j)$ DIRECTLY IN TERMS OF $f_G(g)$ (NO CDF STEP, NO DERIVATIVE) AS FOLLOWS

CONSIDER SMALL INTERVAL $\beta = [j_1, j_2]$ OF WIDTH $\Delta j = j_2 - j_1$ THAT CONTAINS j

ASSUME m IS INCREASING & LET WIDTH OF CORRESPONDING INTERVAL $\alpha = [g_1, g_2]$ BE $\Delta g = g_2 - g_1$

SINCE M IS MONOTONIC, α MUST CONTAIN $g = m^{-1}(j)$

NOW, $P_J(\beta) = \int_{j_1}^{j_2} f_J(x)\,dx \approx f_J(j)\,\Delta j$



& $P_G(\alpha) = \int_{g_1}^{g_2} f_G(\lambda)\,dx \approx f_G(g)\,\Delta g$

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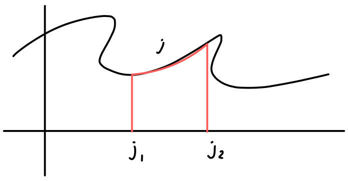

& $P_G(\alpha) = \int_{g_1}^{g_2} f_G(x)\,dx \simeq f_G(g)\,\Delta g$

THESE 2 PROBABILITIES MUST BE EQUAL, SO $f_J(j) \simeq \dfrac{f_G(g)}{\left(\frac{\Delta j}{\Delta g}\right)}$

NOW NOTE THAT $\left(\dfrac{\Delta j}{\Delta g}\right) \longrightarrow m'(g)$ IS $\Delta j \rightarrow 0$

SO THAT $f_J(j) = \dfrac{f_G(g)}{m'(g)}$ FOR m INCREASING

<u>EXERCISE</u> FOR m DECREASING, SHOW THAT $f_J(j) = \dfrac{f_G(g)}{-m^{-1}(g)}$

<u>GENENERAL RESULT</u>

FOR $J = m(G)$ & m STRICTLY MONOTONIC

$$f_J(j) = \frac{f_G(g)}{|m'(g)|} = \frac{f_G(m^{-1}(j))}{|m'(m^{-1}(j))|}$$