Chapter 2. Review of Random Processes

- 1 Random Variables and Error Functions
- 2 Concepts of Random Processes
- 3 Wide-sense Stationary Processes and Transmission over LTI
- 4 White Gaussian Noise Processes

1 Random Variables and Error Functions

Means, Variances, and moments of Random Variables

Let X be a random variable with the density function $f_X(x)$.

- Mean of X

$$\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

- Variance of X

$$\sigma^2 = Var[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

- The second of moment of X

$$m = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

- Relationship

$$\sigma^2 = m - \mu^2$$

- If ***X*** is a discrete random variable, then the above integrals are replaced by the summations.

Correlation and Covariance

- Let X and Y be two random variables, $f_{X,Y}(x, y)$ be their joint density function,
- $\mu_X = E[X]$, $\mu_Y = E[Y]$, their respective means,
- $\sigma_X^2 = Var[X]$ $\sigma_Y^2 = Var[Y]$, variances,
- and $Z = g(X, Y)$ is well-defined. Then Z is also a random variable, and the mean of Z is given by

$$E[g(X,Y)] = \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dx$$

- If $g(X,Y) = XY$, then

$$E[g(X,Y)] = E[XY]$$

is the correlation of X and Y.

- If $g(X,Y) = (X - \mu_X)(Y - \mu_Y)$, then

$$\begin{aligned} Cov(X,Y) &= E[g(X,Y)] \\ &= E[(X - \mu_X)(Y - \mu_Y)] \end{aligned}$$

is the covariance of X and Y.

- If $g(X,Y) = (X - \mu_X)(Y - \mu_Y)/(\sigma_X \sigma_Y)$ then

$$\rho(X,Y) = E[g(X,Y)]$$

is the correlation coefficient.

- Relationship:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

If $\rho = 0$, then X and Y are said to be uncorrelated.

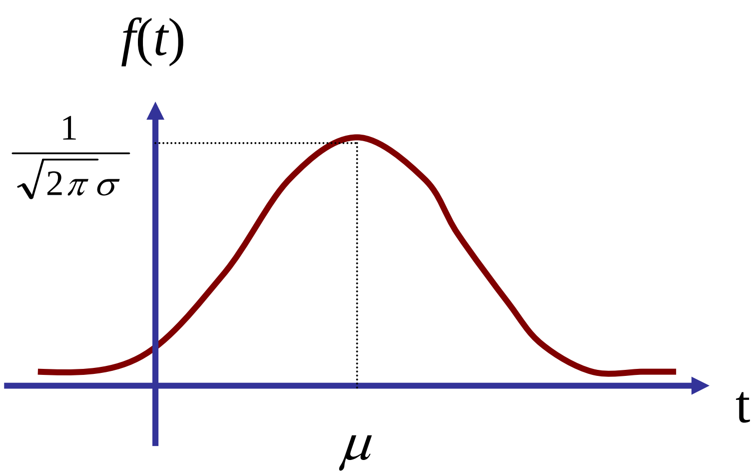
Guassian Random Variables

- $X \sim N(\mu, \sigma)$: X is a gaunssian r.v. with mean μ and variance σ. The pdf of X:

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, \quad -\infty < t < \infty$$

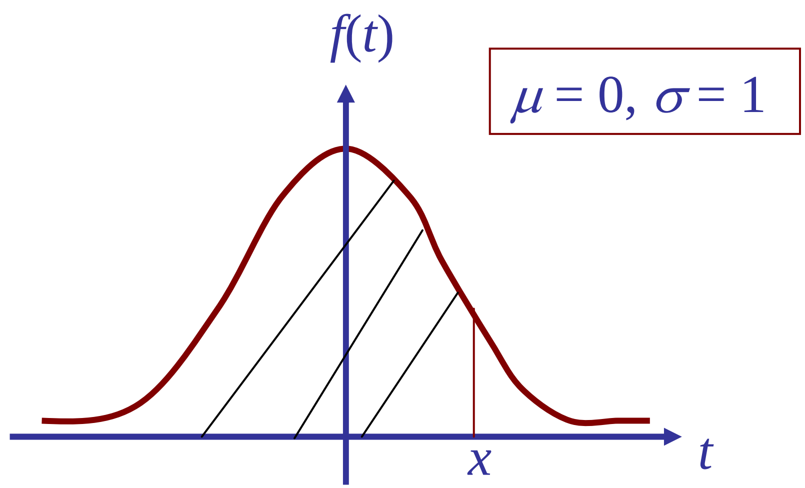
- The cdf of X:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$



- Unit or standard gaussian: $\mu = 0$ and $\sigma = 1$. The cdf of the unit gaussian:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$



$\Phi(x)$ = the shaded area

Computation of $F(x)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Jointly Gaussian Random Variables

Let X and Y be gaussian random variables with means μ_X and μ_y, variances σ_X and σ_y. We say that X and Y have a bivariate Gaussian pdf if the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}S\right]$$

where

$$S = \left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}$$

Property. If $f_{X,Y}(x,y)$ is defined by the above, then the conditional distributions of X given that $Y = y$, and Y given that $X = x$, respectively, are all gaussian distributions with the following parameters listed in (a).

(a) $f_{X|Y}(x \mid Y = y)$: $N(\tilde{\mu}_X(y), \tilde{\sigma}_X^2)$

where

$$\tilde{\mu}_X(y) = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

$$\tilde{\sigma}_X^2 = \sigma_X^2(1 - \rho^2)$$

What are the parameters of

$$f_{Y|X}(y \mid X = x)\,?$$

(b) The parameter ρ is equal to the correlation coefficient of X and Y, i.e.,

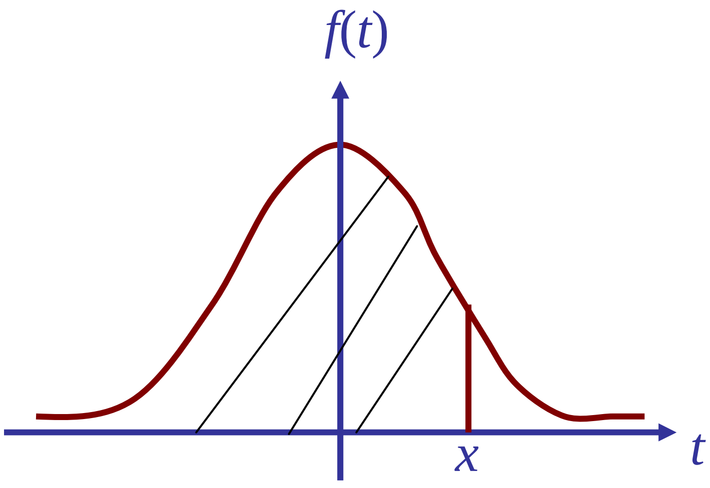
$$\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

(c) X and Y are independent if and only if X and Y are uncorrelated. In other word, X and Y are independent if and only if $\rho = 0$.

$Q(x)$ Function

- The cdf of the unit Gaussian random variable , reproduced here:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

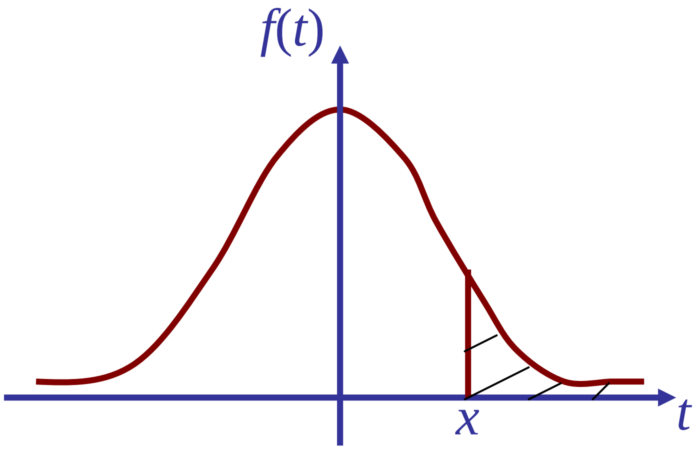


$\Phi(x)$ = the shaded area

$Q(x)$ function is defined by

$$Q(x) = 1 - \Phi(x)$$

which is called the tail probability.



$Q(x)$ = the shaded area

Error Function and the Complementary Error Function

- The error function:

$$erf(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt$$

- The complementary error function:

$$erfc(x) = \int_x^\infty \frac{2}{\sqrt{\pi}} e^{-t^2} dt$$

- Relation:

$$erfc = 1 - erf(x)$$

What are the relationships among $\Phi(x)$, $Q(x)$, erfc(x), or erf(x)?

- $$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$

- $$\Phi(x) = 1 - \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right)$$

- X is a gaunssian r.v. with mean μ and variance σ, then the cdf F(x) of X can be expressed in terms of erf(x):

$$F(x) = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

A few remarks on the definition of random processes:

- A random process is a collection of random variables:

$$\{X(t), t \in T\}$$

where $X(t)$ is a random variable which maps an outcome ξ to a number $X(t, \xi)$ for every outcome of an experiment.

- We will use $X(t)$ to represent a random process omitting , as in the case of random variables, its dependence on ξ.

2 Concepts of Random Processes

$X(t)$ has the following interpretations:

- It is a family (or ensemble) of functions $X(t, \xi)$.

Both t and ξ are variables

- It is a single time function (or a sample function, the realization of the process).

t is variable and ξ is fixed

- $X(t)$ is a random variable equal to the state of the given process at time t.

t is fixed and ξ is variable

- If t and ξ are fixed, then

$X(t)$ is a number

In a communication system, there are two types of random processes:

- For the transmitted signals, they have two properties:

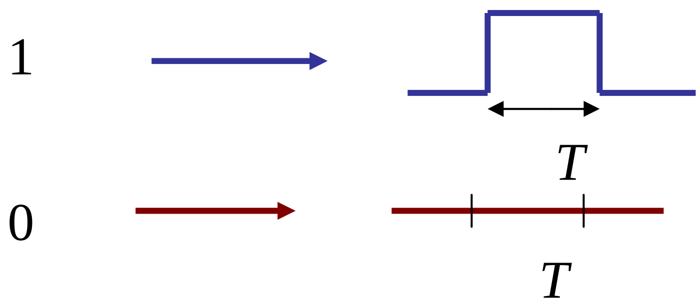
  - the signals are functions of time

  - the signals are random in the sense that before conducting an experiment, it is not possible to describe exactly the waveform that will be observed.

  Thus a transmitted signal is a random process.

- The noise /interference introduced by channel is a random process.

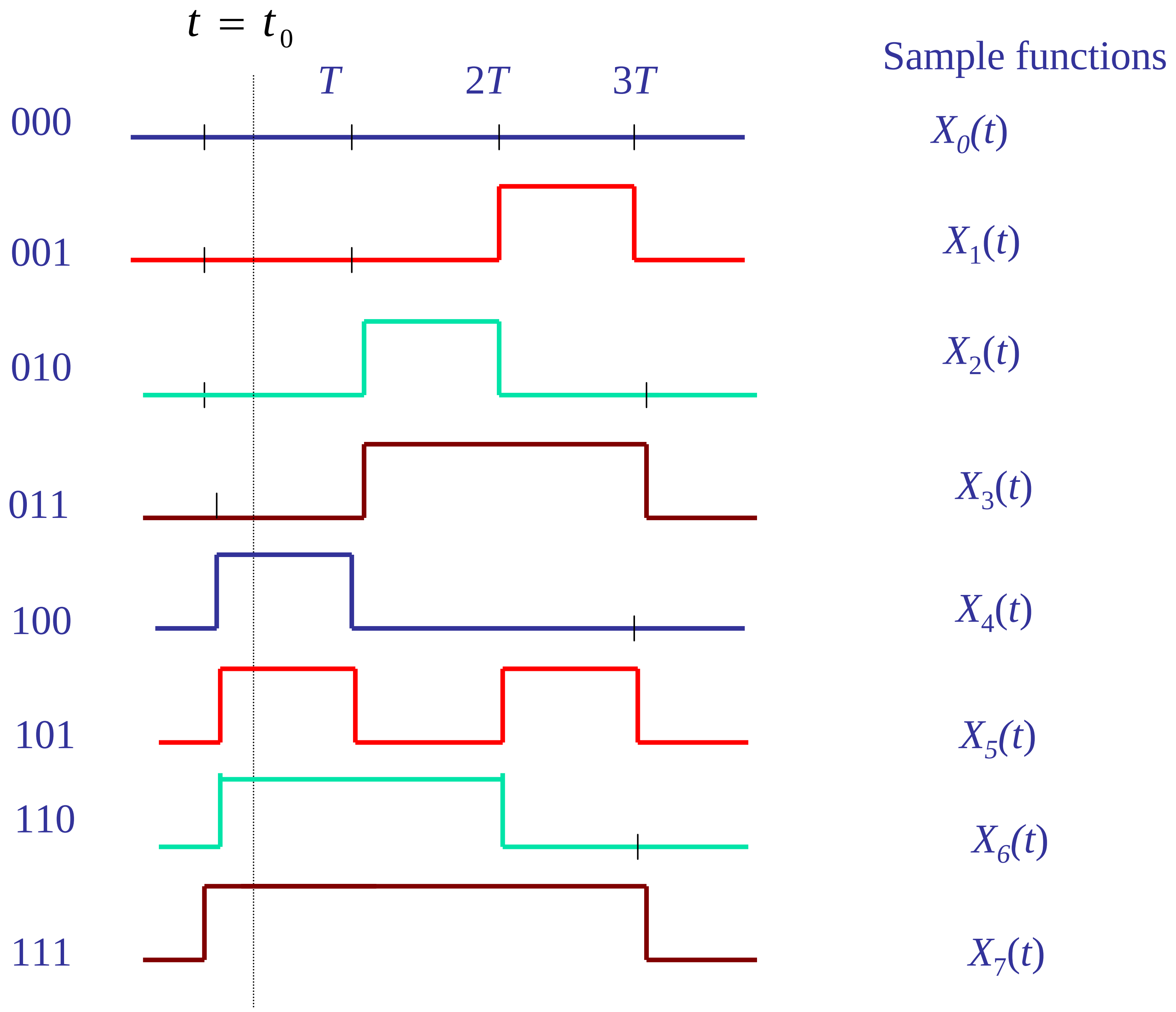
Example 2.1. Experiment: to transmit 3 binary digits to the receiver



T: symbol duration

X(*t*) is a random process or a random signal which is a collection of all the following eight waveforms.

$$\{X_0(t), \ldots, X_7(t)\}$$



$t = t_0$
T
$2T$
$3T$
Sample functions
000
001
010
011
100
101
110
111
$X_0(t)$
$X_1(t)$
$X_2(t)$
$X_3(t)$
$X_4(t)$
$X_5(t)$
$X_6(t)$
$X_7(t)$

Example 2.2. The voltage of an ac generator with random amplitude A and phase *Θ* is given by

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

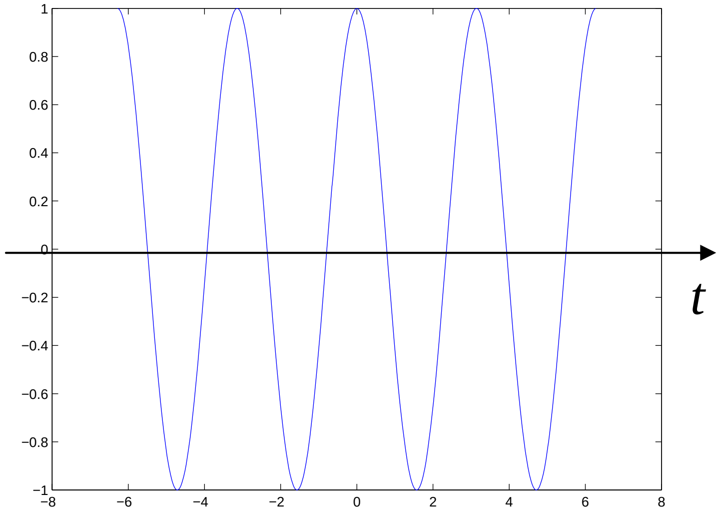
where *Θ* is a random variable uniformly distributed over (0, 2π).

$X(t)$ is a random process which consists of a family of cosine waves and a single sample is the function:

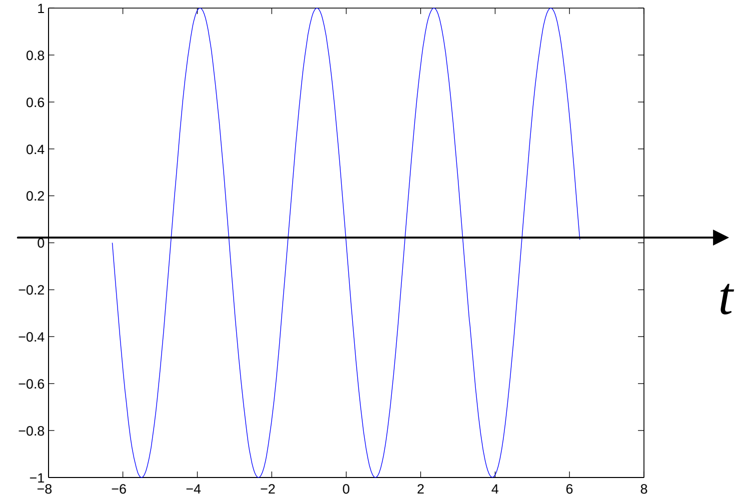
$$X(t,\theta) = A\cos(2\pi f_c t + \theta)$$

The following figures showed samples functions for θ being equal to 0, θ/2, -θ /4, and θ/4.

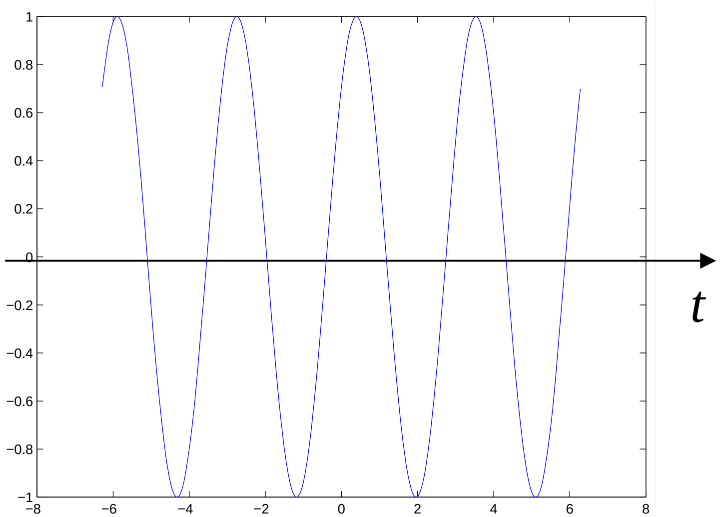
$\theta = 0$



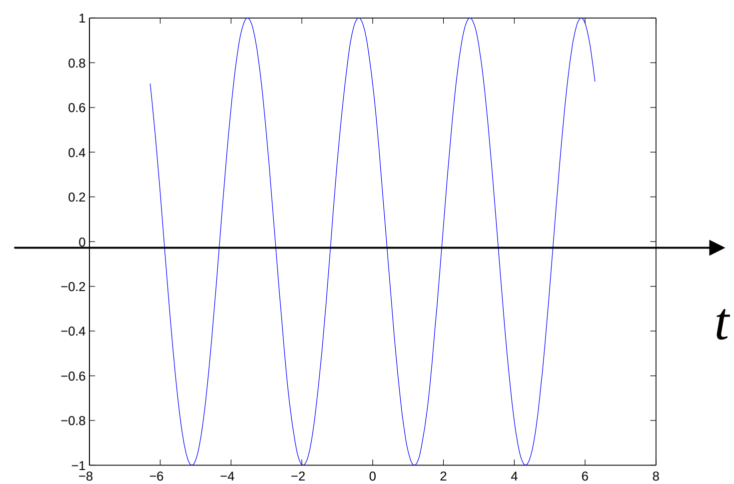
$\theta = \pi/2$



$\theta = -\pi/4$



$\theta = \pi/4$



Difference between the two examples

- The first example consists of a family of functions that cannot be described in terms of a formula.

- In the second example, the random process consists of a family of cosine waves and it is completely specified in terms of the random variable Θ.

Means, autocorrelation and crosscorrelation

- Mean of the random process $X(t)$ is the mean of random variable $X(t)$ at time instant t:

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_{X(t)}(x)dx$$

where $f_X(t)(x)$ is the pdf of $X(t)$ at time instant t.

- Autocorrelation function of $X(t)$ is a function of two variables $t_1 = t$ and $t_2 = t + \tau$,

$$R_X(t, t+\tau) = E[X(t)X(t+\tau)]$$

This is a measure of the degree to which two time samples of the same random process are related.

- Crosscorrelation function of two random processes $X(t)$ abd $Y(t)$ is a function of two variables $t_1 = t$, and $t_2 = t + \tau$,, defined by

$$R_{X,Y}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

3 Wide-Sense Stationary (WSS) Processes and Transmission over LTI

What is a WSS Process?

- A process $X(t)$ is WSS if the following conditions are satisfied:

$$(i)\ \ \mu_X(t) = E[X(t)] = \text{constant}$$

independent of t.

$$(ii)\ R_X(t+\tau, t) = R_X(\tau)$$

depends only on the time difference τ and not on the variables $t_1 = t + \tau$. and $t_2 = t$.

- In other words, a random process $X(t)$ is WSS if its two statistics, its mean and autocorrelation, do not vary with a shift in the time origin.

Example 2.3. Find the mean and autocorrelation function of the random process $X(t)$ (in E.g. 2.2),

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where Θ is uniformly distributed over $[0, 2\pi]$.

$$\text{pdf}: f_\Theta(\theta) = \begin{cases} 1/(2\pi), & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Is $X(t)$ a WSS?

Solution.

According to the definitions,

$$E[X(t)] = \int_0^{2\pi} A\cos(2\pi f_c t + \theta)\frac{1}{2\pi}\,d\theta$$

$$= 0$$

$$R_X(t, t+\tau)$$

$$= E[A\cos(2\pi f_c t + \Theta)A\cos(2\pi f_c(t+\tau) + \Theta)]$$

$$= A^2 E\left[\frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{2}\cos(2\pi f_c(2t+\tau) + 2\Theta)\right]$$

$$= \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Since both the mean and autocorrelation of $X(t)$ do not depend on time t, then $X(t)$ is a WSS process.

Properties of Autocorrelation Function of a WSS process *X*(*t*)

1. $R_X(\tau) = R_X(-\tau)$

Symmetric in τ about zero

2. $|R_X(\tau)| \le R_X(0)$

for all τ,

Maximum value occurred at the origin

3. $R_X(\tau) \leftrightarrow S_X(f)$

Autocorrleation and psd form a pair of the Fourier transform

4. $R_X(0) = E[X(t)^2]$

The value at origin is equal to the average power of the signal

Power Spectral Density (PSD) of a WSS Random Process

For a given WSS process $X(t)$, the psd of $X(t)$ is the Fourier transform of its autocorrelation, i.e.,

$$S_X(f) = \mathcal{F}(R_X(\tau)) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

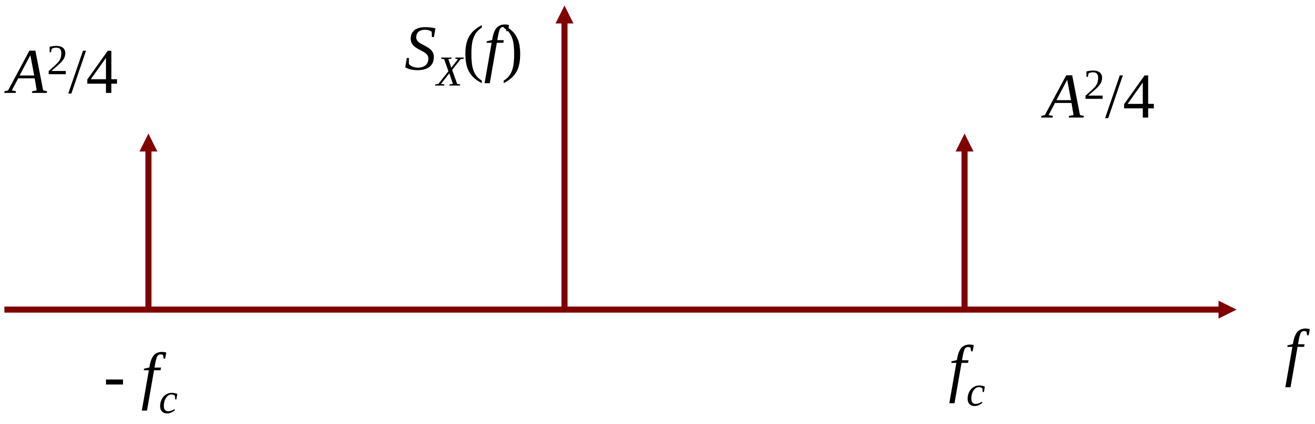
$$R_X(\tau) = \mathcal{F}^{-1}(S_X(f)) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} df$$

For the random process in Example 2.3, we have

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Hence, the psd of $X(t)$ is the Fourier transform of the autocorrelation of $X(t)$, given by

$$S_X(f) = \frac{A^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$



Example 2. 4 Let

$$Y(t) = X(t)\cos(2\pi f_c t + \Theta)$$

where $X(t)$ is a WSS process with psd $S_X(f)$, Θ is uniformly distributed over $[0, 2\pi]$, and $X(t)$ is independent of Θ and

$$\cos(2\pi f_c t + \Theta)$$

Find the psd of $Y(t)$.

Solution.

First we need to show that $Y(t)$ is WSS.

Mean:

$$m_Y(t) = E[Y(t)]$$

$$= E[X(t)\cos(2\pi f_c t + \Theta)]$$

$$= E[X(t)]E[\cos(2\pi f_c t + \Theta)]$$

(by independence)

$$= m_X(t) \cdot 0 = 0$$

(by Example 2.3)

Autocorrelation of $Y(t)$:

$$R_Y(t+\tau, t) = E[Y(t)Y(t+\tau)]$$

$$= E[X(t)\cos(2\pi f_c t + \Theta) X(t+\tau)\cos(2\pi f_c (t+\tau) + \Theta)]$$

$$= E[X(t)X(t+\tau)]E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta)]$$

By Example 2.3.

$$= R_X(\tau)\frac{1}{2}\cos(2\pi f_c \tau) = R_Y(\tau)$$

Hence, $Y(t)$ is WSS. Therefore

$$S_Y(t) = F[R_Y(\tau)] = \frac{1}{4}F[R_X(\tau)(e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau})]$$

$$= \frac{1}{4}[S_X(f - f_c) + S_X(f + f_c)]$$

Properties of PSD

1. $S_X(f) \geq 0$ always real valued

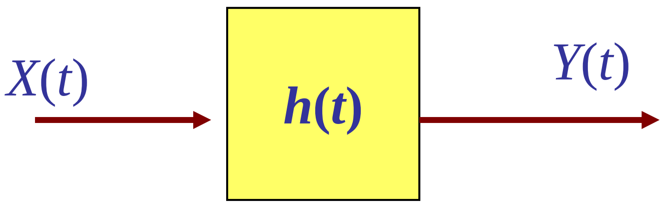
2. $S_X(f) = S_X(-f)$ for $X(t)$ real-valued

3. $S_X(f) \leftrightarrow R_X(\tau)$ a pair of Fourier transform

4. $P = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\,df$ Relationship between average power and psd

Transmission over LTI Systems

- Response of LTI system to a random input $X(t)$:



$$Y(t) = X(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

Properties of the output:

1) If $X(t)$ is WSS, so does $Y(t)$.

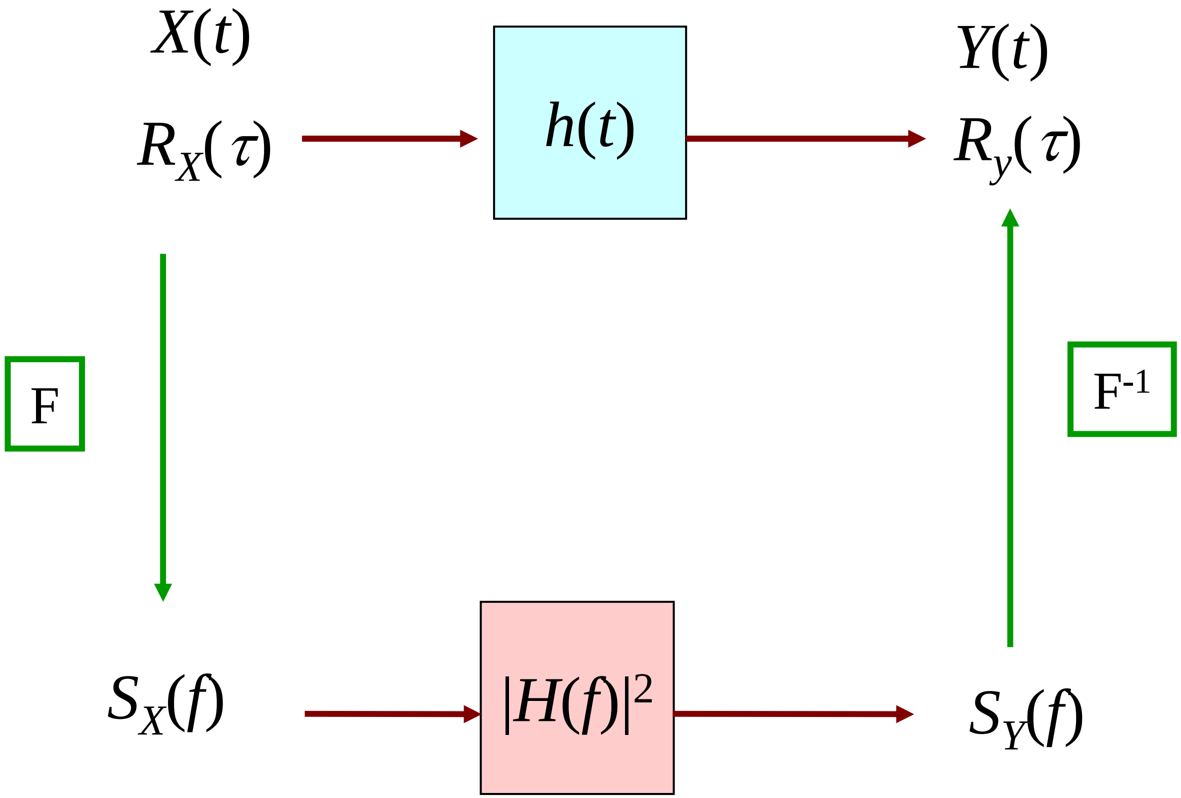
2) Mean: $\mu_Y = \mu_X H(0)$

3) Autocorrelation:

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

4) PSD:

$$S_Y(f) = S_X(f)\,|\,H(f)\,|^2$$

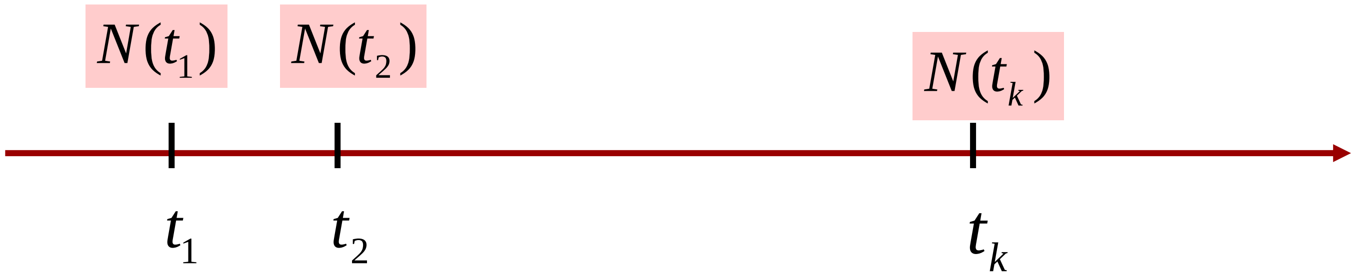

$X(t)$
$R_X(\tau)$
$h(t)$
$Y(t)$
$R_y(\tau)$
F
F^{-1}
$S_X(f)$
$|H(f)|^2$
$S_Y(f)$

4 White Gaussian Noise Processes

Definition. A random process $N(t)$ is a white gaussian noise if $N(t)$ is a WSS process satisfying

(1) $\mu_N = E[N(t)] = 0$ (zero mean)

(2) For any time instants $t_1 < t_2 < \ldots < t_k$, $N(t_1), N(t_2), \ldots, N(t_k)$ are independent gaussian random variables.



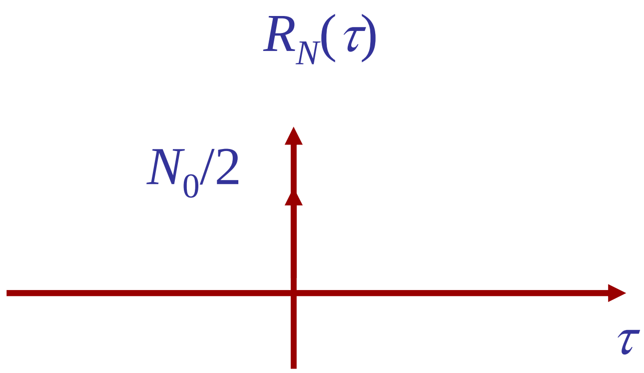
The characteristics of $N(t)$

(1) $N(t)$ is a Gaussian random variable for any time instance t.

(2) Zero mean: $\mu_N = 0$

(3) The autocorrelation function:

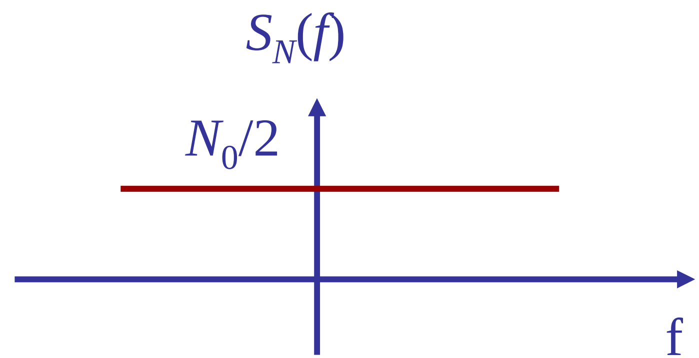
$$R_N(\tau) = \frac{N_0}{2}\delta(\tau), \ -\infty < \tau < \infty$$



(4) The power spectral density is flat:

$$S_N(f) = \frac{N_0}{2}, \ -\infty < f < \infty$$

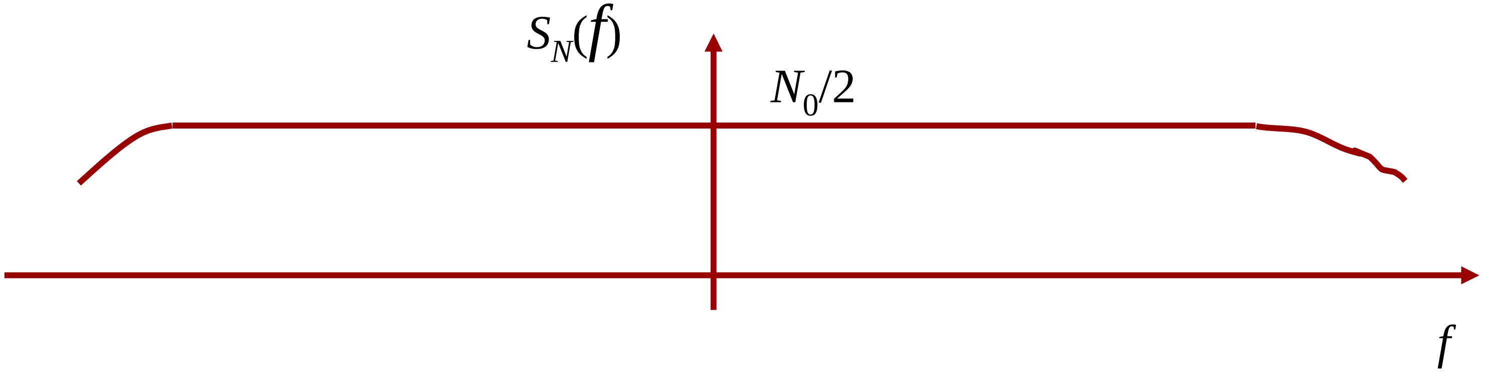
where N_0 is the power per unit bandwidth of $N(t)$.



(5) The average power is infinity:

$$P_N = E[N^2(t)] = \infty$$

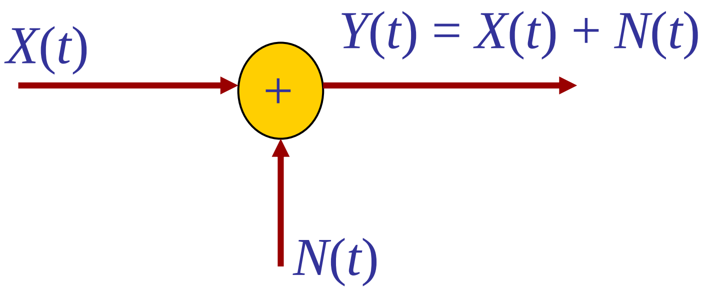
Power spectrum of thermal noise



The above spectrum achieves its maximum at $f = 0$. The spectrum goes to infinity, but the rate of convergence to zero is very slow. From this, we conclude that the thermal noise, though not precisely white, for all practical purpose can be modeled as a white gaussian noise process.

Example 2. 5. Sum with a White Gaussian Noise.

Suppose that $N(t)$ is a white Gaussian noise with power spectrum $N_0/2$. We also assume that $X(t)$ and $N(t)$ are joint WSS and independent. Find the mean and the power spectral density of $Y(t)$.



Solution.

Mean:

$$\mu_Y(t) = E[X(t) + N(t)]$$

$$= E[X(t)] + E[N(t)]$$

$$= \mu_X + 0 = \mu_X$$

Autocorrelation:

$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\big[(X(t) + N(t))(X(t+\tau) + N(t+\tau))\big]$$

$$= R_X(t,\tau) + R_{XN}(t,\tau) + R_{NX}(t,\tau) + R_N(t,\tau)$$

$$= R_X(\tau) + R_N(\tau) = R_Y(\tau)$$

(by independence and joint WSS)

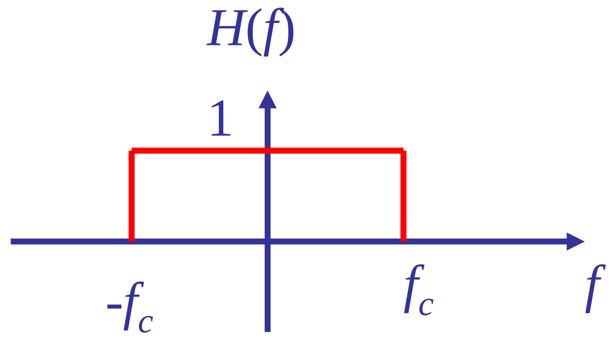
Thus, $Y(t)$ is WSS. The psd of $Y(t)$:

$$S_Y(f) = F(R_Y(\tau)) = F(R_X(\tau) + R_N(\tau))$$

$$= S_X(f) + S_N(f) = S_X(f) + \frac{N_0}{2}$$

Example 2.6 Effect of an Ideal Filter on White Gaussian Noise.

A white gaussian noise $N(t)$ with psd $S_N(f) = \frac{N_0}{2}$ is the input to the ideal low-pass filter shown below.



Find the psd and the autocorrelation of the output process $Y(t)$.
Is the output process $Y(t)$ a white Gaussian noise ?

Solution.

The psd is given by

$$S_Y(f) = S_N(f)\,|H(f)|^2 = \begin{cases} \dfrac{N_0}{2} & \text{for } |f| \le f_c \\ 0 & \text{otherwise} \end{cases}$$

The autocorrelation is the inverse Fourier transform of the psd, thus

$$R_Y(\tau) = F^{-1}(S_Y(f)) = N_0 f_c \operatorname{sin}c(2 f_c \tau)$$

Remark. The ideal low-pass filter transforms the autocorrelation (delta function) of white noise into a sinc function. After filtering, we no longer have white noise. The output signal will have zero correlation with shifted copies of itself, only at shifts of $\tau = n/(2f_c)$ (n not zero).

Example 2.6 Effect of an RC Filter on White Gaussian Noise

A white gaussian noise $N(t)$ with psd $S_N(f) = \frac{N_0}{2}$ is the input to the RC filter with impulse response

$$h(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a = 1/(RC)$.

Find the psd and the autocorrelation of the output process $Y(t)$.

Solution.

The frequency response is:

$$H(f) = F(h(t)) = \frac{1}{a + j2\pi f} \quad \Rightarrow \quad |H(f)|^2 = \frac{1}{a^2 + (2\pi f)^2}$$

Thus

$$S_Y(f) = S_N(f)\,|H(f)|^2 = \frac{N_0}{2} \frac{a^2}{a^2 + (2\pi f)^2}$$

$$R_Y(\tau) = F^{-1}(S_Y(f)) = \frac{N_0 RC}{4} e^{-|\tau|/RC}$$

Again, we no longer have white noise after filtering. The RC filter transforms the input autocorrelation function of white noise into an exponential function.