PROBABILITY & STOCHASTIC PROCESSES — MORGAN BERGEN

SEQUENTIAL EXPERIMENTS — NOV 4 2022

TREE DIAGRAMS

TREE DIAGRAMS DISPLAY THE OUTCOMES OF THE SUBEXPERIMENTS IN A SEQUENTIAL EXPERIMENT.

LABELS OF BRANCHES ARE PROBABILITIES & CONDITIONAL PROBABILITIES

LABELS OF BRANCHES OF THE SECOND SUBEXPERIMENT ARE CONDITIONAL PROBABILITIES OF THE EVENTS IN THE SECOND SUBEXPERIMENT.

LEAFS CORRESPOND TO THE PROBABILITY OF THE SEQUENCE OF SUBEXPERIMENTS

EXAMPLE 2.1

FOR RESISTORS OF A PREVIOUS EXAMPLE WE USE A TO DENOTE THE EVENT THAT A RANDOMLY CHOSEN RESISTOR IS "WITHIN 50 Ω OF THE NOMINAL VALUE.

A "ACCEPTED"

N "NOT ACCEPTABLE"

N IS THE COMPLEMENT OF A

THE EXPERIMENT OF TESTING A RESISTOR CAN BE VIEWED AS A TWO-STEP PROCEDURE

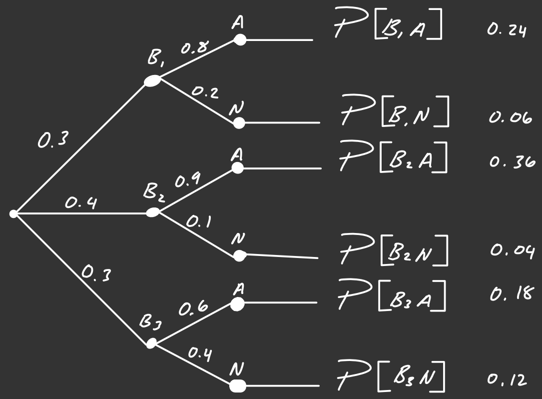
1. IDENTIFY WHICH MACHINE (B_1, B_2, B_3) PRODUCED A RESISTOR
2. WE FIND OUT IF THE RESISTOR IS ACCEPTABLE

DRAW A TREE DIAGRAM FOR THIS SEQ EXP

WHAT IS THE PROBABILITY OF CHOOSING A RESISTOR FROM MACHINE B_2 THAT IS NOT ACCEPTABLE

SEQUENCE 1 — B_1, B_2, B_3

SEQUENCE 2 — A, N



THEOREM 1.7 IS UTILIZED

EX:

$$P[B_1N] = P[B_1 \cap N] = P[N \mid B_1] = \left(\frac{P[N \cap B_1]}{P[B_1]}\right) P[B_1]$$

$$\left(\frac{P[N \cap B_1]}{P[B_1]}\right) P[B_1] = \left(\frac{P[N]\,P[B_1]}{P[B_1]}\right) P[B_1]$$

$$= \left(\frac{(0.3)(0.2)}{(0.3)}\right) (0.3)$$

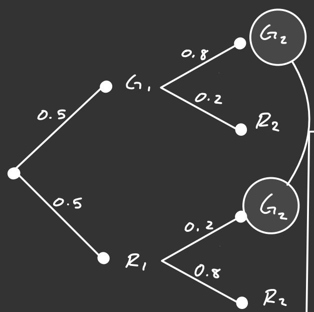
$$= \left(\frac{0.06}{0.3}\right)(0.3)$$

$$P[B_1N] = 0.06$$

EXAMPLE 2.2

TRAFFIC ENGINEERS HAVE COORDINATED THE TIMING OF TWO TRAFFIC LIGHTS TO ENCOURAGE A RUN OF GREEN LIGHTS. TIMING WAS DESIGNED SO THE PROBABILITY 0.8 A DRIVER WILL FIND THE 2ND LIGHT TO HAVE THE SAME COLOR AS THE 1ST. ASSUMING THE 1ST LIGHT IS EQUALLY LIKELY TO BE RED OR GREEN,

1. WHAT IS THE PROBABILITY $P[G_2]$ THAT THE SECOND LIGHT IS GREEN?
2. WHAT IS THE PROBABILITY $P[W]$ THE PROBABILITY THAT YOU WAIT FOR AT LEAST ONE OF THE FIRST TWO LIGHTS?
3. WHAT IS $P[G_1 \mid R_1]$ THE CONDITIONAL PROBABILITY OF A GREEN FIRST LIGHT GIVEN A RED SECOND LIGHT?

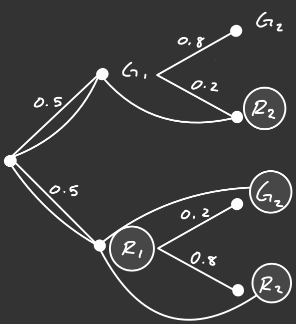


$$P[G_1 G_2] = P[G_1 \cap G_2] = \left(\frac{P[G_1]P[G_2]}{P[G_1]}\right)P[G_1] = \left(\frac{0.5 \cdot 0.8}{0.5}\right)(0.5) = 0.4$$

$$P[R_1 G_2] = P[R_1 \cap G_2] = \left(\frac{P[R_1]P[G_2]}{P[R_1]}\right)P[R_1] = \left(\frac{0.5 \cdot 0.2}{0.5}\right)(0.5) = 0.1$$

$$P[G_2] = P[G_1 G_2] + P[R_1 G_2] = 0.1 + 0.4 = 0.5$$

$$\boxed{P[G_2] = 0.5}$$



$$P[G_1 G_2] = P[G_1 \cap G_2] = \left(\frac{P[G_1]P[G_2]}{P[G_1]}\right)P[G_1] = \left(\frac{0.5 \cdot 0.8}{0.5}\right)(0.5) = 0.4$$

$$P[G_1 R_2] = P[G_1 \wedge R_2] = \left(\frac{P[G_1]P[R_2]}{P[G_1]}\right)P[G_1] = \left(\frac{0.5 \cdot 0.2}{0.5}\right)(0.5) = 0.1$$

$$P[R_1 G_2] = P[R_1 \cap G_2] = \left(\frac{P[R_1]P[G_2]}{P[R_1]}\right)P[R_1] = \left(\frac{0.5 \cdot 0.2}{0.5}\right)(0.5) = 0.1$$

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$$P[W] = P[G_1 R_2] + P[R_1 G_2] + P[R_1 R_2] = 0.6$$

$$P[W] = 1 - P[(G_1 G_2)^c] = 1 - 0.4 = 0.6$$

$$\boxed{P[W] = 0.6}$$

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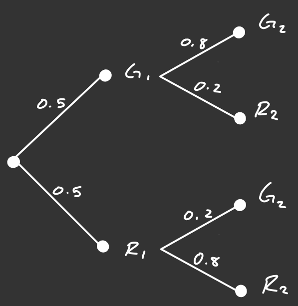
$\boxed{3.}$ WHAT IS $P[G_1 \mid R_1]$ THE CONDITIONAL PROBABILITY OF A GREEN FIRST LIGHT GIVEN A RED SECOND LIGHT?



$$P[G_1 G_2] = P[G_1 \cap G_2] = \left(\frac{P[G_1]P[G_2]}{P[G_1]}\right) P[G_1] = \left(\frac{0.5 \cdot 0.8}{0.5}\right)(0.5) = 0.4$$

$$P[G_1 R_2] = P[G_1 \cap R_2] = \left(\frac{P[G_1]P[R_2]}{P[G_1]}\right) P[G_1] = \left(\frac{0.5 \cdot 0.2}{0.5}\right)(0.5) = 0.1$$

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$$P[R_1 R_2] = P[R_1 \cap R_2] = \left(\frac{P[R_1]P[R_2]}{P[R_1]}\right) P[R_1] = \left(\frac{0.5 \cdot 0.8}{0.5}\right)(0.5) = 0.4$$

$$P[G_1 \mid R_2] = \left(\frac{P[G_1 \cap R_2]}{P[G_1]}\right) = \frac{P[G_1]P[R_2]}{P[G_1]} = \frac{(0.5)(0.2)}{(0.5)} = 0.2$$

$$\boxed{P[G_1 \mid R_2] = 0.2}$$

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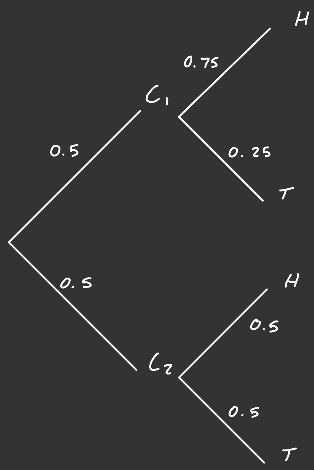
EXAMPLE 2.3

YOU HAVE 2 COINS ONE BIASED & ONE FAIR. COIN 1 IS BIASED. IT COMES UP HEADS W/ PROB OF 3/4, COIN 2 UP HEADS W/ PROB 1/2. SUPPOSE YOU PICK A COIN AT RANDOM & FLIP IT. C_i DENOTES THE EVENT THAT COIN i IS PICKED. LET H & T DENOTE THE POSSIBLE OUTCOMES OF THE FLIP.

1. GIVEN THAT THE OUTCOME OF THE FLIP IS HEAD, WHAT IS $P[C_1|H]$, THE PROBABILITY THAT YOU PICKED THE BIASED COIN?



$$P[C_1 H] = \left(\frac{P[C_1]P[H]}{P[C_1]}\right)P[C_1] = \frac{(0.5)(0.75)}{(0.5)}(0.5) = 0.375$$

$$P[C_1 T] = \left(\frac{P[C_1]P[T]}{P[C_1]}\right)P[C_1] = 0.125$$

$$P[C_2 H] = \left(\frac{P[C_2]P[H]}{P[C_2]}\right)P[C_2] = 0.25$$

$$P[C_2 T] = \left(\frac{P[C_2]P[T]}{P[C_2]}\right)P[C_2] = 0.25$$

$$P[C_1|H] = \frac{P[C_1 H]}{P[C_1 H] + P[C_2 H]} = \frac{0.375}{(0.375)+(0.25)} = 0.6$$

$$\boxed{P[C_1|H] = 0.}$$

2. GIVEN THE OUTCOME IS TAIL, WHAT IS THE PROBABILITY $P[C_1|T]$

$$P[C_1|T] = \frac{P[C_1 T]}{P[C_1 T] + P[C_2 T]} = \frac{(0.125)}{(0.125)+(0.25)} = 0.\overline{33}$$

$$\boxed{P[C_1|T] = 0.\overline{33}}$$

EXAMPLE 2.4

MONTY HALL GAME, A CAR IS BEHIND ONE OF THREE DOORS. YOUR GOAL IS TO SELECT THE DOOR THAT HIDES THE CAR. YOU MAKE A PRELIMINARY SELECTION & THEN A FINAL SELECTION THE GAME PROCEEDS AS FOLLOWS:

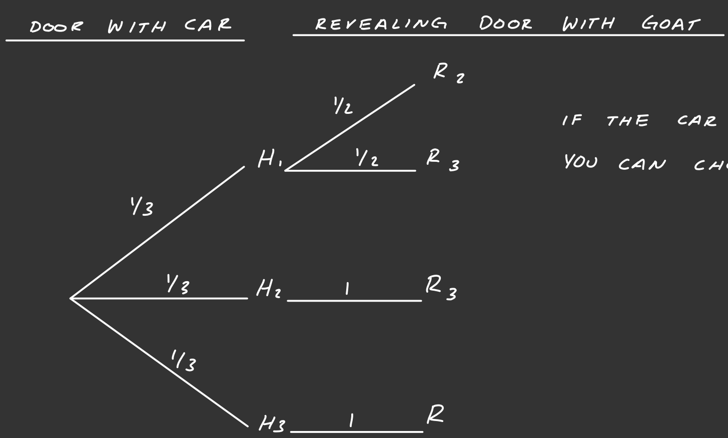
- YOU SELECT A DOOR
- HOST OPENS ONE OF THE TWO DOORS YOU DIDNT SELECT TO REVEAL A GOAT (HE KNOWS WHICH DOOR)
- MONTY THEN ASKS YOU IF YOU WOULD LIKE TO SWITCH YOUR SELECTION TO THE OTHER UNOPENED DOOR
- CHOOSE TO STAY OR SWITCH
- MONTY THEN REVEALS THE PRIZE BEHIND YOUR CHOSEN DOOR.

TO MAX YOUR PROBABILITY $P[C]$ OF WINNING A CAR, IS SWITCHING ...

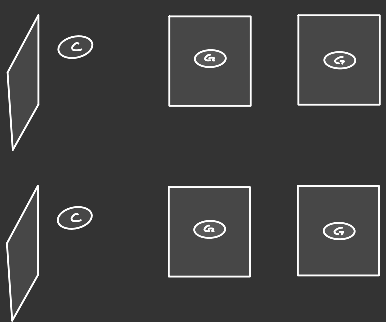
(A) GOOD IDEA
(B) BAD IDEA
(C) NO DIFFERENCE

TO SOLVE THIS WE WILL CONSIDE THE SWITCH AND STAY POLICIES SEPARATELY.
THUS 2 SEPARATE TREE DIAGRAMS.
SWITCH TREE DIAGRAM
STAY TREE DIAGRAM
FIRST WE WILL DESCRIBE WHAT IS THE SAME NO MATTER WHAT POLICY YOU FOLLO
THREE DOORS, 1, 2, 3
LET H_i DENOTE THE EVENT THAT THE CAR IS HIDDEN BEHIND DOOR i, ASSUME YOU CHOOSE DOOR 1

STAY DIAGRAM



IF THE CAR IS BEHIND DOOR 1
YOU CAN CHOOSE TO OPEN DOOR



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- MONTY THEN ASKS YOU IF YOU WOULD LIKE TO SWITCH YOUR SELECTION TO THE OTHER UNOPENED DOOR
- CHOOSE TO STAY OR SWITCH
- MONTY THEN REVEALS THE PRIZE BEHIND YOUR CHOSEN DOOR.

TO MAX YOUR PROBABILITY $P[C]$ OF WINNING A CAR, IS SWITCHING ...

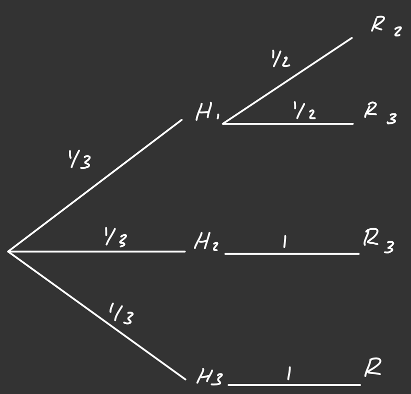
(A) GOOD IDEA

(B) BAD IDEA

(C) NO DIFFERENCE

SWITCH DIAGRAM

CHOOSING DOOR — MONTY REVEALING DOOR WITH GOAT



D1 H_1 WHEN THE CAR IS BEHIND DOOR 1

R_2 MONTY OPENS DOOR 2

D3 G YOU SWITCH TO DOOR 3 AND GET A GOAT

D1 H_2 ?

COUNTING METHODS

IN ALL APPLICATION OF PROBABILITY THEORY IT IS IMPORTANT TO UNDERSTAND THE SAMPLE SPACE OF AN EXPERIMENT. COUNTING METHODS WILL DETERMINE THE NUMBER OF OUTCOMES IN THE SAMPLE SPACE.

EXAMPLE 2.5

CHOOSE 7 CARDS AT RANDOM FROM A DECK OF 52 DIFFERENT CARDS. DISPLAY THE CARDS IN THE ORDER IN WHICH YOU CAN CHOOSE THEM. HOW MANY DIFFERENT SEQUENCES OF CARDS ARE POSSIBLE?

$$\prod_{52}^{46} \ell = 52 \times 51 \times 50 \times \cdots \times 46 = 674,274,182,400 \iff \binom{52}{7} = \frac{52!}{7!(52-7)!}$$

IF AN EXPERIMENT E HAS K SUBEXPERIMENTS $E_1, \ldots, E_K$ WHERE E_i HAS N_i OUTCOMES THEN E HAS $\prod_{i=1}^{K} N_i$ OUTCOMES

$(N)_K$ TO DENOTE THE NUMBER OF POSSIBLE K-PERMUTATIONS OF N DISTINGUISHABLE OBJECTS N DISTINGUISHABLE OBJECTS AND THE EXPERIMENT IS TO CHOOSE A SEQUENCE OF K OF THESE OBJECTS.

$$(N)_K = N(N-1)(N-2)\ldots(N-K+1)$$

THM 2.2 NUMBER OF K-PERMUTATIONS OF N DISTINGUISABLE OBJECTS.

$$(N)_K = N(N-1)(N-2)\cdots(N-K+1)$$

$$(N)_K = \frac{N!}{(N-K)!}$$

THM 2.3

NUMBER OF WAYS TO CHOOSE K OBJECTS OUT OF N DISTINGUISHABLE OBJECTS IS

$$\binom{N}{K} = \begin{cases} \dfrac{N!}{K!(N-K)!}, & K = 0, 1, \ldots, N \\ 0, & \text{OTHERWISE} \end{cases}$$

EXAMPLE 2.10

THERE ARE 4 Q IN A DECK OF 52 CARDS.
YOU ARE GIVEN 7 CARDS AT RANDOM FROM THE DECK.
WHAT IS THE PROBABILITY THAT YOU HAVE NO QUEENS

AFTER $\binom{52}{7}$ OBSERVATION IS TO DETERMINE IF THERE ARE ONE OR MORE QUEENS IN THE SELECTION. THE SAMPLE SPACE $\binom{52}{7}$ POSSIBLE COMBINATIONS OF 7 CARDS

LET $H = \binom{52}{7}$ POSSIBLE COMBINATIONS OF 7 CARDS, EACH WITH PROBABILITY $1/H$

$\exists$ $H_{NQ} = \binom{52-4}{7}$ COMBINATIONS WITH NO QUEENS. THE PROBABILITY OF RECIEVING NO QUEENS IS THE RATIO OF THE NUMBER OF OUTCOMES W/ NO QUEENS : TO THE NUMBER OF OUTCOMES IN THE SAMPLE SPACE. $H_{NQ} / H = 0.5504$ $\therefore$

EX 2.12

LET USB SLOTS A & B. $\forall$ SLOT CAN BE USED FOR CONNECTING A MEMORY CARD (m), CAMERA (c), OR PRINTER (p). IT IS POSSIBLE TO CONNECT TWO MEMORY CARDS, TWO CAMERAS, OR TWO PRINTERS TO THE LAPTOP. HOW MANY WAYS CAN WE USE THE TWO USB SLOTS?

THIS EX CORRESPONDS TO SAMPLING TWO TIMES WITH REPLACEMENT FROM THE SET $\{m, c, p\}$
LET XY DENOTE THE OUTCOME THAT DEVICE TYPE X IS USED IN SLOT A & DEVICE TYPE Y IS USED IN SLOT B.

$$S = \{mm, mc, mp, cm, cc, cp, pm, pc, pp\}$$

FTC := FUNDAMENTAL THEOREM OF COUNTING $\longrightarrow$ $M^N = 3^2$

EXAMPLE 2.15

A CHIP FABRICATION FACILITY PRODUCES MICROPROCESSORS. EACH MICROPROCESSOR IS TESTED TO DETERMINE WHETHER IT RUNS RELIABLY AT AN ACCEPTABLE CLOCK SPEED. A SUB EXPERIMENT TO TEST A MICROPROCESSOR HAS SAMPLE SPACE $S_{SUB} = \{0, 1\}$ TO INDICATE WHETHER THE TEST WAS A FAILURE (0) OR A SUCCESS (1). FOR TEST i, WE RECORD $\{X_i = 0 \;\|\; X_i = 1\}$ TO INDICATE THE RESULT. IN TESTING 4 MICROPROCESSORS, THE OBSERVATION, THE OBSERVATION SEQUENCE $X_1 X_2 X_3 X_4$ IS ONE OF 16 POSSIBLE OUTCOMES

$$\boxed{2^4 = 16}$$

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2.1.5

SUPPOSE THAT FOR THE GENERAL POPULATION 1 IN 5000 PPL CARRY HIV
HIV TEST YIELDS EITHER A POSITIVE (+) OR A NEGATIVE (-)
SUPPOSE THE TEST IS CORRECT 99% OF THE TIME.

WHAT IS $P[-|H]$, THE CONDITIONAL PROBABILITY THAT A PERSON TESTS GIVEN THAT THE PERSON DOES HAVE THE HIV VIRUS?

$$P[-|H] = P[+|H^c] = 1 - P[+|H] = 1 - 99/100 = 0.01$$

$$\boxed{P[-|H] = 0.01}$$

$$P[H] = 1/5000$$
$$P[H^c] = 4999/5000$$
$$\boxed{P[+|H] = 99/100}$$

WHAT IS $P[H|+]$, THE CONDITIONAL PROBABILITY THAT A RANDOMLY CHOSEN PERSON HAS THE HIV VIRUS GIVEN THAT THE PERSON TESTS POSITIVE?
THE PROB THAT A PERSON WHO, HAS THE DISEASE & TESTED POSITIVE IS

WE CANT FIND THIS SO WE MUST MANIPULATE

$$P[H|+] = \frac{P[H \cap +]}{P[H]} = \frac{P[H]\,P[+]}{P[H]}$$

THM 1.1 BAYS THEOREM

WHEN WE HAVE ADVANCE INFORMATION ABOUT $P[A|B]$ BUT WE NEED TO CALCULATE $P[B|A]$ WE DO THE FOLLOWING,

$$P[B|A] = \frac{P[A|B] \cdot P[B]}{P[A]}$$

INTEGRATING THIS THEOREM LEADS US TO THE FOLLOWING,

$$P[+|H] = 99/100$$

$$P[H|+] = \frac{P[+|H]\,P[H]}{P[+]} = \frac{(99/100)(1/5000)}{P[+|H]\,P[H] + P[+|H^c]\,P[H^c]} =$$

$$P[+] = P[+ \cap H] + P[+ \cap H^c]$$
$$= P[+|H]\,P[H] + P[+|H^c]\,P[H^c]$$

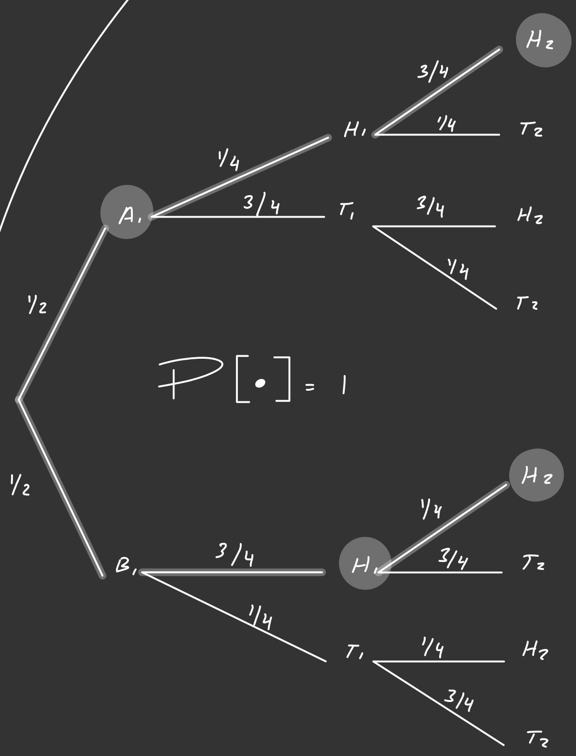
$$= \frac{(99/100)(1/5000)}{(99/100)(1/5000) + (1/100)(4999/5000)}$$

$$= 0.19419$$

$$\boxed{P[H|+] = 0.19419}$$

2.1.7

YOU HAVE 2 BIASED COINS. COIN A COMES UP HEADS WITH PROBABILITY $1/4$, COIN B COMES UP HEADS WITH PROBABILITY $3/4$. HOWEVER YOU DONT KNOW WHICH IS WHICH SO YOU CHOOSE EACH COIN ONCE CHOOSING THE FIRST COIN RANDOMLY. USE H_i & T_i TO DENOTE THE RESULT OF FLIP i LET A_1 BE THE EVENT THAT COIN A WAS FLIPPED FIRST & B_1 BE THE EVENT THAT COIN A WAS FLIPPED FIRST. WHAT IS $P[H_1 H_2]$? ARE H_1 & H_2 INDEPENDENT?



$AH_1BH_2 = H_1H_2 \qquad P[AH_1H_2] = \frac{1}{2}\frac{1}{4}\frac{3}{4} = 3/32$

$AH_1BT_2 = H_1T_2 \qquad P[AH_1T_2] = \frac{1}{2}\frac{1}{4}\frac{1}{4} = 1/32$

$AT_1BH_2 = T_1H_2 \qquad P[AT_1H_2] = \frac{1}{2}\frac{3}{4}\frac{3}{4} = 9/32$

$AT_1BT_2 = T_1T_2 \qquad P[AT_1T_2] = \frac{1}{2}\frac{3}{4}\frac{1}{4} = 3/32$

$BH_1AH_2 = H_1H_2 \qquad P[BH_1H_2] = \frac{1}{2}\frac{3}{4}\frac{1}{4} = 3/32$

$BH_1AT_2 = H_1T_2 \qquad P[BH_1T_2] = \frac{1}{2}\frac{3}{4}\frac{3}{4} = 9/32$

$BT_1AH_2 = T_1H_2 \qquad P[BT_2H_2] = \frac{1}{2}\frac{1}{4}\frac{1}{4} = 1/32$

$BT_1AT_2 = T_1T_2 \qquad P[BT_1T_2] = \frac{1}{2}\frac{1}{4}\frac{3}{4} = 3/32$

THUS $P[H_1H_2] = P[AH_1H_2] + P[BH_1H_2] = 3/32 + 3/32 = 6/32$

$\therefore P[H_1H_2] = 6/32$

H_1 & H_2 WOULD BE INDEPENDENT IFF THERE WAS SAMPLING WITH REPLACEMENT IN ORDER TO DETERMINE THIS WE EXAMINE IF

$$P[H_1]P[H_2] \stackrel{?}{=} P[H_1H_2]$$

$$P[H_1] = P[A_1H_1H_2] + P[A_1H_1T_2] + P[BH_1H_2] + P[BH_1T_2]$$
$$= 3/32 + 1/32 + 3/32 + 9/32$$
$$P[H_1] = 1/2$$

$$P[H_2] = P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2]$$
$$= 3/32 + 9/32 + 3/32 + 1/32$$
$$P[H_2] = 1/2$$

$$P[H_1]P[H_2] \neq P[H_1H_2]$$

$$\boxed{\therefore H_1 \,\&\, H_2 \ ! \text{ INDEP.}}$$

$$(1/2)(1/2) \neq 6/32$$

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2.1.8

NEWBORN W/ DEFECT D W/ PROBABILITY $P[D] = 10^{-4}$. IN A GENERAL EXAM A NEWBORN A HEART ARRHYTHMIA A OCCURS W/ PROB OF 0.99

NO

$P[D] = 10^{-4}$ $\qquad$ $P[\bar{D}] = 0.9999$

$P[A \mid D] = 0.99$

$P[A \mid \bar{D}] = 0.09$

2.3 INDEPENDENT TRIALS

INDEPENDENT TRIALS ARE IDENTICAL SUBEXPERIMENTS IN A SEQUENTIAL EXPERIMENT.

WE START WITH A SUBEXPERIMENT WHERE THERE ARE TWO OUTCOMES: SUCCESS (1) OCCURS WITH A PROBABILITY p OTHERWISE A FAILURE (0) OCCURS WITH A PROBABILITY $1-p$.

$\forall$ RESULTS FROM TRIAL ARE INDEPENDENT. AN OUTCOME OF A COMPLETE EXPERIMENT IS A SEQUENCE OF SUCCESSES & FAILURES.

LET E_{N_0,N_1} DENOTE THE EVENT N_0 FAILURE & N_1 SUCCESSES IN $N = N_0 + N_1$ TRIALS

TO FIND $P[E_{N_0,N_1}]$ WE CONSIDER, NOTE: BINARY SAMPLE SPACE $\rightarrow$ THM 2.6

NUMBER OF OBSERVATIONS N_0 TIMES & $N_1 = N - N_0$ TIMES $= \binom{N}{N_1}$

EXAMPLE 2.19

WHAT IS THE PROBABILITY $P[E_{2,3}]$ TWO FAILURES & THREE SUCCESSES IN FIVE INDEPENDENT TRIALS WITH SUCCESS PROBABILITY p

$$\left.\begin{array}{l} N_0 = 2 \text{ FAILURES} \\ N_1 = 3 \text{ SUCCESSES} \end{array}\right\} \quad N_0 + N_1 = 2 + 3 = 5 = N \text{ TRIALS}$$

$$P[E_{2,3}] = P[E_{N_0=2,\, N_1=3}] = \left|\{\dots\}\right| \Rightarrow \binom{5}{3} = \frac{5!}{3!(5-3)!} = 10 \text{ OUTCOMES}$$

$$= \{11100, 11010, 11001, 10101, 10011, 01110, 01101, 01011, 00111\}$$

$P[\{11100\}] =$ THE PRODUCT OF FIVE PROBABILITIES, EACH RELATED TO ONE SUBEXPERIMENT

OUTCOMES WITH $N_1 = 3$, 3 OF THE $P[\{\}] = p$

OUTCOMES WITH $N_0 = 2$, 2 OF THE $P[\{\}] = (1-p)$

$$P[E_{N_0=2\, N_1=3}] = \binom{5}{3}(1-p)^2 p^3 \iff P[E_{N_0 N_1}] = \binom{N}{N_1}(1-p)^{N_0} p^{N_1}$$

FOR $N = N_0 + N_1$ INDEP. TRIALS WE OBSERVE THAT

- EACH OUTCOME WITH N_0 FAILURES & N_1 SUCCESSES HAVE THE P OF $(1-p)^{N_0} p^{N_1}$
- THERE ARE $\binom{N}{N_0} = \binom{N}{N_1}$ OUTCOMES TO HAVE N_0 FAILURES & N_1 SUCCESSES

THUS THE PROB OF N_1 SUCCESSES IN N INDEPENDENT TRIALS IS THE SUM OF $\binom{N}{N_1}$ TERMS EACH WITH A P OF $(1-p)^{N_0} p^{N_1} = (1-p)^{N-N_1} p^{N_1}$

THM 2.8

PROB OF N_0 FAILURES & N_1 SUCCESSES IN $N = N_0 + N_1$ INDEPENDENT TRIALS IS

$$P[E_{N_0,N_1}] = \binom{N}{N_1}(1-p)^{N-N_1} p^{N_1} = \binom{N}{N_0}(1-p)^{N_0} p^{N-N_0}, \quad N = N_0 + N_1$$

EX 2.20

A RANDOMLY TESTED RESISTOR WAS ACCEPTABLE WITH PROBABILITY $P[A] = 0.78$. IF WE TEST 100 RESISTORS, WHAT IS THE P OF T_i, THE EVENT THAT i RESISTOR TEST ACCEPTABLE?

TESTING EACH RESISTOR IS AN INDEP. TRIAL W/ A SUCCESS OCCURING WHEN A RESISTOR IS ACCEPTABLE

THUS FOR $0 \leq i \leq 100$

$$P[T_i] = \binom{100}{i} (0.78)^i (1-0.78)^{100-i}$$

E.G. $P[T_{78}] \simeq 0.096$

EX 2.21

TO COMMUNICATE ONE BIT OF INFORMATION RELIABLY, PHONES TRANSMIT THE SAME BINARY SYMBOL 5 TIMES. THUS, 00000 := "ZERO" & 11111 := "ONE". RECIEVER DETECTS CORRECT INFO IF THREE OR MORE BINARY SYMBOLS ARE RECIEVED CORRECTLY. WHAT IS THE INFORMATION ERROR PROBABILITY $P[E]$ IF THE BINARY SYMBOL ERROR PROBABILITY IS $q = 0.1$?

WE HAVE 5 TRIALS CORRESPONDING TO THE FIVE TIMES THE BINARY SYMBOL IS SENT

ON $\forall$ TRIAL A SUCCESS (1) OCCURS WHEN A BINARY SYMBOL IS RECIEVED CORRECTLY.

THE P OF SUCCESS IS $p = 1 - q = 0.9$ WHERE $q = P[E] = 0.1$

THE ERROR EVENT OCCURS WHEN THE NUMBER OF SUCCESSES IS STRICTLY LESS THAN THREE

$\Rightarrow$ E $N_1 < 3$

$$P[E] = P[E_{0,5}] + P[E_{1,4}] + P[E_{2,3}]$$

$$P[E_{N_0 N_1}] = \binom{N}{N_1} (1-p)^{N-N_1} p^{N_1} = \binom{N}{N_0} (1-p)^{N_0} p^{N-N_0}$$

$$\cancel{= \binom{N = 0+5}{N_0 = 0} (1-p)^{N_0 = 0} q^{N - N_0 = 5-0}}$$

$$\cancel{= \binom{5}{0} (1-p)^0 q^{5-0} + \binom{5}{1} (1-p)^1}$$

$$P[E] = P[E_{0,5}] + P[E_{1,4}] + P[E_{2,3}]$$

$$= \binom{5}{0} q^5 + \binom{5}{1} p q^4 + \binom{5}{2} p^2 q^3$$

$$= 0.00856$$

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EXAMPLE 2.22

A PACKET PROCESSED BY AN INTERNET ROUTER CARRIES EITHER AUDIO INFO OR WITH P 7/10, VIDEO WITH P 2/10, OR TEXT WITH P 1/10. LET $E_{a,v,t}$ DENOTE THE EVENT THAT THE ROUTER PROCESSES a AUDIO PACKETS, v VIDEO PACKETS, OR t TEXT PACKETS IN SEQUENCE OF 100 PACKETS. IN THIS CASE,

$$P[E_{a,v,t}] = \binom{100}{a,v,t}\left(\frac{7}{10}\right)^a\left(\frac{2}{10}\right)^v\left(\frac{1}{10}\right)^t$$

THM 2.9

A SUBEXPERIMENT HAS SAMPLE SPACE $S_{sub} = \{s_0, \dots, s_{m-1}\}$ WITH $P[s_i] = p_i$ FOR $N = N_0 + \dots + N_{m-1}$ INDEPENDENT TRIALS, THE PROBABILITY OF N_i OCCURENCES OF s_i, $i = 0, 1, \dots, m-1$ IS

$$P[E_{N_0,\dots,N_{m-1}}] = \binom{N}{N_0, \dots, N_{m-1}} p_0^{N_0} \cdots p_{m-1}^{N_{m-1}}$$

PROBLEM O 2.2.3

YOUR CANDY HAS 12 PIECES, THREE PIECES OF EACH FOUR FLAVORS: BERRY, LEMON, ORANGE, & CHERRY ARRANGED IN A RANDOM ORDER IN THE PACK. YOU DRAW THE FIRST THREE PIECES FROM THE PACK

(A) WHAT IS THE PROBABILITY THAT THEY ARE ALL THE SAME FLAVOR?

(B) WHAT IS THE PROBABILITY THAT THEY ARE ALL DIFFERENT FLAVORS?

$B_1, B_2, B_3, L_1, L_2, L_3, O_1, O_2, O_3, C_1, C_2, C_3$

LET B_i, L_i, O_i, C_i DENOTE THE EVENTS THAT THE iTH PIECE IS BERRY, LEMON, ORANGE, & CHERRY RESPECTIVELY. LET F_1 DENOTE THE EVENT THAT ALL THREE PIECES DRAW ARE ALL THE SAME FLAVOR, THUS

$$F_1 = \{B_1B_2B_3, L_1L_2L_3, O_1O_2O_3, C_1C_2C_3\}$$

$$P[F_1] = P[B_1B_2B_3] + P[L_1L_2L_3] + P[O_1O_2O_3] + P[C_1C_2C_3]$$

$$P[B_1B_2B_3] = \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{1}{10} = \frac{1}{220}$$

PROBABILITY THAT THE SEQUENCE / PERMUTATION? IS ALL BBB

$$P[F_1] = \frac{1}{220} + \frac{1}{220} + \frac{1}{220} + \frac{1}{220} = \frac{4}{220} = \frac{1}{55}$$

I DONT UNDERSTAND HOW WE GOT THERE
WHAT THEOREM WAS USE?

COME BACK LATER

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EXAMPLE 2.9

THE NUMBER OF COMBINATIONS OF SEVEN CARDS CHOSEN FROM A DECK IS

$$\binom{52}{7} = \frac{52 \cdot 51 \cdot \ldots \cdot 46}{2 \cdot 3 \cdot \ldots \cdot 7} = 133,784,560$$

THIS IS THE NUMBER OF 7-COMBINATIONS OF 52 OF 52 OBJECTS. BY CONTRAST, WE FOUND IN AN EXAMPLE 7-PERMUTATIONS OF 52-OBJECTS IS 674,274,182,400

IF THERE ARE 11 PLAYERS ON A BASKETBALL TEAM THE LINEUP WILL CONSIST OF FIVE PLAYERS, WHERE THERE ARE $\binom{11}{5} = 462$ POSSIBLE STARTING LINEUPS.

A BASEBALL TEAM HAS 15 FIELD PITCHES & 10 PITCHERS.
EACH FIELD PLAYER CAN TAKE ANY OF THE EIGHT NON PITCHING POSITIONS.

C_1 [15 FIELD PLAYERS $\xrightarrow{\text{FILL}}$ 8 POSITIONS
C_2 [10 PITCHERS $\xrightarrow{\text{FILL}}$ 1 POSITION

$$\binom{N}{K} \Rightarrow \binom{N \text{ CHOOSE}}{K} \Rightarrow \binom{\text{TOTAL DISCRETE OBJECTS}}{\text{TOTAL AMOUNT OF CHOICES}}$$

STARTING LINE UP CONSISTS OF ONE PITCHER & EIGHT FIELD PLAYERS
THUS THE LINEUP POSSIBLITES IS AS FOLLOWS,

$$N = \underbrace{\binom{10}{1}}_{C_2}\underbrace{\binom{15}{8}}_{C_1} = \left(\frac{10!}{1!(10-1)!}\right)\left(\frac{15!}{8!(15-8)!}\right) = 64,350$$

FOR EACH CHOICE OF STARTING LINEUP, THE MANAGER MUST SUBMIT TO THE UMPIRE A BATTING ORDER FOR THE 9 STARTERS. THE NUMBER OF POSSIBLE BATTING ORDERS IS $N \cdot 9! = 23,351,328,000$ SINCE THERE ARE N WAYS TO CHOOSE THE 9 STARTERS, & FOR EACH CHOICE OF 9 STARTERS, THERE ARE $9! = 362,880$ POSSIBLE BATTING ORDERS

EXAMPLE 2.10

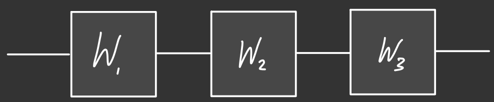
RELIABILITY ANALYSIS

TO FIND THE SUCCESS PROBABILITY OF A COMPLICATED PROCESS WITH COMPONENTS IN SERIES & COMPONENTS IN PARALLEL, IT IS HELPFUL TO CONSIDER A GROUP OF COMPONENTS IN SERIES AS ONE EQUIVALENT COMPONENT & A GROUP OF COMPONENTS IN PARALLEL AS ANOTHER EQUIVALENT COMPONENT.

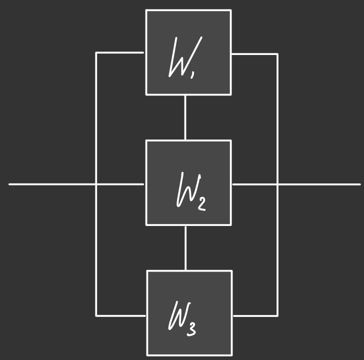
SEQUENTIAL EXPERIMENTS ARE MODELS FOR PRACTICAL PROCESSES THAT DEPEND ON SEVERAL OPERATIONS TO SUCCEED.

WE WILL DESCRIBE AN EXAMPLE IN WHICH ALL OPERATIONS IN A PROCESS SUCCEED WITH PROBABILITY p INDEPENDENT OF THE SUCCESS OR FAILURE OF OTHER COMPONENTS

COMPONENTS IN SERIES



COMPONENTS IN PARALLEL



LET W_i DENOTE THE EVENT THAT COMPONENT i SUCCEEDS.

THERE ARE TWO BASIC TYPES OF OPERATIONS

COMPONENTS IN SERIES - THE OPERATION SUCCEEDS IF ALL OF ITS COMPONENTS SUCCEED

ONE EXAMPLE OF SUCH AN OPERATION IS A SEQUENCE OF COMPUTER PROGRAM IN WHICH EACH PROGRAM AFTER THE FIRST ONE USES THE RESULT OF THE PREVIOUS PROGRAM. THEREFORE THE COMPLETE OPERATION FAILS IF ANY COMPONENT PROGRAM FAILS. WHENEVER THE OPERATION CONSISTS OF K COMPONENTS IN SERIES WE NEED $\forall K$ OPS TO SUCCEED IN ORDER TO HAVE A SUCCESSFUL OPERATION. THE P

$$P[W] = P[W_1 W_2 \cdots W_n] = p \times p \times \cdots \times p = p^n$$

IF THE INDEPENDENT COMPONENTS ARE IN PARALLEL HAVE DIFFERENT SUCCESS PROBABILITIES

$p_1, p_2, \cdots p_n$ T

$$P[W] = P[W_1 W_2 \ldots W_n] = p_1 \cdot p_2 \cdots p_n$$

WITH COMPONENTS IN SERIES THE PROBABILITY OF SUCCESS $<$ THAN WITH THE WEAKEST COMPONENT

COMPONENTS IN PARALLEL, THE OPERATION SUCCEEDS $S(0)$ VS $S(1)$ IS IF $\forall W_n$ WORKS

THIS OPERATION OCCURS WHEN WE INTRODUCE REDUNDANCY TO PROMOTE RELIABILITY. THIS OPERATION IS PREVALENT IN REDUNDANT SYSTEMS.

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RELIABILITY ANALYSIS

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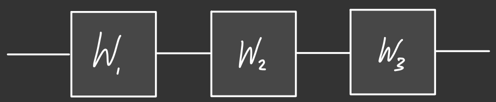
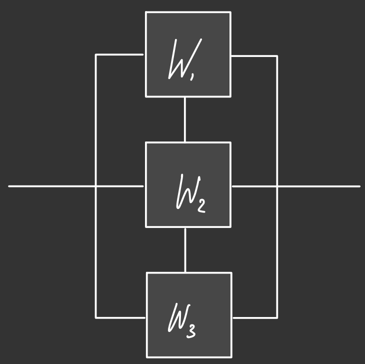
COMPONENTS IN SERIES



$$P[W] = P[W_1 W_2 \cdots W_n]$$

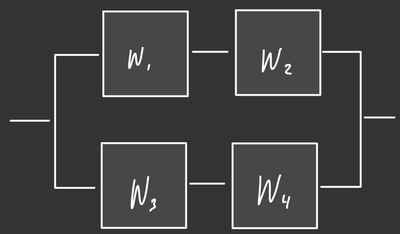
$\forall p_n$

COMPONENTS IN PARALLEL



$$P[W] = 1 - P[W^c]$$
$$= 1 - (1-p)^n$$

INDEPENDENT COMPONENTS IN PARALLEL



$$P[W^c] = P[W_1^c W_2^c \cdots W_n^c]$$
$$= (1-p_1)(1-p_2) \times \cdots \times (1-p_n)$$

LET W_i DENOTE THE EVENT THAT COMPONENT i SUCCEEDS.

$$P[W] = P[W_1 W_2 \cdots W_n] = p \times p \times \cdots \times p = p^n$$

IF THE INDEPENDENT COMPONENTS ARE IN PARALLEL HAVE DIFFERENT SUCCESS PROBABILITIES $p_1, p_2, \cdots p_n$ T

$$P[W] = P[W_1 W_2 \ldots W_n] = p_1 \cdot p_2 \cdots p_n$$

WITH COMPONENTS IN SERIES THE PROBABILITY OF SUCCESS $<$ THAN WITH THE WEAKEST COMPONENT

COMPONENTS IN PARALLEL, THE OPERATION SUCCEEDS $S(0)$ vs $S(1)$ IS IF $\forall W_N$ WORKS
THIS OPERATION OCCURS WHEN WE INTRODUCE REDUNDANCY TO PROMOTE RELIABILITY. THIS OPERATION IS PREVALENT IN REDUNDANT SYSTEMS.

$$P[W^c] = P[W_1^c W_2^c W_3^c \cdots W_n^c] = (1-p)^n$$

THUS THE P THAT PARALLEL SYSTEMS OPERATIONAL SUCCESS IS...

$$P[W] = 1 - P[W^c] = 1 - (1-p^n)$$

PROBABILITY & STOCHASTIC PROCESSES — MORGAN BERGEN

SEQUENTIAL EXPERIMENT — RELIABILITY ANALYSIS — NOV 09 2022

RELIABILITY ANALYSIS

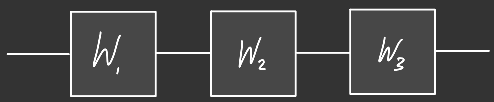
TO FIND THE SUCCESS PROBABILITY OF A COMPLICATED PROCESS WITH COMPONENTS IN SERIES & COMPONENTS IN PARALLEL, IT IS HELPFUL TO CONSIDER A GROUP OF COMPONENTS IN SERIES AS ONE EQUIVALENT COMPONENT & A GROUP OF COMPONENTS IN PARALLEL AS ANOTHER EQUIVALENT COMPONENT.

SEQUENTIAL EXPERIMENTS ARE MODELS FOR PRACTICAL PROCESSES THAT DEPEND ON SEVERAL OPERATIONS TO SUCCEED.

WE WILL DESCRIBE AN EXAMPLE IN WHICH ALL OPERATIONS IN A PROCESS SUCCEED WITH PROBABILITY p INDEPENDENT OF THE SUCCESS OR FAILURE OF OTHER COMPONENTS

WEAK

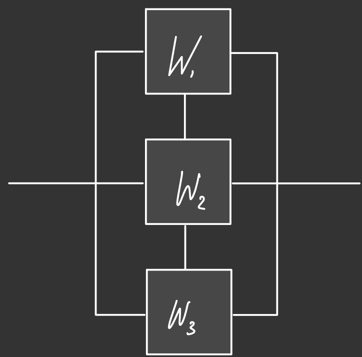
COMPONENTS IN SERIES



$$P[W] = P[W_1 W_2 \cdots W_n]$$

$\forall p_n$

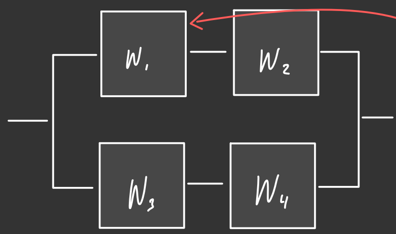
COMPONENTS IN PARALLEL



$$P[W] = 1 - P[W^c]$$
$$= 1 - (1-p)^n$$

STRONG

INDEPENDENT COMPONENTS IN PARALLEL



$$P[W^c] = P[W_1^c W_2^c \cdots W_n^c]$$
$$= (1-p_1)(1-p_2) \times \cdots \times (1-p_n)$$

SIMELTANEOUSLY SEQUENTIAL & PARALLEL

IF THE INDEPENDENT COMPONENTS IN PARALLEL HAVE DIFFERENT SUCCESS PROBABILITIES $p_1, p_2, p_3, \ldots, p_n$

THE OPERATION FAILS WITH PROBABILITY

$$P[W^c] = P[W_1^c W_2^c \cdots W_3^c] = (1-p_1)(1-p_2) \cdots (1-p_n)$$

THE P THAT THE PARALLEL OPERATION SUCCEEDS IS

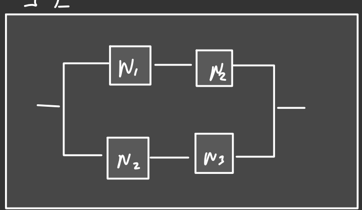
$$P[W] = 1 - P[W^c] = (1-p_1)(1-p_2) \cdots (1-p_n)$$

WITH COMPONENTS IN PARALLEL, THE P THAT AN OPERATION SUCCEEDS IS HIGHER THAN THE PROBABILITY OF SUCCESS OF THE STRONGEST COMPONENT

EX: AN OPT. CONSISTING OF 2 REDUNDANT PARTS, THE FIRST PART HAS TWO COMPONENTS IN SERIES W_1 & W_2, AND THE SECOND PART HAS TWO COMPONENTS IN SERIES W_3 & W_4

$\mathcal{S} := \{W_1, W_{\ldots}, W_4\}$ $\forall W_n$ SUCCEED WITH P OF $p = 0.9$ THUS THE FOLLOWING MUST BE TRUE

Ω



$$P[W_1 W_2] = p_1 \cdot p_2 = p^2 = 0.81$$

?? PAGE 55

POSSIBLE PROOFS

$(A \cup B)^c = A^c \cap B^c$

$A \subset B \quad P[A] \leq P[B]$

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

TEXTBOOK EXAMPLES

ONE OF THE EXERCISES FROM TEXTBOOK

ODD NUMBER PROBLEMS ARE ☐ WILL BE MORE ON

2.2.6

THEOREM 2.3

$$\binom{N}{K} = \frac{(N)_K}{K!} = \frac{N!}{K!(N-K)!}$$

MEANS & VARIANCE

WHATS THE MEAN FOR A UNIFORM CONTINUOUS RANDOM VARIABLES

ONE OR TWO PROBLEMS THAT ARE MORE DIFFICULT FROM THE OTHERS

THAT YOU WOULDNT EXPECT. THE HARD ONE WILL BE AT THE END

TREE DIAGRAMS

DISCRETE

PROBABILITY MASS FUNCTION

MEAN

SECOND MOMENT

COMMON TYPES OF DISCRETE

$\Phi(z) \quad N(0,1)$

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QUIZ 1.5

MONITOR BEHAVIOR IN STORE.

CLASSIFY B — BUYING
N — NOT BUYING
L — STAYING LONG
R — STAYING RAPID

$P[N] = 0.7$

$P[NL] = 0.35$

$P[L] = 0.6$

$P[N] = 0.7 = P[NL] + P[NP]$

$P[L] = 0.6 = P[NL] + P[BL]$

$P[NR] = 0.7 - 0.35 = 0.35$

$P[BL] = 0.6 - 0.35 = 0.25$

$P[BR] = 0.05$

EVENT SPACE FOR 4 POSSIBLE EVENTS REPRESENTED

	N	B
L	0.35	?
R	?	?

	N	B
L	0.35	0.25
R	0.35	?

$P[B \cup L] = P[NL] + P[BL] + P[BR]$
$= 0.35 + 0.25 + 0.05 = 0.65$

$P[N \cup L] = 0.7 + 0.6 - 0.35 = 0.95$

$P[N \cup B] = P[S] = 1$

$P[LR] = P[LL^c] = 0$