EECS 461 PROBABILITY & STATISTICS

HW 09

TUES OCT 25 2022

MORGAN BERGEN

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(1) PROBLEM 5.2.4 CONSIDER THE PROBLEM AS STATED TO BE PART (A) & ADD PART (B) FIND THE MARGINAL PMFs FOR X & Y

FOR TWO INDEPENDENT FLIPS OF A FAIR COIN, LET X EQUAL THE TOTAL NUMBER OF TAILS & LET Y EQUAL THE NUMBER OF HEADS ON THE LAST FLIP. FIND THE PMF $P_{X,Y}(x,y)$

(A/B) RAND VARS X & Y HAVE $(0,1,2)$ & $(0,1)$ THUS THE TABLE REPRESENTS THE JOINT PMF

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0	0	0.25	0.25
$x=1$	0	0.5	0	0.5
$x=2$	0.25	0	0	0
$P_Y(y)$	0.25	0.5	0.25	1

$$P_{X,Y} = \begin{cases} 0.25\,x & , \quad x=0 \text{ \& } y=2 \\ 0.5\,x & , \quad x=1 \text{ \& } y=1 \\ 0.25 & , \quad x=2 \text{ \& } y=0 \\ 0 & , \quad \text{OTHERWISE} \end{cases}$$

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(2)

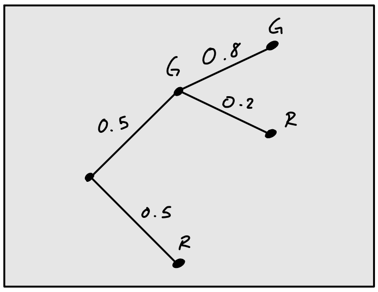
[redacted] Consider two successive traffic lights. Assume the first light is equally likely to be red or green when a random driver approaches it (we are simplifying by assuming there is no amber light). Then assume that, as the driver approaches the second light, the probability that the second light is the same color as the first one was when the driver approached it is 0.8, due to timing coordination. For this pair of lights, let X be the number of green lights that a random driver will encounter when approaching each light. Then let Y be the number of green lights that a random driver will encounter as approaching each light before encountering the first red light.

A • Find the joint PMF of X and Y. Express your answer both as a table and as points (with PMF values) on an X,Y plane. Hint: use a probability tree with outcomes being either a red or green color for each light.

B • Express the probability that the second light is green as a driver approaches in terms of X and Y , then find that probability.

C • Express the probability that at least one light is red as a driver approaches in terms of X and Y , then find that probability

(A) $$P_{X,Y}(x,y) = \begin{cases} 0.5 & , \quad x=0,\ y=0 \\ 0.1 & , \quad x=1,\ y=1 \\ 0.4 & , \quad x=2,\ y=0 \\ 0 & , \quad \text{OTHERWISE} \end{cases}$$



$P_{X,Y}(x,y)$	0	1	$P_Y(y)$
0	0.5	0	0.5
1	0	0.1	0.1
2	0.4	0	0..4
$P_X(x)$	0.9	0.1	1

(B) $P_{X,Y}(x,y) \Rightarrow P_{X,Y}(x=2, y=0) = 0.5$

(C) $P_{X,Y}(x=0, y=0) + P_{X,Y}(x=1, y=1) = 0.5$

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(3) PROBLEM 5.3.4 YOU MAY INITIALLY WANT TO USE A TABLE TO EXPRESS THE MARGINAL PMFs, BUT YOUR FINAL ANSWER SHOULD BE A COMPLETE MATHMATICAL EXPRESSION FOR EACH MARGINAL PMF, THEN USE THOSE MATHMATICAL EXPRESSIONS TO FIND THE MEANS

RANDOM VARIABLES X & Y HAVE JOINT PMF,

$$P_{X,Y}(x,y) = \begin{cases} 1/21 & x = 0, 1, 2, 3, 4, 5 \\ & y = 0, 1, \dots, x \\ 0 & \text{OTHERWISE} \end{cases}$$

FIND THE MARGINAL PMFs $P_X(x)$ & $P_Y(y)$ AND THE EXPECTED VALUES $E[X]$ AND $E[Y]$

$$P_X(x) = \sum_{x=0}^{x} 1/21 \quad \Rightarrow \quad P_X(x) = \frac{x+1}{21}, \quad x = 0, 1, 2, 3, 4, 5$$

$$E[X] = \sum_{x=0}^{5} x P_X(x) = \sum_{x=0}^{5} \frac{x^2 + x}{21} = \frac{1}{21} \sum^{5} [x^2 + x] = \frac{1}{21}[55 + 15] = \frac{70}{21} = \frac{10}{3}$$

$$P_Y(y) = \sum_{y=0}^{5} 6/21 \quad \Rightarrow \quad P_Y(y) = \frac{6-y}{21}, \quad y = 0, 1, 2, 3, 4, 5$$

$$E[Y] = \sum_{y=0}^{5} y P_Y(y) = \sum_{y=0}^{5} \frac{6y - y^2}{21} = \frac{1}{21} \sum_{y=0}^{5} [6y - y^2] = \frac{35}{21} = \frac{5}{3}$$

$P_{X,Y}(x,y)$	0	1	2	3	4	5	$P_X(x)$
0	1/21						1/21
1	1/21	1/21					2/21
2	1/21	1/21	1/21				3/21
3	1/21	1/21	1/21	1/21			4/21
4	1/21	1/21	1/21	1/21	1/21		5/21
5	1/21	1/21	1/21	1/21	1/21	1/21	6/21
$P_Y(y)$	1/21	2/21	3/21	4/21	5/21	6/21	1

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(4) PROBLEM 5.4.2 PARTS A & B ONLY

RANDOM VARIABLES X & Y HAVE JOINT PDF

$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

(A) FIND THE CONSTANT c

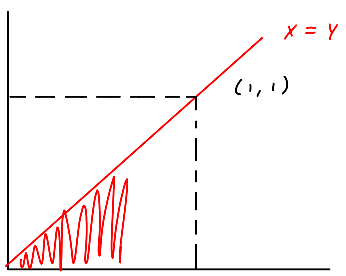
$$\int_0^1\int_0^1 f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1\int_0^1 cxy^2\,dx\,dy = 1 = c\int_0^1 \left[\frac{x^2}{2}\right] y^2 \Big|_0^1 dy =$$

$$1 = c\,\frac{1}{2}\left[\frac{y^3}{3}\right]\Big|_0^1 \Rightarrow 1 = c\left[\frac{1}{2}\right]\left[\frac{1}{3}\right] \Rightarrow 1 = \frac{c}{6} \Rightarrow \boxed{c = 6}$$

REGION OF INTEGRATION AS $x > y$

$x > y$, $y < x < 1$ & $0 < y < 1$

(B) FIND $P[X > Y]$ & $P[Y < X^2]$



$$P[X > Y] = \int_0^1\int_y^1 6xy^2\,dx\,dy = \int_0^1 6y^2\left[\frac{x^2}{2}\right]\Big|_y^1 dy = \int_0^1 6y^2\left[\frac{1}{2} - \frac{y^2}{2}\right] dy = \int_0^1 3y^2 - 3y^4\,dy$$

$$= \left[\frac{3y^3}{3} - \frac{3y^5}{5}\right]\Big|_0^1 = \boxed{\frac{2}{5}}$$

SWITCH ORDER OF INT.

$$P[Y < X^2] = \int_0^{x^2}\int_0^1 6xy^2\,dx\,dy = \int_0^1\int_0^{x^2} 6xy^2\,dy\,dx = \int_0^1 \frac{6xy^3}{3}\Big|_0^{x^2} dx = \int_0^1 2x(x^2 - 0)\,dx$$

$$= \int_0^1 2x^7\,dx = \frac{2x^8}{8}\Big|_0^1 = \frac{2}{8} = \boxed{\frac{1}{4}}$$

$$P[X > Y] \;\&\; P[Y < X^2] = P[X > Y]\cdot P[Y < X^2] = 2/5 \cdot 1/4 = 2/20 = \boxed{1/10}$$

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(5) THE JOINT PDF OF G & H ARE

$f_{G,H}(g,h) = 3g$ FOR $0 \le g \le 1$ & $0 < h \le g$

$f_{G,H}(g,h) = 0$ FOR OTHERWISE

(A) FIND $P[G > 0.5, 0.2 < H]$

$$\int_{0.5}^{h}\int_{0.2}^{g} 3g\,dh\,dg = \int_{0.5}^{1}(3g^2 - 0.6g)\,dg = \frac{3g^3}{3} - \frac{0.6g^2}{2}\Bigg|_{0.5}^{1} = \left(\frac{3(1)^3}{3} - \frac{0.6}{2}\right) - \left(\frac{3(0.5)^3}{3} - \frac{0.6(0.5)^2}{2}\right)$$

$$= (1 - 0.6/2) - (0.05) = 0.7 - 0.05 = \boxed{0.65}$$

(B) FIND THE MARGINAL DISTRIBUTION OF G. EXPRESS IT AS A COMPLETE MATHMATICAL EXPRESSION.

$$f_G(g) = \int_0^g 3g\,dh = 3g^2 \quad \text{SUCH THAT} \quad 0 < g < 1$$

(C) FIND THE MARGINAL DISTRIBUTION OF H. EXPRESS IT " " "

$$f_H(h) = \int_h^1 3g\,dg = \frac{3(1-h^2)}{2} \quad \text{SUCH THAT} \quad 0 < h < 1$$

(D) VERIFY THAT EACH MARGINAL IS VALID (INTEGRATES TO 1)

$$\int_0^1 3g^2\,dg = 3\left(\frac{g^3}{3}\right)\Bigg|_0^1 = 1$$

$$\int_0^1 \frac{3(1-h^2)}{2}\,dh = \frac{3}{2}\int 1 - h^2\,dh = \left(\frac{3}{2}h - \frac{1}{2}h^3\right)\Bigg|_0^1 = 1$$

(E) DETERMINE WHETHER OR NOT G & H ARE INDEP.

$$\frac{9g^2(1-h^2)}{2} \neq 3g, \quad \text{THUS, ...}$$

$f_G(g)\, f_H(h)$ DOES NOT EQUAL $f_{G,H}(g,h)$

$\boxed{\therefore\ g \text{ \& } h \text{ ARE NOT INDEPENDENT}}$

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(6) PROBLEM 5.6.4 OBSERVE INDEPENDENT FLIPS OF A FAIR COIN UNTIL HEAD OCCURS TWICE. LET X_1 EQUAL THE NUMBER OF FLIPS UP TO AND INCLUDING THE FIRST H. LET X_2 EQUAL THE NUMBER OF ADDITIONAL FLIPS TO & INCLUDING THE SECOND H. WHAT ARE $P_{X_1}(x_1)$ & $P_{X_2}(x_2)$. ARE X_1 & X_2 INDEPENDENT? FIND $P_{X_1,X_2}(x_1,x_2)$

PDF OF X_1 : $P_{X_1}(x_1) = (1-p)^{x_1-1} p, x_1 = 1,2,3,4,\ldots$

$p = 1/2$ FOR COIN FLIP

$P_{X_1}(x_1) = 1/2^{x_1}, \quad x_1 = 1, 2, 3, \ldots$

$P_{X_2}(x_2) = 1/2^{x_2}, \quad x_2 = 1, 2, 3, \ldots$

BECAUSE EACH TOSS IS INDEP. BOTH X_1 & X_2 ARE INDEPENDENT FROM ONE ANOTHER

$$P_{X_1,X_2}(x_1,x_2) = P_{X_1}(x_1) P_{X_2}(x_2) = (1/2)^{x_1} \cdot (1/2)^{x_2} = \left(\frac{1}{2}\right)^{x_1+x_2}$$

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(7) TWO INDEPENDENT RVs G & H HAVE THE FOLLOWING PDFs:

$f_G(g) = 1/5$ FOR $5 < g \leq 10$

$= 0$ FOR OTHERWISE

$f_H(n) = K \cdot h^2$ FOR $2 < h \leq 4$ FOR SOME CONSTANT K & 0 OTHERWISE

(A) DETERMINE THE VALUE OF K

$$\int_2^4 f_H(n)\,dh = 1 = \int_2^4 K h^2\,dh = 1 = K\left(\frac{h^3}{3}\right)\Big|_2^4 = 1 \qquad K\frac{56}{3} = 1 \qquad K = \frac{3}{56}$$

(B) FIND THE JOINT PDF OF G & H. BE SURE TO GIVE A COMPLETE DESCRIPTION

$$f_{G,H}(g,h) = f_G(g)\, f_H(n) = (3/280)h^2$$

$$f_{G,H}(g,h) = \begin{cases} (3/280)h^2 & , 5 < g \leq 10 , \ 2 < h \leq 4 \\ 0 & , \text{OTHERWISE} \end{cases}$$

(C) VERIFY THAT THE PDF IS VALID & (INTEGRATES TO 1)

$$1 = \int_5^{10}\int_2^4 \frac{3}{280} h^2\,dh\,dg = \int_5^{10} \frac{3h^3}{840}\Big|_2^4 dg = \int_5^{10} \frac{3(4)^3}{840} - \frac{3(2)^3}{840}\,dg =$$

$$= \int_5^{10} 0.2\,dg = 0.2g\Big|_5^{10} = 0.2(10) - 0.2(5) = 1$$

$1 = 1$ ✓