probability and stochastic processes

link

1. experiments, models, and probabilities

theorem 1.1 demorgan's law related all three basic operations $ (A \cup B)^c = (A^c \cap B^c) \ $

theorem 1.2 for mutually exclusive events A_1 and A_2, $ P[A_1 \cup A_2] = P[A_1] + P[A_2] $

theorem 1.3 If $ A = A_1 \cup A_2 \cup \cdots \cup A_m $ and $ A_i \cap A_j = \emptyset $ for all $ i \neq j $, then

$$ P[A] = \sum_{i=1}^m P[A_i] $$

theorem 1.4 The probability measure $P[.]$ is a function that satisfies the following properties:

1. $ P[\emptyset] = 0 \ $
2. $ P[A^c] = 1 - P[A] \ $
3. For any A and B (not necessarily mutually exclusive), $ P[A \cup B] = P[A] + P[B] - P[A \cap B] $
4. If $ A \subset B,$ $then$ $P[A] \leq P[B] $

Theorem 1.5 The probability of an event $ B = {s_1, s_2, \cdots, s_m} $ is the sum of the probabilities of the outcomes contained in the event:

$$ P[B] = \sum_{i=1}^m P[{s_i}] $$

theorem 1.6 For an experiment with sample space $ S = {s_1, s_2, \cdots, s_n} $ in which each outcomes s_i is equally likely,

$$ P[{s_i}] = \frac{1}{n} \space \space \space 1 ≤ i ≤ n $$

theroem 1.7 A conditional probability measure $ P[A|B] $ has the following properties that correspond to the axioms of probability:

Axiom 1: $ P[A|B] \geq 0 $

Axiom 2: $ P[B|B] = 1 $

Axiom 3: If $ A = A_1 \cup A_2 \cup \cdots \cup A_m $ and $ A_i \cap A_j = \emptyset $ for all $ i \neq j $, then

$$ P[A|B] = P[A_1|B] + P[A_2|B] + \cdots + P[A_m|B] $$

Theorem 1.8 For a partition $ B = {B_1, B_2, \cdots, B_m} $ and any event A in the sample space, let $ C_i = A \cap B_i $ For $ i ≠ j $, the events $ C_i $ and $ C_j $ are mutually exclusive and $ A = C_1 \cup C_2 \cup \cdots $

Theorem 1.9 For any event $ A $ and partition $ {B_1, B_2, \cdots, B_m} $

$$ P[A] = \sum_{i=1}^m P[A \cap B_i $$

Theorem 1.10 Law of total probability

For a partition {$ { B_1, B_2, \cdots, B_m } $} with $ P[B_i] > 0 $ for all $ i $,

$$ P[A] = \sum_{i=1}^m P[A|B_i] P[B_i] $$

Theorem 1.11 Bayes' theorem

$$ P[B|A] = \frac{P[A|B] P[B]}{P[A]} $$

Definition 1.1 Outcome An outcome of an experiment is a possible result of the experiment.

Definition 1.2 Sample space The sample space of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes of the experiment.

Definition 1.3 Event An event is a subset of the sample space.

Definition 1.4 Axioms of Probability A probability measure $ P[.] $ is a function that maps events in the sample spacce to real numbers such that

Axiom 1 For any event $ A $, $ P[A] \geq 0 $

Axiom 2 $ P[S] = 1 $

Axiom 3 For any countable collection $A_1, A_2, \cdots $ of mutually exclusive events,

$$ P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots $$

Definition 1.5 Conditional probability The conditional probability of an event $ A $ given the occurance of the event B is

$$ P[A|B] = {P[AB] \over P[B]}$$

Conditional probability is defined only when $ P[B] > 0 $.

Definition 1.6 Two independent events Two events $ A $ and $ B $ are independent if

$$ P[AB] = P[A]P[B] $$

Definition 1.7 Three Independent Events $ A_1, A_2, A_3 $ are mutually exclusive and independent if and only if

(a) A_1 and $A_2 $ are independent

(b) A_2 and A_3 are independent

(c) A_1 and A_3 are independent

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$

Definition 1.8 More than Two Independent Events

If $n \geq 3$ events $A_1, A_2, \cdots, A_n$ are mutually independent if an only if

(a) all collections of $n - 1$ events chosen from $A_1, A_2, \cdots, A_n$ are mutually independent,

(b) $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2] \cdots P[A_n]$

2. Sequential Experiments

Theorem 2.1

An experiment consists of two subexperiments. If one subexperiment has k outcomes and the other has n outcomes, then the experiment has kn outcomes.

Theorem 2.2

The number of k-permutations of n distinguishable objects is

$$ {n \choose k} = {(n)_k \over k! } = {n! \over {k! (n - k)!}}$$

Theorem 2.4

Given m distinguishable objects, there are m^n ways to choose ith replacement an ordered sample of n objects.

Theorem 2.5

For n repitions of a subexperiment with sample space $S_sub = {s_1, s_2, \cdots, s_m-1}$, the sample space S of the sequential experiment has m^n outcomes.

Theorem 2.6

The number of observation sequences for n subexperiments with sample space $S = {0,1}$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $n \choose n_1$.

Theorem 2.7

For n reptitions of a subexperiment with sample space $S = {s_0, s_1, \cdots, s_m-1}$, the number of length $n = n_0 + n_1 + \cdots + n_{m-1}$ observation sequences with s_i appearing n_i times is

$$ {n \choose n_0, n_1, \cdots, n_{m-1}} = {n! \over {n_0! n_1! \cdots n_{m-1}!}} $$

Theorem 2.8

The probability of n_0 failures and n_1 successes in $n = n_0 + n_1$ independent trials is

$$ P[E_{n_0, n_1}] = {n \choose n_1} (1-p)^{n-n_1} p^n_1 = {n \choose n_0} (1-p)^{n_0} p^{n - n_0} $$

Theorem 2.9

A subexperiment has sample space $S = {s_0, s_1, \cdots, s_m-1}$ with $P[s_i] = p_i$ for $n = n_0 + n_1 + \cdots + n_{m-1}$ independent trials, the probability of n_i occurrences of s_i, $i = 0, 1, \cdots, m-1$ is

$$ P[E_{n_0, n_1, \cdots, n_{m-1}}] = {n \choose n_0, n_1, \cdots, n_{m-1}} p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} $$

Definition 2.1 n choose k

For an integer $n ≥ 0$, we define

$$ {n \choose k} = \begin{cases} {n! \over {k! (n - k)!}} & k = 0, 1, \dots, n, \ 0 & \text{otherwise} \ \end{cases} $$

Definition 2.2 Multinomial coefficient $\space \text{For an integer n ≥ 0, we define }$

$${n \choose n_0, n_1, \dots, n_{m-1}} = {n! \over {n_0! n_1! \cdots n_{m-1}!}}$$

3. Discrete Random Variables

Theorem 3.1

$\text{For a discrete random variable X with PMF} P_X(x), \text{and range} \space S_x: $

$\text{(a) For any x,} \space P_X(x) ≥ 0$

$\text{(b) } \sum_{x \in S_x} P_X(x) = 1$

$\text{(c) For any event} $ $B \subset S_x, \space \text{The probability that X is in the set B is }$

$$P[B] = \sum_{x \in B} P_X(x)$$

Theorem 3.2

$\text{For any discrete random variable X with range} \space S_x = { x_1, x_2, \dots } \space \text{satisfying} \space x_1 ≤ x_2 ≤ \dots $,

$\text{(a) } F_X=(-\infty) = 0 \space \text{and} \space F_X(\infty) = 1 $

$\text{(b) For all } x' ≥ x, F_X(x') ≥ F_X(x) $

$\text{(c) For all } x' > x, F_X(x') > F_X(x) $

$\text{(d) } F_X(x) = F_X(x_i) \text{for all x such that } x_i ≤ x ≤ x_{i+1} $

Theorem 3.3

$ \text{For all b > a, } F_X(b) - F_X(a) = P[a < X ≤ b] $

Theorem 3.4

☑ = p $

Theorem 3.5

☑ = 1/p} $

Theorem 3.6

$ \text{(a) For the binomial (n, p) random variable X of Definition 3.6} $

☑ = np \space $$

$ \text{(b) For the Pascal (k, p) random variable X of Definition 3.7} $

- [x] = k/p $$

$ \text{(c) For the discrete uniform (k, l) random variable X of Definition 3.8} $

- [x] = \frac{k + l}{2} $$

Theorem 3.8

$ \text{Perfom n Bernoulli trials. In each trial, let the probability of success be } {\alpha} / n, \text{where } {\alpha} > 0 \text{is a constant and } n >\alpha.$

$\text{ Let the random variable } K_n \text{ be the number of successes in the n trials. As } n \rightarrow \infty, P_{K_n}(k) \text{ converges to the PMF of a Poisson } (\alpha) \text{random variable. }$

Theorem 3.9

$ \text{For a discrete random variable X, the PMF of Y = g(X) is}$

$$ P_Y(y) = \sum_{x: g(x) = y} P_X(x) $$

Theorem 3.10

$ \text{Given a random variable X with PMF } P_X(x), \text{ and the derived random variable} Y = g(x), \text{ the expected value of Y is }$

$$ E[Y] = \mu_Y = \sum_{x \in S_x} g(x) P_X(x) $$

Theorem 3.11

$ \text{For any random variable X, } $

$$ E[X - \mu_{X}] = 0 $$

Theorem 3.12

$ \text{For any random variable X, }$

- [x] + b $$

Theorem 3.13

$ \text{In the absence of observations, the minimum mean square error estimate random variable X is} $

- [x] $$

Theorem 3.14

- [x] = E[X^2] - \mu^2_X = E[X^2] - (E[X])^2 $$

Theorem 3.15

- [x] $$

Theorem 3.16

- [x] = p(1-p) $
- [x] = ({1-p})/{p^2} $
- [x] = np(1 - p) $
- [x] = k(1 - p)/p^2 $
- [x] = \alpha $
- [x] = (l - k)(l - k + 2)/12 $

Definition 3.1 Random Variable

$\text{A random variable consists of an experiment with a probability measure P[.] defined on a sample space S and a function that assigns a real number to each outcome in the sample spacce of the experiment.}$