

5 MULTIPLE RANDOM VARIABLES

5.6 INDEPENDENT RANDOM VARIABLES

5.7 EXPECTED VALUE OF A FUNCTION OF TWO RANDOM VARIABLES

5.8 COVARIANCE, CORRELATION, AND INDEPENDENCE

5.9 BIVARIATE GAUSSIAN RANDOM VARIABLES

1. RECALL THE AIR SUPPORT ROOF EXAMPLE FROM CLASS, IN WHICH G REPRESENTED INSIDE BAROMETRIC PRESSURE & H REPRESENTED OUTSIDE BAROMETRIC PRESSURE. THE JOINT PDF WAS GIVEN AS:

$$f_{G,H}(g,h) = c/g \quad \text{FOR} \quad 27 \leq h \leq g \leq 33$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

- A. FIND $E[G]$ & $E[H]$ $c = 1.7185$

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h) dh = \int_{-\infty}^{\infty} \frac{c}{g} dh = \frac{c}{g} \int_{h=27}^{h=g} 1 dh = \frac{c}{g} \left[h \right]_{27}^g = \frac{c}{g} (g - 27) = \frac{1.7185}{g} (g - 27) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h) dg = \int_{-\infty}^{\infty} \frac{c}{g} dg = c \int_{g=h}^{g=33} \frac{1}{g} dg = c \int_h^{33} \frac{1}{g} dg = c \left[\log(g) \right]_h^{33} = c \left[\log(33) - \log(h) \right] = -1.7185 \left[\log\left(\frac{33}{h}\right) \right]$$

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(1.7185 - \frac{46.3995}{g} \right), & h \leq g \leq 33 \\ 0, & \text{OTHERWISE} \end{cases} \quad f_H(h) = \begin{cases} -1.7185 \left[\log\left(\frac{33}{h}\right) \right], & 27 \leq h \leq 33 \\ 0, & \text{OTHERWISE} \end{cases}$$



EECS 461 PROBABILITY & STATISTICS
MULTIPLE RANDOM VARIABLES — HOMEWORK 10 5.6 – 5.9
MORGAN BERGEN — OCT 27 2022

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$$f_{G,H}(g,h) = c/g \quad \text{FOR} \quad 27 \leq h \leq g \leq 33 \quad f_{G,H}(g,h) = f_G(g) \cdot f_H(h)$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

A. FIND $E[G]$ & $E[H]$ $c = 1.7185$

C. FIND THE VARIANCE OF BOTH G & H

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h) dh = \int_{-\infty}^{\infty} \frac{c}{g} dh = \frac{c}{g} \int_{h=27}^{h=g} 1 dh = \frac{c}{g} \left[h \right]_{27}^g = \frac{c}{g} (g - 27) = \frac{1.7185}{g} (g - 27) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h) dg = \int_{-\infty}^{\infty} \frac{c}{g} dg = c \int_{g=h}^{g=33} \frac{1}{g} dg = c \int_h^{33} \frac{1}{g} dg = c \left[\log(g) \right]_h^{33} = c \left[\log(33) - \log(h) \right] = 1.7185 \left[\log\left(\frac{33}{h}\right) \right]$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF THE EXPECTED VALUE

$$E[G] = \int_{-\infty}^{\infty} g f_G(g) dg = \int_{g=27}^{g=33} g \left(\frac{c}{g} (27 - g) \right) dg = c \int_{27}^{33} (g - 27) dg = c \int_{27}^{33} g dg - 27c \int_{27}^{33} 1 dg = c \left(\frac{g^2}{2} \right) \Big|_{27}^{33} + -27cg \Big|_{27}^{33}$$

$$= c \left[\frac{33^2}{2} - \frac{27^2}{2} \right] + \left[(-27 \cdot 33c) - (-27 \cdot 27c) \right] = 180c - 162c = 18c = 18 \cdot 1.7185$$

$$E[G] = 1.7185 (18) = 30.933 \quad \therefore E[G] = 30.933$$

$$E[G^2] = \int_{-\infty}^{\infty} g^2 f_G(g) dg = \int_{g=27}^{g=33} g^2 \left(\frac{c}{g} (27 - g) \right) dg = c \int_{27}^{33} (27 - g)g dg =$$

INTEGRATION BY PARTS			
$u = 27 - g$	LOWER BOUND	$u = 27 - 27 = 0$	$= \int_0^{-6} (u - 27) u du = \int_0^{-6} u^2 du + 27 \int_0^{-6} u du = 558$
$du = -dg$	UPPER BOUND	$u = 27 - 33 = -6$	

$$E[G^2] = 1.7185 (558) = 958.923$$

$$VAR[G] = E[G^2] - (E[G])^2 = 958.923 - (30.933)^2 = 2.072511$$

$$\therefore VAR[G] = 2.0725$$

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C. FIND THE VARIANCE OF BOTH G & H

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(\frac{c}{g}\right)^{1.7185} & , \quad 27 \leq g \leq 33 \\ 0 & , \quad \text{OTHERWISE} \end{cases} \quad c = 1.7185$$

$$f_H(h) = \begin{cases} 1.7185 \left[\log\left(\frac{33}{h}\right) \right] & , \quad 27 \leq h \leq 33 \\ 0 & , \quad \text{OTHERWISE} \end{cases}$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF H TO THEN FIND THE EXPECTED VALUE

$$\begin{aligned} E[H] &= \int_{-\infty}^{\infty} h f_H(h) dh = \int_{27}^{33} h \left(c \left[\log(33) - \log(h) \right] \right) dh = c \int_{27}^{33} \left[h \log(33) - h \log(h) \right] dh \\ &= c \left[\left(\log(33) \int_{27}^{33} h dh \right) - \int_{27}^{33} h \log(h) dh \right] = c \left[\frac{1}{2} h^2 \log(33) \Big|_{27}^{33} - \int_{27}^{33} h \log(h) dh \right] \\ &= c \left[\frac{33^2}{2} \log(33) - \frac{27^2}{2} \log(27) - \int_{27}^{33} h \log(h) dh \right] = c \left[\frac{360}{2} \log(33) - \int_{27}^{33} h \log(h) dh \right] \\ &= c \left[\frac{360}{2} \log(33) - \left(\frac{1}{2} h^2 \log(h) \Big|_{27}^{33} - \frac{1}{2} \int_{27}^{33} h dh \right) \right] = c \left[180 \log(33) - \frac{9}{4} (40 + 81 \log(729) - 121 \log(1089)) \right] \\ &= c \left[90 - \frac{729}{4} \log\left(\frac{121}{81}\right) \right], \quad c = 1.7185 \end{aligned}$$

INTEGRATION BY PARTS $\int V du = VU - \int V du \Rightarrow$
LET $V = \log(h)$ $dv = h dh$
 $u = h^2/2$ $du = h^2/2$

$$E[H] \approx 16.855 (1.7185)$$

$$\therefore E[H] \approx 28.9662$$

NEXT WE WILL FIND THE SECOND MOMENT OF H

$$E[H^2] = \int_{-\infty}^{\infty} h^2 f_H(h) dh = \int_{27}^{33} h^2 \left(c \left[\log(33) - \log(h) \right] \right) dh = c \left[\log(33) \int_{27}^{33} h^2 dh - \int_{27}^{33} h^2 \log(h) dh \right] = c \left[\frac{1}{3} h^3 \log(33) \Big|_{27}^{33} - \int_{27}^{33} h^2 \log(h) dh \right] = c \left[5418 \log(33) - \int_{27}^{33} h^2 \log(h) dh \right]$$

INTEGRATION BY PARTS

$$\int u dv = uv - \int v du$$

$$u = \log(h) \quad v = h^3/3$$

$$du = 1/h dh \quad dv = h^2 dh$$

$$= c \left[5418 \log(33) - \left(\frac{1}{3} h^3 \log(h) \Big|_{27}^{33} - \frac{1}{3} \int_{27}^{33} h^2 dh \right) \right] = c \left[5418 \log(33) - \left(-3 (602 + 2187(27) - 3993 \log(33)) \right) \right] = 489.4$$

$$E[H^2] \approx 489.4 (1.7185) \quad \therefore E[H^2] \approx 841.033$$

$$\text{VAR}[H] = E[H^2] - (E[H])^2 = 841.033 - (28.9662)^2 \approx 1.992$$

$$\therefore \text{VAR}[H] \approx 1.992$$

B. FIND $E[G_H]$ AND $\text{COV}[G, H]$

$$E[G_H] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} gh f_{G,H}(g, h) dg dh = \int_{27}^{33} \int_{27}^{33} \left(\frac{c}{g}\right)(gh) dg dh = 6c \log(11/9) = 1080c$$

$$E[G_H] = 1080 (1.7185)$$

$$\therefore E[G_H] = 1855.98$$

$$\begin{aligned} \text{COV}[G_H] &= E[G_H] - E[G] E[H] \\ &= 1855.98 - (30.933 \cdot 28.9662) \end{aligned}$$

$$\therefore \text{COV}[G_H] = 959.9685$$

D. FIND THE VARIANCE OF $(G + H)$

$$\text{VAR}[G + H] = \text{VAR}[G] + \text{VAR}[H] = 2.0725 + 1.992 = 4.0645$$

$$\therefore \text{VAR}[G + H] = 4.0645$$

E. FIND THE CORRELATION COEFFICIENT $\rho_{G,H}$

$$\rho_{G,H} = \frac{\text{COV}[G_H]}{\sqrt{\text{VAR}[G] \text{VAR}[H]}} = \frac{959.9685}{\sqrt{(2.0725)(1.992)}} = 472.460$$

$$\therefore \rho_{G,H} = 472.460$$

FINDING THE MARGINALS
 PRODUCT OF THE MARGINALS

F. FIND $E[G - H]$ AND GIVE A PRACTICAL PHYSICAL INTERPRETATION FOR THIS EXPECTATION

$$E[G - H] \Rightarrow$$

2. MANUFACTURE OF A WIDGET REQUIRES WELDING 2 JOINTS & TIGHTENING 3 BOLTS. LET X BE THE NUMBER OF DEFECTIVE WELDS AND Y BE THE NUMBER OF IMPROPERLY TIGHTENED BOLTS. FROM PAST EXPERIENCE, THE JOINT PMF IS GIVEN BELOW

	$X=0$	$X=1$	$X=2$	$X=3$
$Y=0$	0.840	0.030	0.020	0.010
$Y=1$	0.060	0.010	0.008	0.002
$Y=2$	0.010	0.005	0.004	0.001

- A. FIND THE EXPECTED NUMBER OF DEFECTIVE WELDS, THE EXPECTED NUMBER OF IMPROPERLY TIGHTENED BOLTS, AND THE MEAN TOTAL NUMBER OF MANUFACTURING PROBLEMS

$$E[X] = 0(0.840 + 0.060 + 0.010) + 1(0.030 + 0.010 + 0.005) + 2(0.020 + 0.008 + 0.004) + 3(0.010 + 0.002 + 0.001) \\ = 0(0.91) + 1(0.045) + 2(0.032) + 3(0.012)$$

$$\therefore E[X] = 0.148$$

$$E[X^2] = 0(0.91) + 1^2(0.045) + 2^2(0.032) + 3^2(0.012)$$

$$E[X^2] = 0.2837$$

$$E[Y] = 0(0.84 + 0.03 + 0.02 + 0.01) + 1(0.06 + 0.01 + 0.008 + 0.002) + 2(0.010 + 0.005 + 0.004 + 0.001)$$

$$\therefore E[Y] = 0.12$$

$$E[Y^2] = 1^2(0.08) + 2^2(0.02)$$

$$E[Y^2] = 0.16$$

- B. FIND THE CORRELATION (NOT CORRELATION COEFFICIENT) AND COVARIANCE OF X & Y

$$E[XY] = \sum_{y=0}^2 \sum_{x=0}^3 xy P_{XY}(x, y) = (1)(1)(0.01) + (1)(2)(0.008) + (1)(3)(0.002) + \\ + (1)(2)(0.005) + (2)(2)(0.004) + (3)(2)(0.001)$$

$$\therefore E[XY] = 0.064 \quad \leftarrow \text{CORRELATION}$$

$$\text{COV}[XY] = E[XY] - E[X]E[Y]$$

$$= 0.064 - 0.148(0.12)$$

$$\therefore \text{COV}[XY] = 0.04624 \quad \leftarrow \text{COVARIANCE}$$

- C. FIND THE VARIANCE OF BOTH X AND Y

$$\text{VAR}[X] = E[X^2] - (E[X])^2 \\ = 0.2837 - (0.148^2)$$

$$\therefore \text{VAR}[X] = 0.261796$$

$$\text{VAR}[Y] = E[Y^2] - (E[Y])^2 \\ = 0.16 - (0.12^2)$$

$$\therefore \text{VAR}[Y] = 0.1456$$

2. MANUFACTURE OF A WIDGET REQUIRES WELDING 2 JOINTS & TIGHTENING 3 BOLTS. LET X BE THE NUMBER OF DEFECTIVE WELDS AND Y BE THE NUMBER OF IMPROPERLY TIGHTENED BOLTS. FROM PAST EXPERIENCE, THE JOINT PMF IS GIVEN BELOW

--	$X=0$	$X=1$	$X=2$	$X=3$
$Y=0$	0.840	0.030	0.020	0.010
$Y=1$	0.060	0.010	0.008	0.002
$Y=2$	0.010	0.005	0.004	0.001

$$0.84 + 0.06 + 0.01 = 0.91$$

$$0.84 + 0.03 + 0.02 + 0.01 = .90$$

- D. FIND THE CORRELATION COEFFICIENT $\rho_{X,Y}$

$$\rho_{X,Y} = \frac{\text{COV}[X,Y]}{\sqrt{\text{VAR}[X] \text{VAR}[Y]}} = \frac{0.04624}{\sqrt{(0.2617)(0.1456)}} =$$

$$\therefore \rho_{G,H} =$$

- E. ARE X AND Y INDEPENDENT? JUSTIFY YOUR ANSWER MATHEMATICALLY

NO THE REASON AS TO WHY IS AS FOLLOWS,

$$P_{XY}(x,y) \neq P_X(x) P_Y(y) \quad \text{FOR } \forall x, y \text{ VALUES}$$

$$P_{XY}(0,0) = 0.840 \neq P_X(0) = 0.91 \cdot P_Y(0) = 0.90$$

$$0.84 \neq (0.91 \cdot 0.90)$$

3. THE LENGTH L AND WIDTH W OF A RECTANGLE HAVE JOINT PDF GIVEN BY.

$$f_{L,W}(l,w) = 2e^{-(l+2w)} \text{ FOR } l \geq 0, w \geq 0, \text{ AND } 0 \text{ OTHERWISE}$$

A. FIND THE CORRELATION OF L AND W : $E[LW]$, WHICH IS ALSO THE EXPECTED AREA OF THE RECTANGLE

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy$$

$$E[LW] = \int_0^{\infty} \int_0^{\infty} 2e^{-(l+2w)} lw \, dl \, dw = \int_0^{\infty} w^2 e^{-2w} dw = \boxed{\frac{1}{4}}$$

B. ARE L AND W INDEPENDENT? JUSTIFY YOUR ANSWER MATHEMATICALLY

IN ORDER FOR L & W TO BE INDEPENDENT THE FOLLOWING MUST BE TRUE,

$$f_L(l) = \int_{-\infty}^{\infty} f_{L,W}(l,w) dw = \int_0^{\infty} 2e^{-(l+2w)} dw = 2e^{-l}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{L,W}(l,w) dl = \int_0^{\infty} 2e^{-(l+2w)} dl = e^{-2w}$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} = f_L(l) \cdot f_W(w)$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} = \left(f_L(l) = 2e^{-l} \right) \left(f_W(w) = e^{-2w} \right)$$

$$\boxed{2e^{-(l+2w)} = e^{-l} (2e^{-2w})} \Rightarrow \text{TRUE}$$