EECS 461 Probability and Statistics
Fall Semester 2022
Assignment **#10 COMPLETE NOW** Due **1 November 2022**

Reading: Sections 5.6 - 5.9 in Yates/Goodman (This is the *entire* reading assignment)

Do all of the Quizzes in the Reading assignment but do *not* hand them in. Answers to the Quizzes are on the book's website (search Yates Goodman Wiley)

For all problems from the book, you should use the method(s) from the corresponding section to solve the problem.

1. Recall the air support roof example from class, in which G represented inside barometric pressure and H represented outside barometric pressure. The joint PDF was given as:

   $$f_{G,H}(g,h) = c/g \quad \text{for } 27 \le h \le g \le 33$$

   and 0 otherwise, with c approximately 1.724. The G and H marginals were derived in class. For each part below, show your work and give numerical values for all of your answers.

   a. Find E[G] and E[H].

   b. Find E[GH] and Cov[G,H].

   c. Find the variance of both G and H.

   d. Find the variance of $(G + H)$.

   e. Find the correlation coefficient $\rho_{G,H}$.

   f. Find E[$G - H$] and give a practical physical interpretation for this expectation.

2. Manufacture of a widget requires welding 2 joints and tightening 3 bolts. Let X be the number of defective welds and Y be the number of improperly tightened bolts. From past experience, the joint PMF is given below.

---	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 0$	0.840	0.030	0.020	0.010
$y = 1$	0.060	0.010	0.008	0.002
$y = 2$	0.010	0.005	0.004	0.001

   a. Find the expected number of defective welds, the expected number of improperly tightened bolts, and the mean total number of manufacturing problems.

   b. Find the correlation (not correlation coefficient) and covariance of X and Y.

   c. Find the variance of both X and Y.

   d. Find the correlation coefficient $\rho_{X,Y}$.

e. Are X and Y independent? Justify your answer mathematically.

3. The length L and width W of a rectangle have joint PDF given by: $f_{L,W}(l, w) = 2e^{-(l+2w)}$ for $l \geq 0$, $w \geq 0$ and 0 otherwise.

   a. Find the correlation of L and W: $\mathrm{E}[LW]$, which is also the expected area of the rectangle.

   b. Are L and W independent? Justify your answer mathematically.

4. A random voltage is measured at 2 time instants. Let the RVs X and Y represent those 2 measurements. Both X and Y are Gaussian with mean=0 and variance=4 watts. These 2 measurements are determined to be uncorrelated. Write the joint PDF of the 2 measurements.

5 MULTIPLE RANDOM VARIABLES

5.6 INDEPENDENT RANDOM VARIABLES

5.7 EXPECTED VALUE OF A FUNCTION OF TWO RANDOM VARIABLES

5.8 COVARIANCE, CORRELATION, AND INDEPENDENCE

5.9 BIVARIATE GAUSSIAN RANDOM VARIABLES

1. RECALL THE AIR SUPPORT ROOF EXAMPLE FROM CLASS, IN WHICH G REPRESENTED INSIDE BAROMETRIC PRESSURE & H REPRESENTED OUTSIDE BAROMETRIC PRESSURE. THE JOINT PDF WAS GIVEN AS:

$$f_{G,H}(g,h) = c/g \qquad \text{FOR} \qquad 27 \le h \le g \le 33$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

A. FIND $E[G]$ & $E[H]$ $\quad c = 1.7185$

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{\infty} \frac{c}{g}\,dh = \frac{c}{g}\int_{h=27}^{h=g} 1\,dh = \frac{c}{g}\left[h\Big|_{27}^{g}\right] = \frac{c}{g}\left(g-27\right) = \frac{1.7185}{g}\left(g-27\right) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dg = \int_{-\infty}^{\infty} \frac{c}{g}\,dg = c\int_{g=h}^{g=33} \frac{1}{g}\,dg = c\int_{h}^{33} \frac{1}{g}\,dg = c\left[\text{LOG}(g)\right]\Big|_{h}^{33} = c\left[\text{LOG}(33) - \text{LOG}(h)\right] = -1.7185\left[\text{LOG}\left(\frac{33}{h}\right)\right]$$

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(1.7185 - \dfrac{46.3995}{g}\right), & h \le g \le 33 \\ 0, & \text{OTHERWISE} \end{cases} \qquad f_H(h) = \begin{cases} -1.7185\left[\text{LOG}\left(\dfrac{33}{h}\right)\right], & 27 \le h \le g \\ 0, & \text{OTHERWISE} \end{cases}$$

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MULTIPLE RANDOM VARIABLES — HOMEWORK 10 5.6 – 5.9

MORGAN BERGEN — OCT 27 2022

1. RECALL THE AIR SUPPORT ROOF EXAMPLE FROM CLASS, IN WHICH G REPRESENTED INSIDE BAROMETRIC PRESSURE & H REPRESENTED OUTSIDE BAROMETRIC PRESSURE. THE JOINT PDF WAS GIVEN AS:

$$f_{G,H}(g,h) = c/g \qquad \text{FOR} \qquad 27 \le h \le g \le 33 \qquad\qquad f_{G,H}(g,h) = f_G(g)\cdot f_H(g)$$

AND 0 OTHERWISE, WITH c APPROXIMATELY 1.7185. THE G & H MARGINALS WERE DERIVED IN CLASS FOR EACH PART BELOW, SHOW YOUR WORK & GIVE NUMERICAL VALUES FOR ALL OF YOUR ANSWERS.

A. FIND $E[G]$ & $E[H]$ $\quad c = 1.7185$

C. FIND THE VARIANCE OF BOTH G & H

$$f_G(g) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dh = \int_{-\infty}^{\infty} \frac{c}{g}\,dh = \frac{c}{g}\int_{h=27}^{h=g} 1\,dh = \frac{c}{g}\left[h\Big|_{27}^{g}\right] = \frac{c}{g}(g-27) = \frac{1.7185}{g}(g-27) = 1.7185 - \frac{46.3995}{g}$$

$$f_H(h) = \int_{-\infty}^{\infty} f_{G,H}(g,h)\,dg = \int_{-\infty}^{\infty} \frac{c}{g}\,dg = c\int_{g=h}^{g=33} \frac{1}{g}\,dg = c\int_{h}^{33}\frac{1}{g}\,dg = c\left[\log(g)\right]\Big|_{h}^{33} = c\left[\log(33) - \log(h)\right] = 1.7185\left[\log\left(\frac{33}{h}\right)\right]$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF THE EXPECTED VALUE

$$E[G] = \int_{-\infty}^{\infty} g\,f_G(g)\,dg = \int_{g=27}^{g=33} g\left(\frac{c}{g}(27-g)\right)dg = c\int_{27}^{33}(g-27)\,dg = c\int_{27}^{33} g\,dg - 27c\int_{27}^{33} 1\,dg = c\left(\frac{g^2}{2}\right)\Bigg|_{27}^{33} + -27cg\Big|_{27}^{33}$$

$$= c\left[\frac{33^2}{2} - \frac{27^2}{2}\right] + \left[(-27\cdot 33c) - (-27\cdot 27c)\right] = 180c - 162c = 18c = 18\cdot 1.7185$$

$$E[G] = 1.7185\,(18) = 30.933 \qquad \boxed{\therefore\ E[G] = 30.933}$$

$$E[G^2] = \int_{-\infty}^{\infty} g^2 f_G(g)\,dg = \int_{g=27}^{g=33} g^2\left(\frac{c}{g}(27-g)\right)dg = c\int_{27}^{33}(27-g)g\,dg =$$

INTEGRATION BY PARTS

$u = 27 - g$ $\quad$ LOWER BOUND $\quad u = 27 - 27 = 0$

$du = -dv$ $\quad$ UPPER BOUND $\quad u = 27 - 33 = 6$

$$= \int_{0}^{-6}(u-27)\,u\,du = \int_{0}^{-6} u^2\,du + 27\int_{0}^{-6} u\,du = 558$$

$$E[G^2] = 1.7185\,(558) = 958.923$$

$$\mathrm{VAR}[G] = E[G^2] - \left(E[G]\right)^2 = 958\ 923 - (30.933)^2 = 2.072511 \qquad \boxed{\therefore\ \mathrm{VAR}[G] = 2.0725}$$

C. FIND THE VARIANCE OF BOTH G & H

THUS THE COMPLETE MARGINAL PDFs ARE AS FOLLOWS

$$f_G(g) = \begin{cases} \left(\frac{c}{g}(g-27)\right), & 27 \leq g \leq 33 \quad c = 1.7185 \\ 0, & \text{OTHERWISE} \end{cases} \qquad f_H(h) = \begin{cases} 1.7185\left[\log\left(\frac{33}{h}\right)\right], & 27 \leq h \leq 33 \\ 0, & \text{OTHERWISE} \end{cases}$$

FROM THE MARGINALS WE MUST CALCULATE THE FIRST MOMENT OF H TO THEN FIND THE EXPECTED VALUE

$$E[H] = \int_{-\infty}^{\infty} h f_H(h)\,dh = \int_{27}^{33} h\left(c\left[\log(33) - \log(h)\right]\right)dh = c\int_{27}^{33}\left[h\log(33) - h\log(h)\right]dh = c\left[\left(\log(23)\int_{27}^{33} h\,dh\right) - \int_{27}^{33} h\log(h)\,dh\right] = c\left[\frac{1}{2}h^2\log(23)\Big|_{27}^{33} - \int_{27}^{33} h\log(h)\,dh\right]$$

$$= c\left[\frac{33^2}{2}\log(23) - \frac{27^2}{2}\log(23) - \int_{27}^{33} h\log(h)\,dh\right] = c\left[\frac{360}{2}\log(23) - \int_{27}^{33} h\log(h)\,dh\right] =$$

INTEGRATION BY PARTS $\int v\,du = vu - \int v\,du$ =>

LET $V = \log(h)$ $\quad dv = h\,dh$

$U = h^2/2$ $\quad du = h^2/2$

$$= c\left[\frac{360}{2}\log(23) - \left(\frac{1}{2}h^2\log(h)\Big|_{27}^{33} - \frac{1}{2}\int_{27}^{33} h\,dh\right)\right] = c\left[180\log(23) - \frac{9}{4}\left(40 + 81\log(729) - 121\log(1089)\right)\right] = c\left[90 - \frac{729}{4}\log\left(\frac{121}{81}\right)\right], \quad c = 1.7185$$

$E[H] \simeq 16.855\,(1.7185)$ $\quad \boxed{\therefore E[H] \simeq 28.9662}$

NEXT WE WILL FIND THE SECOND MOMENT OF H

$$E[H^2] = \int_{-\infty}^{\infty} h^2 f_H(h)\,dh = \int_{27}^{33} h^2\left(c\left[\log(33) - \log(h)\right]\right)dh = c\left[\log(33)\int_{27}^{33} h^2\,dh - \int_{27}^{33} h^2\log(h)\,dh\right] = c\left[\frac{1}{3}h^3\log(33)\Big|_{27}^{33} - \int_{27}^{33} h^2\log(h)\,dh\right] = c\left[5418\log(33) - \int_{27}^{33} h^2\log(h)\,dh\right]$$

INTEGRATION BY PARTS

$\int u\,dv = uv - \int v\,du$

$u = \log(h)$ $\quad v = h^3/3$

$du = 1/h\,dh$ $\quad dv = h^2\,dh$

$$= c\left[5418\log(33) - \left(\frac{1}{3}h^3\log(h)\Big|_{27}^{33} - \frac{1}{3}\int_{27}^{33} h^2\,dh\right)\right] = c\left[5418\log(33) - \left(-3\left(602 + 2187(27) - 3993\log(33)\right)\right)\right] = 489.4$$

$E[H^2] \simeq 489.4\,(1.7185)$ $\quad \therefore E[H^2] \simeq 841.033$

$\mathrm{VAR}[H] = E[H^2] - (E[H])^2 = 841.033 - (28.9662)^2 \simeq 1.992$ $\quad \boxed{\therefore \mathrm{VAR}[H] \simeq 1.992}$

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B. FIND $E[GH]$ AND $\mathrm{COV}[G, H]$

$$E[GH] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} gh\, f_{G,H}(g,h)\, dg\, dh = \int_{27}^{33}\int_{27}^{33} \left(\frac{c}{g}\right)(gh)\, dg\, dh = 6c\log(11/9) = 1080c$$

$$E[GH] = 1080\,(1.7185)$$

$$\boxed{\therefore E[GH] = 1855.98}$$

$$\mathrm{cov}[GH] = E[GH] - E[G]\,E[H]$$
$$= 1855.98 - (30.933 \cdot 28.9662)$$

$$\boxed{\therefore \mathrm{Cov}[GH] = 959.9685}$$

D. FIND THE VARIANCE OF $(G + H)$

$$\mathrm{VAR}[G+H] = \mathrm{VAR}[G] + \mathrm{VAR}[H] = 2.0725 + 1.992 = 4.0645$$

$$\boxed{\therefore \mathrm{VAR}[G+H] = 4.0645}$$

E. FIND THE CORRELATION COEFFICIENT $\rho_{G,H}$

$$\rho_{G,H} = \frac{\mathrm{COV}[GH]}{\sqrt{\mathrm{VAR}[G]\,\mathrm{VAR}[H]}} = \frac{959.9685}{\sqrt{(2.0725)(1.992)}} = 472.460$$

$$\boxed{\therefore \rho_{G,H} = 472.460}$$

FINDING THE MARGINALS

PRODUCT OF THE MARGINALS

F. FIND $E[G-H]$ AND GIVE A PRACTICAL PHYSICAL INTERPRETATION FOR THIS EXPECTATION

$E[G-H] \Rightarrow$

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2. MANUFACTURE OF A WIDGET REQUIRES WELDING 2 JOINTS & TIGHTENING 3 BOLTS. LET X BE THE NUMBER OF DEFECTIVE WELDS AND Y BE THE NUMBER OF IMPROPERLY TIGHTENED BOLTS. FROM PAST EXPERIENCE, THE JOINT PMF IS GIVEN BELOW

— — —	X = 0	X = 1	X = 2	X = 3
Y = 0	0.840	0.030	0.020	0.010
Y = 1	0.060	0.010	0.008	0.002
Y = 2	0.010	0.005	0.004	0.001

A. FIND THE EXPECTED NUMBER OF DEFECTIVE WELDS, THE EXPECTED NUMBER OF IMPROPERLY TIGHTENED BOLTS, AND THE MEAN TOTAL NUMBER OF MANUFACTURING PROBLEMS

$$E[X] = 0(0.840 + 0.060 + 0.010) + 1(0.03 + 0.01 + 0.005) + 2(0.020 + 0.008 + 0.004) + 3(0.01 + 0.002 + 0.001)$$
$$= 0(0.91) + 1(0.045) + 2(0.032) + 3(0.013)$$
$$\boxed{\therefore E[X] = 0.148}$$

$$E[X^2] = 0(0.91) + 1^2(0.045) + 2^2(0.032) + 3^2(0.013)$$
$$E[X^2] = 0.2837$$

$$E[Y] = 0(0.84 + 0.03 + 0.02 + 0.01) + 1(0.06 + 0.01 + 0.008 + 0.002) + 2(0.010 + 0.005 + 0.004 + 0.001)$$
$$\boxed{\therefore E[Y] = 0.12}$$

$$E[Y^2] = 1^2(0.08) + 2^2(0.02)$$
$$E[Y^2] = 0.16$$

B. FIND THE CORRELATION (NOT CORRELATION COEFFICIENT) AND COVARIANCE OF X & Y

$$E[XY] = \sum_{y=0}^{2} \sum_{x=0}^{3} xy\, P_{XY}(x,y) = (1)(1)(0.01) + (1)(2)(0.008) + (1)(3)(0.002) + (1)(2)(0.005) + (2)(2)(0.004) + (3)(2)(0.001)$$
$$\boxed{\therefore E[XY] = 0.064}$$ CORRELATION

$$\mathrm{COV}[XY] = E[XY] - E[X]\,E[Y]$$
$$= 0.064 - 0.148(0.12)$$
$$\boxed{\therefore \mathrm{COV}[XY] = 0.04624}$$ COVARIANCE

C. FIND THE VARIANCE OF BOTH X AND Y

$$\mathrm{VAR}[X] = E[X^2] - (E[X])^2$$
$$= 0.2837 - (0.148^2)$$
$$\boxed{\therefore \mathrm{VAR}[X] = 0.261796}$$

$$\mathrm{VAR}[Y] = E[Y^2] - (E[Y])^2$$
$$= 0.16 - (0.12^2)$$
$$\boxed{\therefore \mathrm{VAR}[Y] = 0.1456}$$

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Y = 0	0.840	0.030	0.020	0.010
Y = 1	0.060	0.010	0.008	0.002
Y = 2	0.010	0.005	0.004	0.001

0.84 + 0.06 + 0.01 = 0.91

0.84 + 0.03 + 0.02 + 0.01 = .90

D. FIND THE CORRELATION COEFFICIENT $\rho_{X,Y}$

$$\rho_{X,Y} = \frac{\mathrm{COV}[XY]}{\sqrt{\mathrm{VAR}[X]\,\mathrm{VAR}[Y]}} = \frac{0.04624}{\sqrt{(0.2617)(0.1456)}} =$$

$\therefore \rho_{G,H} =$

E. ARE X AND Y INDEPENDENT? JUSTIFY YOUR ANSWER MATHMATICALLY

NO THE REASON AS TO WHY IS AS FOLLOWS,

$P_{XY}(x, y) \neq P_X(x)\, P_Y(y)$ FOR $! \forall x, y$ VALUES

$P_{XY}(0,0) = 0.840 \neq P_X(0) = 0.91 \cdot P_Y(y) = 0.90$

$0.84 \neq (0.91 \cdot 0.90)$

3. THE LENGTH L AND WIDTH W OF A RECTANGLE HAVE JOINT PDF GIVEN BY.

$$f_{L,W}(l,w) = 2e^{-(l+2w)} \quad \text{FOR } l \geq 0,\ w \geq 0, \text{ AND } 0 \text{ OTHERWISE}$$

A. FIND THE CORRELATION OF L AND W : $E[LW]$, WHICH IS ALSO THE EXPECTED AREA OF THE RECTANGLE

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$E[LW] = \int_0^{\infty}\int_0^{\infty} 2e^{-(l+2w)}\, lw \; dl\,dw = \int_0^{\infty} w^2 e^{-2w}\, dw = \boxed{\frac{1}{4}}$$

B. ARE L AND W INDEPENDENT? JUSTIFY YOUR ANSWER MATHMATICALLY

IN ORDER FOR L & W TO BE INDEPENDENT THE FOLLOWING MUST BE TRUE,

$$f_L(l) = \int_{-\infty}^{\infty} f_{L,W}(l,w)\,dl = \int_0^{\infty} 2e^{-(l+2w)}\,dl = 2e^{-2w}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{L,W}(l,w)\,dw = \int_0^{\infty} 2e^{-(l+2w)}\,dw = e^{-l}$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} = f_L(l)\cdot f_W(w)$$

$$f_{L,W}(l,w) = 2e^{-(l+2w)} = \left(f_L(l) = 2e^{-2w}\right)\left(f_W(w) = e^{-l}\right)$$

$$\boxed{2e^{-(l+2w)} = e^{-l}\left(2e^{-2w}\right)} \Leftarrow \text{TRUE}$$