SFWR ENG 2MX3 Summary

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> *Math objects made using [MathType](); graphs made using [Winplot]().*

Please join GitHub and contribute to this document. There is a guide on how to do this on my GitHub.

CTRL-F (?) to find locations which need to fixed

Table of Contents

Systems

Order of writing symbols, such as exponents: digits-i-variables

There are 3 ways of representing a system:
- Difference Equation
- State Space Equations (ABCD), and
- Impulse Response

Deltas

2 types of deltas (δ):

Kronecker Delta [δ(n)]: discrete time domain, $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}, \int \delta(0) = 1$

Dirac Delta [δ(t)]: continuous time domain, $\delta(t)$, $\delta(0) = \infty$, $\delta(\text{else}) = 0$, $\int\delta(0) = 1$

Also, Heaviside function is not defined at 0, whereas the step function is.

Discrete manipulates functions of n.
Continuous manipulates functions of t.

y = output
Causal system: system = 0 before impulse
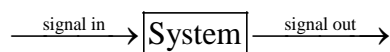
Only use cosine curves; represent sine curves with a phase shift.

If you have a y(n−2) state, but not a y(n−1) state, include a y(n−1) state anyways.

SISO: single-input, single-output
MIMO system: Multiple Inputs Multiple Outputs.

All the systems we examine are "zero-state". Initial state is zero, s(0) = 0

signal in $\longrightarrow$ |System| $\longrightarrow$ signal out

Discrete Systems

If there's no x in the equation, there must be a mistake because there's no input.

- More precise, more theoretical
- Discrete input signals
- Easier to process because it's digital

Continuous System

- e.g. $x(t) = 2\cos(t)$
- Analog

Difference equation

Definition

The first sum is manipulation of input signals and the second part is manipulation of output signals.

$y(n) = \sum_{k=0}^{N} \alpha_k x(n-k) + \sum_{k-1}^{M} \beta_k y(n-k)$, which is interchangeable with:

$H(\omega) = \dfrac{\sum_{k=0}^{\infty} \alpha_k e^{-i\omega k}}{1 - \sum_{k=1}^{\infty} \beta_k e^{-i\omega k}}$ (frequency response)

State Space Equations

ABCD

The **state space equations method** is also known as **ABCD Method**. This is because you represent your states by the four matrices, **A**, **B**, **C**, and **D**. There are 2 state space equations:

- Next state equation
- Output

Given k states, **next state equation**:

$$\mathbf{S}(n+1) = \begin{bmatrix} S_1(n+1) \\ S_2(n+1) \\ \vdots \\ S_k(n+1) \end{bmatrix} = \mathbf{A} \begin{bmatrix} S_1(n) \\ S_2(n) \\ \vdots \\ S_k(n) \end{bmatrix} + \mathbf{B}x(n)$$

Notice how the left side of the equation is the next state of each state? Expanding the matrices would show you how many of each state and the input each state represents.

The output equation is just re-writing the given equation in terms of the states you determined and your inputs:

$$y(n) = \mathbf{C} \begin{bmatrix} S_1(n+1) \\ S_2(n+1) \\ \vdots \\ S_k(n+1) \end{bmatrix} + \mathbf{D}x(n)$$

The state space equation method use matrices of states to represent any equation in a concise way.

States are any part of the given equation that is not y(n) or x(n). That is because the number of x(n)'s (i.e. inputs) are represented by the **B** matrix. On the other hand, the y(n) is not considered a state, but rather, a collection of states. Thus, if your next state is y(n), expand it by equating it to the initial equation of y(n).

e.g.)
$$y(n) = x(n) - y(n-2)$$
Recall that x(n) is already going to be represented by the **B** matrix, so it cannot be a state. However, the $y(n-2)$ *can* be represented by a state. Let's call it S_1:
$$S_1(n) = y(n-2)$$

Now, the next state will be:
$$S_1(n+1) = y(n+1-2) = y(n-1)$$
Hey! There's no matrix for y(n − 1). There are only matrices for y(n) and x(n)! So we should make another state. Think of it like a temporary variable where the value of the variable is passed down, until S_1.

Ok, so let's call this state S_2:
$$S_2(n) = y(n-1)$$

Let's look at the next states of both S_1 and S_2:

$$S_1(n) = y(n-2) \qquad\qquad S_1(n+1) = y(n-1)$$
$$S_2(n) = y(n-1) \qquad\qquad S_2(n+1) = y(n)$$

Remember that we're supposed to represent the next states by a combination of the previous states? So let's change those next states. Recall: when you have y(n), you expand it into the states it represents.
$$S_1(n+1) = y(n-1) = S_2(n)$$
$$S_2(n+1) = y(n) = x(n) - S_1(n)$$

For the next part, you need to understand matrix expansion. For now, I'll re-write each next-state equation individually before combining it to show what's actually going on:

$$S_1(n+1) = 0S_1(n) + 1S_2(n) + 0x(n)$$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} S_1(n) \\ S_2(n) \end{bmatrix} + [0]x(n)$$
$$S_2(n+1) = -1S_1(n) + 0S_2(n) + 1x(n)$$
$$= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} S_1(n) \\ S_2(n) \end{bmatrix} + [1]x(n)$$

Now put those two together to get:
$$\mathbf{S}(n+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} S_1(n) \\ S_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} x(n)$$

The output equation is found the same way:
$$y(n) = -1S_1(n) + 0S_2(n) + x(n)$$
$$= \underbrace{\begin{bmatrix} -1 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} S_1(n) \\ S_2(n) \end{bmatrix} + \underbrace{[1]}_{\mathbf{D}} x(n)$$

You can use these matrices for determining impulse response.

Linear Time Invariant (LTI) **System**: you may flip order of system and delay (?)

Converting to Difference Equation
Euler's equations convert states to the difference equation.
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Compound Interest
An application of the state space equation other than signals is for determining compound interest.

n: billing period
α: interest
x: deposits
$$y(n) = y(n-1) + \alpha y(n-1) + x(n)$$
$$= (1+\alpha)y(n-1) + x(n)$$

Similar to an impulse, let's model someone putting money in a bank and leaving it there forever, where A is the value of an initial deposit:

$$x(n) = \begin{cases} A, & n = 0 \\ 0, & \text{else} \end{cases}$$

$$y(0) = A$$

$$y(1) = (1+\alpha)A$$

$$y(2) = (1+\alpha)^2 A$$

So you can interpolate: $y(n) = (1+\alpha)^n A$

Impulse Response

Impulses can be modeled by the Kronecker delta function.

FIR

Finite Impulse Response (FIR) system: A system that has an impulse response that has a finite duration (is zero at a finite time). One way of seeing if it is finite is if it has no y's in it,

e.g.) State Space Table

y(n) = x(n) + x(n−2)

How we model systems in this course is using a **state space table**. This table analyzes what the output of the given system is when you are given an impulse. The prof usually leaves the labels blank, but this time, I'll leave them in there. Next time, however, they won't be there:

Time	0	1	2	3	4
Input [$x(n)$]	1	0	0	0	0
Output [$y(n)$]	1	0	1	0	0

From this table, you can tell that it's an FIR system, since the output is 0 after $t = 2$.

IIR

Infinite Impulse Response (IIR) system. A system that has an impulse response that has an infinite duration (continues to respond indefinitely). IIR systems generally have y on both sides of the equation

e.g.)

y(n) = x(n) + 0.5y(n−1)

	1	0	0	0	0
	1	½	¼	⅛	1/16

Notice how the system doesn't end? This is why it is called an IIR system. Also, notice how the labels are gone? Get used to it and imagine them there every time you make a table like this.

Impulse Response

Impulse Response is notated as $h(n)$. It is the output, $y(n)$, when the input, $x(n)$ is a combination of shifts of the Kronecker delta function.

To clarify what the equations mean: $\xrightarrow{x(n)}$ $\boxed{h(n)}$ $\xrightarrow{y(n)}$

Remember ABCD representation? Impulse response can be represented by inputting the ABCD matrices:

$$h(n) = \begin{cases} 0, & n < 0 \\ \mathbf{D}, & n = 0 \\ \mathbf{CA}^{n-1}\mathbf{B}, & n > 0 \end{cases}$$

Convolution Sum

$$y = h * x$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

If something appears to have the convolution equation, it <u>will</u> be convolution and thus, you may use its properties.

Impulse response to difference equation

Sometimes your impulse response may be specified. There are 2 ways of converting your impulse response into the difference equation:
- Convolution Sum method
- Table method

e.g.)

We're going to use the same example to demonstrate both methods

$$h(n) = \begin{cases} 0, & \text{else} \\ 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \end{cases}$$

$$x(n) = \delta(n) + \delta(n-2)$$

To simplify the process, take the $h(n)$ and split it up into a sum of diracs:

$$h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

Convolution Equation method

$$\boxed{y(n) = \sum_{i=-\infty}^{\infty} h(n-i)x(i)}$$

Let's plug in some values of n:

$$y(0) = \sum_{i=0}^{\infty} h(0-i)x(i)$$

However, since $h(\text{anything} < 0) = 0$, we only get output when $i = 0$:

$$y(0) = h(0) \cdot x(0)$$
$$= 1 \cdot 1 = 1$$

$$y(1) = \sum_{i=0}^{\infty} h\underbrace{(1-i)}_{0,1}x(i)$$
$$= h(1-0) \cdot x(0) + h(1-1)x(1)$$
$$= (2 \cdot 1) + (1 \cdot 0) = 2$$

$$y(2) = \sum_{i=0}^{\infty} h(2-i)x(i)$$
$$= h(2-0)x(0) + h(2-1)x(1) + h(2-2)x(2)$$
$$= (3 \cdot 1) + (2 \cdot 0) + (1 \cdot 1)$$
$$= 3 + 0 + 1 = 4$$

$$y(3) = \sum_{i=0}^{\infty} h(3-i)x(i)$$
$$= h(3-0)x(0) + h(3-1)x(1) + h(3-2)x(2) + h(3-3)x(3)$$
$$= (0 \cdot 1) + (3 \cdot 0) + (2 \cdot 1) + (1 \cdot 0)$$
$$= 2$$

We're supposed to stop at this point for some reason.

Table method

For each of the impulses in $x(n)$, look at the impulse response. Add all the $h(n)$'s up at the end.

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)}$$

n	0	1	2	3	4	5
$\delta(n)$ [x(0)]	1	0		0	0	0
h(0)	1	2	3	0	0	0
$\delta(n-2)$	0	0	1	0	0	0
h(2)	0	0	1	2	3	0
y(n)	1	2	4	2	3	0

Kemal's Method

Someone else probably thought of this before me, but there's a faster way of directly converting your impulse response to your difference equation:

$$h(n) = \begin{cases} \alpha_0, & n = 0 \\ \vdots \\ \alpha_N, & n = N \end{cases}$$

Plug it into: $y(n) = \sum_{k=0}^{N} \alpha_k x(n-k)$ and expand.

It may be easier to convert the impulse response to an equation of dirac deltas.

If you aren't given an equation for x at this point, assume $x(n) = \delta(n)$. You may then leave it in/convert to dirac delta equations.

Step Response

Step Response: $\text{step}(n-k) = \begin{cases} 0, & n < k \\ 1, & n \geq k \end{cases}$

$y = h * x$

$y(n) = \sum_{k=-\infty}^{\infty} h(k) \text{step}(n-k)$

You can't have $h(k < 0)$, since time starts at 0:

$y(n) = \sum_{k=0}^{\infty} h(k) \text{step}(n-k)$

k can only be as big as n before that iteration of the sum is negative:

$y(n) = \sum_{k=0}^{n} h(k) \cdot 1_{*}$

*step(0…n)

Discrete and Continuous Frequency

	Discrete	Continuous
Period	$p = \dfrac{\text{samples}}{\text{cycle}}$ (must be a natural number)	$p = \dfrac{\text{seconds}}{\text{cycle}}$
Frequency	$f = \dfrac{1}{p} = \dfrac{\text{cycles}}{\text{sample}}$	$f = \dfrac{1}{p} = \dfrac{\text{cycles}}{\text{second}} = \text{Hz}$
Normalized Frequency	$\omega = 2\pi f = \dfrac{\text{radians}}{\text{sample}}$	$\omega = 2\pi f = \dfrac{\text{radians}}{\text{second}}$
Sampling Frequency	$f_s = \dfrac{\text{samples}}{\text{second}}$	
Pitch	$\text{Pitch} = f_s \cdot f_{\text{discrete}} = \dfrac{\text{samples}}{\text{second}} \cdot \dfrac{\text{cycles}}{\text{sample}} = \dfrac{\text{cycles}}{\text{second}} = \text{Hz}$	

Remember that *upper-case* variables (frequency-domain) are *multiplied*: $Y(\omega) = H(\omega) \times X(\omega)$, and *lower-case* variables (time-domain) are *convoluted*: $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$

$$x(n) = k(n+kp)$$
$$= \cos\left(\tfrac{\pi}{4} n\right) \leftarrow \omega = \tfrac{\pi}{4}$$
$$= \tfrac{\pi}{4} n = k 2\pi$$
$$\Rightarrow p = 8$$

Your period is selected such that the coefficient × p = a multiple of 2π. If there is a phase shift, don't worry about it when calculating period, frequency, or normalized frequency.

Often the coefficient can actually be your normalized frequency (remember this to save steps). Choose an ω_0, such as $\omega_0 = \pi/4$ and find the period, such that $\omega_0 \cdot p$ is any multiple of 2π, i.e. $\omega_0 \cdot p = k2\pi$
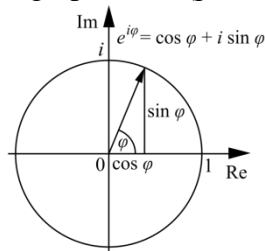
e.g.

If you wanted to normalize $x = \tfrac{3\pi}{4} n$ (should be divisible by 2)

$p = 8$

$$\omega = \frac{2\pi}{p} = \frac{\pi}{4}$$

Complex Numbers

Draw a graph where the x values represent the real dimension and the y values represent the complex dimension. This is actually the graph of e^{ix} (pretend the φ's are actually x's):

Refer to Euler's equations a lot (in the difference equations section).

e.g. $e^{i\frac{\pi}{4}}$

argument ($\measuredangle$): the degree of the point on the circle
The argument of each point is actually the value of x currently being examined, such that
$$\measuredangle e^{ix} = x$$
$$\measuredangle e^{i\frac{\pi}{4}} = \tfrac{\pi}{4}$$

gain: magnitude of the line connecting the origin to the point

Divide by the magnitude on the unit circle. Thus, the point is: $\left(\frac{1}{2}, \frac{i}{2}\right)$

Know your special triangles.

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

Also remember how the unit circle works, such as how $e^{i\pi} = -1$

Summary: numbers can be converted to their complex exponential equivalent by finding the gain and the angle, then num $= \text{gain}\, e^{i\measuredangle}$

$\frac{1}{i} = -i$

Frequency Response

Intro

Frequency Response [H(ω)]: Although H(ω) is actually called the transfer function (if you ever need to Google it), call it frequency response for now

Upper-case variable: frequency domain (i.e. a function of ω)
Lower-case variable: time domain (i.e. a function of t or n)

Formula: $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}} = \dfrac{\sum\limits_{k=0}^{\infty} Y_k e^{-i\omega_0 k}}{1 - \sum\limits_{k=1}^{\infty} X_k e^{-i\omega_0 k}}$

e.g. if $x(n) = e^{i\omega n}$, $y(n) = H(\omega) e^{i\omega n}$

$\measuredangle H(\omega) = $ phase shift of the cosines in H(ω)

The gain is represented by $\|H(\omega)\|$

When your X_k's go to 0, they are known as **poles** or **eigenfrequencies** because they result in an ∞ frequency response.
Eigenfrequency is the frequency that will cause a glass to break
Resonance frequency

Conservative: you don't retain energy (?)

Discrete LTI Signal

Given a discrete-time signal, x(n) and an H(ω), the output is computed in the following way:
Your input may contain multiple terms. Treat each cosine individually.

To understand what variables I am referring to, take each cosine and compare it to this general equation: $\boxed{y(n) = \|H(\omega)\| \cos(\omega n + \sphericalangle H(\omega))}$

Note: if x(n) = a number [such as 5], treat it as if k = 0, so x(n) = 5cos(0n)

But wait! Was that a y on the left side of the equation? Shouldn't that be an x? No. I just showed you the output equation. This is the equation you're going to put your answer in after you have solved for the variables. I showed you this first because your x is going to be very similar to this.

1. Convert all sines in your x(n) to cosines
2. Identify the ω's from each of the equations in x. ω ALWAYS comes from x.

To make it easier to organize the information you have just gotten, make a table with a different row for each cosine. This works because each cosine should have its own ω. To find the H(ω) for each row, plug the ω of that row into H(ω)

Output to Frequency Response: $\boxed{y(n) = \|H(\omega)\| x(n + \sphericalangle H(\omega))}$

e.g. y(DE) 1
Find the output to the difference equation
y(n) + y(n−1) = 2x(n) − 5x(n−2)
Convert to frequency response
$$H(\omega)e^{i\omega n} + H(\omega)e^{i\omega(n-1)} = 2e^{i\omega n} - 5e^{i\omega(n-2)}$$
$$H(\omega)\left(e^{i\omega n} + e^{i\omega n}e^{-i\omega}\right) = 2e^{i\omega n} - 5e^{i\omega n}e^{-2i\omega}$$
$$H(\omega)\left(1 + e^{-i\omega}\right) = 2 - 5e^{-2i\omega}$$
Factor everything out to isolate H(ω):
$$H(\omega) = \frac{2 - 5e^{-2i\omega}}{1 + e^{-i\omega}}$$

e.g. y(H,x) 1

(I'm supposed to use $H(\omega) = \dfrac{1 + e^{-i\omega}}{e^{-i\omega}}$, but I actually did it with the previous H)

Use the frequency response to determine the output to
$$x(n) = \cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) + \sin\left(\pi n + \tfrac{3\pi}{2}\right)$$
$$\boxed{H(\omega) = \|H(\omega)\| e^{i\sphericalangle H(\omega)}}$$

ω	H(ω)	‖H(ω)‖	$\sphericalangle H(\omega)$
0	− 3/2	3/2	−π

| $\pi/2$ | Just plugin ω into H $$\frac{2-5e^{-2i\frac{\pi}{2}}}{1+e^{-i\frac{\pi}{2}}}$$ $$=\frac{2-5(-1)}{1-i}$$ $$=\frac{7}{1-i}$$ $$=\frac{7+7i}{1+1}$$ $$=\frac{7}{2}+\frac{7}{2}i$$ | What is the magnitude of the point, H(ω), on the e^{ix} graph $$\sqrt{3.5^2+3.5^2}=3.5\sqrt{2}$$ | $\frac{7}{2}+\frac{7}{2}i=3.5\sqrt{2}e^{i\sphericalangle H(\omega)}$ $1+i=\sqrt{2}e^{i\sphericalangle H(\omega)}$ $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=e^{i\sphericalangle H(\omega)}$ $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=e^{i\sphericalangle H(\omega)}$ $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}\boxed{=\frac{-7\pi}{4}}$ |
| π | 0 | 0 | 0 |

Where ϕ is the previous phase shift:

$$\boxed{y(n)=\|H(\omega)\|[\cos/\sin](\omega n+\phi+\sphericalangle H(\omega))}$$

$$y(n)=\tfrac{3}{2}\cos(-\pi)=\tfrac{-3}{2}+5\cos\left(\tfrac{\pi}{2}n-\tfrac{7\pi}{4}+\tfrac{\pi}{4}\right)$$

$$y(n)=\tfrac{-3}{2}+\tfrac{7}{\sqrt{2}}\cos\left(\tfrac{\pi}{2}n-\tfrac{3\pi}{2}\right)$$

e.g. y(H,x) 2

$$H(\omega)=\cos(\omega)$$

$$x(n)=2+\cos\left(\tfrac{\pi}{2}n+\tfrac{\pi}{4}\right)+\sin\left(\pi n+\tfrac{3\pi}{2}\right)$$

$$=2\cos(0n)+\cos\left(\tfrac{\pi}{2}n+\tfrac{\pi}{4}\right)+\cos(\pi n+\pi)$$

ω	H(ω)	$\|H(\omega)\|$	$\sphericalangle H(\omega)$
0	1	1	$0\leftarrow\arg(e^{i0})$
$\pi/2$	0	0	$-$
π	-1	1	$-\pi$

<u>The argument will always be negative because you can't look ahead of time.</u>

r = 1; pi <= t <= 2pi
(x,y) = (-1,0)
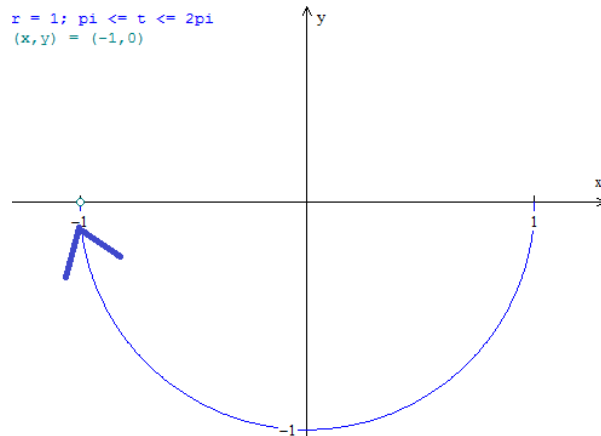
Now that we have this information, plug it into the y-equation from earlier.

$$y(n) = \|H(\omega)\| \cos(\omega n + \phi + \sphericalangle H(\omega))$$

$$y(n) = 1 \cdot 2 + 0 \cdot \cos\left(\tfrac{\pi}{2} n + \tfrac{\pi}{4}\right) + |-1| \cos(\pi n + \pi - \pi)$$

$$= 2 + \cos(\pi n)$$

e.g. y(DE,x) 2

$$y(n) = x(n) + x(n-2)$$

$$x(n) = 1 + \cos\left(\tfrac{\pi}{2} n\right) + \cos\left(\pi n + \tfrac{\pi}{2}\right)$$

Given that each impulse is represented by $e^{i\omega n}$, you could temporarily represent all x's with $e^{i\omega n}$, then plug in the value of x afterwards.

$$x(n) = e^{i\omega n}, \, y(n) = H(\omega) e^{i\omega n}$$

$$H(\omega) e^{i\omega n} = e^{i\omega n} + e^{i\omega(n-2)}$$

$$H(\omega) = \frac{e^{i\omega n} + e^{i\omega n} e^{-2i\omega}}{e^{i\omega n}}$$

$$= 1 + e^{-2i\omega}$$

ω	H(ω)	‖H(ω)‖	$\sphericalangle H(\omega)$		
0	$1+e^0 = 2$	2	0		
$\pi/2$	$1+e^{-i\pi} = 1-1 = 0$	0	−		
π	$1+e^{-2i\pi} = 1+1 = 2$	2	0		

$$y(n) = \|H(\omega)\| x(n + \sphericalangle H(\omega))$$

Plug in the equation of the output to each impulse:

$$y(n) = 2 + 0\cos\left(\tfrac{\pi}{2} n\right) + 2\cos\left(\pi n + \tfrac{\pi}{2}\right) = 2 + 2\cos\left(\pi n + \tfrac{\pi}{2}\right)$$
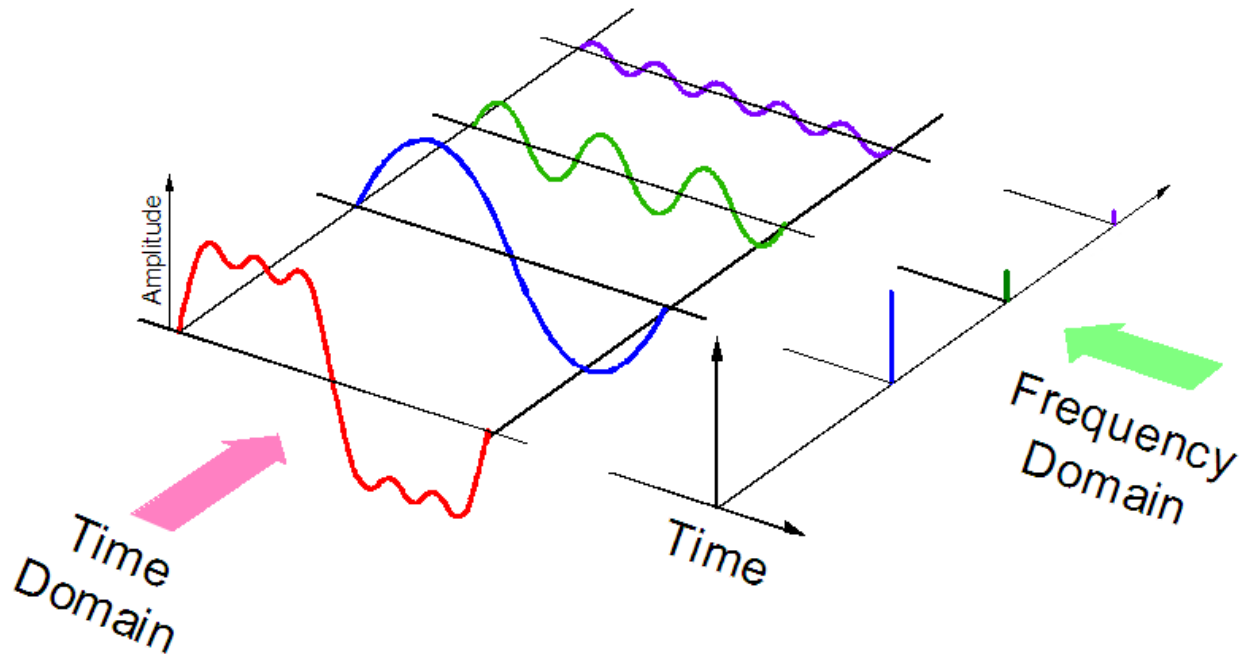
e.g. DE(H)

$$H(\omega) = \tfrac{1}{2} e^{-2i\omega} + \tfrac{1}{2} e^{0i\omega} \leftarrow y = H(\omega) e^{i\omega n}$$

$$H(\omega)e^{i\omega n} = \tfrac{1}{2}e^{i\omega(n-2)} + \tfrac{1}{2}e^{i\omega n}$$
$$y(n) = \tfrac{1}{2}e^{i\omega(n-2)} + \tfrac{1}{2}e^{i\omega n}$$
$$= \tfrac{1}{2}x(n-2) + \tfrac{1}{2}x(n) \leftarrow x(n) = e^{i\omega n}$$

Fourier

Time and Frequency Domain
[Gif demonstration of what the frequency domain is](.).

Impulse Response to Frequency Response
In the Fourier Series, ω is not what it usually is ($2\pi f$), since it changes with each k or n (depending on discrete / continuous). That is why we represent $2\pi f$ by a constant, ω_0. It is calculated the same way as you would calculate ω, normally (i.e. $\omega_0 = 2\pi/p = 2\pi f$).

Although it may seem that X_k is not a function that is affected by frequency, it is because each frequency, ω, is simply a multiple of the fundamental frequency that is chosen, ω_0.
Fundamental frequency [ω_0]: a constant
For discrete frequency: $\omega = \omega_0 \times k$
For continuous frequency: $\omega = \omega_0 \times m$

Forward transform: time $\rightarrow$ frequency domain (look for negative exponents on e)
Transform^{-1}: frequency $\rightarrow$ time domain (look for positive exponents on e)

If h is a function of time, then CTFT; if not, DTFT. i.e. h(t), do CTFT; h(n), do DTFT
The impulse response, $h(n)$, of a system is the $y(n)$ when $x(n)$ is an impulse: $\delta(n)$.

k is the specific frequency you are aiming to get. It ranges from 0 to (p−1)

N = p

Master Chart
Note:

	FS (Fourier Series)	FT (Fourier Transform)
DT (Discrete Time)	$X_k = \dfrac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ink\omega_0}$ $x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$ $x(n) = \dfrac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{i\omega n}\,d\omega$ See more below
CT (Continuous Time)	$X_m = \dfrac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}\,dt$ $x(t) = \sum_{m=-\infty}^{\infty} X_m e^{im\omega_0 t}$ See more below	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$ $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\,d\omega$ Summary: $\text{CTFT}\left(ce^{i\omega_0 t}\right) = c\cdot 2\pi\delta(\omega-\omega_0)$ **essentially, Laplace**

CTFS
Notice how there is no range for the CTFS? That's because it is actually from −∞ to ∞. An easier way to do it than computing it is to convert everything from your x(t) equation into complex exponentials with the form $x(t) = e^{i\omega t}$. Now realize that X_n is the output of impulse response, so

$$y(t) = H(\omega)x(t)$$

$$H(\omega) = \frac{y(t)}{x(t)} = \frac{X_m}{e^{i\omega t}} = X_m e^{-i\omega t} = X_m e^{-i\omega_0 mt}$$

That means the coefficients in front of each of your complex exponentials in your x(t) equation represent a different X_m.

e.g. $x(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \frac{1}{2}e^{i\cdot 1\cdot 1\cdot t} + \frac{1}{2}e^{i\cdot 1(-1)t}$, $X_{m=1}=X_1 = \frac{1}{2}$, $X_{m=-1}=X_{-1} = \frac{1}{2}$.

Notice how n is directly related to ω? That's why each m is known as a frequency.

Sometimes, you can notice that the ω's are multiples of the same ω. This is because the function is periodic.

e.g.)
$\omega_1 = \frac{\pi}{4}, \omega_2 = \frac{\pi}{2}$

$\text{CTFS: } X_m = \frac{1}{p}\int_0^p x(t)e^{-i\omega_0 mt}\,dt$

Taking $X_m = \frac{1}{p}\int_0^p \left(\frac{1}{2}e^{i\frac{\pi}{2}t} + \frac{1}{2}e^{-i\frac{\pi}{2}t} + \frac{1}{2}e^{i\frac{\pi}{4}t} + \frac{1}{2}e^{-i\frac{\pi}{4}t}\right)e^{-i\omega_0 mt}\,dt$,

Choose an ω_0, such as $\omega_0 = \pi/4$ and find the period, such that $\omega_0 \cdot p$ is any multiple of 2π, i.e. $\omega_0 \cdot p = k2\pi$

$p = 8$

$$X_m = \frac{1}{16}\int_0^8 \left(e^{i\frac{\pi}{2}t} + e^{-i\frac{\pi}{2}t} + e^{i\frac{\pi}{4}t} + e^{-i\frac{\pi}{4}t}\right)e^{-i\frac{\pi}{4}mt}\,dt$$

Now, put all the items in terms of ω_0:

$$X_m = \frac{1}{16}\int_0^8 \left(e^{i\frac{\pi}{4}2t} + e^{-i\frac{\pi}{4}2t} + e^{i\frac{\pi}{4}t} + e^{-i\frac{\pi}{4}t}\right)e^{-i\frac{\pi}{4}mt}\,dt$$

Now, expand:

$$X_m = \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(1-m)} + e^{i\frac{\pi}{4}t(2-m)} + e^{i\frac{\pi}{4}t(1+m)} + e^{i\frac{\pi}{4}t(2+m)}\,dt$$

$$\boxed{\begin{aligned}\frac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}\,dt &= \frac{p}{p}x(t)\delta(m) \\ \Rightarrow \frac{1}{p}\int_0^p e^{-im\omega_0 t}\,dt &= \delta(m)\end{aligned}}$$

$$X_m = \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(1-m)}\,dt + \frac{1}{16}\int_0^8 e^{-i\frac{\pi}{4}t(1+m)}\,dt + \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(2-m)}\,dt + \frac{1}{16}\int_0^8 e^{-i\frac{\pi}{4}t(2+m)}\,dt$$

$$= \tfrac{1}{2}\delta(1-m) + \tfrac{1}{2}\delta(1+m) + \tfrac{1}{2}\delta(2-m) + \tfrac{1}{2}\delta(2+m)$$

$$X_1 = \tfrac{1}{2}, X_{-1} = \tfrac{1}{2}, X_2 = \tfrac{1}{2}, X_{-2} = \tfrac{1}{2}$$

DTFS

Don't split your $x(n)$ up into the complex exponential equivalences

k's downward

n's across

$$\text{CTFT}\{h(n) = \delta(n-a)\} = e^{-i\omega a}$$

CTFT/CFT & DTFT

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$$

$$\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega = 2\pi\delta(\omega)$$

$$\boxed{\int_{-\infty}^{\infty}\left(c\cdot e^{i\omega_0 t}\right)dt = c\cdot 2\pi\delta(\omega - \omega_0)}$$

$$\boxed{\int_{-\infty}^{\infty}\left(c\cdot e^{i\omega_0 n}\right)dn = c\cdot 2\pi\delta(\omega - \omega_0)}$$

Since sine and cosine are continuous functions, they can't simply be represented by impulses. Thus, assume 2π-periodicity and add '$-2\pi n$' to the end of the Kronecker Delta:

$$\cos(\omega_0 t) \Rightarrow \pi \sum_{n=0}^{7} \left[\delta(\omega + \omega_0 - 2\pi n) + \delta(\omega - \omega_0 - 2\pi n) \right]$$

$$\sin(\omega_0 t) \Rightarrow \pi i \sum_{n=0}^{7} \left[\delta(\omega + \omega_0 - 2\pi n) - \delta(\omega - \omega_0 - 2\pi n) \right]$$

DFT/FFT

Here's a very simple article that explains what the DFT does. It also gives you an idea of what the other Fourier equations do.

The DFT is actually the most popular transform, because there's an efficient algorithm for it called the **Fast Fourier Transform** (FFT). It's has the same equation as the DTFS, but with different scaling. See the difference:

DTFS	DFT/FFT
$X_k = \dfrac{1}{p} \displaystyle\sum_{m=0}^{p-1} x(m) e^{-imk\omega_0}$	$X_k = \displaystyle\sum_{m=0}^{p-1} x(m) e^{-imk\omega_0}$
$x(n) = \displaystyle\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$	$x(n) = \dfrac{1}{p} \displaystyle\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$

The DFT^{-1} is also known as the **synthesis equation**.

Because the [forward] DFT doesn't divide out the number of points, p, using a higher sampling rate will make the DFT larger for the same input. This is why it's actually better to scale the DFT output by 1/p anyways when looking at it afterwards.
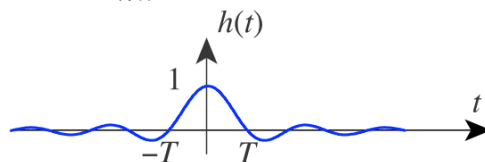
Filter a signal
http://www.falstad.com/fourier/

Sinc Function

Filtered signals are generally represented by the **sinc function**: $y(x) = \dfrac{\sin(x)}{x}$

Normalized sinc function: $y(x) = \dfrac{\sin(\pi x)}{\pi x}$

Fourier of sinc:
$$\forall \, \omega \in \mathbb{R}, \quad X(\omega) = \begin{cases} T & \text{if } |\omega| \le \pi/T \\ 0 & \text{otherwise} \end{cases}$$

Transforming a Signal example

If we are given an equation for a signal and we're trying to put it into a different form, think of it like taking a signal generator and putting some sort of filter on it.

For example,

$$x_1(t) = \begin{cases} \frac{\pi}{a}, & -a \leq t \leq a \\ 0, & \text{else} \end{cases} \Leftrightarrow X(\omega) = 2\pi \frac{\sin(a\omega)}{a\omega}$$

into:

$$x_2(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ 0, & \text{else} \end{cases}$$

In order to do that, we'll need to use the CFT. However, before we can that, we need to figure out our **amplitude** [a] and how we're going to going to manipulate the original signal to get the thing we want.

Since we all know that amplitude is half the period, (p = 2−0 = 2), a = 2/2 = 1, so

$$x_1(t) = \begin{cases} \pi, & -1 \leq t \leq 1 \\ 0, & \text{else} \end{cases} \Leftrightarrow X(\omega) = 2\pi \frac{\sin(\omega)}{\omega}$$

Now, to determine how the transform will be, you need to **scale** and **shift** the function.
To scale, you need to figure out what factor (let's say k) should be multiplied by $x_1(t)$ to get $x_2(t)$

$$x_1(t) = \pi, \qquad -1 \leq t \leq 1$$
$$x_2(t) = x_1(t) \cdot k$$
$$x_2(t) = k\pi$$
$$1 = k\pi$$
$$k = \tfrac{1}{\pi}$$

With that in mind, we now need to shift the function. Let's say N is your shifting factor. When shifting in the <u>time domain</u>, you know that you could easily add or subtract, like $x(t) \Rightarrow x(t - N)$. However, we will need to translate that to the <u>frequency domain</u>, which is different. For this to mirror in the frequency domain, you need to multiply $X(\omega)$ by $e^{-i\omega N}$.

If $x(n) \xrightarrow{\text{CFT}} X(\omega)$,

$$x(n - N) \xrightarrow{\text{CFT}} X(\omega) \cdot e^{-i\omega N}$$

(skip the proof)

Proof

Take $x(t) = 2\pi\delta(t)$. Well remember this function from the CTFT section?

$$\int_{-\pi}^{\pi} e^{i\omega(t-N)} dt = 2\pi\delta(t - N)$$

Yeah, so N is our shift. Let's expand that equation

$$\int_{-\pi}^{\pi} e^{i\omega t} e^{-i\omega N} dt = 2\pi\delta(t-N)$$

$$e^{-i\omega N} \underbrace{\int_{-\pi}^{\pi} e^{i\omega t} dt}_{\substack{2\pi\delta(t) \\ x(t)}} = 2\pi\delta(t-N)$$

Putting it together

We have a range from −1 to 1 and we want a range from 0 to 2. This means we want to use $x(t - 1)$ to bring it one second from the past.

Therefore, your final equation will be:

$$\boxed{X(\omega) = e^{-i\omega N} kX(\omega)}$$

Or specifically for this example:

$$X(\omega) = e^{-i\omega 1} \frac{1}{\not\pi} 2 \not\pi \frac{\sin(\omega)}{\omega}$$

$$= e^{-i\omega} 2 \frac{\sin(\omega)}{\omega}$$

Sawtooth

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$

Since the range is 0-1, you know $p = 1$

$\omega_0 = 2\pi/p = 2\pi/1 = 2\pi$

Use the <u>CTFS</u> to find X_m. I'll just use a range of 0 to 1, since that's the only range that x is not 0.

$$X_m = \frac{1}{1} \int_0^1 t e^{-i 2\pi m t} dt$$

Since k is an integer, the ω will always be a multiple of 2π. In the complex domain, $\omega + 2\pi m = \omega$. So $2 \cdot \pi \cdot m = 2\pi = 0$

So we can equate the above equation to:

$$X_m = \int_0^1 t e^{-i0t} dt = \int_0^1 t dt$$

Since an integral is the area under the curve, where the curve is a triangle,

$$X_m = \int_0^1 t dt = \frac{1}{2}$$

Filter Design

When designing a filter, use tilde ($\tilde{H}$) to denote a prototype of your filter.

Note: "removal of frequency component at a point" means H(point) = 0, so you just pretend the value is a root at that time.

Constant signal means that the magnitude of the frequency response will be at the gain most of the time, such as |H(0)|

a.k.a. band-stop filter
$$H\left(\tfrac{\pi}{2}\right)=0$$
$$H\left(0\right)=1$$

1) Set up the equation such that you have $\tilde{H}\left(\omega\right)=\prod\left(e^{-i\omega}-e^{-i\{\text{roots}\}}\right)$ because when $\omega=$ the root, your bracket will $=0$. In our case, your only root is at $\pi/2$, so:
$$\tilde{H}\left(\omega\right)=\left(e^{-i\omega}-e^{-i\frac{\pi}{2}}\right)$$

2) These systems require **Conjugate complex symmetry (CCS)** (When do things require CCS?) You need it so that your answer is real. If you don't have it, your output will be incorrect. Thus, when you have a value, you need to reflect it across the real axis

$$\cos\left(x\right)=\tfrac{1}{2}e^{i\omega}+\tfrac{1}{2}e^{-i\omega}$$

To allow for conjugate complex symmetry, we'll add a second part to the prototype:
$$\tilde{H}\left(\omega\right)=\left(e^{-i\omega}-e^{-i\frac{\pi}{2}}\right)\underbrace{\left(e^{-i\omega}-e^{i\frac{\pi}{2}}\right)}_{\text{allows CCS}},$$
$$=e^{-i2\omega}+1$$
To account for this, add a second initial condition, the complex conjugate, $H\left(\tfrac{-\pi}{2}\right)=0$.

3) Now plug in your other initial condition:
Although $H\left(0\right)=1$, $\tilde{H}\left(0\right)=2$.
To account for the difference, insert a constant of ½:
$$H\left(\omega\right)=\tfrac{1}{2}\left(e^{-i2\omega}+1\right)$$

Sampling
sampling: $x\left(t\right)\xrightarrow{\text{sampling}}x\left(n\right)$; think Riemann sum

Unfortunately, when you do sampling, you lose information because you're taking the value at each point and storing it in digital form.

sampling period (a.k.a. sampling interval) [Δt]: time between signals

sampling frequency (a.k.a. sampling rate): 1/Δt, Hz

[ω] (radians/second): 2π×f

Sample a continuous signal into a discrete signal by plugging the equation of the continuous signal into the discrete

$$x_2(n) = x_1(\Delta t \cdot n)$$

e.g.

$$x_1(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$$
$$\Rightarrow x_2(n) = \cos(\omega_1 \cdot \Delta t \cdot n) + \cos(\omega_2 \cdot \Delta t \cdot n)$$

Impulse contains all frequencies evenly.

If you are sampling a given frequency band, $p = \dfrac{f_s}{f_{band}}$

Z and Laplace Transform

If you aren't given a value for x, assume it's a delta function.

e.g. Practice 3, question 3 $\ddot{y} + 4\dot{y} - 2y = x$

1) Laplace:
$$s^2 Y(s) + 4sY(s) - 2Y(s) = X(s)$$

2) Isolate for Y:
$$Y(s) = \frac{1}{s^2 + 4s - 2} X(s)$$
Poles of the transfer function are the coefficients of X(s) function.
Since you assume x(t) = δ(t), so X(s) = 1

3)
$$H(s) = \frac{1}{s^2 + 4s - 2} \cdot 1$$
4) Quadratic equation OR factor to find the poles
$$= \frac{1}{\left(s - \left(-2 + \sqrt{6}\right)\right)\left(s - \left(-2 - \sqrt{6}\right)\right)}$$
This region of convergence **doesn't** include the imaginary axis.

The region of convergence of $H(s)$ must include the imaginary axis in order for the system to be stable.

$H(s)$ is:
1. Called the **transfer function**, and is
2. The Laplace transform of the impulse response, h(t).

Z-transform

Kind of like the laplace transform but for discrete functions.

Block Diagrams

$\oplus$ ALWAYS has 2 inputs, one output. If anything else, ignore. Output = sum of inputs.
When a wire splits up into 2, multiply outputs of the junction by the input.

When a wire encounters a delay or an amplifier or anything without a junction, multiply your current value by the object.

Feedback loop: jump ahead, then look back. Also,

$$H(\omega) = \frac{H_1(\omega)}{1 - H_1(\omega)H_2(\omega)}$$

D = delay = $e^{-i\omega}$
Bonus: D is technically a function, so keep it on the left of things that depend on it, similar to d/dt