SFWR ENG 2MX3 Summary

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Math objects made using MathType; graphs made using Winplot.

Please join GitHub and contribute to this document. There is a guide on how to do this on my GitHub.

CTRL-F (?) to find locations which need to fixed

Table of Contents

Note: Order of writing symbols, such as exponents: digits-i-variables

Systems

There are 3 ways of representing a system:

- Difference Equation
- State Space Equations (ABCD), and
- Impulse Response

y = output
Causal system: a system that deals with current and past inputs and outputs. You cannot predict the future, i.e. system = 0 before impulse
Unless otherwise stated, all systems we'll be dealing with will be causal

It's often easier to only use cosine curves and represent sine curves with a phase shift.

SISO: single-input, single-output
MIMO system: Multiple Inputs Multiple Outputs.

A visual way of representing systems is using a block diagram.

Discrete manipulates functions of *n*.
Continuous manipulates functions of *t*.

Discrete Systems

If there's no x in the equation, there must be a mistake because there's no input.

- More precise, more theoretical
- Discrete input signals
- Easier to store because it's digital

Continuous System

- easier to process and calculate by hand
- e.g. $x(t) = 2\cos(t)$
- a.k.a. Analog

Deltas

2 types of deltas (δ):

Kronecker Delta [δ(n)]: discrete time domain, $\delta(n)=\begin{cases}1, & n=0\\ 0, & \text{else}\end{cases}, \int \delta(0)=1$

Dirac Delta [δ(t)]: continuous time domain, $\delta(t)=\begin{cases}\infty, & n=0\\ 0, & \text{else}\end{cases}, \int \delta(0)=1$

They generally are used to signify an impulse at one point in time.

State Space Equations

Difference Equation

The **difference equation** is a way of representing a system.

- First sum is of manipulations of previous input signals
- Second sum is of manipulations of previous output signals
- You only deal with previous inputs and outputs, since it's impossible to predict the future in a causal system
- β begins at 1 and not 0 is because y(n−0) = y(n), which is already on the left side of the equation.

$$y(n)=\sum_{k=0}^{N}\alpha_k x(n-k)+\sum_{k=1}^{M}\beta_k y(n-k)$$

We represent our equation by a sum of α's and β's. We call each of these a **state**. We store each of these states in a column matrix, **S**, such that the n^{th} state is S(n).

Linear Time Invariant (LTI) **System**:

- Given a sinusoidal input, the output of an LTI system will be a sinusoid with the same frequency, but possibly different phase and amplitude
- The *nextState* function, (S(n+1)), is an $N\times(N+M)$ matrix, and the *output* function is a $K\times(N+M)$ matrix
- You may flip order of system and delay when building the system

zero-state: s(0) = 0
All the systems we examine are "zero-state" because they are causal

ABCD

The **state space equations method** is also known as **ABCD Method**. This is because you represent your states by the four matrices, **A**, **B**, **C**, and **D**. There are 2 state space equations:

- Next state equation
- Output

Given *k* states, **next state equation**:

$$\mathbf{S}(n+1)=\mathbf{AS}(n)+\mathbf{B}x(n)$$

$$\begin{bmatrix} S_1(n+1) \\ S_2(n+1) \\ \vdots \\ S_k(n+1) \end{bmatrix} = \mathbf{A}\begin{bmatrix} S_1(n) \\ S_2(n) \\ \vdots \\ S_k(n) \end{bmatrix} + \mathbf{B}x(n)$$

Notice how the left side of the equation is the next state of each state? Expanding the matrices would show you how many of each state and the input each state represents.

The output equation is just re-writing the given equation in terms of the states you determined and your inputs:

$$y(n)=\mathbf{C}\begin{bmatrix} S_1(n+1) \\ S_2(n+1) \\ \vdots \\ S_k(n+1) \end{bmatrix} + \mathbf{D}x(n)$$

The state space equation method use matrices of states to represent any equation in a concise way.

If your next state is y(n), expand it by equating it to the initial equation of y(n).

Note: if you have a y(n−k) state, you <u>must</u> also have y(n−(1…k)) states. If you don't, make them.

e.g.)

Let's put the following equation into ABCD: $y(n)=x(n)-y(n-2)$

To begin, we need to determine what the states will be. Recall that x(n) is already going to be represented by the **B** matrix, so it cannot be a state. However, $y(n-2)$ *can* be represented by a state. Let's call it S_1:

$S_1(n)=y(n-2)$

Now, we'll find the next states:

$S_1(n+1)=y(n+1-2)=y(n-1)$

Our goal is to put all the states in terms of either previous states or x(n). Unfortunately, y(n−1) isn't either of those. This shows the significance in making your own states. In the future, figure out what states you'll need before you start finding the next states. You need the extra states as holders of the output, until it reaches the state that is supposed to use it.

Ok, so let's call this new state S_2:

$S_2(n)=y(n-1)$

Let's look at the next states of both S_1 and S_2:

$$S_1(n)=y(n-2) \qquad S_1(n+1)=y(n-1)$$
$$S_2(n)=y(n-1) \qquad S_2(n+1)=y(n)$$

Remember how we're supposed to represent the next states by a combination of the previous states? So change them so they are represented by previous states.

$$S_1(n+1)=y(n-1)=S_2(n)$$
$$S_2(n+1)=y(n)=x(n)-S_1(n)$$

For the next part, you need to understand matrix expansion. For now, I'll re-write each next-state equation individually before combining it to show what's actually going on:

$$\begin{aligned}S_1(n+1)&=0S_1(n)+1S_2(n)+0x(n)\\&=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}S_1(n)\\S_2(n)\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}x(n)\end{aligned}$$

$$\begin{aligned}S_2(n+1)&=-1S_1(n)+0S_2(n)+1x(n)\\&=\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}S_1(n)\\S_2(n)\end{bmatrix}+\begin{bmatrix}1\end{bmatrix}x(n)\end{aligned}$$

Now put those two together to get:

$$\mathbf{S}(n+1)=\underbrace{\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}S_1(n)\\S_2(n)\end{bmatrix}+\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\mathbf{B}}x(n)$$

The output equation is found the same way:

$$\begin{aligned}y(n)&=-1S_1(n)+0S_2(n)+x(n)\\&=\underbrace{\begin{bmatrix}-1 & 0\end{bmatrix}}_{\mathbf{C}}\begin{bmatrix}S_1(n)\\S_2(n)\end{bmatrix}+\underbrace{\begin{bmatrix}1\end{bmatrix}}_{\mathbf{D}}x(n)\end{aligned}$$

You can use these matrices for determining impulse response.

Compound Interest

An application of the state space equation other than signals is for determining compound interest.

n: billing period
α: interest
x: deposits

$$\begin{aligned}y(n)&=y(n-1)+\alpha y(n-1)+x(n)\\&=(1+\alpha)y(n-1)+x(n)\end{aligned}$$

Similar to an impulse, let's model someone putting money in a bank and leaving it there forever, where A is the value of an initial deposit:

$$x(n)=\begin{cases}A, & n=0\\ 0, & \text{else}\end{cases}$$

$$y(0)=A$$

$$y(1)=(1+\alpha)A$$

$$y(2)=(1+\alpha)^2 A$$

So you can interpolate: $y(n)=(1+\alpha)^n A$

Modeling Systems

FIR

Finite Impulse Response (FIR) system: A system that has an impulse response that has finite duration (is zero at a finite time). If there are no y's in it, it has to be finite

Output

To model the output of a system in the form of the difference equation, it is easiest to use a **state space table**. This table analyzes what the output of the given system is when you are given an impulse as your input. The prof usually leaves the labels blank, but this time, I'll leave them in there. Next time, however, they won't be there.

e.g. y(n) = x(n) + x(n−2)

Time	0	1	2	3	4
Input [$x(n)$]	1	0	0	0	0
Output [$y(n)$]	1	0	1	0	0

From this table, you can tell that it's an FIR system, since the output is 0 after t = 2.

IIR

Infinite Impulse Response (IIR) system. A system that has an impulse response that has an infinite duration (continues to respond indefinitely). IIR systems generally have y on both sides of the equation

e.g.)

y(n) = x(n) + 0.5y(n−1)

	1	0	0	0	0
	1	½	¼	⅛	1/16

Notice how the system doesn't end? This is why it is called an IIR system. Also, notice how the labels are gone? Get used to it and imagine them there every time he makes a table like this.

Impulse Response

Impulses can be modeled by the Kronecker delta function.

Impulse response [h]: the output to a system that has been affected by some external change, given an impulse. Since it is the output given the input is an impulse, the actual output, y, can be determined by a function of x convoluted with h.

Remember ABCD representation? The ABCD matrices can be used to define the transfer function:

$$h(n) = \begin{cases} 0, & n < 0 \\ \mathbf{D}, & n = 0 \\ \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}, & n > 0 \end{cases}$$

Convolution Sum

For some of the methods of solving for impulse response, you'll need to use the convolution sum. If an equation is in the form of the convolution equation, you may use the properties of the convolution sum.

$$y = h * x$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\,x(k) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

To clarify what the equations mean: $\xrightarrow{x(n)} \boxed{h(n)} \xrightarrow{y(n)}$

IR to DE

Sometimes your impulse response may be specified. There are 3 ways of converting your impulse response into the difference equation:

- Convolution Sum method
- Table method
- My Method

We're going to use the same example to demonstrate both methods

$$h(n) = \begin{cases} 0, & \text{else} \\ 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \end{cases}$$

$x(n) = \delta(n) + \delta(n - 2)$

Table method

To simplify the process, take the $h(n)$ and split it up into a sum of deltas/impulses:

$$h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

For each of the impulses in $x(n)$, look at the impulse response. Add all the $h(n)$'s up at the end.

$$\boxed{y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k)}$$

n	0	1	2	3	4	5
δ(n) [x(0)]	1	0		0	0	0
h(0)	1	2	3	0	0	0
δ(n−2)	0	0	1	0	0	0
h(2)	0	0	1	2	3	0
y(n)	1	2	4	2	3	0

Convolution

$$\boxed{y(n)=\sum_{i=-\infty}^{\infty}h(n-i)x(i)}$$

Let's plug in some values of *n*:

$$y(0)=\sum_{i=0}^{\infty}h(0-i)x(i)$$

However, since *h*(anything < 0) = 0, we only get output when *i* = 0:

$$\begin{aligned}y(0)&=h(0)\cdot x(0)\\&=1\cdot 1=1\end{aligned}$$

$$\begin{aligned}y(1)&=\sum_{i=0}^{\infty}h\underbrace{(1-i)}_{0,1}x(i)\\&=h(1-0)\cdot x(0)+h(1-1)x(1)\\&=(2\cdot 1)+(1\cdot 0)=2\end{aligned}$$

$$\begin{aligned}y(2)&=\sum_{i=0}^{\infty}h(2-i)x(i)\\&=h(2-0)x(0)+h(2-1)x(1)+h(2-2)x(2)\\&=(3\cdot 1)+(2\cdot 0)+(1\cdot 1)\\&=3+0+1=4\end{aligned}$$

$$\begin{aligned}y(3)&=\sum_{i=0}^{\infty}h(3-i)x(i)\\&=h(3-0)x(0)+h(3-1)x(1)+h(3-2)x(2)+h(3-3)x(3)\\&=(0\cdot 1)+(3\cdot 0)+(2\cdot 1)+(1\cdot 0)\\&=2\end{aligned}$$

Kemal's Method

Someone else probably thought of this before me, but there's a faster way of directly converting your impulse response to your difference equation:

$$h(n) = \begin{cases} \alpha_0, & n = 0 \\ \vdots & \\ \alpha_N, & n = N \end{cases}$$

Plug it into: $y(n) = \sum_{k=0}^{N} \alpha_k x(n-k)$ and expand.

It may be easier to convert the impulse response to an equation of dirac deltas.

If you aren't given an equation for *x* at this point, assume $x(n) = \delta(n)$. You may then leave it in/convert to dirac delta equations.

Step Response

The step response is the impulse response when the input is the Heaviside function, the unit step function

Step Response: $\text{step}(n-k) = \begin{cases} 0, & n < k \\ 1, & n \geq k \end{cases}$

$$y = h * x$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\text{step}(n-k)$$

You can't have $h(k < 0)$, since time starts at 0:

$$y(n) = \sum_{k=0}^{\infty} h(k)\text{step}(n-k)$$

k can only be as big as n before that iteration of the sum is negative:

$$y(n) = \sum_{k=0}^{n} h(k) \cdot \underbrace{1}_{*} \leftarrow \text{*step}(0 \ldots n)$$

Discrete and Continuous Frequency

	Discrete	Continuous
Period	$p = \frac{\text{samples}}{\text{cycle}}$ (must be a natural number)	$p = \frac{\text{seconds}}{\text{cycle}}$
Frequency	$f = \frac{1}{p} = \frac{\text{cycles}}{\text{sample}}$	$f = \frac{1}{p} = \frac{\text{cycles}}{\text{second}} = \text{Hz}$
Normalized Frequency	$\omega = 2\pi f = \frac{\text{radians}}{\text{sample}}$	$\omega = 2\pi f = \frac{\text{radians}}{\text{second}}$
Sampling Frequency	$f_s = \frac{\text{samples}}{\text{second}}$	
Pitch	$\text{Pitch} = f_s \cdot f_{\text{discrete}} = \frac{\text{samples}}{\text{second}} \cdot \frac{\text{cycles}}{\text{sample}} = \frac{\text{cycles}}{\text{second}} = \text{Hz}$	

Remember that *upper-case* variables (frequency-domain) are *multiplied*: $Y(\omega) = H(\omega) \times X(\omega)$,

and *lower-case* variables (time-domain) are *convoluted*: $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,\mathrm{d}\tau$

Period

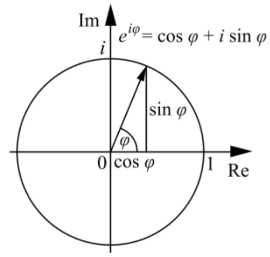
$$
\begin{aligned}
x(n) &= k(n+kp) \\
&= \cos\left(\tfrac{\pi}{4}n\right) \leftarrow \omega = \tfrac{\pi}{4} \\
&= \tfrac{\pi}{4}n = k2\pi \\
&\Rightarrow p = 8
\end{aligned}
$$

Your period is selected such that the coefficient × p = a multiple of 2π. If there is a phase shift, don't worry about it when calculating period, frequency, or normalized frequency.

Often the coefficient can actually be your normalized frequency (remember this to save steps). Choose an ω_0, such as $\omega_0 = \pi/4$ and find the period, such that $\omega_0 \cdot p$ is any multiple of 2π, i.e. $\omega_0 \cdot p = k2\pi$

Complex Numbers

Draw a graph where the x values represent the real dimension and the y values represent the complex plane. This is actually the graph of e^{ix} (pretend the φ's are actually x's):



You will frequently need to refer to Euler's equations.

argument ($\measuredangle$): the angle of the line that connects the point to the origin on the complex plane
The argument of each point is actually the value of x currently being examined, such that
$\measuredangle e^{ix} = x$
e.g. $\measuredangle e^{i\frac{\pi}{4}} = \frac{\pi}{4}$

gain: magnitude of the line connecting the origin to the point

Divide by the magnitude on the unit circle. Thus, the point is: $\left(\frac{1}{2}, \frac{i}{2}\right)$
Know your special triangles.
$\sqrt{1^2 + 1^2} = \sqrt{2}$

How the complex plane works, such as how $e^{-i\frac{\pi}{4}} = 1 - i$

Summary: numbers can be converted to their complex exponential equivalent by finding the gain and the angle, then num = gain $e^{i\measuredangle}$

$\frac{1}{i} = -i$

Euler

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Frequency Response

Intro

Upper-case variable: frequency domain (i.e. a function of ω)
Lower-case variable: time domain (i.e. a function of t or n)

Frequency Response [H(ω)]: how a system scales the amplitude and shifts the phase of an input of multiple periodic signals
It is the Fourier equivalent of the Transfer Function

Formula: $H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}=\frac{\sum_{k=0}^{\infty}Y_k e^{-i\omega_0 k}}{1-\sum_{k=1}^{\infty}X_k e^{-i\omega_0 k}}$

Note that the last equation comes from the difference equation:

$$H(\omega)=\frac{\sum_{k=0}^{\infty}\alpha_k e^{-i\omega k}}{1-\sum_{k=1}^{\infty}\beta_k e^{-i\omega k}}$$

e.g. if $x(n)=e^{i\omega n}, y(n)=H(\omega)e^{i\omega n}$

$\sphericalangle H(\omega)$ = phase shift of the cosines in H(ω)

The gain is represented by $\left\|H(\omega)\right\|$

When your X_k's go to 0, they are known as **poles** or **eigenfrequencies** because they result in an ∞ frequency response.
Eigenfrequency is the frequency that will cause a glass to break
Resonance frequency

Conservative: you don't retain energy (?)

Discrete LTI Signal

Given a discrete-time signal, x(n) and an H(ω), the output is computed in the following way:
Your input may contain multiple terms. Treat each cosine individually.

To understand what variables I am referring to, take each cosine and compare it to this general equation: $\boxed{y(n)=\left\|H(\omega)\right\|\cos\left(\omega n+\sphericalangle H(\omega)\right)}$

Note: if x(n) = a number [such as 5], treat it as if k = 0, so x(n) = 5cos(0n)

But wait! Was that a y on the left side of the equation? Shouldn't that be an x? No. I just showed you the output equation. This is the equation you're going to put your answer in after you have solved for the variables. I showed you this first because your x is going to be very similar to this.

1. Convert all sines in your x(n) to cosines
2. Identify the ω's from each of the equations in x. ω ALWAYS comes from x.

To make it easier to organize the information you have just gotten, make a table with a different row for each cosine. This works because each cosine should have its own ω. To find the H(ω) for each row, plug the ω of that row into H(ω)

Output to Frequency Response: $\boxed{y(n)=\left\|H(\omega)\right\|x\left(n+\sphericalangle H(\omega)\right)}$

e.g. y(DE) 1

Find the output to the difference equation
y(n) + y(n−1) = 2x(n) − 5x(n−2)
Convert to frequency response

$$H(\omega)e^{i\omega n}+H(\omega)e^{i\omega(n-1)}=2e^{i\omega n}-5e^{i\omega(n-2)}$$

$$H(\omega)\left(e^{i\omega n}+e^{i\omega n}e^{-i\omega}\right)=2e^{i\omega n}-5e^{i\omega n}e^{-2i\omega}$$

$$H(\omega)\left(1+e^{-i\omega}\right)=2-5e^{-2i\omega}$$

Factor everything out to isolate H(ω):

$$H(\omega)=\frac{2-5e^{-2i\omega}}{1+e^{-i\omega}}$$

e.g. y(H,x) 1

(I'm supposed to use $H(\omega)=\frac{1+e^{-i\omega}}{e^{-i\omega}}$, but I actually did it with the previous H)

Use the frequency response to determine the output to

$$x(n)=\cos\left(\tfrac{\pi}{2}n+\tfrac{\pi}{4}\right)+\sin\left(\pi n+\tfrac{3\pi}{2}\right)$$

$$\boxed{H(\omega)=\|H(\omega)\|e^{i\sphericalangle H(\omega)}}$$

ω	H(ω)	‖H(ω)‖	$\sphericalangle H(\omega)$
0	− 3/2	3/2	−π
π/2	Just plugin ω into H $\frac{2-5e^{-2i\frac{\pi}{2}}}{1+e^{-i\frac{\pi}{2}}}$ $=\frac{2-5(-1)}{1-i}$ $=\frac{7}{1-i}$ $=\frac{7+7i}{1+1}$ $=\frac{7}{2}+\frac{7}{2}i$	What is the magnitude of the point, H(ω), on the e^{ix} graph $\sqrt{3.5^2+3.5^2}=3.5\sqrt{2}$	$\frac{7}{2}+\frac{7}{2}i=3.5\sqrt{2}e^{i\sphericalangle H(\omega)}$ $1+i=\sqrt{2}e^{i\sphericalangle H(\omega)}$ $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=e^{i\sphericalangle H(\omega)}$ $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=e^{i\sphericalangle H(\omega)}$ $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}\boxed{=\frac{-7\pi}{4}}$
π	0	0	0

Where ϕ is the previous phase shift:

$$\boxed{y(n)=\|H(\omega)\|[\cos/\sin]\left(\omega n+\phi+\sphericalangle H(\omega)\right)}$$

$$y(n)=\tfrac{3}{2}\cos(-\pi)=\tfrac{-3}{2}+5\cos\left(\tfrac{\pi}{2}n-\tfrac{7\pi}{4}+\tfrac{\pi}{4}\right)$$

$$y(n)=\tfrac{-3}{2}+\tfrac{7}{\sqrt{2}}\cos\left(\tfrac{\pi}{2}n-\tfrac{3\pi}{2}\right)$$

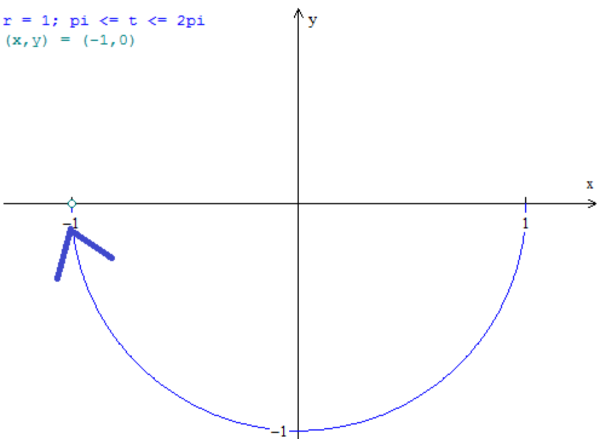
e.g. y(H,x) 2

$$H(\omega) = \cos(\omega)$$

$$x(n) = 2 + \cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) + \sin\left(\pi n + \tfrac{3\pi}{2}\right)$$
$$= 2\cos(0n) + \cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) + \cos(\pi n + \pi)$$

ω	H(ω)	‖H(ω)‖	$\sphericalangle H(\omega)$
0	1	1	0 ←arg(e^{i0})
π/2	0	0	–
π	−1	1	−π

<u>The argument will always be negative because you can't look ahead of time</u>.



Now that we have this information, plug it into the y-equation from earlier.

$$y(n) = \|H(\omega)\| \cos(\omega n + \phi + \sphericalangle H(\omega))$$
$$y(n) = 1 \cdot 2 + 0 \cdot \cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right) + |-1| \cos(\pi n + \pi - \pi)$$
$$= 2 + \cos(\pi n)$$

e.g. y(DE,x) 2

$$y(n) = x(n) + x(n-2)$$
$$x(n) = 1 + \cos\left(\tfrac{\pi}{2}n\right) + \cos\left(\pi n + \tfrac{\pi}{2}\right)$$

Given that each impulse is represented by $e^{i\omega n}$, you could temporarily represent all x's with $e^{i\omega n}$, then plug in the value of x afterwards.

$$x(n) = e^{i\omega n}, y(n) = H(\omega)e^{i\omega n}$$
$$H(\omega)e^{i\omega n} = e^{i\omega n} + e^{i\omega(n-2)}$$
$$H(\omega) = \frac{e^{i\omega n} + e^{i\omega n}e^{-2i\omega}}{e^{i\omega n}}$$
$$= 1 + e^{-2i\omega}$$

ω	H(ω)	‖H(ω)‖	$\sphericalangle H(\omega)$		
0	$1+e^{0}=2$	2	0		
π/2	$1+e^{-i\pi}=1-1=0$	0	–		
π	$1+e^{-2i\pi}=1+1=2$	2	0		

$$y(n)=\|H(\omega)\|x(n+\sphericalangle H(\omega))$$

Plug in the equation of the output to each impulse:

$$y(n)=2+0\cos\left(\tfrac{\pi}{2}n\right)+2\cos\left(\pi n+\tfrac{\pi}{2}\right)=2+2\cos\left(\pi n+\tfrac{\pi}{2}\right)$$

e.g. DE(H)

$$H(\omega)=\tfrac{1}{2}e^{-2i\omega}+\tfrac{1}{2}e^{0i\omega}\leftarrow y=H(\omega)e^{i\omega n}$$

$$H(\omega)e^{i\omega n}=\tfrac{1}{2}e^{i\omega(n-2)}+\tfrac{1}{2}e^{i\omega n}$$

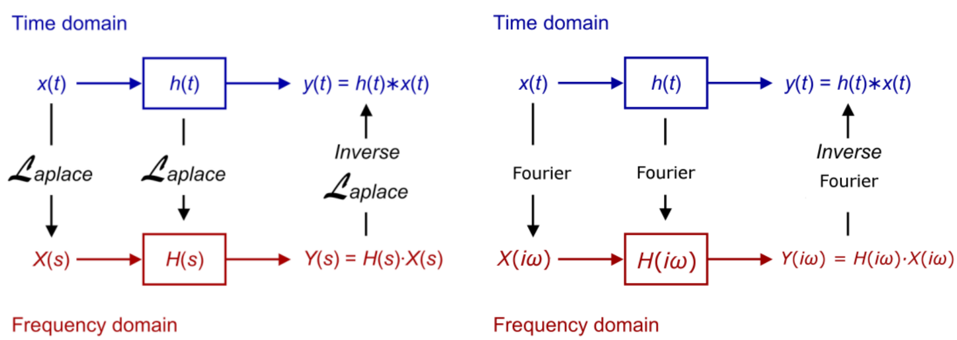
$$\begin{aligned}y(n)&=\tfrac{1}{2}e^{i\omega(n-2)}+\tfrac{1}{2}e^{i\omega n}\\&=\tfrac{1}{2}x(n-2)+\tfrac{1}{2}x(n)\leftarrow x(n)=e^{i\omega n}\end{aligned}$$

Fourier

Do you remember Laplace? Fourier is almost the same (source):

$$\mathcal{L}\{f(t)\}=F(s)=\int_{0}^{\infty}f(t)e^{-st}\mathrm{d}t$$

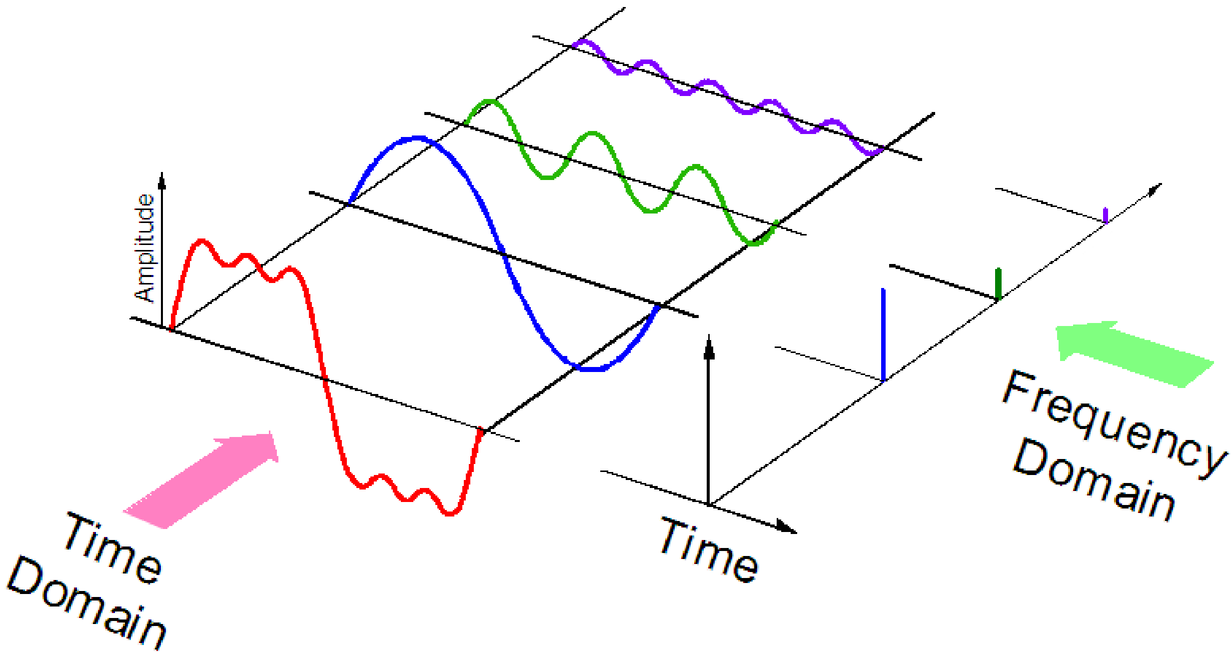
$$\mathcal{F}\{f(t)\}=F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\mathrm{d}t$$



In this course Fourier will be useful when converting between the time and frequency.

Time and Frequency Domain

Gif demonstration of what the frequency domain is.



Impulse Response to Frequency Response

In the Fourier Series, ω is not what it usually is ($2\pi f$), since it changes with each k or n (depending on discrete / continuous). That is why we represent $2\pi f$ by a constant, ω_0. It is calculated the same way as you would calculate ω, normally (i.e. $\omega_0 = 2\pi/p = 2\pi f$).

Although it may seem that X_k is not a function that is affected by frequency, it is because each frequency, ω, is simply a multiple of the fundamental frequency that is chosen, ω_0.
Fundamental frequency [ω_0]: a constant
For discrete frequency: $\omega = \omega_0 \times k$
For continuous frequency: $\omega = \omega_0 \times m$

Forward transform: time → frequency domain (look for negative exponents on e)
Transform^{-1}: frequency → time domain (look for positive exponents on e)

If h is a function of time, then CTFT; if not, DTFT. i.e. h(t), do CTFT; h(n), do DTFT
The impulse response, $h(n)$, of a system is the $y(n)$ when $x(n)$ is an impulse: $\delta(n)$.

k is the specific frequency you are aiming to get. It ranges from 0 to (p−1)

N = p

Master Chart

Note: DFT/FFT is below

	FS (Fourier Series)	**FT (Fourier Transform)**
DT (Discrete Time)	$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ink\omega_0}$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$

	$x(n)=\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$	$x(n)=\frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{i\omega n}\mathrm{d}\omega$ See more below
CT (Continuous Time)	$X_m=\frac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}\mathrm{d}t$ $x(t)=\sum_{m=-\infty}^{\infty} X_m e^{im\omega_0 t}$ See more below	$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-i\omega t}\mathrm{d}t$ $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\mathrm{d}\omega$ Summary: $\mathrm{CTFT}\left(ce^{i\omega_0 t}\right)=c\cdot 2\pi\delta(\omega-\omega_0)$ **essentially, Laplace**

CTFS

Notice how there is no range for the CTFS? That's because it is actually from $-\infty$ to ∞. An easier way to do it than computing it is to convert everything from your x(t) equation into complex exponentials (i.e. form $e^{i\omega t}$). Now realize that X_n is the output of impulse response, so

$y(t)=H(\omega)x(t)$

$H(\omega)=\frac{y(t)}{x(t)}=\frac{X_m}{e^{i\omega t}}=X_m e^{-i\omega t}=X_m e^{-i\omega_0 mt}$

That means the coefficients in front of each of your complex exponentials in your x(t) equation represent a different X_m.

e.g. $x(t)=\frac{1}{2}e^{it}+\frac{1}{2}e^{-it}=\frac{1}{2}e^{i\cdot 1\cdot 1\cdot t}+\frac{1}{2}e^{i\cdot 1(-1)t}$, $X_{m=1}=X_1=½$, $X_{m=-1}=X_{-1}=½$.

Notice how *n* is directly related to ω? That's why each *m* is known as a frequency.

Sometimes, you can notice that the ω's are multiples of the same ω. This is because the function is periodic.

e.g.)

$\omega_1=\frac{\pi}{4},\omega_2=\frac{\pi}{2}$

CTFS: $X_m=\frac{1}{p}\int_0^p x(t)e^{-i\omega_0 mt}\mathrm{d}t$

Taking $X_m=\frac{1}{p}\int_0^p\left(\frac{1}{2}e^{i\frac{\pi}{2}t}+\frac{1}{2}e^{-i\frac{\pi}{2}t}+\frac{1}{2}e^{i\frac{\pi}{4}t}+\frac{1}{2}e^{-i\frac{\pi}{4}t}\right)e^{-i\omega_0 mt}\mathrm{d}t$,

Choose an ω_0, such as $\omega_0 = \pi/4$ and find the period, such that $\omega_0\cdot p$ is any multiple of 2π, i.e. $\omega_0\cdot p = k2\pi$

p = 8

$X_m=\frac{1}{16}\int_0^8\left(e^{i\frac{\pi}{2}t}+e^{-i\frac{\pi}{2}t}+e^{i\frac{\pi}{4}t}+e^{-i\frac{\pi}{4}t}\right)e^{-i\frac{\pi}{4}mt}\mathrm{d}t$

Now, put all the items in terms of ω_0:

$X_m=\frac{1}{16}\int_0^8\left(e^{i\frac{\pi}{4}2t}+e^{-i\frac{\pi}{4}2t}+e^{i\frac{\pi}{4}t}+e^{-i\frac{\pi}{4}t}\right)e^{-i\frac{\pi}{4}mt}\mathrm{d}t$

Now, expand:

$$X_m = \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(1-m)} + e^{i\frac{\pi}{4}t(2-m)} + e^{i\frac{\pi}{4}t(1+m)} + e^{i\frac{\pi}{4}t(2+m)}\mathrm{d}t$$

$$\boxed{\begin{gathered}\frac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}\mathrm{d}t = \frac{p}{p}x(t)\delta(m)\\ \Rightarrow \frac{1}{p}\int_0^p e^{-im\omega_0 t}\mathrm{d}t = \delta(m)\end{gathered}}$$

$$\begin{aligned}X_m &= \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(1-m)}\mathrm{d}t + \frac{1}{16}\int_0^8 e^{-i\frac{\pi}{4}t(1+m)}\mathrm{d}t + \frac{1}{16}\int_0^8 e^{i\frac{\pi}{4}t(2-m)}\mathrm{d}t + \frac{1}{16}\int_0^8 e^{-i\frac{\pi}{4}t(2+m)}\mathrm{d}t\\ &= \tfrac{1}{2}\delta(1-m) + \tfrac{1}{2}\delta(1+m) + \tfrac{1}{2}\delta(2-m) + \tfrac{1}{2}\delta(2+m)\end{aligned}$$

$$X_1 = \tfrac{1}{2}, X_{-1} = \tfrac{1}{2}, X_2 = \tfrac{1}{2}, X_{-2} = \tfrac{1}{2}$$

DTFS

Don’t split your $x(n)$ up into the complex exponential equivalences
k’s downward
n’s across

$$\text{CTFT}\{h(n) = \delta(n-a)\} = e^{-i\omega a}$$

CTFT/CFT & DTFT

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty} e^{i\omega t}\mathrm{d}\omega = 2\pi\delta(\omega)$$

$$\boxed{\int_{-\infty}^{\infty}\left(c\cdot e^{i\omega_0 t}\right)\mathrm{d}t = c\cdot 2\pi\delta(\omega-\omega_0)}$$

$$\boxed{\int_{-\infty}^{\infty}\left(c\cdot e^{i\omega_0 n}\right)\mathrm{d}n = c\cdot 2\pi\delta(\omega-\omega_0)}$$

Since sine and cosine are continuous functions, they can’t simply be represented by impulses. Thus, assume 2π-periodicity and add ‘$-2\pi n$’ to the end of the Kronecker Delta:

$$\cos(\omega_0 t) \Rightarrow \pi\sum_{n=0}^{7}\left[\delta(\omega+\omega_0-2\pi n) + \delta(\omega-\omega_0-2\pi n)\right]$$

$$\sin(\omega_0 t) \Rightarrow \pi i\sum_{n=0}^{7}\left[\delta(\omega+\omega_0-2\pi n) - \delta(\omega-\omega_0-2\pi n)\right]$$

DFT/FFT

Here’s a very simple article that explains what the DFT does. It also gives you an idea of what the other Fourier equations do.

The DFT is actually the most popular transform, because there's an efficient algorithm for it called the **Fast Fourier Transform** (FFT). It’s has the same equation as the DTFS, but with different scaling. See the difference:

DTFS	**DFT/FFT**
$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ink\omega_0}$	$X_k = \sum_{n=0}^{p-1} x(n)e^{-ink\omega_0}$
$x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$	$x(n) = \frac{1}{p}\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$

The DFT^{-1} is also known as the **synthesis equation**.

Because the [forward] DFT doesn't divide out the number of points, *p*, using a higher sampling rate will make the DFT larger for the same input. This is why it's actually better to scale the DFT output by 1/p anyways when looking at it afterwards.

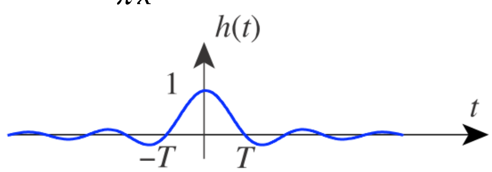
Filter a signal

http://www.falstad.com/fourier/

Sinc Function

Filtered signals are generally represented by the **sinc function**: $y(x) = \frac{\sin(x)}{x}$

Normalized sinc function: $y(x) = \frac{\sin(\pi x)}{\pi x}$



Fourier of sinc:

$$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = \begin{cases} T & \text{if } |\omega| \le \pi/T \\ 0 & \text{otherwise} \end{cases}$$

Transforming a Signal example

If we are given an equation for a signal and we’re trying to put it into a different form, think of it like taking a signal generator and putting some sort of filter on it.

For example,

$$x_1(t) = \begin{cases} \frac{\pi}{a}, & -a \le t \le a \\ 0, & \text{else} \end{cases} \Leftrightarrow X(\omega) = 2\pi \frac{\sin(a\omega)}{a\omega}$$

into:

$$x_2(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{else} \end{cases}$$

In order to do that, we'll need to use the CFT. However, before we can that, we need to figure out our **amplitude** [a] and how we're going to going to manipulate the original signal to get the thing we want.

Since we all know that amplitude is half the period, (p = 2−0 = 2), a = 2/2 = 1, so

$$x_1(t) = \begin{cases} \pi, & -1 \le t \le 1 \\ 0, & \text{else} \end{cases} \Leftrightarrow X(\omega) = 2\pi \frac{\sin(\omega)}{\omega}$$

Now, to determine how the transform will be, you need to **scale** and **shift** the function.
To scale, you need to figure out what factor (let's say k) should be multiplied by $x_1(t)$ to get $x_2(t)$

$$x_1(t) = \pi, \qquad -1 \le t \le 1$$

$$x_2(t) = x_1(t) \cdot k$$

$$x_2(t) = k\pi$$

$$1 = k\pi$$

$$k = \tfrac{1}{\pi}$$

With that in mind, we now need to shift the function. Let's say N is your shifting factor. When shifting in the time domain, you know that you could easily add or subtract, like $x(t) \Rightarrow x(t - N)$. However, we will need to translate that to the frequency domain, which is different. For this to mirror in the frequency domain, you need to multiply $X(\omega)$ by $e^{-i\omega N}$.

If $x(n) \xrightarrow{\text{CFT}} X(\omega)$,

$$x(n-N) \xrightarrow{\text{CFT}} X(\omega) \cdot e^{-i\omega N}$$

(skip the proof)

Proof

Take $x(t) = 2\pi\delta(t)$. Well remember this function from the CTFT section?

$$\int_{-\pi}^{\pi} e^{i\omega(t-N)} \mathrm{d}t = 2\pi\delta(t-N)$$

Yeah, so N is our shift. Let's expand that equation

$$\int_{-\pi}^{\pi} e^{i\omega t} e^{-i\omega N} \mathrm{d}t = 2\pi\delta(t-N)$$

$$e^{-i\omega N} \underbrace{\int_{-\pi}^{\pi} e^{i\omega t} \mathrm{d}t}_{\underbrace{2\pi\delta(t)}_{x(t)}} = 2\pi\delta(t-N)$$

Putting it together

We have a range from −1 to 1 and we want a range from 0 to 2. This means we want to use $x(t - 1)$ to bring it one second from the past.

Therefore, your final equation will be:

$$\boxed{X(\omega) = e^{-i\omega N} kX(\omega)}$$

Or specifically for this example:

$$X(\omega) = e^{-i\omega 1} \frac{1}{\cancel{\pi}} 2\cancel{\pi} \frac{\sin(\omega)}{\omega}$$
$$= e^{-i\omega} 2 \frac{\sin(\omega)}{\omega}$$

Sawtooth

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$

Since the range is 0-1, you know p = 1
$\omega_0 = 2\pi/p = 2\pi/1 = 2\pi$
Use the CTFS to find X_m. I'll just use a range of 0 to 1, since that's the only range that *x* is not 0.

$$X_m = \frac{1}{1}\int_0^1 te^{-i2\pi mt}\mathrm{d}t$$

Since k is an integer, the ω will always be a multiple of 2π. In the complex domain, ω+2πm = ω.
So 2·π·m = 2π = 0
So we can equate the above equation to:

$$X_m = \int_0^1 te^{-i0t}\mathrm{d}t = \int_0^1 t\mathrm{d}t$$

Since an integral is the area under the curve, where the curve is a triangle,

$$X_m = \int_0^1 t\mathrm{d}t = \frac{1}{2}$$

Filter Design

When designing a filter, use tilde ($\tilde{H}$) to denote a prototype of your filter.

Note: "removal of frequency component at a point" means H(point) = 0, so you just pretend the value is a root at that time.

Constant signal means that the magnitude of the frequency response will be at the gain most of the time, such as |H(0)|

e.g. Notch filter

a.k.a. band-stop filter

$$H\left(\tfrac{\pi}{2}\right) = 0$$
$$H(0) = 1$$

1) Set up the equation such that you have $\tilde{H}(\omega) = \prod\left(e^{-i\omega} - e^{-i\{\text{roots}\}}\right)$ because when ω = the root, your bracket will = 0. In our case, your only root is at π/2, so:

$$\tilde{H}(\omega) = \left(e^{-i\omega} - e^{-i\frac{\pi}{2}}\right)$$

2) These systems require **Conjugate complex symmetry (CCS)** (When do things require CCS?) You need it so that your answer is real. If you don't have it, your output will be incorrect. Thus, when you have a value, you need to reflect it across the real axis

$\cos(x) = \frac{1}{2}e^{i\omega} + \frac{1}{2}e^{-i\omega}$

To allow for conjugate complex symmetry, we'll add a second part to the prototype:

$$\tilde{H}(\omega) = \left(e^{-i\omega} - e^{-i\frac{\pi}{2}}\right)\underbrace{\left(e^{-i\omega} - e^{i\frac{\pi}{2}}\right)}_{\text{allows CCS}},$$
$$= e^{-i2\omega} + 1$$

To account for this, add a second initial condition, the complex conjugate, $H\left(\frac{-\pi}{2}\right) = 0$.

3) Now plug in your other initial condition:

Although $H(0) = 1, \tilde{H}(0) = 2$.

To account for the difference, insert a constant of ½:

$H(\omega) = \frac{1}{2}\left(e^{-i2\omega} + 1\right)$

Sampling

sampling: $x(t) \xrightarrow{\text{sampling}} x(n)$; think Riemann sum

Unfortunately, when you do sampling, you lose information because you're taking the value at each point and storing it in digital form.

sampling period (a.k.a. sampling interval) [Δt]: time between signals

sampling frequency (a.k.a. sampling rate): 1/Δt, Hz

[ω] (radians/second): 2π×f

Sample a continuous signal into a discrete signal by plugging the equation of the continuous signal into the discrete

$x_2(n) = x_1(\Delta t \cdot n)$

e.g.

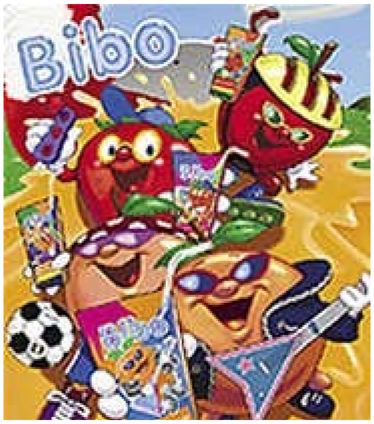
$x_1(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$

$\Rightarrow x_2(n) = \cos(\omega_1 \cdot \Delta t \cdot n) + \cos(\omega_2 \cdot \Delta t \cdot n)$

Impulse contains all frequencies evenly.

If you are sampling a given frequency band, $p = \dfrac{f_s}{f_{\text{band}}}$

BIBO Stability



Bounded-Input, Bounded Output (BIBO) Stability: Given $|x(t)| \leq M$ for all t (i.e. bounded input), $|y(t)| \leq N$ for all t (bounded output)
N can depend on M

For LTI systems, if h(t) is stable $\Leftrightarrow \int |h(t)| \mathrm{d}t < \infty$

$$|y(t)| = x(t) * h(t)$$
$$= \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \mathrm{d}\tau \right|$$
$$\leq \int_{-\infty}^{\infty} |x(\tau) h(t-\tau)| \mathrm{d}\tau$$
$$\leq \int_{-\infty}^{\infty} M \, |h(t-\tau)| \mathrm{d}\tau$$

x is bounded by M, so it's ok if you replace it with M:

$$\leq M \int_{-\infty}^{\infty} |h(t-\tau)| \mathrm{d}\tau$$

The shift doesn't change the integral from (−∞,∞), so let u = t−τ:

$$= M \int_{-\infty}^{\infty} |h(u)| \mathrm{d}u$$

Note: $\left| \int f(t) \mathrm{d}t \right| = \int |f(t)| \mathrm{d}t$

Z and Laplace Transform

[σ]: Real Part of s
$s = \sigma + i\omega$

If you aren't given a value for x, assume it's a delta function.

Transfer function [$H(s)$]: the Laplace transform of the impulse response, $h(t)$.

e.g. Practice 3, question 3 $\ddot{y}+4\dot{y}-2y=x$

1) Laplace:
$s^2Y(s)+4sY(s)-2Y(s)=X(s)$

2) Isolate for Y:
$$Y(s)=\frac{1}{s^2+4s-2}X(s)$$
Poles of the transfer function are the coefficients of X(s) function.
Since you assume x(t) = δ(t), so X(s) = 1

3)
$$H(s)=\frac{1}{s^2+4s-2}\cdot 1$$
4) Quadratic equation OR factor to find the poles
$$=\frac{1}{\left(s-\left(-2+\sqrt{6}\right)\right)\left(s-\left(-2-\sqrt{6}\right)\right)}$$
This region of convergence **doesn't** include the imaginary axis.

The region of convergence of $H(s)$ must include the imaginary axis in order for the system to be stable.

Z-transform

Kind of like the Laplace transform but for discrete functions.

Block Diagrams

⊕ ALWAYS has 2 inputs, one output. If anything else, ignore. Output = sum of inputs.
When a wire splits up into 2, multiply outputs of the junction by the input.

When a wire encounters a delay or an amplifier or anything without a junction, multiply your current value by the object.

Feedback loop: jump ahead, then look back. Also,
$$H(\omega)=\frac{H_1(\omega)}{1-H_1(\omega)H_2(\omega)}$$

D = delay = $e^{-i\omega}$
Bonus: D is technically a function, so keep it on the left of things that depend on it, similar to d/dt