Linear Optimization Assignment 2
Report

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Question 1)

Implementing simplex method from scratch

1. Tabular minimization simplex method has been implemented in python. It takes in the cost vector c, the coefficient matrix A and the constants b and converts them to tabular form.
2. At each iteration, it checks if further optimization is required by checking the reduced cost vector entries. If any of the reduced cost vector entries is less than 0, then we optimize further.
3. If optimization is required, we find the entering and the leaving variable.
4. To find the entering variable, we find the index of the column with the minimum cost vector entry.
5. To find the leaving variable, we find the row with the minimum value of bi/aij.
6. Elementary row operations are performed to make the entering row as identity.
7. When all the reduced cost vector entries become positive no further optimization is possible. We print the optimal solution and the optimal value.

An example from the lectures using simplex method implemented from scratch-

$$
\text{minimize} - x_1 - 2x_2 - x_3
$$
s.t.
$$
\begin{cases}
2x_1 + x_2 - x_3 \le 2 \\
2x_1 - x_2 + 5x_3 \le 6 \\
4x_1 + x_2 + x_3 \le 6 \\
x_i \ge 0, i = 1, 2, 3
\end{cases}
$$

Example

[10]
```
c = [-1,-2,-1,0,0,0]
A = [
    [2,1,-1,1,0,0],
    [2,-1,5,0,1,0],
    [4,1,1,0,0,1]
]
b = [2,6,6]
```

```
simplex(A,b,c,print_iterations=True)
```

```
Iteration 1
[[ 2.  1. -1.  1.  0.  0.  2.]
 [ 2. -1.  5.  0.  1.  0.  6.]
 [ 4.  1.  1.  0.  0.  1.  6.]
 [-1. -2. -1.  0.  0.  0.  0.]]

Iteration 2
[[ 2.  1. -1.  1.  0.  0.  2.]
 [ 4.  0.  4.  1.  1.  0.  8.]
 [ 2.  0.  2. -1.  0.  1.  4.]
 [ 3.  0. -3.  2.  0.  0.  4.]]

Iteration 3
[[ 3.    1.    0.    1.25  0.25  0.    4.  ]
 [ 1.    0.    1.    0.25  0.25  0.    2.  ]
 [ 0.    0.    0.   -1.5  -0.5   1.    0.  ]
 [ 6.    0.    0.    2.75  0.75  0.   10.  ]]

The optimal solution is [0. 4. 2. 0. 0. 0.]
The optimal value is -10.0
```

Question 2)

LPs have been formulated for flow networks 1 and 2 below:

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Primal for network 1

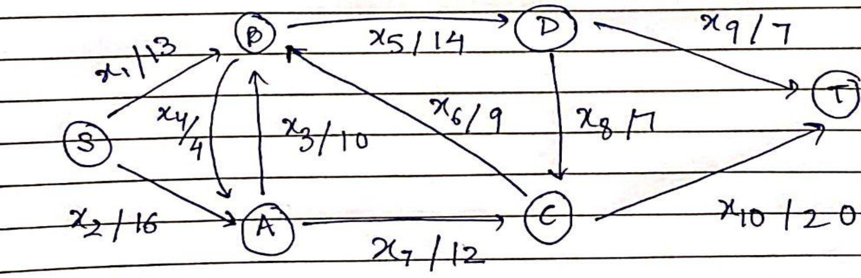
Max $z = x_1 + x_2$

Flow constraints

$$x_2 + x_4 - x_3 - x_7 = 0$$
$$x_1 + x_3 + x_6 - x_4 - x_5 = 0$$
$$x_7 + x_8 - x_6 - x_{10} = 0$$
$$x_5 - x_8 - x_9 = 0$$

Arc constraints

$$x_1 \leq 13$$
$$x_2 \leq 16$$
$$x_3 \leq 10$$
$$x_4 \leq 4$$
$$x_5 \leq 14$$
$$x_6 \leq 9$$
$$x_7 \leq 12$$
$$x_8 \leq 7$$
$$x_9 \leq 7$$
$$x_{10} \leq 20$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 0$$



DELTA Notebook

Primal for Network 2

Converting the flow network matrix to an LP, we get

$$\max \ Z = x_1 + x_2 + x_3$$

s.t. Flow constraints:

$$x_1 + x_6 + x_{18} - x_4 - x_5 = 0$$
$$x_2 - x_6 - x_8 = 0$$
$$x_3 + x_8 + x_{12} - x_9 - x_{10} - x_{11} = 0$$
$$x_9 + x_{15} + x_{19} - x_{12} - x_{13} - x_{14} = 0$$
$$x_4 - x_{15} - x_{17} = 0$$
$$x_5 - x_{18} - x_{19} - x_{20} = 0$$
$$x_{10} + x_{13} + x_{25} - x_{21} - x_{22} - x_{23} - x_{24} = 0$$
$$x_{11} + x_{21} + x_{23} + x_{30} - x_{25} - x_{26} - x_{27} = 0$$
$$x_{17} + x_{22} - x_{28} - x_{29} = 0$$
$$x_{23} + x_{26} - x_{30} - x_{31} = 0$$

Arc constraints:

$x_1 \leq 11$, $x_2 \leq 15$, $x_3 \leq 10$, $x_4 \leq 18$, $x_5 \leq 4$, $x_6 \leq 3$

$x_7 \leq 8$, $x_8 \leq 5$, $x_9 \leq 6$, $x_{10} \leq 3$, $x_{11} \leq 11$, $x_{12} \leq 4$, $x_{13} \leq 17$

$x_{14} \leq 6$, $x_{15} \leq 3$, $x_{16} \leq 16$, $x_{17} \leq 13$, $x_{18} \leq 12$, $x_{19} \leq 4$, $x_{20} \leq 21$, $x_{21} \leq 4$

$x_{22} \leq 4$, $x_{23} \leq 4$, $x_{24} \leq 3$, $x_{25} \leq 4$, $x_{26} \leq 5$, $x_{27} \leq 4$, $x_{28} \leq 7$, $x_{29} \leq 9$

$x_{30} \leq 2$, $x_{31} \leq 15$

and $x_i \geq 0 \ \forall \, i \in [1, 31]$

Question 3)

Duals of the above LPs have been calculated below:

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Dual for network 1.

$$\min\ 13y_5 + 16y_6 + 10y_7 + 4y_8 + 14y_9 + 9y_{10} + 12y_{11} + 7y_{12} + 7y_{13} + 20y_{14}$$

Subject to

$$y_2 + y_5 \geq 1$$
$$y_1 + y_6 \geq 1$$
$$-y_1 + y_2 + y_7 \geq 0$$
$$y_1 - y_2 + y_8 \geq 0$$
$$-y_2 + y_4 + y_9 \geq 0$$
$$y_2 - y_3 + y_{10} \geq 0$$
$$-y_1 + y_3 + y_{11} \geq 0$$
$$y_3 - y_4 + y_{12} \geq 0$$
$$-y_4 + y_{13} \geq 0$$
$$-y_3 + y_{14} \geq 0$$

y_1, y_2, y_3, y_4 are unrestricted

$y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14} \geq 0$

DUAL OF NETWORK 2

$$\text{MIN } Z = 11y_1 + 15y_2 + 10y_3 + 18y_4 + 4y_5 + 3y_6 + 8y_7 + 5y_8 + 6y_9 + 3y_{10} + 11y_{11} + 4y_{12} + 17y_{13} + 6y_{14} + 3y_{15} + 16y_{16} + 13y_{17} + 12y_{18} + 4y_{19} + 21y_{20} + 4y_{21} + 4y_{22} + 4y_{23} + 3y_{24} + 4y_{25} + 5y_{26} + 4y_{27} + 7y_{28} + 9y_{29} + 2y_{30} + 15y_{31}$$

SUBJECT TO:

$y_1 + y_{32} \geq 1$

$y_2 + y_{33} \geq 1$

$y_3 + y_{34} \geq 1$

$y_4 - y_{32} + y_{36} \geq 0$

$y_5 - y_{32} + y_{37} \geq 0$

$y_6 + y_{32} - y_{33} \geq 0$

$y_7 \geq 0$

$y_8 + y_{33} + y_{34} \geq 0$

$y_9 - y_{34} + y_{35} \geq 0$

$y_{10} - y_{34} + y_{38} \geq 0$

$y_{11} - y_{34} + y_{39} \geq 0$

$y_{12} + y_{34} - y_{35} \geq 0$

$y_{13} - y_{35} + y_{38} \geq 0$

$y_{14} - y_{35} \geq 0$

$y_{15} + y_{35} - y_{36} \geq 0$

$y_{16} \geq 0$

$y_{17} - y_{36} + y_{40} \geq 0$

$y_{18} + y_{32} - y_{37} \geq 0$

$y_{19} + y_{35} - y_{37} \geq 0$

$y_{20} - y_{37} \geq 0$

$y_{21} - y_{38} + y_{39} \geq 0$

$y_{22} - y_{38} + y_{40} \geq 0$

$y_{23} - y_{38} + y_{39} + y_{41} \geq 0$

$$y_{24} - y_{38} \geq 0$$

$$y_{25} + y_{38} - y_{39} \geq 0$$

$$y_{26} - y_{39} + y_{41} \geq 0$$

$$y_{27} - y_{39} \geq 0$$

$$y_{28} - y_{40} \quad \cancel{y_{41}} \geq 0$$

$$y_{29} - y_{40} \geq 0$$

$$y_{30} + y_{39} - y_{41} \geq 0$$

$$y_{31} - y_{41} \geq 0$$

AND, $y_1, y_2 \cdots y_{31} \geq 0$

$y_{32} \cdots y_{41} \in \mathbb{R}$.

Question 4)

Solving LPP using simplex from scratch

The following results are obtained on solving LPP using simplex from scratch

Primal Network 1

```
The optimal solution is [13. 13.  1.  0. 14.  0. 12.  7.  7. 19.]
The optimal value is 26.0
```

x1 = 13, x2 = 13, x3 = 1, x4 = 0, x5 = 14, x6 = 0, x7 = 12, x8 = 7, x9 = 7, x10 = 19

Primal Network 2

```
The optimal solution is [11.  8. 10. 10.  4.  3.  8.  5.  6.  3.  7.  1.
0.  6.  1. 16.  9.  0.
  0.  4.  0.  0.  0.  3.  0.  3.  4.  0.  9.  0.  3.]
The optimal value is 29.0
```

Dual Network 1

```
The optimal solution is [1. 1. 0. 1. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0.]
The optimal value is 26.0
```

Dual Network 2

```
The optimal solution is [1. 0. 1. 0. 0. 1. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.
0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
The optimal value is 29.0
```

Question 5)

LPs and their duals have been solved in AMPL

Flow Network 1

The LP of this network was formulated above, and consequently file Network_1_LP.run was made to display it's solution in AMPL. Running this file displayed the below output:

```
ampl: include Network_1_LP.run
CPLEX 20.1.0.0: optimal solution; objective 26
1 dual simplex iterations (0 in phase I)
x1 = 10
x2 = 16
x3 = 4
x4 = 0
x5 = 14
x6 = 0
x7 = 12
x8 = 7
x9 = 7
x10 = 19
z = 26
```

Flow Network 1 - Dual

The Dual of above LP was formulated above, and consequently file Network_1_LP_Dual.run was made to display it's solution in AMPL. Running this file displayed the below output:

```
ampl: include Network_1_LP_Dual.run
CPLEX 20.1.0.0: optimal solution; objective 26
6 dual simplex iterations (0 in phase I)
y1 = 1
y2 = 1
y3 = 0
y4 = 1
y5 = 0
y6 = 0
y7 = 0
y8 = 0
y9 = 0
y10 = 0
y11 = 1
y12 = 1
y13 = 1
y14 = 0
z = 26
```

As we can see, the optimal value derived from primal and dual LPs are same.

This means that for network 1, min-cut = max flow = 26

Flow Network 2

The LP of this network was formulated above, and consequently file Network_2_LP.run was made to display it's solution in AMPL. Running this file displayed the below output:

```
ampl: include Network_2_LP.run
CPLEX 20.1.0.0: optimal solution; objective 29
8 dual simplex iterations (0 in phase I)
x1 = 11
x2 = 8
x3 = 10
x4 = 10
x5 = 4
x6 = 3
x7 = 0
x8 = 5
x9 = 6
x10 = 0
x11 = 9
x12 = 0
x13 = 3
x14 = 6
x15 = 3
x16 = 0
x17 = 7
x18 = 0
x19 = 0
x20 = 4
x21 = 0
x22 = 2
x23 = 0
x24 = 3
x25 = 2
x26 = 3
x27 = 4
x28 = 0
x29 = 9
x30 = 0
x31 = 3
z = 29
```

Flow Network 2 - Dual

The Dual of above LP was formulated above, and consequently file Network_2_LP_Dual.run was made to display it's solution in AMPL. Running this file displayed the below output:

```
ampl: include Network_2_LP_Dual.run
CPLEX 20.1.0.0: optimal solution; objective 29
15 dual simplex iterations (0 in phase I)
y1 = 1
y2 = 0
y3 = 1
y4 = 0
y5 = 0
y6 = 1
y7 = 0
y8 = 1
y9 = 0
y10 = 0
y11 = 0
y12 = 0
y13 = 0
y14 = 0
y15 = 0
y16 = 0
y17 = 0
y18 = 0
y19 = 0
y20 = 0
y21 = 0
y22 = 0
y23 = 0
y24 = 0
y25 = 0
y26 = 0
y27 = 0
y28 = 0
y29 = 0
y30 = 0
y31 = 0
```

```
y32 = 0
y33 = 1
y34 = 0
y35 = 0
y36 = 0
y37 = 0
y38 = 0
y39 = 0
y40 = 0
y41 = 0
z = 29
```

As we can see, the optimal value derived from primal and dual LPs are same.

This means that for network 2, min-cut = max flow = 29

Also, the optimal solutions generated of all the above LPs are integral, which means flow through each edge is an integer, as required.