Q1)
ans) a)

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SI - Assingment - 1

Q.
ans) $x_i \overset{iid}{\sim} Gamma(r, \lambda)$ $(r=5, \lambda=2)$

Now if $X \sim Gamma(r, \lambda)$
then

$$f(x) = \frac{\lambda^r x_n^{r-1}}{\Gamma(r)} \exp(-\lambda x_n)$$

$$\Rightarrow L(\theta) = \prod_{i=1}^{n} \frac{\lambda^r x_n^{r-1}}{\Gamma(r)} \exp(-\lambda x_n)$$

(Likelihood function)

$$\Rightarrow L(\theta) = \frac{(\lambda^{nr}) \cdot \left(\prod_{i=1}^{n} x_i\right)^{r-1}}{(\Gamma(r))^n} \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

Now,
log-likelihood function $\rightarrow$ $l(\theta) = \log(L(\theta))$

$$= n \cdot r \cdot \log\lambda + (r-1) \cdot \sum_{i=1}^{n} \log(x_i) - n\log(\Gamma(r)) - \lambda \cdot \sum_{i=1}^{n} x_i$$

For maximum,

$$\frac{d\,l(\theta)}{dr} = 0 \quad \text{and} \quad \frac{d\,l(\theta)}{d\lambda} = 0$$

$\rightarrow$ firstly $\frac{d\,l(\theta)}{d\lambda} = 0$

$$\Rightarrow \frac{n\gamma}{\lambda} + 0 - 0 - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \lambda = \frac{n\gamma}{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{r}{\bar{x}}}$$

~~Now, for~~ $\cancel{\frac{d\,l(\theta)}{dr} = 0}$

$$\Rightarrow \cancel{n \log \lambda + \sum_{i=1}^{n} x_i - \frac{n}{\Gamma(r)} \frac{d\Gamma(r)}{dr} = 0}$$

To check maximum

$$\Rightarrow \frac{d^2 l(\theta)}{d\lambda^2} = \frac{-n\gamma}{\lambda^2}$$

$$= -\frac{n(\bar{x})^2}{r}$$

$n, \bar{x}, r$ are +ve so $\frac{d^2 L(\theta)}{d\lambda^2} < 0$

Now, $\frac{d\,l(\theta)}{dr} = 0$

$$\Rightarrow n \log \lambda + \sum_{i=1}^{n} x_i - \frac{n}{\Gamma(r)} \frac{d\Gamma(r)}{dr} = 0$$

Sub λ & diff further leads to $\rightarrow$

$$\ln(r) - \underbrace{\psi(r)}_{\text{digamma}} = \ln\left(\bar{x}\right) - \frac{\sum^{n} \ln(x_i)}{n}$$

b)

The eq^n can only be solved iteratively using complex numerical analysis, beyond the scope of our purview, so we leave the solution at this point.

Now ∀ method of moments estimation

$$X_i \sim Gamma(r, \lambda)$$

$$E[X] = \frac{r}{\lambda} \qquad E[X^2] = \frac{r(r+1)}{\lambda} \quad \leftarrow Var(x) + (E(x))^2$$

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{r}{\lambda} \qquad \frac{\sum x_i^2}{n} = \frac{r(r+1)}{\lambda} \quad — ①$$

Put $r = \lambda \frac{\sum x}{n}$ in ②

$$\frac{\lambda \bar{x}(\lambda \bar{x} + 1)}{\lambda^2} = \frac{\sum x_i^2}{n}$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{\bar{x}}{\frac{\sum x_i^2}{n} - \bar{x}^2}}$$

$$\Rightarrow \boxed{\hat{r} = \hat{\lambda}\bar{x} = \frac{\bar{x}^2}{\frac{\sum x_i^2}{n} - \bar{x}^2}}$$

For the maximum likelihood estimate, R routine 'optim()' is used to maximise the log likelihood of the random sample drawn from the Gamma distribution.

The following Maximum likelihood estimates for r = 5 and λ = 2 were obtained:
$\hat{r}_{MLE} = 5.018516$
$\hat{\lambda}_{MLE} = 2.061471$

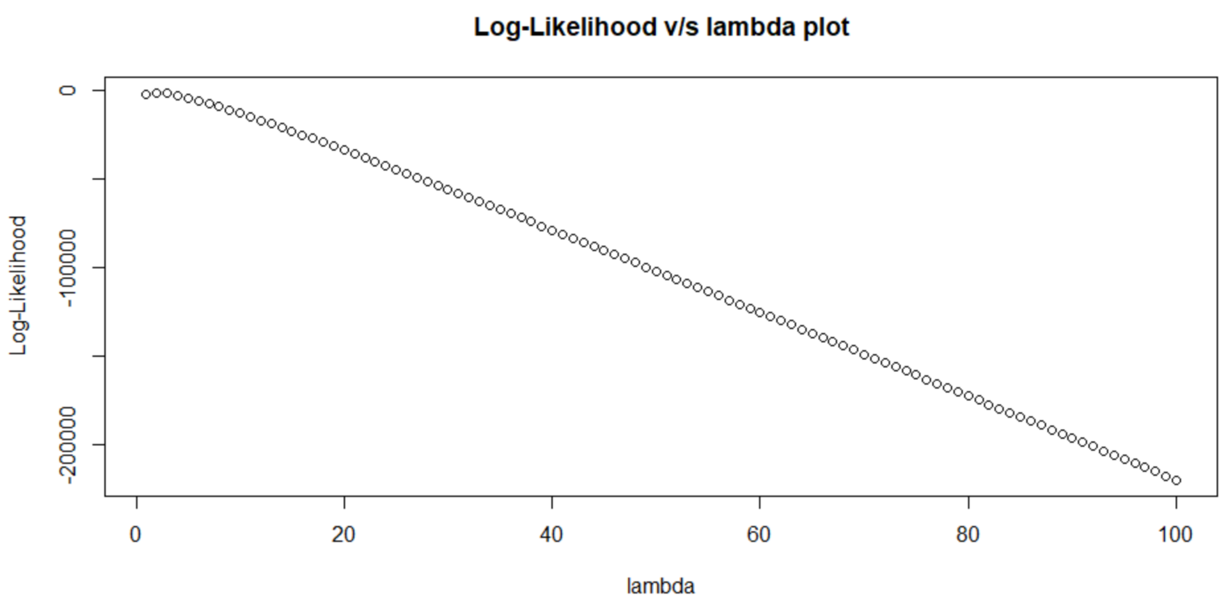
The value of log-likelihood at this value of r and λ is L(r,λ) = -1432.231

The MLE using Method of Moments where r = $(\text{mean})^2$/variance and λ = mean/variance

$\hat{r}_{MLE} = 5.018560$
$\hat{\lambda}_{MLE} = 2.061468$

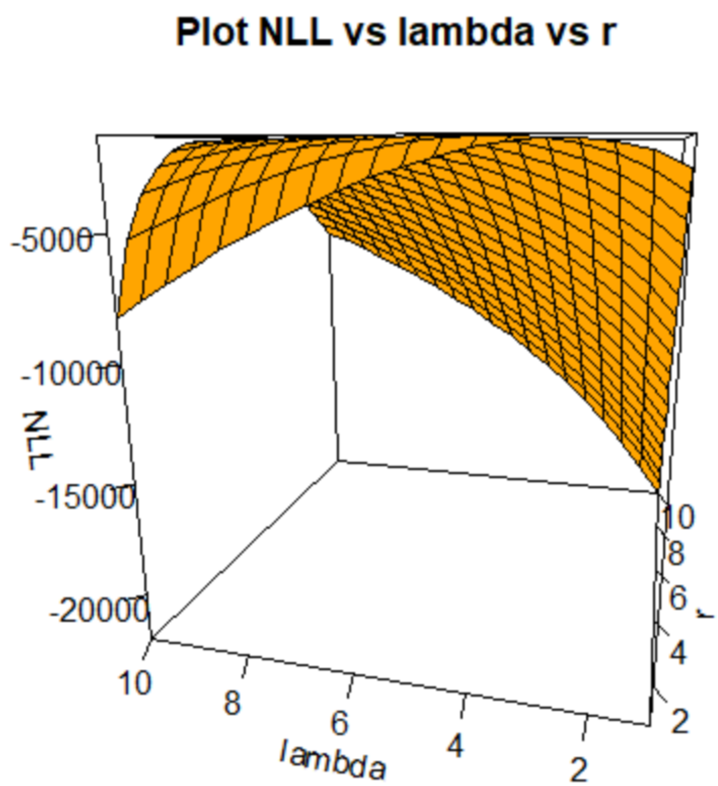
The value of log-likelihood at this value of r and λ is L(r,λ) = -1432.231

c) Below is the plot of Log-Likelihood with varied λ and constant r = 5. We observe that the log-likelihood function value is maximum at λ ≃ 2, which agrees with the theoretical value and the value obtained using the optimiser function (optim()).
Log-Likelihood value is plotted on the y-axis. While the x-axis contains various values of λ from 1 to 100 with a jump of 1.



Conclusion: The maximum likelihood estimate of the unknown parameter λ (i.e 2.061468) calculated using the R routine 'optim()' is nearly the same as the given value (≃ 2) which is also verified using the above plot.

d) Below is a 3-d plot of NLL with various values of lambda and r. We observe that the below plot reaches the maximum value at λ ≃ 2 and r ≃ 5 which is consistent with theoretically obtained values.
NLL is plotted on the z-axis, r is plotted on the x-axis and lambda is plotted on the y-axis both varying between 1 to 10 at a step = 0.5



Conclusion: Maximum reached graphically using R routines 'optim()' and 'persp()' gives us approximately the same value of r (≃ 5)and λ (≃ 2) as theoretically obtained.

Q2)
ans) a)

Q2.
ans) a) $X \overset{iid}{\sim} \text{Normal}(\mu, \sigma)$

then $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2} \frac{(x_i-\mu)^2}{\sigma^2}\right)$$

$\mu \to$ mean
$\sigma^2 \to$ variance

$$L(\theta) = \prod_{i=1}^{n} f(x)$$

(Likelihood function) $$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-1}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}\right)$$

$\Rightarrow$ $$\ell(\theta) = \log(L(\theta))$$
LLF
$$= -\frac{n}{2}\log(2\pi\sigma^2) + \frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\forall$ max, $\frac{d\ell(\theta)}{d\mu} = 0$ and $\frac{d\ell(\theta)}{d\sigma} = 0$

$\rightarrow$ Firstly $\frac{d\ell(\theta)}{d\sigma} = 0$

$\Rightarrow$ $$0 - \frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 = 0$$

$\Rightarrow$ $$\sigma = \sqrt{\frac{\sum(x_i-\mu)^2}{n}}$$

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To check $\forall$ maxima

$$\frac{d^2 l(\theta)}{d^2\sigma} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\left(\Sigma(x_i-\mu)^2\right)$$

substituting

$$= \frac{n^2}{\Sigma(x_i-\mu)^2} - \frac{3n^2}{\Sigma(x_i-\mu)^2}$$

$$= \frac{-2n^2}{\Sigma(x_i-\mu)^2} < 0$$

$\therefore$ maxima achieved.

So, $\boxed{\sigma^2_{MLE} = \frac{\Sigma(x_i-\mu)^2}{n}}$

Now, $\frac{dl(\theta)}{d\mu} = 0$

$\Rightarrow$ first substituting σ^2_{MLE} in $l(\theta)$ & then differentiating w.r.t μ

$\Rightarrow$ we'll get

$$\mu = \frac{\Sigma x_i}{n}$$

To check max,

$$\frac{d^2 l(\theta)}{d\mu^2} = \frac{-n}{\sigma^2} < 0$$

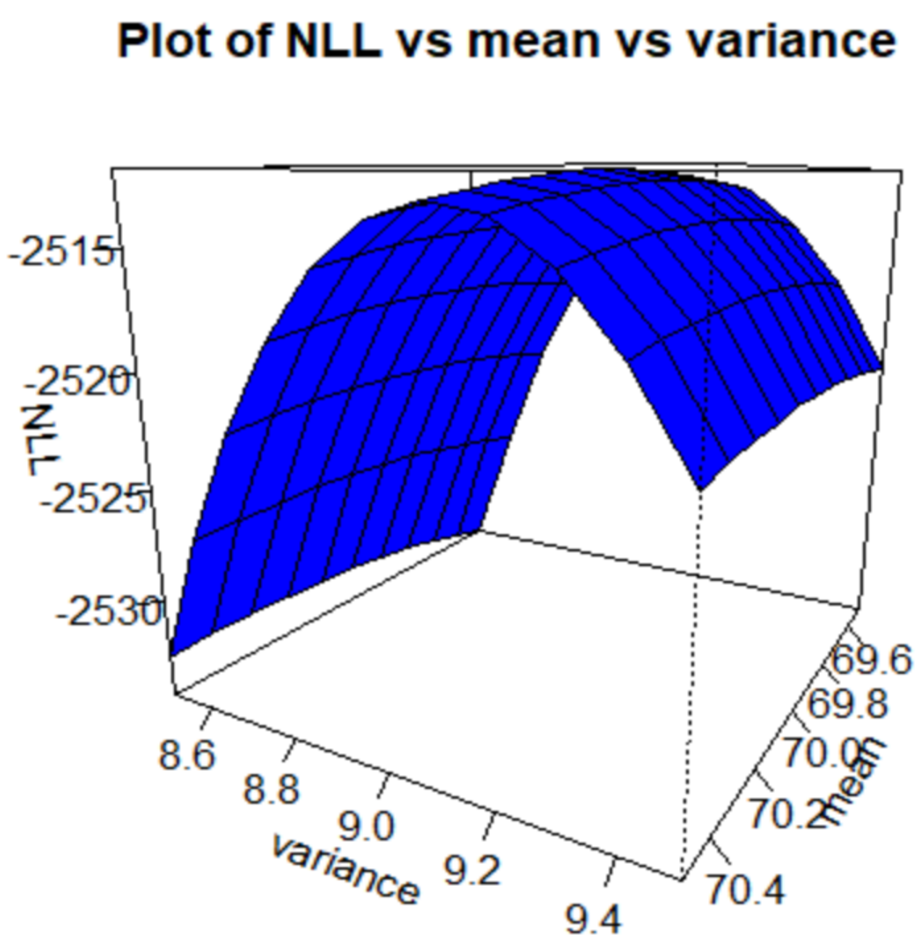
So, max attained.

$$\boxed{\hat{\mu}_{MLE} = \frac{\Sigma x_i}{n}}$$

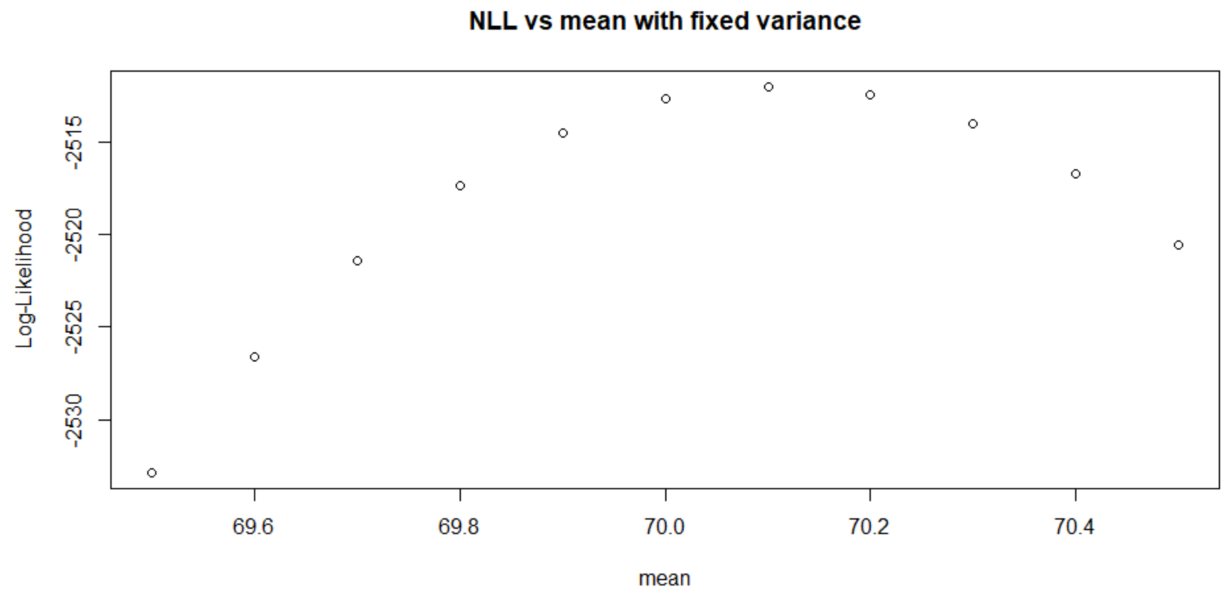
For the maximum likelihood estimate, R routine 'optim()' is used to maximise the log likelihood of the random sample drawn from the Normal distribution.
The following Maximum likelihood estimates for μ and σ^2 were obtained:
$\hat{\mu}_{MLE} = 70.109648$
$\sigma^2_{MLE} = 8.898797$

The value of log-likelihood at this value of μ and σ^2 is $L(\mu, \sigma^2) = -2512.004$

b) In the below 3-d plot, NLL is plotted against varied values of μ and σ^2.
We observe that the below plot reaches the maximum value at $\mu \simeq 70.10$ and $\sigma^2 \simeq 8.89$ which is consistent with theoretically obtained values.
NLL is plotted on the z-axis, μ is plotted on the x-axis varying between 69.5 and 70.5 with step = 0.1 and σ^2 is plotted on the y-axis varying between 8.5 and 9.5 with step = 0.1

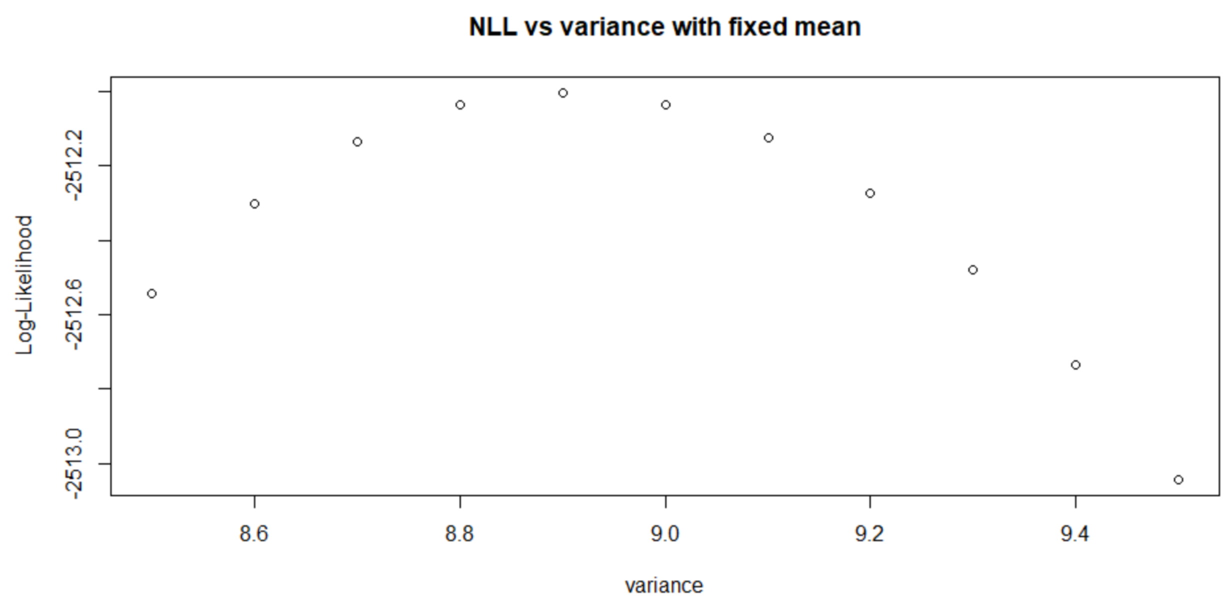


For a deeper explanation, I have provided two 2-d plots providing a clearer picture of maximum.
In the first plot, variance is fixed at the estimated value and mean is varied -



As we can see, we attain maximum at $\mu \simeq 70.10$, consistent with the theoretical value.

Similarly, in the second plot, the mean is fixed at the estimated value, and variance is varied.



As we can see, we attain maximum at $\sigma^2 \simeq 8.89$, consistent with the theoretical value.

Conclusion: Maximum reached graphically using R routines 'optim()' and 'persp()' gives us approximately the same value of μ ($\simeq 70.10$) and σ^2 ($\simeq 8.89$) as theoretically obtained.