

Algebraic Geometry

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Syllabus

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0 Brief review of classical algebraic geometry and motivation for scheme theory

The following are the main references for the course.

- R Hartshorne, Algebraic geometry, 1977
- U Goertz and T Wedhorn, Algebraic geometry I, 2010
- R Vakil, The rising sea: foundations of algebraic geometry, 2017

Lecture 1
Friday
09/10/20

0.1 Classical algebraic geometry

Throughout this discussion, we take the base field k to be algebraically closed. An **affine variety** $V \subseteq \mathbb{A}^n(k)$, where, once one has chosen coordinates, $\mathbb{A}^n(k) = k^n$, is given by the vanishing of polynomials $f_1, \dots, f_r \in k[X_1, \dots, X_n]$. If $I = \langle f_1, \dots, f_r \rangle \triangleleft k[X_1, \dots, X_n]$ is any ideal, we set

$$\mathbb{V}(I) = \{z \in \mathbb{A}^n \mid \forall f \in I, f(z) = 0\}.$$

First set $\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^*$ with **homogeneous coordinates** $(x_0 : \dots : x_n)$. A **projective variety** $V \subseteq \mathbb{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \dots, F_r \in k[X_0, \dots, X_n]$. If I is the ideal generated by the homogeneous ideals F_i , that is if $F \in I$ then so are all its homogeneous parts, we set

$$\mathbb{V}(I) = \{z \in \mathbb{P}^n \mid \forall F \in I \text{ homogeneous, } F(z) = 0\}.$$

If $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$, set

$$\mathbb{I}(V) = \{f \in k[X_1, \dots, X_n] \mid \forall x \in V, f(x) = 0\}.$$

Observe that $\mathbb{V}(\mathbb{I}(V)) = V$, by tautology, and $\mathbb{I}(\mathbb{V}(I)) \supseteq \sqrt{I}$, which is obvious. Recall that the **radical** \sqrt{I} of the ideal I is defined by $f \in \sqrt{I}$ if and only if there exists $m > 0$ such that $f^m \in I$. **Hilbert's Nullstellensatz** states that, noting $k = \bar{k}$, $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. The **coordinate ring** is

$$k[V] = k[X_1, \dots, X_n] / \mathbb{I}(V).$$

This may be regarded as the ring of polynomial functions on V , and it is a finitely generated reduced k -algebra. Recall that a **k -algebra** is a commutative ring containing k as a subring. It is **finitely generated** if it is the quotient of a polynomial ring over k , and **reduced** if $a^m = 0$ implies that $a = 0$.

0.2 Why schemes?

A better question is what is wrong with varieties?

- With varieties, always work over algebraically closed fields. For example, let $I = \langle X^2 + Y^2 + 1 \rangle \subseteq \mathbb{R}[X, Y]$. Then $\mathbb{V}(I) = \emptyset$, but I is a prime ideal, hence radical, so $\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[X, Y] \neq I$.
- Number theory? Diophantine equations. If $I \subseteq \mathbb{Z}[X_1, \dots, X_n]$ is an ideal, have $\mathbb{V}(I) \subseteq \mathbb{Z}^n$. For example, $X^n + Y^n = Z^n$.
- Why should we only consider radical, or prime, ideals? For example, a natural situation is

$$X_1 = \mathbb{V}(X - Y^2) \subseteq \mathbb{A}^2, \quad X_2 = \mathbb{V}(X) \subseteq \mathbb{A}^2.$$

Then $X_1 \cap X_2 = \mathbb{V}(X - Y^2, X)$. Note $I = \langle X - Y^2, X \rangle = \langle X, Y^2 \rangle$ is not a radical ideal, because $Y \notin I$ and $Y^2 \in I$ so $Y \notin \sqrt{I}$. Recall the coordinate ring of X_i is $k[X_i] = k[X, Y]/I_i$. Then $k[X_1 \cap X_2] = k[X, Y]/\langle X, Y^2 \rangle \cong k[Y]/\langle Y^2 \rangle$. So thinking of the coordinate ring of $X_1 \cap X_2$ as functions on $X_1 \cap X_2$, we have a function Y whose square is zero, but is not itself zero.

0.3 Categorical philosophy

What is a point? In the category of sets, objects are sets, and if A and B are sets, then morphisms are $\text{Hom}(A, B)$, the set of maps $f : A \rightarrow B$. Let $*$ be a one-element set. Then the elements of any set X are in one-to-one correspondence with $\text{Hom}(*, X)$. In the category of affine varieties, objects are affine varieties and morphisms are $\text{Hom}(X, Y) = \text{Hom}_{k\text{-alg}}(k[Y], k[X])$. In this category, a point is a single point with coordinate ring k . Giving a morphism

$$\{\text{point}\} \rightarrow X = \mathbb{V}(I) \subseteq \mathbb{A}^n, \quad I \subseteq k[X_1, \dots, X_n],$$

for I a radical ideal, is the same as giving a homomorphism

$$\begin{aligned} \phi : k[X] = k[X_1, \dots, X_n]/I &\longrightarrow k \\ X_i &\longmapsto a_i \end{aligned}$$

Note that ϕ vanishes in I if and only if $f(a_1, \dots, a_n) = 0$ for all $f \in I$, which is if and only if $(a_1, \dots, a_n) \in \mathbb{V}(I) = X$. Note ϕ is surjective, and hence $\ker \phi$ is a maximal ideal. With k algebraically closed, the maximal ideals at $k[X]$ are all of the form $\langle X_1 - a_1, \dots, X_n - a_n \rangle$ for $(a_1, \dots, a_n) \in X$, a consequence of Hilbert's Nullstellensatz. That is, there exists one-to-one correspondences

$$\{\text{points of } X\} \longleftrightarrow \{k\text{-algebra homomorphisms } \phi : k[X] \rightarrow k\} \longleftrightarrow \{\text{maximal ideals of } k[X]\}.$$

0.4 Solutions over non-algebraically closed fields

What if k is not algebraically closed? We may want to consider solutions not just in $k^n = \mathbb{A}^n$ but $(k')^n$ for k' any field extension of k . That is, we may consider k -algebra homomorphisms

$$\begin{aligned} \phi : k[X] = k[X_1, \dots, X_r]/I &\longrightarrow k' \\ X_i &\longmapsto a_i \end{aligned}$$

This gives a tuple $(a_1, \dots, a_n) \in (k')^n$ with $f(a_1, \dots, a_n) = 0$ for all $f \in I$. Then ϕ need not be surjective, so can only say the image of ϕ is a subring of a field, hence an integral domain. Thus $\ker \phi$ is a prime ideal, and maximal if and only if $\text{im } \phi$ is a field.

Example. The \mathbb{R} -algebra homomorphism

$$\begin{aligned} \phi : \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle &\longrightarrow \mathbb{C} \\ X &\longmapsto 0 \\ Y &\longmapsto i \end{aligned}$$

is surjective with kernel $\langle X, Y^2 + 1 \rangle$, since $\mathbb{R}[Y] / \langle Y^2 + 1 \rangle \cong \mathbb{C}$. This is a maximal ideal but is not of the form $\langle X - a, Y - b \rangle$ for $(a, b) \in \mathbb{R}^2$. If instead we considered the map

$$\begin{aligned} \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle &\longrightarrow \mathbb{C} \\ X &\longmapsto 0 \\ Y &\longmapsto -i \end{aligned},$$

we get the same kernel. That is, $(0, i)$ and $(0, -i)$ are solutions to $X^2 + Y^2 + 1 = 0$, but they correspond to the same maximal ideal. In fact, this maximal ideal corresponds to a Galois orbit of $\text{Gal}(\mathbb{C}/\mathbb{R})$ of solutions.

There are more exotic points by taking even bigger fields.

Example. Let $k(X)$ be the field of fractions of $k[X] = \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle$. There is an inclusion

$$\begin{aligned} k[X] &\longrightarrow k(X) \\ f &\longmapsto \frac{f}{1} \\ (X, Y) &\longmapsto (X, Y) \end{aligned}.$$

The kernel of this map is zero. This gives a solution to the equation $X^2 + Y^2 + 1 = 0$ with coordinates in the field $k(X)$. This solution is $(X, Y) \in \mathbb{A}^2(k(X))$.

The moral is that once we start looking at solutions to equation over any field, then we get maps $k[X] \rightarrow k'$ with kernel not necessarily maximal. What about solutions over rings?

Example. Let $A = \mathbb{Z}[X_1, \dots, X_n]/I$, and let R be any commutative ring. We define an R -valued point of $\text{Spec } A$ to be a ring homomorphism

$$\begin{array}{ccc} A & \longrightarrow & R \\ X_i & \longmapsto & r_i \end{array}.$$

Then $f(r_1, \dots, r_n) = 0$ for all $f \in I$. This gives a lot of flexibility. For example,

- $R = \mathbb{Z}$ gives diophantine equations,
- $R = \mathbb{F}_p$ gives solutions modulo p , and
- $R = \mathbb{Q}$ gives rational solutions.

Take this to its logical conclusion. Let A be a ring, where all rings are commutative in this course. Given A , we hope for some geometric object $\text{Spec } A$, the **spectrum** of A . For a ring R , the set of R -valued points of X is

$$X(R) = \text{Hom}_{\text{ring}}(A, R).$$

A morphism $X = \text{Spec } A \rightarrow Y = \text{Spec } B$ should be the same thing as giving a morphism $\phi: B \rightarrow A$. Define the category of **affine schemes** to be the opposite category to the category of rings. Define a **scheme** to be something which is locally isomorphic to an affine scheme. By analogy, a **manifold** is a topological space with an open cover $\{U_i\}$ with each U_i homeomorphic to an open subset of \mathbb{R}^n . To make sense of the definition of schemes, we need a lot of language.

0.5 Spectrum of a ring

Definition. Let A be a ring. Then

$$\text{Spec } A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ a prime ideal}\}.$$

For $I \subseteq A$ an ideal, define

$$\mathbb{V}(I) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime, } \mathfrak{p} \supseteq I\}.$$

Proposition 0.1. *The sets $\mathbb{V}(I)$ form the closed sets of a topology on $\text{Spec } A$, called the **Zariski topology**.*

Proof.

- $\mathbb{V}(A) = \emptyset$.
- $\mathbb{V}(0) = \text{Spec } A$.
- If $\{I_i\}_{i \in J}$ is a collection of ideals, then

$$\mathbb{V}\left(\sum_{i \in J} I_i\right) = \bigcap_{i \in J} \mathbb{V}(I_i).$$

- Claim that

$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2).$$

\supseteq Obvious.

\subseteq If $\mathfrak{p} \supseteq I_1 \cap I_2$ is prime, then $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$. See Atiyah-Macdonald, Proposition 1.11.ii. ¹

□

Example. Let $A = k[X_1, \dots, X_n]$ with k algebraically closed and $I \subseteq A$ an ideal. Then the maximal ideals \mathfrak{m} of A containing I are in one-to-one correspondence with the zero set of I in $\mathbb{A}^n(k)$, so

$$\left\{ \langle X_1 - a_1, \dots, X_n - a_n \rangle \supseteq I, a_i \in k \right\} \quad \rightsquigarrow \quad \left\{ (a_1, \dots, a_n) \in \mathbb{V}(I) \subseteq \mathbb{A}^n(k) \right\}.$$

The new $\mathbb{V}(I)$ now extends this notion of zero set by including possible other prime ideals.

Example. If k is a field, $\text{Spec } k = \{0\}$, so the topological space cannot see the field.

We fix this by also thinking about what functions are on these spaces.

¹Exercise: try to prove without looking up

1 Sheaves

Fix a topological space X .

1.1 Sheaves

Definition. A **presheaf** \mathcal{F} on X consists of the following data.

- For every open set $U \subseteq X$ an abelian group $\mathcal{F}(U)$.
- Whenever given an inclusion $V \subseteq U \subseteq X$, a **restriction map** $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$, a homomorphism, such that
 - $\rho_{UU} = \text{id}_{\mathcal{F}(U)}$, and
 - if $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark. Can think of a presheaf as a contravariant functor from the category of open sets of X , the category whose objects are open subsets of X and whose morphisms are inclusions of open sets, to the category of abelian groups. Can replace the category of abelian groups with any desired category, such as commutative rings.

Definition. A **morphism of presheaves** $f : \mathcal{F} \rightarrow \mathcal{G}$ is a collection of homomorphisms $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ such that for all $V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

is commutative.

Definition. A presheaf \mathcal{F} is a **sheaf** if it satisfies the following additional axioms.

- S1. If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = \rho_{UU_i}(s) = 0$ for all i , then $s = 0$.
- S2. If U and $\{U_i\}$ are as in S1 and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j , then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all i .

Remark.

- If \mathcal{F} is a sheaf, then $\emptyset \subseteq X$ is covered by the empty covering, and hence $\mathcal{F}(\emptyset) = 0$.
- S1 and S2 together can be described as saying, given U and $\{U_i\}_{i \in I}$,

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \xrightleftharpoons[\beta_2]{\beta_1} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where

$$\alpha(s) = (s|_{U_i})_{i \in I}, \quad \beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})_{i,j}, \quad \beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}.$$

Exactness means

- α is injective, which is S1,
- $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and
- for any $(s_i) \in \prod_{i \in I} \mathcal{F}(U_i)$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}(U)$ with $\alpha(s) = (s_i)$, which is S2.

1.2 Examples

Example.

- Let X be any topological space, and let

$$\mathcal{F}(U) = \{\text{continuous functions } U \rightarrow \mathbb{R}\}.$$

This is a sheaf, by

$$\begin{aligned} \rho_{UV} : \mathcal{F}(U) &\longrightarrow \mathcal{F}(V) \\ f &\longmapsto f|_V \end{aligned}.$$

S1. A continuous function is zero if it is zero on every open set of a cover.

S2. Continuous functions can be glued.

- Let $X = \mathbb{C}$ with the Euclidean topology, and let

$$\mathcal{F}(U) = \{f : U \rightarrow \mathbb{C} \mid f \text{ is a bounded analytic function}\}.$$

This is a presheaf. It satisfies S1, and does not satisfy S2. For example, consider the cover $\{U_i\}_{i \in \{1,2,\dots\}}$ of \mathbb{C} given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$ and

$$\begin{aligned} s_i : U_i &\longrightarrow \mathbb{C} \\ z &\longmapsto z \end{aligned}.$$

Note if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. But if we glue we get the function $z : \mathbb{C} \rightarrow \mathbb{C}$, which is not bounded. Note $\mathcal{F}(\mathbb{C}) = \mathbb{C}$.

- Take any group G and set $\mathcal{F}(U) = G$ for any open set U . This is called the **constant presheaf**. This is not a sheaf. Let $U = U_1 \sqcup U_2$. If we wanted a sheaf,

$$\begin{array}{ccc} \mathcal{F}(U_1) = G & & \mathcal{F}(U_2) = G \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(U_1 \cap U_2) = \mathcal{F}(\emptyset) = 0 & \end{array},$$

so if S2 is satisfied, would want $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ to glue. We would then want to have $\mathcal{F}(U) = G \times G$. Now give G the discrete topology, and define instead

$$\mathcal{F}(U) = \{f : U \rightarrow G \text{ continuous}\},$$

that is f is locally constant. That is, if $x \in U$, there exists a neighbourhood $x \in V \subseteq U$ with $f|_V$ constant. This is called the **constant sheaf** and if U is non-empty and connected, then $\mathcal{F}(U) = G$.

- If X is an algebraic variety, and $U \subseteq X$ is a Zariski open subset, define

$$\mathcal{O}_X(U) = \{f : U \rightarrow k \mid f \text{ regular function}\}.$$

Roughly f is **regular** means that every point of U has an open neighbourhood on which f is expressed as a ratio of polynomials g/h with h non-vanishing on the neighbourhood. Then \mathcal{O}_X is a sheaf, called the **structure sheaf** of X .

1.3 Stalks

Definition. Let \mathcal{F} be a presheaf on X . Let $p \in X$. Then the **stalk** of \mathcal{F} at p is

$$\mathcal{F}_p = \{(U, s) \mid U \subseteq X \text{ is an open neighbourhood of } p, s \in \mathcal{F}(U)\} / \equiv,$$

where $(U, s) \equiv (V, s')$ if there exists $W \subseteq U \cap V$ also a neighbourhood of p such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a **germ**.

Remark. $\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$.

Note that a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of presheaves induces a morphism

$$f_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p \\ (U, s) \longmapsto (U, f_U(s)) .$$

Proposition 1.1. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$.*

Proof.

\implies Obvious.

\impliedby Assume f_p is an isomorphism for all $p \in X$. Need to show that $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an isomorphism for all $U \subseteq X$, as then we can define $(f^{-1})_U = (f_U)^{-1}$. Check that with this definition, $(f^{-1})_U$ is compatible with restriction maps, hence f^{-1} is a morphism of sheaves.²

- f_U is injective. Suppose $s \in \mathcal{F}(U)$, and $f_U(s) = 0$. Then for all $p \in U$, $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. Since f_p is injective, $(U, s) = 0$ in \mathcal{F}_p . That is, there exists an open neighbourhood V_p of p in U such that $s|_{V_p} = 0$. Since $\{V_p\}_{p \in U}$ cover U , we see by S1 that $s = 0$.
- f_U is surjective. Let $t \in \mathcal{G}(U)$ and write $t_p = (U, t) \in \mathcal{G}_p$. Since f_p is surjective, there exists $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. That is, there exists $V_p \subseteq U$ an open neighbourhood of p , and a germ (V_p, s_p) such that $(V_p, f_{V_p}(s_p)) \equiv (U, t)$. By shrinking V_p if necessary, we can assume that $t|_{V_p} = f_{V_p}(s_p)$. Now on $V_p \cap V_q$,

$$f_{V_p \cap V_q}(s_p|_{V_p \cap V_q} - s_q|_{V_p \cap V_q}) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0,$$

and hence by injectivity of $f_{V_p \cap V_q}$ already proved, we have $s_p|_{V_p \cap V_q} = s_q|_{V_p \cap V_q}$. By S2 the s_p 's glue to give an element $s \in \mathcal{F}(U)$ with $s|_{V_p} = s_p$, for all $p \in U$. Now

$$f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(s_p) = t|_{V_p} .$$

By S1, applied to $f_U(s) - t$, we get $f_U(s) = t$. Thus f_U is surjective.

□

1.4 Sheafification

Theorem 1.2 (Sheafification). *Given a presheaf \mathcal{F} , there exists a sheaf \mathcal{F}^+ and a morphism $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$ satisfying the following universal property. For any sheaf \mathcal{G} and morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $\phi^+ \circ \theta = \phi$, so*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ & \searrow \phi & \downarrow \phi^+ \\ & & \mathcal{G} \end{array}$$

The pair (\mathcal{F}^+, θ) is unique up to unique isomorphism, and is called the **sheafification** of \mathcal{F} .

Proof. See example sheet 1. The idea is to make \mathcal{F}^+ look like functions. Define

$$\mathcal{F}^+(U) = \left\{ s : U \rightarrow \bigsqcup_{p \in U} \mathcal{F}_p \mid \begin{array}{l} \forall p \in U, s(p) \in \mathcal{F}_p, \\ \forall p \in U, \exists p \in V \subseteq U, \exists t \in \mathcal{F}(V), \forall q \in V, s(q) = (V, t) \in \mathcal{F}_q \end{array} \right\} .$$

Then

$$\begin{array}{ccc} \theta_U : \mathcal{F}(U) & \longrightarrow & \mathcal{F}^+(U) \\ s & \longmapsto & (p \mapsto (U, s) \in \mathcal{F}_p) . \end{array}$$

□

Exercise. A recommendation is to do all exercises in chapter II.1 of Hartshorne.

²Exercise

1.5 Kernels, cokernels, and images

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Definition. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on a space X . We define the following.

- The **presheaf kernel** of f , $\ker f$, is the presheaf given by $(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$.
- The **presheaf cokernel** $\operatorname{coker} f$ is the presheaf given by $(\operatorname{coker} f)(U) = \operatorname{coker}(f_U) = \mathcal{G}(U) / \operatorname{im} f_U$.
- The **presheaf image** $\operatorname{im} f$ is the presheaf given by $(\operatorname{im} f)(U) = \operatorname{im} f_U$.

Exercise. Check that these are presheaves, that is restrictions work.

Remark 1.3. If $f : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is also a sheaf.

Proof. S1 is certainly satisfied. If $s \in (\ker f)(U) \subseteq \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for all U_i in a cover of U so $s = 0$ by S1 for \mathcal{F} . Given $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ an open cover of U , and with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ by S2 for \mathcal{F} . But $f_U(s) = 0$ since $f_U(s)|_{U_i} = f_{U_i}(s|_{U_i}) = f_{U_i}(s_i) = 0$ so by S1, $f_U(s) = 0$. \square

Example. Let $X = \mathbb{P}^1$, or think of the Riemann sphere. Let $P, Q \in X$ be distinct points. Let \mathcal{G} be the sheaf of regular functions on X , or think of the sheaf of holomorphic functions. Let \mathcal{F} be the sheaf of regular functions on X which vanish at P and Q . Note $\mathcal{F}(U) = \mathcal{G}(U)$ if $U \cap \{P, Q\} = \emptyset$. Let $U = \mathbb{P}^1 \setminus \{P\}$ and $V = \mathbb{P}^1 \setminus \{Q\}$. Note $\mathcal{F}(\mathbb{P}^1) = 0$ and $\mathcal{G}(\mathbb{P}^1) = k$, because regular functions on \mathbb{P}^1 are constants. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be the obvious inclusion. Then

$$\begin{aligned} (\operatorname{coker} f)(\mathbb{P}^1) &= k, & (\operatorname{coker} f)(U) &= \mathcal{G}(U) / \mathcal{F}(U) = k[X] / \langle X \rangle = k, \\ (\operatorname{coker} f)(V) &= k, & (\operatorname{coker} f)(U \cap V) &= \mathcal{G}(U \cap V) / \mathcal{F}(U \cap V) = 0. \end{aligned}$$

If S2 holds, then we would need to have $(\operatorname{coker} f)(\mathbb{P}^1) = k \oplus k$. This is not a bug, but a feature.

Definition. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves.

- The **sheaf kernel** $\ker f$ of f is just the presheaf kernel.
- The **sheaf cokernel** is the sheaf associated to the presheaf cokernel of f .
- The **sheaf image** is the sheaf associated to the presheaf image of f .

\mathcal{F} is a **subsheaf** of \mathcal{G} if we have inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ for all U compatible with restrictions.

Exercise. The sheaf image $\operatorname{im} f$ is a subsheaf of \mathcal{G} .

We say f is **injective** if $\ker f = 0$. We say f is **surjective** if $\operatorname{im} f = \mathcal{G}$. We say a sequence of morphisms of sheaves

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \rightarrow \dots$$

is **exact** if $\ker f^{i+1} = \operatorname{im} f^i$ for all i . If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, we write \mathcal{F}/\mathcal{F}' for the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) / \mathcal{F}'(U)$. That is, this is the cokernel of the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$. A warning is if $f : \mathcal{F} \rightarrow \mathcal{G}$ is surjective, we do not necessarily have $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ surjective for all U .

Lemma 1.4. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then for all $p \in X$,

$$(\ker f)_p = \ker(f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p), \quad (\operatorname{im} f)_p = \operatorname{im} f_p.$$

Proof. Have a map

$$\begin{array}{ccc} (\ker f)_p & \longrightarrow & \ker f_p \subseteq \mathcal{F}_p \\ (U, s) & \longmapsto & (U, s) \end{array}.$$

If $s \in (\ker f)(U) = \ker f_U$ represents a germ $(U, s) \in (\ker f)_p$, then $(U, s) \in \mathcal{F}_p$, and $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. So $(U, s) \in \ker f_p$.

- **Injective.** If $(U, s) = 0$ in \mathcal{F}_p , there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, s|_V) = (V, 0) = 0$ in $(\ker f)_p$.
- **Surjective.** If $(U, s) \in \ker f_p$, then $(U, f_U(s)) = 0$ in \mathcal{G}_p . That is, there exists a neighbourhood $V \subseteq U$ of p such that $0 = f_U(s)|_V = f_V(s|_V)$. Thus $s|_V \in (\ker f)(V)$, and $(V, s|_V) \in (\ker f)_p$, and $(V, s|_V)$ maps to the same element in $\ker f_p$ represented by (U, s) .

Let $\text{im}' f$ be the presheaf image. An easy fact is if \mathcal{F} is a presheaf with associated sheaf \mathcal{F}^+ , then $\mathcal{F}_p \cong \mathcal{F}_p^+$ for all $p \in X$.³ Thus $(\text{im } f)_p = (\text{im}' f)_p$, so need to show $(\text{im}' f)_p \cong \text{im } f_p$. Define a map by

$$\begin{aligned} (\text{im}' f)_p &\longrightarrow \text{im } f_p \\ (U, s) &\longmapsto (U, s) \end{aligned}$$

- **Injective.** If $(U, s) = 0$ in \mathcal{G}_p then there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, 0)$ in $(\text{im}' f)_p$.
- **Surjective.** If $(U, s) \in \text{im } f_p$, then there exists $(V, t) \in \mathcal{F}_p$ with $(U, s) = f_p(V, t) = (V, f_V(t))$, so after shrinking U and V if necessary, then we can take $U = V$ and $f_U(t) = s$. Then $(U, s) \in (\text{im}' f)_p$.

□

Proposition 1.5. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then*

1. *f is injective if and only if $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ is injective for all p , and*
2. *f is surjective if and only if $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ is surjective for all p .*

Proof.

1. f_p is injective for all p if and only if $\ker f_p = 0$ for all p , if and only if $(\ker f)_p = 0$ for all p , if and only if $\ker f = 0$,⁴ which is if and only if f is injective.
2. f_p is surjective for all p if and only if $\text{im } f_p = \mathcal{G}_p$ for all p , if and only if $(\text{im } f)_p = \mathcal{G}_p$ for all p , if and only if $\text{im } f = \mathcal{G}$,⁵ which is if and only if f is surjective.

□

Remark. Given $f : \mathcal{F} \rightarrow \mathcal{G}$, in fact $\mathcal{G}/\text{im } f \cong \text{coker } f$.⁶

1.6 Passing between spaces

Let $f : X \rightarrow Y$ be a continuous map between topological spaces, \mathcal{F} a sheaf on X , and \mathcal{G} a sheaf on Y . Define $f_*\mathcal{F}$ by, for $U \subseteq Y$

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

Exercise. Check $f_*\mathcal{F}$ is a sheaf on Y .

Define $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \subseteq X \mapsto \{(V, s) \mid V \supseteq f(U), V \text{ open}, s \in \mathcal{G}(V)\} / \sim,$$

where $(V, s) \sim (V', s')$ if there exists $W \subseteq V \cap V'$ such that $f(U) \subseteq W$, and $s|_W = s'|_W$.

Example. If $f : \{p\} \rightarrow X$ is an inclusion of a point, then $f^{-1}\mathcal{G} = \mathcal{G}_p$. This is a group but defines a sheaf on a one-point space. More generally, if $\iota : Z \hookrightarrow X$ is an inclusion of a subset with induced topology, we often write

$$\mathcal{F}|_Z = \iota^{-1}\mathcal{F}.$$

If Z is open in X , then this is easy, since if $U \subseteq Z$ then $\mathcal{F}|_Z(U) = \mathcal{F}(U)$.

Remark. If $s \in \mathcal{F}(U)$ we say s is a **section** of \mathcal{F} over U . We often write

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}),$$

thinking of $\Gamma(U, \cdot)$ as a functor from the category of sheaves on a space X to the category of abelian groups.

³Exercise: check

⁴Exercise: check by S1

⁵Exercise: check using $\text{im } f \subseteq \mathcal{G}$

⁶Exercise

2 Schemes

Want to construct a sheaf \mathcal{O} on $\text{Spec } A$, analogous to the sheaf of regular functions on a variety, and \mathcal{O} will be a sheaf of rings. That is, $\mathcal{O}(U)$ will be a ring for each open set U and restriction maps will be ring homomorphisms.

2.1 Localisation of a ring

Importantly recall the following. Let A be a ring, where all rings are commutative with unity, and $S \subseteq A$ be a multiplicatively closed subset, that is $1 \in S$ and if $s_1, s_2 \in S$ then $s_1 s_2 \in S$. We define a ring

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\} / \sim,$$

where $(a, s) \sim (a', s')$ if there exists $s'' \in S$ such that $s''(as' - a's) = 0$. Then $S^{-1}A$ is called the **localisation of A at S** . Note that we write a/s for the equivalence class of (a, s) . The usual equivalence relation on fractions is $a/s = a'/s'$ if and only if $as' = a's$. We need the extra possibility of killing $as' - a's$ with s'' if A is not an integral domain.

Example.

- Take $f \in A$ and $S = \{1, f, \dots\} \subseteq A$. Then we write $A_f = S^{-1}A$. These will correspond to open subsets.
- If $\mathfrak{p} \subseteq A$ is a prime ideal and $S = A \setminus \mathfrak{p}$, then
 - $1 \in S$, and
 - $a, b \in S$ and $ab \in \mathfrak{p}$ is a contradiction by definition of prime ideals, so $ab \in S$.

Then $A_{\mathfrak{p}} = S^{-1}A$ is the **localisation of A at \mathfrak{p}** . These will correspond to stalks.

2.2 Construction of the structure sheaf

Let

$$\mathcal{O}(U) = \left\{ s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mid \begin{array}{l} \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ \forall \mathfrak{p} \in U, \exists \mathfrak{p} \in V \subseteq U \text{ open}, \exists a, f \in A, \forall \mathfrak{q} \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}} \end{array} \right\}.$$

Proposition 2.1. For any $\mathfrak{p} \in \text{Spec } A$, $\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$.

Proof. Have a map

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}} \\ (U, s) & \longmapsto & s(\mathfrak{p}) \end{array}.$$

- Surjective. Any element of $A_{\mathfrak{p}}$ can be written as a/f for some $a \in A$ and $f \notin \mathfrak{p}$. Then $\mathbb{D}(f) = \text{Spec } A \setminus \mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \notin \mathfrak{p}\}$, since $\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\}$. Now a/f defines an element of $\mathcal{O}(\mathbb{D}(f))$ given by

$$\begin{array}{ccc} s & : & \mathbb{D}(f) \longrightarrow A_{\mathfrak{q}} \\ \mathfrak{q} & \longmapsto & \frac{a}{f} \end{array},$$

and in particular, $s(\mathfrak{p}) = a/f \in A_{\mathfrak{p}}$.

- Injective. Let $\mathfrak{p} \in U \subseteq \text{Spec } A$ and $s \in \mathcal{O}(U)$ with $s(\mathfrak{p}) = 0$ in $A_{\mathfrak{p}}$. Want to show $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$. By shrinking U if necessary, we can assume that s is given by $a, f \in A$ with $s(\mathfrak{q}) = a/f$ for all $\mathfrak{q} \in U$. In particular $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$. Thus $a/f = 0/1$ in $A_{\mathfrak{p}}$ so there exists $h \in A \setminus \mathfrak{p}$ such that $0 = h \cdot (a \cdot 1 - f \cdot 0) = h \cdot a$ in A . Now let $V = U \cap \mathbb{D}(h)$. Then $(V, s|_V) = 0$, since for $\mathfrak{q} \in V$, $s|_V(\mathfrak{q}) = s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$ and $h \cdot a = 0$, and $h \in A \setminus \mathfrak{q}$ so $h \cdot a = 0$ implies $a/f = 0/1$ in $A_{\mathfrak{q}}$. Thus $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$.

□

Proposition 2.2. For any $f \in A$, $\mathcal{O}(\mathbb{D}(f)) = A_f$.

In particular, as $\text{Spec } A = \mathbb{D}(1)$, the **global sections** of \mathcal{O} is $\mathcal{O}(\text{Spec } A) = A_1 = A$.

Proof. Let

$$\begin{aligned} \psi : A_f &\longrightarrow \mathcal{O}(\mathbb{D}(f)) \\ \frac{a}{f^n} &\longmapsto \left(\mathfrak{p} \in \mathbb{D}(f) \mapsto \frac{a}{f^n} \in A_{\mathfrak{p}} \right), \end{aligned}$$

since $f \notin \mathfrak{p}$ implies that $f^n \notin \mathfrak{p}$ for all $n \geq 0$.

- **Injective.** If $\psi(a/f^n) = 0$, then for all $\mathfrak{p} \in \mathbb{D}(f)$, $a/f^n = 0$ in $A_{\mathfrak{p}}$, that is there exists $h \in A \setminus \mathfrak{p}$ such that $h \cdot a = 0$ in A . Let $I = \{g \in A \mid g \cdot a = 0\}$, the **annihilator** of a . So $h \in I$ and $h \notin \mathfrak{p}$, so $I \not\subseteq \mathfrak{p}$. This is true for all $\mathfrak{p} \in \mathbb{D}(f)$, so $\mathbb{V}(I) \cap \mathbb{D}(f) = \emptyset$. Thus $f \in \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p} = \sqrt{I}$, the radical, so $f^m \in I$ for some $m > 0$. Thus $f^m \cdot a = 0$, so $a/f^n = 0$ in A_f . Thus ψ is injective.
- **Surjective.** Let $s \in \mathcal{O}(\mathbb{D}(f))$. Cover $\mathbb{D}(f)$ with open sets V_i on which s is represented as a_i/g_i with $a_i, g_i \in A$ such that $g_i \notin \mathfrak{p}$ whenever $\mathfrak{p} \in V_i$. Thus $V_i \subseteq \mathbb{D}(g_i)$. By question 1 on example sheet 1, the sets of the form $\mathbb{D}(h)$ form a base for the Zariski topology on $\text{Spec } A$. Thus we can assume $V_i = \mathbb{D}(h_i)$ for some $h_i \in A$. Since $\mathbb{D}(h_i) \subseteq \mathbb{D}(g_i)$, we have $\mathbb{V}(h_i) \supseteq \mathbb{V}(g_i)$, so $\sqrt{\langle h_i \rangle} \subseteq \sqrt{\langle g_i \rangle}$, so $h_i^n \in \langle g_i \rangle$ for some n , say $h_i^n = c_i g_i$, so $a_i/g_i = c_i a_i/h_i^n$. Now replace h_i by h_i^n , since this does not change open sets because in general $\mathbb{D}(h_i) = \mathbb{D}(h_i^n)$, and replace a_i by $c_i a_i$. The situation so far is that we can assume $\mathbb{D}(f)$ is covered by sets $\mathbb{D}(h_i)$ such that s is represented by a_i/h_i on $\mathbb{D}(h_i)$. Claim that $\mathbb{D}(f)$ can be covered by a finite number of the $\mathbb{D}(h_i)$, that is $\mathbb{D}(f)$ is quasi-compact. Since

$$\begin{aligned} \mathbb{D}(f) \subseteq \bigcup_i \mathbb{D}(h_i) &\iff \mathbb{V}(f) \supseteq \bigcap_i \mathbb{V}(h_i) = \mathbb{V}\left(\sum_i \langle h_i \rangle\right) &\iff f \in \bigcap_{\mathfrak{p} \in \mathbb{V}(\sum_i \langle h_i \rangle)} \mathfrak{p} \\ &\iff f \in \sqrt{\sum_i \langle h_i \rangle} &\iff \exists n, f^n \in \sum_i \langle h_i \rangle, \end{aligned}$$

we can write $f^n = \sum_{i \in I} b_i h_i$ for some finite index set I . Thus reversing this argument, $\mathbb{D}(f) \subseteq \bigcup_{i \in I} \mathbb{D}(h_i)$. We now pass to this finite subcover $\{\mathbb{D}(h_i)\}$. On $\mathbb{D}(h_i) \cap \mathbb{D}(h_j) = \mathbb{D}(h_i h_j)$, note a_i/h_i and a_j/h_j both represent s , so by injectivity shown in the last lecture, $a_i h_j / h_i h_j = a_i / h_i = a_j / h_j = a_j h_i / h_i h_j$ in $A_{h_i h_j}$. Thus for some n , $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A . We can pick an n sufficiently large to work for all pairs i and j . Rewriting, $h_j^{n+1} (h_i^n a_i) - h_i^{n+1} (h_j a_j) = 0$. Replace each h_i by h_i^{n+1} and a_i by $h_i^n a_i$, since $a_i/h_i = a_i h_i^n / h_i^{n+1}$. Thus we can assume that s is still represented on $\mathbb{D}(h_i)$ by a_i/h_i but also for each i and j have $h_i a_j = h_j a_i$. Note $f^n = \sum_i b_i h_i$ for $b_i \in A$, since $\{\mathbb{D}(h_i)\}$ cover $\mathbb{D}(f)$. Let $a = \sum_i b_i a_i$. Then for any j , $h_j a = \sum_i b_i a_i h_j = \sum_i b_i a_j h_i = f^n a_j$. Thus $a/f^n = a_j/h_j$ on $\mathbb{D}(h_j)$. Thus $\psi(a/f^n) = s$, so ψ is surjective. \square

We now have a topological space $\text{Spec } A$ equipped with a sheaf of rings \mathcal{O} .

2.3 Ringed spaces

Definition. A **ringed space** is a pair (X, \mathcal{O}_X) where

- X is a topological space, and
- \mathcal{O}_X is a sheaf of rings on X .

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is the following data.

- $f : X \rightarrow Y$ a continuous map.
- $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ a morphism of sheaves of rings, that is for each $U \subseteq Y$ open, we have a ring homomorphism $f_U^\# : \mathcal{O}_Y(U) \rightarrow (f_* \mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$.

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Example.

- Let X be a topological space, and let \mathcal{O}_X be the sheaf of continuous \mathbb{R} -valued functions. Then if (Y, \mathcal{O}_Y) is similarly defined, given $f : X \rightarrow Y$, we get $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ defined by

$$\begin{aligned} f_U^\# : \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \phi &\longmapsto \phi \circ f \end{aligned} .$$

- Let X be a variety, and let \mathcal{O}_X be the sheaf of regular functions on X . A morphism of varieties $f : X \rightarrow Y$ is a continuous map inducing

$$\begin{aligned} f_U^\# : \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \phi &\longmapsto \phi \circ f \end{aligned} .$$

A ring is **local** if it has a unique maximal ideal.

Definition. A **locally ringed space** (X, \mathcal{O}_X) is a ringed space such that $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$. A **morphism** $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of **locally ringed spaces** is a morphism of ringed spaces such that the induced homomorphism $f_p^\# : \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$ is a **local homomorphism** for all $p \in X$.

- The map is defined by ⁷

$$\begin{aligned} f_p^\# : \mathcal{O}_{Y,f(p)} &\longrightarrow \mathcal{O}_{X,p} \\ (U, s) &\longmapsto (f^{-1}(U), f_U^\#(s)) \end{aligned} .$$

- A ring homomorphism $\phi : (A, \mathfrak{m}_A) \rightarrow (B, \mathfrak{m}_B)$ is **local** if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where \mathfrak{m}_A is the maximal ideal of A . Note that $\phi(A \setminus \mathfrak{m}_A) = \phi(A^*) \subseteq B^* = B \setminus \mathfrak{m}_B$, where A^* is the set of invertible elements of A . Thus $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ always.

Example. In the case of varieties, $\mathcal{O}_{X,p}$ has a unique maximal ideal

$$\{(U, f) \in \mathcal{O}_X(U) \mid f(p) = 0\} / \sim .$$

If $f(p) \neq 0$, f is nowhere vanishing on some neighbourhood of p , so after shrinking U , we can invert f . The local homomorphism condition just follows from the pull-back $\phi \circ f$ of a function ϕ vanishing at $f(p)$ vanishes at p .

2.4 Affine schemes

The key example $(\text{Spec } A, \mathcal{O})$ is a locally ringed space, which we call an affine scheme.

Theorem 2.3. *The category of affine schemes with locally ringed morphisms is equivalent to the opposite category of rings.*

Need to show that

- if $\phi : A \rightarrow B$ is a ring homomorphism, we obtain an induced morphism $(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$, and
- any morphism of affine schemes as locally ringed spaces arises in this way.

Proof.

- Given a ring homomorphism $\phi : A \rightarrow B$, define

$$\begin{aligned} f : \text{Spec } B &\longrightarrow \text{Spec } A \\ \mathfrak{p} &\longmapsto \phi^{-1}(\mathfrak{p}) \end{aligned} .$$

Note $\phi^{-1}(\mathfrak{p})$ is prime, since if $ab \in \phi^{-1}(\mathfrak{p})$, then $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, thus either $\phi(a) \in \mathfrak{p}$ or $\phi(b) \in \mathfrak{p}$, and hence either $a \in \phi^{-1}(\mathfrak{p})$ or $b \in \phi^{-1}(\mathfrak{p})$. Then f is continuous, since

$$\begin{aligned} f^{-1}(\mathbb{V}(I)) &= f^{-1}(\{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \supseteq I\}) = \{\mathfrak{q} \in \text{Spec } B \mid f(\mathfrak{q}) \supseteq I\} \\ &= \{\mathfrak{q} \in \text{Spec } B \mid \phi^{-1}(\mathfrak{q}) \supseteq I\} = \{\mathfrak{q} \in \text{Spec } B \mid \mathfrak{q} \supseteq \phi(I)\} = \mathbb{V}(\phi(I)) . \end{aligned}$$

⁷Exercise: check well-defined

We need to construct $f^\# : \mathcal{O}_{\text{Spec } A} \rightarrow f_* \mathcal{O}_{\text{Spec } B}$. For $\mathfrak{p} \in \text{Spec } B$, we obtain a natural homomorphism

$$\begin{array}{ccc} \phi_{\mathfrak{p}} : A_{\phi^{-1}(\mathfrak{p})} & \longrightarrow & B_{\mathfrak{p}} \\ \frac{a}{s} & \longmapsto & \frac{\phi(a)}{\phi(s)} \end{array}.$$

Note $\phi_{\mathfrak{p}}$ is a local homomorphism, since the maximal ideal $\mathfrak{p}B_{\mathfrak{p}}$ of $B_{\mathfrak{p}}$ is generated by the image of \mathfrak{p} under the map

$$\begin{array}{ccc} B & \longrightarrow & B_{\mathfrak{p}} \\ b & \longmapsto & \frac{b}{1} \end{array},$$

and the maximal ideal $\phi^{-1}(\mathfrak{p})A_{\phi^{-1}(\mathfrak{p})}$ of $A_{\phi^{-1}(\mathfrak{p})}$ is generated by the image of $\phi^{-1}(\mathfrak{p})$ under the map

$$\begin{array}{ccc} A & \longrightarrow & A_{\phi^{-1}(\mathfrak{p})} \\ a & \longmapsto & \frac{a}{1} \end{array},$$

so have a commutative diagram

$$\begin{array}{ccccc} \phi^{-1}(\mathfrak{p}) & \subset & A & \xrightarrow{\phi} & B & \supset & \mathfrak{p} \\ & & \downarrow & & \downarrow & & \\ f(\mathfrak{p})A_{f(\mathfrak{p})} & \subset & A_{\phi^{-1}(\mathfrak{p})} & \xrightarrow{\phi_{\mathfrak{p}}} & B_{\mathfrak{p}} & \supset & \mathfrak{p}B_{\mathfrak{p}} \end{array},$$

thus $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = \phi^{-1}(\mathfrak{p})A_{\phi^{-1}(\mathfrak{p})}$. Given $V \subseteq \text{Spec } A$ open, we may define

$$\begin{array}{ccc} f_V^\# : \mathcal{O}_{\text{Spec } A}(V) & \longrightarrow & \mathcal{O}_{\text{Spec } B}(f^{-1}(V)) \\ (\mathfrak{p} \in V \mapsto s(\mathfrak{p}) \in A_{\mathfrak{p}}) & \longmapsto & (\mathfrak{q} \in f^{-1}(V) \mapsto \phi_{\mathfrak{q}}(s(f(\mathfrak{q}))) \in B_{\mathfrak{q}}) \end{array}.$$

Note that we need to check the local coherence part of the definition of \mathcal{O} . That is, if s is locally given by a/h , then $f_V^\#(s)$ is locally given by $\phi(a)/\phi(h)$. This gives the desired map $f^\# : \mathcal{O}_{\text{Spec } A} \rightarrow f_* \mathcal{O}_{\text{Spec } B}$, and the induced map on stalks $f_{\mathfrak{p}}^\# : \mathcal{O}_{\text{Spec } A, f(\mathfrak{p})} \rightarrow \mathcal{O}_{\text{Spec } B, \mathfrak{p}}$ agrees with $\phi_{\mathfrak{p}} : A_{\phi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$, by construction. Hence $(f, f^\#)$ is a morphism of locally ringed spaces.

2. Now suppose given a morphism $(f, f^\#) : \text{Spec } B \rightarrow \text{Spec } A$ of locally ringed spaces. Take

$$\phi = f_{\text{Spec } A}^\# : \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) = A \rightarrow \Gamma(\text{Spec } B, \mathcal{O}_{\text{Spec } B}) = B.$$

We need to show ϕ gives rise to $(f, f^\#)$. We have $f_{\mathfrak{p}}^\# : \mathcal{O}_{\text{Spec } A, f(\mathfrak{p})} = A_{f(\mathfrak{p})} \rightarrow \mathcal{O}_{\text{Spec } B, \mathfrak{p}} = B_{\mathfrak{p}}$ a local homomorphism. This is compatible with the corresponding map on global sections, that is

$$\begin{array}{ccc} \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) & \xrightarrow{f_{\text{Spec } A}^\#} & \Gamma(\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\text{Spec } A, f(\mathfrak{p})} & \xrightarrow{f_{\mathfrak{p}}^\#} & \mathcal{O}_{\text{Spec } B, \mathfrak{p}} \end{array}$$

is commutative. That is, we have a commutative diagram

$$\begin{array}{ccccc} f(\mathfrak{p}) & \subset & A & \xrightarrow{\phi} & B & \supset & \mathfrak{p} \\ & & \downarrow & & \downarrow & & \\ f(\mathfrak{p})A_{f(\mathfrak{p})} & \subset & A_{f(\mathfrak{p})} & \xrightarrow{f_{\mathfrak{p}}^\#} & B_{\mathfrak{p}} & \supset & \mathfrak{p}B_{\mathfrak{p}} \end{array}.$$

Then $(f_{\mathfrak{p}}^\#)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p})A_{f(\mathfrak{p})}$ since $f_{\mathfrak{p}}^\#$ is a local homomorphism, and by commutativity of the diagram, $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Thus f is induced by ϕ , and $f_{\mathfrak{p}}^\# = \phi_{\mathfrak{p}}$. So $f^\#$ is as constructed previously. \square

Remark. Demanding $(f, f^\#)$ was a morphism of locally ringed spaces was crucial to make the proof work.

Definition. An **affine scheme** is a locally ringed space isomorphic, in the category of locally ringed spaces, to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A . A **scheme** is a locally ringed space (X, \mathcal{O}_X) with an open cover $\{(U_i, \mathcal{O}_X|_{U_i})\}$ with each $(U_i, \mathcal{O}_X|_{U_i})$ an affine scheme, where $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U_i$ open. A **morphism of schemes** is a morphism of locally ringed spaces.

Example. Let k be a field. Then $\operatorname{Spec} k = (\{0\}, k)$.

- What does giving a morphism $f : \operatorname{Spec} k \rightarrow X$ to a scheme mean? First, this selects a point $x \in X$, the image of f . Second, we get a local ring homomorphism $f_x^\# : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\operatorname{Spec} k,0} = k$, that is $(f_x^\#)^{-1}(0) = \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$. Thus we get a factorisation $f_x^\# : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x \rightarrow k$, where $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a field, written as $\kappa(x)$, called the **residue field** of X at x . Thus f induces an inclusion $\kappa(x) \hookrightarrow k$. Conversely, given such an inclusion $\iota : \kappa(x) \hookrightarrow k$ of fields, we get a scheme morphism by defining $f(0) = x$, and

$$\begin{array}{ccc} f^\# : \mathcal{O}_X & \longrightarrow & f_*k \\ s & \longmapsto & \iota(s(x)) \end{array}, \quad s(x) \in \mathcal{O}_{X,x}.$$

The moral is that giving a morphism $f : \operatorname{Spec} k \rightarrow X$ is equivalent to giving a point $x \in X$ and an inclusion $\iota : \kappa(x) \rightarrow k$. Note that if $X = \operatorname{Spec} A$, giving $\operatorname{Spec} k \rightarrow \operatorname{Spec} A$ is equivalent to giving a homomorphism $A \rightarrow k$, which we viewed at the beginning of the course as a k -valued point on $\operatorname{Spec} A$.

- What does giving $X \rightarrow \operatorname{Spec} k$ mean? No information in the continuous map, but need also a map $f^\# : k \rightarrow f_*\mathcal{O}_X$, that is a map $k \rightarrow \Gamma(\operatorname{Spec} k, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. That is, $\Gamma(X, \mathcal{O}_X)$ carries a k -algebra structure. Note this induces k -algebra structures on $\mathcal{O}_X(U)$ for all U via the composition $k \rightarrow \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U)$ and similarly all stalks $\mathcal{O}_{X,p}$ are also k -algebras. We say X is a **scheme defined over k** . For example, in affine varieties, consider $A = k[X_1, \dots, X_n]/I$ with $I = \sqrt{I}$. Then $\operatorname{Spec} A$ is our replacement for $\mathbb{V}(I) \subseteq \mathbb{A}_k^n$, viewing $\operatorname{Spec} A$ as a scheme over k . If $k \subseteq k'$ is a field extension, a **k' -valued point** of X/k is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k' & \xrightarrow{\quad} & X \\ & \searrow & \swarrow \\ & \operatorname{Spec} k & \end{array}.$$

We write $X(k')$ for the set of such morphisms.

Remark. It is rare in algebraic geometry to work with schemes alone, but rather always working over a base scheme.

Fix a base scheme S . Define **Sch**/ S to be the category whose objects are morphisms $T \rightarrow S$ and morphisms are commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\quad} & T' \\ & \searrow & \swarrow \\ & S & \end{array}.$$

We will frequently work with **Sch**/ $k = \mathbf{Sch}/\operatorname{Spec} k$. Given $T \rightarrow S$ and $X \rightarrow S$ objects in **Sch**/ S , a **T -valued point** of $X \rightarrow S$ is a morphism $T \rightarrow X$ over S , so

$$\begin{array}{ccc} T & \xrightarrow{\quad} & X \\ & \searrow & \swarrow \\ & S & \end{array},$$

and we write $X(T)$ for the set of T -valued points. The **Yoneda philosophy** is that $X(T)$ for all T determines X .

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Example. Fix a field k , and let $D = \operatorname{Spec} k[t] / \langle t^2 \rangle = (\{t\}, k[t] / \langle t^2 \rangle)$. Then t does not make sense as k -valued function anymore, as $t^2 = 0$. Let X be any scheme over k . What is $X(D)$? Given $f : D \rightarrow X$ a morphism of schemes over k , we get a point $x \in X$ as the image of f and a local homomorphism

$$\begin{aligned} f_x^\# : \mathcal{O}_{X,x} &\longrightarrow k[t] / \langle t^2 \rangle \\ \mathfrak{m}_x &\longmapsto \langle t \rangle \end{aligned}.$$

Note that \mathfrak{m}_x^2 maps to zero, hence we get a k -linear map $\mathfrak{m}_x / \mathfrak{m}_x^2 \rightarrow \langle t \rangle \cong k$ as a k -vector space. We also have a composed surjective k -algebra homomorphism $\mathcal{O}_{X,x} \rightarrow k[t] / \langle t \rangle \cong k$ with kernel \mathfrak{m}_x , and hence we have $\kappa(x) = \mathcal{O}_{X,x} / \mathfrak{m}_x \cong k$. So we get

- a k -valued point x with residue field k , and
- a k -vector space map $\mathfrak{m}_x / \mathfrak{m}_x^2 \rightarrow k$, that is an element of $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$, the dual vector space.

Then $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$ is called the **Zariski tangent space** to X at x . Think of D as a point plus an arrow.

Example. Glued schemes are a special case of a question on example sheet 1. Suppose given two schemes X_1 and X_2 and open subsets $U_i \subseteq X_i$. Recall U_i is also a locally ringed space $(U_i, \mathcal{O}_{X_i}|_{U_i})$, and in fact U_i is then a scheme. Given an isomorphism $f : U_1 \xrightarrow{\sim} U_2$, can glue X_1 and X_2 along U_1 and U_2 to get a scheme X with an open cover $\{X_1, X_2\}$, so $X = X_1 \sqcup X_2 / \sim$ such that $x_1 \in U_1 \sim x_2 \in U_2$ if $f(x_1) = x_2$, and need to define \mathcal{O}_X . Now take $\mathbb{A}_k^n = \operatorname{Spec} k[X_1, \dots, X_n]$, so $\mathbb{A}_k^1 = \operatorname{Spec} k[X]$. Take $X_1 = X_2 = \mathbb{A}_k^1$.

- Glue $U_1 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(X) \subseteq X_1$ and $U_2 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(X) \subseteq X_2$ via the identity map. This is the affine line with doubled origin.
- Could instead glue U_1 and U_2 via the map given by $X \mapsto X^{-1}$, where $U_1 = \operatorname{Spec} k[X]_X = U_2$ and

$$\begin{aligned} k[X]_X &\longrightarrow k[X]_X \\ X &\longmapsto X^{-1} \end{aligned}$$

induces an isomorphism $U_1 \rightarrow U_2$. When we glue, we get the projective line over k , \mathbb{P}_k^1 .

2.5 Projective schemes

Let S be a graded ring, that is

$$S = \bigoplus_{d \geq 0} S_d,$$

with S_d an abelian group, and product law satisfies $S_d \cdot S_{d'} \subseteq S_{d+d'}$.

Example. $S = k[X_0, \dots, X_n]$, and S_d is the space of polynomials which are homogeneous of degree d , that is spanned by monomials of degree d .

We write

$$S_+ = \bigoplus_{d \geq 1} S_d,$$

which we call the **irrelevant ideal**.

Definition. $I \subseteq S$ is a **homogeneous ideal** if I is generated by its homogeneous elements, that is elements in S_d for various d .

Definition. Let

$$\operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \text{ is homogeneous, } \mathfrak{p} \not\supseteq S_+\}.$$

For $I \subseteq S$ a homogeneous ideal, set

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}.$$

Exercise. Check the $\mathbb{V}(I)$ form the closed sets of a topology on $\operatorname{Proj} S$.

Notation. For $\mathfrak{p} \in \text{Proj } S$, let

$$T = \{f \in S \setminus \mathfrak{p} \mid f \text{ is homogeneous}\}.$$

Then T is a multiplicatively closed subset of S , and let $S_{(\mathfrak{p})} \subseteq T^{-1}S$ be the subring of elements of degree zero, that is written in the form s/s' with $s \in S$ homogeneous and $s' \in T$ with $\deg s = \deg s'$. For $f \in S$ homogeneous, we write $S_{(f)} \subseteq S_f$ for the subset of elements of degree zero.

Can now define a sheaf \mathcal{O} on $\text{Proj } S$. For $U \subseteq \text{Proj } S$ open, set

$$\mathcal{O}(U) = \left\{ s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \left| \begin{array}{l} \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \\ \forall \mathfrak{p} \in U, \exists \mathfrak{p}' \in V \subseteq U \text{ open, } \exists a, f \in S, \forall \mathfrak{q} \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = \frac{a}{f} \in S_{(\mathfrak{q})} \end{array} \right. \right\},$$

where a and f are homogeneous of the same degree. As before, $\mathcal{O}_{\mathfrak{p}} = S_{(\mathfrak{p})}$.⁸ Is the locally ringed space $(\text{Proj } S, \mathcal{O})$ a scheme?

Notation. If $f \in S$ is homogeneous, then we write

$$\mathbb{D}_+(f) = \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\},$$

which is an open set and $\mathbb{D}_+(f) = \text{Proj } S \setminus \mathbb{V}(f)$.

Proposition 2.4. $(\mathbb{D}_+(f), \mathcal{O}|_{\mathbb{D}_+(f)}) \cong \text{Spec } S_{(f)}$ as locally ringed spaces. Further, the open sets $\mathbb{D}_+(f)$ for $f \in S_+$ cover $\text{Proj } S$. Hence $(\text{Proj } S, \mathcal{O})$ is a scheme.

Proof. Will be on example sheet 2. □

Definition. If A is a ring, define

$$\mathbb{P}_A^n = \text{Proj } A[X_0, \dots, X_n].$$

Example. If k is an algebraically closed field, consider $\mathbb{P}_k^1 = \text{Proj } k[X_0, X_1]$. The closed points, that is points \mathfrak{p} such that $\{\mathfrak{p}\}$ is closed, correspond to maximal elements of $\text{Proj } S$.⁹ These maximal elements are ideals of the form $\langle aX_0 - bX_1 \rangle$. The only maximal homogeneous ideal of $k[X_0, X_1]$ is $\langle X_0, X_1 \rangle = S_+$, since any maximal ideal is of the form $\langle X_0 - a_0, X_1 - a_1 \rangle$. The other prime ideals of $k[X_0, X_1]$ are principal, that is of the form $\langle f \rangle$ with f irreducible or $f = 0$. For $\langle f \rangle$ to be homogeneous, f must be homogeneous. Any such polynomial splits into linear factors, all homogeneous, so in order for f to be irreducible it must be linear. Note we have a one-to-one correspondence between

$$\begin{aligned} \{\langle aX_0 - bX_1 \rangle \mid a, b \in k, a, b \text{ not both zero}\} &\longrightarrow (k^2 \setminus \{(0, 0)\})/k^* \\ \langle aX_0 - bX_1 \rangle &\longmapsto (b : a) \end{aligned},$$

where k^* acts by $(a, b) \mapsto (\lambda a, \lambda b)$ for $\lambda \in k^*$. The conclusion is that the closed points of \mathbb{P}_k^1 are in one-to-one correspondence with points of $(k^2 \setminus \{(0, 0)\})/k^*$. More generally, the closed points of \mathbb{P}_k^n are in one-to-one correspondence with points of $(k^{n+1} \setminus \{0\})/k^*$. Can see this by making use of the open cover $\{\mathbb{D}_+(X_i) \mid 0 \leq i \leq n\}$,¹⁰ which is an open cover since $\mathfrak{p} \notin \mathbb{D}_+(X_i)$ for any i implies that $X_i \in \mathfrak{p}$ for all i , so $S_+ \subseteq \mathfrak{p}$ and so $\mathfrak{p} \notin \text{Proj } S$.

Example. Let $S = k[X_0, \dots, X_n]$, but grade by $\deg X_i = w_i$, where w_0, \dots, w_n are positive integers. Define $\mathbb{WP}^n(w_0, \dots, w_n) = \text{Proj } S$, the **weighted projective space**. For example, $\mathbb{WP}^2(1, 1, 2)$ has an open cover $\{\mathbb{D}_+(X_i) \mid 0 \leq i \leq 2\}$. Consider $\mathbb{D}_+(X_2) = \text{Spec } S_{(X_2)}$. Note

$$S_{(X_2)} = k\left[\frac{X_0^2}{X_2}, \frac{X_0X_1}{X_2}, \frac{X_1^2}{X_2}\right] \cong k[U, V, W] / \langle UW - V^2 \rangle \subseteq S_{X_2},$$

so $\text{Spec } S_{(X_2)}$ is a quadric cone with a singular point. Similarly, $\mathbb{D}_+(X_0)$ and $\mathbb{D}_+(X_1)$ are both isomorphic to \mathbb{A}_k^2 .

⁸Exercise: check

⁹Exercise: check

¹⁰Exercise: good exercise

Example. Let $M = \mathbb{Z}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$. Let $\Delta \subseteq M_{\mathbb{R}}$ be a compact convex lattice polytope. That is, there exists a finite set $V \subseteq M$ such that Δ is the convex hull of V , that is the smallest convex set containing V . Let

$$C(\Delta) = \{(m, r) \in M_{\mathbb{R}} \oplus \mathbb{R} \mid m \in r\Delta, r \geq 0\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

Here $r\Delta = \{rm \mid m \in \Delta\}$. This is the **cone over Δ** . Let

$$S = k[C(\Delta) \cap (M \oplus \mathbb{Z})] = \bigoplus_{P \in C(\Delta) \cap (M \oplus \mathbb{Z})} kZ^P,$$

with multiplication given by $Z^P Z^{P'} = Z^{P+P'}$, since $C(\Delta) \cap (M \oplus \mathbb{Z})$ is a monoid, that is it is closed under addition and contains zero. This makes S into a ring, and it is graded by $\deg Z^{(m,r)} = r$. Define $\mathbb{P}_{\Delta} = \text{Proj } S$. This is called a **projective toric variety**.

- Let Δ be the convex hull of $\{0, e_1, \dots, e_n\}$ with e_1, \dots, e_n the standard basis of $M = \mathbb{Z}^n$. Check that $S = k[X_0, \dots, X_n]$ with standard grading $X_0 = Z^{(0,1)}$ and $X_i = Z^{(e_i,1)}$.¹¹ So $\mathbb{P}_{\Delta} = \mathbb{P}_k^n$.
- Let $n = 2$, and let Δ be the convex hull of $\{(0,0), (1,0), (0,1), (1,1)\}$. In S , the degree d monomials are $\{Z^{(a,b,d)} \mid 0 \leq a \leq d, 0 \leq b \leq d\}$. Any of these can be written as a product of monomials of degree one, that is the monomials $X = Z^{(0,0,1)}$, $Y = Z^{(1,0,1)}$, $W = Z^{(0,1,1)}$, and $T = Z^{(1,1,1)}$. Thus $S = k[X, Y, W, T] / \langle XT - YW \rangle$. So $\text{Proj } S$ can be thought of as a quadric surface in \mathbb{P}_k^3 .

2.6 Open and closed subschemes

Definition. An **open subscheme** of a scheme X is a scheme $(U, \mathcal{O}_X|_U)$ for $U \subseteq X$ an open subset. Note that this is a scheme because from question 1 and question 11 on the first example sheet, open affine subsets of X form a basis for the topology on X . An **open immersion** is a morphism $f : X \rightarrow Y$ which induces an isomorphism of X with an open subscheme of Y . A **closed immersion** $f : X \rightarrow Y$ is a morphism which is a homeomorphism onto a closed subset of Y , and the induced morphism $f^{\#} : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is surjective. A **closed subscheme** of Y is an equivalence class of closed immersions, where

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & X' \\ & \searrow & \swarrow \\ & Y & \end{array}$$

are equivalent if there exists an isomorphism i making the diagram commute.

Example.

- Let $Y = \text{Spec } A$, let $I \subseteq A$ be an ideal, and let $X = \text{Spec } A/I$. Note the map of schemes induced by the quotient map $A \rightarrow A/I$ identifies $\text{Spec } A/I$ with $\mathbb{V}(I) \subseteq \text{Spec } A$. Thus $f : X \rightarrow Y$, induced by $A \rightarrow A/I$, satisfies the first condition of being a closed immersion. Note that $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is surjective on stalks. For $\mathfrak{p} \in \mathbb{V}(I)$, $\mathcal{O}_{Y,\mathfrak{p}} = A_{\mathfrak{p}}$ and $(f_*\mathcal{O}_X)_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$ since all open sets in X are of the form $U \cap X$ for U an open set of Y and $\mathcal{O}_{X,\mathfrak{p}} = (A/I)_{\mathfrak{p}/I}$. Certainly $A_{\mathfrak{p}} \rightarrow (A/I)_{\mathfrak{p}/I}$ is surjective.
- Let $\text{Spec } k[X, Y] / \langle X \rangle \rightarrow \text{Spec } k[X, Y] = \mathbb{A}^2$. This gives a closed subscheme structure to the set $\mathbb{V}(X)$. Note $\mathbb{V}(X^2, XY) = \mathbb{V}(X)$. This gives a closed immersion $\text{Spec } k[X, Y] / \langle X^2, XY \rangle \rightarrow \mathbb{A}^2$. This gives a different closed subscheme structure on $\mathbb{V}(X)$. Note these two subschemes are isomorphic away from the origin, which we can see by looking at $\mathbb{D}(Y) \subseteq \text{Spec } k[X, Y] / \langle X \rangle$, where

$$\mathbb{D}(Y) \cong \text{Spec } (k[X, Y] / \langle X \rangle)_Y = \text{Spec } k[Y]_Y.$$

Looking at $\mathbb{D}(Y) \subseteq \text{Spec } k[X, Y] / \langle X^2, XY \rangle$,

$$\mathbb{D}(Y) \cong \text{Spec } (k[X, Y] / \langle X^2, XY \rangle)_Y \cong \text{Spec } k[X, Y]_Y / \langle X \rangle \cong \text{Spec } k[Y]_Y.$$

¹¹Exercise

2.7 Fibre products

Let \mathcal{C} be a category and

$$\begin{array}{ccc} & Y & \\ & \downarrow g & \\ X & \xrightarrow{f} & Z \end{array}$$

be a diagram in \mathcal{C} . Then the **fibre product**, if it exists, is an object W equipped with morphisms $p : W \rightarrow X$ and $q : W \rightarrow Y$ such that $f \circ p = g \circ q$ satisfying the following universal property. For any W' equipped with maps $p' : W' \rightarrow X$ and $q' : W' \rightarrow Y$ such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : W' \rightarrow W$ making the diagram

$$\begin{array}{ccccc} W' & & \xrightarrow{q'} & & Y \\ & \searrow \exists! h & & \searrow q & \\ & & W & \xrightarrow{q} & Y \\ & \searrow p' & \downarrow p & & \downarrow g \\ & & X & \xrightarrow{f} & Z \end{array}$$

commute, that is $p \circ h = p'$ and $q \circ h = q'$. Note that if the fibre product exists, it is unique up to unique isomorphism.

Example. Let \mathcal{C} be the category of sets. Then

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

It will be helpful to think about the fibre product, and more generally other universal properties, via the Yoneda lemma.

Definition. Let \mathcal{C} be a category. Write h_X for the contravariant functor

$$\begin{array}{rcl} h_X : & \mathcal{C} & \longrightarrow \mathbf{Set} \\ & Y & \longmapsto \mathrm{Hom}(Y, X) \\ f : Y \rightarrow Z & \longmapsto & (\phi \in \mathrm{Hom}(Z, X) \mapsto \phi \circ f \in \mathrm{Hom}(Y, X)) \end{array}.$$

Recall that a **natural transformation** between contravariant functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, written as $T : \mathcal{C} \rightarrow \mathcal{D}$, consists of the data $T(X) : F(X) \rightarrow G(X)$ for all $X \in \mathrm{Ob} \mathcal{C}$ such that for all $f : X \rightarrow Y$ in \mathcal{C}

$$\begin{array}{ccc} F(X) & \xleftarrow{F(f)} & F(Y) \\ T(X) \downarrow & & \downarrow T(Y) \\ G(X) & \xleftarrow{G(f)} & G(Y) \end{array}$$

is commutative.

Lemma 2.5 (Yoneda's lemma). *The set of natural transformations between $h_X : \mathcal{C} \rightarrow \mathbf{Set}$ and $G : \mathcal{C} \rightarrow \mathbf{Set}$ is $G(X)$.*

Proof. Given $\eta \in G(X)$, we need to define a map

$$\begin{array}{ccc} h_X(Y) = \mathrm{Hom}(Y, X) & \longrightarrow & G(Y) \\ f & \longmapsto & G(f)(\eta) \end{array},$$

for all objects $Y \in \mathcal{C}$. Check that this defines a natural transformation $h_X \rightarrow G$.¹² Conversely, given $T : h_X \rightarrow G$ a natural transformation, take $\eta = T(X)(\mathrm{id}_X)$. Check that these two maps are inverse to each other.¹³ \square

Corollary 2.6. *The set of natural transformations $h_X \rightarrow h_Y$ is $h_Y(X) = \mathrm{Hom}(X, Y)$.*

¹²Exercise

¹³Exercise

Definition. A contravariant functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is said to be **representable** if $F \cong h_X$ for some $X \in \text{Ob } \mathcal{C}$.

Lots of questions in algebraic geometry are about representability of functors. Redefining, the fibre product in a category \mathcal{C} is an object which represents the functor

$$T \mapsto \text{Hom}(T, X) \times_{\text{Hom}(T, Z)} \text{Hom}(T, Y),$$

since an element of the set $\text{Hom}(T, X) \times_{\text{Hom}(T, Z)} \text{Hom}(T, Y)$ is a commutative diagram

$$\begin{array}{ccccc} T & & & & \\ & \searrow q & & & \\ & & W & \dashrightarrow & Y \\ & \searrow p & \downarrow & & \downarrow g \\ & & X & \xrightarrow{f} & Z \end{array}$$

The advantage of using Yoneda is that we can check identities using fibre products using identities of fibre products of sets.

Example. In \mathbf{Set} ,

$$\begin{aligned} (A \times_B C) \times_C D &\longleftrightarrow A \times_B D \\ ((a, c), d) &\longmapsto (a, d) \\ ((a, f(d)), d) &\longleftarrow (a, d) \end{aligned}, \quad f : D \rightarrow C.$$

Then we have two functors

$$\begin{array}{ccc} T & \longrightarrow & (h_A(T) \times_{h_B(T)} h_C(T)) \times_{h_C(T)} h_D(T) \\ & \searrow & \downarrow \sim \\ & & h_A(T) \times_{h_B(T)} h_D(T) \end{array},$$

and natural transformations showing those functors are isomorphic, and hence represent isomorphic objects.

Theorem 2.7. *Fibre products exist in the category of schemes.*

Proof. Will construct $X \times_S Y$ for various cases, bootstrapping up to the general case.

Step 1. Let $X = \text{Spec } A$, let $Y = \text{Spec } B$, and let $S = \text{Spec } R$, so

$$\begin{array}{ccc} & Y & \\ & \downarrow & \\ X & \longrightarrow & S \end{array} \iff \begin{array}{ccc} & B & \\ & \uparrow & \\ A & \longleftarrow & R \end{array}.$$

Push-outs exist in the category of rings, so

$$\begin{array}{ccccc} C & & & & \\ & \nwarrow p'_2 & & & \\ & & A \otimes_R B & \xleftarrow{p_2} & B \\ & \nwarrow \exists! h & \uparrow p_1 & & \uparrow g \\ & & A & \xleftarrow{f} & R \end{array}$$

where $p_1(a) = a \otimes 1$ and $p_2(b) = 1 \otimes b$. Here h is defined by $h(a \otimes b) = p'_1(a)p'_2(b)$. Check well-defined. ¹⁴ Thus $\text{Spec } A \otimes_R B$ is $\text{Spec } A \times_{\text{Spec } R} \text{Spec } B$ in the category of affine schemes.

¹⁴Exercise

If T is an arbitrary scheme, then giving a morphism $T \rightarrow \operatorname{Spec} A$ is the same as giving a morphism $A \rightarrow \Gamma(T, \mathcal{O}_T)$, by question 12, example sheet 1. Thus giving a commutative diagram

$$\begin{array}{ccc} T & & \operatorname{Spec} B \\ & \searrow & \downarrow \\ & \operatorname{Spec} A & \longrightarrow \operatorname{Spec} R \end{array}$$

is equivalent to

$$\begin{array}{ccc} \Gamma(T, \mathcal{O}_T) & \xleftarrow{\exists! h} & A \otimes_R B \longleftarrow B \\ & \nwarrow & \uparrow g \\ & A & \xleftarrow{f} R \end{array}$$

and $h : A \otimes_R B \rightarrow \Gamma(T, \mathcal{O}_T)$ induces a map $T \rightarrow \operatorname{Spec} A \otimes_R B$. Thus $\operatorname{Spec} A \otimes_R B$ is the fibre product $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ in the category of schemes.

Step 2. Will construct more general fibre products by gluing of schemes using question 14 on example sheet 1. We also glue morphisms, so if X and Y are schemes, $\{U_i\}$ an open cover of X , and we are given morphisms $f_i : U_i \rightarrow Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then we obtain $f : X \rightarrow Y$ such that $f|_{U_i} = f_i$. The argument is given in the examples class.

Step 3. If $X, Y \rightarrow S$ are given and $U \subseteq X$ is open, suppose that $X \times_S Y$ exists, with projections $p_1 : X \times_S Y \rightarrow X$ and $p_2 : X \times_S Y \rightarrow Y$. Then $p_1^{-1}(U)$ is $U \times_S Y$. By commutativity of the diagram

$$\begin{array}{ccccc} T & & & & \\ & \searrow \exists! h' & & & \\ & p_1^{-1}(U) \hookrightarrow X \times_S Y \longrightarrow Y & & & \\ & \downarrow & \downarrow & \downarrow & \\ & U \hookrightarrow X \longrightarrow S & & & \end{array}$$

the image of h' must be contained in $p_1^{-1}(U)$. Thus h' factors through $p_1^{-1}(U) \hookrightarrow X \times_S Y$ giving the unique map h , so the universal property holds for $p_1^{-1}(U)$.

Step 4. Suppose $\{X_i\}$ is an open cover of X and $X_i \times_S Y$ exists for each i . Then $X \times_S Y$ exists. Let $X_{ij} = X_i \cap X_j$, and let $U_{ij} = p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y$. By step 3, $U_{ij} = X_{ij} \times_S Y$. By the universal property of fibre products there exists a unique isomorphism $\phi_{ij} : U_{ij} \rightarrow U_{ji}$. Check these gluing maps ϕ_{ij} satisfy the requirements of question 14 on example sheet 1.¹⁵ Thus we can glue the $X_i \times_S Y$ via ϕ_{ij} 's to get a scheme $X \times_S Y$, but need to check it satisfies the fibre product axioms. So suppose given

$$\begin{array}{ccc} T & \xrightarrow{p'_2} & Y \\ & \searrow p'_1 & \downarrow \\ & X & \longrightarrow S \end{array}$$

¹⁵Exercise: check

Let $T_i = (p'_1)^{-1}(X_i)$, so get a morphism $\theta_i : T_i \rightarrow X_i \times_S Y \hookrightarrow X \times_S Y$, where $X_i \times_S Y \hookrightarrow X \times_S Y$ is an open immersion by construction. On $T_i \cap T_j$ these maps agree since they factor through $X_{ij} \times_S Y \subseteq X_i \times_S Y$ and $X_{ji} \times_S Y \subseteq X_j \times_S Y$ and by the universal property they agree. Thus using step 2, we can glue the θ_i 's to get $\theta : T \rightarrow X \times_S Y$.

Step 5. Using step 4 and 1 we may construct $X \times_S Y$ when S and Y are affine. Repeating for Y , we obtain $X \times_S Y$ when S is affine, and X and Y are arbitrary.

Step 6. Let X, Y, S be arbitrary, take an open affine cover $\{S_i\}$ of S , let $f : X \rightarrow S$ and $g : Y \rightarrow S$, and let $X_i = f^{-1}(S_i)$ and $Y_i = g^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists and $X_i \times_{S_i} Y_i = X_i \times_S Y_i$.¹⁶ Use the same gluing argument as before, to get $X \times_S Y$.

□

2.8 Fibres of morphisms

The philosophy in **Set** is

$$\begin{array}{ccc} f^{-1}(y) = \{y\} \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \{y\} & \longrightarrow & Y \end{array}$$

Given $f : X \rightarrow Y$ a morphism and $y \in Y$, let $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of y , so we get a morphism $\text{Spec } \kappa(y) \rightarrow Y$ with image y . Then we define

$$X_y = \text{Spec } \kappa(y) \times_Y X$$

to be the **scheme-theoretic fibre** of f at y .

Example. Let $f : X = \text{Spec } k[X] \rightarrow Y = \text{Spec } k[t]$ be induced by

$$\begin{array}{ccc} k[t] & \longrightarrow & k[X] \\ t & \longmapsto & X^2 \end{array}.$$

For $y = \langle t - a \rangle \subseteq k[t]$ and $a \in k$, $\kappa(y) = k[t]/\langle t - a \rangle \cong k$. If B is an A -algebra then $A/I \otimes_A B = B/IB$, so

$$X_y = \text{Spec } \kappa(y) \otimes_{k[t]} k[X] = \text{Spec } k[X] / \langle X^2 - a \rangle.$$

If $a \neq 0$ and $\text{ch } k \neq 2$, we obtain either X_y consists of two distinct points, if $\sqrt{a} \in k$, or a single point if $\sqrt{a} \notin k$. If $a = 0$, we get $\text{Spec } k[X] / \langle X^2 \rangle$.

Remark.

- In general, it is hard to calculate fibre products, since $X \times_S Y$ is not the set-theoretic fibre product in general. For example, $\mathbb{A}_k^1 \times_{\text{Spec } k} \mathbb{A}_k^1 = \text{Spec } k[X] \otimes_k k[Y] = \text{Spec } k[X, Y] = \mathbb{A}_k^2$.
- If we are interested only in varieties, such as schemes over a field k , the usual product of varieties $X \times Y$ corresponds to $X \times_{\text{Spec } k} Y$. More generally, if we are working in the category **Sch**/ S , the natural product is $X \times_S Y$.
- Given schemes S and T with a morphism $T \rightarrow S$, we get a functor

$$\begin{array}{ccc} \mathbf{Sch}/S & \longrightarrow & \mathbf{Sch}/T \\ (X \rightarrow S) & \longmapsto & (X \times_S T \rightarrow T) \end{array}.$$

This functor is called **base-change**.

¹⁶Exercise: check, immediate from universal property

Example. Consider a scheme X over $\operatorname{Spec} \mathbb{Z}$, such as $X = \operatorname{Proj} \mathbb{Z}[X, Y, Z] / I \rightarrow \operatorname{Spec} \mathbb{Z}$ where $I = \langle X^n + Y^n - Z^n \rangle$. May consider base-changes

- $\operatorname{Spec} \mathbb{F}_p \rightarrow \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p = \operatorname{Proj} \mathbb{F}_p[X, Y, Z] / I$,
- $\operatorname{Spec} \mathbb{Q} \rightarrow \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \rightarrow \mathbb{Q}$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q} = \operatorname{Proj} \mathbb{Q}[X, Y, Z] / I$, or
- $\operatorname{Spec} \mathbb{C} \rightarrow \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \rightarrow \mathbb{C}$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C} = \operatorname{Proj} \mathbb{C}[X, Y, Z] / I \subseteq \mathbb{P}_{\mathbb{C}}^2$.

2.9 Properties of schemes and morphisms of schemes

See example sheet 2 for more details or your favourite algebraic geometry text.

Definition. A scheme X is **integral** if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

Definition. A scheme X is **reduced** if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

Definition. A scheme X is **irreducible** if the underlying topological space X is irreducible, that is if $X = X_1 \cup X_2$ with $X_1, X_2 \subseteq X$ closed, then either $X_1 = X$ or $X_2 = X$.

Example. Let $X = \operatorname{Spec} k[X, Y] / \langle XY \rangle$.

- X is not integral because $\Gamma(X, \mathcal{O}_X) = k[X, Y] / \langle XY \rangle$ is not an integral domain, since $XY = 0$.
- X is reduced.
- X is not irreducible, since $X = \mathbb{V}(X) \cup \mathbb{V}(Y)$.

Theorem 2.8. X is integral if and only if X is reduced and irreducible.

Definition. Let X be a scheme. It is **locally Noetherian** if there exists a cover $\{U_i\}$ of X with $U_i = \operatorname{Spec} A_i$ affine and A_i Noetherian. Then X is **Noetherian** if the cover may be taken to be finite.

Example. $\operatorname{Spec} k[X_1, X_2, \dots]$ with a countable number of variables is not locally Noetherian.

Not obvious, but can show that X is locally Noetherian if and only if, if $U \subseteq X$ is affine and $U = \operatorname{Spec} A$, then A is Noetherian.

Definition. A morphism $f : X \rightarrow Y$ of schemes is **locally of finite type** if there is a covering of Y by affine open sets $\{V_i = \operatorname{Spec} B_i\}$ such that for each i , $f^{-1}(V_i)$ can be covered by affine open sets $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, where each A_{ij} is a finitely generated B_i -algebra. We say f is of **finite type** if for each i , the cover $\{U_{ij}\}$ may be taken to be finite.

Definition. Let k be an algebraically closed field. A **variety over k** is a scheme X over $\operatorname{Spec} k$ which is integral and $X \rightarrow \operatorname{Spec} k$ is of finite type, that is X can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$ with A_i a finitely generated k -algebra.

The A_i must be integral domains, so $A_i = k[X_1, \dots, X_n] / I$ where I is a prime ideal. Note that this still allows a non-Hausdorff scheme $\mathbb{A}^1 \cup \mathbb{A}^1$ obtained by gluing $\mathbb{D}(X) \subseteq \mathbb{A}^1$ to $\mathbb{D}(X) \subseteq \mathbb{A}^1$.

Example. Let $X_i = \operatorname{Spec} k[X_i, Y_i] / \langle X_i Y_i \rangle$ for $i \in \mathbb{Z}$. Glue X_i to X_{i+1} along open subsets $U_{i,i+1} \subseteq X_i$ given by $\mathbb{D}(X_i)$ and $U_{i+1,i} \subseteq X_{i+1}$ given by $\mathbb{D}(Y_{i+1})$ via the map

$$\begin{aligned} U_{i+1,i} = \operatorname{Spec} k[Y_{i+1}]_{Y_{i+1}} &\longrightarrow U_{i,i+1} = \operatorname{Spec} k[X_i]_{X_i} \\ Y_{i+1} &\longmapsto \frac{1}{X_i} \end{aligned} .$$

Doing this for all i , we get an infinite chain of \mathbb{P}^1 's. Note $\{X_i\}$ forms an open cover of X but has no finite subcover. Not quasi-compact, only locally of finite type over $\operatorname{Spec} k$.

2.10 Separated and proper morphisms

Remark. A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed.

Example. Let X be \mathbb{R} with doubled origin in the usual Euclidean topology. Then $X \times X$ is \mathbb{R}^2 with doubled axes and four origins. Then Δ only contains two origins but other origins are in the closure of Δ .

Definition. Let $f : X \rightarrow Y$ be a morphism of schemes, and $\Delta : X \rightarrow X \times_Y X$ be the morphism induced by the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \text{id}_X \searrow & & \downarrow f \\ & X & \xrightarrow{f} Y \end{array}$$

We say f is **separated** if Δ is a closed immersion.

Theorem 2.9 (Valuative criterion for separatedness). *Let $f : X \rightarrow Y$ be a morphism and X Noetherian. Then f is separated if and only if the following condition holds. For any field k and any valuation ring $R \subseteq k$, that is for any $x \in k$ such that $x \neq 0$ either $x \in R$ or $x^{-1} \in R$, let $T = \text{Spec } R$ and $U = \text{Spec } k$, and $\iota : U \rightarrow T$ be the morphism induced by the inclusion $R \hookrightarrow k$. Given a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad} & X \\ \downarrow \iota & \swarrow \iota' & \downarrow f \\ T & \xrightarrow{\quad} & Y \end{array}$$

then there exists at most one morphism $\iota' : T \rightarrow X$ making the diagram commute.

The intuition is if R is a valuation ring, it has a zero prime ideal and a unique maximal ideal, and $\overline{\{0\}} = \mathbb{V}(0) = \text{Spec } R = T$.

Remark. We may now define a variety over a field k as a scheme X which is integral and finite type, separated over $\text{Spec } k$.

Definition. A morphism $f : X \rightarrow Y$ is **proper** if it is separated, of finite type, and universally closed. That is, for any morphism $Y' \rightarrow Y$ the induced projection $X \times_Y Y' \rightarrow Y'$ is a closed map, that is the image of a closed set is closed.

Example.

- $\mathbb{P}_k^n = \text{Proj } k[X_0, \dots, X_n] \rightarrow \text{Spec } k$ is proper.
- $\mathbb{A}_k^1 \rightarrow \text{Spec } k$ is not proper. Consider the base-change by $\mathbb{A}_k^1 \rightarrow \text{Spec } k$. Let

$$\begin{array}{ccc} p_2 : \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 = \mathbb{A}_k^2 = \text{Spec } k[X] \otimes_k k[Y] = \text{Spec } k[X, Y] & \longrightarrow & \mathbb{A}_k^1 = \text{Spec } k[t] \\ (x, y) & \longmapsto & y \end{array}$$

This is not a closed map. For example, $p_2(\mathbb{V}(XY - 1)) = \mathbb{D}(t)$, which is open and not closed.

Theorem 2.10 (Valuative criterion for properness). *Let $f : X \rightarrow Y$ be a finite type morphism with X Noetherian. Then f is proper if as in the criterion for separatedness, whenever given a diagram*

$$\begin{array}{ccc} \text{Spec } k = U & \xrightarrow{\quad} & X \\ \downarrow & \swarrow \exists! g & \downarrow f \\ \text{Spec } R = T & \xrightarrow{\quad} & Y \end{array}$$

there exists a unique morphism $g : T \rightarrow X$ making the diagram commute.

Example. Projective varieties, that is closed subvarieties in \mathbb{P}_k^n , are proper over $\text{Spec } k$.

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3 Sheaves of \mathcal{O}_X -modules

The idea is to go from the notion of an A -module M to the notion of an \mathcal{O}_X -module \mathcal{F} .

Definition. Let (X, \mathcal{O}_X) be a ringed space. A **sheaf of \mathcal{O}_X -modules** is a sheaf of abelian groups \mathcal{F} on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$ -module, compatible with restriction, that is if $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, then $s|_V \cdot m|_V = (s \cdot m)|_V$ for $V \subseteq U$. A **morphism of sheaves of \mathcal{O}_X -modules** $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of abelian groups such that for all $U \subseteq X$, $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules.

- Kernels, cokernels, and images of morphisms of sheaves of \mathcal{O}_X -modules are sheaves of \mathcal{O}_X -modules.
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the group of \mathcal{O}_X -module homomorphisms $\{\phi : \mathcal{F} \rightarrow \mathcal{G}\}$. This is an $\mathcal{O}_X(X)$ -module. Then $U \mapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, which is an \mathcal{O}_X -module, is a sheaf of \mathcal{O}_X -modules, written $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the **sheaf hom**.
- If \mathcal{F} and \mathcal{G} are \mathcal{O}_X -modules, we denote by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.
- Push-forwards and pull-backs. For modules, let $\phi : A \rightarrow B$ be a homomorphism of rings, let M be a B -module, and let N be an A -module. Then M is also an A -module such that

$$a \cdot m = \phi(a) \cdot m, \quad a \in A, \quad m \in M,$$

and $B \otimes_A N$ is a B -module via

$$b \cdot (b' \otimes n) = bb' \otimes n, \quad b \in B, \quad b' \otimes n \in B \otimes_A N.$$

Given $f : X \rightarrow Y$ a morphism of ringed spaces, so $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$, if \mathcal{F} is a sheaf of \mathcal{O}_X -modules and \mathcal{G} is a sheaf of \mathcal{O}_Y -modules, then the following holds.

- $f_*\mathcal{F}$ is naturally a sheaf of $f_*\mathcal{O}_X$ -modules, since $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ which is a $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$ -module, and hence $f_*\mathcal{F}$ is an \mathcal{O}_Y -module via $f^\#$.
- $f^{-1}\mathcal{G}$ is naturally a sheaf of $f^{-1}\mathcal{O}_Y$ -modules. But $f^\#$ induces the adjoint map $f^\# : f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$, by question 10 on example sheet 1. Define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This is a sheaf of \mathcal{O}_X -modules.

Definition. A sheaf of \mathcal{O}_X -modules is **free** if it is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ for some index set I . If $\#I = r < \infty$, then we say \mathcal{F} has **rank** r . A sheaf \mathcal{F} is **locally free** of rank r if there exists an open cover $\{U_i\}$ on X such that $\mathcal{F}|_{U_i}$ is free of rank r for each i . Then \mathcal{F} is a **line bundle** if it is rank one. Often more generally, one might refer to a rank r locally free sheaf as a rank r **vector bundle**.

If $S \subseteq A$ is a multiplicatively closed subset, then

$$S^{-1}M = \left\{ \frac{m}{a} \mid a \in S, m \in M \right\} / \sim,$$

where $m/a \sim m'/a'$ if and only if there exists $b \in S$ such that $b(ma' - m'a) = 0$. Also, $S^{-1}M = M \otimes_A S^{-1}A$.

Example. Let $X = \text{Spec } A$ be an affine scheme, and let M be an A -module. For $\mathfrak{p} \in \text{Spec } A$, we have the localisation $M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$. Define a sheaf \widetilde{M} on $\text{Spec } A$ by

$$\widetilde{M}(U) = \left\{ s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \mid \begin{array}{l} \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ \forall \mathfrak{p} \in U, \exists \mathfrak{p} \in V \subseteq U \text{ open, } \exists m \in M, \exists s \in A, \forall \mathfrak{q} \in V, s \notin \mathfrak{q}, s(\mathfrak{q}) = \frac{m}{s} \end{array} \right\}.$$

Example. $\widetilde{A} = \mathcal{O}_{\text{Spec } A}$.

Proposition 3.1.

- $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$.
- $\widetilde{M}(\mathbb{D}(f)) = M_f$.
- $\Gamma(\text{Spec } A, \widetilde{M}) = M$.

Proof. Exactly as the corresponding statements for $\mathcal{O}_{\text{Spec } A}$. □