Local Fields

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Michaelmas 2020

Syllabus

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1 Basic theory

Lecture 1 Friday 09/10/20

How can we find solutions to Diophantine equations? Let $f(x_1, \dots, x_r) \in \mathbb{Z}[x_1, \dots, x_r]$ be a polynomial with integer coefficients. What are integer or rational solutions to $f(x_1, \dots, x_r) = 0$? Finding solutions to Diophantine equations in general is a very difficult problem. Consider a related but much simpler problem of solving the congruences

$$f(x_1, \dots, x_r) \equiv 0 \mod p, \qquad \dots, \qquad f(x_1, \dots, x_r) \equiv 0 \mod p^n, \qquad \dots.$$

Now this is just a finite computation, since modulo primes there are only finitely many choices for solutions, so this is a much easier problem. Local fields give a way to package all this information together.

1.1 Absolute values

Definition 1.1.1. Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that,

1. $|x| = 0$ if and only if $x = 0$,
2. $|xy| = |x||y|$ for all $x, y \in K$, and
3. the triangle inequality $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

We say $(K, |\cdot|)$ is a **valued field**.

Example.

- Let $K = \mathbb{R}, \mathbb{C}$ with the usual absolute value. Write $|\cdot|_\infty$ for this absolute value.
- Let K be any field. The **trivial absolute value** on K is defined by

$$|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}.$$

  Ignore this case in this course.

- Let $K = \mathbb{Q}$ and p a prime. For $0 \neq x \in \mathbb{Q}$, write $x = p^n (a/b)$, where $a, b \in \mathbb{Z}$ such that $(a, p) = 1$ and $(b, p) = 1$. The **p-adic absolute value** is defined to be

$$|x|_p = \begin{cases} 0 & x = 0 \\ p^{-n} & x = p^n \dfrac{a}{b} \end{cases}.$$

  Axiom 1 is clear. Write $y = p^m (c/d)$. Axiom 2 is

$$|xy|_p = \left| p^{m+n} \frac{ac}{bd} \right|_p = p^{-m-n} = |x|_p |y|_p.$$

  Without loss of generality $m \geq n$. Axiom 3 is

$$|x + y|_p = \left| p^n \frac{ad + p^{m-n} bc}{bd} \right|_p = |p^n|_p \left| \frac{ad + p^{m-n} bc}{bd} \right|_p \leq p^{-n} = \max\left(|x|_p, |y|_p \right).$$

An absolute value on K induces a metric $\mathrm{d}(x, y) = |x - y|$ on K, hence induces a topology on K.

Exercise. $+$ and $\cdot$ are continuous.

Definition 1.1.2. Let $|\cdot|$ and $|\cdot|'$ be absolute values on a field K. We say $|\cdot|$ and $|\cdot|'$ are **equivalent** if they induce the same topology. An equivalence class of absolute values is called a **place**.

Proposition 1.1.3. *Let $|\cdot|$ and $|\cdot|'$ be non-trivial absolute values on K. The following are equivalent.*

1. *$|\cdot|$ and $|\cdot|'$ are equivalent.*
2. *$|x| < 1$ if and only if $|x|' < 1$ for all $x \in K$.*
3. *There exists $c \in \mathbb{R}_{>0}$ such that $|x|^c = |x|'$ for all $x \in K$.*

Proof.

$1 \implies 2$. $|x| < 1$ if and only if $x^n \to 0$ with respect to $|\cdot|$, if and only if $x^n \to 0$ with respect to $|\cdot|'$, if and only if $|x|' < 1$.

$2 \implies 3$. Let $a \in K^\times$ such that $|a| < 1$, which exists since $|\cdot|$ is non-trivial. We need to show that

$$\frac{\log|x|}{\log|a|} = \frac{\log|x|'}{\log|a|'}, \qquad x \in K^\times.$$

Assume $\log|x| / \log|a| < \log|x|' / \log|a|'$. Choose $m, n \in \mathbb{Z}$ such that

$$\frac{\log|x|}{\log|a|} < \frac{m}{n} < \frac{\log|x|'}{\log|a|'}.$$

Then we have $n \log|x| < m \log|a|$ and $n \log|x|' > m \log|a|'$, so $|x^n/a^m| < 1$ and $|x^n/a^m|' > 1$, a contradiction. Similarly for $\log|x| / \log|a| > \log|x|' / \log|a|'$.

$3 \implies 1$. Clear.

□

This course is mainly interested in the following types of absolute values.

Definition 1.1.4. An absolute value $|\cdot|$ on K is said to be **non-archimedean** if it satisfies the **ultrametric inequality**

$$|x + y| \le \max(|x|, |y|).$$

If $|\cdot|$ is not non-archimedean, then it is **archimedean**.

Example.

- $|\cdot|_\infty$ on $\mathbb{R}$ is archimedean.
- $|\cdot|_p$ is a non-archimedean absolute value on $\mathbb{Q}$.

Lemma 1.1.5 (All triangles are isosceles)**.** *Let $(K, |\cdot|)$ be a non-archimedean valued field and $x, y \in K$. If $|x| < |y|$, then $|x - y| = |y|$.*

Fact.

- $|1| = |-1| = 1$.
- $|-y| = |y|$.

Proof. $|x - y| \le \max(|x|, |y|) = |y|$, and $|y| \le \max(|x|, |x - y|)$, so $|y| \le |x - y|$. □

Convergence is easier for non-archimedean $|\cdot|$.

Proposition 1.1.6. *Let $(K, |\cdot|)$ be non-archimedean and $(x_n)_{n=1}^\infty$ a sequence in K. If $|x_n - x_{n+1}| \to 0$, then $(x_n)_{n=1}^\infty$ is Cauchy. In particular, if K is in addition complete, then $(x_n)_{n=1}^\infty$ converges.*

Proof. For $\epsilon > 0$, choose N such that $|x_n - x_{n+1}| < \epsilon$ for all $n > N$. Then for $N < n < m$,

$$|x_n - x_m| = |(x_n - x_{n+1}) + \cdots + (x_{m-1} - x_m)| < \epsilon,$$

so $(x_n)_{n=1}^\infty$ is Cauchy. □

Example. Let $p = 5$. Construct a sequence $(x_n)_{n=1}^{\infty}$ such that

1. $x_n^2 + 1 \equiv 0 \mod 5^n$, and
2. $x_n \equiv x_{n+1} \mod 5^n$,

as follows. Take $x_1 = 2$. Suppose have constructed x_n. Let $x_n^2 + 1 = a5^n$ and set $x_{n+1} = x_n + b5^n$. Then

$$x_{n+1}^2 + 1 = x_n^2 + 2bx_n5^n + b^25^{2n} + 1 = a5^n + 2x_nb5^n + b^25^{2n} \equiv (a + 2x_nb)\,5^n \mod 5^{n+1}.$$

We choose b such that $a + 2x_nb \equiv 0 \mod 5$. Then we have $x_{n+1}^2 + 1 \equiv 0 \mod 5^{n+1}$ as desired. By 2, $(x_n)_{n=1}^{\infty}$ is Cauchy. Suppose $x_n \to L \in \mathbb{Q}$. Then $x_n^2 \to L^2$. But by 1, $x_n^2 \to -1$, so $L^2 = -1$, a contradiction. Thus $(\mathbb{Q}, |\cdot|_5)$ is not complete.

Definition 1.1.7. The **p-adic numbers** $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Remark. By analogy, $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$.

Let K be a non-archimedean valued field. For $x \in K$ and $r \in \mathbb{R}_{>0}$, define

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$$\mathrm{B}(x, r) = \{y \in K \mid |x - y| < r\}, \qquad \overline{\mathrm{B}}(x, r) = \{y \in K \mid |x - y| \le r\}.$$

Lemma 1.1.8. *Let $(K, |\cdot|)$ be non-archimedean.*

1. *If $z \in \mathrm{B}(x, r)$, then $\mathrm{B}(z, r) = \mathrm{B}(x, r)$, so open balls do not have centres.*
2. *If $z \in \overline{\mathrm{B}}(x, r)$, then $\overline{\mathrm{B}}(z, r) = \overline{\mathrm{B}}(x, r)$.*
3. *$\mathrm{B}(x, r)$ is closed.*
4. *$\overline{\mathrm{B}}(x, r)$ is open.*

Proof.

1. Let $y \in \mathrm{B}(x, r)$. Then $|x - y| < r$, so $|z - y| = |(z - x) + (x - y)| \le \max(|z - x|, |x - y|) < r$. Thus $\mathrm{B}(x, r) \subseteq \mathrm{B}(z, r)$. The reverse inclusion follows by symmetry.
2. Same as 1.
3. Let $y \notin \mathrm{B}(x, r)$. If $z \in \mathrm{B}(x, r) \cap \mathrm{B}(y, r)$, then $\mathrm{B}(x, r) = \mathrm{B}(z, r) = \mathrm{B}(y, r)$, so $y \in \mathrm{B}(x, r)$, a contradiction. Thus $\mathrm{B}(x, r) \cap \mathrm{B}(y, r) = \emptyset$.
4. If $z \in \overline{\mathrm{B}}(x, r)$, then $\mathrm{B}(z, r) \subseteq \overline{\mathrm{B}}(z, r) = \overline{\mathrm{B}}(x, r)$, by 2.

□

1.2 Valuation rings

Definition 1.2.1. Let K be a field. A **valuation** on K is a function $v : K^\times \to \mathbb{R}$ such that

- $v(xy) = v(x) + v(y)$, and
- $v(x + y) \ge \min(v(x), v(y))$.

Fix $0 < \alpha < 1$. If v is a valuation on K, then

$$|x| = \begin{cases} \alpha^{v(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

determines a non-archimedean absolute value. Conversely, a non-archimedean absolute value determines a valuation $v(x) = \log_a |x|$.

Remark.

- We ignore the trivial valuation $v(x) = 0$ for all $x \in K^\times$, which corresponds to the trivial absolute value.
- Say v_1 and v_2 are **equivalent** if there exists $c \in \mathbb{R}_{>0}$ such that $v_1(x) = cv_2(x)$ for all $x \in K^\times$.

Example.

- Let $K = \mathbb{Q}$. Then $\mathrm{v}_p(x) = -\log_p |x|_p$ is the **p-adic valuation**.
- Let k be a field, and let $K = k(t) = \operatorname{Frac} k[t]$ be the **rational function field**. Then
$$v\left(t^n \frac{f(t)}{g(t)}\right) = n, \qquad f, g \in k[t], \qquad f(0), g(0) \neq 0$$
is the **t-adic valuation**.
- Let $K = k((t)) = \operatorname{Frac} k[[t]] = \left\{\sum_{i=n}^{\infty} a_i t^i \;\middle|\; a_i \in k,\ n \in \mathbb{Z}\right\}$ be the **field of formal Laurent series** over k. Then
$$v\left(\sum_i a_i t^i\right) = \min\{i \mid a_i \neq 0\}$$
is the t-adic valuation on K.

Definition 1.2.2. Let $(K, |\cdot|)$ be a non-archimedean valued field. The **valuation ring** of K is defined to be
$$\mathcal{O}_K = \overline{\mathrm{B}}(0,1) = \{x \in K \mid |x| \leq 1\} = \left\{x \in K^\times \;\middle|\; v(x) \geq 0\right\} \cup \{0\}.$$

Proposition 1.2.3.

1. *$\mathcal{O}_K$ is an open subring of K.*
2. *The subsets $\{x \in K \mid |x| \leq r\}$ and $\{x \in K \mid |x| < r\}$ for $r \leq 1$ are open ideals in $\mathcal{O}_K$.*
3. *$\mathcal{O}_K^\times = \{x \in K \mid |x| = 1\}$.*

Proof.

1. By last lecture, $|1| = 1$, so $1 \in \mathcal{O}_K$. Since $|0| = 0$, $0 \in \mathcal{O}_K$. Since $|-1| = 1$, $|-x| = |x|$. Thus if $x \in \mathcal{O}_K$, then $-x \in \mathcal{O}_K$. If $x, y \in \mathcal{O}_K$, then $|x+y| \leq \max(|x|, |y|) \leq 1$, so $x + y \in \mathcal{O}_K$. If $x, y \in \mathcal{O}_K$, then $|xy| = |x||y| \leq 1$, so $xy \in \mathcal{O}_K$. Thus $\mathcal{O}_K$ is a ring. Since $\mathcal{O}_K = \overline{\mathrm{B}}(0,1)$ it is open.
2. Similar to 1.
3. Note that $|x|\left|x^{-1}\right| = \left|xx^{-1}\right| = 1$. Thus $|x| = 1$ if and only if $\left|x^{-1}\right| = 1$, if and only if $x, x^{-1} \in \mathcal{O}_K$, if and only if $x \in \mathcal{O}_K^\times$.

$\square$

Notation.

- $\mathfrak{m} = \{x \in \mathcal{O}_K \mid |x| < 1\}$ is a maximal ideal of $\mathcal{O}_K$.
- $k = \mathcal{O}_K / \mathfrak{m}$ is the **residue field**.

A ring is **local** if it has a unique maximal ideal.

Exercise. R is local if and only if $R \setminus R^\times$ is an ideal.

Corollary 1.2.4. *$\mathcal{O}_K$ is a local ring with unique maximal ideal $\mathfrak{m}$.*

Example.

- Let $K = k((t))$. Then $\mathcal{O}_K = k[[t]]$, $\mathfrak{m} = \langle t \rangle$, and the residue field is k.
- Let $K = \mathbb{Q}$ with $|\cdot|_p$. Then $\mathcal{O}_K = \mathbb{Z}_{\langle p \rangle}$, $\mathfrak{m} = p\mathbb{Z}_{\langle p \rangle}$, and $k = \mathbb{F}_p$.

Definition 1.2.5. Let $v : K^\times \to \mathbb{R}$ be a valuation. If $v(K^\times) \cong \mathbb{Z}$, we say v is a **discrete valuation**, and K is said to be a **discretely valued field**. An element $\pi \in \mathcal{O}_K$ is a **uniformiser** if $v(\pi) > 0$ and $v(\pi)$ generates $v(K^\times)$.

Example.

- $K = \mathbb{Q}$ with the p-adic valuation.
- $K = k(t)$ with the t-adic valuation.

Remark. If v is a discrete valuation, we can replace it with an equivalent one such that $v(K^\times) = \mathbb{Z} \subseteq \mathbb{R}$. Such v are called **normalised valuations**. Then $v(\pi) = 1$ for π a uniformiser.

Lemma 1.2.6. *Let v be a valuation on K. The following are equivalent.*

1. *v is discrete.*
2. *$\mathcal{O}_K$ is a PID.*
3. *$\mathcal{O}_K$ is Noetherian.*
4. *$\mathfrak{m}$ is principal.*

Proof.

$1 \implies 2$. Let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. Let $x \in I$ such that $v(x) = \min\{v(a) \mid a \in I\}$ which exists since v is discrete. Then $x\mathcal{O}_K = \{a \in \mathcal{O}_K \mid v(a) \geq v(x)\} \subseteq I$, and hence $x\mathcal{O}_K = I$ by definition of x.

$2 \implies 3$. Clear.

$3 \implies 4$. Write $\mathfrak{m} = \mathcal{O}_K x_1 + \dots + \mathcal{O}_K x_n$. Without loss of generality $v(x_1) \leq \dots \leq v(x_n)$. Then $\mathfrak{m} = \mathcal{O}_K x_1$.

$4 \implies 1$. Let $\mathfrak{m} = \mathcal{O}_K \pi$ for some $\pi \in \mathcal{O}_K$ and let $c = v(\pi)$. Then if $v(x) > 0$, $x \in \mathfrak{m}$ and hence $v(x) \geq c$. Thus $v(K^\times) \cap (0, c) = \emptyset$. Since $v(K^\times)$ is a subgroup of $(\mathbb{R}, +)$, we have $v(K^\times) = c\mathbb{Z}$.

□

Lemma 1.2.7. *Let v be a discrete valuation on K and $\pi \in \mathcal{O}_K$ a uniformiser. For all $x \in K^\times$, there exist $n \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$ such that $x = \pi^n u$. In particular $K = \mathcal{O}_K[1/x]$ for any $x \in \mathfrak{m}$ and hence $K = \operatorname{Frac} \mathcal{O}_K$.*

Proof. For $x \in K^\times$, let n such that $v(x) = nv(\pi) = v(\pi^n)$, then $v(x\pi^{-n}) = 0$, so $u = x\pi^{-n} \in \mathcal{O}_K^\times$. □

Definition 1.2.8. A ring R is called a **discrete valuation ring (DVR)** if it is a PID with exactly one non-zero prime ideal, necessarily maximal.

Lemma 1.2.9.

1. *Let v be a discrete valuation on K. Then $\mathcal{O}_K$ is a DVR.*
2. *Let R be a DVR. Then there exists a valuation v on $K = \operatorname{Frac} R$ such that $R = \mathcal{O}_K$.*

Proof.

1. $\mathcal{O}_K$ is a PID by Lemma 1.2.6. Let $0 \neq I \subseteq \mathcal{O}_K$ be an ideal, then $I = \langle x \rangle$. If $x = \pi^n u$ for π a uniformiser, then $\langle x \rangle$ is prime if and only if $n = 1$ and $I = \langle \pi \rangle = \mathfrak{m}$.
2. Let R be a DVR with maximal ideal $\mathfrak{m}$. Then $\mathfrak{m} = \langle \pi \rangle$ for some $\pi \in R$. By unique factorisation of PIDs, we may write any $x \in R \setminus \{0\}$ uniquely as $\pi^n u$ for $n \geq 0$ and $u \in R^\times$. Then any $y \in K \setminus \{0\}$ can be written uniquely as $\pi^m u$ for $u \in R^\times$ and $m \in \mathbb{Z}$. Define $v(\pi^m u) = m$. It is easy to check v is a valuation and $\mathcal{O}_K = R$.

□

Example.

- $\mathbb{Z}_{\langle p \rangle}$ is a DVR, the valuation ring of $|\cdot|_p$ on $\mathbb{Q}$.
- The ring of formal power series $k[[t]] = \left\{\sum_{n \geq 0} a_n t^n \;\middle|\; a_n \in k\right\}$ is a DVR, the valuation ring for the t-adic absolute value on $k((t))$.
- Non-example. Let $K = k(t)$ be the rational function field, and let $K' = K\left(t^{1/2}, t^{1/4}, \dots\right)$. Then the t-adic valuation extends to K', and $v\left(t^{1/2^n}\right) = 1/2^n$ is not discrete.

1.3 The p-adic numbers

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Recall that $\mathbb{Q}_p$ is defined to be the completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$. By example sheet 1, $\mathbb{Q}_p$ is a field, $|\cdot|_p$ extends to $\mathbb{Q}_p$, and the associated valuation is discrete, so $\mathbb{Q}_p$ is a discretely valued field.

Definition 1.3.1. The ring of **p-adic integers** $\mathbb{Z}_p$ is the valuation ring

$$\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \,\middle|\, |x|_p \le 1\right\}.$$

Fact.

- $\mathbb{Z}_p$ is a DVR with maximal ideal $p\mathbb{Z}_p$.
- The non-zero ideals in $\mathbb{Z}_p$ are $p^n\mathbb{Z}_p$ for $n \in \mathbb{N}$.

Proposition 1.3.2. *$\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$. In particular $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to $|\cdot|_p$.*

Proof. Need to show $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ and $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ is open, $\mathbb{Z}_p \cap \mathbb{Q}$ is dense in $\mathbb{Z}_p$. Then

$$\mathbb{Z}_p \cap \mathbb{Q} = \left\{x \in \mathbb{Q} \,\middle|\, |x|_p \le 1\right\} = \left\{\frac{a}{b} \in \mathbb{Q} \,\middle|\, p \nmid b\right\} = \mathbb{Z}_{\langle p \rangle},$$

the localisation at $\langle p \rangle$. Thus it suffices to show $\mathbb{Z}$ is dense in $\mathbb{Z}_{\langle p \rangle}$. Let $a/b \in \mathbb{Z}_{\langle p \rangle}$ for $a, b \in \mathbb{Z}$ and $p \nmid b$. For $n \in \mathbb{N}$, choose $y_n \in \mathbb{Z}$ such that $by_n \equiv a \mod p^n$. Then $y_n \to a/b$ as $n \to \infty$. In particular, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, which is complete. □

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets or groups or rings together with homomorphisms $\phi_n : A_{n+1} \to A_n$, the **transition maps**. The **inverse limit** of $(A_n)_{n=1}^{\infty}$ is the set or group or ring

$$\varprojlim_n A_n = \left\{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n \,\middle|\, \phi_n(a_{n+1}) = a_n\right\},$$

so

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{\phi_n} & A_n & \xrightarrow{\phi_{n-1}} & A_{n-1} \\ a_{n+1} & \longmapsto & a_n & \longmapsto & a_{n-1} \end{array}.$$

Fact. If A_n is a group or ring, then $\varprojlim_n A_n$ is a group or ring.

Let $\theta_m : \varprojlim_n A_n \to A_m$ denote the natural projection. The inverse limit satisfies the following universal property.

Proposition 1.3.3. *Let $\left((A_n)_{n=1}^{\infty}, (\phi_n)_{n=1}^{\infty}\right)$ as above. Then for any set or group or ring B together with homomorphisms $\psi_n : B \to A_n$ such that*

$$\begin{tikzcd} B \arrow[r, "\psi_{n+1}"] \arrow[dr, "\psi_n"'] & A_{n+1} \arrow[d, "\phi_n"] \\ & A_n \end{tikzcd}$$

commutes for all n, there is a unique homomorphism $\psi : B \to \varprojlim_n A_n$ such that $\theta_n \circ \psi = \psi_n$.

Proof. Define

$$\begin{array}{cccc} \psi & : & B & \longrightarrow & \prod_{n=1}^{\infty} A_n \\ & & b & \longmapsto & \prod_{n=1}^{\infty} \psi_n(b) \end{array}.$$

Then $\psi_n = \phi_n \circ \psi_{n+1}$ implies that $\psi(b) \in \varprojlim_n A_n$. The map is clearly unique, determined by $\psi_n = \phi_n \circ \psi_{n+1}$, and is a homomorphism of rings. □

Definition 1.3.4. Let R be a ring and $I \subseteq R$ an ideal. The ***I*-adic completion** of R is the ring

$$\widehat{R} = \varprojlim_n R/I^n,$$

where $\phi_n : R/I^{n+1} \to R/I^n$ is the natural projection. Note there is a natural map $\iota : R \to \widehat{R}$ by the universal property. We say that R is ***I*-adically complete** if ι is an isomorphism.

Fact. $\ker\left(\iota : R \to \widehat{R}\right) = \bigcap_{n=1}^{\infty} I^n$.

Let $(K,|\cdot|)$ be a non-archimedean valued field and $\pi \in \mathcal{O}_K$ such that $|\pi| < 1$.

Proposition 1.3.5. *Assume K is complete.*

1. *Then $\mathcal{O}_K \cong \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$, so $\mathcal{O}_K$ is π-adically complete.*
2. *If in addition K is discretely valued and π is a uniformiser, then every $x \in \mathcal{O}_K$ can be written uniquely as $x = \sum_{i=0}^{\infty} a_i\pi^i$ for $a_i \in A$, where A is a set of coset representatives for $k = \mathcal{O}_K/\pi\mathcal{O}_K$. Moreover, any series $\sum_{i=0}^{\infty} a_i\pi^i$ converges to an element in $\mathcal{O}_K$.*

Proof.

1. Let $\iota : \mathcal{O}_K \to \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$. Since $\bigcap_{n=1}^{\infty} \pi^n\mathcal{O}_K = \{0\}$, ι is injective. Let $(x_n)_{n=1}^{\infty} \in \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$ and for each n, choose $y_n \in \mathcal{O}_K$ a lift of $x_n \in \mathcal{O}_K/\pi^n\mathcal{O}_K$. Let v be the valuation on K normalised such that $v(\pi) = 1$, then $v(y_n - y_{n+1}) \geq n$, since $y_n - y_{n+1} \in \pi^n\mathcal{O}_K$, so $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{O}_K$. But $\mathcal{O}_K$ is complete, since $\mathcal{O}_K \subseteq K$ is closed, so $y_n \to y$, and y maps to $(x_n)_{n=1}^{\infty}$. Thus ι is surjective.
2. Let $x \in \mathcal{O}_K$. Choose a_i inductively. Choose $a_0 \in A$ such that $a_0 \equiv x \mod \pi$. Suppose have chosen $a_0, \dots, a_k$ such that $\sum_{i=0}^{k} a_i\pi^i \equiv x \mod \pi^{k+1}$. Then $\sum_{i=0}^{k} a_i\pi^i - x = c\pi^{k+1}$ for $c \in \mathcal{O}_K$. Choose $a_{k+1} \equiv -c \mod \pi$. Then $\sum_{i=0}^{k+1} a_i\pi^i \equiv x \mod \pi^{k+2}$, so $\sum_{i=0}^{\infty} a_i\pi^i = x$. For uniqueness, assume $\sum_{i=0}^{\infty} a_i\pi^i = \sum_{i=0}^{\infty} b_i\pi^i \in \mathcal{O}_K$. Then let n be minimal such that $a_n \neq b_n$. Then $\sum_{i=0}^{\infty} a_i\pi^i \not\equiv \sum_{i=0}^{\infty} b_i\pi^i \mod \pi^{n+1}$, a contradiction.

∎

A warning is if $(K,|\cdot|)$ is not discretely valued, $\mathcal{O}_K$ is not necessarily $\mathfrak{m}$-adically complete.

Corollary 1.3.6. *If K is as in Proposition 1.3.5.2, then every $x \in K$ can be written uniquely as $\sum_{i=n}^{\infty} a_i\pi^i$ for $a_i \in A$. Conversely any such expression defines an element of K.*

Proof. Use $K = \mathcal{O}_K[1/\pi]$. ∎

Corollary 1.3.7.

1. *$\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.*
2. *Every element of $\mathbb{Q}_p$ can be written uniquely as $\sum_{i=n}^{\infty} a_ip^i$ for $a_i \in \{0, \dots, p-1\}$.*

Proof.

1. By Proposition 1.3.5, it suffices to show that $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$. Let $f_n : \mathbb{Z} \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ be the natural map. We have $\ker f_n = \left\{x \in \mathbb{Z} \;\middle|\; |x|_p \leq p^{-n}\right\} = p^n\mathbb{Z}$, so $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ is injective. Let $\overline{c} \in \mathbb{Z}_p/p^n\mathbb{Z}_p$, and $c \in \mathbb{Z}_p$ a lift. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, can choose $x \in \mathbb{Z}$ such that $x \in c + p^n\mathbb{Z}_p$, which is open in $\mathbb{Z}_p$, so $f_n(x) = \overline{c}$. Thus $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ is surjective.
2. Follows from Corollary 1.3.6 noting that $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$.

∎

Example.

- $1/(1-p) = 1 + p + \dots \in \mathbb{Q}_p$.
- Let $K = k((t))$ with the t-adic valuation. Then $\mathcal{O}_K = k[[t]] = \varprojlim_n k[[t]]/\langle t^n\rangle$. Moreover $\mathcal{O}_K$ is the t-adic completion of $k[t]$.

2 Complete valued fields

2.1 Hensel's lemma

Lecture 4 Friday 16/10/20

For complete valued fields, there is a nice way to produce solutions in $\mathcal{O}_K$ to certain equations from solutions modulo $\mathfrak{m}$.

Theorem 2.1.1 (Hensel's lemma version 1)**.** *Let $(K, |\cdot|)$ be a complete discretely valued field. Let $f(X) \in \mathcal{O}_K[X]$ and assume there exists $a \in \mathcal{O}_K$ such that $|f(a)| < |f'(a)|^2$, where $f'(a)$ is the **formal derivative** such that if $f(X) = X^n$ then $f'(X) = nX^{n-1}$. Then there exists a unique $x \in \mathcal{O}_K$ such that $f(x) = 0$ and $|x - a| < |f'(a)|$.*

Proof. Let $\pi \in \mathcal{O}_K$ be a uniformiser and let $r = v(f'(a))$. We construct a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{O}_K$ such that

1. $f(x_n) \equiv 0 \mod \pi^{n+2r}$, and
2. $x_{n+1} \equiv x_n \mod \pi^{n+r}$.

Take $x_1 = a$, then $f(x_1) \equiv 0 \mod \pi^{1+2r}$. Suppose have constructed $x_1, \dots, x_n$ satisfying 1 and 2. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2. Since $x_n \equiv x_1 \mod \pi^{1+r}$, $v(f'(x_n)) = r$ and hence $f(x_n)/f'(x_n) \equiv 0 \mod \pi^{n+r}$ by 1. It follows that $x_{n+1} \equiv x_n \mod \pi^{n+r}$ so 2 holds.

1. Note that for X and Y indeterminates,

   $$f(X+Y) = f_0(X) + f_1(X)Y + \dots, \qquad f_i(X) \in \mathcal{O}_K[X], \qquad f_0(X) = f(X), \qquad f_1(X) = f'(X).$$

   Thus

   $$f(x_{n+1}) = f(x_n) + f'(x_n)c + \dots, \qquad c = -\frac{f(x_n)}{f'(x_n)}.$$

   Since $c \equiv 0 \mod \pi^{n+r}$ and $v(f_i(x_n)) \geq 0$, we have $f(x_{n+1}) \equiv f(x_n) + f'(x_n)c \equiv 0 \mod \pi^{n+2r+1}$, so 1 holds.

This gives the construction of $(x_n)_{n=1}^\infty$.

- By property 2, $(x_n)_{n=1}^\infty$ is Cauchy, so let $x \in \mathcal{O}_K$ such that $x_n \to x$. Then $f(x) = \lim_{n\to\infty} f(x_n) = 0$ by 1. Moreover 2 implies $a = x_1 \equiv x_n \mod \pi^{1+r}$ for all n, so $a \equiv x \mod \pi^{1+r}$, so $|x - a| < |f'(a)|$. This proves existence.

- For uniqueness, suppose x' also satisfies $f(x') = 0$ and $|x' - a| < |f'(a)|$. Set $\delta = x' - x \neq 0$. Then $|x' - a| < |f'(a)|$, $|x - a| < |f'(a)|$, and the ultrametric inequality implies $|\delta| = |x - x'| < |f'(a)| = |f'(x)|$. But

  $$0 = f(x') = f(x + \delta) = \underbrace{f(x)}_{=0} + f'(x)\delta + \underbrace{\dots}_{|\cdot| \leq |\delta|^2}.$$

  Hence $|f'(x)\delta| \leq |\delta|^2$, so $|f'(x)| \leq |\delta|$, a contradiction.

$\square$

Corollary 2.1.2. *Let $(K, |\cdot|)$ be a complete discretely valued field. Let $f(X) \in \mathcal{O}_K[X]$ and $\overline{c} \in k = \mathcal{O}_K/\mathfrak{m}$ a simple root of $\overline{f}(X) = f(X) \mod \mathfrak{m} \in k[X]$. Then there exists a unique $x \in \mathcal{O}_K$ such that $f(x) = 0$ and $x \equiv \overline{c} \mod \mathfrak{m}$.*

Proof. Apply Theorem 2.1.1 to a lift $c \in \mathcal{O}_K$ of $\overline{c}$. Then $|f(c)| < |f'(c)|^2 = 1$ since $\overline{c}$ is a simple root. $\square$

Example. $f(X) = X^2 - 2$ has a simple root modulo seven. Thus $\sqrt{2} \in \mathbb{Z}_7 \subseteq \mathbb{Q}_7$.

Corollary 2.1.3.

$$\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2 \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & p > 2 \\ (\mathbb{Z}/2\mathbb{Z})^3 & p = 2 \end{cases}.$$

Proof.

$p > 2$. Let $b \in \mathbb{Z}_p^\times$. Applying Corollary 2.1.2 to $f(X) = X^2 - b$, we find that $b \in \left(\mathbb{Z}_p^\times\right)^2$ if and only if $b \in \left(\mathbb{F}_p^\times\right)^2$. Thus $\mathbb{Z}_p^\times / \left(\mathbb{Z}_p^\times\right)^2 \cong \mathbb{F}_p^\times / \left(\mathbb{F}_p^\times\right)^2 \cong \mathbb{Z}/2\mathbb{Z}$ since $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$. We have an isomorphism $\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}$ given by $(u, n) \mapsto up^n$. Thus $\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2 \cong (\mathbb{Z}/2\mathbb{Z})^2$.

$p = 2$. Let $b \in \mathbb{Z}_2^\times$. Consider $f(X) = X^2 - b$. Then $f'(X) = 2X \equiv 0 \mod 2$. Let $b \equiv 1 \mod 8$. Then $|f(1)|_2 \le 2^{-3} < |f'(1)|_2^2 = 2^{-2}$. By Hensel's lemma, $f(X)$ has a root in $\mathbb{Z}_2$, so $b \in \left(\mathbb{Z}_2^\times\right)^2$ if and only if $b \equiv 1 \mod 8$. Thus $\mathbb{Z}_2^\times / \left(\mathbb{Z}_2^\times\right)^2 \cong (\mathbb{Z}/8\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z})^2$. Again using $\mathbb{Q}_2^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}$, we find that $\mathbb{Q}_2^\times / \left(\mathbb{Q}_2^\times\right)^2 \cong (\mathbb{Z}/2\mathbb{Z})^3$.

$\square$

Remark. The proof of Hensel's lemma uses the iteration $x_{n+1} = x_n - f(x_n)/f'(x_n)$, the non-archimedean analogue of the Newton-Raphson method.

For later applications, we need the following version of Hensel's lemma.

Theorem 2.1.4 (Hensel's lemma version 2)**.** *Let $(K, |\cdot|)$ be a complete discretely valued field and $f(X) \in \mathcal{O}_K[X]$. Suppose $\overline{f}(X) = f(X) \mod \mathfrak{m} \in k[X]$ factorises as $\overline{f}(X) = \overline{g}(X)\overline{h}(X)$ in $k[X]$, with $\overline{g}(X)$ and $\overline{h}(X)$ coprime. Then there is a factorisation $f(X) = g(X)h(X)$ in $\mathcal{O}_K[X]$, with $\overline{g}(X) = g(X) \mod \mathfrak{m}$, $\overline{h}(X) = h(X) \mod \mathfrak{m}$, and $\deg \overline{g} = \deg g$.*

Proof. Example sheet 1. $\square$

Corollary 2.1.5. *Let $f(X) = a_nX^n + \dots + a_0 \in K[X]$ with $a_0, a_n \neq 0$. If $f(X)$ is irreducible, then $|a_i| \le \max(|a_0|, |a_n|)$ for all i.*

Proof. Upon scaling, we may assume $f(X) \in \mathcal{O}_K[X]$ with $\max_i(|a_i|) = 1$. Thus we need to show that $\max(|a_0|, |a_n|) = 1$. If not, let r be minimal such that $|a_r| = 1$, then $0 < r < n$. Thus we have $\overline{f}(X) = X^r(a_r + \dots + a_nX^{n-r}) \mod \mathfrak{m}$. Then Theorem 2.1.4 implies $f(X) = g(X)h(X)$ and $0 < \deg g < n$. $\square$

2.2 Teichmüller lifts

Lecture 5 Monday 19/10/20

Recall that in lecture 3 every element of $x \in \mathbb{Q}_p$ can be written as $x = \sum_{i=n}^\infty a_ip^i$ for $a_i \in A = \{0, \dots, p-1\}$, but $\mathbb{F}_p \to A \subseteq \mathbb{Z}_p$ does not respect any algebraic structure. It turns out there is a natural choice of coset representatives in many cases which does respect some algebraic structure.

Definition 2.2.1. A ring R of characteristic p is a **perfect ring** if the Frobenius $x \mapsto x^p$ is an automorphism of R. A field of characteristic p is a **perfect field** if it is perfect as a ring.

Remark. Since $\operatorname{ch} R = p$, $(x+y)^p = x^p + y^p$, so Frobenius is a ring homomorphism.

Example.

- $\mathbb{F}_{p^n}$ and $\overline{\mathbb{F}_p}$ are perfect fields.
- $\mathbb{F}_p[t]$ is not perfect, since t is not in the image of Frobenius.
- $\mathbb{F}_p\left(t^{1/p^\infty}\right) = \mathbb{F}_p\left(t, t^{1/p}, \dots\right)$ is a perfect field, the **perfection** of $\mathbb{F}_p(t)$. The t-adic absolute value extends to $\mathbb{F}_p\left(t^{1/p^\infty}\right)$, and the completion of $\mathbb{F}_p\left(t^{1/p^\infty}\right)$ is a **perfectoid field**.

Fact. A field k is perfect if and only if any finite extension of k is separable.

Theorem 2.2.2. *Let $(K,|\cdot|)$ be a complete discretely valued field such that $k = \mathcal{O}_K/\mathfrak{m}$ is a perfect field of characteristic p. Then there exists a unique map $[\cdot] : k \to \mathcal{O}_K$ such that*

1. *$a \equiv [a] \mod \mathfrak{m}$ for all $a \in k$, and*
2. *$[ab] \equiv [a]\,[b] \mod \mathfrak{m}$ for all $a, b \in k$.*

Moreover if $\operatorname{ch} \mathcal{O}_K = p$, $[\cdot]$ is a ring homomorphism.

Definition 2.2.3. The element $[a] \in \mathcal{O}_K$ constructed in Theorem 2.2.2 is called the **Teichmüller lift** of a.

The following is the idea of the proof. Let $\alpha \in \mathcal{O}_K$ be any lift of $a \in k$. Then α is well-defined up to $\pi\mathcal{O}_K$. Let $\beta \in \mathcal{O}_K$ be a lift of $a^{1/p}$. We claim that β is a better lift. Why? Let $\beta' \in \mathcal{O}_K$ be another lift of $a^{1/p}$, then $\beta = \beta' + \pi u$ for $u \in \mathcal{O}_K$, so

$$\beta^p = (\beta' + \pi u)^p = \beta'^p + \underbrace{\sum_{i=1}^{p} \binom{p}{i} \beta'^{p-i} (\pi u)^i}_{\in \pi^2 \mathcal{O}_K},$$

using $p \in \langle \pi \rangle$, so β^p is well-defined up to $\pi^2\mathcal{O}_K$. Repeat this process to get better and better lifts.

Lemma 2.2.4. *Let $(K,|\cdot|)$ be as in Theorem 2.2.2, and fix $\pi \in \mathcal{O}_K$ a uniformiser. Let $x, y \in \mathcal{O}_K$ such that $x \equiv y \mod \pi^k$ for $k \geq 1$. Then $x^p \equiv y^p \mod \pi^{k+1}$.*

Proof. Let $x = y + u\pi^k$ for $u \in \mathcal{O}_K$. Then

$$x^p = \sum_{i=0}^{p} \binom{p}{i} \left(u\pi^k\right)^i y^{p-i} = y^p + pu\pi^k y^{p-1} + \sum_{i=2}^{p} \binom{p}{i} \left(u\pi^k\right)^i y^{p-i}.$$

Since $\mathcal{O}_K/\pi\mathcal{O}_K$ has characteristic p, we have $p \in \langle \pi \rangle$. Thus $pu\pi^k y^{p-1} \in \pi^{k+1}\mathcal{O}_K$. For $i \geq 2$, $\left(u\pi^k\right)^i \in \pi^{k+1}\mathcal{O}_K$, so $x^p \equiv y^p \mod \pi^{k+1}$. ☐

Proof of Theorem 2.2.2. Let $a \in k$. For each $i \geq 0$ we choose a lift $y_i \in \mathcal{O}_K$ of a^{1/p^i}, and we define

$$x_i = y_i^{p^i}.$$

Then $x_i \equiv y_i^{p^i} \equiv \left(a^{1/p^i}\right)^{p^i} \equiv a \mod \pi$. We claim that $(x_i)_{i=1}^{\infty}$ is a Cauchy sequence, and its limit $x_i \to x$ is independent of the choice of y_i.

- By construction $y_i \equiv y_{i+1}^p \mod \pi$. By Lemma 2.2.4 and induction on k, we have $y_i^{p^k} \equiv y_{i+1}^{p^{k+1}} \mod \pi^{k+1}$, and hence $x_i \equiv x_{i+1} \mod \pi^{i+1}$, by taking $k = i$, so $|x_i - x_{i+1}| \to 0$. Then $(x_i)_{i=1}^{\infty}$ is Cauchy, so $x_i \to x \in \mathcal{O}_K$.
- Suppose $(x_i')_{i=1}^{\infty}$ arises from another choice of y_i' lifting a^{1/p^i}. Then x_i' is Cauchy, and $x_i' \to x' \in \mathcal{O}_K$. Let
$$x_i'' = \begin{cases} x_i & i \text{ even} \\ x_i' & i \text{ odd} \end{cases}.$$
Then x_i'' arises from lifting
$$y_i'' = \begin{cases} y_i & i \text{ even} \\ y_i' & i \text{ odd} \end{cases}.$$
Then $(x_i'')_{i=1}^{\infty}$ is Cauchy and $x_i'' \to x$ and $x_i'' \to x'$, so $x = x'$, hence x is independent of y_i.

We define $[a] = x$.

1. $x \equiv a \mod \pi$, so 1 is satisfied.
2. We let $b \in k$ and we choose $u_i \in \mathcal{O}_K$ a lift of b^{1/p^i}, and let $z_i = u_i^{p^i}$. Then $\lim_{i\to\infty} z_i = [b]$. Now $u_i y_i$ is a lift of $(ab)^{1/p^i}$, hence
$$[ab] = \lim_{i\to\infty} x_i z_i = \lim_{i\to\infty} x_i \lim_{i\to\infty} z_i = [a]\,[b]\,,$$
so 2 is satisfied.

If $\operatorname{ch} \mathcal{O}_K = p$, $y_i + u_i$ is a lift of $a^{1/p^i} + b^{1/p^i} = (a+b)^{1/p^i}$. Then

$$[a+b] = \lim_{i\to\infty} (y_i + u_i)^{p^i} = \lim_{i\to\infty} \left(y_i^{p^i} + u_i^{p^i}\right) = \lim_{i\to\infty} (x_i + z_i) = [a] + [b].$$

It is easy to check that $[0] = 0$ and $[1] = 1$, so $[\cdot]$ is a ring homomorphism. For uniqueness, let $\phi : k \to \mathcal{O}_K$ be another such map. Then for $a \in k$, $\phi\left(a^{1/p^i}\right)$ is a lift of a^{1/p^i}, it follows that

$$[a] = \lim_{i\to\infty} \phi\left(a^{1/p^i}\right)^{p^i} = \lim_{i\to\infty} \phi(a) = \phi(a).$$

□

Example 2.2.5. Let $K = \mathbb{Q}_p$, and let $[\cdot] : \mathbb{F}_p \to \mathbb{Z}_p$. If $a \in \mathbb{F}_p^\times$, then $[a]^{p-1} = \left[a^{p-1}\right] = [1] = 1$, so $[a]$ is a $(p-1)$-th root of unity.

More generally is the following.

Lemma 2.2.6. *Let $(K,|\cdot|)$ be a complete discretely valued field. If $k = \mathcal{O}_K/\mathfrak{m} \subseteq \overline{\mathbb{F}_p}$, $[a] \in \mathcal{O}_K^\times$ is a root of unity.*

Proof. If $a \in k$, then $a \in \mathbb{F}_{p^n}$ for some n, so $[a]^{p^n-1} = \left[a^{p^n-1}\right] = [1] = 1$. □

Theorem 2.2.7. *Let $(K,|\cdot|)$ be a complete discretely valued field such that k is perfect with $\operatorname{ch} k = p > 0$. Then $K \cong k((t))$.*

Proof. Since $K = \operatorname{Frac} \mathcal{O}_K$, it suffices to show $\mathcal{O}_K \cong k[[t]]$. Fix $\pi \in \mathcal{O}_K$ a uniformiser, let $[\cdot] : k \to \mathcal{O}_K$ be the Teichmüller map, and define

$$\begin{array}{cccc} \phi & : & k[[t]] & \longrightarrow & \mathcal{O}_K \\ & & \sum_{i=0}^\infty a_i t^i & \longmapsto & \sum_{i=0}^\infty [a_i] \pi^i \end{array}.$$

Then ϕ is a ring homomorphism since $[\cdot]$ is a ring homomorphism and it is a bijection by Proposition 1.3.5.2. □

2.3 Extensions of complete valued fields

Lecture 6 Wednesday 21/10/20

Theorem 2.3.1. *Let $(K,|\cdot|)$ be a complete non-archimedean discretely valued field and L/K a finite extension of degree n.*

1. *$|\cdot|$ extends uniquely to an absolute value $|\cdot|_L$ on L defined by*

$$|y|_L = \left|\mathrm{N}_{L/K}(y)\right|^{\frac{1}{n}}, \qquad y \in L.$$

2. *L is complete with respect to $|\cdot|_L$.*

Recall that if L/K is finite,

$$\begin{array}{cccc} \mathrm{N}_{L/K} & : & L & \longrightarrow & K \\ & & y & \longmapsto & \det_K(\cdot y) \end{array},$$

where $\cdot y : L \to L$ is the K-linear map induced by multiplication by y.

Fact.

- $\mathrm{N}_{L/K}(xy) = \mathrm{N}_{L/K}(x)\,\mathrm{N}_{L/K}(y)$.
- Let $X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X]$ be the minimal polynomial of $y \in L$. Then $\mathrm{N}_{L/K}(y) = \pm a_0^m$ for $m \geq 1$.

Definition 2.3.2. Let $(K,|\cdot|)$ be a non-archimedean valued field and V a vector space over K. *A* **norm** on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying

- $\|x\| = 0$ if and only if $x = 0$,
- $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in K$ and $x \in V$, and
- $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in V$.

Example. If V is finite dimensional and $e_1, \dots, e_n$ is a basis of V, the **sup norm** on V is defined by

$$\|x\|_{\mathrm{sup}} = \max_i |x_i|, \qquad x = \sum_{i=1}^n x_i e_i.$$

Exercise. $\|\cdot\|_{\mathrm{sup}}$ is a norm.

Definition 2.3.3. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are **equivalent** if there exists $C, D > 0$ such that

$$C\|x\|_1 \leq \|x\|_2 \leq D\|x\|_1, \qquad x \in V.$$

Fact. A norm defines a topology on V, and equivalent norms induce the same topology.

Proposition 2.3.4. *Let $(K,|\cdot|)$ be complete non-archimedean and V a finite dimensional vector space over K. Then V is complete with respect to $\|\cdot\|_{\mathrm{sup}}$.*

Proof. Let $(v_i)_{i=1}^\infty$ be a Cauchy sequence in V and $e_1, \dots, e_n$ a basis for V. Write $v_i = \sum_{j=1}^n x_j^i e_j$. Then $\left(x_j^i\right)_{i=0}^\infty$ is a Cauchy sequence in K. Let $x_j^i \to x_j \in K$, then $v_i \to v = \sum_{j=1}^n x_j e_j$. ☐

Theorem 2.3.5. *Let $(K,|\cdot|)$ be complete non-archimedean and V a finite dimensional vector space over K. Then any two norms on V are equivalent. In particular V is complete with respect to any norm.*

Proof. Since equivalence defines an equivalence relation on the set of norms, it suffices to show any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{\mathrm{sup}}$. Let $e_1, \dots, e_n$ be a basis for V, and set $D = \max_i \|e_i\|$. Then for $x = \sum_{i=1}^n x_i e_i$, we have

$$\|x\| \leq \max_i \|x_i e_i\| = \max_i |x_i| \|e_i\| \leq D \max_i |x_i| = D\|x\|_{\mathrm{sup}}.$$

To find C such that $C\|\cdot\|_{\mathrm{sup}} \leq \|\cdot\|$, we induct on $n = \dim V$.

$n = 1$. $\|x\| = \|x_1 e_1\| = |x_1| \|e_1\|$ so take $C = \|e_1\|$, since $|x_1| = \|x\|_{\mathrm{sup}}$.

$n > 1$. Set $V_i = \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n \rangle$. By induction, V_i is complete with respect to $\|\cdot\|$, hence closed. Then $e_i + V_i$ is closed for all i, and hence $S = \bigcup_{i=1}^n (e_i + V_i)$ is a closed subset not containing zero. Thus there exists $C > 0$ such that $\mathrm{B}(0, C) \cap S = \emptyset$ where $\mathrm{B}(0, C) = \{x \in V \mid \|x\| < C\}$. Let $x = \sum_{i=1}^n x_i e_i$ and suppose $|x_j| = \max_i |x_i|$. Then $\|x\|_{\mathrm{sup}} = |x_j|$, and $(1/x_j) x \in S$. Thus $\|(1/x_j) x\| \geq C$, so $\|x\| \geq C|x_j| = C\|x\|_{\mathrm{sup}}$.

The completeness of V follows since V is complete with respect to $\|\cdot\|_{\mathrm{sup}}$. ☐

Definition 2.3.6. Let $R \subseteq S$ be rings.

- We say $s \in S$ is **integral** over R if there exists a monic polynomial $f(X) \in R[X]$ such that $f(s) = 0$.
- The **integral closure** $R^{\mathrm{Int}\, S}$ of R inside S is defined to be

$$R^{\mathrm{Int}\, S} = \{s \in S \mid s \text{ is integral over } R\}.$$

- We say R is **integrally closed** in S if $R^{\mathrm{Int}\, S} = R$.

Proposition 2.3.7. *$R^{\mathrm{Int}\, S}$ is a subring of S. Moreover $R^{\mathrm{Int}\, S}$ is integrally closed in S.*

Proof. Example sheet 2. ☐

Lemma 2.3.8. *Let $(K,|\cdot|)$ be a non-archimedean valued field. Then $\mathcal{O}_K$ is integrally closed in K.*

Proof. Let $x \in K$ be integral over $\mathcal{O}_K$, and without loss of generality $x \neq 0$. Let $f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in \mathcal{O}_K[X]$ such that $f(x) = 0$. Then $x = -a_{n-1} - \dots - a_0/x^{n-1}$. If $|x| > 1$, we have $\left|-a_{n-1} - \dots - a_0/x^{n-1}\right| \leq 1$, a contradiction. Thus $|x| \leq 1$, so $x \in \mathcal{O}_K$. ☐

Proof of Theorem 2.3.1.

1. We show $|\cdot|_L = |\mathrm{N}_{L/K}(\cdot)|$ satisfies the three axioms in the definition of absolute values.

    1. $|y|_L = 0$ if and only if $|\mathrm{N}_{L/K}(y)| = 0$, if and only if $\mathrm{N}_{L/K}(y) = 0$, if and only if $y = 0$, by property of $\mathrm{N}_{L/K}$.
    2. $|y_1 y_2|_L = |\mathrm{N}_{L/K}(y_1 y_2)| = |\mathrm{N}_{L/K}(y_1)\mathrm{N}_{L/K}(y_2)| = |\mathrm{N}_{L/K}(y_1)||\mathrm{N}_{L/K}(y_2)| = |y_1|_L |y_2|_L$.
    3. Set $\mathcal{O}_L = \{y \in L \mid |y|_L \le 1\}$. Claim that $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ inside L.
        - Let $0 \ne y \in \mathcal{O}_L$ and let $f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in K[X]$ be the minimal polynomial of y. By property of $\mathrm{N}_{L/K}$, there exists $m \ge 1$ such that $\mathrm{N}_{L/K}(y) = \pm a_0^m$. By Corollary 2.1.5, we have $|a_i| \le \max\left(|\mathrm{N}_{L/K}(y)|^{1/m}, 1\right) = 1$, since $|\mathrm{N}_{L/K}(y)| \le 1$. Thus $a_i \in \mathcal{O}_K$ for all i, so $f \in \mathcal{O}_K[X]$, so y is integral over $\mathcal{O}_K$.
        - Conversely let $y \in L$ be integral over $\mathcal{O}_K$. Again by property of $\mathrm{N}_{L/K}$, we have
        $$\mathrm{N}_{L/K}(y) = \left(\prod_{\sigma : L \to \overline{K}} \sigma(y)\right)^d, \qquad d \ge 1,$$
        where $\overline{K}$ is an algebraic closure of K and σ runs over K-algebra homomorphisms. For all such $\sigma : L \to \overline{K}$, $\sigma(y)$ is integral over $\mathcal{O}_K$. Thus $\mathrm{N}_{L/K}(y) \in K$ is integral over $\mathcal{O}_K$. By Lemma 2.3.8, $\mathrm{N}_{L/K}(y) \in \mathcal{O}_K$, so $|\mathrm{N}_{L/K}(y)| \le 1$, so $y \in \mathcal{O}_L$.

    Thus $\mathcal{O}_K^{\mathrm{Int}\, L} = \mathcal{O}_L$ and proves the claim. Now we prove 3. Let $x, y \in L$. Without loss of generality assume $|x|_L \le |y|_L$, then $|x/y|_L \le 1$, so $x/y \in \mathcal{O}_L$. Since $1 \in \mathcal{O}_L = \mathcal{O}_K^{\mathrm{Int}\, L}$, we have $1 + x/y \in \mathcal{O}_L$ and hence $|1 + x/y|_L \le 1$, so $|x + y|_L \le |y|_L = \max(|y|_L, |x|_L)$. Thus 3 is satisfied. If $|\cdot|_L'$ is another absolute value on L extending $|\cdot|$, then note that $|\cdot|_L$ and $|\cdot|_L'$ are norms on L. By Theorem 2.3.5, $|\cdot|_L'$ and $|\cdot|_L$ induce the same topology on L, so $|\cdot|_L' = |\cdot|_L^c$ for some $c > 0$. Since $|\cdot|_L'$ extends $|\cdot|$, we have $c = 1$.

2. Since $|\cdot|_L$ defines a norm on K, Theorem 2.3.5 implies L is complete with respect to $|\cdot|_L$.

□

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Corollary 2.3.9. *Let $(K, |\cdot|)$ be a complete non-archimedean discretely valued field and L/K a finite extension. Then*

1. *L is discretely valued with respect to $|\cdot|_L$, and*
2. *$\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.*

Proof.

1. Let v be a valuation on K, and let v_L be a valuation on L such that v_L extends v. If $y \in L^\times$, then $|y|_L = |\mathrm{N}_{L/K}(y)|^{1/n}$ for $n = [L : K]$, so $v_L(y) = (1/n)\, v\left(\mathrm{N}_{L/K}(y)\right)$. Thus $v_L(L^\times) \subseteq (1/n)\, v(K^\times)$, so v_L is discrete.
2. Proved in in the last lecture.

□

Corollary 2.3.10. *Let $(K, |\cdot|)$ be a complete non-archimedean discretely valued field and $\overline{K}/K$ an algebraic closure. Then $|\cdot|$ extends to a unique absolute value $|\cdot|_{\overline{K}}$ on $\overline{K}$.*

Proof. If $x \in \overline{K}$, then $x \in L$ for some L/K finite. Define $|x|_{\overline{K}} = |x|_L$. Well-defined, that is independent of L, by the uniqueness in Theorem 2.3.1. The axioms for $|\cdot|_{\overline{K}}$ to be an absolute value can be checked over finite extensions. Uniqueness is clear. □

Remark. $|\cdot|_{\overline{K}}$ on $\overline{K}$ is never discrete. For example, if $K = \mathbb{Q}_p$, then $\sqrt[n]{p} \in \overline{\mathbb{Q}_p}$ for all $n \in \mathbb{N}_{>0}$, so $\mathrm{v}_p\left(\sqrt[n]{p}\right) = (1/n)\, \mathrm{v}_p(p) = 1/n$. Then $\overline{\mathbb{Q}_p}$ is not complete with respect to $|\cdot|_{\overline{\mathbb{Q}_p}}$. By example sheet 2, if $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}_p}$ with respect to $|\cdot|_{\overline{\mathbb{Q}_p}}$, then $\mathbb{C}_p$ is algebraically closed.

3 Local fields

Definition 3.0.1. Let $(K,|\cdot|)$ be a valued field. Then K is a **local field** if it is complete and locally compact.

Example. $\mathbb{R}$ and $\mathbb{C}$ are local fields.

3.1 Non-archimedean local fields

Proposition 3.1.1. *Let $(K,|\cdot|)$ be a non-archimedean complete valued field. The following are equivalent.*

1. *K is locally compact.*
2. *$\mathcal{O}_K$ is compact.*
3. *v is discrete and $k = \mathcal{O}_K/\mathfrak{m}$ is finite.*

Proof.

$1 \implies 2$. Let $U \ni 0$ be a compact neighbourhood of zero. Then there exists $x \in \mathcal{O}_K$ such that $x\mathcal{O}_K \subseteq U$. Since $x\mathcal{O}_K$ is closed, $x\mathcal{O}_K$ is compact, so $\mathcal{O}_K$ is compact, since $x^{-1} : x\mathcal{O}_K \to \mathcal{O}_K$ is homeomorphism.

$2 \implies 1$. If $\mathcal{O}_K$ is compact, then $a + \mathcal{O}_K$ compact for all $a \in K$, so K is locally compact.

$2 \implies 3$. Let $x \in \mathfrak{m}$, and $A_x \subseteq \mathcal{O}_K$ be a set of coset representatives for $\mathcal{O}_K/x\mathcal{O}_K$. Then

$$\mathcal{O}_K = \bigcup_{y \in A_x} (y + x\mathcal{O}_K)$$

is a disjoint open cover, so A_x is finite by compactness of $\mathcal{O}_K$, so $\mathcal{O}_K/x\mathcal{O}_K$ is finite, so $\mathcal{O}_K/\mathfrak{m}$ is finite. Suppose v is not discrete. Let $x = x_1, x_2, \dots$ such that $v(x_1) > v(x_2) > \dots > 0$. Then $x_1\mathcal{O}_K \subsetneq x_2\mathcal{O}_K \subsetneq \dots \subsetneq \mathcal{O}_K$. But $\mathcal{O}_K/x\mathcal{O}_K$ is finite so can only have finitely many subgroups, a contradiction.

$3 \implies 2$. Since $\mathcal{O}_K$ is a metric space, it suffices to show $\mathcal{O}_K$ is sequentially compact. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{O}_K$ and fix $\pi \in \mathcal{O}_K$ a uniformiser in $\mathcal{O}_K$. Since $\pi^i\mathcal{O}_K/\pi^{i+1}\mathcal{O}_K \cong k$, $\mathcal{O}_K/\pi^i\mathcal{O}_K$ is finite for all i, since $\mathcal{O}_K \supseteq \dots \supseteq \pi^i\mathcal{O}_K$. Since $\mathcal{O}_K/\pi\mathcal{O}_K$ is finite, there exists $a_1 \in \mathcal{O}_K/\pi\mathcal{O}_K$ and a subsequence $(x_{1,n})_{n=1}^{\infty}$ such that $x_{1,n} \equiv a_1 \mod \pi$. We define $y_1 = x_{1,1}$. Since $\mathcal{O}_K/\pi^2\mathcal{O}_K$ is finite, there exists $a_2 \in \mathcal{O}_K/\pi^2\mathcal{O}_K$ and a subsequence $(x_{2,n})_{n=1}^{\infty}$ of $(x_{1,n})_{n=1}^{\infty}$ such that $x_{2,n} \equiv a_2 \mod \pi^2$. Define $y_2 = x_{2,2}$. Continuing in this fashion, we obtain sequences $(x_{i,n})_{n=1}^{\infty}$ for $i = 1, 2, \dots$ such that

- $(x_{i+1,n})_{n=1}^{\infty}$ is a subsequence of $(x_{i,n})_{n=1}^{\infty}$, and
- for any i, there exists $a_i \in \mathcal{O}_K/\pi^i\mathcal{O}_K$ such that $x_{i,n} \equiv a_i \mod \pi^i$ for all n.

Then necessarily $a_i \equiv a_{i+1} \mod \pi^i$ for all i. Now choose $y_i = x_{ii}$. This defines a subsequence $(y_n)_{n=1}^{\infty}$. Moreover $y_i \equiv a_i \equiv a_{i+1} \equiv y_{i+1} \mod \pi^i$. Thus y_i is Cauchy, hence converges by completeness.

$\square$

Example.

- $\mathbb{Q}_p$ is a local field.
- $\mathbb{F}_p((t))$ is a local field.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets or groups or rings and $\phi_n : A_{n+1} \to A_n$ homomorphisms.

Definition 3.1.2. Assume A_n is finite. The **profinite topology** on $A = \varprojlim_n A_n$ is the weakest topology on A such that $A \to A_n$ is continuous for all n, where A_n are equipped with the discrete topology.

Fact. $A = \varprojlim_n A_n$ with profinite topology is compact, totally disconnected, and Hausdorff.

Proposition 3.1.3. *Let K be a local field. Under the isomorphism $\mathcal{O}_K \cong \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$ for $\pi \in \mathcal{O}_K$ a uniformiser, the topology on $\mathcal{O}_K$ coincides with the profinite topology.*

Proof. One checks that the sets

$$B = \{a + \pi^n \mathcal{O}_K \mid n \in \mathbb{N}_{\geq 1},\ a \in A_{\pi^n}\},$$

where A_{π^n} is a set of coset representatives for $\mathcal{O}_K/\pi^n\mathcal{O}_K$, is a basis of open sets in both topologies. For $|\cdot|$, this is clear. For the profinite topology, $\mathcal{O}_K \to \mathcal{O}_K/\pi^n\mathcal{O}_K$ is continuous if and only if $a + \pi^n\mathcal{O}_K$ is open for all $a \in A_{\pi^n}$. Thus B is a basis for the profinite topology. □

Remark. This gives another proof that $\mathcal{O}_K$ is compact.

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Lemma 3.1.4. *Let K be a non-archimedean local field and L/K a finite extension. Then L is a local field.*

Proof. By Theorem 2.3.1, L is complete and discretely valued. It suffices to show $k_L = \mathcal{O}_L/\mathfrak{m}_L$ is finite. Let $\alpha_1, \dots, \alpha_n$ be a basis for L as a K-vector space. The sup norm $\|\cdot\|_{\sup}$ is equivalent to $|\cdot|_L$ implies there exists $r > 0$ such that $\mathcal{O}_L \subseteq \left\{x \in L \;\middle|\; \|x\|_{\sup} \leq r\right\}$. Take $a \in K$ such that $|a| \geq r$, then $\mathcal{O}_L \subseteq \bigoplus_{i=1}^n a\alpha_i\mathcal{O}_K$, so $\mathcal{O}_L$ is finitely generated as a module over $\mathcal{O}_K$. Thus k_L is finitely generated over k. □

Theorem 3.1.5. *Let K be a local field. Then either*

- *$K \cong \mathbb{R}$ or $K \cong \mathbb{C}$,*
- *K is a finite extension of $\mathbb{Q}_p$, or*
- *$K \cong \mathbb{F}_{p^n}((t))$ for p prime and $n \geq 1$.*

Definition 3.1.6. A discretely valued field $(K, |\cdot|)$ has **equal characteristic** if $\operatorname{ch} K = \operatorname{ch} k$. Otherwise it has **mixed characteristic**.

Example. $\operatorname{ch} \mathbb{Q}_p = 0$ and $\operatorname{ch} \mathbb{F}_p = p$, so $\mathbb{Q}_p$ has mixed characteristic.

Note that if K is a non-archimedean local field, $\operatorname{ch} k = p > 0$ and hence K has equal characteristic if $\operatorname{ch} K = p$, or mixed characteristic if $\operatorname{ch} K = 0$.

Theorem 3.1.7. *Let K be a non-archimedean local field of equal characteristic $p > 0$. Then $K \cong \mathbb{F}_{p^n}((t))$ for some $n \geq 1$.*

Proof. K is complete discretely valued and $\operatorname{ch} K > 0$. Moreover $k \cong \mathbb{F}_{p^n}$ is finite, hence perfect. By Theorem 2.2.7, $K \cong \mathbb{F}_{p^n}((t))$. □

3.2 Witt vectors*

For motivation, consider $\mathbb{Z}_p$. Let $x = \sum_{i=0}^{\infty} [x_i] p^i \in \mathbb{Z}_p$ and $y = \sum_{i=0}^{\infty} [y_i] p^i \in \mathbb{Z}_p$ for $x_i, y_i \in \mathbb{F}_p$. Suppose $x + y = s = \sum_{i=0}^{\infty} [s_i] p^i$. Can we write s_i in terms of x_j and y_j? Reducing modulo p we obtain

$$x_0 + y_0 = s_0 \in \mathbb{F}_p,$$

so s_0 is determined by x_0 and y_0. What about s_1? Reducing modulo p^2, $[x_0]+[y_0]+p[x_1]+p[y_1] \equiv [s_0]+p[s_1] \mod p^2$, so

$$p[s_1] \equiv [x_0] + [y_0] - [s_0] + p[x_1] + p[y_1] \mod p^2,$$

and $[x_0] + [y_0] - [s_0] \in p\mathbb{Z}_p$. So we need $[x_0] + [y_0] - [s_0]$ modulo p^2. Note $\left[x_0^{1/p}\right] + \left[y_0^{1/p}\right] \equiv \left[s_0^{1/p}\right] \mod p$, so by Lemma 2.2.4

$$[s_0] \equiv \left(\left[x_0^{\frac{1}{p}}\right] + \left[y_0^{\frac{1}{p}}\right]\right)^p \equiv [x_0] + [y_0] + \sum_{d=1}^{p-1} \binom{p}{d} \left[x_0^{\frac{d}{p}}\right] \left[y_0^{\frac{p-d}{p}}\right] \mod p^2.$$

Thus

$$s_1 = x_1 + y_1 - \sum_{d=1}^{p-1} \frac{1}{p} \binom{p}{d} \left[x_0^{\frac{d}{p}}\right] \left[y_0^{\frac{p-d}{p}}\right].$$

Can find similar expressions for $s_2, s_3, \dots$. Witt noticed the general pattern.

Definition 3.2.1. The **n-th Witt polynomial** w_n is defined by

$$\mathrm{w}_n\left(X_0, \ldots, X_n\right)=\sum_{i=0}^n p^i X_i^{p^{n-i}} \in \mathbb{Z}\left[X_0, \ldots, X_n\right].$$

Define $\mathrm{S}_n \in \mathbb{Q}\left[X_0, Y_0, \ldots, X_n, Y_n\right]$ inductively by the equation

$$\mathrm{w}_n\left(\mathrm{S}_0, \ldots, \mathrm{S}_n\right)=\mathrm{w}_n\left(X_0, \ldots, X_n\right)+\mathrm{w}_n\left(Y_0, \ldots, Y_n\right),$$

where the only term containing S_n is $p^n \mathrm{S}_n$.

Fact (Witt)**.** $\mathrm{S}_n \in \mathbb{Z}\left[X_0, Y_0, \ldots, X_n, Y_n\right]$.

Example. $\mathrm{S}_0=X_0+Y_0$ and

$$\mathrm{S}_1=X_1+Y_1+\sum_{d=1}^{p-1} \frac{1}{p}\binom{p}{d} X_0^d Y_0^{p-d}.$$

Theorem 3.2.2. *Suppose that*

$$\sum_{i=0}^{\infty}\left[x_i\right] p^i+\sum_{i=0}^{\infty}\left[y_i\right] p^i=\sum_{i=0}^{\infty}\left[s_i\right] p^i \in \mathbb{Z}_p.$$

Then we have

$$s_n=\mathrm{S}_n\left(x_0^{\frac{1}{p^n}}, y_0^{\frac{1}{p^n}}, \ldots, x_n, y_n\right).$$

Proof. Example sheet 2. A hint is Lemma 2.2.4. ☐

Similarly, defines $\mathrm{Z}_n \in \mathbb{Q}\left[X_0, Y_0, \ldots, X_n, Y_n\right]$ by

$$\mathrm{w}_n\left(\mathrm{Z}_0, \ldots, \mathrm{Z}_n\right)=\mathrm{w}_n\left(X_0, \ldots, X_n\right) \mathrm{w}_n\left(Y_0, \ldots, Y_n\right),$$

Fact (Witt)**.** $\mathrm{Z}_n \in \mathbb{Z}\left[X_0, Y_0, \ldots, X_n, Y_n\right]$.

We have

$$\sum_{i=0}^{\infty}\left[x_i\right] p^i \sum_{i=0}^{\infty}\left[y_i\right] p^i=\sum_{i=0}^{\infty}\left[z_i\right] p^i,$$

where

$$z_n=\mathrm{Z}_n\left(x_0^{\frac{1}{p^n}}, y_0^{\frac{1}{p^n}}, \ldots, x_n, y_n\right).$$

The conclusion is that the ring structure on $\mathbb{Z}_p$ can be reconstructed from the arithmetic of $\mathbb{F}_p$.

Definition 3.2.3. A ring A is a **strict p-ring** if it is p-adically complete, p is not a zero divisor in A, and A/pA is a perfect ring of characteristic p.

Theorem 3.2.4 (Existence of Witt vectors)**.** *Let R be a perfect ring of characteristic p.*

1. *There exists a strict p-ring $\mathrm{W}(R)$, called the* ***Witt vectors*** *of R, such that $\mathrm{W}(R)/p\mathrm{W}(R) \cong R$ which is unique up to isomorphism.*

2. *If R' is another perfect ring and $f: R \rightarrow R'$ is a ring homomorphism. Then there exists a unique ring homomorphism $F: \mathrm{W}(R) \rightarrow \mathrm{W}(R')$ such that the diagram*

$$\begin{tikzcd}
\mathrm{W}(R) \arrow[r, "F"] \arrow[d] & \mathrm{W}(R') \arrow[d] \\
R \arrow[r, "f"'] & R'
\end{tikzcd}$$

   commutes, so $\mathrm{W}(R)$ is the mixed characteristic analogue of $R[[t]]$.

Proof. See Rabinoff's The theory of Witt vectors.

1. Define
$$\mathrm{W}(R) = \{(a_n)_{n=0}^\infty \mid a_n \in R\}.$$
Define addition and multiplication by $(a_n)_{n=0}^\infty + (b_n)_{n=0}^\infty = (s_n)_{n=0}^\infty$ and $(a_n)_{n=0}^\infty (b_n)_{n=0}^\infty = (z_n)_{n=0}^\infty$ where
$$s_n = \mathrm{S}_n(a_0, b_0, \dots, a_n, b_n), \qquad z_n = \mathrm{Z}_n(a_0, b_0, \dots, a_n, b_n).$$
Check this defines a ring structure. For $a = (a_0, a_1, \dots) \in \mathrm{W}(R)$, we compute
$$pa = (0, a_0^p, a_1^p, \dots),$$
so p is not a zero divisor. Moreover
$$\mathrm{W}(R)/p^i\mathrm{W}(R) = \left\{(a_n)_{n=0}^{i-1} \;\middle|\; a_n \in R\right\}.$$
Compute explicitly
$$\mathrm{W}(R) \cong \varprojlim_i \mathrm{W}(R)/p^i\mathrm{W}(R).$$

2. For $f : R \to R'$, define
$$\begin{array}{rccc} F & : & \mathrm{W}(R) & \longrightarrow & \mathrm{W}(R') \\ & & (a_0, a_1, \dots) & \longmapsto & (f(a_0), f(a_1), \dots) \end{array}.$$

$\square$

Remark. If $R = \mathbb{F}_p$, then $\mathrm{W}(\mathbb{F}_p) \cong \mathbb{Z}_p$. The isomorphism is given by
$$(a_0, a_1, \dots) \mapsto \sum_{i=0}^\infty \left[a_i^{\frac{1}{p^i}}\right] p^i.$$

Proposition 3.2.5. *Let $(K, |\cdot|)$ be a complete discretely valued field such that $p \in \mathcal{O}_K$ is a uniformiser and $k = \mathcal{O}_K/\mathfrak{m}$ is perfect. Then $\mathcal{O}_K \cong \mathrm{W}(k)$.*

Proof. By uniqueness of $\mathrm{W}(k)$, it suffices to check that $\mathcal{O}_K$ is a strict p-ring. This is clear from properties of $\mathcal{O}_K$. $\square$

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Remark. Let k be a perfect field. If $K = \mathrm{Frac}\,\mathrm{W}(k)$, then K is a complete discretely valued field with $\mathcal{O}_K \cong \mathrm{W}(k)$ and $p = \mathrm{ch}\,k \in \mathcal{O}_K$ is a uniformiser.

Proposition 3.2.6. *Let $(K, |\cdot|)$ be a complete discretely valued field with $k = \mathcal{O}_K/\mathfrak{m}$ perfect of characteristic p, then $\mathcal{O}_K$ is finite over $\mathrm{W}(k)$.*

Proof. Consider the subset $R \subseteq \mathcal{O}_K$ defined by
$$R = \left\{\sum_{i=0}^\infty [a_i]\, p^i \;\middle|\; a_i \in k\right\}.$$
Calculating as in the example of $\mathbb{Z}_p$ shows that $R \cong \mathrm{W}(k)$. Let π be a uniformiser in $\mathcal{O}_K$ and let $e \in \mathbb{N}$ such that $ev(\pi) = v(p)$. Let
$$M = \bigoplus_{i=0}^{e-1} \pi^i R \subseteq \mathcal{O}_K,$$
an R-submodule. Since $\sum_{n=0}^\infty [x_n]\, \pi^n \equiv \sum_{n=0}^{e-1} [x_n]\, \pi^n \mod p$, M generates $\mathcal{O}_K/p\mathcal{O}_K$ as an R-module, so $\mathcal{O}_K = M + p\mathcal{O}_K$. Iterating, $\mathcal{O}_K = M + \dots + p^{m-1}M + p^m\mathcal{O}_K = M + p^m\mathcal{O}_K$, so $M \to \mathcal{O}_K/p^m\mathcal{O}_K$ is surjective for all m. Then since $M \cong \varprojlim_n M/p^nM$, we have $M \to \mathcal{O}_K \cong \varprojlim_n \mathcal{O}_K/p^n\mathcal{O}_K$ is surjective. Thus $M = \mathcal{O}_K$. $\square$

Theorem 3.2.7. *Let K be a non-archimedean local field of mixed characteristic. Then K is a finite extension of $\mathbb{Q}_p$.*

Proof. Let $k = \mathbb{F}_{p^n}$ for some prime p. Then by Proposition 3.2.6, K is a finite extension of $\operatorname{Frac} \mathrm{W}(\mathbb{F}_{p^n})$. It suffices to show that $\mathrm{W}(\mathbb{F}_{p^n})$ is finite over $\mathbb{Z}_p$. Let $e_1, \dots, e_n \in \mathbb{F}_{p^n}$ be a basis of $\mathbb{F}_{p^n}$ as an $\mathbb{F}_p$-vector space, and we write

$$M = \bigoplus_{i=1}^{n} \mathrm{W}(\mathbb{F}_p)[e_i] \subseteq \mathrm{W}(\mathbb{F}_{p^n}),$$

a $\mathrm{W}(\mathbb{F}_p)$-submodule. For $x = \sum_{i=0}^{\infty} [x_i] p^i \in \mathrm{W}(\mathbb{F}_{p^n})$, let $x_0 = \sum_{i=1}^{n} \lambda_i e_i$ for $\lambda_i \in \mathbb{F}_p$. Then $x - \sum_{i=1}^{n} [\lambda_i][e_i] \in p\mathrm{W}(\mathbb{F}_{p^n})$, since $[\lambda_i] \in \mathrm{W}(\mathbb{F}_p)$ by commutativity of

$$\begin{tikzcd} \mathbb{F}_p \arrow[r, "{[\cdot]}"] \arrow[d] & \mathrm{W}(\mathbb{F}_p) \arrow[d] \\ \mathbb{F}_{p^n} \arrow[r, "{[\cdot]}"'] & \mathrm{W}(\mathbb{F}_{p^n}) \end{tikzcd},$$

so $\mathrm{W}(\mathbb{F}_{p^n}) = M + p\mathrm{W}(\mathbb{F}_{p^n})$. Arguing as in Proposition 3.2.6 shows $M = \mathrm{W}(\mathbb{F}_{p^n})$. ☐

3.3 Classification of local fields

We consider the archimedean case.

Lemma 3.3.1. *An absolute value $|\cdot|$ on a field is non-archimedean if and only if $|n|$ is bounded for all $n \in \mathbb{Z}$.*

Proof.

$\implies$ Since $|-1| = 1, |-n| = |n|$, thus it suffices to show that $|n|$ is bounded for $n \geq 1$. Then $|n| = |1 + \dots + 1| \leq 1$.

$\impliedby$ Suppose $|n| \leq B$ for all $n \in \mathbb{Z}$. Let $x, y \in K$ with $|x| \leq |y|$. Then we have

$$|x+y|^m = \left| \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i} \right| \leq \sum_{i=0}^{m} \left| \binom{m}{i} x^i y^{m-i} \right| \leq |y|^m (m+1) B.$$

Taking m-th roots gives

$$|x+y| \leq |y| |(m+1) B|^{\frac{1}{m}},$$

and $|(m+1) B|^{1/m} \to 1$ as $m \to \infty$. Thus $|x+y| \leq |y| = \max(|x|, |y|)$.

☐

Corollary 3.3.2. *If $(K, |\cdot|)$ is a valued field with $\operatorname{ch} K > 0$, then K is non-archimedean.*

Theorem 3.3.3 (Ostrowski's theorem)**.** *Any non-trivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or the p-adic absolute value $|\cdot|_p$ for some prime p.*

Proof.

Case 1. $|\cdot|$ is archimedean. We fix $b > 1$ an integer such that $|b| > 1$, which exists by Lemma 3.3.1. Let $a > 1$ be an integer and write b^n in base a, so $b^n = c_m a^m + \dots + c_0$ for $0 \leq c_i < a$. Let $B = \max_{0 \leq c < a} |c|$, then we have $|b^n| \leq (m+1) B \max(|a|^m, 1)$, so

$$|b| \leq ((n \log_a b + 1) B)^{\frac{1}{n}} \max\left(|a|^{\log_a b}, 1\right),$$

and $((n \log_a b + 1) B)^{1/n} \to 1$ as $n \to \infty$, so $|b| \leq \max\left(|a|^{\log_a b}, 1\right)$. Then $|a| > 1$ and

$$|b| \leq |a|^{\log_a b}. \tag{1}$$

Switching the roles of a and b, we obtain

$$|a| \leq |b|^{\log_b a}. \tag{2}$$

By (1) and (2),

$$\frac{\log|a|}{\log a} = \frac{\log|b|}{\log b} = \lambda \in \mathbb{R}_{>0},$$

using $\log_a b = \log b / \log a$, so $|a| = a^\lambda$ for all $a \in \mathbb{Z}$ such that $a > 1$, so $|x| = |x|_\infty^\lambda$ for all $x \in \mathbb{Q}$. Hence $|\cdot|$ is equivalent to $|\cdot|_\infty$.

Case 2. $|\cdot|$ is non-archimedean. As in Lemma 3.3.1, we have $|n| \leq 1$ for all $n \in \mathbb{Z}$. Since $|\cdot|$ is non-trivial, there exists $n \in \mathbb{Z}_{>1}$ such that $|n| < 1$. Write $n = p_1^{e_1} \dots p_r^{e_r}$, a decomposition into prime factors. Then $|p| < 1$ for some $p \in \{p_1, \dots, p_r\}$. Suppose $|q| < 1$ for some prime q such that $q \neq p$. Write $1 = rp + sq$ for $r, s \in \mathbb{Z}$. Then $1 = |rp + sq| \leq \max(|rp|, |sq|) < 1$, a contradiction. Thus $|p| = \alpha < 1$ and $|q| = 1$ for all primes $q \neq p$, so $|\cdot|$ is equivalent to $|\cdot|_p$.

□

Theorem 3.3.4. *Let $(K, |\cdot|)$ be an archimedean local field. Then $K = \mathbb{R}$ or $K = \mathbb{C}$ and $|\cdot|$ is equivalent to the usual absolute value $|\cdot|_\infty$.*

Proof. If $\operatorname{ch} K > 0$, then K is non-archimedean by Corollary 3.3.2. Therefore $\operatorname{ch} K = 0$, and hence $\mathbb{Q} \subseteq K$. Since $|\cdot|$ is archimedean, $|\cdot||_{\mathbb{Q}}$ is equivalent to $|\cdot|_\infty$ by Ostrowski. Therefore, since K is complete, we have $\mathbb{R} \subseteq K$.

- We first consider the case $\mathbb{C} \subseteq K$. Then by uniqueness of extensions of absolute values, $|\cdot||_{\mathbb{C}}$ is equivalent to $|\cdot|_\infty$. Suppose $\alpha \in K \setminus \mathbb{C}$. Then $f(x) = |x - \alpha|$ is a continuous function on $\mathbb{C}$, hence attains a lower bound at $b \in \mathbb{C}$ say, since $\mathbb{C} \subseteq K$ is closed. Set $\beta = \alpha - b$ and we let $c \in \mathbb{C}$ such that $0 < |c| < |\beta|$. We have $|\beta - a| \geq |\beta|$ for all $a \in \mathbb{C}$. Hence

$$\frac{|\beta - c|}{|\beta|} \leq \frac{|\beta - c|}{|\beta|} \prod_{\zeta^n = 1,\ \zeta \neq 1} \frac{|\beta - \zeta c|}{|\beta|} = \frac{|\beta^n - c^n|}{|\beta|^n} = \left|1 - \left(\frac{c}{\beta}\right)^n\right| \to 1,$$

as $n \to \infty$, since $|c/\beta| < 1$ implies that $(c/\beta)^n \to 0$. Then $|\beta - c| \leq |\beta|$, so $|\beta - c| = |\beta|$. Replacing β by $\beta - c$ and iterating, we obtain $|\beta - mc| = |\beta|$ for all $m \in \mathbb{N}$, so

$$|m||c| = |mc| \leq |\beta - mc| + |\beta| = 2|\beta|.$$

This contradicts Lemma 3.3.1, hence $K = \mathbb{C}$.

- Now suppose K does not contain $\mathbb{C}$. Define $L = K(i)$ where $i^2 = -1$. Can extend $|\cdot|$ to an absolute value $|\cdot|_L$ on L given by

$$|a + ib|_L = \sqrt{|a|^2 + |b|^2}, \qquad a, b \in K.$$

Applying the above argument gives $K(i) = L = \mathbb{C}$, hence $K = \mathbb{R}$.

□

Proof of Theorem 3.1.5.

- $|\cdot|$ archimedean is Theorem 3.3.4.
- $|\cdot|$ non-archimedean and $\operatorname{ch} K = 0$ is Theorem 3.2.7.
- $|\cdot|$ non-archimedean and $\operatorname{ch} K > 0$ is Theorem 3.1.7.

□

3.4 Global fields

Lecture 10
Friday
30/10/20

Definition 3.4.1. A **global field** is a field which is either

- an algebraic number field, or
- a **global function field**, the rational function field of an algebraic curve over a finite field, or equivalently a finite extension of $\mathbb{F}_p(t)$.

We mainly focus on the number field. We show that local fields are completions of global fields.

Lemma 3.4.2. *Let $(K,|\cdot|)$ be a complete discretely valued field and L/K a Galois extension and $|\cdot|_L$ the unique extension of $|\cdot|$ to L. Then for $x \in L$ and $\sigma \in \mathrm{Gal}(L/K)$, we have $|\sigma(x)|_L = |x|_L$.*

Proof. Since $x \mapsto |\sigma(x)|_L$ is also another absolute value on L extending $|\cdot|$ on K, Lemma 3.4.2 follows from uniqueness of $|\cdot|_L$. $\square$

Lemma 3.4.3 (Krasner's lemma)**.** *Let $(K,|\cdot|)$ a complete discretely valued field. Let $f(X) \in K[X]$ be a separable irreducible polynomial with roots $\alpha_1, \dots, \alpha_n \in \overline{K}$, the separable closure of K. Suppose $\beta \in \overline{K}$ with $|\beta - \alpha_1| < |\beta - \alpha_i|$ for $i = 2, \dots, n$. Then $\alpha_1 \in K(\beta)$.*

Proof. Let $L = K(\beta)$ and $L' = L(\alpha_1, \dots, \alpha_n)$. Then L'/L is a Galois extension. Let $\sigma \in \mathrm{Gal}(L'/L)$. We have $|\beta - \sigma(\alpha_1)| = |\sigma(\beta - \alpha_1)| = |\beta - \alpha_1|$, by Lemma 3.4.2. Thus $\sigma(\alpha_1) = \alpha_1$, so $\alpha_1 \in K(\beta)$. $\square$

Proposition 3.4.4 (Nearby polynomials define the same extension)**.** *Let $(K,|\cdot|)$ be a complete discretely valued field and $f(X) = \sum_{i=0}^n a_i X^i \in \mathcal{O}_K[X]$ be a separable irreducible monic polynomial. Let $\alpha \in \overline{K}$ be a root of f. Then there exists $\epsilon > 0$ such that for any $g(X) = \sum_{i=0}^n b_i X^i \in \mathcal{O}_K[X]$ monic with $|a_i - b_i| < \epsilon$, there exists a root β of $g(X)$ such that $K(\alpha) = K(\beta)$.*

Proof. Let $\alpha = \alpha_1, \dots, \alpha_n \in \overline{K}$ be the roots of f which are necessarily distinct. Then $f'(\alpha) \neq 0$. We choose ϵ sufficiently small such that $|g(\alpha_1)| < |f'(\alpha_1)|^2$ and $|f'(\alpha_1) - g'(\alpha_1)| < |f'(\alpha_1)|$. Then we have $|g(\alpha_1)| < |f'(\alpha_1)|^2 = |g'(\alpha_1)|^2$. By Hensel's lemma applied to the field $K(\alpha_1)$, there exists $\beta \in K(\alpha_1)$ such that $g(\beta) = 0$ and $|\beta - \alpha_1| < |g'(\alpha_1)|$. Then

$$|g'(\alpha_1)| = |f'(\alpha_1)| = \prod_{i=2}^n |\alpha_1 - \alpha_i| \le |\alpha_1 - \alpha_i|, \qquad i = 2, \dots, n,$$

using $|\alpha_1 - \alpha_i| \le 1$. Since $|\beta - \alpha_1| < |g'(\alpha_1)| = |f'(\alpha_1)| \le |\alpha_1 - \alpha_i| = |\beta - \alpha_i|$ for $i = 2, \dots, n$, by Krasner's lemma, $\alpha \in K(\beta)$, so $K(\alpha) = K(\beta)$. $\square$

Theorem 3.4.5. *Let K be a local field, then K is the completion of a global field.*

Proof.

Case 1. $|\cdot|$ is archimedean. Then $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ and $\mathbb{C}$ is the completion of $\mathbb{Q}(i)$ with respect to $|\cdot|_\infty$.

Case 2. $|\cdot|$ is non-archimedean of equal characteristic. Then $K \cong \mathbb{F}_q((t))$, so K is the completion of $\mathbb{F}_q(t)$ with respect to the t-adic absolute value.

Case 3. $|\cdot|$ is non-archimedean of mixed characteristic. Then $K \cong \mathbb{Q}_p(\alpha)$ for α a root of a monic irreducible polynomial $f(X) \in \mathbb{Z}_p[X]$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we choose $g(X) \in \mathbb{Z}[X]$ as in Proposition 3.4.4. Then $K = \mathbb{Q}_p(\beta)$ for β a root of $g(X)$. Since $\beta \in \overline{\mathbb{Q}}$, we have $\mathbb{Q}(\beta) \subseteq \mathbb{Q}_p(\beta) = K$, so K is the completion of $\mathbb{Q}(\beta)$.

$\square$

4 Dedekind domains

The global analogue of a DVR is a Dedekind domain.

4.1 Dedekind domains

Definition 4.1.1. A **Dedekind domain** is a ring R such that

- R is a Noetherian integral domain,
- R is integrally closed in $\operatorname{Frac} R$, and
- Every non-zero prime ideal is maximal.

Example.

- The ring of integers in a number field is a Dedekind domain.
- Any PID, hence DVR, is a Dedekind domain.

Theorem 4.1.2. *A ring R is a DVR if and only if R is a Dedekind domain with exactly one non-zero prime ideal.*

Lemma 4.1.3. *Let R be a Noetherian ring and $I \subseteq R$ a non-zero ideal. Then there exist non-zero prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subseteq R$ such that $\mathfrak{p}_1 \dots \mathfrak{p}_r \subseteq I$.*

Proof. Suppose not. Since R is Noetherian, we may choose I maximal without this property. Then I is not prime, so there exists $x, y \in R \setminus I$ such that $xy \in I$. Let $I_1 = I + \langle x \rangle$ and $I_2 = I + \langle y \rangle$. Then by maximality of I, there exists $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ prime ideals such that $\mathfrak{p}_1 \dots \mathfrak{p}_r \subseteq I_1$ and $\mathfrak{q}_1 \dots \mathfrak{q}_s \subseteq I_2$, so $\mathfrak{p}_1 \dots \mathfrak{p}_r \mathfrak{q}_1 \dots \mathfrak{q}_s \subseteq I_1 I_2 \subseteq I$, a contradiction. □

Lemma 4.1.4. *Let R be an integral domain which is integrally closed in $K = \operatorname{Frac} R$. Let $I \subseteq R$ be a non-zero finitely generated ideal and $x \in K$. Then if $xI \subseteq I$, we have $x \in R$.*

Proof. Let $I = \langle c_1, \dots, c_n \rangle$. We write $xc_i = \sum_{i=1}^n a_{ij} c_i$ for some $a_{ij} \in R$. Let A be the matrix $A = (a_{ij})_{1 \le i,j \le n}$ and set $B = x\mathrm{I}_n - A \in \operatorname{Mat}_{n \times n} K$. Then $B \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}^\intercal = 0$ in K^n. Multiplying by the adjugate matrix for B, $(\det B)\, \mathrm{I}_n \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}^\intercal = 0$, so $\det B = 0$. But $\det B$ is a monic polynomial in x with coefficients in R. Thus x is integral over R, so $x \in R$. □

Proof of Theorem 4.1.2.

$\implies$ Clear.

$\impliedby$ We need to show R is a PID. The assumption implies R is a local ring with unique maximal ideal $\mathfrak{m}$.

Step 1. $\mathfrak{m}$ is principal. Let $0 \neq x \in \mathfrak{m}$. By Lemma 4.1.3, $\langle x \rangle \supseteq \mathfrak{m}^n$ for some $n \geq 1$. Let n be minimal such that $\langle x \rangle \supseteq \mathfrak{m}^n$, then we may choose $y \in \mathfrak{m}^{n-1} \setminus \langle x \rangle$. Set $\pi = x/y$. Then we have $y\mathfrak{m} \subseteq \mathfrak{m}^n \subseteq \langle x \rangle$, so $\pi^{-1}\mathfrak{m} \subseteq R$. If $\pi^{-1}\mathfrak{m} \subseteq \mathfrak{m}$, then $\pi^{-1} \in R$ by Lemma 4.1.4 and $y \in \langle x \rangle$, a contradiction. Hence $\pi^{-1}\mathfrak{m} = R$, so $\mathfrak{m} = \pi R$ is principal.

Step 2. R is a PID. Let $I \subseteq R$ be a non-zero ideal. Consider the sequence of ideals $I \subseteq \pi^{-1} I \subseteq \dots$ in K. Then $\pi^{-k} I \neq \pi^{-(k+1)} I$ for all k by Lemma 4.1.4. Therefore since R is Noetherian, we may choose n maximal such that $\pi^{-n} I \subseteq R$. If $\pi^{-n} I \subseteq \mathfrak{m} = \langle \pi \rangle$, then $\pi^{-(n+1)} I \subseteq R$, a contradiction. Thus $\pi^{-n} I = R$, so $I = \langle \pi^n \rangle$.

□