Algebraic Topology

Lectured by Prof Ivan Smith
Typed by David Kurniadi Angdinata

Michaelmas 2020

Syllabus

Contents

0 Introduction

0.1 Connectedness

Lecture 1 Friday 09/10/20

Algebraic topology concerns the connectivity properties of topological spaces.

Definition. A space X is **path-connected** if for $p, q \in X$, there exists $\gamma : [0,1] \to X$ continuous with $\gamma(0) = p$ and $\gamma(1) = q$.

Example. $\mathbb{R}$ is path-connected, and $\mathbb{R} \setminus \{0\}$ is not.

Corollary 0.1 (Intermediate value theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x < y$ satisfy $f(x) > 0$ and $f(y) > 0$ then f takes the value zero on $[x, y]$.*

Proof. Otherwise $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ disconnect $[x, y]$, a contradiction. □

Definition. Let X and Y be topological spaces. Maps $f_0, f_1 : Y \to X$ are **homotopic** if there exists $F : Y \times [0,1] \to X$ continuous such that $F|_{Y \times \{0\}} = f_0$ and $F|_{Y \times \{1\}} = f_1$. Write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$.

Exercise. $\simeq$ is an equivalence relation on the set of maps from Y to X.

Note that X is **path-connected** if and only if every two maps $\{\text{point}\} \to X$ are homotopic. Let

$$\mathrm{S}^n = \left\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\right\},$$

so $\mathrm{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition. X is **simply-connected** if every two maps $\mathrm{S}^1 \to X$ are homotopic.

Example. $\mathbb{R}^2$ is simply-connected, and $\mathbb{R}^2 \setminus \{0\}$ is not. From complex analysis you know $\gamma : \mathrm{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ has a **winding number** or **degree** $\deg \gamma \in \mathbb{Z}$, for which

- if $\gamma_n(t) = e^{2\pi i n t}$ then $\deg \gamma_n = n$, and
- $\deg \gamma_1 = \deg \gamma_2$ if $\gamma_1 \simeq \gamma_2$.

For differentiable γ, $\deg \gamma = \int_\gamma \frac{1}{z} \, \mathrm{d}z$.

Corollary 0.2 (Fundamental theorem of algebra). *Every non-constant complex polynomial has a root.*

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be non-constant, and without loss of generality monic. Suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$. Let

$$\gamma_R(t) = f\left(Re^{2\pi i t}\right),$$

so $\gamma_R : \mathrm{S}^1 \to \mathbb{R}^2 \setminus \{0\}$. Since γ_0 is constant, $\deg \gamma_0 = 0$, so $\deg \gamma_R = 0$ for all R. But take $R \gg \sum_i |a_i|$. Let

$$f_s(z) = z^n + s\left(a_1 z^{n-1} + \dots + a_n\right), \qquad 0 \le s \le 1.$$

On the circle $|z| = R$, $f_s(z) \neq 0$ for all s. So if

$$\gamma_{R,s}(t) = f_s\left(Re^{2\pi i t}\right),$$

then $\gamma_{R,1} = \gamma_R$, which has degree zero from before, and $\gamma_{R,0} : t \mapsto R^n e^{2\pi i n t}$, which has degree $n \neq 0$, a contradiction. □

Definition. X is k**-connected** if every two maps $\mathrm{S}^i \to X$ are homotopic whenever $i \le k$.

Example. $\mathbb{R}^n$ is $(n-1)$-connected, and $\mathbb{R}^n \setminus \{0\}$ is not. Maps $\mathrm{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ have a homotopy invariant degree in $\mathbb{Z}$, where the degree of the inclusion is one and the degree of the constant map is zero. You may well not have seen this, and we will prove it later.

Corollary 0.3 (Brouwer's theorem)**.** *Any map* $f : \overline{\mathrm{B}^n} = \{x \in \mathbb{R}^n \mid \|x\| \le 1\} \to \overline{\mathrm{B}^n}$ *has a fixed point.*

Proof. Suppose f has no fixed point. Let

$$\gamma_R(v) = Rv - f(Rv), \qquad 0 \le R \le 1, \qquad v \in \mathrm{S}^{n-1} = \partial\overline{\mathrm{B}^n}.$$

Since f has no fixed point, γ_R takes values in $\mathbb{R}^n \setminus \{0\}$. Since γ_0 is constant, $\deg\gamma_0 = 0$, so $\deg\gamma_1 = 0$ by homotopy invariance. Let

$$\gamma_{1,s}(v) = v - sf(v), \qquad 0 \le s \le 1.$$

Then $\gamma_{1,1} = \gamma_1$, and $\operatorname{im}\gamma_{1,s} \subseteq \mathbb{R}^n \setminus \{0\}$ as $\|v\| = 1$ and $\|sf(v)\| = |s|\|f(v)\| < 1$ if $s < 1$, so $\deg\gamma_{1,0} = \deg\gamma_{1,1}$. The inclusion has $\deg\gamma_{1,0} = 1$ and $\deg\gamma_{1,1} = 0$ from above, a contradiction. ☐

0.2 Homotopy

Definition. $f : X \to Y$ is a **homotopy equivalence** if there exists $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Then g is a **homotopy inverse** for f, and $\simeq$ is an equivalence relation on spaces.

Example. If X and Y are homeomorphic they are trivially homotopy equivalent, by taking $g = f^{-1}$.

Example. $\mathbb{R}^n \setminus \{0\} \simeq \mathrm{S}^{n-1}$. Let

$$\begin{array}{cccc} f & : & \mathbb{R}^n \setminus \{0\} & \longrightarrow & \mathrm{S}^{n-1} \\ & & v & \longmapsto & \dfrac{v}{\|v\|} \end{array},$$

and let $g : \mathrm{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion. Then $f \circ g = \mathrm{id}_{\mathrm{S}^{n-1}}$ and $g \circ f \simeq_F \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy

$$F(t, v) = tv + (1-t)\frac{v}{\|v\|}.$$

Example. $\{0\} \simeq \mathbb{R}^n$ is a homotopy equivalence. [1] If $X \simeq \{\text{point}\}$ we say X is **contractible**.

Algebraic topology is the study of topological spaces up to homotopy equivalence. The idea is that homeomorphism is too delicate a relation. Homotopy equivalence keeps track of essential topological information. More precisely, we assign

$$\{\text{spaces}\} \to \{\text{groups}\}, \qquad \{\text{maps of spaces}\} \to \{\text{homomorphism of groups}\},$$

so we get algebraic invariants. They are defined for all spaces, but have more structure and use or interest for nicer spaces. The classical first attempt is homotopy theory. One can concatenate loops γ and τ by

$$(\gamma * \tau)(t) = \begin{cases} \gamma(2t) & t \le \frac{1}{2} \\ \tau(1-2t) & t \ge \frac{1}{2} \end{cases}.$$

This is a well-defined operation on the **fundamental group**

$$\pi_1(X, x_0) = \left\{\text{maps } \gamma : \mathrm{S}^1 \to X \;\middle|\; \gamma(0) = x_0 \text{ fixed}\right\} / (\simeq \text{ preserving } x_0).$$

Similarly, the n**-th homotopy group** is

$$\pi_n(X, x_0) = \{\text{based maps } \mathrm{S}^n \to X \text{ at } x_0\} / \simeq.$$

The issue is that they are very hard to compute, such as $\pi_n\left(\mathrm{S}^2, x_0\right)$ not known for all n. There is no simply-connected **manifold**, a space X locally homeomorphic to $\mathbb{R}^n$, of dimension greater than zero, with $\pi_n(X)$ known for all n. So we will do something else, homology and cohomology. It is algebraically harder to set up, but the computational gain is worth it. Note that computing cohomology of harder spaces, such as the space of diffeomorphisms of some manifold or the space of embeddings of one manifold into another, is still very hard.

Remark.

- Algebraic topology is all about being able to compute. It is important to do lots of examples.
- Our nice spaces are manifolds and indeed smooth manifolds. There is some overlap with differential geometry which will be useful, not essential but advised.

[1] Exercise: check

1 Definition and examples

Lecture 2
Monday
12/10/20

We will define invariants of spaces in two stages.

- Associate to X a chain or cochain complex.
- Take the homology or cohomology of that complex.

1.1 Chain and cochain complexes

Definition. A **chain complex** $(C_\bullet, \partial)$ is a sequence of abelian groups and homomorphisms

$$\dots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots,$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n. We write $\partial^2 = 0$, and ∂ is the **differential** or **boundary map**. The **homology groups** $\mathrm{H}(C_\bullet, \partial)$ are the graded groups

$$\mathrm{H}_n(C_\bullet) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Definition. A **cochain complex** $(C^\bullet, \partial)$ is a sequence of abelian groups and homomorphisms

$$\dots \to C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \to \dots,$$

such that $\partial^n \circ \partial^{n-1} = 0$ for all n. We write $\partial^2 = 0$, and ∂ is still the **differential** or **boundary map**. The **cohomology groups** $\mathrm{H}(C^\bullet, \partial)$ are

$$\mathrm{H}^n(C^\bullet) = \ker \partial^n / \operatorname{im} \partial^{n-1}.$$

Elements of $\ker(\partial : C_n \to C_{n-1})$ are **cycles**. Elements of $\operatorname{im}(\partial : C_{n+1} \to C_n)$ are **boundaries**. Elements of $\ker\left(\partial : C^n \to C^{n+1}\right)$ are **cocycles**. Elements of $\operatorname{im}\left(\partial : C^{n-1} \to C^n\right)$ are **coboundaries**. Write all ∂_i and ∂^i as ∂, or occasionally $\partial_\bullet$ and $\partial^\bullet$. Elements of $\mathrm{H}_\bullet(C_\bullet)$ are **homology classes** and of $\mathrm{H}^\bullet(C^\bullet)$ are **cohomology classes**.

Definition. A **chain map** between chain complexes $(C_\bullet, \partial)$ and $(D_\bullet, \partial)$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that for all n the diagram

$$\begin{tikzcd}
\dots \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & D_n \arrow[r, "\partial"'] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

commutes. That is, $f_{n-1} \circ \partial_n^{C_\bullet} = \partial_n^{D_\bullet} \circ f_n$.

Exercise. Define a **cochain map** of cochain complexes.

Lemma 1.1. *A chain map $f : C_\bullet \to D_\bullet$ induces homomorphisms $(f_*)_n : \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet)$ for each n.*

Proof. Let $[a] \in \mathrm{H}_n(C_\bullet)$, so a is represented by a cycle $\alpha \in C_n$, where $\partial(\alpha) = 0$. Then $\partial(f_n(\alpha)) = f_{n-1}(\partial(\alpha)) = 0$, so $f_n(\alpha)$ is a cycle. Define $(f_*)_n([a]) = [f_n(\alpha)] \in \mathrm{H}_n(D_\bullet)$. We made a choice of representing the cycle α. But if $[a]$ is represented by α and α', then $\alpha - \alpha' \in \operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)$. Say $\alpha - \alpha' = \partial(\tau)$. Then $f_n(\alpha) - f_n(\alpha') = f_n(\alpha - \alpha') = f_n(\partial(\tau)) = \partial(f_{n+1}(\tau))$, so $[f_n(\alpha)] = [f_n(\alpha') + \partial(f_{n+1}(\tau))] = [f_n(\alpha')]$ as $[\operatorname{im} \partial] = 0$ in $\mathrm{H}_n(D_\bullet)$. So $(f_*)_n$ is well-defined, and it is easy to see it is a homomorphism. ☐

Exercise. If $C_\bullet, D_\bullet, E_\bullet$ are chain complexes and $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$ are chain maps then $\{g_n \circ f_n : C_n \to E_n\}_n$ defines a chain map. Also

$$(g \circ f)_* = g_* \circ f_*, \qquad (\mathrm{id}_{C_\bullet})_* = \mathrm{id}_{\mathrm{H}_\bullet(C_\bullet)} \tag{1}$$

The goal is to associate to a space X chain complexes $\mathrm{C}_\bullet(X)$ and cochain complexes $\mathrm{C}^\bullet(X)$ such that a map $f : X \to Y$ yields chain maps $f : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(Y)$ and cochain maps $f : \mathrm{C}^\bullet(Y) \to \mathrm{C}^\bullet(X)$. Then (1) will say we have a functor

$$\begin{array}{rcl} \mathbf{Top} & \longrightarrow & \mathbf{Ab} \\ X & \longmapsto & \mathrm{H}_\bullet(X) \end{array},$$

from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms. Our complexes $\mathrm{C}_\bullet$ and $\mathrm{C}^\bullet$ will have the benefit that they are intrinsic but will be huge and unwieldy. We will

- prove structure theorems to help compute, and
- find smaller complexes later for nice spaces, such as CW-complexes.

1.2 Singular homology and cohomology

Definition. The **standard simplex** is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \forall i,\ t_i \geq 0,\ \sum_i t_i = 1 \right\}.$$

The ***i*-th face** of Δ^n is

$$\Delta_i^n = \{ \underline{t} \in \Delta^n \mid t_i = 0 \}.$$

Note that there exists a canonical homeomorphism

$$\begin{array}{rcrcl} \delta_i & : & \Delta^{n-1} & \longrightarrow & \Delta_i^n \subseteq \Delta^n \\ & & (t_0, \dots, t_{n-1}) & \longmapsto & (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \end{array}.$$

Definition. If X is a space, a **singular n-simplex** in X is a map $\sigma : \Delta^n \to X$. The **singular chain complex** $(\mathrm{C}_\bullet(X), \partial)$ has

$$\mathrm{C}_n(X) = \left\{ \sum_{i=1}^N n_i \sigma_i \;\middle|\; N < \infty,\ n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \right\},$$

the free abelian group on the singular n-simplices in X, and

$$\begin{array}{rcrcl} \partial & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n-1}(X) \\ & & \sigma & \longmapsto & \sum_{i=0}^n (-1)^i (\sigma \circ \delta_i) \end{array},$$

extended linearly.

Example. Δ^0 is a point, Δ^1 is a line, Δ^2 is a triangle, and Δ^3 is a tetrahedron.

Note that $n+1$ ordered points $\{v_i\}_{0 \leq i \leq n} \subseteq \mathbb{R}^{n+1}$ determine an n-simplex if $\{v_i - v_0 \mid 1 \leq i \leq n\}$ are linearly independent, by taking their convex hull, and

$$\begin{array}{rcrcl} \sigma & : & \Delta^n & \longrightarrow & \mathbb{R}^{n+1} \\ & & \underline{t} & \longmapsto & \sum_{i=0}^n t_i v_i \end{array}.$$

We orient the edges $v_i \to v_j$ if $i < j$. Write $[v_0, \dots, v_n]$ for this n-simplex, then

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i \left. \sigma \right|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]},$$

where the index $\widehat{v_i}$ is omitted.

Lemma 1.2. $\partial^2 = 0$.

Proof.

$$\partial\left(\partial\left(\sigma\right)\right) = \sum_{j<i} \left(-1\right)^i \left(-1\right)^j \sigma|_{[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]} + \sum_{j>i} \left(-1\right)^i \left(-1\right)^{j-1} \sigma|_{[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]}.$$

Exchange i and j and the two terms cancel. ☐

Definition. The **singular homology** of X is

$$\mathrm{H}_\bullet\left(X\right) = \mathrm{H}_\bullet\left(X;\mathbb{Z}\right) = \mathrm{H}\left(\mathrm{C}_\bullet\left(X\right),\partial\right).$$

Trivially this is a homeomorphism invariant of X, since we only used the notion of continuous maps to X to define it.

Definition. The **singular cochain complex** $\left(\mathrm{C}^\bullet\left(X\right),\partial^*\right)$ has

$$\mathrm{C}^n\left(X\right) = \mathrm{Hom}\left(\mathrm{C}_n\left(X\right),\mathbb{Z}\right),$$

and

$$\begin{array}{cccc} \partial^* & : & \mathrm{C}^n\left(X\right) & \longrightarrow & \mathrm{C}^{n+1}\left(X\right) \\ & & \psi & \longmapsto & \left(\sigma \mapsto \psi\left(\partial\left(\sigma\right)\right)\right) \end{array}, \qquad \sigma \in \mathrm{C}_{n+1}\left(X\right),$$

which is adjoint to ∂.

Then $\partial^*\left(\partial^*\left(\psi\right)\right)\left(\sigma\right) = \partial^*\left(\psi\right)\left(\partial\left(\sigma\right)\right) = \psi\left(\partial\left(\partial\left(\sigma\right)\right)\right) = 0$, so $\left(\partial^*\right)^2 = 0$ and this is a cochain complex.

Definition. The **singular cohomology** of X is

$$\mathrm{H}^\bullet\left(X;\mathbb{Z}\right) = \mathrm{H}\left(\mathrm{C}^\bullet\left(X\right),\partial^*\right).$$

The following is the rough idea.

- $\partial^2 = 0$ implies that the boundary of the boundary vanishes.
- $\mathrm{H}_i\left(X\right)$ will probe i-dimensional holes or regions in X.
- $\mathrm{H}^i\left(X\right)$ will be a rule associating an integer to an i-dimensional region of X.

Note that $\mathrm{H}^\bullet\left(X;\mathbb{Z}\right) \ncong \mathrm{Hom}\left(\mathrm{H}_\bullet\left(X\right),\mathbb{Z}\right)$ in general. Lecture 3 Wednesday 14/10/20

Remark. Let $f : X \to Y$ be continuous. If $\sigma : \Delta^n \to X$ then $f \circ \sigma : \Delta^n \to Y$, so f gives a homomorphism $\left(f_\#\right)_n : \mathrm{C}_n\left(X\right) \to \mathrm{C}_n\left(Y\right)$. Also $f \circ \left(\sigma|_{\Delta_i^n}\right) \equiv \left(f \circ \sigma\right)|_{\Delta_i^n}$, since $f \circ \left(\sigma \circ \delta_i\right) = \left(f \circ \sigma\right) \circ \delta_i$. Thus

$$\begin{array}{ccccc} f_\# & : & \mathrm{C}_\bullet\left(X\right) & \longrightarrow & \mathrm{C}_\bullet\left(Y\right) \\ & & \sigma & \longmapsto & f \circ \sigma \end{array}$$

is a chain map such that

$$\begin{tikzcd} \mathrm{C}_n\left(X\right) \arrow[r, "\partial"] \arrow[d, "\left(f_\#\right)_n"'] & \mathrm{C}_{n-1}\left(X\right) \arrow[d, "\left(f_\#\right)_{n-1}"] \\ \mathrm{C}_n\left(Y\right) \arrow[r, "\partial"'] & \mathrm{C}_{n-1}\left(Y\right) \end{tikzcd},$$

which gives homomorphisms

$$f_* : \mathrm{H}_\bullet\left(X\right) \to \mathrm{H}_\bullet\left(Y\right),$$

that is $\left(f_*\right)_n : \mathrm{H}_n\left(X\right) \to \mathrm{H}_n\left(Y\right)$ for each n. By the exercise,

$$\left(\left(f \circ g\right)_*\right)_n = \left(f_*\right)_n \circ \left(g_*\right)_n, \qquad \left(\left(\mathrm{id}_{\mathrm{C}_\bullet\left(X\right)}\right)_*\right)_n = \mathrm{id}_{\mathrm{H}_n\left(X\right)}.$$

Note that $f : X \to Y$ induces a cochain map

$$\begin{array}{ccccc} f^\# & : & \mathrm{C}^\bullet\left(Y\right) & \longrightarrow & \mathrm{C}^\bullet\left(X\right) \\ & & \psi & \longmapsto & \left(\sigma \mapsto \psi\left(f \circ \sigma\right)\right) \end{array},$$

and homomorphisms

$$f^* : \mathrm{H}^\bullet\left(Y\right) \to \mathrm{H}^\bullet\left(X\right),$$

so cohomology is contravariant.

1.3 Basic examples

What can we compute?

Lemma 1.3. *Let X be a point. Then*

$$\mathrm{H}_i\left(\{point\}\right) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & otherwise \end{cases}.$$

Proof. For each $n \geq 0$, there exists a unique n-simplex $\sigma_n : \Delta^n \to \{\text{point}\}$ in X, the constant map. Then $\partial\left(\sigma_1\right) = \sigma_1 \circ \delta_0 - \sigma_1 \circ \delta_1 = \sigma_0 - \sigma_0 = 0$ and $\partial\left(\sigma_2\right) = \sigma_2 \circ \delta_0 - \sigma_2 \circ \delta_1 + \sigma_2 \circ \delta_2 = \sigma_1 - \sigma_1 + \sigma_1 = \sigma_1$, and

$$\partial\left(\sigma_n\right) = \begin{cases} \sigma_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

So $\mathrm{C}_\bullet\left(\{\text{point}\}\right)$ is

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{C}_3\left(\{\text{point}\}\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{C}_2\left(\{\text{point}\}\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{C}_1\left(\{\text{point}\}\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{C}_0\left(\{\text{point}\}\right) \arrow[d, phantom, "\cong" rotate=90] \\ \dots \arrow[r, dashed, "1"'] & \mathbb{Z} \arrow[r, dashed, "0"'] & \mathbb{Z} \arrow[r, dashed, "1"'] & \mathbb{Z} \arrow[r, dashed, "0"'] & \mathbb{Z} \end{tikzcd}.$$

Now check the result. ∎

Exercise.

$$\mathrm{H}^i\left(\{\text{point}\}\right) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

There is basically only one other computation we can do from the definitions.

Lemma 1.4. *If $X = \coprod_{\alpha \in I} X_\alpha$ is a disjoint union of path-components,*

$$\mathrm{H}_i\left(X\right) \cong \bigoplus_{\alpha \in I} \mathrm{H}_i\left(X_\alpha\right).$$

Proof. Any continuous map $\sigma : \Delta^i \to X$ has image in one X_α and then all the faces of σ lie in the same X_α, so

$$\mathrm{C}_\bullet\left(X\right) = \bigoplus_\alpha \mathrm{C}_\bullet\left(X_\alpha\right),$$

compatibly with the differential. ∎

Lemma 1.5. *If X is path-connected and non-empty,*

$$\mathrm{H}_0\left(X\right) \cong \mathbb{Z}.$$

We sometimes write $\pi_0\left(X\right)$ for the set of path-components of X.

Proof. Define the **augmentation**

$$\begin{array}{cccccc} \epsilon & : & \mathrm{C}_0\left(X\right) & \longrightarrow & \mathbb{Z} \\ & & \sum_i n_i \sigma_i & \longmapsto & \sum_i n_i \end{array},$$

where $\sigma_i : \{\text{point}\} \to X$ are 0-simplices in X. Since $X \neq \emptyset$, ϵ is onto. If $\tau = [v_0, v_1] : \Delta^1 \to X$, then $\epsilon\left(\partial\left(\tau\right)\right) = \epsilon\left(v_1 - v_0\right) = 0$. So $\mathrm{im}\left(\partial : \mathrm{C}_1\left(X\right) \to \mathrm{C}_0\left(X\right)\right) \subseteq \ker \epsilon$, so ϵ defines $\mathrm{H}_0\left(X\right) = \mathrm{C}_0\left(X\right) / \mathrm{im}\, \partial \to \mathbb{Z}$. So far we did not use path-connectivity. But suppose $\sum_i n_i \sigma_i \in \ker \epsilon$. Fix a basepoint $p \in X$. For all i pick

$$\begin{array}{ccccc} \tau_i & : & \Delta^i \cong [0, 1] & \longrightarrow & X \\ & & 1 & \longmapsto & \sigma_i \\ & & 0 & \longmapsto & p \end{array}.$$

Then $\partial\left(\sum_i n_i \tau_i\right) = \sum_i n_i \sigma_i - \left(\sum_i n_i\right) p = \sum_i n_i \sigma_i$, as $\sum_i n_i \sigma_i \in \ker \epsilon$, so $\ker \epsilon \subseteq \mathrm{im}\, \partial$ and $\epsilon : \mathrm{H}_0\left(X\right) \xrightarrow{\sim} \mathbb{Z}$. ∎

1.4 Structural theorems

The following is an informal picture. Let X be an annulus, and let $\sigma : \Delta^1 \to X$ be a 1-simplex, which happens to be a closed loop $[0,1] \to X$ going around the inner circle. Recall that σ has $\partial(\sigma) = \sigma(1) - \sigma(0) = 0$, so σ defines $[\sigma] \in \mathrm{H}_1(X)$. We would hope this is non-zero, as we cannot see a way to fill in σ with 2-simplices, in contrast to a 1-simplex $\tau : \Delta^1 \cong [0,1] \to X$ away from the inner circle. But $\mathrm{C}_i(X)$ is uncountably generated for all i and very hard to control. A question is how do we rule out all configurations of 2-simplices, or other representatives for $[\sigma] \in \mathrm{H}_i(X)$? Informally, in the realm of nice spaces, there is nothing else you can compute from the definition. Homology and cohomology is rendered useful by a collection of structural theorems. We will state these, and see how to use them, and then return to prove them later.

Theorem 1.6 (Homotopy invariance)**.** *If $f : X \to Y$ and $g : X \to Y$ are homotopic, then*

$$f_* = g_* : \mathrm{H}_\bullet(Y) \to \mathrm{H}_\bullet(Y), \qquad f^* = g^* : \mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(Y).$$

Corollary 1.7. *If $X \simeq Y$ then $\mathrm{H}_\bullet(X) \cong \mathrm{H}_\bullet(Y)$ and $\mathrm{H}^\bullet(X) \cong \mathrm{H}^\bullet(Y)$.*

Proof. There exist $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, so $(f_*)^{-1} = g_*$ are isomorphisms. ☐

Thus cohomology is insensitive to inessential deformations of a space.

Corollary 1.8. *For every n,*

$$\mathrm{H}_\bullet(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \textit{otherwise} \end{cases},$$

and similarly for $\mathrm{H}^\bullet(\mathbb{R}^n)$.

Definition. An **exact sequence** is a chain or cochain complex with vanishing homology or cohomology, so

$$\dots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots,$$

such that $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for all n.

- Given homomorphisms
$$A \xrightarrow{f} B \xrightarrow{g} C,$$
say this is **exact at** B if $\ker g = \operatorname{im} f$.

- If
$$0 \to A \xrightarrow{f} B \to 0$$
is exact, $A \cong_f B$.

- A **short exact sequence** is one of shape
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Example. If

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n \to 0,$$

possibly $A = \mathbb{Z} \oplus \mathbb{Z}/n$, and

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto (1,0)} \mathbb{Z} \oplus \mathbb{Z}/n \xrightarrow{(0,1) \mapsto 1} \mathbb{Z}/n \to 0$$

or $A = \mathbb{Z}$, and

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{p \mapsto p \mod n} \mathbb{Z}/n \to 0.$$

See question sheet 1.

Theorem 1.9 (Mayer-Vietoris). *If $X = A \cup B$ with A and B open, there are **Mayer-Vietoris boundary homomorphisms** $\partial_{\mathrm{MV}} : \mathrm{H}_{i+1}(X) \to \mathrm{H}_i(A \cap B)$, yielding a **long exact sequence***

$$\dots \to \mathrm{H}_{i+1}(X) \xrightarrow{\partial_{\mathrm{MV}}} \mathrm{H}_i(A \cap B) \xrightarrow{\left((i_A)_*, (i_B)_*\right)} \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \xrightarrow{(j_A)_* - (j_B)_*} \mathrm{H}_i(X) \to \dots,$$

where

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"'] & A \arrow[d, hook, "j_A"] \\ B \arrow[r, hook, "j_B"'] & X \end{tikzcd}.$$

The Mayer-Vietoris boundary homomorphism is defined algebraically and is not associated to a map of spaces.

Remark. Suppose $\sigma \in \mathrm{C}_{i+1}(X)$ is a cycle, so $\partial(\sigma) = 0$, and $\sigma = \alpha + \beta$ for chains $\alpha \in \mathrm{C}_{i+1}(A)$ and $\beta \in \mathrm{C}_{i+1}(B)$. Then $\partial(\alpha) = -\partial(\beta)$ and $\partial_{\mathrm{MV}}([\sigma]) = [\partial(\alpha)]$, since $\partial(\alpha) \in A \cap B$.

Remark. The Mayer-Vietoris sequence is natural, so if $X = A \cup B$ and $Y = C \cup D$ and $f : X \to Y$ has $f(A) \subseteq C$ and $f(B) \subseteq D$ then there are homomorphisms of exact sequences

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_{i+1}(X) \arrow[r, "\partial_{\mathrm{MV}}"] \arrow[d, "f_*"] & \mathrm{H}_i(A \cap B) \arrow[r] \arrow[d, "f_*"] & \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \arrow[r] \arrow[d, "f_*"] & \mathrm{H}_i(X) \arrow[r] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r] & \mathrm{H}_{i+1}(Y) \arrow[r, "\partial_{\mathrm{MV}}"'] & \mathrm{H}_i(C \cap D) \arrow[r] & \mathrm{H}_i(C) \oplus \mathrm{H}_i(D) \arrow[r] & \mathrm{H}_i(Y) \arrow[r] & \dots \end{tikzcd},$$

such that all squares commute.

Remark. There is a Mayer-Vietoris sequence in cohomology, which is also natural. There are $\partial_{\mathrm{MV}}^* : \mathrm{H}^i(A \cap B) \to \mathrm{H}^{i+1}(X)$ such that

$$\dots \to \mathrm{H}^i(X) \xrightarrow{(j_A^*, j_B^*)} \mathrm{H}^i(A) \oplus \mathrm{H}^i(B) \xrightarrow{i_A^* - i_B^*} \mathrm{H}^i(A \cap B) \xrightarrow{\partial_{\mathrm{MV}}} \mathrm{H}^{i+1}(X) \to \dots$$

is exact, where

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"'] & A \arrow[d, hook, "j_A"] \\ B \arrow[r, hook, "j_B"'] & X \end{tikzcd}.$$

1.5 The sphere

Proposition 1.10.

$$\mathrm{H}_i(\mathrm{S}^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{H}^i(\mathrm{S}^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $\mathrm{S}^1 = X = A \cup B$ where A and B are open intervals such that $A \cap B$ are two disjoint open intervals, so $A \simeq \{\text{point}\} \simeq B$ and $A \cap B \simeq \{\text{point } p\} \sqcup \{\text{point } q\}$. By homotopy invariance,

$$\mathrm{H}_\bullet(\mathbb{R}) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases},$$

so we know $\mathrm{H}_\bullet(A)$, $\mathrm{H}_\bullet(B)$, and $\mathrm{H}_\bullet(A \cap B)$. Mayer-Vietoris for $i \geq 2$ gives

$$\begin{array}{ccccc} \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) & \longrightarrow & \mathrm{H}_i(\mathrm{S}^1) & \longrightarrow & \mathrm{H}_{i-1}(A \cap B) \\ \cong & & & & \cong \\ 0 & & & & 0 \end{array}.$$

Check that $\mathrm{H}_i\left(\mathrm{S}^1\right)=0$. [2] Mayer-Vietoris for $i=0,1$ gives

$$\begin{tikzcd}
\mathrm{H}_1(A)\oplus\mathrm{H}_1(B) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}_1\left(\mathrm{S}^1\right) \arrow[r] & \mathrm{H}_0(A\cap B) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}_0(A)\oplus\mathrm{H}_0(B) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}_0\left(\mathrm{S}^1\right) \arrow[d, phantom, "\cong"] \\
0 & & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, dashed, "\alpha"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, dashed, "\beta"'] & \mathbb{Z}
\end{tikzcd}.$$

Recall that $\mathrm{H}_0(Z)$ is free abelian on $\pi_0(Z)$, the set of path-components, and indeed is generated by $\sigma:\{\text{point}\}\to Z$, for any choice of point in each component. So

$$\alpha=\left(\left(i_A\right)_*,\left(i_B\right)_*\right) \quad : \quad \begin{array}{ccc} \mathbb{Z}\langle p\rangle\oplus\mathbb{Z}\langle q\rangle & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} \\ (a,b) & \longmapsto & (a+b,a+b) \end{array},$$

and

$$\beta=\left(j_A\right)_*-\left(j_B\right)_* \quad : \quad \begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (u,v) & \longmapsto & u-v \end{array}.$$

By exactness, $\mathrm{H}_1\left(\mathrm{S}^1\right)\cong\ker\alpha\cong\mathbb{Z}$, generated by $(1,-1)\equiv(p,-q)\in\mathrm{H}_0(A)\oplus\mathrm{H}_0(B)$. ☐

Lecture 4 Friday 16/10/20

The same method as for computing $\mathrm{H}_\bullet\left(\mathrm{S}^1\right)$ shows the following.

Proposition 1.11.

$$\mathrm{H}_j\left(\mathrm{S}^n\right)\cong\begin{cases}\mathbb{Z} & j=0,n \\ 0 & \textit{otherwise}\end{cases}, \qquad \mathrm{H}^j\left(\mathrm{S}^n\right)\cong\begin{cases}\mathbb{Z} & j=0,n \\ 0 & \textit{otherwise}\end{cases}.$$

This time let us do the cohomology computation.

Proof. Let $\mathrm{S}^n=A\cup B$ where $A\cong B\cong\mathbb{R}^n$ and $A\cap B\cong\mathrm{S}^{n-1}\times(0,1)\simeq\mathrm{S}^{n-1}$. By homotopy invariance and induction, we know $\mathrm{H}^\bullet(A)$, $\mathrm{H}^\bullet(B)$, and $\mathrm{H}^\bullet(A\cap B)$. Mayer-Vietoris now gives

$$\begin{tikzcd}
\mathrm{H}^i\left(\mathbb{R}^n\right)\oplus\mathrm{H}^i\left(\mathbb{R}^n\right) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^i\left(\mathrm{S}^{n-1}\right) \arrow[r] & \mathrm{H}^{i+1}\left(\mathrm{S}^n\right) \arrow[r] & \mathrm{H}^{i+1}\left(\mathbb{R}^n\right)\oplus\mathrm{H}^{i+1}\left(\mathbb{R}^n\right) \arrow[d, phantom, "\cong"] \\
0 & & & 0
\end{tikzcd},$$

so $\mathrm{H}^i\left(\mathrm{S}^{n-1}\right)\xrightarrow{\sim}\mathrm{H}^{i+1}\left(\mathrm{S}^n\right)$ for all $i>0$. For $i=0,1$,

$$\begin{tikzcd}
\mathrm{H}^0\left(\mathrm{S}^n\right) \arrow[r] & \mathrm{H}^0\left(\mathbb{R}^n\right)\oplus\mathrm{H}^0\left(\mathbb{R}^n\right) \arrow[r] & \mathrm{H}^0\left(\mathrm{S}^{n-1}\right) \arrow[r] & \mathrm{H}^1\left(\mathrm{S}^n\right) \arrow[r] & \mathrm{H}^1\left(\mathbb{R}^n\right)\oplus\mathrm{H}^1\left(\mathbb{R}^n\right) \arrow[d, phantom, "\cong"] \\
& & & & 0
\end{tikzcd}.$$

We showed before that for path-connected X, $\mathrm{H}_0(X)\cong\mathbb{Z}$ is generated by $\sigma:\{\text{point}\}\to X\in\mathrm{C}_0(X)$. By question sheet 1, $\mathrm{H}^0(X)\cong\mathbb{Z}$ is generated by

$$\psi \quad : \quad \begin{array}{ccc} \mathrm{C}_0(X) & \longrightarrow & \mathbb{Z} \\ \sigma & \longmapsto & 1 \end{array}, \qquad \sigma:\{\text{point}\}\to X.$$

If $n>1$, S^{n-1} is connected. So

$$\begin{tikzcd}
\mathrm{H}^0\left(\mathrm{S}^n\right) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^0\left(\mathbb{R}^n\right)\oplus\mathrm{H}^0\left(\mathbb{R}^n\right) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^0\left(\mathrm{S}^{n-1}\right) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^1\left(\mathrm{S}^n\right) \arrow[r] & \mathrm{H}^1\left(\mathbb{R}^n\right)\oplus\mathrm{H}^1\left(\mathbb{R}^n\right) \arrow[d, phantom, "\cong"] \\
\mathbb{Z} \arrow[r, dashed] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, dashed, "\alpha"'] & \mathbb{Z} & & 0
\end{tikzcd},$$

where $\alpha(p,q)=p+q$ is onto, so $\mathrm{H}^1\left(\mathrm{S}^n\right)=0$, and now we have computed enough to complete the induction. ☐

Corollary 1.12. *$\mathbb{R}^m\cong\mathbb{R}^n$ if and only if $m=n$.*

Proof. If $\mathbb{R}^m\cong\mathbb{R}^n$, then $\mathrm{S}^{m-1}\simeq\mathbb{R}^m\setminus\{0\}\cong\mathbb{R}^n\setminus\{0\}\simeq\mathrm{S}^{n-1}$, so $\mathrm{S}^{m-1}\simeq\mathrm{S}^{n-1}$. Thus $\mathrm{H}_\bullet\left(\mathrm{S}^{m-1}\right)\cong\mathrm{H}_\bullet\left(\mathrm{S}^{n-1}\right)$, so $m=n$. ☐

This homeomorphism invariance of dimension was an early success of the subject. Recall there are space-filling curves $\phi:[0,1]\to[0,1]^2$ that are continuous and surjective.

[2] Exercise

1.6 Degrees

Lemma 1.13. *Assume $n > 0$. A map $f : \mathrm{S}^n \to \mathrm{S}^n$ has a **degree** $\deg f \in \mathbb{Z}$ and if $g \simeq f$, then $\deg g = \deg f$.*

Proof. f induces $(f_*)_n : \mathrm{H}_n(\mathrm{S}^n) \cong \mathbb{Z} \to \mathrm{H}_n(\mathrm{S}^n) \cong \mathbb{Z}$, which is multiplication by an integer. This defines $\deg f$. If $g \simeq f$, then $g_* = f_*$. A caveat is to use the same isomorphism on both sides and make sure $\deg f$ is defined and not just up to sign. $\square$

Exercise. Check that $\deg(f \circ g) = \deg f \cdot \deg g$.

Example. $\deg \mathrm{id} = 1$, so if f is a homeomorphism, $\deg f \in \{\pm 1\}$.

Example. The degree of the constant map is zero, since the constant map

$$\begin{array}{cccc} f & : & \mathrm{S}^n & \longrightarrow & \mathrm{S}^n \\ & & x & \longmapsto & p \end{array}$$

factorises as $\mathrm{S}^n \to \{\text{point}\} \to \mathrm{S}^n$, so

$$\begin{tikzcd} \mathrm{H}_n(\mathrm{S}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_n(\{\text{point}\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_n(\mathrm{S}^n) \arrow[d, phantom, "\cong" rotate=-90] \\ \mathbb{Z} \arrow[r, dashed] & 0 \arrow[r, dashed] & \mathbb{Z} \end{tikzcd}$$

factorises through the zero group.

Note that combining with $\mathrm{S}^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$, this fills in details, modulo homotopy invariance and Mayer-Vietoris, for results from the first lecture on Brouwer's theorem.

Lemma 1.14. *Let $\mathrm{O}(k) = \{A \in \mathrm{Mat}_k \mathbb{R} \mid AA^\intercal = \mathrm{I}\}$. A matrix $A \in \mathrm{O}(n+1)$, which acts on $\mathrm{S}^n \subseteq \mathbb{R}^{n+1}$, acts on $\mathrm{H}_n(\mathrm{S}^n)$ by multiplication by $\det A$.*

Proof. $\mathrm{O}(n+1)$ has two path-connected components, so by homotopy invariance of degree, it suffices to show reflection in a hyperplane has degree -1. Let $H = \mathrm{S}^{n-1}$ be a hyperplane, let L be an invariant hemisphere, and let $H' = \partial L \cap H$. Note that a reflection $\mathrm{r}_H : \mathrm{S}^n \to \mathrm{S}^n$ in H induces a reflection $\mathrm{r}_{H'} : \partial L = \mathrm{S}^{n-1} \to \partial L = \mathrm{S}^{n-1}$ in H'. We computed $\mathrm{H}_\bullet(\mathrm{S}^n)$ by Mayer-Vietoris, using the decomposition which is r_H-invariant. By the naturality of Mayer-Vietoris,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_n(\mathrm{S}^n) \arrow[r, "\sim"] \arrow[d, "\mathrm{r}_H"'] & \mathrm{H}_{n-1}\left(\mathrm{S}^{n-1}\right) \arrow[r] \arrow[d, "\mathrm{r}_{H'}"] & 0 \\ 0 \arrow[r] & \mathrm{H}_n(\mathrm{S}^n) \arrow[r, "\sim"'] & \mathrm{H}_{n-1}\left(\mathrm{S}^{n-1}\right) \arrow[r] & 0 \end{tikzcd},$$

so inductively, it suffices to treat the case $n = 1$. So consider a circle $\mathrm{S}^1 = A \cup B$ where $p, q \in A \cap B$. Our former Mayer-Vietoris computation of $\mathrm{H}_\bullet\left(\mathrm{S}^1\right)$ gave

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_1\left(\mathrm{S}^1\right) \arrow[r] & \mathrm{H}_0(p \sqcup q) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \arrow[d, phantom, "\cong" rotate=-90] \\ & & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed, "\alpha"'] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd},$$

and $\mathrm{H}_1\left(\mathrm{S}^1\right) = \ker \alpha \cong \mathbb{Z}\langle(1, -1)\rangle$ is generated by $p - q$. So as r_H exchanges p and q it acts on $\mathrm{H}_1\left(\mathrm{S}^1\right)$ by -1. $\square$

Corollary 1.15.

1. *The antipodal map*

$$\begin{array}{cccc} \mathrm{a}_n & : & \mathrm{S}^n & \longrightarrow & \mathrm{S}^n \\ & & x & \longmapsto & -x \end{array}$$

   has degree $(-1)^{n+1}$.

2. *If $f : \mathrm{S}^n \to \mathrm{S}^n$ has no fixed point, $f \simeq \mathrm{a}_n$.*

3. *If G acts freely on S^{2k}, then $G \leq \mathbb{Z}/2$.*

Proof.

1. $\mathrm{a}_n : \mathrm{S}^n \to \mathrm{S}^n$ is a composition of $n+1$ reflections $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.
2. We will show if $f(x) \neq g(x)$ for all x, then $f \simeq \mathrm{a}_n \circ g$. Consider
$$\phi_t : x \mapsto \frac{tf(x) - (1-t)g(x)}{\|tf(x) - (1-t)g(x)\|}, \qquad 0 \leq t \leq 1.$$
Note that $tf(x) + (1-t)g(x) \neq 0$ or $t = \frac{1}{2}$ and $f(x) = g(x)$, a contradiction. So $f = \phi_1 \simeq \phi_0 = \mathrm{a}_n \circ g$.
3. Question sheet 1.

$\square$

We borrow a concept from differential topology. A **vector field** on S^n is a map $v : \mathrm{S}^n \to \mathbb{R}^{n+1}$ such that for all $x \in \mathrm{S}^n$, the Euclidean inner product on $\mathbb{R}^{n+1}$ has $\langle x, v(x) \rangle = 0$. Note that this is a global section of the tangent bundle $\mathrm{TS}^n \to \mathrm{S}^n$.

Proposition 1.16 (Hairy ball theorem)**.** S^n *has a nowhere-vanishing vector field if and only if* n *is odd.*

Proof. If $n = 2k-1$, set
$$v(x_1, y_1, \dots, x_k, y_k) = (-y_1, x_1, \dots, -y_k, x_k).$$
Suppose n is even, and for contradiction that such v exists. So $v/\|v\| : \mathrm{S}^n \to \mathrm{S}^n$. Consider
$$v_t(x) = (\cos t)\, x + (\sin t) \frac{v}{\|v\|}(x).$$
Then $|v_t(x)| = 1$ for all t, and $v_0 = \mathrm{id}$ and $v_\pi = -\mathrm{id} = \mathrm{a}_n$, so $\mathrm{id}_{\mathrm{S}^n} \simeq \mathrm{a}_n$. Thus $\deg \mathrm{id} = \deg \mathrm{a}_n$, so $1 = (-1)^{n+1}$. $\square$

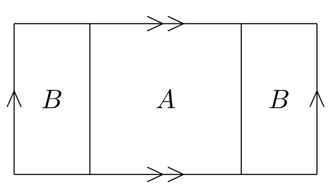
1.7 The Klein bottle

Lecture 5 Monday 19/10/20

We should do one computation which involves knowing the maps, not just on $\mathrm{H}_0(X)$, in an exact sequence, and not just that the sequence is exact. The **Klein bottle** K is obtained from gluing two Möbius bands together.

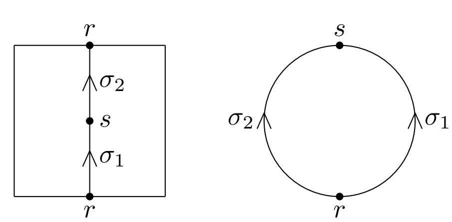
Lemma 1.17.
$$\mathrm{H}_j(K; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & j = 1 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Apply Mayer-Vietoris to K



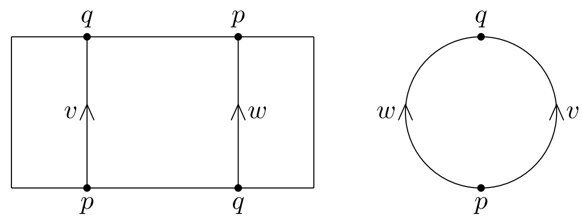
,

where $A \simeq \mathrm{S}^1$ is a Möbius band



,

and $B \simeq \mathrm{S}^1$ is a similar Möbius band, such that $A \cap B \simeq \mathrm{S}^1$ is



.

The essential part of the long exact sequence is

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathrm{H}_2(K) & \longrightarrow & \mathrm{H}_1(A \cap B) & \xrightarrow{\psi} & \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) & \longrightarrow & \mathrm{H}_1(K) & \xrightarrow{0} & \mathrm{H}_0(A \cap B) & \longrightarrow & \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \\
& & & & \cong & & \cong & & & & \cong & & \cong \\
& & & & \mathbb{Z} & \dashrightarrow & \mathbb{Z} \oplus \mathbb{Z} & & & & \mathbb{Z} & \xrightarrow[p \mapsto (p,p)]{} & \mathbb{Z} \oplus \mathbb{Z}
\end{array}.$$

By exactness, $\mathrm{H}_1(K) = (\mathbb{Z} \oplus \mathbb{Z}) / \operatorname{im} \psi$ and $\mathrm{H}_2(K) \cong \ker \psi$. The key claim is that $\psi(1) = (2,2)$ and note $(\mathbb{Z} \oplus \mathbb{Z}) / \langle 2,2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2$. For this, $A \cap B$ is homotopy equivalent to the boundary circle of the central Möbius band, so $\mathrm{H}_1(A \cap B) = \mathbb{Z}\langle v + w \rangle$, and A is homotopy equivalent to the core circle of the central Möbius band, so $\mathrm{H}_1(A) = \mathbb{Z}\langle \sigma_1 + \sigma_2 \rangle$. Thus $\psi : v \mapsto \sigma_1 + \sigma_2$ and $\psi : w \mapsto \sigma_1 + \sigma_2$. □

Remark. We could define

$$\mathrm{C}_k(X; G) = \left\{ \sum_i a_i \sigma_i \;\middle|\; a_i \in G, \; \sigma_i : \Delta^k \to X \right\},$$

for any abelian group G, with the same differential ∂, which gives $\mathrm{H}_\bullet(X; G)$, the **singular homology with coefficients in** G.

Example.

$$\mathrm{H}_j\left(\mathrm{S}^1; \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Z}/2 & j = 0, 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{H}_i(\{\text{point}\}; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In the previous sequence, if we compute $\mathrm{H}_\bullet(K; \mathbb{Z}/2)$, get

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}_2(K; \mathbb{Z}/2) & \longrightarrow & \mathrm{H}_1(A \cap B; \mathbb{Z}/2) & \xrightarrow{\psi} & \mathrm{H}_1(A; \mathbb{Z}/2) \oplus \mathrm{H}_1(B; \mathbb{Z}/2) \\
& & & & \cong & & \cong \\
& & & & \mathbb{Z}/2 & \xrightarrow[1 \mapsto (2,2) \equiv (0,0)]{} & \mathbb{Z}/2 \oplus \mathbb{Z}/2
\end{array},$$

so ψ vanishes for $\mathrm{H}_\bullet(-; \mathbb{Z}/2)$ and

$$\mathrm{H}_i(K; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z}/2 & i = 2 \\ 0 & \text{otherwise} \end{cases}.$$

It is also instructive to think about cohomology in this example, where $K = A \cup B$ for $A, B \simeq \mathrm{S}^1$ and $A \cap B \simeq \mathrm{S}^1$ as before. So the interesting parts of the cohomology Mayer-Vietoris sequences look like

$$\begin{array}{ccccccccc}
\mathrm{H}^1(K) & \xrightarrow{(j_A^*, j_B^*)} & \mathrm{H}^1(A) \oplus \mathrm{H}^1(B) & \xrightarrow{i_A^* - i_B^*} & \mathrm{H}^1(A \cap B) & \longrightarrow & \mathrm{H}^2(K) & \longrightarrow & 0 \\
& & \cong & & \cong & & & & \\
& & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\psi]{} & \mathbb{Z} & & & &
\end{array}.$$

Check that this ψ is $(a, b) \mapsto 2(a - b)$. [3] So $\mathrm{H}^2(K) \cong \mathbb{Z}/2$. For contrast, $\mathrm{H}_2(K) = 0$ if we use $\mathbb{Z}$ coefficients.

Remark. There were many ways we could have cut up K. In some cases, some decompositions will give easier algebra than others.

[3] Exercise

2 Structural theorems

Now we should pay some debts.

2.1 Chain homotopy

Let $C_\bullet$ and $D_\bullet$ be chain complexes.

Definition. Chain maps $f : C_\bullet \to D_\bullet$ and $g : C_\bullet \to D_\bullet$ are **chain homotopic** if there exist $P_n : C_n \to D_{n+1}$ such that

$$P_{n-1} \circ \partial_n^{C_\bullet} \pm \partial_{n+1}^{D_\bullet} \circ P_n = f_n - g_n,$$

so

$$\begin{tikzcd} \ldots \arrow[r] & C_n \arrow[r, "\partial"] \arrow[dl, "P_n"'] \arrow[d, dashed] & C_{n-1} \arrow[r] \arrow[dl, "P_{n-1}"] & \ldots \\ \ldots \arrow[r] & D_{n+1} \arrow[r, "\partial"'] & D_n \arrow[r] & \ldots \end{tikzcd}.$$

Lemma 2.1. *If $f : C_\bullet \to D_\bullet$ and $g : C_\bullet \to D_\bullet$ are chain homotopic, then*

$$(f_*)_i = (g_*)_i : \mathrm{H}_i(C_\bullet, \partial) \to \mathrm{H}_i(D_\bullet, \partial),$$

for all i, that is chain homotopic maps induce the same map on homology.

Recall we are trying to prove if $f \simeq g : X \to Y$, then $f_* = g_* : \mathrm{H}_\bullet(X) \to \mathrm{H}_\bullet(Y)$. So it will be sufficient to show $f_\#, g_\# : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(Y)$ are chain homotopic.

Proof. Let

$$\begin{tikzcd} & C_n \arrow[r, "\partial"] \arrow[dl, "P_n"'] \arrow[d, dashed] & C_{n-1} \arrow[dl, "P_{n-1}"] \\ D_{n+1} \arrow[r, "\partial"'] & D_n & \end{tikzcd},$$

such that $P_{n-1} \circ \partial \pm \partial \circ P_n = f_n - g_n$. Let $\alpha \in C_n$ be a cycle, so $\partial(\alpha) = 0$. So $\partial(f_n(\alpha)) = f_{n-1}(\partial(\alpha)) = 0$, so $(f_*)_n([\alpha]) = [f_n(\alpha)]$. So

$$f_n(\alpha) - g_n(\alpha) = (f_n - g_n)(\alpha) = P_{n-1}(\partial(\alpha)) \pm \partial(P_n(\alpha)) = \partial(P_n(\alpha)) \in \operatorname{im} \partial,$$

so $[f_n(\alpha)] = [g_n(\alpha)] \in \mathrm{H}_n(D_\bullet)$. $\square$

Exercise. Chain homotopy is an equivalence relation on chain complexes and chain maps.

2.2 Homotopy invariance

Theorem 2.2 (Homotopy invariance, version 2). *If $f \simeq g : X \to Y$ then*

$$f_\# \simeq g_\# : (\mathrm{C}_\bullet(X), \partial) \to (\mathrm{C}_\bullet(Y), \partial)$$

are chain homotopic.

Proof. If $f \simeq g$, then there exists $F : X \times [0,1] \to Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$. So if

$$\begin{array}{cccccc} \iota_0 & : & X & \longrightarrow & X \times [0,1] \\ & & x & \longmapsto & (x, 0) \end{array}, \qquad \begin{array}{cccccc} \iota_1 & : & X & \longrightarrow & X \times [0,1] \\ & & x & \longmapsto & (x, 1) \end{array},$$

then $f = F \circ \iota_0$ and $g = F \circ \iota_1$, so $f_\# = g_\#$ if $(\iota_0)_\# = (\iota_1)_\#$ and it suffices to prove that $(\iota_0)_\# \simeq (\iota_1)_\# : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X \times [0,1])$, so Y is out of the picture. So want $P_n : \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X \times [0,1])$. The idea is that P_n is a **prism operator**

$$\begin{array}{rcl} \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n+1}(X \times [0,1]) \\ \sigma : \Delta^n \to X & \longmapsto & \text{linear combination of simplices for } \sigma \times \mathrm{id} : \Delta^n \times [0,1] \to X \times [0,1] \end{array}.$$

It gives an universal way of cutting up $\Delta^n \times [0,1]$ into $(n+1)$-simplices. The equation

$$\partial \circ P \pm P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$$

says that the boundary of the prism is the prism on the boundary plus the top minus the bottom. The details of the proof are not very illuminating, so we will be quite terse. Label the base of the prism by $[v_0, \dots, v_n]$ and the top $[w_0, \dots, w_n]$. Claim that $\sigma_{n+1}^i = [v_0, \dots, v_i, w_i, \dots, w_n]$ is an $(n+1)$-simplex, and

$$\Delta^n \times [0,1] = \bigcup_{i=0}^n \sigma_{n+1}^i.$$

We will not prove this, so see Hatcher. Define

$$\begin{array}{rccl} P_n & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n+1}(X \times [0,1]) \\ & & \sigma & \longmapsto & \sum_{i=0}^n (-1)^i \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,v_i,w_i,\dots,w_n]} = \sum_{i=0}^n (-1)^i \left((\sigma \times \mathrm{id}) \circ \sigma_{n+1}^i\right) \end{array}.$$

Claim that $\partial \circ P + P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$. Well,

$$\begin{aligned} \partial(P_n(\sigma)) &= \sum_{j \le i} (-1)^i (-1)^j \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j \ge i} (-1)^i (-1)^{j+1} \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]} \\ &= \left.(\sigma \times \mathrm{id})\right|_{[\widehat{v_0},w_0,\dots,w_n]} - \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,v_n,\widehat{w_n}]} \\ &\quad + \sum_{j < i} (-1)^i (-1)^j \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} \left.(\sigma \times \mathrm{id})\right|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}, \end{aligned}$$

since the $i = j$ terms cancel in pairs except for $i = j = 0$, the top, and $i = j = n$, the bottom. Check that the latter sums are $-P_n(\partial(\sigma))$, [4] which is routine but unenlightening. □

Remark. If $C^\bullet$ and $D^\bullet$ are cochain complexes, $f \simeq g$ are **cochain homotopic** if there exist $P^i : C^i \to D^{i-1}$ such that

$$\partial^* \circ P \pm P \circ \partial^* = f - g,$$

so

$$\begin{tikzcd} & \dots \arrow[r] & C^i \arrow[r, "\partial^i"] \arrow[dl, "P^i"'] \arrow[d, dashed] & C^{i+1} \arrow[r] \arrow[dl, "P^{i+1}"] & \dots \\ \dots \arrow[r] & D^{i-1} \arrow[r, "\partial^{i+1}"'] & D^i \arrow[r] & \dots & \end{tikzcd}.$$

Check that [5]

$$f^* = g^* : \mathrm{H}^\bullet(C^\bullet) \to \mathrm{H}^\bullet(D^\bullet).$$

Then $P_n : \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X \times [0,1])$ has dual

$$P^n : \mathrm{Hom}(\mathrm{C}_{n+1}(X \times [0,1]), \mathbb{Z}) = \mathrm{C}^{n+1}(X \times [0,1]) \to \mathrm{Hom}(\mathrm{C}_n(X), \mathbb{Z}) = \mathrm{C}^n(X),$$

and $\partial \circ P + P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$ implies that

$$\partial^* \circ P + P \circ \partial^* = \iota_1^\# - \iota_0^\#,$$

so cohomology is also homotopy invariant.

[4] Exercise
[5] Exercise

2.3 The long exact sequence

Lecture 6 Wednesday 21/10/20

We have made various computations using homotopy invariance, which we have proved, and Mayer-Vietoris, which we have not. Before addressing that, we need some more algebra. Recall that a short exact sequence is an exact sequence of the shape

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0, \qquad \operatorname{im} \alpha = \ker \beta.$$

Definition. A **short exact sequence of chain complexes** is a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d, "\partial"] & B_n \arrow[r, "\beta"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] \arrow[d] & B_{n-1} \arrow[r, "\beta"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd},$$

such that all squares commute, and the columns are chain complexes and the rows are exact, so $\operatorname{im} \alpha = \ker \beta$ and $\partial^2 = 0$. Write

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0.$$

Proposition 2.3. *If*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes, there is a boundary map $\delta : \mathrm{H}_n(C_\bullet) \to \mathrm{H}_{n-1}(A_\bullet)$ fitting into a ***long exact sequence*** *on homology*

$$\dots \to \mathrm{H}_n(A_\bullet) \xrightarrow{(\alpha_*)_n} \mathrm{H}_n(B_\bullet) \xrightarrow{(\beta_*)_n} \mathrm{H}_n(C_\bullet) \xrightarrow{\delta} \mathrm{H}_{n-1}(A_\bullet) \to \dots.$$

Proof. By diagram chasing, we will construct δ, and the proof of exactness is relegated to question sheet 1. Let

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d] & B_n \arrow[r, "\beta"] \arrow[d] & C_n \arrow[r] \arrow[d] \arrow[dll, dashed] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] \arrow[d] & B_{n-1} \arrow[r, "\beta"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "\alpha"] \arrow[d] & B_{n-2} \arrow[r, "\beta"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}.$$

Let $c_n \in C_n$ be a cycle, so $\partial(c_n) = 0$, representing $[c_n] \in \mathrm{H}_n(C_\bullet)$. Since β is onto, there exists $b_n \in B_n$ such that $\beta(b_n) = c_n$. Since the top right square commutes, $\beta(\partial(b_n)) = \partial(\beta(b_n)) = \partial(c_n) = 0$. Since the middle sequence is exact, $\partial(b_n) \in \ker\beta = \operatorname{im}\alpha$, so $\partial(b_n) = \alpha(a_{n-1})$. Since the bottom left square commutes, $\alpha(\partial(a_{n-1})) = \partial(\alpha(a_{n-1})) = \partial^2(b_n) = 0$. Then α is one-to-one, so $\alpha(\partial(a_{n-1})) = 0$ implies that $\partial(a_{n-1}) = 0$, and set

$$\delta([c_n]) = [a_{n-1}].$$

Check δ is well-defined.

- Given c_n, we chose b_n. If $\beta(b_n') = c_n$, $b_n - b_n' \in \ker\beta = \operatorname{im}\alpha$, so $b_n' = b_n + \alpha(a_n)$ for some $a_n \in A_n$, and $\partial(b_n') = \partial(b_n) + \partial(\alpha(a_n)) = \alpha(a_{n-1} + \partial(a_n))$, so $[a_{n-1}] \in \mathrm{H}_{n-1}(A_\bullet)$ is unchanged.
- If $[c_n] = [c_n']$, then $c_n - c_n' \in \operatorname{im}\partial$, say $c_n' = c_n + \partial(c_{n+1})$. Pick b_{n+1} such that $\beta(b_{n+1}) = c_{n+1}$ and then $b_n \mapsto b_n + \partial(b_{n+1})$ and $\partial(b_n)$ is unchanged, so get the same a_{n-1}.

So δ is well-defined and it is easy to see it is a homomorphism. In the resulting

$$\dots \to \mathrm{H}_n(A_\bullet) \xrightarrow{(\alpha_*)_n} \mathrm{H}_n(B_\bullet) \xrightarrow{(\beta_*)_n} \mathrm{H}_n(C_\bullet) \xrightarrow{\delta} \mathrm{H}_{n-1}(A_\bullet) \to \dots,$$

should check exactness at all three kinds of terms, that is $\operatorname{im}\beta_* \subseteq \ker\delta$ and $\ker\delta \subseteq \operatorname{im}\beta_*$, etc, so six inclusions in total. [6] $\square$

For this piece of algebra to be useful, we need a source of short exact sequences of chain complexes.

Example. Recall if G is an abelian group,

$$\mathrm{C}_k(X;G) = \left\{ \sum_i a_i\sigma_i \;\middle|\; a_i \in G,\ \sigma_i : \Delta^k \to X \right\},$$

which gives $\mathrm{H}_\bullet(X;G)$, the singular homology with coefficients in G. Note that if

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is a short exact sequence of groups,

$$0 \to \mathrm{C}_\bullet(X;G_1) \to \mathrm{C}_\bullet(X;G_2) \to \mathrm{C}_\bullet(X;G_3) \to 0$$

is a short exact sequence of chain complexes. The resulting $\delta : \mathrm{H}_n(X;G_3) \to \mathrm{H}_{n-1}(X;G_1)$ is a **Bockstein homomorphism**. For example,

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{p \mapsto p \mod n} \mathbb{Z}/n \to 0, \qquad 0 \to \mathbb{Z}/n \xrightarrow{\cdot n} \mathbb{Z}/n^2 \xrightarrow{p \mapsto p \mod n} \mathbb{Z}/n \to 0$$

give the **classical Bockstein homomorphisms**

$$\mathrm{H}_p(X;\mathbb{Z}/n) \to \mathrm{H}_{p-1}(X;\mathbb{Z}), \qquad \mathrm{H}_p(X;\mathbb{Z}/n) \to \mathrm{H}_{p-1}(X;\mathbb{Z}/n).$$

We will revisit these later, probably.

2.4 Relative homology

Example. Let $A \subseteq X$ be a subspace. We have an inclusion $\mathrm{C}_\bullet(A) \hookrightarrow \mathrm{C}_\bullet(X)$ compatible with boundary maps, since if $\sigma : \Delta^i \to A \subseteq X$, then $\sigma \circ \delta_i : \Delta^{i-1} \to A$ too. Define

$$\mathrm{C}_\bullet(X,A) = \mathrm{C}_\bullet(X) / \mathrm{C}_\bullet(A),$$

so

$$0 \to \mathrm{C}_\bullet(A) \to \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X,A) \to 0$$

is a short exact sequence of chain complexes.

Definition. $\mathrm{H}_\bullet(\mathrm{C}_\bullet(X,A), \partial)$ is denoted $\mathrm{H}_\bullet(X,A)$, or $\mathrm{H}_\bullet(X,A;G)$, the **relative homology** of (X,A).

[6] Exercise: do this

Lemma 2.4. *If $f : (X, A) \to (Y, B)$ is a map of pairs, that is $f : X \to Y$ satisfies $f(A) \subseteq B$, then f induces $(f_*)_i : \mathrm{H}_i(X, A) \to \mathrm{H}_i(Y, B)$ for all i.*

Proof. Elementary. $\square$

The long exact sequence

$$\dots \to \mathrm{H}_i(A) \to \mathrm{H}_i(X) \to \mathrm{H}_i(X, A) \to \mathrm{H}_{i-1}(A) \to \dots$$

is called the **long exact sequence of the pair** (X, A).

Remark.

- Cycles in $\mathrm{C}_\bullet(X, A)$ are chains in X whose boundary lies in A.
- You might expect that things in A do not matter for $\mathrm{C}_\bullet(X, A)$, as we quotient all simplices in A. A precise version of that intuition is excision.

Theorem 2.5 (Excision)**.** *Let X be a space, $A \subseteq X$ a subspace, and Z a subspace such that $\overline{Z} \subseteq \mathring{A}$. Then the inclusion $\iota : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ is an isomorphism on relative homology, so for all n,*

$$(\iota_*)_n : \mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} \mathrm{H}_n(X, A).$$

We will prove excision and Mayer-Vietoris together next time. For now, let us see how this helps us understand relative homology.

Remark. Naturality under maps, homotopy invariance, the relative homology long exact sequence, and excision are the key tools of homology and cohomology. Much of what we will do will be built from these.

Lemma 2.6 (Five lemma)**.** *Suppose*

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "\alpha"] & B \arrow[r, "\partial"] \arrow[d, "\beta"] & C \arrow[r, "\partial"] \arrow[d, "\gamma"] & D \arrow[r, "\partial"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "\partial'"'] & B' \arrow[r, "\partial'"'] & C' \arrow[r, "\partial'"'] & D' \arrow[r, "\partial'"'] & E' \end{tikzcd}$$

is a commuting diagram of abelian groups with exact rows. If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then so is γ.

Proof. More diagram chasing. We will show γ is one-to-one, and you check it is onto. Let $c \in C$ have $\gamma(c) = 0$. Then $\delta(\partial(c)) = \partial'(\gamma(c)) = 0$ so $\partial(c) \in \ker \delta$, and δ is an isomorphism so $\partial(c) = 0$. Since the rows are exact, $c \in \ker \partial = \operatorname{im} \partial$, so $c = \partial(b)$ for $b \in B$. Then $\partial'(\beta(b)) = \gamma(\partial(b)) = \gamma(c) = 0$, so $\beta(b) \in \ker \partial' = \operatorname{im} \partial'$, and $\beta(b) = \partial'(a')$. Since α is an isomorphism, there exists $a \in A$ such that $\alpha(a) = a'$. Now $\beta(\partial(a)) = \partial'(\alpha(a)) = \partial'(a') = \beta(b)$ so $\partial(a) - b \in \ker \beta$, and β is an isomorphism so $b = \partial(a)$. Thus $c = \partial(b) = \partial^2(a) = 0$ and c is one-to-one. $\square$

Corollary 2.7. *If $f : (X, A) \to (Y, B)$ is a map of pairs, and any two of the induced homomorphisms*

$$\mathrm{H}_\bullet(X) \to \mathrm{H}_\bullet(Y), \qquad \mathrm{H}_\bullet(A) \to \mathrm{H}_\bullet(B), \qquad \mathrm{H}_\bullet(X, A) \to \mathrm{H}_\bullet(Y, B)$$

are isomorphisms, then so is the third.

Proof. Apply the five lemma to

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_i(A) \arrow[r] \arrow[d] & \mathrm{H}_i(X) \arrow[r] \arrow[d] & \mathrm{H}_i(X, A) \arrow[r] \arrow[d] & \mathrm{H}_{i-1}(A) \arrow[r] \arrow[d] & \mathrm{H}_{i-1}(X) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \mathrm{H}_i(B) \arrow[r] & \mathrm{H}_i(Y) \arrow[r] & \mathrm{H}_i(Y, B) \arrow[r] & \mathrm{H}_{i-1}(B) \arrow[r] & \mathrm{H}_{i-1}(Y) \arrow[r] & \dots \end{tikzcd}.$$

$\square$

2.5 Reduced homology and good pairs

We need two definitions to proceed. The first looks a bit odd, but be patient.

Definition. If X is a space, and $x_0 \in X$ is a basepoint, the **reduced homology** is

$$\widetilde{\mathrm{H}_i}(X) = \mathrm{H}_i(X, x_0).$$

Exercise. The long exact sequence of a pair shows

$$\widetilde{\mathrm{H}_0}(X) \oplus \mathbb{Z} \cong \mathrm{H}_0(X), \qquad \widetilde{\mathrm{H}_i}(X) \cong \mathrm{H}_i(X), \qquad i > 0.$$

Definition. A pair (X, A) is **good** if $A \subseteq X$ is closed and is a deformation retract of an open neighbourhood $A \subseteq U \subseteq X$, that is there exists $H : [0,1] \times U \to U$ such that

- $H|_{\{0\}\times U} = \mathrm{id}$ and $H|_{\{1\}\times U}$ has image in A, and
- H is fixed on A, so for all $t \in [0,1]$ and $a \in A$, $H(t, a) = a$.

So you can squeeze U back onto A without moving A. If X, and hence U, is Hausdorff, A is automatically closed.

Proposition 2.8. *If (X, A) is good, the natural map $(X, A) \to (X/A, A/A)$ induces isomorphisms*

$$\mathrm{H}_\bullet(X, A) \xrightarrow{\sim} \widetilde{\mathrm{H}_\bullet}(X/A).$$

Proof. Note that homotopy invariance and the five lemma show inclusion defines isomorphisms

$$\mathrm{H}_\bullet(A) \xrightarrow{\sim} \mathrm{H}_\bullet(U), \qquad \mathrm{H}_\bullet(X, A) \xrightarrow{\sim} \mathrm{H}_\bullet(X, U).$$

The inclusion $A/A = \{\text{point}\} \hookrightarrow U/A$ is a deformation retract and in particular a homotopy equivalence, so

$$\mathrm{H}_\bullet(X/A, A/A) \xrightarrow{\sim} \mathrm{H}_\bullet(X/A, U/A)$$

is also an isomorphism by the five lemma. Consider

$$\begin{tikzcd}
\mathrm{H}_\bullet(X, A) \arrow[r, "\sim", "\text{Homotopy}"'] \arrow[d] & \mathrm{H}_\bullet(X, U) & \mathrm{H}_\bullet(X \setminus A, U \setminus A) \arrow[l, "\sim"', "\text{Excision}"] \arrow[d] \\
\mathrm{H}_\bullet(X/A, A/A) \arrow[r, "\text{Homotopy}", "\sim"'] & \mathrm{H}_\bullet(X/A, U/A) & \mathrm{H}_\bullet((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \arrow[l, "\text{Excision}"', "\sim"]
\end{tikzcd},$$

where the vertical maps collapse A. Then the right vertical map is a homeomorphism of pairs, since $X \setminus A \cong (X/A) \setminus (A/A)$. So the right vertical map is an isomorphism and hence the left vertical map is an isomorphism. ◻

Remark. The **tubular neighbourhood theorem** of differential topology, which we will discuss more later, implies that if X is a smooth manifold and $A \subseteq X$ is a compact smooth submanifold, (X, A) is a good pair.

Example.

$$\mathrm{H}_j(\mathrm{D}^n, \partial\mathrm{D}^n) \cong \widetilde{\mathrm{H}_j}(\mathrm{D}^n/\partial\mathrm{D}^n) = \widetilde{\mathrm{H}_j}(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & j = n \\ 0 & \text{otherwise} \end{cases}.$$

Example. Let S^1 be the equator. Then

$$\mathrm{H}_j\left(\mathrm{S}^2, \mathrm{S}^1\right) \cong \widetilde{\mathrm{H}_j}\left(\mathrm{S}^2 \vee \mathrm{S}^2\right) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & j = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Remark. If M is a manifold and $x \in M$, by excision with $Z = M \setminus \{\text{open disc neighbourhood of } x\}$ and homotopy invariance or directly from the long exact sequence of a pair,

$$\mathrm{H}_j(M, M \setminus \{x\}) \cong \mathrm{H}_j(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathrm{H}_j(\mathrm{D}^n, \partial\mathrm{D}^n) \cong \begin{cases} \mathbb{Z} & j = n = \dim_{\mathbb{R}} M \\ 0 & \text{otherwise} \end{cases}.$$

2.6 Mayer-Vietoris and excision

Lecture 7 Friday 23/10/20

We have stated two major properties of homology and cohomology without proof, Mayer-Vietoris and excision. Recall that we also saw if

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes, then there exists a long exact sequence in homology

$$\dots \to \mathrm{H}_i(A_\bullet) \to \mathrm{H}_i(B_\bullet) \to \mathrm{H}_i(C_\bullet) \to \mathrm{H}_{i-1}(A_\bullet) \to \dots.$$

Mayer-Vietoris will be a consequence of this.

Definition. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subsets of X with the property that $X = \bigcup_{\alpha \in I} \mathring{U}_\alpha$, such as an open cover. Then

$$\mathrm{C}_j^{\mathcal{U}}(X) = \left\{ \sum_i a_i \sigma_i \;\middle|\; a_i \in \mathbb{Z},\ \sigma_i : \Delta^j \to X,\ \exists \alpha(i) \in I,\ \operatorname{im} \sigma_i \subseteq U_{\alpha(i)} \right\}$$

is the **subcomplex** of $(\mathrm{C}_\bullet(X), \partial)$ generated by simplices each of which lie wholly inside some set in $\mathcal{U}$.

Note that

$$\begin{array}{ccc} \mathrm{C}_\bullet(X) & \longrightarrow & \mathrm{C}_{\bullet-1}(X) \\ \cup & & \cup \\ \mathrm{C}_\bullet^{\mathcal{U}}(X) & \longrightarrow & \mathrm{C}_{\bullet-1}^{\mathcal{U}}(X) \end{array},$$

since $\mathrm{C}_\bullet^{\mathcal{U}}(X)$ is preserved by ∂ so is a subcomplex.

Proposition 2.9 (Small simplices theorem)**.** *The inclusion $\mathrm{C}_\bullet^{\mathcal{U}}(X) \hookrightarrow \mathrm{C}_\bullet(X)$ induces an isomorphism on homology.*

Remark. Suppose $f : X \to Y$ sends each element of $\mathcal{U}$ into some element of $\mathcal{V}$, the corresponding cover of Y. Then f induces $f_\# : \mathrm{C}_\bullet^{\mathcal{U}}(X) \to \mathrm{C}_\bullet^{\mathcal{V}}(Y)$.

Example (Mayer-Vietoris)**.** Let $\mathcal{U} = \{A, B\}$ for $A, B \subseteq X$ open. Then there is an obvious short exact sequence of chain complexes

$$0 \to \mathrm{C}_\bullet(A \cap B) \xrightarrow{\sigma \mapsto (\sigma, \sigma)} \mathrm{C}_\bullet(A) \oplus \mathrm{C}_\bullet(B) \xrightarrow{(u,v) \mapsto u - v} \mathrm{C}_\bullet^{\mathcal{U}}(X) \to 0,$$

which is onto since $\mathrm{C}_\bullet^{\mathcal{U}}(X)$ only contains simplices lying in A or B. The associated long exact sequence is the Mayer-Vietoris sequence, using small simplices to identify $\mathrm{H}_\bullet\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)\right) \xrightarrow{\sim} \mathrm{H}_\bullet(\mathrm{C}_\bullet(X))$. Note also the construction of the ∂ map in the long exact sequence associated to a short exact sequence of complexes does reproduce our earlier description of ∂_{MV}. Also the naturality of Mayer-Vietoris under maps $f : X \to Y$ such that $f(A) \subseteq C$ and $f(B) \subseteq D$ is just the naturality of $\mathrm{C}_\bullet^{\mathcal{U}}(X) \to \mathrm{C}_\bullet^{\mathcal{V}}(Y)$.

Example (Excision)**.** Recall we have $Z, A \subseteq X$ and $\overline{Z} \subseteq \mathring{A}$. Let $B = X \setminus Z$ and let $\mathcal{U} = \{A, B\}$, so the interiors of A and B do cover X. Note that

$$\mathrm{C}_n^{\mathcal{U}}(X) / \mathrm{C}_n(A) \cong \mathrm{C}_n(B) / \mathrm{C}_n(A \cap B)$$

is the free abelian group on simplices in B not wholly contained in A. The short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_\bullet(A) & \longrightarrow & \mathrm{C}_\bullet^{\mathcal{U}}(X) & \longrightarrow & \mathrm{C}_\bullet^{\mathcal{U}}(X) / \mathrm{C}_\bullet(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_\bullet(A) & \longrightarrow & \mathrm{C}_\bullet(X) & \longrightarrow & \mathrm{C}_\bullet(X) / \mathrm{C}_\bullet(A) & \longrightarrow & 0 \end{array},$$

and the natural map of short exact sequences give a map of long exact sequences

$$\begin{array}{ccccccccccc} \dots \to & \mathrm{H}_i(A) & \to & \mathrm{H}_i\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)\right) & \to & \mathrm{H}_i\left(\mathrm{C}_\bullet^{\mathcal{U}}(X) / \mathrm{C}_\bullet(A)\right) & \to & \mathrm{H}_{i-1}(A) & \to & \mathrm{H}_{i-1}\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)\right) & \to \dots \\ & \downarrow{=} & & \sim\downarrow{\text{small simplices}} & & \downarrow{\phi} & & \sim\downarrow{\text{small simplices}} & & \downarrow{=} & \\ \dots \to & \mathrm{H}_i(A) & \longrightarrow & \mathrm{H}_i(X) & \longrightarrow & \mathrm{H}_i(X, A) & \longrightarrow & \mathrm{H}_{i-1}(A) & \longrightarrow & \mathrm{H}_{i-1}(X) & \longrightarrow \dots \end{array}.$$

So by the five lemma, ϕ is an isomorphism, so

$$\mathrm{C}_\bullet^{\mathcal{U}}(X)/\mathrm{C}_\bullet(A) \hookrightarrow \mathrm{C}_\bullet(X)/\mathrm{C}_\bullet(A)$$

is an isomorphism on homology. So

$$\begin{aligned}\mathrm{H}_\bullet(X,A) = \mathrm{H}_\bullet(\mathrm{C}_\bullet(X)/\mathrm{C}_\bullet(A)) &\cong \mathrm{H}_\bullet\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)/\mathrm{C}_\bullet(A)\right) \\ &\cong \mathrm{H}_\bullet(\mathrm{C}_\bullet(B)/\mathrm{C}_\bullet(A\cap B)) = \mathrm{H}_\bullet(B, A\cap B) = \mathrm{H}_\bullet(X\setminus Z, A\setminus Z),\end{aligned}$$

proving excision.

2.7 Small simplices theorem

So it just remains to prove the small simplices theorem that $\mathrm{C}_\bullet^{\mathcal{U}}(X) \hookrightarrow \mathrm{C}_\bullet(X)$ is an isomorphism on homology. The key geometric ingredient is to divide simplices into smaller simplices.

Definition. The **barycentre**, or centre of mass, of Δ^n is

$$\mathrm{b}_n = \frac{(1,\dots,1)}{n+1}.$$

A **barycentric subdivision** is the following three-step procedure.

- Subdivide the boundary.
- Add the barycentre.
- Cone off from the barycentre to the subdivided boundary.

Definition. If $\sigma : \Delta^i \to \Delta^n \in \mathrm{C}_i(\Delta^n)$,

$$\begin{array}{rccc}\mathrm{Cone}_i^{\Delta^n}(\sigma) & : & \Delta^{i+1} & \longrightarrow & \Delta^n \\ & & (t_0,\dots,t_{i+1}) & \longmapsto & t_0\mathrm{b}_n + (1-t_0)\,\sigma\left(\dfrac{(t_1,\dots,t_{i+1})}{1-t_0}\right)\end{array}.$$

So, extended linearly, $\mathrm{Cone}_i^{\Delta^n} : \mathrm{C}_i(\Delta^n) \to \mathrm{C}_{i+1}(\Delta^n)$.

Exercise.

$$\partial\left(\mathrm{Cone}_i^{\Delta^n}(\sigma)\right) = \begin{cases}\sigma - \mathrm{Cone}_{i-1}^{\Delta^n}(\partial(\sigma)) & i > 0 \\ \sigma - \epsilon(\sigma)\,\mathrm{b}_n & i = 0\end{cases},$$

where

$$\begin{array}{rccc}\epsilon & : & \mathrm{C}_0(\Delta^n) & \longrightarrow & \mathbb{Z} \\ & & \sum_i n_i p_i & \longmapsto & \sum_i n_i\end{array}$$

is the augmentation.

Definition. Define

$$\begin{array}{rccc}\mathrm{c} & : & \mathrm{C}_\bullet(\Delta^n) & \longrightarrow & \mathrm{C}_\bullet(\Delta^n) \\ & & \sigma & \longmapsto & \begin{cases}\epsilon(\sigma)\,\mathrm{b}_n & \text{on } \mathrm{C}_0(\Delta^n) \\ 0 & \text{on } \mathrm{C}_i(\Delta^n),\ i > 0\end{cases}\end{array}.$$

Then

$$\partial \circ \mathrm{Cone}^{\Delta^n} + \mathrm{Cone}^{\Delta^n} \circ \partial = \mathrm{id}_{\mathrm{C}_\bullet(\Delta^n)} - \mathrm{c}.$$

Definition. A collection of chain maps $\phi^X : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X)$, defined for all spaces X, is **natural** if for all $f : X \to Y$,

$$f_\# \circ \phi^X = \phi^Y \circ f_\#.$$

Similarly for a collection $P : \mathrm{C}_\bullet(X) \to \mathrm{C}_{\bullet+1}(X)$ of chain homotopies between natural ϕ^X and ψ^X.

Definition. Define

$$\phi_0^X = \mathrm{id}_{\mathrm{C}_0(X)}, \qquad \begin{array}{rccl} \phi_n^X & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X) \\ & & \sigma & \longmapsto & \sigma_\#\left(\mathrm{Cone}_{n-1}^{\Delta^n}\left(\phi_{n-1}^{\Delta^n}\left(\partial\left(\iota_n\right)\right)\right)\right) \end{array},$$

where $\iota_n : \Delta^n \to \Delta^n \in \mathrm{C}_n(\Delta^n)$ is the identity, so $\partial(\iota_n) \in \mathrm{C}_{n-1}(\Delta^n)$.

Since $\sigma : \Delta^n \to X$ is $\sigma \circ \iota_n : \Delta^n \to \Delta^n \to X$, this is natural, since

$$\phi_n^X(\sigma) = \phi_n^X(\sigma_\#(\iota_n)) = \sigma_\#\left(\phi_n^{\Delta^n}(\iota_n)\right).$$

The idea is that we know how to subdivide Δ^n, so know how to subdivide any simplex in X.

Definition. Similarly, define

$$\begin{array}{rccl} P_n^X & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n+1}(X) \\ & & \sigma & \longmapsto & \sigma_\#\left(\mathrm{Cone}_n^{\Delta^n}\left(\phi_n^{\Delta^n}(\iota_n) - \iota_n - P_{n-1}^{\Delta^n}\left(\partial\left(\iota_n\right)\right)\right)\right) \end{array}.$$

This decomposes the prism $\Delta^n \times [0,1]$ by joining $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$ to the barycentre of $\Delta^n \times \{1\}$.

Fact. $\phi^X : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X)$ is a natural chain map, and $P^X : \mathrm{C}_\bullet(X) \to \mathrm{C}_{\bullet+1}(X)$ is a natural chain homotopy from ϕ^X to the identity, that is

$$\partial \circ P_n^X + P_{n-1}^X \circ \partial = \phi_n^X - \mathrm{id}_{\mathrm{C}_n(X)}.$$

We will not prove this.

Ok, now we know how to divide simplices.

Lemma 2.10. *If $[v_0, \dots, v_n] \subseteq \mathbb{R}^{n+1}$ is a simplex, then each simplex of its barycentric division has Euclidean diameter at most $n/(n+1)$ the Euclidean diameter of $[v_0, \dots, v_n]$.*

Corollary 2.11.

1. *If $\sigma \in \mathrm{C}_n^{\mathcal{U}}(X)$, $\phi_n^X(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$.*
2. *If $\sigma \in \mathrm{C}_n(X)$, there exists $k \gg 0$ such that $\left(\phi_n^X\right)^k(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$.*

Proof.

1. Obvious.
2. σ is a finite sum of simplices, so it suffices to prove the result for one $\sigma : \Delta^n \to X$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Now $\left\{\sigma^{-1}\left(\mathring{U}_\alpha\right)\right\}_{\alpha \in I}$ is an open cover of Δ^n, so has a Lebesgue number, that is there exists $\epsilon > 0$ such that any open ϵ-ball in Δ^n lies in some $\sigma^{-1}(U_\alpha)$. Now pick $k \gg 0$ such that $(n/(n+1))^k \ll \epsilon$.

□

Proof of Proposition 2.9. Let $U : \mathrm{H}_\bullet\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)\right) \to \mathrm{H}_\bullet(X)$ be the natural map.

- If $[c] \in \mathrm{H}_n(X)$, there exists k such that $\left(\phi_n^\times\right)^k(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. Since $\phi^X \simeq \mathrm{id}$, $\left(\phi^X\right)^k \simeq \mathrm{id}$, so there exists F such that $\partial \circ F + F \circ \partial = \left(\phi^X\right)^k - \mathrm{id}$. Then $\left(\phi^X\right)^k(c) = c + \mathrm{im}\,\partial$, so U is onto.
- If $U([c]) = 0$ for $[c] \in \mathrm{H}_n\left(\mathrm{C}_\bullet^{\mathcal{U}}(X)\right)$ and $z \in \mathrm{C}_{n+1}(X)$ has $\partial(z) = c$, there exists k such that $\left(\phi_{n+1}^X\right)^k(z) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$ and $\left(\phi_{n+1}^X\right)^k(z) - z = (\partial \circ F + F \circ \partial)(z)$, so

$$c = \partial(z) = \partial\left(\left(\phi_{n+1}^X\right)^k(z)\right) - \partial\left(F\left(\partial(z)\right)\right) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X),$$

since $\partial(z) \in \mathrm{C}_n^{\mathcal{U}}(X)$ and F is natural. Then $c \in \mathrm{im}\left(\partial : \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\right)$, so $[c] = 0$ and U is one-to-one.

□