

Algebraic Topology

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Syllabus

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0 Introduction

0.1 Connectedness

Lecture 1
Friday
09/10/20

Algebraic topology concerns the connectivity properties of topological spaces.

Definition. A space X is **path-connected** if for $p, q \in X$, there exists $\gamma : [0, 1] \rightarrow X$ continuous with $\gamma(0) = p$ and $\gamma(1) = q$.

Example. \mathbb{R} is path-connected, and $\mathbb{R} \setminus \{0\}$ is not.

Corollary 0.1 (Intermediate value theorem). *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $x < y$ satisfy $f(x) > 0$ and $f(y) < 0$ then f takes the value zero on $[x, y]$.*

Proof. Otherwise $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ disconnect $[x, y]$, a contradiction. \square

Definition. Let X and Y be topological spaces. Maps $f_0, f_1 : Y \rightarrow X$ are **homotopic** if there exists $F : Y \times [0, 1] \rightarrow X$ continuous such that $F|_{Y \times \{0\}} = f_0$ and $F|_{Y \times \{1\}} = f_1$. Write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$.

Exercise. \simeq is an equivalence relation on the set of maps from Y to X .

Note that X is **path-connected** if and only if every two maps $\{\text{point}\} \rightarrow X$ are homotopic. Let

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\},$$

so $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition. X is **simply-connected** if every two maps $S^1 \rightarrow X$ are homotopic.

Example. \mathbb{R}^2 is simply-connected, and $\mathbb{R}^2 \setminus \{0\}$ is not. From complex analysis you know $\gamma : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ has a **winding number** or **degree** $\deg \gamma \in \mathbb{Z}$, for which

- if $\gamma_n(t) = e^{2\pi i n t}$ then $\deg \gamma_n = n$, and
- $\deg \gamma_1 = \deg \gamma_2$ if $\gamma_1 \simeq \gamma_2$.

For differentiable γ , $\deg \gamma = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz$.

Corollary 0.2 (Fundamental theorem of algebra). *Every non-constant complex polynomial has a root.*

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be non-constant, and without loss of generality monic. Suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$. Let

$$\gamma_R(t) = f(Re^{2\pi i t}),$$

so $\gamma_R : S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$. Since γ_0 is constant, $\deg \gamma_0 = 0$, so $\deg \gamma_R = 0$ for all R . But take $R \gg \sum_i |a_i|$. Let

$$f_s(z) = z^n + s(a_1 z^{n-1} + \dots + a_n), \quad 0 \leq s \leq 1.$$

On the circle $|z| = R$, $f_s(z) \neq 0$ for all s . So if

$$\gamma_{R,s}(t) = f_s(Re^{2\pi i t}),$$

then $\gamma_{R,1} = \gamma_R$, which has degree zero from before, and $\gamma_{R,0} : t \mapsto R^n e^{2\pi i n t}$, which has degree $n \neq 0$, a contradiction. \square

Definition. X is **k -connected** if every two maps $S^i \rightarrow X$ are homotopic whenever $i \leq k$.

Example. \mathbb{R}^n is $(n-1)$ -connected, and $\mathbb{R}^n \setminus \{0\}$ is not. Maps $S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ have a homotopy invariant degree in \mathbb{Z} , where the degree of the inclusion is one and the degree of the constant map is zero. You may well not have seen this, and we will prove it later.

Corollary 0.3 (Brouwer's theorem). *Any map $f : \overline{B^n} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \rightarrow \overline{B^n}$ has a fixed point.*

Proof. Suppose f has no fixed point. Let

$$\gamma_R(v) = Rv - f(Rv), \quad 0 \leq R \leq 1, \quad v \in S^{n-1} = \partial \overline{B^n}.$$

Since f has no fixed point, γ_R takes values in $\mathbb{R}^n \setminus \{0\}$. Since γ_0 is constant, $\deg \gamma_0 = 0$, so $\deg \gamma_1 = 0$ by homotopy invariance. Let

$$\gamma_{1,s}(v) = v - sf(v), \quad 0 \leq s \leq 1.$$

Then $\gamma_{1,1} = \gamma_1$, and $\text{im } \gamma_{1,s} \subseteq \mathbb{R}^n \setminus \{0\}$ as $\|v\| = 1$ and $\|sf(v)\| = |s|\|f(v)\| < 1$ if $s < 1$, so $\deg \gamma_{1,0} = \deg \gamma_{1,1}$. The inclusion has $\deg \gamma_{1,0} = 1$ and $\deg \gamma_{1,1} = 0$ from above, a contradiction. \square

0.2 Homotopy

Definition. $f : X \rightarrow Y$ is a **homotopy equivalence** if there exists $g : Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$. Then g is a **homotopy inverse** for f , and \simeq is an equivalence relation on spaces.

Example. If X and Y are homeomorphic they are trivially homotopy equivalent, by taking $g = f^{-1}$.

Example. $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. Let

$$\begin{aligned} f : \mathbb{R}^n \setminus \{0\} &\longrightarrow S^{n-1} \\ v &\longmapsto \frac{v}{\|v\|}, \end{aligned}$$

and let $g : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion. Then $f \circ g = \text{id}_{S^{n-1}}$ and $g \circ f \simeq_F \text{id}_{\mathbb{R}^n \setminus \{0\}}$ via the homotopy

$$F(t, v) = tv + (1-t) \frac{v}{\|v\|}.$$

Example. $\{0\} \simeq \mathbb{R}^n$ is a homotopy equivalence.¹ If $X \simeq \{\text{point}\}$ we say X is **contractible**.

Algebraic topology is the study of topological spaces up to homotopy equivalence. The idea is that homeomorphism is too delicate a relation. Homotopy equivalence keeps track of essential topological information. More precisely, we assign

$$\{\text{spaces}\} \rightarrow \{\text{groups}\}, \quad \{\text{maps of spaces}\} \rightarrow \{\text{homomorphism of groups}\},$$

so we get algebraic invariants. They are defined for all spaces, but have more structure and use or interest for nicer spaces. The classical first attempt is homotopy theory. One can concatenate loops γ and τ by

$$(\gamma * \tau)(t) = \begin{cases} \gamma(2t) & t \leq \frac{1}{2} \\ \tau(1-2t) & t \geq \frac{1}{2} \end{cases}.$$

This is a well-defined operation on the **fundamental group**

$$\pi_1(X, x_0) = \{\text{maps } \gamma : S^1 \rightarrow X \mid \gamma(0) = x_0 \text{ fixed}\} / (\simeq \text{ preserving } x_0).$$

Similarly, the **n -th homotopy group** is

$$\pi_n(X, x_0) = \{\text{based maps } S^n \rightarrow X \text{ at } x_0\} / \simeq.$$

The issue is that they are very hard to compute, such as $\pi_n(S^2, x_0)$ not known for all n . There is no simply-connected **manifold**, a Hausdorff second countable space X locally homeomorphic to \mathbb{R}^n , of dimension greater than zero, with $\pi_n(X)$ known for all n . So we will do something else, homology and cohomology. It is algebraically harder to set up, but the computational gain is worth it. Note that computing cohomology of harder spaces, such as the space of diffeomorphisms of some manifold or the space of embeddings of one manifold into another, is still very hard.

Remark.

- Algebraic topology is all about being able to compute. It is important to do lots of examples.
- Our nice spaces are manifolds and indeed smooth manifolds. There is some overlap with differential geometry which will be useful, not essential but advised.

¹Exercise: check

1 Singular homology and cohomology

Lecture 2
Monday
12/10/20

We will define invariants of spaces in two stages.

- Associate to X a chain or cochain complex.
- Take the homology or cohomology of that complex.

1.1 Chain and cochain complexes

Definition. A **chain complex** (C_\bullet, ∂) is a sequence of abelian groups and homomorphisms

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots,$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n . We write $\partial^2 = 0$, and ∂ is the **differential** or **boundary map**. The **homology groups** $H_\bullet(C_\bullet, \partial)$ are the graded groups

$$H_n(C_\bullet) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Definition. A **cochain complex** (C^\bullet, ∂) is a sequence of abelian groups and homomorphisms

$$\cdots \rightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \rightarrow \cdots,$$

such that $\partial^n \circ \partial^{n-1} = 0$ for all n . We write $\partial^2 = 0$, and ∂ is still the **differential** or **boundary map**. The **cohomology groups** $H^\bullet(C^\bullet, \partial)$ are

$$H^n(C^\bullet) = \ker \partial^n / \operatorname{im} \partial^{n-1}.$$

Elements of $\ker(\partial : C_n \rightarrow C_{n-1})$ are **cycles**. Elements of $\operatorname{im}(\partial : C_{n+1} \rightarrow C_n)$ are **boundaries**. Elements of $\ker(\partial : C^n \rightarrow C^{n+1})$ are **cocycles**. Elements of $\operatorname{im}(\partial : C^{n-1} \rightarrow C^n)$ are **coboundaries**. Write all ∂_i and ∂^i as ∂ , or occasionally ∂_\bullet and ∂^\bullet . Elements of $H_\bullet(C_\bullet)$ are **homology classes** and of $H^\bullet(C^\bullet)$ are **cohomology classes**.

Definition. A **chain map** between chain complexes (C_\bullet, ∂) and (D_\bullet, ∂) is a sequence of homomorphisms $f_n : C_n \rightarrow D_n$ such that for all n the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \cdots \\ & & f_n \downarrow & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & D_n & \xrightarrow{\partial} & D_{n-1} & \longrightarrow & \cdots \end{array}$$

commutes. That is, $f_{n-1} \circ \partial_n^{C_\bullet} = \partial_{n-1}^{D_\bullet} \circ f_n$.

Exercise. Define a **cochain map** of cochain complexes.

Lemma 1.1. A chain map $f : C_\bullet \rightarrow D_\bullet$ induces homomorphisms $(f_*)_n : H_n(C_\bullet) \rightarrow H_n(D_\bullet)$ for each n .

Proof. Let $[a] \in H_n(C_\bullet)$, so a is represented by a cycle $\alpha \in C_n$, where $\partial(\alpha) = 0$. Then $\partial(f_n(\alpha)) = f_{n-1}(\partial(\alpha)) = 0$, so $f_n(\alpha)$ is a cycle. Define $(f_*)_n([a]) = [f_n(\alpha)] \in H_n(D_\bullet)$. We made a choice of representing the cycle α . But if $[a]$ is represented by α and α' , then $\alpha - \alpha' \in \operatorname{im}(\partial_{n+1} : C_{n+1} \rightarrow C_n)$. Say $\alpha - \alpha' = \partial(\tau)$. Then $f_n(\alpha) - f_n(\alpha') = f_n(\alpha - \alpha') = f_n(\partial(\tau)) = \partial(f_{n+1}(\tau))$, so $[f_n(\alpha)] = [f_n(\alpha') + \partial(f_{n+1}(\tau))] = [f_n(\alpha')]$ as $[\operatorname{im} \partial] = 0$ in $H_n(D_\bullet)$. So $(f_*)_n$ is well-defined, and it is easy to see it is a homomorphism. \square

Exercise. If $C_\bullet, D_\bullet, E_\bullet$ are chain complexes and $f : C_\bullet \rightarrow D_\bullet$ and $g : D_\bullet \rightarrow E_\bullet$ are chain maps then $\{g_n \circ f_n : C_n \rightarrow E_n\}_n$ defines a chain map. Also

$$(g \circ f)_* = g_* \circ f_*, \quad (\operatorname{id}_{C_\bullet})_* = \operatorname{id}_{H_\bullet(C_\bullet)} \quad (1)$$

The goal is to associate to a space X chain complexes $C_\bullet(X)$ and cochain complexes $C^\bullet(X)$ such that a map $f : X \rightarrow Y$ yields chain maps $f : C_\bullet(X) \rightarrow C_\bullet(Y)$ and cochain maps $f : C^\bullet(Y) \rightarrow C^\bullet(X)$. Then (1) will say we have a functor

$$\begin{array}{ccc} \mathbf{Top} & \longrightarrow & \mathbf{Ab} \\ X & \longmapsto & H_\bullet(X) \end{array} ,$$

from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms. Our complexes C_\bullet and C^\bullet will have the benefit that they are intrinsic but will be huge and unwieldy. We will

- prove structure theorems to help compute, and
- find smaller complexes later for nice spaces, such as CW-complexes.

1.2 Singular homology and cohomology

Definition. The **standard simplex** is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \forall i, t_i \geq 0, \sum_i t_i = 1 \right\}.$$

The i -th **face** of Δ^n is

$$\Delta_i^n = \{ \underline{t} \in \Delta^n \mid t_i = 0 \}.$$

Note that there exists a canonical homeomorphism

$$\begin{array}{ccc} \delta_i : & \Delta^{n-1} & \longrightarrow \Delta_i^n \subseteq \Delta^n \\ & (t_0, \dots, t_{n-1}) & \longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \end{array} .$$

Definition. If X is a space, a **singular n -simplex** in X is a map $\sigma : \Delta^n \rightarrow X$. The **singular chain complex** $(C_\bullet(X), \partial)$ has

$$C_n(X) = \left\{ \sum_{i=1}^N n_i \sigma_i \mid N < \infty, n_i \in \mathbb{Z}, \sigma_i : \Delta^n \rightarrow X \right\},$$

the free abelian group on the singular n -simplices in X , and

$$\begin{array}{ccc} \partial : & C_n(X) & \longrightarrow C_{n-1}(X) \\ & \sigma & \longmapsto \sum_{i=0}^n (-1)^i (\sigma \circ \delta_i) \end{array} ,$$

extended linearly.

Example. Δ^0 is a point, Δ^1 is a line, Δ^2 is a triangle, and Δ^3 is a tetrahedron.

Note that $n+1$ ordered points $\{v_i\}_{0 \leq i \leq n} \subseteq \mathbb{R}^{n+1}$ determine an n -simplex if $\{v_i - v_0 \mid 1 \leq i \leq n\}$ are linearly independent, by taking their convex hull, and

$$\begin{array}{ccc} \sigma : & \Delta^n & \longrightarrow \mathbb{R}^{n+1} \\ & \underline{t} & \longmapsto \sum_{i=0}^n t_i v_i \end{array} .$$

We orient the edges $v_i \rightarrow v_j$ if $i < j$. Write $[v_0, \dots, v_n]$ for this n -simplex, then

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]},$$

where the index $\widehat{v_i}$ is omitted.

Lemma 1.2. $\partial^2 = 0$.

Proof.

$$\partial(\partial(\sigma)) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]}.$$

Exchange i and j and the two terms cancel. □

Definition. The **singular homology** of X is

$$H_\bullet(X) = H_\bullet(X; \mathbb{Z}) = H_\bullet(C_\bullet(X), \partial).$$

Trivially this is a homeomorphism invariant of X , since we only used the notion of continuous maps to X to define it.

Definition. The **singular cochain complex** $(C^\bullet(X), \partial^*)$ has

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z}),$$

and

$$\begin{aligned} \partial^* : C^n(X) &\longrightarrow C^{n+1}(X) \\ \psi &\longmapsto (\sigma \mapsto \psi(\partial(\sigma))) \end{aligned}, \quad \sigma \in C_{n+1}(X),$$

which is adjoint to ∂ .

Then $\partial^*(\partial^*(\psi))(\sigma) = \partial^*(\psi)(\partial(\sigma)) = \psi(\partial(\partial(\sigma))) = 0$, so $(\partial^*)^2 = 0$ and this is a cochain complex.

Definition. The **singular cohomology** of X is

$$H^\bullet(X; \mathbb{Z}) = H^\bullet(C^\bullet(X), \partial^*).$$

The following is the rough idea.

- $\partial^2 = 0$ implies that the boundary of the boundary vanishes.
- $H_i(X)$ will probe i -dimensional holes or regions in X .
- $H^i(X)$ will be a rule associating an integer to an i -dimensional region of X .

Note that $H^\bullet(X; \mathbb{Z}) \not\cong \text{Hom}(H_\bullet(X), \mathbb{Z})$ in general.

Remark. Let $f : X \rightarrow Y$ be continuous. If $\sigma : \Delta^n \rightarrow X$ then $f \circ \sigma : \Delta^n \rightarrow Y$, so f gives a homomorphism $(f_\#)_n : C_n(X) \rightarrow C_n(Y)$. Also $f \circ (\sigma|_{\Delta_i^n}) \equiv (f \circ \sigma)|_{\Delta_i^n}$, since $f \circ (\sigma \circ \delta_i) = (f \circ \sigma) \circ \delta_i$. Thus

$$\begin{aligned} f_\# : C_\bullet(X) &\longrightarrow C_\bullet(Y) \\ \sigma &\longmapsto f \circ \sigma \end{aligned}$$

is a chain map such that

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ (f_\#)_n \downarrow & & \downarrow (f_\#)_{n-1} \\ C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) \end{array}$$

which gives homomorphisms

$$f_* : H_\bullet(X) \rightarrow H_\bullet(Y),$$

that is $(f_*)_n : H_n(X) \rightarrow H_n(Y)$ for each n . By the exercise,

$$((f \circ g)_*)_n = (f_*)_n \circ (g_*)_n, \quad ((\text{id}_{C_\bullet(X)})_*)_n = \text{id}_{H_n(X)}.$$

Note that $f : X \rightarrow Y$ induces a cochain map

$$\begin{aligned} f^\# : C^\bullet(Y) &\longrightarrow C^\bullet(X) \\ \psi &\longmapsto (\sigma \mapsto \psi(f \circ \sigma)) \end{aligned},$$

and homomorphisms

$$f^* : H^\bullet(Y) \rightarrow H^\bullet(X),$$

so cohomology is contravariant.

Lecture 3
Wednesday
14/10/20

1.3 Basic examples

What can we compute?

Lemma 1.3. *Let X be a point. Then*

$$H_i(\{\text{point}\}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. For each $n \geq 0$, there exists a unique n -simplex $\sigma_n : \Delta^n \rightarrow \{\text{point}\}$ in X , the constant map. Then $\partial(\sigma_1) = \sigma_1 \circ \delta_0 - \sigma_1 \circ \delta_1 = \sigma_0 - \sigma_0 = 0$ and $\partial(\sigma_2) = \sigma_2 \circ \delta_0 - \sigma_2 \circ \delta_1 + \sigma_2 \circ \delta_2 = \sigma_1 - \sigma_1 + \sigma_1 = \sigma_1$, and

$$\partial(\sigma_n) = \begin{cases} \sigma_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

So $C_\bullet(\{\text{point}\})$ is

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_3(\{\text{point}\}) & \longrightarrow & C_2(\{\text{point}\}) & \longrightarrow & C_1(\{\text{point}\}) & \longrightarrow & C_0(\{\text{point}\}) \\ & & \parallel & & \parallel & & \parallel & & \parallel \\ \dots & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \end{array}.$$

Now check the result. □

Exercise.

$$H^i(\{\text{point}\}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

There is basically only one other computation we can do from the definitions.

Lemma 1.4. *If $X = \bigsqcup_{\alpha \in I} X_\alpha$ is a disjoint union of path-components,*

$$H_i(X) \cong \bigoplus_{\alpha \in I} H_i(X_\alpha).$$

Proof. Any continuous map $\sigma : \Delta^i \rightarrow X$ has image in one X_α and then all the faces of σ lie in the same X_α , so

$$C_\bullet(X) = \bigoplus_{\alpha} C_\bullet(X_\alpha),$$

compatibly with the differential. □

Lemma 1.5. *If X is path-connected and non-empty,*

$$H_0(X) \cong \mathbb{Z}.$$

We sometimes write $\pi_0(X)$ for the set of path-components of X .

Proof. Define the **augmentation**

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_i n_i \sigma_i &\longmapsto \sum_i n_i, \end{aligned}$$

where $\sigma_i : \{\text{point}\} \rightarrow X$ are 0-simplices in X . Since $X \neq \emptyset$, ϵ is onto. If $\tau = [v_0, v_1] : \Delta^1 \rightarrow X$, then $\epsilon(\partial(\tau)) = \epsilon(v_1 - v_0) = 0$. So $\text{im}(\partial : C_1(X) \rightarrow C_0(X)) \subseteq \ker \epsilon$, so ϵ defines $H_0(X) = C_0(X) / \text{im } \partial \rightarrow \mathbb{Z}$. So far we did not use path-connectivity. But suppose $\sum_i n_i \sigma_i \in \ker \epsilon$. Fix a basepoint $p \in X$. For all i pick

$$\begin{aligned} \tau_i : \Delta^1 &\cong [0, 1] \longrightarrow X \\ 1 &\longmapsto \sigma_i \\ 0 &\longmapsto p \end{aligned}$$

Then $\partial(\sum_i n_i \tau_i) = \sum_i n_i \sigma_i - (\sum_i n_i) p = \sum_i n_i \sigma_i$, as $\sum_i n_i \sigma_i \in \ker \epsilon$, so $\ker \epsilon \subseteq \text{im } \partial$ and $\epsilon : H_0(X) \xrightarrow{\sim} \mathbb{Z}$. □

1.4 Structural theorems

The following is an informal picture. Let X be an annulus, and let $\sigma : \Delta^1 \rightarrow X$ be a 1-simplex, which happens to be a closed loop $[0, 1] \rightarrow X$ going around the inner circle. Recall that σ has $\partial(\sigma) = \sigma(1) - \sigma(0) = 0$, so σ defines $[\sigma] \in H_1(X)$. We would hope this is non-zero, as we cannot see a way to fill in σ with 2-simplices, in contrast to a 1-simplex $\tau : \Delta^1 \cong [0, 1] \rightarrow X$ away from the inner circle. But $C_i(X)$ is uncountably generated for all i and very hard to control. A question is how do we rule out all configurations of 2-simplices, or other representatives for $[\sigma] \in H_i(X)$? Informally, in the realm of nice spaces, there is nothing else you can compute from the definition. Homology and cohomology are rendered useful by a collection of structural theorems. We will state these, and see how to use them, and then return to prove them later.

Theorem 1.6 (Homotopy invariance). *If $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are homotopic, then*

$$f_* = g_* : H_\bullet(Y) \rightarrow H_\bullet(Y), \quad f^* = g^* : H^\bullet(Y) \rightarrow H^\bullet(Y).$$

Corollary 1.7. *If $X \simeq Y$ then $H_\bullet(X) \cong H_\bullet(Y)$ and $H^\bullet(X) \cong H^\bullet(Y)$.*

Proof. There exist $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$, so $(f_*)^{-1} = g_*$ are isomorphisms. \square

Thus homology and cohomology are insensitive to inessential deformations of a space.

Corollary 1.8. *For every n ,*

$$H_\bullet(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases},$$

and similarly for $H^\bullet(\mathbb{R}^n)$.

Definition. An **exact sequence** is a chain or cochain complex with vanishing homology or cohomology, so

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots,$$

such that $\ker \partial_n = \text{im } \partial_{n+1}$ for all n .

- Given homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

say this is **exact at B** if $\ker g = \text{im } f$.

- If

$$0 \rightarrow A \xrightarrow{f} B \rightarrow 0$$

is exact, $A \cong_f B$.

- A **short exact sequence** is one of shape

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0.$$

Example. If

$$0 \rightarrow \mathbb{Z} \rightarrow A \rightarrow \mathbb{Z}/n \rightarrow 0,$$

possibly $A = \mathbb{Z} \oplus \mathbb{Z}/n$, and

$$0 \rightarrow \mathbb{Z} \xrightarrow{1 \mapsto (1,0)} \mathbb{Z} \oplus \mathbb{Z}/n \xrightarrow{(0,1) \mapsto 1} \mathbb{Z}/n \rightarrow 0$$

or $A = \mathbb{Z}$, and

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{p \mapsto p \bmod n} \mathbb{Z}/n \rightarrow 0.$$

See question sheet 1.

Theorem 1.9 (Mayer-Vietoris). *If $X = A \cup B$ with A and B open, there are **Mayer-Vietoris boundary homomorphisms** $\partial_{MV} : H_{i+1}(X) \rightarrow H_i(A \cap B)$, yielding a **long exact sequence***

$$\cdots \rightarrow H_{i+1}(X) \xrightarrow{\partial_{MV}} H_i(A \cap B) \xrightarrow{((i_A)_*, (i_B)_*)} H_i(A) \oplus H_i(B) \xrightarrow{(j_A)_* - (j_B)_*} H_i(X) \rightarrow \cdots,$$

where

$$\begin{array}{ccc} A \cap B & \xhookrightarrow{i_A} & A \\ i_B \downarrow & & \downarrow j_A \\ B & \xhookrightarrow{j_B} & X \end{array}.$$

The Mayer-Vietoris boundary homomorphism is defined algebraically and is not associated to a map of spaces.

Remark. Suppose $\sigma \in C_{i+1}(X)$ is a cycle, so $\partial(\sigma) = 0$, and $\sigma = \alpha + \beta$ for chains $\alpha \in C_{i+1}(A)$ and $\beta \in C_{i+1}(B)$. Then $\partial(\alpha) = -\partial(\beta)$ and $\partial_{MV}([\sigma]) = [\partial(\alpha)]$, since $\partial(\alpha) \in A \cap B$.

Remark. The Mayer-Vietoris sequence is natural, so if $X = A \cup B$ and $Y = C \cup D$ and $f : X \rightarrow Y$ has $f(A) \subseteq C$ and $f(B) \subseteq D$ then there are homomorphisms of exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{i+1}(X) & \xrightarrow{\partial_{MV}} & H_i(A \cap B) & \longrightarrow & H_i(A) \oplus H_i(B) \longrightarrow H_i(X) \longrightarrow \cdots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \cdots & \longrightarrow & H_{i+1}(Y) & \xrightarrow{\partial_{MV}} & H_i(C \cap D) & \longrightarrow & H_i(C) \oplus H_i(D) \longrightarrow H_i(Y) \longrightarrow \cdots \end{array},$$

such that all squares commute.

Remark. There is a Mayer-Vietoris sequence in cohomology, which is also natural. There are $\partial_{MV}^* : H^i(A \cap B) \rightarrow H^{i+1}(X)$ such that

$$\cdots \rightarrow H^i(X) \xrightarrow{(j_A^*, j_B^*)} H^i(A) \oplus H^i(B) \xrightarrow{i_A^* - i_B^*} H^i(A \cap B) \xrightarrow{\partial_{MV}^*} H^{i+1}(X) \rightarrow \cdots$$

is exact, where

$$\begin{array}{ccc} A \cap B & \xhookrightarrow{i_A} & A \\ i_B \downarrow & & \downarrow j_A \\ B & \xhookrightarrow{j_B} & X \end{array}.$$

1.5 The sphere

Proposition 1.10.

$$H_i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}, \quad H^i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $S^1 = X = A \cup B$ where A and B are open intervals such that $A \cap B$ are two disjoint open intervals, so $A \simeq \{\text{point}\} \simeq B$ and $A \cap B \simeq \{\text{point}\} \sqcup \{\text{point}\} = \{p\} \sqcup \{q\}$. By homotopy invariance,

$$H_\bullet(\mathbb{R}) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases},$$

so we know $H_\bullet(A)$, $H_\bullet(B)$, and $H_\bullet(A \cap B)$. Mayer-Vietoris for $i \geq 2$ gives

$$\begin{array}{ccccc} H_i(A) \oplus H_i(B) & \longrightarrow & H_i(S^1) & \longrightarrow & H_{i-1}(A \cap B) \\ \parallel & & & & \parallel \\ 0 & & & & 0 \end{array}.$$

Check that $H_i(S^1) = 0$.² Mayer-Vietoris for $i = 0, 1$ gives

$$\begin{array}{ccccccc} H_1(A) \oplus H_1(B) & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \longrightarrow H_0(S^1) \\ \downarrow \cong & & & & \downarrow \cong & & \downarrow \cong \\ 0 & & & & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\alpha]{\quad} & \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[\beta]{\quad} \mathbb{Z} \end{array}$$

Recall that $H_0(Z)$ is free abelian on $\pi_0(Z)$, the set of path-components, and indeed is generated by $\sigma : \{\text{point}\} \rightarrow Z$, for any choice of point in each component. So

$$\alpha = ((i_A)_*, (i_B)_*) : \mathbb{Z}\langle p \rangle \oplus \mathbb{Z}\langle q \rangle \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) \longmapsto (a + b, a + b),$$

and

$$\beta = (j_A)_* - (j_B)_* : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \\ (u, v) \longmapsto u - v.$$

By exactness, $H_1(S^1) \cong \ker \alpha \cong \mathbb{Z}$, generated by $(1, -1) \equiv (p, -q) \in H_0(A) \oplus H_0(B)$. □

The same method as for computing $H_\bullet(S^1)$ shows the following.

Proposition 1.11.

$$H_j(S^n) \cong \begin{cases} \mathbb{Z} & j = 0, n \\ 0 & \text{otherwise} \end{cases}, \quad H^j(S^n) \cong \begin{cases} \mathbb{Z} & j = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

This time let us do the cohomology computation.

Proof. Let $S^n = A \cup B$ where $A \cong B \cong \mathbb{R}^n$ and $A \cap B \cong S^{n-1} \times (0, 1) \simeq S^{n-1}$. By homotopy invariance and induction, we know $H^\bullet(A)$, $H^\bullet(B)$, and $H^\bullet(A \cap B)$. Mayer-Vietoris now gives

$$\begin{array}{ccccccc} H^i(\mathbb{R}^n) \oplus H^i(\mathbb{R}^n) & \longrightarrow & H^i(S^{n-1}) & \longrightarrow & H^{i+1}(S^n) & \longrightarrow & H^{i+1}(\mathbb{R}^n) \oplus H^{i+1}(\mathbb{R}^n) \\ \downarrow \cong & & & & & & \downarrow \cong \\ 0 & & & & & & 0 \end{array},$$

so $H^i(S^{n-1}) \xrightarrow{\sim} H^{i+1}(S^n)$ for all $i > 0$. For $i = 0, 1$,

$$\begin{array}{ccccccc} H^0(S^n) & \longrightarrow & H^0(\mathbb{R}^n) \oplus H^0(\mathbb{R}^n) & \longrightarrow & H^0(S^{n-1}) & \longrightarrow & H^1(S^n) \longrightarrow H^1(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n) \\ & & & & & & \downarrow \cong \\ & & & & & & 0 \end{array}.$$

We showed before that for path-connected X , $H_0(X) \cong \mathbb{Z}$ is generated by $\sigma : \{\text{point}\} \rightarrow X \in C_0(X)$. By question sheet 1, $H^0(X) \cong \mathbb{Z}$ is generated by

$$\psi : C_0(X) \longrightarrow \mathbb{Z} \\ \sigma \longmapsto 1, \quad \sigma : \{\text{point}\} \rightarrow X.$$

If $n > 1$, then S^{n-1} is connected. So

$$\begin{array}{ccccccc} H^0(S^n) & \longrightarrow & H^0(\mathbb{R}^n) \oplus H^0(\mathbb{R}^n) & \longrightarrow & H^0(S^{n-1}) & \longrightarrow & H^1(S^n) \longrightarrow H^1(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\alpha]{\quad} & \mathbb{Z} & & 0 \end{array},$$

where $\alpha(p, q) = p + q$ is onto, so $H^1(S^n) = 0$, and now we have computed enough to complete the induction. □

Corollary 1.12. $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.

Proof. If $\mathbb{R}^m \cong \mathbb{R}^n$, then $S^{m-1} \simeq \mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, so $S^{m-1} \simeq S^{n-1}$. Thus $H_\bullet(S^{m-1}) \cong H_\bullet(S^{n-1})$, so $m = n$. □

This homeomorphism invariance of dimension was an early success of the subject. Recall there are space-filling curves $\phi : [0, 1] \rightarrow [0, 1]^2$ that are continuous and surjective.

²Exercise

1.6 Degrees

Lemma 1.13. Assume $n > 0$. A map $f : S^n \rightarrow S^n$ has a **degree** $\deg f \in \mathbb{Z}$ and if $g \simeq f$, then $\deg g = \deg f$.

Proof. f induces $(f_*)_n : H_n(S^n) \cong \mathbb{Z} \rightarrow H_n(S^n) \cong \mathbb{Z}$, which is multiplication by an integer. This defines $\deg f$. If $g \simeq f$, then $g_* = f_*$. A caveat is to use the same isomorphism on both sides and make sure $\deg f$ is defined and not just up to sign. \square

Exercise. Check that $\deg(f \circ g) = \deg f \cdot \deg g$.

Example. $\deg \text{id} = 1$, so if f is a homeomorphism, $\deg f \in \{\pm 1\}$.

Example. The degree of the constant map is zero, since the constant map

$$\begin{array}{ccc} f & : & S^n \longrightarrow S^n \\ & & x \longmapsto p \end{array}$$

factorises as $S^n \rightarrow \{\text{point}\} \rightarrow S^n$, so

$$\begin{array}{ccccc} H_n(S^n) & \longrightarrow & H_n(\{\text{point}\}) & \longrightarrow & H_n(S^n) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & \xrightarrow{\quad \quad \quad} & 0 & \xrightarrow{\quad \quad \quad} & \mathbb{Z} \end{array}$$

factorises through the zero group.

Note that combining with $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$, this fills in details, modulo homotopy invariance and Mayer-Vietoris, for results from the first lecture on Brouwer's theorem.

Lemma 1.14. Let $O(k) = \{A \in \text{Mat}_k \mathbb{R} \mid AA^T = I\}$. A matrix $A \in O(n+1)$, which acts on $S^n \subseteq \mathbb{R}^{n+1}$, acts on $H_n(S^n)$ by multiplication by $\det A$.

Proof. $O(n+1)$ has two path-connected components, so by homotopy invariance of degree, it suffices to show reflection in a hyperplane has degree -1 . Let $H = S^{n-1}$ be a hyperplane, let L be an invariant hemisphere, and let $H' = \partial L \cap H$. Note that a reflection $r_H : S^n \rightarrow S^n$ in H induces a reflection $r_{H'} : \partial L = S^{n-1} \rightarrow \partial L = S^{n-1}$ in H' . We computed $H_\bullet(S^n)$ by Mayer-Vietoris, using the decomposition which is r_H -invariant. By the naturality of Mayer-Vietoris,

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(S^n) & \xrightarrow{\sim} & H_{n-1}(S^{n-1}) & \longrightarrow & 0 \\ & & \downarrow r_H & & \downarrow r_{H'} & & \\ 0 & \longrightarrow & H_n(S^n) & \xrightarrow{\sim} & H_{n-1}(S^{n-1}) & \longrightarrow & 0 \end{array},$$

so inductively, it suffices to treat the case $n = 1$. So consider a circle $S^1 = A \cup B$ where $p, q \in A \cap B$. Our former Mayer-Vietoris computation of $H_\bullet(S^1)$ gave

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(p \sqcup q) & \longrightarrow & H_0(A) \oplus H_0(B) \\ & & & & \downarrow \cong & & \downarrow \cong \\ & & & & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\quad \quad \quad} & \mathbb{Z} \oplus \mathbb{Z} \end{array},$$

and $H_1(S^1) = \ker \alpha \cong \mathbb{Z} \langle (1, -1) \rangle$ is generated by $p - q$. So as r_H exchanges p and q it acts on $H_1(S^1)$ by -1 . \square

Corollary 1.15.

1. The antipodal map

$$\begin{array}{ccc} a_n & : & S^n \longrightarrow S^n \\ & & x \longmapsto -x \end{array}$$

has degree $(-1)^{n+1}$.

2. If $f : S^n \rightarrow S^n$ has no fixed point, then $f \simeq a_n$.

3. If G acts freely on S^{2k} , then $G \leq \mathbb{Z}/2$.

Proof.

1. $a_n : S^n \rightarrow S^n$ is a composition of $n + 1$ reflections $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$.
2. We will show if $f(x) \neq g(x)$ for all x , then $f \simeq a_n \circ g$. Consider

$$\phi_t : x \mapsto \frac{tf(x) - (1-t)g(x)}{\|tf(x) - (1-t)g(x)\|}, \quad 0 \leq t \leq 1.$$

Note that $tf(x) + (1-t)g(x) \neq 0$ or $t = \frac{1}{2}$ and $f(x) = g(x)$, a contradiction. So $f = \phi_1 \simeq \phi_0 = a_n \circ g$.

3. Question sheet 1.

□

We borrow a concept from differential topology. A **vector field** on S^n is a map $v : S^n \rightarrow \mathbb{R}^{n+1}$ such that for all $x \in S^n$, the Euclidean inner product on \mathbb{R}^{n+1} has $\langle x, v(x) \rangle = 0$. Note that this is a global section of the tangent bundle $TS^n \rightarrow S^n$.

Proposition 1.16 (Hairy ball theorem). S^n has a nowhere-vanishing vector field if and only if n is odd.

Proof. If $n = 2k - 1$, set

$$v(x_1, y_1, \dots, x_k, y_k) = (-y_1, x_1, \dots, -y_k, x_k).$$

Suppose n is even, and for contradiction that such v exists. So $v/\|v\| : S^n \rightarrow S^n$. Consider

$$v_t(x) = (\cos t)x + (\sin t) \frac{v}{\|v\|}(x).$$

Then $|v_t(x)| = 1$ for all t , and $v_0 = \text{id}$ and $v_\pi = -\text{id} = a_n$, so $\text{id}_{S^n} \simeq a_n$. Thus $\deg \text{id} = \deg a_n$, so $1 = (-1)^{n+1}$. □

1.7 The Klein bottle

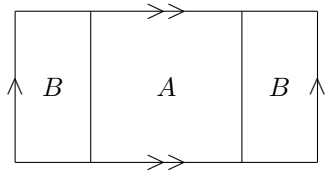
We should do one computation which involves knowing the maps, not just on $H_0(X)$, in an exact sequence, and not just that the sequence is exact. The **Klein bottle** K is obtained from gluing two Möbius bands together.

Lecture 5
Monday
19/10/20

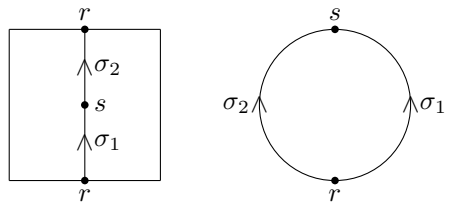
Lemma 1.17.

$$H_j(K; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

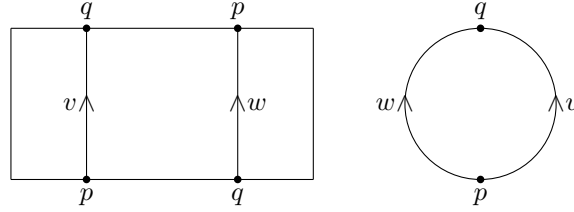
Proof. Apply Mayer-Vietoris to K



where $A \simeq S^1$ is a Möbius band



and $B \simeq S^1$ is a similar Möbius band, such that $A \cap B \simeq S^1$ is



The essential part of the long exact sequence is

$$\begin{array}{ccccccc}
 0 \longrightarrow H_2(K) \longrightarrow H_1(A \cap B) & \xrightarrow{\psi} & H_1(A) \oplus H_1(B) & \longrightarrow & H_1(K) & \xrightarrow{0} & H_0(A \cap B) \longrightarrow H_0(A) \oplus H_0(B) \\
 \parallel & & \parallel & & & & \parallel \\
 \mathbb{Z} & \xrightarrow{\quad \quad \quad} & \mathbb{Z} \oplus \mathbb{Z} & & & & \mathbb{Z} \xrightarrow[p \mapsto (p,p)]{\quad \quad \quad} \mathbb{Z} \oplus \mathbb{Z}
 \end{array}$$

By exactness, $H_1(K) = (\mathbb{Z} \oplus \mathbb{Z}) / \text{im } \psi$ and $H_2(K) \cong \ker \psi$. The key claim is that $\psi(1) = (2, 2)$ and note $(\mathbb{Z} \oplus \mathbb{Z}) / \langle 2, 2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2$. For this, $A \cap B$ is homotopy equivalent to the boundary circle of the central Möbius band, so $H_1(A \cap B) = \mathbb{Z} \langle v + w \rangle$, and A is homotopy equivalent to the core circle of the central Möbius band, so $H_1(A) = \mathbb{Z} \langle \sigma_1 + \sigma_2 \rangle$. Thus $\psi : v \mapsto \sigma_1 + \sigma_2$ and $\psi : w \mapsto \sigma_1 + \sigma_2$. \square

Remark. We could define

$$C_k(X; G) = \left\{ \sum_i a_i \sigma_i \mid a_i \in G, \sigma_i : \Delta^k \rightarrow X \right\},$$

for any abelian group G , with the same differential ∂ , which gives $H_\bullet(X; G)$, the **singular homology with coefficients in G** .

Example.

$$H_j(S^1; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & j = 0, 1 \\ 0 & \text{otherwise} \end{cases}, \quad H_i(\{\text{point}\}; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In the previous sequence, if we compute $H_\bullet(K; \mathbb{Z}/2)$, get

$$\begin{array}{ccccccc}
 0 \longrightarrow H_2(K; \mathbb{Z}/2) \longrightarrow H_1(A \cap B; \mathbb{Z}/2) & \xrightarrow{\psi} & H_1(A; \mathbb{Z}/2) \oplus H_1(B; \mathbb{Z}/2) & & & & \\
 \parallel & & \parallel & & & & \\
 \mathbb{Z}/2 & \xrightarrow[1 \mapsto (2,2) \equiv (0,0)]{\quad \quad \quad} & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & & & &
 \end{array}$$

so ψ vanishes for $H_\bullet(-; \mathbb{Z}/2)$ and

$$H_i(K; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z}/2 & i = 2 \\ 0 & \text{otherwise} \end{cases}.$$

It is also instructive to think about cohomology in this example, where $K = A \cup B$ for $A, B \simeq S^1$ and $A \cap B \simeq S^1$ as before. So the interesting parts of the cohomology Mayer-Vietoris sequences look like

$$\begin{array}{ccccccc}
 H^1(K) & \xrightarrow{(j_A^*, j_B^*)} & H^1(A) \oplus H^1(B) & \xrightarrow{i_A^* - i_B^*} & H^1(A \cap B) & \longrightarrow & H^2(K) \longrightarrow 0 \\
 \parallel & & \parallel & & \parallel & & \\
 \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\psi]{\quad \quad \quad} & \mathbb{Z} & & & &
 \end{array}$$

Check that this ψ is $(a, b) \mapsto 2(a - b)$.³ So $H^2(K) \cong \mathbb{Z}/2$. For contrast, $H_2(K) = 0$ if we use \mathbb{Z} coefficients.

Remark. There were many ways we could have cut up K . In some cases, some decompositions will give easier algebra than others.

³Exercise

2 Structural theorems

Now we should pay some debts.

2.1 Chain homotopy

Let C_\bullet and D_\bullet be chain complexes.

Definition. Chain maps $f : C_\bullet \rightarrow D_\bullet$ and $g : C_\bullet \rightarrow D_\bullet$ are **chain homotopic** if there exist $P_n : C_n \rightarrow D_{n+1}$ such that

$$P_{n-1} \circ \partial_n^{C_\bullet} \pm \partial_{n+1}^{D_\bullet} \circ P_n = f_n - g_n,$$

so

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \cdots \\ & & \searrow P_n & \downarrow & \swarrow P_{n-1} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{\partial} & D_n & \longrightarrow & \cdots \end{array}.$$

Lemma 2.1. If $f : C_\bullet \rightarrow D_\bullet$ and $g : C_\bullet \rightarrow D_\bullet$ are chain homotopic, then

$$(f_*)_i = (g_*)_i : H_i(C_\bullet, \partial) \rightarrow H_i(D_\bullet, \partial),$$

for all i , that is chain homotopic maps induce the same map on homology.

Recall we are trying to prove if $f \simeq g : X \rightarrow Y$, then $f_* = g_* : H_\bullet(X) \rightarrow H_\bullet(Y)$. So it will be sufficient to show $f_\#, g_\# : C_\bullet(X) \rightarrow C_\bullet(Y)$ are chain homotopic.

Proof. Let

$$\begin{array}{ccc} & C_n & \xrightarrow{\partial} C_{n-1} \\ & \searrow P_n & \downarrow \\ D_{n+1} & \xrightarrow{\partial} & D_n \end{array} \quad \swarrow P_{n-1},$$

such that $P_{n-1} \circ \partial \pm \partial \circ P_n = f_n - g_n$. Let $\alpha \in C_n$ be a cycle, so $\partial(\alpha) = 0$. So $\partial(f_n(\alpha)) = f_{n-1}(\partial(\alpha)) = 0$, so $(f_*)_n([\alpha]) = [f_n(\alpha)]$. So

$$f_n(\alpha) - g_n(\alpha) = (f_n - g_n)(\alpha) = P_{n-1}(\partial(\alpha)) \pm \partial(P_n(\alpha)) = \partial(P_n(\alpha)) \in \text{im } \partial,$$

so $[f_n(\alpha)] = [g_n(\alpha)] \in H_n(D_\bullet)$. □

Exercise. Chain homotopy is an equivalence relation on chain complexes and chain maps.

2.2 Proof of homotopy invariance

Theorem 2.2 (Homotopy invariance, version 2). If $f \simeq g : X \rightarrow Y$ then

$$f_\# \simeq g_\# : (C_\bullet(X), \partial) \rightarrow (C_\bullet(Y), \partial)$$

are chain homotopic.

Proof. If $f \simeq g$, then there exists $F : X \times [0, 1] \rightarrow Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$. So if

$$\begin{array}{ccc} \iota_0 : X & \longrightarrow & X \times [0, 1] \\ x & \longmapsto & (x, 0) \end{array}, \quad \begin{array}{ccc} \iota_1 : X & \longrightarrow & X \times [0, 1] \\ x & \longmapsto & (x, 1) \end{array},$$

then $f = F \circ \iota_0$ and $g = F \circ \iota_1$, so $f_\# = g_\#$ if $(\iota_0)_\# = (\iota_1)_\#$ and it suffices to prove that $(\iota_0)_\# \simeq (\iota_1)_\# : C_\bullet(X) \rightarrow C_\bullet(X \times [0, 1])$, so Y is out of the picture. So want $P_n : C_n(X) \rightarrow C_{n+1}(X \times [0, 1])$. The idea is that P_n is a **prism operator**

$$\begin{array}{ccc} C_n(X) & \longrightarrow & C_{n+1}(X \times [0, 1]) \\ \sigma : \Delta^n \rightarrow X & \longmapsto & \text{linear combination of simplices for } \sigma \times \text{id} : \Delta^n \times [0, 1] \rightarrow X \times [0, 1] \end{array}.$$

It gives an universal way of cutting up $\Delta^n \times [0, 1]$ into $(n + 1)$ -simplices. The equation

$$\partial \circ P \pm P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$$

says that the boundary of the prism is the prism on the boundary plus the top minus the bottom. The details of the proof are not very illuminating, so we will be quite terse. Label the base of the prism by $[v_0, \dots, v_n]$ and the top $[w_0, \dots, w_n]$. Claim that $\sigma_{n+1}^i = [v_0, \dots, v_i, w_i, \dots, w_n]$ is an $(n + 1)$ -simplex, and

$$\Delta^n \times [0, 1] = \bigcup_{i=0}^n \sigma_{n+1}^i.$$

We will not prove this, so see Hatcher. Define

$$\begin{aligned} P_n : C_n(X) &\longrightarrow C_{n+1}(X \times [0, 1]) \\ \sigma &\longmapsto \sum_{i=0}^n (-1)^i (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, w_n]} = \sum_{i=0}^n (-1)^i ((\sigma \times \text{id}) \circ \sigma_{n+1}^i) . \end{aligned}$$

Claim that $\partial \circ P + P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$. Well,

$$\begin{aligned} \partial(P_n(\sigma)) &= \sum_{j \leq i} (-1)^i (-1)^j (\sigma \times \text{id})|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{j+1} (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]} \\ &= (\sigma \times \text{id})|_{[\widehat{v_0}, w_0, \dots, w_n]} - (\sigma \times \text{id})|_{[v_0, \dots, v_n, \widehat{w_n}]} \\ &\quad + \sum_{j < i} (-1)^i (-1)^j (\sigma \times \text{id})|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]} , \end{aligned}$$

since the $i = j$ terms cancel in pairs except for $i = j = 0$, the top, and $i = j = n$, the bottom. Check that the latter sums are $-P_n(\partial(\sigma))$,⁴ which is routine but unenlightening. \square

Remark. If C^\bullet and D^\bullet are cochain complexes, then $f \simeq g$ are **cochain homotopic** if there exist $P^i : C^i \rightarrow D^{i-1}$ such that

$$\partial^* \circ P \pm P \circ \partial^* = f - g,$$

so

$$\begin{array}{ccccccc} \dots & \xrightarrow{\quad} & C^i & \xrightarrow{\partial^i} & C^{i+1} & \xrightarrow{\quad} & \dots \\ & & \swarrow P^i & \downarrow & \swarrow P^{i+1} & & \\ \dots & \xrightarrow{\quad} & D^{i-1} & \xrightarrow{\partial^{i+1}} & D^i & \xrightarrow{\quad} & \dots \end{array}$$

Check that⁵

$$f^* = g^* : H^\bullet(C^\bullet) \rightarrow H^\bullet(D^\bullet).$$

Then $P_n : C_n(X) \rightarrow C_{n+1}(X \times [0, 1])$ has dual

$$P^n : \text{Hom}(C_{n+1}(X \times [0, 1]), \mathbb{Z}) = C^{n+1}(X \times [0, 1]) \rightarrow \text{Hom}(C_n(X), \mathbb{Z}) = C^n(X),$$

and $\partial \circ P + P \circ \partial = (\iota_1)_\# - (\iota_0)_\#$ implies that

$$\partial^* \circ P + P \circ \partial^* = \iota_1^\# - \iota_0^\#,$$

so cohomology is also homotopy invariant.

⁴Exercise

⁵Exercise

2.3 The long exact sequence

We have made various computations using homotopy invariance, which we have proved, and Mayer-Vietoris, which we have not. Before addressing that, we need some more algebra. Recall that a short exact sequence is an exact sequence of the shape

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0, \quad \text{im } \alpha = \ker \beta.$$

Definition. A short exact sequence of chain complexes is a diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{n+1} & \xrightarrow{\alpha} & B_{n+1} & \xrightarrow{\beta} & C_{n+1} \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & A_n & \xrightarrow{\alpha} & B_n & \xrightarrow{\beta} & C_n \longrightarrow 0, \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{\alpha} & B_{n-1} & \xrightarrow{\beta} & C_{n-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

such that all squares commute, and the columns are chain complexes and the rows are exact, so $\text{im } \alpha = \ker \beta$ and $\partial^2 = 0$. Write

$$0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0.$$

Proposition 2.3. *If*

$$0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$$

is a short exact sequence of chain complexes, there is a boundary map $\delta : H_n(C_\bullet) \rightarrow H_{n-1}(A_\bullet)$ fitting into a long exact sequence on homology

$$\cdots \rightarrow H_n(A_\bullet) \xrightarrow{(\alpha_*)_n} H_n(B_\bullet) \xrightarrow{(\beta_*)_n} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \rightarrow \cdots$$

Proof. By diagram chasing, we will construct δ , and the proof of exactness is relegated to question sheet 1. Let

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_n & \xrightarrow{\alpha} & B_n & \xrightarrow{\beta} & C_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{\alpha} & B_{n-1} & \xrightarrow{\beta} & C_{n-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{n-2} & \xrightarrow{\alpha} & B_{n-2} & \xrightarrow{\beta} & C_{n-2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

(Note: A dashed arrow labeled α points from B_n to A_{n-1} in the original image.)

Let $c_n \in C_n$ be a cycle, so $\partial(c_n) = 0$, representing $[c_n] \in H_n(C_\bullet)$. Since β is onto, there exists $b_n \in B_n$ such that $\beta(b_n) = c_n$. Since the top right square commutes, $\beta(\partial(b_n)) = \partial(\beta(b_n)) = \partial(c_n) = 0$. Since the middle sequence is exact, $\partial(b_n) \in \ker \beta = \text{im } \alpha$, so $\partial(b_n) = \alpha(a_{n-1})$. Since the bottom left square commutes, $\alpha(\partial(a_{n-1})) = \partial(\alpha(a_{n-1})) = \partial^2(b_n) = 0$. Then α is one-to-one, so $\alpha(\partial(a_{n-1})) = 0$ implies that $\partial(a_{n-1}) = 0$, and set

$$\delta([c_n]) = [a_{n-1}].$$

Check δ is well-defined.

- Given c_n , we chose b_n . If $\beta(b'_n) = c_n$, then $b_n - b'_n \in \ker \beta = \text{im } \alpha$, so $b'_n = b_n + \alpha(a_n)$ for some $a_n \in A_n$, and $\partial(b'_n) = \partial(b_n) + \partial(\alpha(a_n)) = \alpha(a_{n-1} + \partial(a_n))$, so $[a_{n-1}] \in H_{n-1}(A_\bullet)$ is unchanged.
- If $[c_n] = [c'_n]$, then $c_n - c'_n \in \text{im } \partial$, say $c'_n = c_n + \partial(c_{n+1})$. Pick b_{n+1} such that $\beta(b_{n+1}) = c_{n+1}$ and then $b_n \mapsto b_n + \partial(b_{n+1})$ and $\partial(b_n)$ is unchanged, so get the same a_{n-1} .

So δ is well-defined and it is easy to see it is a homomorphism. In the resulting

$$\cdots \rightarrow H_n(A_\bullet) \xrightarrow{(\alpha_*)} H_n(B_\bullet) \xrightarrow{(\beta_*)} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \rightarrow \cdots,$$

should check exactness at all three kinds of terms, that is $\text{im } \beta_* \subseteq \ker \delta$ and $\ker \delta \subseteq \text{im } \beta_*$, etc, so six inclusions in total. ⁶ \square

For this piece of algebra to be useful, we need a source of short exact sequences of chain complexes.

Example. Recall if G is an abelian group,

$$C_k(X; G) = \left\{ \sum_i a_i \sigma_i \mid a_i \in G, \sigma_i : \Delta^k \rightarrow X \right\},$$

which gives $H_\bullet(X; G)$, the singular homology with coefficients in G . Note that if

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$$

is a short exact sequence of groups,

$$0 \rightarrow C_\bullet(X; G_1) \rightarrow C_\bullet(X; G_2) \rightarrow C_\bullet(X; G_3) \rightarrow 0$$

is a short exact sequence of chain complexes. The resulting $\delta : H_n(X; G_3) \rightarrow H_{n-1}(X; G_1)$ is a **Bockstein homomorphism**. For example,

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{p \mapsto p \pmod n} \mathbb{Z}/n \rightarrow 0, \quad 0 \rightarrow \mathbb{Z}/n \xrightarrow{\cdot n} \mathbb{Z}/n^2 \xrightarrow{p \mapsto p \pmod n} \mathbb{Z}/n \rightarrow 0$$

give the **classical Bockstein homomorphisms**

$$H_p(X; \mathbb{Z}/n) \rightarrow H_{p-1}(X; \mathbb{Z}), \quad H_p(X; \mathbb{Z}/n) \rightarrow H_{p-1}(X; \mathbb{Z}/n).$$

We will revisit these later, probably.

2.4 Relative homology

Example. Let $A \subseteq X$ be a subspace. We have an inclusion $C_\bullet(A) \hookrightarrow C_\bullet(X)$ compatible with boundary maps, since if $\sigma : \Delta^i \rightarrow A \subseteq X$, then $\sigma \circ \delta_i : \Delta^{i-1} \rightarrow A$ too. Define

$$C_\bullet(X, A) = C_\bullet(X) / C_\bullet(A),$$

so

$$0 \rightarrow C_\bullet(A) \rightarrow C_\bullet(X) \rightarrow C_\bullet(X, A) \rightarrow 0$$

is a short exact sequence of chain complexes.

Definition. $H_\bullet(C_\bullet(X, A), \partial)$ is denoted $H_\bullet(X, A)$, or $H_\bullet(X, A; G)$, the **relative homology** of (X, A) .

⁶Exercise: do this

Lemma 2.4. *If $f : (X, A) \rightarrow (Y, B)$ is a map of pairs, that is $f : X \rightarrow Y$ satisfies $f(A) \subseteq B$, then f induces $(f_*)_i : H_i(X, A) \rightarrow H_i(Y, B)$ for all i .*

Proof. Elementary. □

The long exact sequence

$$\cdots \rightarrow H_i(A) \rightarrow H_i(X) \rightarrow H_i(X, A) \rightarrow H_{i-1}(A) \rightarrow \cdots$$

is called the **long exact sequence of the pair** (X, A) .

Remark.

- Cycles in $C_\bullet(X, A)$ are chains in X whose boundary lies in A .
- You might expect that things in A do not matter for $C_\bullet(X, A)$, as we quotient all simplices in A . A precise version of that intuition is excision.

Theorem 2.5 (Excision). *Let X be a space, $A \subseteq X$ a subspace, and Z a subspace such that $\bar{Z} \subseteq \mathring{A}$. Then the inclusion $\iota : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ is an isomorphism on relative homology, so for all n ,*

$$(\iota_*)_n : H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A).$$

We will prove excision and Mayer-Vietoris together next time. For now, let us see how this helps us understand relative homology.

Remark. Naturality under maps, homotopy invariance, the relative homology long exact sequence, and excision are the key tools of homology and cohomology. Much of what we will do will be built from these.

Lemma 2.6 (5-lemma). *Suppose*

$$\begin{array}{ccccccccc} A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \xrightarrow{\partial} & D & \xrightarrow{\partial} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \xrightarrow{\partial'} & B' & \xrightarrow{\partial'} & C' & \xrightarrow{\partial'} & D' & \xrightarrow{\partial'} & E' \end{array}$$

is a commuting diagram of abelian groups with exact rows. If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then so is γ .

Proof. More diagram chasing. We will show γ is one-to-one, and you check it is onto.⁷ Let $c \in C$ have $\gamma(c) = 0$. Then $\delta(\partial(c)) = \partial'(\gamma(c)) = 0$ so $\partial(c) \in \ker \delta$, and δ is an isomorphism so $\partial(c) = 0$. Since the rows are exact, $c \in \ker \partial = \text{im } \partial$, so $c = \partial(b)$ for $b \in B$. Then $\partial'(\beta(b)) = \gamma(\partial(b)) = \gamma(c) = 0$, so $\beta(b) \in \ker \partial' = \text{im } \partial'$, and $\beta(b) = \partial'(a')$. Since α is an isomorphism, there exists $a \in A$ such that $\alpha(a) = a'$. Now $\beta(\partial(a)) = \partial'(\alpha(a)) = \partial'(a') = \beta(b)$ so $\partial(a) - b \in \ker \beta$, and β is an isomorphism so $b = \partial(a)$. Thus $c = \partial(b) = \partial^2(a) = 0$ and c is one-to-one. □

Corollary 2.7. *If $f : (X, A) \rightarrow (Y, B)$ is a map of pairs, and any two of the induced homomorphisms*

$$H_\bullet(X) \rightarrow H_\bullet(Y), \quad H_\bullet(A) \rightarrow H_\bullet(B), \quad H_\bullet(X, A) \rightarrow H_\bullet(Y, B)$$

are isomorphisms, then so is the third.

Proof. Apply the 5-lemma to

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_i(A) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, A) & \longrightarrow & H_{i-1}(A) & \longrightarrow & H_{i-1}(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_i(B) & \longrightarrow & H_i(Y) & \longrightarrow & H_i(Y, B) & \longrightarrow & H_{i-1}(B) & \longrightarrow & H_{i-1}(Y) & \longrightarrow & \cdots \end{array}$$

□

⁷Exercise

2.5 Reduced homology and good pairs

We need two definitions to proceed. The first looks a bit odd, but be patient.

Definition. If X is a space, and $x_0 \in X$ is a basepoint, the **reduced homology** is

$$\widetilde{H}_i(X) = H_i(X, x_0).$$

Exercise. The long exact sequence of a pair shows

$$\widetilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X), \quad \widetilde{H}_i(X) \cong H_i(X), \quad i > 0.$$

Definition. A pair (X, A) is **good** if $A \subseteq X$ is closed and is a deformation retract of an open neighbourhood $A \subseteq U \subseteq X$, that is there exists $H : [0, 1] \times U \rightarrow U$ such that

- $H|_{\{0\} \times U} = \text{id}$ and $H|_{\{1\} \times U}$ has image in A , and
- H is fixed on A , so for all $t \in [0, 1]$ and $a \in A$, $H(t, a) = a$.

So you can squeeze U back onto A without moving A . If X , and hence U , is Hausdorff, then A is automatically closed.

Proposition 2.8. *If (X, A) is good, the natural map $(X, A) \rightarrow (X/A, A/A)$ induces isomorphisms*

$$H_\bullet(X, A) \xrightarrow{\sim} \widetilde{H}_\bullet(X/A).$$

Proof. Note that homotopy invariance and the 5-lemma show inclusion defines isomorphisms

$$H_\bullet(A) \xrightarrow{\sim} H_\bullet(U), \quad H_\bullet(X, A) \xrightarrow{\sim} H_\bullet(X, U).$$

The inclusion $A/A = \{\text{point}\} \hookrightarrow U/A$ is a deformation retract and in particular a homotopy equivalence, so

$$H_\bullet(X/A, A/A) \xrightarrow{\sim} H_\bullet(X/A, U/A)$$

is also an isomorphism by the 5-lemma. Consider

$$\begin{array}{ccccc} H_\bullet(X, A) & \xrightarrow[\sim]{\text{Homotopy}} & H_\bullet(X, U) & \xleftarrow[\sim]{\text{Excision}} & H_\bullet(X \setminus A, U \setminus A) \\ \downarrow & & & & \downarrow \\ H_\bullet(X/A, A/A) & \xrightarrow[\sim]{\text{Homotopy}} & H_\bullet(X/A, U/A) & \xleftarrow[\sim]{\text{Excision}} & H_\bullet((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \end{array},$$

where the vertical maps collapse A . Then the right vertical map is a homeomorphism of pairs, since $X \setminus A \cong (X/A) \setminus (A/A)$. So the right vertical map is an isomorphism and hence the left vertical map is an isomorphism. \square

Remark. The **tubular neighbourhood theorem** of differential topology, which we will discuss more later, implies that if X is a smooth manifold and $A \subseteq X$ is a compact smooth submanifold, (X, A) is a good pair.

Example.

$$H_j(D^n, \partial D^n) \cong \widetilde{H}_j(D^n / \partial D^n) = \widetilde{H}_j(S^n) = \begin{cases} \mathbb{Z} & j = n \\ 0 & \text{otherwise} \end{cases}.$$

Example. Let S^1 be the equator. Then

$$H_j(S^2, S^1) \cong \widetilde{H}_j(S^2 \vee S^2) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & j = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Remark. If M is a manifold and $x \in M$, by excision with $Z = M \setminus \{\text{open disc neighbourhood of } x\}$ and homotopy invariance or directly from the long exact sequence of a pair,

$$H_j(M, M \setminus \{x\}) \cong H_j(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_j(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z} & j = n = \dim_{\mathbb{R}} M \\ 0 & \text{otherwise} \end{cases}.$$

2.6 Proof of Mayer-Vietoris and excision

We have stated two major properties of homology and cohomology without proof, Mayer-Vietoris and excision. Recall that we also saw if

$$0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$$

is a short exact sequence of chain complexes, then there exists a long exact sequence in homology

$$\cdots \rightarrow H_i(A_\bullet) \rightarrow H_i(B_\bullet) \rightarrow H_i(C_\bullet) \rightarrow H_{i-1}(A_\bullet) \rightarrow \cdots$$

Mayer-Vietoris will be a consequence of this.

Definition. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subsets of X with the property that $X = \bigcup_{\alpha \in I} U_\alpha$, such as an open cover. Then

$$C_j^\mathcal{U}(X) = \left\{ \sum_i a_i \sigma_i \mid a_i \in \mathbb{Z}, \sigma_i : \Delta^j \rightarrow X, \exists \alpha(i) \in I, \text{im } \sigma_i \subseteq U_{\alpha(i)} \right\}$$

is the **subcomplex** of $(C_\bullet(X), \partial)$ generated by simplices each of which lie wholly inside some set in \mathcal{U} .

Note that

$$\begin{array}{ccc} C_\bullet(X) & \longrightarrow & C_{\bullet-1}(X) \\ \cup & & \cup \\ C_\bullet^\mathcal{U}(X) & \longrightarrow & C_{\bullet-1}^\mathcal{U}(X) \end{array},$$

since $C_\bullet^\mathcal{U}(X)$ is preserved by ∂ so is a subcomplex.

Proposition 2.9 (Small simplices theorem). *The inclusion $C_\bullet^\mathcal{U}(X) \hookrightarrow C_\bullet(X)$ induces an isomorphism on homology.*

Remark. Suppose $f : X \rightarrow Y$ sends each element of \mathcal{U} into some element of \mathcal{V} , the corresponding cover of Y . Then f induces $f_\# : C_\bullet^\mathcal{U}(X) \rightarrow C_\bullet^\mathcal{V}(Y)$.

Example (Mayer-Vietoris). Let $\mathcal{U} = \{A, B\}$ for $A, B \subseteq X$ open. Then there is an obvious short exact sequence of chain complexes

$$0 \rightarrow C_\bullet(A \cap B) \xrightarrow{\sigma \mapsto (\sigma, \sigma)} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{(u, v) \mapsto u - v} C_\bullet^\mathcal{U}(X) \rightarrow 0,$$

which is onto since $C_\bullet^\mathcal{U}(X)$ only contains simplices lying in A or B . The associated long exact sequence is the Mayer-Vietoris sequence, using small simplices to identify $H_\bullet(C_\bullet^\mathcal{U}(X)) \xrightarrow{\sim} H_\bullet(C_\bullet(X))$. Note also the construction of the ∂ map in the long exact sequence associated to a short exact sequence of complexes does reproduce our earlier description of ∂_{MV} . Also the naturality of Mayer-Vietoris under maps $f : X \rightarrow Y$ such that $f(A) \subseteq C$ and $f(B) \subseteq D$ is just the naturality of $C_\bullet^\mathcal{U}(X) \rightarrow C_\bullet^\mathcal{V}(Y)$.

Example (Excision). Recall we have $Z, A \subseteq X$ and $\bar{Z} \subseteq \mathring{A}$. Let $B = X \setminus Z$ and let $\mathcal{U} = \{A, B\}$, so the interiors of A and B do cover X . Note that

$$C_n^\mathcal{U}(X) / C_n(A) \cong C_n(B) / C_n(A \cap B)$$

is the free abelian group on simplices in B not wholly contained in A . The short exact sequences of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet^\mathcal{U}(X) & \longrightarrow & C_\bullet^\mathcal{U}(X) / C_\bullet(A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X) / C_\bullet(A) \longrightarrow 0 \end{array},$$

and the natural map of short exact sequences give a map of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & H_i(A) & \rightarrow & H_i(C_\bullet^\mathcal{U}(X)) & \rightarrow & H_i(C_\bullet^\mathcal{U}(X) / C_\bullet(A)) & \rightarrow & H_{i-1}(A) & \rightarrow & H_{i-1}(C_\bullet^\mathcal{U}(X)) \rightarrow \cdots \\ & & \downarrow = & & \downarrow \sim \text{ss} & & \downarrow \phi & & \downarrow \sim \text{ss} & & \downarrow = \\ \cdots & \rightarrow & H_i(A) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, A) & \longrightarrow & H_{i-1}(A) & \longrightarrow & H_{i-1}(X) \longrightarrow \cdots \end{array}.$$

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So by the 5-lemma, ϕ is an isomorphism, so

$$C_{\bullet}^{\mathcal{U}}(X)/C_{\bullet}(A) \hookrightarrow C_{\bullet}(X)/C_{\bullet}(A)$$

is an isomorphism on homology. So

$$\begin{aligned} H_{\bullet}(X, A) &= H_{\bullet}(C_{\bullet}(X)/C_{\bullet}(A)) \cong H_{\bullet}(C_{\bullet}^{\mathcal{U}}(X)/C_{\bullet}(A)) \\ &\cong H_{\bullet}(C_{\bullet}(B)/C_{\bullet}(A \cap B)) = H_{\bullet}(B, A \cap B) = H_{\bullet}(X \setminus Z, A \setminus Z), \end{aligned}$$

proving excision.

2.7 Proof of small simplices theorem

So it just remains to prove the small simplices theorem that $C_{\bullet}^{\mathcal{U}}(X) \hookrightarrow C_{\bullet}(X)$ is an isomorphism on homology. The key geometric ingredient is to divide simplices into smaller simplices.

Definition. The **barycentre**, or centre of mass, of Δ^n is

$$b_n = \frac{(1, \dots, 1)}{n+1}.$$

A **barycentric subdivision** is the following three-step procedure.

- Subdivide the boundary.
- Add the barycentre.
- Cone off from the barycentre to the subdivided boundary.

Definition. If $\sigma : \Delta^i \rightarrow \Delta^n \in C_i(\Delta^n)$,

$$\begin{aligned} \text{Cone}_i^{\Delta^n}(\sigma) : \quad \Delta^{i+1} &\longrightarrow \Delta^n \\ (t_0, \dots, t_{i+1}) &\longmapsto t_0 b_n + (1 - t_0) \sigma \left(\frac{(t_1, \dots, t_{i+1})}{1 - t_0} \right). \end{aligned}$$

So, extended linearly, $\text{Cone}_i^{\Delta^n} : C_i(\Delta^n) \rightarrow C_{i+1}(\Delta^n)$.

Exercise.

$$\partial \left(\text{Cone}_i^{\Delta^n}(\sigma) \right) = \begin{cases} \sigma - \text{Cone}_{i-1}^{\Delta^n}(\partial(\sigma)) & i > 0 \\ \sigma - \epsilon(\sigma) b_n & i = 0 \end{cases},$$

where

$$\begin{aligned} \epsilon : C_0(\Delta^n) &\longrightarrow \mathbb{Z} \\ \sum_i n_i p_i &\longmapsto \sum_i n_i \end{aligned}$$

is the augmentation.

Definition. Define

$$\begin{aligned} c : C_{\bullet}(\Delta^n) &\longrightarrow C_{\bullet}(\Delta^n) \\ \sigma &\longmapsto \begin{cases} \epsilon(\sigma) b_n & \text{on } C_0(\Delta^n) \\ 0 & \text{on } C_i(\Delta^n), i > 0 \end{cases}. \end{aligned}$$

Then

$$\partial \circ \text{Cone}^{\Delta^n} + \text{Cone}^{\Delta^n} \circ \partial = \text{id}_{C_{\bullet}(\Delta^n)} - c.$$

Definition. A collection of chain maps $\phi^X : C_{\bullet}(X) \rightarrow C_{\bullet}(X)$, defined for all spaces X , is **natural** if for all $f : X \rightarrow Y$,

$$f_{\#} \circ \phi^X = \phi^Y \circ f_{\#}.$$

Similarly for a collection $P : C_{\bullet}(X) \rightarrow C_{\bullet+1}(X)$ of chain homotopies between natural ϕ^X and ψ^X .

Definition. Define

$$\begin{aligned} \phi_n^X : C_n(X) &\longrightarrow C_n(X) \\ \phi_0^X = \text{id}_{C_0(X)}, \quad \sigma &\longmapsto \sigma_{\#} \left(\text{Cone}_{n-1}^{\Delta^n} (\phi_{n-1}^{\Delta^n} (\partial(\iota_n))) \right), \end{aligned}$$

where $\iota_n : \Delta^n \rightarrow \Delta^n \in C_n(\Delta^n)$ is the identity, so $\partial(\iota_n) \in C_{n-1}(\Delta^n)$.

Since $\sigma : \Delta^n \rightarrow X$ is $\sigma \circ \iota_n : \Delta^n \rightarrow \Delta^n \rightarrow X$, this is natural, since

$$\phi_n^X(\sigma) = \phi_n^X(\sigma_{\#}(\iota_n)) = \sigma_{\#}(\phi_n^{\Delta^n}(\iota_n)).$$

The idea is that we know how to subdivide Δ^n , so know how to subdivide any simplex in X .

Definition. Similarly, define

$$\begin{aligned} P_n^X : C_n(X) &\longrightarrow C_{n+1}(X) \\ \sigma &\longmapsto \sigma_{\#} \left(\text{Cone}_n^{\Delta^n} (\phi_n^{\Delta^n}(\iota_n) - \iota_n - P_{n-1}^{\Delta^n}(\partial(\iota_n))) \right). \end{aligned}$$

This decomposes the prism $\Delta^n \times [0, 1]$ by joining $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$ to the barycentre of $\Delta^n \times \{1\}$.

Fact. $\phi^X : C_{\bullet}(X) \rightarrow C_{\bullet}(X)$ is a natural chain map, and $P^X : C_{\bullet}(X) \rightarrow C_{\bullet+1}(X)$ is a natural chain homotopy from ϕ^X to the identity, that is

$$\partial \circ P_n^X + P_{n-1}^X \circ \partial = \phi_n^X - \text{id}_{C_n(X)}.$$

We will not prove this.

Ok, now we know how to divide simplices.

Lemma 2.10. *If $[v_0, \dots, v_n] \subseteq \mathbb{R}^{n+1}$ is a simplex, then each simplex of its barycentric division has Euclidean diameter at most $n/(n+1)$ the Euclidean diameter of $[v_0, \dots, v_n]$.*

Corollary 2.11.

1. If $\sigma \in C_n^{\mathcal{U}}(X)$, then $\phi_n^X(\sigma) \in C_n^{\mathcal{U}}(X)$.
2. If $\sigma \in C_n(X)$, there exists $k \gg 0$ such that $(\phi_n^X)^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

Proof.

1. Obvious.
2. σ is a finite sum of simplices, so it suffices to prove the result for one $\sigma : \Delta^n \rightarrow X$. Let $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$. Now $\left\{ \sigma^{-1} \left(U_{\alpha}^{\circ} \right) \right\}_{\alpha \in I}$ is an open cover of Δ^n , so has a Lebesgue number, that is there exists $\epsilon > 0$ such that any open ϵ -ball in Δ^n lies in some $\sigma^{-1}(U_{\alpha})$. Now pick $k \gg 0$ such that $(n/(n+1))^k \ll \epsilon$.

□

Proof of Proposition 2.9. Let $U : H_{\bullet}(C_{\bullet}^{\mathcal{U}}(X)) \rightarrow H_{\bullet}(X)$ be the natural map.

- If $[c] \in H_n(X)$, there exists k such that $(\phi_n^X)^k(c) \in C_n^{\mathcal{U}}(X)$. Since $\phi^X \simeq \text{id}$, $(\phi^X)^k \simeq \text{id}$, so there exists F such that $\partial \circ F + F \circ \partial = (\phi^X)^k - \text{id}$. Then $(\phi^X)^k(c) = c + \text{im } \partial$, so U is onto.
- If $U([c]) = 0$ for $[c] \in H_n(C_{\bullet}^{\mathcal{U}}(X))$ and $z \in C_{n+1}(X)$ has $\partial(z) = c$, there exists k such that $(\phi_{n+1}^X)^k(z) \in C_{n+1}^{\mathcal{U}}(X)$ and $(\phi_{n+1}^X)^k(z) - z = (\partial \circ F + F \circ \partial)(z)$, so

$$c = \partial(z) = \partial \left((\phi_{n+1}^X)^k(z) \right) - \partial(F(\partial(z))) \in C_{n+1}^{\mathcal{U}}(X),$$

since $\partial(z) \in C_n^{\mathcal{U}}(X)$ and F is natural. Then $c \in \text{im}(\partial : C_{n+1}^{\mathcal{U}}(X) \rightarrow C_n^{\mathcal{U}}(X))$, so $[c] = 0$ and U is one-to-one.

□

3 Cellular homology and cohomology

Singular homology and cohomology are defined for all topological spaces, but we are mostly interested in nice spaces. In particular, we have seen $H_\bullet(S^n)$, $H_\bullet(\text{Klein})$, $H_\bullet(\Sigma_g)$, and $H^\bullet(\mathbb{CP}^n)$ are all finite rank, even though $C_\bullet(X)$ is vast in each case. Our next goal is to develop a computational shortcut which makes this manifest.

3.1 Cell complexes

Definition. A **cell complex**, or **CW complex**, is a space obtained inductively as follows.

- X_0 is a discrete set, such as a finite set.
- Given X_{k-1} ,

$$X_k = X_{k-1} \cup \bigcup_{i \in I_k} D_i^k,$$

for I_k an indexing set and $D_i^k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ a closed disc, called **k -cells**, attached via $\partial D_i^k = S^{k-1} \rightarrow X_{k-1}$, so $X_{k-1} \sqcup \bigsqcup_{i \in I_k} D_i^k \rightarrow X_k$ is the quotient map identifying ∂D_i^k and its image.

- $X = \bigcup_{k \geq 0} X_k$ with the **weak** topology, where $U \subseteq X$ is open if and only if $U \cap X_k$ is open in X_k for all k .

Example.

- $S^n = \{\text{point}\} \cup D^n$ attached via the constant map $\partial D^n \rightarrow \{\text{point}\}$.
- $S^n = \{\text{point}\} \cup \{\text{point}\} \cup D_{\alpha_1}^1 \cup D_{\alpha_2}^1 \cup D_{\beta_1}^2 \cup D_{\beta_2}^2$.
- T^2 has one 0-cell, two 1-cells, and one 2-cell.
- Σ_2 has one 0-cell, four 1-cells, and one 2-cell.
- The **wedge product**. If X and Y are cell complexes, then $X \vee Y = \langle X \sqcup Y \rangle / x_0 \sim y_0$ where $x_0 \in X_0$ and $y_0 \in Y_0$.

Notation. Let X be a cell complex. The D_i^k are k -cells.

- X_k is the **k -skeleton** of X .
- If there exists N such that $X = X_N$, then X is a **finite-dimensional** cell complex.
- If $X = X_N$ and $I_j < \infty$ for all j , then X is a **finite** cell complex. Then X is compact.
- $X = \bigsqcup_{k \geq 0} \mathring{D}_\alpha^k$ is the disjoint union of its open cells C_α^k as attaching maps take ∂D_α^k to X_{k-1} .
- A **subcomplex** $A \subseteq X$ is a closed subspace which is a union of cells of X . Note that given a cell complex X , you cannot throw out a random bunch of cells to get a subcomplex. There may be later cells that try to attach to things you are throwing out.

3.2 Point-set digression

Let $X = \bigcup_{n \geq 0} X_n$ be a cell complex.

Exercise. $A \subseteq X$ is open, or closed, if and only if $(\phi_\alpha^n)^{-1}(A) \subseteq D_\alpha^n$ is open, or closed, for all α , where

$$\phi_\alpha^n : D_\alpha^n \hookrightarrow X_{n-1} \sqcup \bigsqcup_{\alpha} D_\alpha^n \twoheadrightarrow X_n \hookrightarrow X$$

is the **characteristic** map of the cell, so $\phi_\alpha^n|_{\partial D_\alpha^n}$ is the attaching map.

Let $A \subseteq X$. We build an open neighbourhood $N_\epsilon(A)$ of A inductively. Let $N_\epsilon^0(A) = A \cap X_0$. Given $N_\epsilon^n(A) \subseteq X_n$ an open neighbourhood of $A \cap X_n$, define $N_\epsilon^{n+1}(A)$ by specifying

$$\begin{aligned} (\phi_\alpha^{n+1})^{-1}(N_\epsilon^{n+1}(A)) = & \left(\text{open } \epsilon\text{-neighbourhood of } (\phi_\alpha^n)^{-1}(A) \setminus \partial D_\alpha^{n+1} \subseteq D_\alpha^{n+1} \setminus \partial D_\alpha^{n+1} \right) \\ & \cup \left((1 - \epsilon, 1] \times (\phi_\alpha^n)^{-1}(N_\epsilon^n(A)) \right), \end{aligned}$$

where ϵ depends on α , and $(1 - \epsilon, 1]$ is the radial spherical coordinate on D_α^{n+1} and $(\phi_\alpha^n)^{-1}(N_\epsilon^n(A))$ is the angular coordinate on ∂D_α^{n+1} . Then $N_\epsilon(A) = \bigcup_{n \geq 0} N_\epsilon^n(A)$ is open, as it is open in every cell.

Proposition 3.1. *Cell complexes are Hausdorff and locally contractible. So connected if and only if path-connected.*

Proof. For a proof, see Hatcher, appendix A. □

Fact. A compact smooth manifold, perhaps with boundary, is homotopy equivalent to a finite cell complex. And given $N \subseteq M$ a properly embedded submanifold, there exists a cell structure on M making N a subcomplex. Can drop smoothness, but there are nice proofs using Morse theory if you have it.

Lemma 3.2. *If X is a cell complex and $A \subseteq X$ is a subcomplex, then (X, A) is a good pair.*

Proof. See Hatcher. Again, point-set rather than algebraic topology. □

Corollary 3.3. $H_\bullet(X, A) \cong \widetilde{H}_\bullet(X/A)$.

Corollary 3.4. *If $X = \bigcup_{k \geq 0} X_k$ is a cell complex,*

$$H_i(X_k, X_{k-1}) = \begin{cases} \bigoplus_{\alpha \in I_k} \mathbb{Z} & i = k \\ 0 & \text{otherwise} \end{cases}$$

is free abelian on the set of k -cells in X .

Proof. $X_{k-1} \subseteq X_k$ is a subcomplex, so

$$H_\bullet(X_k, X_{k-1}) \cong \widetilde{H}_\bullet(X_k/X_{k-1}) \cong \widetilde{H}_\bullet\left(\bigvee_{\alpha \in I_k} S_\alpha^k\right),$$

as $\partial D_\alpha^k \rightarrow X_{k-1}$ for all k -cells and X_{k-1} is collapsed to a point. Choose $x_\alpha \in S_\alpha^k$ for all α . Then $(\bigsqcup_\alpha S_\alpha^k, \bigsqcup_\alpha \{x_\alpha\})$ is a good pair and $\bigsqcup_\alpha S_\alpha^k / \bigsqcup_\alpha \{x_\alpha\} = \bigvee_\alpha S_\alpha^k$, so

$$H_\bullet(X_k, X_{k-1}) \cong H_\bullet\left(\bigsqcup_\alpha S_\alpha^k, \bigsqcup_\alpha \{x_\alpha\}\right) = \bigoplus_\alpha H_\bullet(S_\alpha^k, \{x_\alpha\}) = \bigoplus_\alpha \widetilde{H}_\bullet(S_\alpha^k).$$

□

Proposition 3.5. *If $Z \subseteq X$ is compact, there exists N such that $Z \subseteq X_N$.*

Proof. We will show Z meets only finitely many cells of X . Suppose for contradiction there exists $S = \{x_0, x_1, \dots\} \subseteq Z$ such that $x_i \in e_i$ and the cells $\{e_i\}$ are pairwise distinct. Claim that S is closed in X . Well, $S \cap X_0$ is closed in X_0 , a discrete space. Inductively, if $S \cap X_{n-1}$ is closed in X_{n-1} and $\phi_\alpha^n : D_\alpha^n \rightarrow X_n$ is an n -cell, $(\phi_\alpha^n|_{\partial D_\alpha^n})^{-1}(S) \subseteq S_\alpha^{n-1}$ is closed. Then

$$(\phi_\alpha^n)^{-1}(S) = (\phi_\alpha^n|_{\partial D_\alpha^n})^{-1}(S) \cup \{\text{at most one point}\} \subseteq D_\alpha^n$$

is closed, since X is the disjoint union of interiors of cells, so S meets each cell of X_n in a closed set, so $S \subseteq X_n$ is closed. Same for $S' \subseteq S$, so S is discrete. Thus S is finite. □

Proposition 3.6. Let $X = \bigcup_{k \geq 0} X_k$ be a cell complex.

1. $H_k(X_n) = 0$ for all $k > n$.
2. The inclusion $X_n \hookrightarrow X$ induces an isomorphism $H_j(X_n) \xrightarrow{\sim} H_j(X)$ for all $j < n$.

Proof.

1. If $k > n$, the long exact sequence of a pair (X_n, X_{n-1}) gives

$$\begin{array}{ccccccc} H_{k+1}(X_n, X_{n-1}) & \longrightarrow & H_k(X_{n-1}) & \longrightarrow & H_k(X_n) & \longrightarrow & H_k(X_n, X_{n-1}) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array},$$

so $H_k(X_n) \cong \dots \cong H_k(X_0) = 0$, since X_0 is a discrete set.

2. The same sequence as before

$$H_{k+1}(X_n, X_{n-1}) \rightarrow H_k(X_{n-1}) \rightarrow H_k(X_n) \rightarrow H_k(X_n, X_{n-1})$$

with $k < n - 1$ shows $H_k(X_{n-1}) \cong \dots \cong H_k(X_N)$ for all $N > n - 1$. If X is finite-dimensional, we are done. In general, if $\alpha \in H_k(X)$, then α is represented by a finite union of simplices, which is compact. If $Z \subseteq X$ is compact, there exists N such that $Z \subseteq X_N$. So $\alpha \in \text{im}(\phi_N : H_k(X_N) \rightarrow H_k(X))$ for all $N \gg 0$. Similarly, if a cycle $\alpha = \sum_i a_i \sigma_i \in H_k(X)$ bounds a $(k+1)$ -chain in X , that $(k+1)$ -chain lives in some $X_{N'}$ for $N' \gg 0$, so $[\alpha] = 0 \in H_k(X_{N'})$, that is ϕ_N is one-to-one for $N \gg 0$. □

Corollary 3.7. Let X be a finite-dimensional cell complex of dimension n . Then

$$H_j(X) = 0, \quad j > n.$$

3.3 Cellular homology

We are still computing from the huge chain groups $C_\bullet(X)$. The goal is to describe a model for $H_\bullet(X)$ starting from a much smaller chain complex.

Definition. Let $X = \bigcup_{n \geq 0} X_n$ be a cell complex. Let

$$C_k^{\text{cell}}(X) = H_k(X_k, X_{k-1}).$$

This is free abelian on the k -cells. Then

$$\begin{array}{ccccccc} & & \dots & & \dots & & \\ & & \searrow & & \nearrow & & \\ & & & H_k(X_k) & & & \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ \text{LES of } (X_{k+1}, X_k) & & & & & \text{LES of } (X_k, X_{k-1}) & \\ \dots \rightarrow H_{k+1}(X_{k+1}, X_k) & \xrightarrow{\partial_{k+1}^{\text{cell}}} & H_k(X_k, X_{k-1}) & \xrightarrow{\partial_k^{\text{cell}}} & H_{k-1}(X_{k-1}, X_{k-2}) & \rightarrow \dots \\ & \nearrow & \searrow & & \nearrow & & \searrow \\ & & \text{LES of } (X_k, X_{k-1}) & & \text{LES of } (X_{k-1}, X_{k-2}) & & \\ & & & H_{k-1}(X_{k-1}) & & & \\ & & \nearrow & & \searrow & & \\ & & \dots & & \dots & & \end{array}$$

Observe that $\partial_k^{\text{cell}} \circ \partial_{k+1}^{\text{cell}} = 0$ since we have two consecutive maps from one long exact sequence. This is the **cellular chain complex** of $X = \bigcup_{n \geq 0} X_n$. Note that it depends on the chosen cell structure. Write $H_\bullet^{\text{cell}}(X)$ for $H_\bullet(C_\bullet^{\text{cell}}(X), \partial_\bullet^{\text{cell}})$.

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Proposition 3.8.

$$H_{\bullet}^{\text{cell}}(X) \cong H_{\bullet}(X).$$

Proof. Recall that $H_j(X_k) = 0$ if $j > k$ and $X_k \hookrightarrow X$ is an isomorphism on homology for $j < k$. Then

$$\begin{array}{ccccccc}
 & & & & H_k(X_{k+1}, X_k) = 0 & & \\
 & & & & \nearrow & & \\
 & & H_k(X_{k+1}) \cong H_k(X) & & & & \\
 & \nearrow & & \searrow & & & \\
 0 = H_k(X_{k-1}) & & H_k(X_k) & & & & \\
 & \nearrow \partial_{k+1} & & \searrow (i_k)_* & & & \\
 H_{k+1}(X_{k+1}, X_k) & \xrightarrow{\partial_{k+1}^{\text{cell}}} & H_k(X_k, X_{k-1}) & \xrightarrow{\partial_k^{\text{cell}}} & H_{k-1}(X_{k-1}, X_{k-2}) & & \\
 & & \searrow \partial_k & & \nearrow (i_{k-1})_* & & \\
 & & H_{k-1}(X_{k-1}) & & & & \\
 & & \nearrow & & & & \\
 & & 0 = H_{k-1}(X_{k-2}) & & & &
 \end{array}$$

so

$$\begin{aligned}
 H_k(X) &\cong H_k(X_{k+1}) \cong H_k(X_k) / \text{im } \partial_{k+1} \cong \text{im } (i_k)_* / \text{im } ((i_k)_* \circ \partial_{k+1}) && \text{since } (i_k)_* \text{ is injective} \\
 &\cong \ker \partial_k / \text{im } \partial_{k+1}^{\text{cell}} \cong \ker ((i_{k-1})_* \circ \partial_k) / \text{im } \partial_{k+1}^{\text{cell}} && \text{since } (i_{k-1})_* \text{ is injective} \\
 &= \ker \partial_k^{\text{cell}} / \text{im } \partial_{k+1}^{\text{cell}} = H_k^{\text{cell}}(X).
 \end{aligned}$$

□

Remark. If X and Y are cell complexes and $f : X \rightarrow Y$ is a map, in general f does not induce maps $C_{\bullet}^{\text{cell}}(X) \rightarrow C_{\bullet}^{\text{cell}}(Y)$. Ok if f is **cellular**, so f takes a k -skeleton X_k into a k -skeleton Y_k , for all k .

The following are immediate.

Corollary 3.9. *Let X be a finite cell complex.*

- $H_k(X)$ is a finitely generated abelian group of rank at most n_k , the number of k -cells.
- If $H_k(X) \neq 0$, every cell structure on X must have at least $\text{rk } H_k(X)$ distinct k -cells.
- If X admits a cell structure with only even-dimensional cells, $H_{\bullet}(X) \cong C_{\bullet}^{\text{cell}}(X)$ for this cell structure.
- $H_{\bullet}(X; \mathbb{F})$ is a finite-dimensional vector space over the field \mathbb{F} , such as \mathbb{Q} .

Example. Let

$$\begin{aligned}
 \mathbb{CP}^n &= \{\text{lines in } \mathbb{C}^{n+1}\} = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* = S^{2n+1} / S^1 \\
 &= \{[z_0 : \dots : z_n] \mid (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}, \forall \lambda \in \mathbb{C}^*, (z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)\} \\
 &= \{z_0 = 0\} \cup \{[1 : z_1 : \dots : z_n] \mid (z_1, \dots, z_n) \in \mathbb{C}^n\} \cong \mathbb{CP}^{n-1} \cup \mathbb{C}^n,
 \end{aligned}$$

where $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is the unit sphere and $S^1 : (z_1, \dots, z_{n+1}) \mapsto (\lambda z_1, \dots, \lambda z_{n+1})$ for $\lambda \in S^1$. The attaching map is

$$\begin{aligned}
 S^{2n+1} &\longrightarrow \mathbb{CP}^n \\
 (z_1, \dots, z_{n+1}) &\longmapsto [z_1 : \dots : z_{n+1}].
 \end{aligned}$$

So, inductively in n , \mathbb{CP}^n has a cell structure with one $2n$ -cell for all n , so

$$H_{\bullet}(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \bullet = 0, 2, \dots, 2n-2, 2n \\ 0 & \text{otherwise} \end{cases}.$$

See example sheet 1.

Remark. Grassmannians

$$\text{Gr}(k; \mathbb{C}^n) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^n\}$$

also have cell structures with only even-dimensional cells.

Exercise. $\mathbb{RP}^n = S^n \setminus \{\pm 1\} = \mathbb{RP}^{n-1} \cup D^n$ has a cell structure with one cell in each degree $0 \leq i \leq n$.

3.4 Degrees

How do we compute $\partial_n^{\text{cell}} : \bigoplus_{\alpha \in I_n} \mathbb{Z} \rightarrow \bigoplus_{\beta \in I_{n-1}} \mathbb{Z}$? That is, want values $d_{\alpha\beta} \in \mathbb{Z}$ such that

$$\partial_n^{\text{cell}}(e_\alpha^n) = \sum_{\beta} d_{\alpha\beta} e_\beta^{n-1}.$$

Note we have an attaching map $\phi_\alpha^n : \partial(e_\alpha^n) = S^{n-1} \rightarrow X_{n-1}$.

Lemma 3.10. $d_{\alpha\beta}$ is the degree of

$$f_{\alpha\beta} : S_\alpha^{n-1} \xrightarrow{\phi_\alpha^n} X_{n-1} \rightarrow X_{n-1}/X_{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \xrightarrow{\text{collapse}} S_\beta^{n-1}.$$

Remark. For this degree to be well-defined, not just up to sign, need generators for $H_{n-1}(S_\alpha^{n-1})$ and $H_{n-1}(S_\beta^{n-1})$, such as identifying $S^{n-1} = \partial D^n \subseteq \mathbb{R}^n$.

Proof. By chasing,

$$\begin{array}{ccccc} H_n(D_\alpha^n, \partial D_\alpha^n) & \xrightarrow[\sim]{\text{LES}} & H_{n-1}(\partial D_\alpha^n) & \xrightarrow{\deg f_{\alpha\beta}} & \widetilde{H}_{n-1}(S_\beta^{n-1}) \\ \downarrow (\phi_\alpha^n)_* & & \downarrow (\phi_\alpha^n|_{\partial D_\alpha^n})_* & & \uparrow \text{collapse} \\ H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X_{n-1}) & \xrightarrow{\mathbb{R}} & \widetilde{H}_{n-1}(\bigvee_\gamma S_\gamma^{n-1}) \\ & \searrow \partial_n^{\text{cell}} & \downarrow (i_{n-1})_* & & \uparrow \\ & & H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow[\text{Excision}]{\sim} & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \end{array} \quad , \quad \begin{array}{ccccc} 1 & \longrightarrow & 1 & \longrightarrow & d_{\alpha\beta} \\ \downarrow e_\alpha & & & & \uparrow \\ & & \sum_\gamma d_{\alpha\gamma} e_\gamma & \longrightarrow & \sum_\gamma d_{\alpha\gamma} e_\gamma \end{array} ,$$

so $d_{\alpha\beta} = \deg f_{\alpha\beta}$ as claimed. \square

For this to be useful, we need to be able to compute degrees. Let $f : S^n \rightarrow S^n$. Assume that there exists $y \in S^n$ such that $f^{-1}(y) = \{x_1, \dots, x_m\}$ is finite. Pick neighbourhoods $U_i \in U_i$ and $y \in V$ homeomorphic to \mathbb{R}^n such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and $f|_{U_i} : U_i \rightarrow V \subseteq S^n$.

Definition. The **local degree** is

$$\deg_{x_i} f = H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z} \rightarrow H_n(V, V \setminus \{y\}) \cong \mathbb{Z}.$$

Note that by excision and the long exact sequence,

$$H_n(U_i, U_i \setminus \{x_i\}) \cong H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(S^n) \cong \mathbb{Z}.$$

By fixing this, $\deg_{x_i} f$ is well-defined.

Lemma 3.11. Under the assumption,

$$\deg f = \sum_{i=1}^m \deg_{x_i} f.$$

Remark. If $f : S^n \rightarrow S^n$ is smooth, then $f^{-1}(y)$ is finite if y is a regular value for f , and by Sard's theorem, almost all values, in particular a dense set, are regular.

Proof.

$$\begin{array}{ccc}
 H_n(S^n) & \xrightarrow{\deg f} & H_n(S^n) \\
 \text{LES} \downarrow \sim & & \sim \downarrow \text{LES} \\
 H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{y\}) \\
 \text{Excision} \uparrow \sim & & \sim \uparrow \text{Excision} \\
 H_n(\bigsqcup_i U_i, \bigsqcup_i U_i \setminus \{x_i\}) \cong \bigoplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow{(f|_{U_i})} & H_n(V, V \setminus \{y\})
 \end{array}
 \quad , \quad
 \begin{array}{ccc}
 1 & \xrightarrow{\quad} & \deg f \\
 \downarrow & & \downarrow \\
 (1, \dots, 1) & \rightarrow & \sum_i \deg_{x_i} f
 \end{array}$$

which implies the result. \square

Example. Let $p(z) = z^k + a_{k-1}z^{k-1} + \dots + a_0$ be a complex polynomial. Then p extends to a map $\hat{p}: \mathbb{C} \cup \{\infty\} = S^2 \rightarrow S^2$ of degree $\deg \hat{p} = k$. As in lecture 1, show $\hat{p} \simeq (q: z \mapsto z^k)$. Now $q^{-1}(1) = \{\zeta_1, \dots, \zeta_k\}$ and near each ζ_i , q is a local homeomorphism. And the different local homeomorphisms differ by rotation, so the local degrees at ζ_i are all equal.

Exercise. The Klein bottle K has

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_2^{\text{cell}}(K) & \longrightarrow & C_1^{\text{cell}}(K) & \longrightarrow & C_0^{\text{cell}}(K) \longrightarrow 0 \\
 & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\
 & & \mathbb{Z} & \xrightarrow{1 \mapsto (0,2)} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}
 \end{array}$$

so $H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and $H_2(K; \mathbb{Z}) = 0$, again.

Example. The real projective space $\mathbb{RP}^n = D^n \cup \mathbb{RP}^{n-1} = D^n \cup \dots \cup D^1 \cup \{\text{point}\}$ has one cell of each degree $0 \leq i \leq n$, so

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_n^{\text{cell}}(\mathbb{RP}^n) & \longrightarrow & \dots & \longrightarrow & C_0^{\text{cell}}(\mathbb{RP}^n) \longrightarrow 0 \\
 & & \downarrow \mathbb{R} & & & & \downarrow \mathbb{R} \\
 & & \mathbb{Z} & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & \mathbb{Z}
 \end{array}$$

Then

$$\eta: \partial D^k \xrightarrow{\phi} \mathbb{RP}^{k-1} \rightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$$

is two-to-one and the local maps differ by the antipodal map, so $\partial_k^{\text{cell}} = 1 + (-1)^k$. Thus

$$C_{\bullet}^{\text{cell}}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} & n \text{ even} \\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} & n \text{ odd} \end{cases}$$

so

$$H_{\bullet}(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ \mathbb{Z}/2 & 0 < \bullet < n, \bullet \text{ odd} \\ \mathbb{Z} & \bullet = n, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Exercise. Let Σ_g be a $4g$ -gon with edge identifications $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. Then

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_2^{\text{cell}}(\Sigma_g) & \longrightarrow & C_1^{\text{cell}}(\Sigma_g) & \longrightarrow & C_0^{\text{cell}}(\Sigma_g) \longrightarrow 0 \\
 & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\
 & & \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z}^{2g} & \xrightarrow{\quad} & \mathbb{Z}
 \end{array}$$

which vanishes as for Klein and as $H_0(\Sigma_g) = \mathbb{Z}$.

3.5 Cellular cohomology

There is also **cellular cohomology**

$$C_{\text{cell}}^i(X) = H^i(X_i, X_{i-1}), \quad \partial_{\text{cell}}^*: H^i(X_i, X_{i-1}) \rightarrow H^{i+1}(X_{i+1}, X_i),$$

and $H_{\text{cell}}^{\bullet}(X) \cong H^{\bullet}(X)$ by the exactly analogous argument as for homology.

Lemma 3.12. $C_{\text{cell}}^{\bullet}(X) = \text{Hom}(C_{\bullet}^{\text{cell}}(X), \mathbb{Z})$ and ∂_{cell}^* is the adjoint of ∂^{cell} .

See question sheet 2 for related results, in particular the existence of the h -maps.

Proof. If $(\cdot)^{\vee}$ is the linear dual, then

$$\begin{array}{ccccc}
 & & \partial_{\text{cell}}^* & & \\
 & \nearrow & & \searrow & \\
 H^i(X_i, X_{i-1}) & \xrightarrow{i^*} & H^i(X_i) & \xrightarrow{\partial^*} & H^{i+1}(X_{i+1}, X_i) \\
 \sim \downarrow h & & \downarrow h & & \sim \downarrow h \\
 \text{Hom}(H_i(X_i, X_{i-1}), \mathbb{Z}) & \xrightarrow{i_*^{\vee}} & \text{Hom}(H_i(X_i), \mathbb{Z}) & \xrightarrow{\partial^{\vee}} & \text{Hom}(H_{i+1}(X_{i+1}, X_i), \mathbb{Z}) \\
 & \searrow & & \nearrow & \\
 & & \partial_{\text{cell}}^{\vee} & &
 \end{array}$$

A direct check shows the outer h maps are isomorphisms, so just need the diagram to commute. The left hand square commutes by naturality of h and the right hand square commutes by naturality of homology and cohomology long exact sequences. \square

Proposition 3.13. Let X be a finite cell complex. Then

$$H^i(X; \mathbb{Z}) \cong (H_i(X; \mathbb{Z}) / \text{tors}) \oplus \text{tors } H_{i-1}(X; \mathbb{Z}).$$

Note that for an abelian group G ,

$$\text{tors } G = \{\text{elements of finite order}\} = \{\text{all } \mathbb{Z}/k\text{-summands} \mid k \geq 2\} \leq G.$$

Proof. This is now just algebra. Let C_{\bullet} be a chain complex of free finitely generated abelian groups, and $C^{\bullet} = \text{Hom}(C_{\bullet}, \mathbb{Z})$ the dual complex. Then

$$H^i(C^{\bullet}) = (H_i(C_{\bullet}) / \text{tors}) \oplus \text{tors } H_{i-1}(C_{\bullet}).$$

The key idea is to break C_{\bullet} into a collection of short exact sequences

$$0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0, \quad 0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n(C_{\bullet}) \rightarrow 0,$$

where $Z_i = \ker(\partial_i : C_i \rightarrow C_{i-1})$ and $B_i = \text{im}(\partial_{i+1} : C_{i+1} \rightarrow C_i)$. In the former sequence, all the terms are free finitely generated abelian groups, so we can non-canonically split these and write $C_n \cong Z_n \oplus B_{n-1}$ for all n , so there exists $\alpha_n : B_{n-1} \rightarrow C_n$ such that $\partial_n \circ \alpha_n = \text{id}$. Then C_{\bullet} becomes

$$\cdots \rightarrow Z_{n+1} \oplus B_n \xrightarrow{B_n \rightarrow Z_n} Z_n \oplus B_{n-1} \xrightarrow{B_{n-1} \rightarrow Z_{n-1}} Z_{n-1} \oplus B_{n-2} \rightarrow \cdots, \quad (2)$$

that is C_{\bullet} breaks into a sum of complexes of the form

$$0 \rightarrow B_n \xrightarrow{A_n} Z_n \rightarrow 0.$$

By the Smith normal form, there exists a \mathbb{Z} -linear change of basis such that A_n has matrix

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_k & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \quad \begin{cases} d_1 \mid d_2 \\ \vdots \\ d_{k-1} \mid d_k \end{cases},$$

so (2) breaks into a direct sum of complexes of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0, \quad 0 \rightarrow \mathbb{Z} \xrightarrow{d_i} \mathbb{Z} \rightarrow 0.$$

Now the relation $H^i(C^{\bullet}) = (H_i(C_{\bullet}) / \text{tors}) \oplus \text{tors } H_{i-1}(C_{\bullet})$ is obvious, by just checking in these two examples. \square

Remark. For abelian groups H and G set

$$\text{Ext}^1(H, G) = \{\text{short exact sequences } 0 \rightarrow G \rightarrow J \rightarrow H \rightarrow 0\} / \sim,$$

where two are equivalent if the obvious thing happens. That is, there exists ϕ making the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & J_1 & \longrightarrow & H \longrightarrow 0 \\ & & \parallel & & \downarrow \exists \phi & & \parallel \\ 0 & \longrightarrow & G & \longrightarrow & J_2 & \longrightarrow & H \longrightarrow 0 \end{array}$$

commute, so ϕ is an isomorphism by the 5-lemma. The **universal coefficient theorem** says there are split exact sequences

$$0 \rightarrow \text{Ext}^1(H_{n-1}(X; \mathbb{Z}), G) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X; \mathbb{Z}), G) \rightarrow 0.$$

We will not prove this.

3.6 Euler characteristic

Recall that if X is a finite cell complex, we saw $H_i(X)$ is finitely generated for all i , and $H_i(X) = 0$ if $i > \dim X$. So $\bigoplus_i H_i(X; \mathbb{Q})$ is a finite-dimensional graded \mathbb{Q} -vector space.

Definition. The **Euler characteristic** of a finite cell complex X is

$$\chi(X) = \sum_{k \geq 0} (-1)^k \text{rk}_{\mathbb{Z}} H_k(X; \mathbb{Z}).$$

Lemma 3.14.

$$\chi(X) = \sum_{k \geq 0} (-1)^k N_k,$$

where N_k is the number of k -cells in X .

Proof. N_k is the rank of $C_k^{\text{cell}}(X)$ and we just observed we have short exact sequences

$$0 \rightarrow B_k \rightarrow Z_k \rightarrow H_k(X) \rightarrow 0, \quad 0 \rightarrow Z_k \rightarrow C_k \rightarrow B_{k-1} \rightarrow 0.$$

Then

$$\text{rk } H_k(X) = \text{rk } Z_k - \text{rk } B_k = z_k - b_k, \quad \text{rk } C_k = \text{rk } Z_k + \text{rk } B_{k-1} = z_k + b_{k-1},$$

so

$$\begin{aligned} \sum_{k \geq 0} (-1)^k \text{rk}_{\mathbb{Z}} H_k(X; \mathbb{Z}) &= \sum_{k \geq 0} (-1)^k (z_k - b_k) = \sum_{k \geq 1} (-1)^k (z_k - (N_{k+1} - z_{k+1})) \\ &= \sum_{k \geq 0} (-1)^k N_{k+1} + z_0 = \sum_{k \geq 0} (-1)^k N_k. \end{aligned}$$

□

Remark. If \mathbb{F} is a field,

$$\chi(X) = \sum_{k \geq 0} (-1)^k \dim_{\mathbb{F}} H_k(X; \mathbb{F}).$$

Example.

- $S^4 \not\cong \mathbb{CP}^2$ as $\chi(S^4) = 2$ and $\chi(\mathbb{CP}^2) = 3$, since S^4 has one 0-cell and one 4-cell and \mathbb{CP}^2 has one 4-cell, one 2-cell, and one 0-cell.
- $\chi(\Sigma_g) = 2 - 2g$, since Σ_g has one 0-cell, $2g$ distinct 1-cells, and one 2-cell.
- If A and B are finite cell complexes, then $A \times B$ has a product cell structure such that the open i -cells are of the form $(j\text{-cell in } A) \times ((i-j)\text{-cell in } B)$, so $\chi(A \times B) = \chi(A) \chi(B)$.
- If $X = A \cup B$ is a union of two subcomplexes, then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

4 Generalised homology theories

4.1 Eilenberg-Steenrod axioms

Definition. An assignment $(X, A) \mapsto h_\bullet(X, A) = \bigoplus_{i \in \mathbb{Z}} h_i(X, A)$ of graded abelian groups to pairs of topological spaces and subspaces is called a **generalised homology theory** if it satisfies the following.

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- Functoriality. A map $f : (X, A) \rightarrow (Y, B)$ induces a degree-preserving homomorphism

$$f_* : h_\bullet(X, A) \rightarrow h_\bullet(Y, B),$$

such that $\text{id}_* = \text{id}$ and $(f \circ g)_* = f_* \circ g_*$.

- Homotopy invariance. If $f \simeq g$, through maps of pairs, then $f_* = g_*$.
- Long exact sequence. If $h_i(X) = h_i(X, \emptyset)$, then there exists $\partial : h_i(X, A) \rightarrow h_{i-1}(A)$ such that

$$\cdots \rightarrow h_i(A) \xrightarrow{\iota_*} h_i(X) \xrightarrow{\iota_*} h_i(X, A) \xrightarrow{\partial} h_{i-1}(A) \rightarrow \cdots$$

is exact and natural.

- Excision. If $\overline{Z} \subseteq \mathring{A}$, then

$$\iota_* : h_\bullet(X \setminus Z, A \setminus Z) \xrightarrow{\sim} h_\bullet(X, A).$$

- Unions. If $X = \bigsqcup_\alpha X_\alpha$, then

$$\bigoplus (\iota_\alpha)_* : \bigoplus_\alpha h_\bullet(X_\alpha) \xrightarrow{\sim} h_\bullet\left(\bigsqcup_\alpha X_\alpha\right).$$

These axioms are usually called the **Eilenberg-Steenrod axioms**.

One sometimes restricts attention to pairs (X, A) which are not too pathological, so for example components and path-components agree. The axioms let us formalise the idea that homology of cell complexes is quite constrained or computable.

Definition. If h_\bullet and k_\bullet are generalised homotopy theories, a **natural transformation** $\Phi : h_\bullet \rightarrow k_\bullet$ comprises homomorphisms $\Phi_{X,A} : h_\bullet(X, A) \rightarrow k_\bullet(X, A)$ for all (X, A) , which are compatible with all the structure.

Example. If $f : (X, A) \rightarrow (Y, B)$ then

$$\begin{array}{ccc} h_\bullet(X, A) & \xrightarrow{f_*} & h_\bullet(Y, B) \\ \Phi_{X,A} \downarrow & & \downarrow \Phi_{Y,B} \\ k_\bullet(X, A) & \xrightarrow{f_*} & k_\bullet(Y, B) \end{array}$$

commutes, and similarly for maps of long exact sequence of pairs and excision or union isomorphisms.

4.2 Cellular example

Proposition 4.1. Let h_\bullet and k_\bullet be generalised homology theories defined on the class of pairs (X, A) where X is homotopy equivalent to a cell complex and $A \subseteq X$ to a subcomplex. Suppose $\Phi : h_\bullet \rightarrow k_\bullet$ is a natural transformation. If $\Phi : h_\bullet(\{\text{point}\}) \xrightarrow{\sim} k_\bullet(\{\text{point}\})$ is an isomorphism for $X = \{\text{point}\}$, and $A = \emptyset$, then $\Phi_{(X,A)} : h_\bullet(X, A) \xrightarrow{\sim} k_\bullet(X, A)$ for all finite-dimensional (X, A) in this class.

Then $h_\bullet(\{\text{point}\})$ is called the **coefficient** of the generalised homology theory.

Notation. Call this class of (X, A) the **cellular pairs**.

Proof. Induct on $\dim X$. If $\dim X = 0$, then $X = \{\text{discrete set}\} = X_0$, so the result follows from unions. So inductively suppose $\Phi_{(X,A)}$ is an isomorphism whenever $\dim X \leq n-1$, and suppose $X = X_n$ is n -dimensional. Consider

$$\begin{array}{ccccccccccc} \dots & \rightarrow & h_{i+1}(X, X_{n-1}) & \rightarrow & h_i(X_{n-1}) & \rightarrow & h_i(X) & \rightarrow & h_i(X, X_{n-1}) & \rightarrow & h_{i-1}(X_{n-1}) & \rightarrow & \dots \\ & & \downarrow \Phi & & \sim \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi & & \sim \downarrow \Phi & & \\ \dots & \rightarrow & k_{i+1}(X, X_{n-1}) & \rightarrow & k_i(X_{n-1}) & \rightarrow & k_i(X) & \rightarrow & k_i(X, X_{n-1}) & \rightarrow & k_{i-1}(X_{n-1}) & \rightarrow & \dots \end{array}$$

By the 5-lemma, if $\Phi_{(X, X_{n-1})}$ is an isomorphism for all i , then $\Phi_{(X, \emptyset)}$ is an isomorphism for all i . Apply excision, by replacing X_{n-1} by a neighbourhood $N_\epsilon(X_{n-1})$ which does not change $h_\bullet(X, A)$ by homotopy invariance then excise X_{n-1} , to show

$$h_\bullet(X, X_{n-1}) = h_\bullet(X_n, X_{n-1}) \cong h_\bullet\left(\bigsqcup_\alpha D_\alpha^n, \bigsqcup_\alpha \partial D_\alpha^n\right),$$

where the union is over n -cells. By unions,

$$h_\bullet\left(\bigsqcup_\alpha D_\alpha^n, \bigsqcup_\alpha \partial D_\alpha^n\right) \cong \bigoplus_\alpha h_\bullet(D_\alpha^n, \partial D_\alpha^n),$$

and similarly for k_\bullet , so it suffices to prove $\Phi_{(D_\alpha^n, \partial D_\alpha^n)}$ is an isomorphism. But now

$$\begin{array}{ccccccccccc} \dots & \rightarrow & h_i(\partial D^n) & \rightarrow & h_i(D^n) & \rightarrow & h_i(D^n, \partial D^n) & \rightarrow & h_{i-1}(\partial D^n) & \rightarrow & h_{i-1}(D^n) & \rightarrow & \dots \\ & & \sim \downarrow \text{Induction} & & \sim \downarrow \text{Homotopy} & & \downarrow \phi & & \sim \downarrow \text{Induction} & & \sim \downarrow \text{Homotopy} & & \\ \dots & \rightarrow & k_i(\partial D^n) & \rightarrow & k_i(D^n) & \rightarrow & k_i(D^n, \partial D^n) & \rightarrow & k_{i-1}(\partial D^n) & \rightarrow & k_{i-1}(D^n) & \rightarrow & \dots \end{array}$$

so by the 5-lemma, ϕ is an isomorphism as required. Inductively this shows $\Phi_{(X, \emptyset)}$ if $\dim X = n$, and then the 5-lemma and the long exact sequence shows $\Phi_{(X, A)}$ is an isomorphism if $\dim X = n$. So we are done for finite-dimensional cellular pairs. \square

The result also holds for infinite-dimensional cellular pairs, but we will not need this.

Example. Note that for $h_\bullet(X, A) = H_\bullet(X, A)$ we know $H_i(X_k) \rightarrow H_i(X)$ is onto once $k > i$, so easy to reduce to the finite-dimensional case.

A warning is that the axioms do not determine $h_\bullet(X, A)$ from $h_\bullet(\{\text{point}\})$ formally, and it is rather that naturally related theories have the same indeterminacy.

4.3 Generalisations

Remark. A **generalised cohomology theory** $(X, A) \mapsto h^\bullet(X, A)$ is similar, and has

- contravariant functoriality, so $f : (X, A) \rightarrow (Y, B)$ induces

$$f^* : h^\bullet(Y, B) \rightarrow h^\bullet(X, A),$$

- homotopy invariance,
- long exact sequence

$$\dots \rightarrow h^\bullet(X, A) \rightarrow h^\bullet(X) \rightarrow h^\bullet(A) \xrightarrow{\partial} h^{\bullet+1}(X, A) \rightarrow \dots,$$

- excision, and
- unions, which is a direct product

$$h^\bullet\left(\bigsqcup_\alpha X_\alpha\right) \cong \prod_\alpha h^\bullet(X_\alpha).$$

Remark. There are uninterestingly different generalised homology theories, such as $(X, A) \mapsto H_\bullet(X, A) \otimes_{\mathbb{Z}} R$ for your favourite graded group R , but interestingly different ones are not obtained from chain complexes.

- In lecture 1 we briefly mentioned homotopy groups $\pi_i(X)$. If

$$\Sigma X = (X \times [0, 1]) / \{(x, 0) \sim \{\text{point}\}, (x, 1) \sim \{\text{point}\} \mid x \in X\}$$

is the **suspension**, then there exist maps

$$\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X) \rightarrow \dots,$$

and these eventually become isomorphisms, so the **stable homotopy group** is

$$\pi_i^{\text{st}} = \lim_k \pi_{i+k}(\Sigma^k X).$$

Then $\pi_\bullet^{\text{st}}(\{\text{point}\})$ is unknown, and determining it is one of the major open problems of mathematics.

- In K-theory, $K_\bullet(X)$ is another generalised homology theory, built out of vector bundles, which we will discuss. Probably developed in the homotopy theory course.

Remark. Different generalised homology theories do not come from chain complexes, but the existence of different chain complexes is still deep and important.

- **Čech cochains.** Fix a cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of a space X . Let

$$S_k = \{(a_0, \dots, a_k) \in A^{k+1} \mid U_{a_0} \cap \dots \cap U_{a_k} \neq \emptyset\},$$

let $\check{C}^k(X, \mathcal{U})$ be the maps from S_k to \mathbb{Z} , and let

$$\begin{aligned} \partial : \check{C}^k(X, \mathcal{U}) &\longrightarrow \check{C}^{k+1}(X, \mathcal{U}) \\ \psi &\longmapsto \left((a_0, \dots, a_{k+1}) \mapsto \sum_{i=0}^{k+1} \psi(a_0, \dots, \widehat{a_i}, \dots, a_{k+1}) \right). \end{aligned}$$

Then $\partial^2 = 0$, by the same proof as singular cohomology, which gives $\check{H}^\bullet(X, \mathcal{U})$. Now set

$$\check{H}^\bullet(X) = \lim_{\mathcal{U}} \check{H}^\bullet(X, \mathcal{U}),$$

the limit over finer and finer covers.

- **Morse cochains.** Take M a compact C^∞ -manifold and $f : M \rightarrow \mathbb{R}$ smooth with non-degenerate critical points, so if $df|_x = 0$, then $d^2f|_x$ is non-degenerate. The index of x is the number of negative eigenvalues of $d^2f|_x$. Let

$$C_{\text{Morse}}^k(f) = \bigoplus_{x \text{ critical in } f \text{ of index } k} \mathbb{Z}.$$

There exists ∂_{Morse} , counting flow lines $\dot{\gamma} = -\nabla f \circ \gamma$ for $\gamma : \mathbb{R} \rightarrow M$, such that $H_{\text{Morse}}^\bullet(f) \cong H^\bullet(M)$.

5 Cup-products

Up to now we developed homology and cohomology in parallel, and we will use Mayer-Vietoris, excision, etc freely for cohomology too. But there is a key difference, which will in some sense dominate the rest of the course. Cohomology is a ring.

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5.1 The cohomology ring

Definition. If $\phi \in C^k(X)$ and $\psi \in C^l(X)$, their **cup-product** $\phi \cup \psi \in C^{k+l}(X)$ is defined by

$$(\phi \cup \psi)([v_0, \dots, v_{k+l}]) = \phi([v_0, \dots, v_k]) \psi([v_k, \dots, v_{k+l}]),$$

so feed the front face of the simplex to ϕ and the back face to ψ .

Notation. We will typically just write $\phi \cdot \psi$ rather than $\phi \cup \psi$, but still call it cup-product.

Lemma 5.1. If $\partial^* : C^\bullet(X) \rightarrow C^{\bullet+1}(X)$ is the coboundary operator in $C^\bullet(X)$, then

$$\partial^*(\phi \cdot \psi) = (\partial^*\phi) \cdot \psi + (-1)^k \phi \cdot (\partial^*\psi), \quad \phi \in C^k(X), \quad \psi \in C^l(X).$$

Note that sometimes write

$$\partial^*(\phi \cdot \psi) = (\partial^*\phi) \cdot \psi + (-1)^{|\phi|} \phi \cdot (\partial^*\psi),$$

where $|\phi|$ is the degree of ϕ , assumed homogeneous.

Proof. Note that

$$((\partial^*\phi) \cdot \psi)([v_0, \dots, v_{k+l+1}]) = \sum_{i=0}^{k+1} (-1)^i \phi([v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]) \psi([v_{k+1}, \dots, v_{k+l+1}]), \quad (3)$$

$$\left((-1)^k \phi \cdot (\partial^*\psi) \right)([v_0, \dots, v_{k+l+1}]) = \phi([v_0, \dots, v_k]) \sum_{i=k}^{k+l+1} (-1)^i \phi([v_k, \dots, \widehat{v}_i, \dots, v_{k+l+1}]), \quad (4)$$

where $(-1)^k$ on the left hand side is absorbed here. The last term of (3) and the first term of (4) cancel, since one has $(-1)^{k+1}$ and one $(-1)^k$. The remaining terms give

$$(\phi \cdot \psi) \left(\sum_{i=0}^{k+l+1} (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_{k+l+1}] \right) = (\partial^*(\phi \cdot \psi))([v_0, \dots, v_{k+l+1}]).$$

□

Corollary 5.2. Cup-product descends to cohomology, that is it induces

$$H^k(X) \times H^l(X) \rightarrow H^{k+l}(X). \quad (5)$$

This makes $H^\bullet(X)$ a graded unital ring.

Proof. Let $\phi \in C^k(X)$ and $\psi \in C^l(X)$ be closed. Then $\partial^*(\phi \cdot \psi) = (\partial^*\phi) \cdot \psi + (-1)^k \phi \cdot (\partial^*\psi) = 0$, so set $[\phi] \cup [\psi] = [\phi \cup \psi]$, an element of $H^{k+l}(X)$. If we change ϕ to $\phi + \partial^*\alpha$ for $\alpha \in C^{k-1}(X)$, then $(\phi + \partial^*\alpha) \cdot \psi = \phi \cdot \psi + (\partial^*\alpha) \cdot \psi = \phi \cdot \psi + \partial^*(\alpha \cdot \psi)$, using $\partial^*\psi = 0$, so $[\phi \cdot \psi]$ does not depend on the choice of cocycle representative for $[\phi]$, and changing the representative for $[\psi]$ is similar. So (5) is well-defined, on cohomology. Let $1 \in C^0(X)$ which is defined by $1(p) = 1 \in \mathbb{Z}$ for all $p \in X$, the generators of $C_0(X)$. Then $(\partial^*1)([v_0, v_1]) = 1(v_0) - 1(v_1) = 0$, so $\partial^*1 = 0$. Thus $[1] \in H^0(X)$, and

$$(\phi \cdot 1)([v_0, \dots, v_k]) = \phi([v_0, \dots, v_k]) \cdot 1(v_k) = \phi([v_0, \dots, v_k]),$$

$$(1 \cdot \psi)([v_0, \dots, v_l]) = 1(v_0) \cdot \psi([v_0, \dots, v_l]) = \psi([v_0, \dots, v_l]),$$

so $[1]$ is a unit. □

Recall that for an abelian group G ,

$$C_j(X; G) = C_j(X; \mathbb{Z}) \otimes G = \left\{ \sum_i a_i \sigma_i \mid a_i \in G, \sigma_i : \Delta^j \rightarrow X \right\}, \quad C^j(X; G) = \text{Hom}_{\mathbb{Z}}(C_j(X; \mathbb{Z}), G),$$

so $C^\bullet(X; R)$ is a ring whenever the coefficient group $G = R$ is a ring. Then $H^\bullet(X; R)$ is a ring if R is a ring and unital if R is unital.

Proposition 5.3.

- *Cup-product is associative, at the chain level, and so on cohomology, so*

$$\phi \cdot (\psi \cdot \tau) = (\phi \cdot \psi) \cdot \tau \in C^{k+l+r}(X), \quad \phi \in C^k(X), \quad \psi \in C^l(X), \quad \tau \in C^r(X).$$

- *If $f : X \rightarrow Y$, then $f^\# : C^\bullet(Y) \rightarrow C^\bullet(X)$ satisfies*

$$f^\#(\phi \cdot \psi) = (f^\# \phi) \cdot (f^\# \psi),$$

which is immediate from the definitions, so $f^ : H^\bullet(Y) \rightarrow H^\bullet(X)$ is a unital ring homomorphism.*

- *Cross-product is*

$$\begin{aligned} \times : H^i(Y) \times H^j(Z) &\longrightarrow H^{i+j}(Y \times Z), & Y &\xleftarrow{p_Y} Y \times Z \xrightarrow{p_Z} Z. \\ (\phi, \psi) &\longmapsto p_Y^* \phi \cup p_Z^* \psi \end{aligned}$$

If $Y = Z = X$ and the diagonal is

$$\begin{aligned} \Delta : X &\longrightarrow X \times X \\ x &\longmapsto (x, x) \end{aligned}$$

cup-product is

$$\cup : H^k(X) \times H^l(X) \xrightarrow{\times} H^{k+l}(X \times X) \xrightarrow{\Delta^*} H^{k+l}(X),$$

so the existence of Δ and the contravariance of cohomology are key.

Great, we have a product. But, as with original definition of homology and cohomology, there is little we can immediately compute.

Example. $H^\bullet(\{\text{point}\}) \cong \mathbb{Z}$ in degree zero, with its usual ring structure, and

$$H^\bullet(S^n) \cong \begin{cases} \mathbb{Z} & \bullet = 0, n \\ 0 & \text{otherwise} \end{cases},$$

so $H^\bullet(S^n) \cong \mathbb{Z}[x] / \langle x^2 \rangle$ for $|x| = n$.

Example. Let X and Y be cell complexes with basepoints $x_0 \in X$ and $y_0 \in Y$, and let $X \vee Y = (X \sqcup Y) / x_0 \sim y_0$. Then

$$\widetilde{H}^\bullet(X \vee Y) \cong \widetilde{H}^\bullet(X) \oplus \widetilde{H}^\bullet(Y)$$

is a ring isomorphism. Indeed,

$$X \xleftarrow{p_X} X \vee Y \xrightarrow{p_Y} Y, \quad X \xrightarrow{\iota_X} X \vee Y \xleftarrow{\iota_Y} Y$$

induce ring homomorphisms

$$H^\bullet(X) \xrightarrow{p_X^*} H^\bullet(X \vee Y) \xleftarrow{p_Y^*} H^\bullet(Y), \quad H^\bullet(X) \xleftarrow{\iota_X^*} H^\bullet(X \vee Y) \xrightarrow{\iota_Y^*} H^\bullet(Y),$$

and Mayer-Vietoris gives

$$H^\bullet(X \vee Y) \xrightarrow{(\iota_X^*, \iota_Y^*)} H^\bullet(X \vee Y) \rightarrow H^\bullet(\{\text{point}\}),$$

which shows

$$p_X^* \oplus p_Y^* : \widetilde{H}^\bullet(X) \oplus \widetilde{H}^\bullet(Y) \rightarrow \widetilde{H}^\bullet(X \vee Y)$$

is an isomorphism additively.

5.2 Key features

From the definitions, not sure we can do much else. We need two key features to get going. We will state them now and prove one of them later.

Proposition 5.4 (Graded commutativity). $H^\bullet(X)$ is **graded commutative**, or **skew-commutative**, so

$$\phi \cdot \psi = (-1)^{kl} \psi \cdot \phi, \quad \phi \in H^k(X), \quad \psi \in H^l(X).$$

Note that this is not true at chain level, only on cohomology.

Example. Suppose

$$H^\bullet(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \bullet = 0, 3, 6 \\ 0 & \text{otherwise} \end{cases}.$$

For degree reasons, the only possible interesting product is $H^3(X; \mathbb{Z}) \times H^3(X; \mathbb{Z}) \rightarrow H^6(X; \mathbb{Z})$, but if $H^3(X) = \mathbb{Z}\theta$, then $\theta \cdot \theta = -\theta \cdot \theta$ as $(-1)^{|\theta|} = -1$, so $2\theta \cdot \theta = 0$. Then $\theta \cdot \theta = 0$, since no 2-torsion in $H^6(X; \mathbb{Z})$. For example, $S^3 \vee S^6$ is such a space.

Let A and B be abelian groups. Then $A \otimes B$ is characterised by the universal property

$$\begin{array}{ccc} A \times B & \xrightarrow{\text{Bilinear}} & C \\ \downarrow \exists! & \nearrow \exists \text{ Linear} & \\ A \otimes B & & \end{array}$$

where C is an abelian group. Concretely, it is generated by symbols $a \otimes b$ such that

$$(a + a') \otimes b = a \otimes b + a' \otimes b, \quad a \otimes (b + b') = a \otimes b + a \otimes b'.$$

Example.

$$\begin{aligned} \mathbb{Z} \otimes A &= A, & \mathbb{Z}/n \otimes A &= A/nA, & (A \otimes B) \otimes C &\cong A \otimes (B \otimes C), & A \otimes B &\cong B \otimes A, \\ \left(\bigoplus_i A_i \right) \otimes B &\cong \bigoplus_i (A_i \otimes B), & f : A \rightarrow A', \, g : B \rightarrow B' &\implies f \otimes g : A \otimes A' \rightarrow B \otimes B'. \end{aligned}$$

Remark. If A and B are modules over a commutative ring R ,

$$A \otimes_R B = (A \otimes B) / \{ra \otimes b = a \otimes rb \mid a \in A, b \in B, r \in R\}.$$

Theorem 5.5 (Künneth theorem). Let Y be a cell complex such that $H^i(Y)$ is free for all i . Then cross-product

$$\bigoplus_{k+l=n} H^k(X) \otimes H^l(Y) \rightarrow H^n(X \times Y)$$

is an isomorphism whenever X is a finite cell complex.

Remark.

- Cross-product $H^i(X) \times H^j(Y) \rightarrow H^{i+j}(X \times Y)$ is bilinear, but bilinear maps are rarely homomorphisms, so natural to pass to tensor product.
- For R a commutative ring, if $H^j(Y; R)$ is a free R -module for all j , then

$$\bigoplus_{k+l=n} H^k(X; R) \otimes_R H^l(Y; R) \xrightarrow{\sim} H^n(X \times Y; R).$$

Note that if R is a field, $H^j(Y; R)$ is free.

- Write

$$\begin{aligned} \times : H^\bullet(X; R) \otimes_R H^\bullet(Y; R) &\longrightarrow H^\bullet(X \times Y; R) \\ a \otimes b &\longmapsto a \times b \end{aligned}$$

for Künneth. Note that this is a ring homomorphism where $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$.⁸

⁸Exercise: check

Example. The **exterior algebra** $\Lambda(x_i \mid x_i \in I)$ is free on generators $\{x_i\}$ subject to skew-commutativity, so $H^\bullet(S^1) \cong \mathbb{Z}[x] / \langle x^2 \rangle = \Lambda(x)$ for $|x| = 1$. By cellular cohomology,

$$H^\bullet(T^2) = \begin{cases} \mathbb{Z} & \bullet = 0, 2 \\ \mathbb{Z}^2 & \bullet = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \begin{aligned} H^0(T^2) &\cong H^0(S^1) \otimes H^0(S^1), \\ H^1(T^2) &\cong H^1(S^1) \otimes H^0(S^1) \oplus H^0(S^1) \otimes H^1(S^1), \\ H^2(T^2) &\cong H^1(S^1) \otimes H^1(S^1). \end{aligned}$$

Let $H^1(T^2)$ be generated by $x_1 \otimes 1$ and $1 \otimes x_2$. For degree reasons, the only possible interesting product is

$$\begin{aligned} H^1(T^2) \times H^1(T^2) &\longrightarrow H^2(T^2) \\ (x_1 \otimes 1, 1 \otimes x_2) &\longmapsto x_1 x_2 = x_1 \times x_2 \end{aligned}$$

and $x_1 x_2 = -x_2 x_1$ by skew-commutativity, so the only non-zero products are those of $\Lambda(x_1, x_2)$.

Corollary 5.6. $H^\bullet(T^n) = \Lambda^\bullet H^1(T^n)$ is the exterior algebra on n degree one generators.

Example. Label $1 \in H^0(\Sigma_g)$ for the unit and $u \in H^2(\Sigma_g)$ for a generator in

$$H^\bullet(\Sigma_g) \cong \begin{cases} \mathbb{Z} & \bullet = 0, 2 \\ \mathbb{Z}^{2g} & \bullet = 1 \end{cases}.$$

The ring structure is

$$\mathbb{Z}\langle x_1, \dots, x_g, y_1, \dots, y_g \mid x_i x_j = 0 = y_i y_j, x_i y_j = \delta_{ij} u \rangle,$$

and note $y_i x_j = -x_j y_i$ by skew-commutativity. Consider

$$\Sigma_g \xrightarrow{\pi} \bigvee_{i=1}^g T^2 \xleftarrow{p} \bigsqcup_{i=1}^g T^2.$$

Check that π^* and p^* are isomorphisms on degree one cohomology, so x_i and y_j define classes in $H^1(\bigvee_i T^2)$ and $H^1(\Sigma_g)$.⁹ On degree two cohomology,

$$\begin{array}{ccccc} H^2(\Sigma_g) & \xleftarrow{\pi^*} & H^2(\bigvee_{i=1}^g T^2) & \xrightarrow{p^*} & H^2(\bigsqcup_{i=1}^g T^2) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xleftarrow{\sum_i q_i \leftarrow (q_1, \dots, q_g)} & \mathbb{Z}^g & \xrightarrow{\sim} & \mathbb{Z}^g \end{array},$$

by symmetry on the T_i^2 . Now the result follows. If $i \neq j$, then x_i and x_j come from disjoint copies of T^2 so $x_i x_j = 0$, and similarly $x_i y_j = 0$ and $y_i y_j = 0$. If $i = j$, then $x_i x_i = 0$ and $y_i y_i = 0$, and $x_i y_i = u$ is the fixed generator of $H^2(T^2)$ for the i -th copy of T^2 .

As $H^n(S^n) \cong \mathbb{Z} \cong H^n(T^n)$ we can define the **degree** of maps $S^n \rightarrow T^n$ or $T^n \rightarrow S^n$ via induced maps on cohomology, well-defined up to sign.

Corollary 5.7. *There is no map $f : S^n \rightarrow T^n$ of non-zero degree if $n > 1$.*

Proof. Let $x_1 \dots x_n$ be the generator of $H^n(T^n)$. Then

$$\begin{aligned} f^* : H^1(T^n) &\longrightarrow H^1(S^n) = 0 \\ x_i &\longmapsto 0 \end{aligned},$$

so $f^*(x_1 \dots x_n) = \prod_i f^*(x_i) = 0$. □

Exercise. In contrast, there exists $f : T^n \rightarrow S^n$ of degree one.

⁹Exercise

5.3 Proof of Kunneth theorem

Recall

$$C^k(X, A) = \{\phi \in C^k(X) \mid \forall \sigma \in C_k(A) \subseteq C_k(X), \phi(\sigma) = 0\}.$$

If $\phi \in C^k(X, A)$ and $\psi \in C^l(X)$ then $\phi \cdot \psi \in C^{k+l}(X, A)$, since

$$(\phi \cdot \psi)([v_0, \dots, v_{k+l}]) = \phi([v_0, \dots, v_k]) \psi([v_k, \dots, v_{k+l}]) = 0.$$

So there is a **relative cup-product**

$$H^\bullet(X, A) \otimes H^\bullet(X) \rightarrow H^\bullet(X, A),$$

and in particular $H^\bullet(X, A)$ is a graded ring. Note that this is typically not unital. Analogously, cross-product defines $C^k(X, A) \otimes C^l(Y) \rightarrow C^{k+l}(X \times Y, A \times Y)$ and a **relative cross-product**

$$H^\bullet(X, A) \otimes H^\bullet(Y) \rightarrow H^\bullet(X \times Y, A \times Y).$$

We will use this to pay one of our debts.

Proof of Theorem 5.5. We consider the associations

$$(X, A) \mapsto h^\bullet(X, A) = H^\bullet(X, A) \otimes H^\bullet(Y), \quad (X, A) \mapsto k^\bullet(X, A) = H^\bullet(X \times Y, A \times Y),$$

functors of (X, A) , with Y fixed. Relative cross-product defines $\Phi : h^\bullet(X, A) \rightarrow k^\bullet(X, A)$, and

$$\Phi_{\{\text{point}\}} : h^\bullet(\{\text{point}\}) \otimes H^\bullet(Y) \rightarrow H^\bullet(\{\text{point}\} \times Y) = k^\bullet(\{\text{point}\})$$

is an isomorphism. So by our discussion with axioms of how generalised cohomology theories behave for finite cell complexes, it suffices to prove

1. h^\bullet and k^\bullet are generalised cohomology theories, and
2. Φ is a natural transformation, or entwines all the structure.

Then $\Phi_{(X,A)}$ will be an isomorphism for all (X, A) and we will be done.

1. All generalised cohomology theory axioms are immediate for k^\bullet from properties of cohomology. For h^\bullet , they are clear except the long exact sequence of a pair and unions. For those, use two algebraic facts.
 - Tensoring with a free module preserves exactness.
 - $(\prod_\alpha M_\alpha) \otimes N = \prod_\alpha (M_\alpha \otimes N)$ if N is finitely generated and free.
2. So we need $\Phi : H^\bullet(X, A) \otimes H^\bullet(Y) \rightarrow H^\bullet(X \times Y, A \times Y)$ to be compatible with homotopy invariance, long exact sequence, excision, etc. Well, cross-product is natural for maps of spaces, so homotopy invariance and excision are fine. So basically just need to check

$$\begin{array}{ccc} H^k(A) \otimes H^l(Y) & \xrightarrow{\delta \otimes \text{id}} & H^{k+1}(X, A) \otimes H^l(Y) \\ \times \downarrow & & \downarrow \times \\ H^{k+l}(A \times Y) & \xrightarrow{\delta} & H^{k+l+1}(X \times Y, A \times Y) \end{array}$$

commutes. To define δ , for $\phi \in C^k(A)$ a cocycle, so $\partial^*(\phi) = 0$, extend ϕ to $\widehat{\phi} \in C^k(X)$ a cochain, and set $\delta(\phi) = \partial^*(\widehat{\phi})$. Note that this does vanish on simplices in A . If $\psi \in C^l(Y)$ is a cocycle, then $\widehat{\phi} \times \psi$ does extend $\phi \times \psi$ using $\partial^*(\psi) = 0$, and this is what we need.

□

5.4 Proof of graded commutativity

The other debt is graded commutativity. Since not true at chain level, can expect proof to be painful.

Proof.

Sketch 1. Let $\epsilon_n = (-1)^{n(n+1)/2}$ and

$$\rho : \begin{array}{ccc} C_n(X) & \longrightarrow & C_n(X) \\ [v_0, \dots, v_n] & \longmapsto & \epsilon_n [v_n, \dots, v_0] \end{array},$$

where ϵ_n is the sign of the element of the n -th symmetric group needed to reorder vertices we indicated. Claim that ρ is a chain map, chain homotopic to the identity. Given this,

$$(\rho^* \phi \cdot \rho^* \psi)([v_0, \dots, v_{k+l}]) = \phi(\epsilon_k [v_k, \dots, v_0]) \psi(\epsilon_l [v_{k+l}, \dots, v_k]),$$

$$(\rho^*(\psi \cdot \phi))([v_0, \dots, v_{k+l}]) = \epsilon_{k+l} \psi([v_k, \dots, v_0]) \phi([v_k, \dots, v_0]),$$

so $\epsilon_k \epsilon_l \rho^* \phi \cdot \rho^* \psi = \epsilon_{k+l} \rho^*(\psi \cdot \phi)$ and $\epsilon_{k+l} = (-1)^{kl} \epsilon_k \epsilon_l$. But $\rho^* \simeq \text{id}$, so $\rho^* = \text{id}$ on cohomology, so $[\phi] \cdot [\psi] = (-1)^{kl} [\psi] \cdot [\phi]$ on cohomology. So just need to claim that ρ is a chain map, chain homotopic to the identity. To see ρ is a chain map, compute directly

$$\partial(\rho([v_0, \dots, v_n])) = \epsilon_n \sum_i (-1)^i [v_n, \dots, \widehat{v_{n-i}}, \dots, v_0],$$

$$\rho(\partial([v_0, \dots, v_n])) = \rho\left(\sum_i (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]\right) = \epsilon_{n-1} \sum_i (-1)^{n-i} [v_n, \dots, \widehat{v_{n-i}}, \dots, v_0],$$

and $\epsilon_n = (-1)^n \epsilon_{n-1}$. To show $\rho : C_n(X) \rightarrow C_n(X)$ is chain homotopic to the identity, use a twisted prism $P : C_n(X) \rightarrow C_{n+1}(X)$ such that $\partial \circ P + P \circ \partial = \rho - \text{id}$. See the prism operator from the proof of homotopy invariance, but reverse order of vertices on the top. If $\pi : \Delta^n \times [0, 1] \rightarrow \Delta^n$ is the projection,

$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i} (\sigma \circ \pi)|_{[v_0, \dots, v_i, w_n, \dots, w_i]}.$$

Compare to the earlier prism operator. In fact, this does the job, since

$$\partial(P(\sigma)) + P(\partial(\sigma)) = \epsilon_n [w_n, \dots, w_0] - [v_0, \dots, v_n].$$

Sketch 2. Let

$$\Delta : \begin{array}{ccc} C_{k+l}(X) & \longrightarrow & C_k(X) \otimes C_l(X) \\ [v_0, \dots, v_{k+l}] & \longmapsto & [v_0, \dots, v_k] \otimes [v_k, \dots, v_{k+l}] \end{array}$$

and

$$\tilde{\Delta} : \begin{array}{ccc} C_{k+l}(X) & \longrightarrow & C_k(X) \otimes C_l(X) \\ [v_0, \dots, v_{k+l}] & \longmapsto & [v_l, \dots, v_{k+l}] \otimes [v_0, \dots, v_l] (-1)^{kl} \end{array}$$

be chain maps. Then

$$\phi \cdot \psi = \cdot_{\mathbb{Z}} \circ (\phi \otimes \psi) \circ \Delta, \quad (-1)^{kl} \psi \cdot \phi = \cdot_{\mathbb{Z}} \circ (\phi \otimes \psi) \circ \tilde{\Delta}.$$

Claim that there is a unique natural chain map $C_{\bullet}(X) \rightarrow C_{\bullet}(X) \otimes C_{\bullet}(X)$ up to chain homotopy equivalence, so Δ and $\tilde{\Delta}$ agree on homology. By naturality, it suffices to prove this for Δ^n itself. But $C_{\bullet}(\Delta^n)$ and $C_{\bullet}(\Delta^n) \otimes C_{\bullet}(\Delta^n)$ are free resolutions of \mathbb{Z} , by the technique of acyclic models, so

$$H_{\bullet}(C_{\bullet}(\Delta^n)) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases}, \quad H_{\bullet}(C_{\bullet}(\Delta^n) \otimes C_{\bullet}(\Delta^n)) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Any two free resolutions of \mathbb{Z} are chain homotopy equivalent. □

5.5 Lyusternik-Schnirelmann categories

Cohomology $H^\bullet(X)$ is a ring. That is useful because $f : X \rightarrow Y$ induces $f^* : H^\bullet(X) \rightarrow H^\bullet(Y)$ a ring homomorphism but it also gives rise to new kinds of invariants.

Definition. The **cup-length** of a space X is

$$\text{cl } X = \max \{N \mid \exists \alpha_i, 1 \leq i \leq N, \alpha_i \in H^{>0}(X), \alpha_1 \cdots \alpha_N \neq 0\}.$$

Similarly, one could for example define the minimal number of elements of $H^\bullet(X)$ needed to generate it as a ring.

Example. $\text{cl } S^n = 1$ and $\text{cl } T^n = \text{cl } (S^1 \times \cdots \times S^1) = n$.

Fix a space X . We define a function

$$\begin{aligned} \nu & : \{\text{subsets of } X\} \longrightarrow \mathbb{N} \cup \{\infty\} \\ A & \longmapsto \nu(A) \end{aligned},$$

the **Lyusternik-Schnirelmann category** of A , where $\nu(A)$ is the least N such that A can be covered by N open sets $U_i \subseteq X$ such that the inclusion maps $U_i \hookrightarrow X$ are homotopic to constant maps. In particular $U_i \simeq \{\text{point}\}$ is contractible. Let $\nu(A) = \infty$ if A cannot be covered by any finite collection of such sets, and let $\nu(\emptyset) = 0$ by convention.

Remark. We usually only discuss ν for spaces X that admit some cover $\mathcal{U} = \{U_i \mid i \in I\}$ by open sets such that $U_i \hookrightarrow X$ is nullhomotopic. Note that if X is also compact, then ν is finite-valued.

Example. Any compact manifold has finite category, and $\nu(S^n) = 2$.

By example sheet 2, $\text{cl } X < \nu(X)$. Let X be a manifold. Then $\nu : \{\text{subsets of } X\} \rightarrow \mathbb{N} \cup \{\infty\}$ satisfies

- if $A \subseteq X$, then there exists a open neighbourhood $A \subseteq U \subseteq X$ such that $\nu(A) = \nu(U)$,
- if $A \subseteq B$, then $\nu(A) \leq \nu(B)$,
- $\nu(A \cup B) \leq \nu(A) + \nu(B)$,
- $\nu(\emptyset) = 0$ and $\nu(\{\text{point}\}) = 1$, and
- ν is a homeomorphism invariant, so if $f : X \xrightarrow{\sim} X$, then $\nu(A) = \nu(f(A))$.

Theorem 5.8. *Let M be a connected, closed smooth manifold. Any smooth function $f : M \rightarrow \mathbb{R}$ has at least $1 + \text{cl } M$ critical points.*

Proof. We will show that if f has finitely many critical points, then there are at least $\nu(M)$ of them. Since $\text{cl } M < \nu(M)$, we win. The proof will use some ideas from differential topology and geometry, and is therefore a digression from the course proper, so not examinable. Pick a Riemannian metric g on M , so we have the downwards gradient vector field $-\nabla f$ of f such that $\langle \nabla f, y \rangle_g = \text{d}f(y)$ for all $y \in \Gamma(TM)$. As M is compact, there is an associated flow $\{\phi_t\}$ of M . Call the set

$$\text{Crit } f = \{f(p) \mid p \text{ is a critical value of } f\}.$$

Note that if $c : I \rightarrow M$ is a curve, $\langle \nabla f, \frac{dc}{dt} \rangle_g = \frac{d}{dt}(f \circ c)$, so ∇f points away from $f^{-1}(t)$. Let

$$M^c = f^{-1}((-\infty, c]), \quad c_j = \sup \{c \mid \nu(M^c) < j\},$$

so $c_1 = \min f$ and $c_N = \max f$ where $N = \nu(M)$. Claim that $c_j \in \text{Crit } f$ and either $c_j < c_{j+1}$ or $f^{-1}(c_j)$ contains infinitely many critical points of f , so if the number of critical values of f is finite, $c_1 < \cdots < c_N$ and we win.

- $c_j \in \text{Crit } f$. This follows from the following property of flows. If $c \in \mathbb{R} \setminus \text{Crit } f$, there exist $t, \delta > 0$ such that $\phi^t(M^{c+\delta}) \subseteq M^{c-\delta}$. The flow is by homeomorphisms, so

$$\nu(M^{c+\delta}) = \nu(\phi^t(M^{c+\delta})) \leq \nu(M^{c-\delta}),$$

so $c \notin \{c_j\} = \{\sup\{c \mid \nu(M^c) < j\}\}$.

- Suppose $|f^{-1}(c_j) \cap \text{Crit } f| < \infty$ is finite. Note that if $\Sigma \subseteq M$ for M a connected manifold is finite, there exists an open $\mathbb{R}^n \cong U \subseteq M$ with $\Sigma \subseteq U$, so $\nu(\Sigma) \leq \nu(U) = 1$. So as $\phi^t(M^{c_j+\delta} \setminus U) \subseteq M^{c_j-\delta}$ for suitable t and δ ,

$$\nu(M^{c_j+\delta}) \leq \nu(M^{c_j+\delta} \setminus U) + 1 \leq \nu(M^{c_j-\delta}) + 1 < j + 1,$$

by definition of c_j . Thus $c_{j+1} \geq c_j + \delta > c_j$, as required. The upshot is if there exists finite many critical points, there are at least $N = \nu(M)$ of them.

□

Corollary 5.9. *Every $f : T^n \rightarrow \mathbb{R}$ has at least $n + 1$ critical points.*

Proof. $\text{cl } T^n = n$.

□

Remark. Morse theory is about studying $H^\bullet(M)$ via $\text{Crit } f$ where f has non-degenerate critical points. The fact that $H^\bullet(M) \cong H_{\text{Morse}}^\bullet(f)$ implies if f has non-degenerate critical points, it has at least $\sum_j \text{rk } H^j(M)$ such. The Lyusternik-Schnirelmann bound is weaker but has no non-degeneracy hypotheses.

6 Vector bundles

Lecture 15
Wednesday
11/11/20

Our goal is to understand cup-product better, and eventually the cohomology rings of manifolds. But we will get there by a roundabout route.

6.1 Vector bundles

Definition. Let B be a space. A **vector bundle** $E \rightarrow B$ of rank d is a family of vector spaces $\{E_b\}_{b \in B}$ and a topology on the disjoint union $E = \bigsqcup_{b \in B} E_b$ such that

- the projection $p : E \rightarrow B$ is continuous, and
- there is **local triviality**, so for all $b \in B$ there exists an open $U \ni b$ and a **local trivialisation** such that the diagram

$$\begin{array}{ccc} p^{-1}(U) = E|_U & \xrightarrow{\psi} & U \times \mathbb{R}^d \\ & \searrow p & \downarrow \pi_1 \\ & & U \end{array}$$

commutes and $\psi : E_y = p^{-1}(y) \rightarrow \{y\} \times \mathbb{R}^n$ is a linear isomorphism for all $y \in U$.

Notation. E is the **total space** and B is the **base space**. The E_b are the **fibres**. A map $s : B \rightarrow E$ such that $p \circ s = \text{id}_B$ is a **section** of E . There is the **zero-section**, which sends

$$\begin{array}{ccc} B & \longrightarrow & E_b \\ b & \longmapsto & 0 \end{array}.$$

Note the zero section $\iota : B \rightarrow E$ and the projection $p : E \rightarrow B$ are inverse homotopy equivalences, that is $p \circ \iota = \text{id}_B$ and $\iota \circ p \simeq \text{id}_E$. The **trivial** rank d vector bundle is $(E = B \times \mathbb{R}^d, p = \pi_1)$.

Example. The Möbius strip is a non-trivial rank one bundle over S^1 , and $S^1 \times \mathbb{R}$ is a trivial rank one bundle over S^1 .

The following are operations on vector bundles.

- **Pullback.** If $\pi : E \rightarrow X$ is a vector bundle and $f : Y \rightarrow X$, then $\pi : f^*E \rightarrow Y$ is defined by

$$f^*E = \{(e, y) \in E \times Y \mid \pi(e) = f(y)\},$$

and $\pi = \pi_2$. So $(f^*E)_y = E_{f(y)}$.

- **Whitney sum.** If $\pi_1 : E \rightarrow X$ and $\pi_2 : F \rightarrow X$ are vector bundles, $E \oplus F \rightarrow X$ has

$$E \oplus F = \{(e, f) \in E \times F \mid \pi_1(e) = \pi_2(f)\},$$

so there exists an induced projection to X . So $(E \oplus F)_x = E_x \oplus F_x$.

Note that both pullback and Whitney sum

- take trivial bundles to trivial bundles, and
- commute with restriction to open sets in X .

That is, $E|_U \oplus F|_U = (E \oplus F)|_U$ and $f^*(E|_U) = (f^*E)|_{f^{-1}(U)}$, so pullback and Whitney sum preserve the property of being locally trivial. More generally, anything you can do to a vector space, you can do to a vector bundle. Given E and F there is $E \otimes F$, the dual bundle E^* , exterior powers of E , etc, with the fibres given by the corresponding vector space operations.

Definition. A **subbundle** $F \subseteq E$ is a subspace such that for all $x \in X$ there exists a local trivialisation of E , so $x \in U$ and

$$\begin{array}{ccc} E|_U & \xrightarrow{\psi} & U \times \mathbb{R}^d \\ \cup & & \cup \\ F|_U = F \cap \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^k \end{array},$$

for $\mathbb{R}^k \subseteq \mathbb{R}^d$. Then $\pi|_F : F \rightarrow X$ is also a vector bundle now of rank k . If $F \subseteq E$ is a subbundle there is **quotient bundle** with fibre E_x/F_x . Note that vector bundles $p : E \rightarrow X$ and $p' : E' \rightarrow X$ are **isomorphic** if there exists

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{g} & X \end{array},$$

where ϕ and g are homeomorphisms, such that $\phi|_{E_x} : E_x \xrightarrow{\sim} E'_{g(x)}$ are linear isomorphisms for all x . Some people insist $g = \text{id}$.

The following is another viewpoint. Let $E \rightarrow B$ be a vector bundle and $\{U_\alpha\}_{\alpha \in A}$ a trivialising open cover. For $\alpha, \beta \in A$,

$$\begin{array}{ccc} E|_{U_\alpha \cap U_\beta} & \xrightarrow{\psi_\alpha} & (U_\alpha \cap U_\beta) \times \mathbb{R}^d \\ \psi_\beta \downarrow & \swarrow & \downarrow \psi_\beta \circ \psi_\alpha^{-1} \\ (U_\alpha \cap U_\beta) \times \mathbb{R}^d & \xrightarrow{\psi_\beta \circ \psi_\alpha^{-1}} & \text{GL}(d; \mathbb{R}) \end{array}.$$

The functions $\psi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$ satisfy the **cocycle condition**

$$\psi_{\alpha\alpha} = \text{id}, \quad \psi_{\alpha\beta} = (\psi_{\beta\alpha})^{-1}, \quad \psi_{\alpha\gamma} \circ \psi_{\gamma\beta} \circ \psi_{\beta\alpha} = \text{id}.$$

We can build

$$E = \left(\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{R}^d \right) / \sim,$$

where $(x, v) \sim (x, \psi_{\beta\alpha}(x)(v))$ for all $x \in U_\alpha \cap U_\beta$. So given a cover $\{U_\alpha\}_{\alpha \in A}$ and matrix-valued functions $\{\psi_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \text{GL}(d; \mathbb{R})\}$ satisfying the cocycle condition there is an associated vector bundle.

Example. If $E \rightarrow B$ and $F \rightarrow B$ are given, $E \otimes F$ is the bundle with underlying set $\bigsqcup_b E_b \otimes F_b$ and topologised via the transition functions $\psi_{\beta\alpha}^E \otimes \psi_{\beta\alpha}^F : U_\alpha \cap U_\beta \rightarrow \text{GL}(d_1 d_2; \mathbb{R})$, where $\text{rk } E = d_1$ and $\text{rk } F = d_2$.

Example. If M is a smooth manifold, the **tangent bundle** TM , of rank $n = \dim_{\mathbb{R}} M$, is defined with respect to an atlas $\left\{ (U_\alpha, \phi_\alpha : U_\alpha \xrightarrow{\sim} \mathbb{R}^n) \right\}$ of charts for M by the transition matrices $\psi_{\beta\alpha}$ of partial derivatives of $\phi_{\beta\alpha}$, where

$$\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

are diffeomorphisms of open sets. The cocycle condition is implied by the chain rule.

Example. If M is a smooth manifold and $\iota : N \hookrightarrow M$ is a smooth submanifold, $TN \subseteq \iota^* TM = TM|_N$ is a subbundle. The quotient $\nu_{N/M} = \iota^* TM / TN$ is the **normal bundle** of N in M .

6.2 The tautological bundle over the Grassmannian

Example. Let

$$X = \text{Gr}(k; \mathbb{R}^n) = \{k\text{-dimensional subspace of } \mathbb{R}^n\},$$

the **Grassmannian**, defining X to be a quotient of $\text{GL}(n; \mathbb{R})$, or $X = \text{O}(n)/\text{O}(k) \times \text{O}(n-k)$ since any subspace has an orthonormal basis so now consider changes of basis which do not change the subspace. The **tautological bundle** $E \rightarrow \text{Gr}(k; \mathbb{R}^n)$ has fibre at x the subspace $\langle x \rangle \subseteq \mathbb{R}^n$, where

$$E = \{(x, v) \in \text{Gr}(k; \mathbb{R}^n) \times \mathbb{R}^n \mid v \in \langle x \rangle\}.$$

Sometimes we write E_{taut} .

Lemma 6.1. E_{taut} is locally trivial.

Proof. Pick an inner product \langle, \rangle on \mathbb{R}^n . For $x \in X$,

$$U = \{y \in X \mid E_y \cap E_x^\perp = \{0\}\}$$

is an open neighbourhood of x . Let

$$\begin{aligned} \psi &: E|_U \longrightarrow U \times E_x = U \times \mathbb{R}^k \\ (y, \xi) &\longmapsto (y, \pi_{\langle x \rangle}(\xi)) \end{aligned},$$

where $\pi_{\langle x \rangle} : \mathbb{R}^n \rightarrow \langle x \rangle$ is the orthogonal projection. For all $y \in U$, $\pi_{\langle x \rangle}|_{E_y} : E_y \xrightarrow{\sim} E_x = \langle x \rangle$ by definition of U . \square

Note that there is an obvious notion of a complex vector bundle, where $E_y \cong \mathbb{C}^d$ for all y and transition maps are valued in $\text{GL}(d; \mathbb{C})$, and a tautological bundle $E \rightarrow \text{Gr}(k; \mathbb{C}^n)$. Thus there is a tautological **line bundle**, a vector bundle of rank one, $\mathcal{L}_{\text{taut}} = \mathcal{L} \rightarrow \mathbb{RP}^n$ with fibres \mathbb{R} and $\mathcal{L}_{\text{taut}} = \mathcal{L} \rightarrow \mathbb{CP}^n$ with fibres \mathbb{C} .

Lemma 6.2. Let X be compact and Hausdorff, or more generally paracompact and Hausdorff. If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X , there is a subordinate **partition of unity** $\{\lambda_\alpha : X \rightarrow \mathbb{R}_{\geq 0}\}_{\alpha \in A}$ such that

- $\text{supp } \lambda_\alpha = \{x \in X \mid \lambda_\alpha(x) \neq 0\} \subseteq U_\alpha$,
- for all $x \in X$, $\#\{i \mid x \in \text{supp } \lambda_i\} < \infty$, and
- for all $x \in X$, $\sum_{\alpha \in A} \lambda_\alpha(x) = 1$.

We will not prove this.

Definition. An **inner product** on a vector bundle $E \rightarrow X$ is a map $\lambda : E \otimes E \rightarrow \mathbb{R}$ such that for all $x \in X$, $\lambda_x : E_x \otimes E_x \rightarrow \mathbb{R}$ is an inner product on E_x .

Lemma 6.3. A vector bundle $p : E \rightarrow X$ over a compact Hausdorff space admits an inner product. Moreover, E is globally generated by sections, so for all $x \in X$ and $\xi_x \in E_x$, there exists $s : X \rightarrow E$ a section such that $s(x) = \xi_x$, so $p \circ s = \text{id}_X$.

Proof. Fix a trivialising open cover $\{U_\alpha\}_{\alpha \in A}$ for E . Fix an inner product \langle, \rangle on \mathbb{R}^d for $d = \text{rk } E$. Via $\psi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{R}^d$, \langle, \rangle gives \langle, \rangle_α an inner product on $E|_{U_\alpha}$. If $\{\lambda_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, for $u \otimes v \in E \otimes E$ set

$$\lambda(u \otimes v) = \langle u, v \rangle = \sum_{\alpha \in A} \lambda_\alpha(p(u)) \langle u, v \rangle_\alpha. \quad (6)$$

Note that $\langle u, v \rangle_\alpha$ is only defined if $p(u) = p(v) \in U_\alpha$. But if this is not true, $\lambda_\alpha(p(u)) = 0$, so that is ok. Note (6) is a finite sum. It is easy to check this is an inner product. Similarly for global generation. If $x \in U_\alpha$ and $\xi_x \in E_x$, pick a section s_α of $E|_{U_\alpha}$ such that $s_\alpha(x) = \xi_x$, such as a constant section with respect to the isomorphism

$$\begin{aligned} E|_{U_\alpha} &\longrightarrow U_\alpha \times \mathbb{R}^d \\ \xi_x &\longmapsto (x, v) \end{aligned}.$$

Now let $s = \sum_\alpha \lambda_\alpha s_\alpha$, a section of E . \square

Corollary 6.4. Let X be compact Hausdorff and $E \rightarrow X$ a vector bundle of rank d . Then there exists $N > d$ and $f : X \rightarrow \text{Gr}(d; \mathbb{R}^N)$ such that

$$E \cong f^* E_{\text{taut}}. \quad (7)$$

We say the tautological bundle and the Grassmannian are **universal** for rank d bundles.

Remark. There is a lot of choice here, such as of N . If $\text{Gr}(d; \mathbb{R}^\infty) = \bigcup_{N \geq d} \text{Gr}(d; \mathbb{R}^N)$ then there is a bijection

$$\begin{aligned} \{\text{homotopy classes of maps } X \rightarrow \text{Gr}(d; \mathbb{R}^\infty)\} &\longrightarrow \{\text{rank } d \text{ vector bundles over } X\} / \cong \\ f &\longmapsto f^* E_{\text{taut}} \end{aligned}.$$

See problem sheet 3.

Proof. By compactness of X , there exists a finite set $\{s_1, \dots, s_N\}$ of sections of E such that for all $x \in X$, $\{s_1(x), \dots, s_N(x)\}$ spans $E_x \cong \mathbb{R}^d$. Fix an inner product \langle, \rangle on E and consider

$$\begin{aligned} \alpha : E &\longrightarrow X \times \mathbb{R}^N \\ (x, \xi) &\longmapsto (x, \langle s_1(x), \xi \rangle, \dots, \langle s_N(x), \xi \rangle) \end{aligned}$$

Then α embeds E as a subbundle of a trivial bundle. We then define

$$\begin{aligned} f : X &\longrightarrow \text{Gr}(d; \mathbb{R}^N) \\ x &\longmapsto \alpha(E_x) \subseteq \mathbb{R}^N \end{aligned}$$

and (7) holds by construction. \square

Note that this also shows that if X is compact Hausdorff and $E \rightarrow X$, there exists $F \rightarrow X$ another subbundle such that $E \oplus F$ is a trivial bundle, and $F_x = \alpha(E_x)^\perp$ with respect to $\langle, \rangle_{\mathbb{R}^N}$.

6.3 Thom classes and Euler classes

Lecture 16
Friday
13/11/20

Last time we discussed vector bundles $E \rightarrow X$. Since $E \simeq X$, it appears as if the cohomology of E has no new information. But in fact vector bundles are ubiquitous in part because they give rise to distinguished elements of cohomology. Note if $E \rightarrow X$ has rank n , $H^n(E_x, E_x \setminus \{0\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Definition. A rank n vector bundle $E \rightarrow X$ is **oriented** if for all $x \in X$ we fix a generator $\epsilon_x \in H^n(E_x, E_x \setminus \{0\})$ and these vary locally trivially, so if $x \in U \subseteq X$ is a trivialising open neighbourhood, then $E_y \xrightarrow{\sim} E_x$ sends $\epsilon_y \mapsto \epsilon_x$ for all $y \in U$. Thus

$$\begin{array}{ccc} E|_U & \xrightarrow{\sim} & U \times E_x \\ \cup & & \cup \\ E_y & \xrightarrow{\sim} & \{y\} \times E_x \end{array}$$

Notation. $E^\# = E \setminus \{\text{zero-section } X \subseteq E\}$ and $E_x^\# = E_x \setminus \{0\}$.

Note that can also make sense of E being R -orientable or R -oriented for a coefficient ring R . Note if $R = \mathbb{F}_2$, every E is \mathbb{F}_2 -oriented, as $H^\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{F}_2) \cong \mathbb{Z}/2$ has a unique generator.

Remark.

- If $E \rightarrow X$ is defined by transition cocycles $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(n; \mathbb{R})$ and $\text{im } g_{\alpha\beta} \subseteq \text{GL}^+(n; \mathbb{R})$ with positive determinant then E is orientable.
- If M is a smooth manifold, M is orientable if and only if TM is orientable.

Theorem 6.5 (Thom isomorphism theorem). *Let $\pi : E \rightarrow X$ be an oriented vector bundle of rank n .*

- $H^k(E, E^\#) = 0$ for $k < n$.
- *There exists a unique element $u_E \in H^n(E, E^\#)$ such that the restriction, the pullback with respect to $(E_x, E_x \setminus \{0\}) \hookrightarrow (E, E^\#)$, is $u_E|_{E_x} = \epsilon_x$ for all x .*
- *The map*

$$\begin{aligned} H^k(X) &\longrightarrow H^{k+n}(E, E^\#) \\ \alpha &\longmapsto \pi^* \alpha \cdot u_E \end{aligned}$$

is an isomorphism for all k .

Definition. $u_E \in H^n(E, E^\#)$ is the **Thom class** of E . Under

$$H^n(E, E^\#) \xrightarrow{\text{LES}} H^n(E) \xrightarrow{(\text{zero-section})^*} H^n(X),$$

u_E maps to e_E , the **Euler class** of E .

The upshot is an oriented vector bundle $E \rightarrow X$ defines a class $e_E \in H^{rk E}(X)$. Note that if $E \rightarrow X$ is oriented, and $f : Y \rightarrow X$, then $f^*E \rightarrow Y$ inherits an orientation via $(E_y, E_y \setminus \{0\}) \xrightarrow{\sim} (E_{f(y)}, E_{f(y)} \setminus \{0\})$. These isomorphisms vary locally trivially. Now the uniqueness part of the Thom isomorphism says

$$u_{f^*E} = \hat{f}^* u_E, \quad \hat{f} : (f^*E, f^*E^\#) \rightarrow (E, E^\#).$$

Definition. If \mathcal{P} is a property, an assignment

$$\begin{array}{ccc} \{\mathcal{P} \text{ vector bundles over } X\} & \longrightarrow & H^\bullet(X) \\ E & \longmapsto & c(E) \end{array},$$

such that $c(f^*E) = f^*c(E)$ for all $f : X \rightarrow Y$ is called a **characteristic class** of \mathcal{P} vector bundles.

Example. The Euler class is a characteristic class for oriented vector bundles.

Characteristic classes give a global measure of the non-triviality of a vector bundle. Note if $E = X \times \mathbb{R}^d$ is trivial, $E = f^*E_{\text{triv}}$ for $\mathbb{R}^d = E_{\text{triv}}$, so $c(E) = f^*c(E_{\text{triv}}) \in f^*H^\bullet(\{\text{point}\})$ is zero in $H^\bullet(X)$.

Lemma 6.6. *If an oriented vector bundle $\pi : E \rightarrow X$ has a nowhere zero section, $e_E = 0$.*

Proof. Suppose $s : X \rightarrow E$ has image in $E^\# = E \setminus \{\text{zero-section}\}$. We have

$$\begin{array}{ccccc} X & \xrightarrow{s} & E^\# \subseteq E & \xrightarrow{\pi} & X \\ & \searrow & \text{id}_X & \nearrow & \end{array},$$

so $e_E \in \text{im}(H^k(E^\#) \rightarrow H^k(X))$ for $k = \text{rk } E$. But

$$\begin{array}{ccccc} & & u_E \mapsto 0 & & \\ & \searrow & & \nearrow & \\ H^k(E, E^\#) & \longrightarrow & H^k(E) & \longrightarrow & H^k(E^\#) \\ & \searrow u_E \mapsto e_E & \downarrow \iota^* & \swarrow s^* & \\ & & H^k(X) & & \end{array},$$

so $e_E = 0$. □

A caveat is that this is not quite as good as it looks.

Lemma 6.7. *If $E \rightarrow X$ is oriented and $\text{rk } E = d$ is odd, $2e_E = 0$.*

Proof. The map

$$\begin{array}{ccc} \alpha & : & E \longrightarrow E \\ & & v \longmapsto -v \end{array}$$

acts by -1 on $H^d(E_x, E_x \setminus \{0\})$ as it is a composition of d reflections, so $\alpha^*u_E = -u_E$. But if $s_0 : X \rightarrow E$ is the zero-section, $\alpha \circ s_0 = s_0$ and $u_E|_{s_0(X)} = e_E \in H^d(X)$. Combine these ingredients. □

So if $H^d(X)$ has no 2-torsion, $e_E = 0$.

6.4 The Gysin sequence

Let $E \rightarrow X$ be a vector bundle of rank d . Suppose E admits an inner product, such as if X is compact Hausdorff. The **sphere bundle** is

$$S(E) = \{e \in E \mid \langle e, e \rangle = 1\}.$$

Note $S(E) \hookrightarrow E^\# = E \setminus \{\text{zero-section}\}$ is a homotopy equivalence, so $S(E)$ is independent of the inner product \langle, \rangle up to homotopy equivalence. The map $S(E) \rightarrow X$ is a fibre bundle with fibre S^{d-1} . In general a **fibre bundle** $p : Z \rightarrow X$ with fibre F is a map such that for all $x \in X$ there exists an open $U \subseteq X$ and local trivialisations

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\psi} & U \times F \\ & \searrow p \quad \swarrow \pi_1 & \\ & U & \end{array},$$

such that $\psi = p^{-1}(t) \xrightarrow{\sim} \{t\} \times F$ for all $t \in U$.

Remark. A vector bundle is not a fibre bundle with fibre \mathbb{R}^d .

Example.

- If $X = \mathbb{RP}^n$ and \mathcal{L} is tautological, then $S(\mathcal{L}) = S^n \rightarrow \mathbb{RP}^n$ has fibre $S^0 = \{p, q\}$.
- If $X = \mathbb{CP}^n$ and \mathcal{L} is the tautological complex line bundle, then $S(\mathcal{L}) = S^{2n+1} \rightarrow \mathbb{CP}^n$ has fibre S^1 .

Consider the long exact sequence of $(E, E^\#)$,

$$\cdots \rightarrow H^i(E, E^\#) \rightarrow H^i(E) \rightarrow H^i(E^\#) \rightarrow H^{i+1}(E, E^\#) \rightarrow \cdots$$

But we can use the Thom isomorphism and homotopy invariance to rewrite this as

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{i+d}(E, E^\#) & \longrightarrow & H^{i+d}(E) & \longrightarrow & H^{i+d}(E^\#) \longrightarrow H^{i+d+1}(E, E^\#) \longrightarrow \cdots \\ & & \sim \uparrow \text{Thom} & & \sim \uparrow \text{Homotopy} & & \sim \uparrow \text{Homotopy} & & \sim \uparrow \text{Thom} \\ \cdots & \longrightarrow & H^i(X) & \xrightarrow{\phi} & H^{i+d}(X) & \longrightarrow & H^{i+d}(S(E)) & \longrightarrow & H^{i+1}(X) \longrightarrow \cdots \end{array}$$

The map

$$\begin{array}{ccc} \phi & : & H^i(X) \longrightarrow H^{i+d}(X) \\ & & \alpha \longmapsto \alpha \cdot e_E \end{array}$$

is cup-product with the Euler class of E , which is

$$\alpha \xrightarrow{\text{Thom}} \pi^* \alpha \cdot u_E \xrightarrow{\text{LES}} (\pi^* \alpha \cdot u_E)|_E \xrightarrow{s_0^*} s_0^* \pi^* \alpha \cdot u_E|_X = \alpha \cdot e_E,$$

where $s_0 : X \rightarrow E$ is the zero-section.

Definition. The **Gysin sequence** of the oriented vector bundle $E \rightarrow X$ is the long exact sequence

$$\cdots \rightarrow H^i(X) \xrightarrow{\cdot e_E} H^{i+d}(X) \rightarrow H^{i+d}(S(E)) \rightarrow H^{i+1}(X) \rightarrow \cdots,$$

where $d = \text{rk } E$.

The latter map is sometimes called integration over the fibre.

Remark. Recall relative cup-product is $H^i(X, A) \oplus H^i(X) \rightarrow H^{i+j}(X, A)$. For any pair (X, A) , the long exact sequence

$$\cdots \rightarrow H^i(X, A) \rightarrow H^i(X) \rightarrow H^i(X) \rightarrow H^{i+1}(X, A) \rightarrow \cdots$$

is a long exact sequence of $H^\bullet(X)$ -modules, that is the maps in the exact sequence commute with cup-product by $H^\bullet(X)$.

Corollary 6.8. *The Gysin sequence is a long exact sequence of left $H^\bullet(X)$ -modules.*

Proof. See problem sheet 3. □

Example. If $\mathcal{L} \rightarrow \mathbb{CP}^n$ is tautological, then $\mathcal{L}_v = \langle v \rangle \subseteq \mathbb{C}^{n+1}$, so

$$\mathcal{L} = \{(u, v) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \mid u \in \langle v \rangle\}.$$

Claim that any complex vector bundle is canonically \mathbb{Z} -oriented, as $\text{GL}(n; \mathbb{C}) \hookrightarrow \text{GL}(2n; \mathbb{R})$ lands in matrices of positive determinant. Also, $S(\mathcal{L}) \cong S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Gysin for $i \leq 2n - 2$ gives

$$\begin{array}{ccccccc} H^{i+1}(S^{n+1}) & \longrightarrow & H^i(\mathbb{CP}^n) & \xrightarrow{\cdot e_{\mathcal{L}}} & H^{i+2}(\mathbb{CP}^n) & \longrightarrow & H^{i+2}(S^{2n+1}) \\ \downarrow \mathbb{R} & & & & & & \downarrow \mathbb{R} \\ 0 & & & & & & 0 \end{array},$$

and

$$H^\bullet(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \bullet = 0, 2, \dots, 2n-2, 2n \\ 0 & \text{otherwise} \end{cases},$$

so $H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[x] / \langle x^{n+1} \rangle$ for $|x| = 2$ where $x = e_{\mathcal{L}}$.

Example. The **Stiefel manifold** is

$$V_k(\mathbb{C}^n) = \{\text{ordered } k\text{-tuples of orthonormal vectors in } \mathbb{C}^n\} \subseteq \mathbb{C}^n \times \cdots \times \mathbb{C}^n.$$

There is a tautological bundle $E \rightarrow V_k(\mathbb{C}^n)$ such that $E|_{\{e_1, \dots, e_k\}} = \langle e_1, \dots, e_k \rangle$.

Exercise. E is locally trivial.

Proposition 6.9.

$$H^\bullet(V_k(\mathbb{C}^n)) = \Lambda(a_{2n-2k+1}, a_{2n-2k+3}, \dots, a_{2n-3}, a_{2n-1})$$

is the exterior algebra on generators $a_i \in H^i(V_k(\mathbb{C}^n))$, that is free except for skew-commutativity.

Proof. Induct on k .

- $V_1(\mathbb{C}^n) \cong S^{2n-1}$ and $H^\bullet(S^{2n-1}) = \Lambda(a_{2n-1}) = \mathbb{Z}[a_{2n-1}] / \langle a_{2n-1}^2 \rangle$.
- Suppose the result holds for $V_k(\mathbb{C}^n)$. There is a forgetful map

$$\begin{array}{ccc} V_{k+1}(\mathbb{C}^n) & \longrightarrow & V_k(\mathbb{C}^n) \\ \{e_1, \dots, e_{k+1}\} & \longmapsto & \{e_1, \dots, e_k\} \end{array},$$

which shows $V_{k+1}(\mathbb{C}^n) = S(F)$, where $F = E^\perp \rightarrow V_k(\mathbb{C}^n)$ with fibre the Hermitian orthogonal complement to $E_x \subseteq \mathbb{C}^n$. Note that F is a rank $n - k$ complex bundle, so $e_F \in H^{2n-2k}(V_k(\mathbb{C}^n)) = 0$, by induction, since $H^\bullet(V_k(\mathbb{C}^n)) = \Lambda(a_{2n-2k+1}, a_{2n-2k+3}, \dots, a_{2n-3}, a_{2n-1})$. Gysin gives

$$0 \xrightarrow{e_F} H^i(V_k(\mathbb{C}^n)) \rightarrow H^i(V_{k-1}(\mathbb{C}^n)) \xrightarrow{\lambda} H^{i-2n+2k+1}(V_k(\mathbb{C}^n)) \xrightarrow{e_F} 0.$$

Choose $a_{2n-2k-1} \in H^{2n-2k-1}(V_{k+1}(\mathbb{C}^n))$ such that $\lambda(a_{2n-2k-1}) = 1 \in H^0(V_k(\mathbb{C}^n))$. Then

$$\begin{array}{ccc} H^\bullet(V_k(\mathbb{C}^n)) \oplus H^\bullet(V_k(\mathbb{C}^n)) & \longrightarrow & H^\bullet(V_{k+1}(\mathbb{C}^n)) \\ (u, v) & \longmapsto & u + v \cdot a_{2n-2k-1} \end{array}$$

is a $H^\bullet(V_k(\mathbb{C}^n))$ -module isomorphism. □

Note that $V_n(\mathbb{C}^n) = U(n)$ is the unitary group. So $H^\bullet(U(n)) = \Lambda(a_1, a_3, \dots, a_{2n-3}, a_{2n-1})$ and $U(n)$ has the same cohomology ring as $S^1 \times S^3 \times \cdots \times S^{2n-3} \times S^{2n-1}$.

Remark. Let

$$b_i(U(n)) = \text{rk } H^i(U(n); \mathbb{Z}),$$

the i -th **Betti number**. Then

$$\sum_{i \geq 0} b_i(U(n)) t^i = \prod_{i=1}^n (1 + t^{2i-1}).$$

Quite often generating functions for cohomology have nice properties.