Profinite Groups and Group Cohomology

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Lent 2020

Syllabus

Contents

0 Introduction

Lecture 1
Thursday
21/01/21

A question is, when are things different?

- $\mathbb{Z}$ is in bijection with $\mathbb{Q}$, by writing down a bijection.
- $\mathbb{Q}$ is not in bijection with $\mathbb{R}$, by diagonalisation.

A solution is to try to find an invariant, which is

- easier to compute,
- computable, and
- preserved under isomorphism.

Example.

- Cardinality of a set.
- Dimension and base field of a vector space, which is complete.
- For an algebraic field extension K over $\mathbb{Q}$, the degree $[K:\mathbb{Q}]$ and the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
- For a topological space X, compactness, connectedness, simplicial homology groups $\mathrm{H}_\bullet(X)$, and the fundamental group $\pi_1(X)$.

Theorem 0.0.1. *There is no algorithm that decides whether a finite presentation represents the trivial group.*

Finite groups we can decide.

- List all the finite quotients of a group.
- If you have two such lists, you can compare.
- If two groups have different sets of finite quotients, they are not isomorphic.

How often does this work?

- Combine all the finite quotients into one object to study, the **profinite completion**, which is a limit of the finite groups.
- More generally, a limit of finite groups is called a **profinite group**.

Example.

- In Galois theory, let $K = \bigcup_{N\in\mathbb{N}} K_N$ be the extension of $\mathbb{Q}$ adjoining all p^N-th roots of unity for p a fixed prime and $N \in \mathbb{N}$, which gives a short exact sequence of Galois groups

$$\mathrm{Gal}(K/K_N) \to \mathrm{Gal}(K/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(K_N/\mathbb{Q}).$$

  Then $\mathrm{Gal}(K_N/\mathbb{Q}) = \left(\mathbb{Z}/p^N\mathbb{Z}\right)^\times$ and $\mathrm{Gal}(K/\mathbb{Q}) = \varprojlim_N \left(\mathbb{Z}/p^N\mathbb{Z}\right)^\times = \mathbb{Z}_p^\times$.
- In algebraic geometry, étale fundamental groups are profinite groups.

The second part of the course is **group cohomology**, which is another invariant, with the following applications.

- Can tell if a group is free for some profinite groups.
- Given a group G and an abelian group A, group cohomology tells us how many groups E exist such that $A \triangleleft E$ and $E/A = G$.

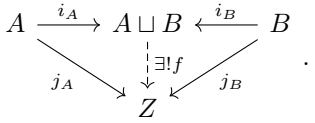
1 Inverse limits

1.1 Categories and limits

Let A and B be sets. How to combine into one thing? The disjoint union $A \sqcup B$ has inclusion maps $i_A : A \hookrightarrow A \sqcup B$ and $i_B : B \hookrightarrow A \sqcup B$, and for any other set Z, with functions $j_A : A \to Z$ and $j_B : B \to Z$ there is a unique function defined by

$$\begin{array}{rccc} f & : & A \sqcup B & \longrightarrow & Z \\ & & a & \longmapsto & j_A(a) \\ & & b & \longmapsto & j_B(b) \end{array},$$

such that $f \circ i_A = j_A$ and $f \circ i_B = j_B$, so



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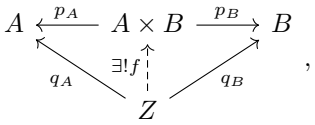
The product $A \times B$ comes with $p_A : A \times B \to A$ and $p_B : A \times B \to B$ such that



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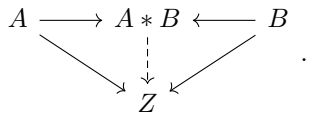
where $f(z) = (q_A(z), q_B(z))$. Reversed all arrows, so there is a duality, and disjoint union is a coproduct. What about groups, and group homomorphisms? The product still works, but the disjoint union is not a group. The coproduct is the free product $A * B$ such that



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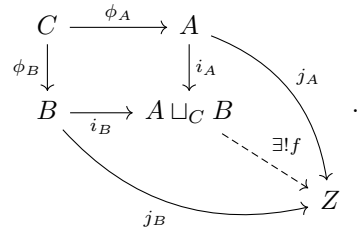
More generally is the pushout. Given groups A, B, and C, and homomorphisms $\phi_A : C \to A$ and $\phi_B : C \to B$, the **pushout** $A \sqcup_C B$ is



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Definition. A **category** $\mathcal{C}$ consists of

- a collection of **objects** $\mathrm{Obj}\,\mathcal{C}$,
- a collection of **morphisms** or **arrows** $\mathrm{Mor}\,\mathcal{C}$, such that each $f \in \mathrm{Mor}\,\mathcal{C}$ has a **domain** $X \in \mathrm{Obj}\,\mathcal{C}$ and a **codomain** $Y \in \mathrm{Obj}\,\mathcal{C}$ written as $f : X \to Y$,
- for all objects $X \in \mathrm{Obj}\,\mathcal{C}$, you have $\mathrm{id}_X : X \to X$, and
- if $f : X \to Y$ and $g : Y \to Z$, we have a defined composition $g \circ f : X \to Z$,

such that

- if $f : X \to Y$, then $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$, and
- if $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Example.

- In **Set**, objects are sets and morphisms are functions.
- In **Grp**, objects are groups and morphisms are group homomorphisms.
- In $\mathbf{Grp}_{\text{fin}}$, objects are finite groups.
- In $\mathbf{Grp}_{\text{inj}}$, morphisms are injective group homomorphisms.

Definition. A **partial ordering** on a set J is a binary relation $\leq$ such that

- $i \leq i$,
- if $i \leq j$ and $j \leq i$, then $i = j$, and
- if $i \leq j$ and $j \leq k$, then $i \leq k$.

A **poset** is a pair $(J, \leq)$, which is a **total ordering** if for all $i, j \in J$ either $i \leq j$ or $j \leq i$. The **poset category** $\mathcal{J}$ has objects $\operatorname{Obj}\mathcal{J} = J$ and morphisms $\operatorname{Mor}\mathcal{J} = \{i \to j \mid i \leq j\}$.

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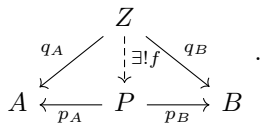
Definition. Let $\mathcal{C}$ be a category. A **product** of $A, B \in \operatorname{Obj}\mathcal{C}$ is an object P, equipped with morphisms $p_A : P \to A$ and $p_B : P \to B$, such that for all $Z \in \operatorname{Obj}\mathcal{C}$ and for all $q_A : Z \to A$ and $q_B : Z \to B$, there exists a unique $f : Z \to P$ such that $p_A \circ f = q_A$ and $p_B \circ f = q_B$, so



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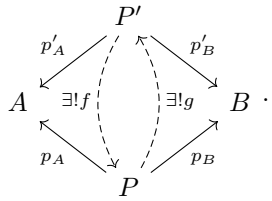
Definition. Objects A and B in a category $\mathcal{C}$ are **isomorphic** if there exist $f : A \to B$ and $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proposition 1.1.1. *If a product of A and B in $\mathcal{C}$ exists, then it is unique up to a unique isomorphism.*

Proof. Let (P, p_A, p_B) and (P', p'_A, p'_B) be products. Then



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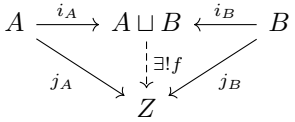
Consider $f \circ g : P \to P$. Then $p_A \circ f \circ g = p'_A \circ g = p_A$ and $p_B \circ f \circ g = p'_B \circ g = p_B$. By uniqueness, $f \circ g = \mathrm{id}_P$. Similarly, $g \circ f = \mathrm{id}_{P'}$. □

Notation. Define $P = A \times B$.

Definition. Let $\mathcal{C}$ be a category and $A, B \in \operatorname{Obj}\mathcal{C}$. Then a **coproduct** is an object $A \sqcup B$, together with maps $i_A : A \to A \sqcup B$ and $i_B : B \to A \sqcup B$, with the universal property



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Products are examples of limits and coproducts are examples of colimits.

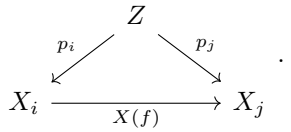
Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ associates an object $F(X) \in \operatorname{Obj} \mathcal{D}$ to each $X \in \operatorname{Obj} \mathcal{C}$, and a morphism $F(f) : F(X) \to F(Y)$ for each $f : X \to Y$ in $\mathcal{C}$, such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
- $F(g \circ f) = F(g) \circ F(f)$.

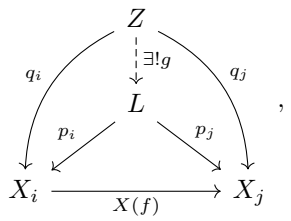
Definition. Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A **diagram of shape $\mathcal{J}$ in $\mathcal{C}$** is a functor $X : \mathcal{J} \to \mathcal{C}$. Often write $X(j) = X_j$, for $j \in \operatorname{Obj} \mathcal{J}$.

Very often, $\mathcal{J}$ is a poset category. In that case, if $i \leq j$, there exists a unique arrow $f : i \to j$ and then denote $X(f) = \phi_{ij}$.

Definition. A **cone** on a diagram $X : \mathcal{J} \to \mathcal{C}$ is an object $Z \in \operatorname{Obj} \mathcal{C}$, together with maps $p_j : Z \to X_j = X(j)$ for all $j \in \operatorname{Obj} \mathcal{J}$ such that for all $f : i \to j$, $X(f) \circ p_i = p_j$, so



A **limit** of a diagram $X : \mathcal{J} \to \mathcal{C}$ is a cone L, with morphisms p_j, such that for any cone Z, with morphisms q_j, there is a unique $g : Z \to L$ such that $p_j \circ f = q_j$, for all $j \in \operatorname{Obj} \mathcal{J}$, so



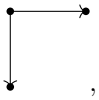
for $f : i \to j$. **Colimits** are as limits, but arrows are reversed.

Example.

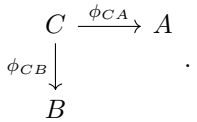
- If $\mathcal{J}$ is the category

  then a diagram of shape $\mathcal{J}$ is a pair of objects. The limit is the product and the colimit is the coproduct.

- If $\mathcal{J}$ is the category

  then a diagram of shape $\mathcal{J}$ in **Grp** would be

  The colimit is the pushout.

Proposition 1.1.2. *Limits and colimits are unique up to unique isomorphism.*

1.2 Inverse limits and profinite groups

Let G be a group. Let $\mathcal{N}$ be the poset category whose objects are $\{N \triangleleft_{\mathrm{f}} G\}$, where $N \triangleleft_{\mathrm{f}} G$ are finite index, with ordering $N_1 \leq N_2$ if and only if $N_1 \subseteq N_2$. There is a diagram of shape $\mathcal{N}$ in **Grp**,

$$\begin{array}{rccc} X & : & \mathcal{N} & \longrightarrow & \mathbf{Grp} \\ & & N & \longmapsto & X_N = G/N \end{array}.$$

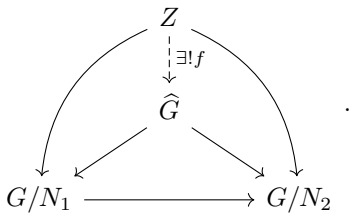
If $N_1 \leq N_2$, then $X(N_1 \to N_2)$ is the quotient map $\phi_{N_1 N_2} : G/N_1 \to G/N_2$, the transition maps.

Definition. Let G be a group. The **profinite completion** of G is the limit of this diagram, denoted $\widehat{G}$.

Then G comes with projections $p_N : \widehat{G} \to G/N$ for all $N \triangleleft_{\mathrm{f}} G$ such that

- if $N_1 \subseteq N_2$, then $\phi_{N_1 N_2} \circ p_{N_1} = p_{N_2}$, and
- if Z is a group, with $q_N : Z \to G/N$ such that $\phi_{N_1 N_2} \circ q_{N_1} = q_{N_2}$, there exists a unique $f : Z \to \widehat{G}$ such that $p_N \circ f = q_N$ for all N.

Thus



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In particular, $Z = G$ works, so there is a unique morphism $\iota_G : G \to \widehat{G}$, the **canonical morphism**, such that the diagrams commute.

Definition. A poset $(J, \leq)$ is an **inverse system** if for all $i, j \in J$ there exists $k \in J$ such that $k \leq i$ and $k \leq j$. An **inverse system of groups** consists of an inverse system $(J, \leq)$ and a diagram of shape $\mathcal{J}$ in **Grp**, so $G : \mathcal{J} \to \mathbf{Grp}$. Thus an inverse system is a group G_j for all $j \in J$ and transition maps $\phi_{ij} : G_i \to G_j$ if $i \leq j$ such that $\phi_{ii} = \mathrm{id}$ and $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all $i \leq j \leq k$. The **inverse limit** of this inverse system of groups G_j is the limit of this diagram, denoted $\varprojlim_j G_j$.

Definition. A **profinite group** is the inverse limit of an inverse system of groups, all of which are finite.

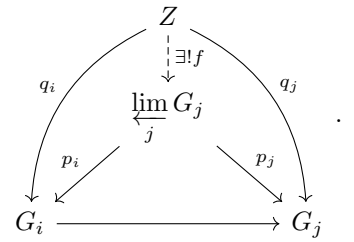
Proposition 1.2.1. *Let $(G_j)_{j \in J}$ be an inverse system of groups. Then the inverse limit exists, and is given by the explicit description*

$$\varprojlim_j G_j = \left\{ (g_j)_{j \in J} \in \prod_{j \in J} G_j \;\middle|\; \forall i \leq j,\ \phi_{ij}(g_i) = g_j \right\}.$$

Proof. This is a group. We have $p_j : \varprojlim_j G_j \to G_j$, restricted from $\prod_{j \in J} G_j \to G_j$, so $g_j = p_j\left((g_j)_{j \in J}\right)$. Take a cone Z on the system. Define

$$\begin{array}{rccc} f & : & Z & \longrightarrow & \varprojlim_j G_j \\ & & z & \longmapsto & (q_j(z))_{j \in J} \end{array}.$$

Then $\phi_{ij}(q_i(z)) = q_j(z)$, so



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□

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Definition. Let $(G_j)_{j\in J}$ be an inverse system of finite groups. Give each G_j the discrete topology. Give $\prod_j G_j$ the product topology. Then $\varprojlim_j G_j \leq \prod_j G_j$ gets the subspace topology.

Proposition 1.2.2. *$\varprojlim_j G_j$ is compact Hausdorff.*

Proof. $\prod_j G_j$ is Hausdorff and compact, by Tychonoff's theorem. Each condition $\phi_{ij}(g_i) = g_j$ is a closed condition, since $\prod_{j\in J} G_j \to G_i \times G_j$, so $\varprojlim_j G_j$ is closed in $\prod_j G_j$. □

Proposition 1.2.3. *Let $(X_j)_{j\in J}$ be an inverse system of non-empty finite sets. Then $\varprojlim_j X_j$ is non-empty.*

Proof. Use the finite intersection property. Let $I_1 \subseteq J$ be a finite subset. Define

$$Y_{I_1} = \left\{ (x_j) \in \prod_j X_j \;\middle|\; \forall i,j \in I_1,\ \forall i \leq j,\ \phi_{ij}(x_i) = x_j \right\} \subseteq \prod_j X_j,$$

a closed subset of the product. Since J is an inverse system and I_1 is finite, there exists $k \in J$ such that $k \leq i$ for all $i \in I_1$. Choose $x_k \in X_k \neq \emptyset$. Define $x_j = \phi_{kj}(x_k)$ for all $j \geq k$. Choose x_j arbitrarily elsewhere. This gives $x = (x_j) \in \prod_{j\in J} X_j$, which lies in Y_{I_1}, since if $i,j \in I_1$ such that $i \leq j$ then

$$x_j = \phi_{kj}(x_k) = \phi_{ij}(\phi_{ki}(x_k)) = \phi_{ij}(x_i).$$

So Y_{I_1} is non-empty. Then $Y_{I_1} \cap \cdots \cap Y_{I_n} \supseteq Y_{I_1 \cup \cdots \cup I_n} \neq \emptyset$. By the finite intersection property, since $\prod_j X_j$ is compact, $\bigcap_{I_1} Y_{I_1} = \varprojlim_j X_j$ is non-empty. □

Proposition 1.2.4. *Let J be a countable set and let $(X_j)_{j\in J}$ be a family of finite sets. Then $X = \prod_{j\in J} X_j$ is **metrisable**, so the metric topology equals to the other topology.*

Proof. Without loss of generality $J = \mathbb{N}$. Give each X_n the discrete metric d_n, where

$$d_n(x_n, y_n) = \begin{cases} 0 & x_n = y_n \\ 1 & x_n \neq y_n \end{cases}, \qquad x_n, y_n \in X_n.$$

Define

$$d((x_n),(y_n)) = \sum_{n=1}^{\infty} \frac{1}{3^n} d_n(x_n, y_n), \qquad (x_n),(y_n) \in \prod_n X_n.$$

We need to show this gives the product topology. Let $f : (X, \tau_{\text{product}}) \to (X, d)$ be the identity function. A basis for the metric topology are open balls $\mathrm{B}(x, 1/3^n)$ for $x \in X$ and $n \in \mathbb{N}$. Then $d((x_n),(y_n)) < 1/3^m$ if and only if $x_n = y_n$ for all $n \leq m$, and

$$f^{-1}\left(\mathrm{B}\left((x_n), \frac{1}{3^m}\right)\right) = \{(y_n) \mid \forall n \leq m,\ y_n = x_n\} = \bigcap_{n=1}^{m} p_n^{-1}(\{x_n\}), \qquad p_n : \prod_n X_n \to X_n$$

is open in the product topology. So f is continuous, so a homeomorphism. □

Proposition 1.2.5. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Lemma 1.2.6. *Let G be a finitely generated group. For each $n \in \mathbb{N}$, there are only finitely many subgroups of index n.*

Proof. For a subgroup $H \leq G$ of index n, we get a homomorphism $G \to \mathcal{S}_n$, since by labelling cosets $H, \ldots, g_n H$ by symbols $1, \ldots, n$, G permutes these right cosets by $g \cdot g_i H = (gg_i) H$ and H is recovered from this as the stabiliser of 1. So there are at most as many subgroups H as homomorphisms to $\mathcal{S}_n$, and there are only finitely many. □

Corollary 1.2.7. *If G is finitely generated, the inverse system $\mathcal{N} = \{N \lhd_{\mathrm{f}} G\}$ is countable.*

Proposition 1.2.8. *Let G be a profinite group. Then G is a **topological group**, so*

$$\begin{array}{rccc} \mathrm{m} : & G\times G & \longrightarrow & G \\ & (g,h) & \longmapsto & gh \end{array}, \qquad \begin{array}{rccc} \mathrm{i} : & G & \longrightarrow & G \\ & g & \longmapsto & g^{-1} \end{array}$$

are continuous.

Definition. Let G and H be topological groups. We say G and H are **isomorphic as topological groups** if and only if there exists $f : G \to H$ which is both an isomorphism of groups and a homeomorphism.

Recall that if G and H are profinite, this is the same as there exists f a continuous isomorphism.

Proposition 1.2.9. *Let H be a topological group and $G = \varprojlim_j G_j$ be an inverse limit of finite groups. Let $p_j : G \to G_j$ be the projection maps. A homomorphism $f : H \to G$ is continuous if and only if each map $f_j = p_j \circ f$ is continuous.*

Proof. $f : H \to G \leq \prod_j G_j$. This is continuous if and only if all f_j are continuous, by definition of the product topology. □

Proposition 1.2.10. *Let $f : H \to G_j$ be a homomorphism from a topological group to a finite group, with the discrete topology. Then f is continuous if and only if* $\ker f$ *is open in H.*

Proof. If f is continuous then $\ker f = f^{-1}(\{1\})$ is open. Assume $f^{-1}(\{1\})$ is open. Then $f^{-1}(\{g\})$ is open for all $g \in G$, since multiplication is continuous and $f^{-1}(\{g\}) = hf^{-1}(\{1\})$ for some $h \in H$. Taking unions, the preimage of any set in G_j is open in H, so f is continuous. □

Proposition 1.2.11. *Let G be a compact topological group. A subgroup of G is open if and only if it is closed and of finite index.*

Proposition 1.2.12. *Let $(G_j)_{j\in J}$ be an inverse system of finite groups. If $G = \varprojlim_j G_j$, then the open subgroups $U_j = \ker(p_j : G \to G_j)$ form a **basis of open neighbourhoods** of the identity $1 \in G$, so if $V \subseteq G$ is any open set with $1 \in V$, then there exists j such that $U_j \subseteq V$.*

Proof. Let $V \ni 1$ be open. By definition of the product topology,

$$V \supseteq p_{j_1}^{-1}(X_{j_1}) \cap \cdots \cap p_{j_n}^{-1}(X_{j_n}) \supseteq p_{j_1}^{-1}(\{1\}) \cap \cdots \cap p_{j_n}^{-1}(\{1\}) = U_{j_1} \cap \cdots \cap U_{j_n}.$$

for $X_{j_i} \subseteq G_{j_i}$. There exists k such that $k \leq j_i$. Since $p_{j_i} = \phi_{kj_i} \circ p_k$, $\ker p_k = U_k \subseteq U_{p_{j_i}} = \ker p_{j_i}$ for all i. Thus $V \supseteq U_k$. □

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Corollary 1.2.13. *If $g = (g_j)_{j\in J} \in G$, then the open cosets $gU_j = p_j^{-1}(\{g_j\})$ form a neighbourhood base at g, so for all open set $V \ni g$, there exists $j \in J$ such that $gU_j \subseteq V$.*

Proof. Continuity of multiplication. □

Corollary 1.2.14. *A subset $X \subseteq G$ is dense if and only if $p_j(X) = p_j(G)$ for all $j \in J$.*

Proof. Suppose X is not dense. There exists a non-empty open set V such that $V \cap X = \emptyset$. Pick $g \in V$. There exists $j \in J$ such that $p_j^{-1}(\{g_j\}) = gU_j \subseteq V$, where $g_j = p_j(g)$. Then $g_j \in p_j(G)$. But for any $x \in X$, $p_j(x) \neq g_j$, otherwise $x \in p_j^{-1}(\{g_j\}) = gU_j \subseteq V$, so $p_j(X) \neq p_j(G)$. Assume X is dense. Then $p_j(X) \subseteq p_j(G)$ is obvious. If $g_j \in p_j(G)$, then $p_j^{-1}(\{g_j\})$ is a non-empty open set, so there exists $x \in X \cap p_j^{-1}(\{g_j\})$, then $p_j(x) = g_j$. So $g_j \in p_j(X)$, so $p_j(X) = p_j(G)$. □

Corollary 1.2.15. *Let Y be a compact topological space and let $f : Y \to G$ be a continuous function. Then f is surjective if and only if $p_j(f(Y)) = p_j(G)$ for all $j \in J$.*

Proof. $p_j(f(Y)) = p_j(G)$ if and only if $f(Y)$ is dense, if and only if $f(Y) = G$, since $f(Y)$ is closed. □

Proposition 1.2.16. *Let G be a profinite group and $X \subseteq G$ be a subset. Then the closure of X is*

$$\overline{X} = \bigcap_{N \leq_{\mathrm{o}} G} XN,$$

where $N \leq_{\mathrm{o}} G$ are open subgroups.

Proof. XN is a union of cosets, hence it is open and closed in G. So $\overline{X} \subseteq XN$ for all $N \leq_{\mathrm{o}} G$, so $\overline{X} \subseteq \bigcap_{N \leq_{\mathrm{o}} G} XN$. Take $g \notin \overline{X}$. There exists an open $V \subseteq G$ such that $g \in V$ but $X \cap V = \emptyset$. Then there exists $j \in J$ such that $V \supseteq gU_j$ for $N = U_j = \ker p_j$. Then $g \notin XN$, since if $g = xn$ for $x \in X$ and $n \in N = U_j$ then $x = gn^{-1} \in gN = gU_j \subseteq V$, a contradiction. Thus $g \notin \bigcap_N XN$, so $\bigcap_N XN \subseteq \overline{X}$. □

Proposition 1.2.17. *Let G be a profinite group and let $\mathcal{U}$ be a collection of open normal subgroups which form a neighbourhood base at the identity. Then*

$$G \cong \varprojlim_{U \in \mathcal{U}} G/U,$$

as topological groups, where G/U are finite groups.

Proof. The quotient maps $G \twoheadrightarrow G/U$ are a cone on the inverse system, so we get a well-defined homomorphism $f : G \to \varprojlim_U G/U$. Then

- f is continuous, since compositions with projection maps are continuous,
- f is surjective, since $G \twoheadrightarrow G/U$ are surjective, and
- f is injective, since if $g \in G \setminus \{1\}$, there exists an open subset V such that $1 \in V$ and $g \notin V$ and there exists $U \in \mathcal{U}$ such that $1 \in U \subseteq V$, then $g \notin \ker(G \to G/U)$, so $g \notin \ker f$.

□

1.3 Change of inverse system

Definition. Let $(J, \leq)$ be an inverse system. A **cofinal subsystem** of J is a subset $I \subseteq J$ such that for all $j \in J$ there exists $i \in I$ such that $i \leq j$.

Then I is an inverse system.

Example. If $k \in J$, then the set

$$J_{\leq k} = \{j \in J \mid j \leq k\},$$

the **principal cofinal subsystem**, is cofinal in J.

Proposition 1.3.1. *Let $(G_j)_{j \in J}$ be an inverse system of finite groups, and let $I \subseteq J$ be cofinal. Then $H = \varprojlim_{i \in I} G_i$ is topologically isomorphic to $G = \varprojlim_{j \in J} G_j$.*

Proof. The projection map $\prod_{j \in J} G_j \to \prod_{i \in I} G_i$ is a continuous homomorphism, and it restricts to $f : G \to H$. Check that f is bijective.

- Injective. Take $g = (g_j)_{j \in J} \in G$. Assume $f(g) = 1$, so $g_i = p_i(f(g)) = 1$ for all $i \in I$. For any $j \in J$, there exists $i \in I$ such that $i \leq j$. Then $g_j = \phi_{ij}(g_i) = \phi_{ij}(1) = 1$. So $g = 1$.
- Surjective. Let $h = (h_i)_{i \in I} \in H$ for $h_i \in G_i$. Define $g = (g_j) \in \prod_{j \in J} G_j$ by setting $g_j = \phi_{ij}(h_i)$ for some $i \in I$ such that $i \leq j$. If $i_1 \leq j$ and $i_2 \leq j$, there exists $i_0 \in I$ such that $i_0 \leq i_1$ and $i_0 \leq i_2$, then

  $$\phi_{i_1 j}(h_{i_1}) = \phi_{i_1 j}(\phi_{i_0 i_1}(h_{i_0})) = \phi_{i_0 j}(h_{i_0}) = \phi_{i_2 j}(\phi_{i_0 i_2}(h_{i_0})) = \phi_{i_2 j}(h_{i_2}).$$

  It also follows that $g \in G$, since if $j_1 \leq j_2$, choose $i \in I$ such that $i \leq j_1$, then

  $$g_{j_2} = \phi_{ij_2}(h_i) = \phi_{j_1 j_2}(\phi_{ij_1}(h_i)) = \phi_{j_1 j_2}(g_{j_1}).$$

  Finally, $f(g) = h$, since $g_i = \phi_{ii}(h_i) = h_i$ for all $i \in I$.

□

Definition. An inverse system of groups is **surjective** if all transition maps are surjective.

Proposition 1.3.2. *Let $(X_j)_{j\in J}$ be an inverse system of finite sets where all transition maps are surjective. Then the projection maps $p_j : \varprojlim_j X_j \to X_j$ are surjective.*

Proposition 1.3.3. *Let $(G_j)_{j\in J}$ be an inverse system of finite groups. Then there exists an inverse system $\left(G'_j\right)_{j\in J}$ such that $G'_j \leq G_j$, with surjective transition maps, such that $\varprojlim_j G_j = \varprojlim_j G'_j$.*

Proof. Let $p_j : G = \varprojlim_j G_j \to G_j$ be the projection. Define $G'_j = p_j(G)$. Since $\phi_{ij} \circ p_i = p_j$, $\left(G'_j\right)$ is an inverse system with $\phi_{ij}|_{G'_i} : G'_i \to G'_j$, and $\phi_{ij}|_{G'_i}$ is surjective. If $g = (g_j) \in G$ then $g_j = p_j(g) \in G'_j$, so $g \in \varprojlim_j G'_j \leq G \leq \prod_j G_j$. Thus $\varprojlim_j G'_j = G$. □

Definition. An inverse system $(J, \leq)$ is **linearly ordered** if there exists a bijection $f : J \to \mathbb{N}$ such that $i \leq j$ if and only if $f(i) \geq f(j)$, the **wrong-way ordering** on $\mathbb{N}$.

Thus cofinal if and only if increasing subsequence.

Proposition 1.3.4. *If J is a countable inverse system, with no **global minimum**, so there does not exist $m \in J$ such that $m \leq j$ for all j, then J has a linearly ordered cofinal subsystem.*

2 Profinite groups

2.1 The p-adic integers

Lecture 5
Saturday
30/01/21

Let p be a prime. Consider

$$\cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

The **ring of p-adic integers** is

$$\mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}.$$

Thus $\alpha \in \mathbb{Z}_p$ is a sequence $(a_n)_{n\in\mathbb{N}}$ of integers modulo p^n for $a_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that $a_n \equiv a_m \mod p^m$ whenever $n \geq m$, since $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, and

$$\begin{array}{rccl} p_n & : & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z} \\ & & \alpha & \longmapsto & a_n = \alpha \mod p^n \end{array}.$$

Given $a \in \mathbb{Z}$, setting $a_n = a \mod p^n$ gives an element $\iota(a) \in \mathbb{Z}_p$ for $\iota : \mathbb{Z} \to \mathbb{Z}_p$. Then ι is injective, since if $a \in \mathbb{Z}$, and $p^n > |a|$ then $a \not\equiv 0 \mod p^n$, so $\iota(a) \neq 0$ in $\mathbb{Z}_p$. Often $\mathbb{Z} \leq \mathbb{Z}_p$.

Definition. Let $\alpha = (a_n), \beta = (b_n) \in \mathbb{Z}_p$. If $\alpha = \beta$ then $\mathrm{d}(\alpha, \beta) = 0$. If $\alpha \neq \beta$, take the smallest n such that $a_n \neq b_n$, and set $\mathrm{d}(\alpha, \beta) = p^{-n}$, the **$p$-adic metric on** $\mathbb{Z}_p$. The restriction of d to $\iota(\mathbb{Z})$ is the **p-adic metric on** $\mathbb{Z}$.

Thus α and β are close if (a_n) and (b_n) agree modulo p^n for all but large n. Since

$$\mathrm{B}(0, r) = \left\{\alpha = (a_n) \;\middle|\; \forall n \leq -\log_p r,\ a_n = 0\right\} = \ker\left(\mathbb{Z}_p \to \mathbb{Z}/p^{\lfloor -\log_p r \rfloor}\mathbb{Z}\right),$$

open balls are the subgroups $p^n\mathbb{Z}_p \leq \mathbb{Z}_p$.

- $\iota(\mathbb{Z})$ is dense in this metric. Let $\alpha = (a_n) \in \mathbb{Z}_p$ and $\epsilon > 0$. Take $n > -\log_p \epsilon$, and choose $a \in \mathbb{Z}$ such that $a \equiv a_n \mod p^n$. Then $\mathrm{d}(\alpha, \iota(a)) \leq p^{-n} < \epsilon$.
- The p-adic metric on $\mathbb{Z}$ is not complete, since $a_n = 1 + \cdots + p^n$ does not converge in $\mathbb{Z}$, but does converge in $\mathbb{Z}_p$.
- The p-adic metric on $\mathbb{Z}_p$ is complete. Let $\alpha^{(k)} = \left(a_n^{(k)}\right)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Z}_p$. For all n there exists K_n such that for all $k, l \geq K_n$, we have $\mathrm{d}\left(\alpha^{(k)}, \alpha^{(l)}\right) \leq p^{-n}$, so $a_n^{(k)} = a_n^{(l)}$ for all $k, l \geq K_n$ so for fixed n, $a_n^{(k)}$ is eventually a constant b_n. Then $\beta = (b_n) \in \mathbb{Z}_p$, and $\alpha^{(k)} \to \beta$ in $\mathbb{Z}_p$.

Thus $\mathbb{Z}_p$ is a completion of $\mathbb{Z}$, but is not the profinite completion of $\mathbb{Z}$.

Definition. Let p be a prime. A **p-group** is a finite group of order p^n for $n \geq 0$. A **pro p-group** is an inverse limit of p-groups.

Definition. Let G be a group and p prime. The set of normal subgroups $N \triangleleft G$ such that $[G : N] = p^n$ for some n form an inverse system $\mathcal{N}_p$. Since $G/N_1 \times G/N_2$ are p-groups, $N_1 \cap N_2 = \ker(G \to G/N_1 \times G/N_2)$ is a p-group. The **pro-p completion** is

$$\widehat{G_{(p)}} = \varprojlim_{N\in\mathcal{N}_p} G/N,$$

where $G/N_1 \to G/N_2$ if $N_1 \leq N_2$.

Proposition 2.1.1. *The additive group $\mathbb{Z}_p$ is abelian and torsionfree.*

Proof. $\mathbb{Z}_p \leq \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is abelian. Let $\alpha = (a_n) \in \mathbb{Z}_p \setminus \{0\}$. Suppose $m\alpha = 0$ for $m \in \mathbb{Z}$. We want $m = 0$. Assume $m = p^r s$ for s coprime to p. Then $\alpha \neq 0$, so there exists n such that $a_n \neq 0$. Consider a_{n+r}. Then $0 \equiv m a_{n+r} \equiv p^r a_{n+r} s \mod p^{n+r}$, so $p^n \mid a_{n+r} s$. Thus $p^n \mid a_{n+r}$, so $a_n \equiv a_{n+r} \equiv 0 \mod p^n$, a contradiction. □

Proposition 2.1.2. *The ring $\mathbb{Z}_p$ has no zero-divisors.*

Proof. Exercise. [1] □

2.2 The profinite completion of the integers

The **profinite completion of the integers** is

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z},$$

where $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ whenever $n\mathbb{Z} \leq m\mathbb{Z}$, which is if and only if $m \mid n$, so $n = mr$.

Theorem 2.2.1 (Chinese remainder theorem)**.** *There is an isomorphism of topological rings*

$$\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p.$$

Proof. Each natural number n is written as a product of prime powers $n = \prod_{p \text{ prime}} p^{e_p(n)}$. The classical CRT gives natural isomorphisms

$$\begin{array}{rccc} f_n & : & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \displaystyle\prod_{p \text{ prime}} \mathbb{Z}/p^{e_p(n)}\mathbb{Z} \\ & & 1 & \longmapsto & (1, \ldots, 1) \end{array},$$

and commutative diagrams

$$\begin{array}{ccc} \mathbb{Z}/mn\mathbb{Z} & \xrightarrow[\sim]{f_{mn}} & \displaystyle\prod_p \mathbb{Z}/p^{e_p(mn)}\mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow[f_n]{\sim} & \displaystyle\prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z} \end{array}.$$

Passing to inverse limits,

$$\begin{array}{ccc} \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\sim} & \varprojlim_n \displaystyle\prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z} \\ \cap & & \cap \\ \displaystyle\prod_n \mathbb{Z}/n\mathbb{Z} & \xrightarrow[\sim]{} & \displaystyle\prod_n \prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z} \end{array}.$$

The natural continuous surjections

$$\prod_p \mathbb{Z}_p \twoheadrightarrow \prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z}$$

form a cone on the inverse system $\left\{\prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z}\right\}$, so there exists

$$f : \prod_p \mathbb{Z}_p \twoheadrightarrow \varprojlim_n \prod_p \mathbb{Z}/p^{e_p(n)}\mathbb{Z},$$

which is continuous by Proposition 1.2.9, surjective by Corollary 1.2.15, and injective since every non-trivial element of $\prod_p \mathbb{Z}_p$ is non-trivial in some quotient $\mathbb{Z}/p^e\mathbb{Z}$. So f is a topological isomorphism as required. □

Corollary 2.2.2. *The abelian group $\widehat{\mathbb{Z}}$ is torsionfree abelian.*

Corollary 2.2.3. *The ring $\widehat{\mathbb{Z}}$ is not an integral domain.*

Proof. Any product of non-trivial rings $R_1 \times R_2$ has zero-divisors, since $(r_1, 0) \cdot (0, r_2) = (0, 0)$. An element of $\widehat{\mathbb{Z}}$ is a zero-divisor if and only if it is zero in some $\mathbb{Z}_p$-factor. □

Elements of $\iota(\mathbb{Z})$ are not zero divisors in $\widehat{\mathbb{Z}}$.

[1] Exercise

2.3 Profinite matrix groups

Lecture 6
Tuesday
02/02/21

For a commutative ring R, we have

$$\mathrm{Mat}_{N\times M} R = \{N \times M \text{ matrices with elements in } R\}.$$

If $N = M$, we have a ring structure, where addition and multiplication are given by the usual formula. There exists a determinant function $\det : \mathrm{Mat}_{N\times N} R \to R$. Then

$$\mathbb{Z}_p^{NM} \cong \mathrm{Mat}_{N\times M} \mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathrm{Mat}_{N\times M} \mathbb{Z}/p^n\mathbb{Z}.$$

By continuity of ring operations on $\mathbb{Z}_p$, addition and multiplication on matrices are continuous, and $\det : \mathrm{Mat}_{N\times N} \mathbb{Z}_p \to \mathbb{Z}_p$ is continuous. Since $\mathbb{Z}_p$ is an integral domain, it has a field of fractions $\mathbb{Q}_p$, so you can do linear algebra over $\mathbb{Q}_p$. A matrix over $\mathbb{Q}_p$ has an inverse over $\mathbb{Q}_p$ if and only if its determinant is non-zero, and a matrix over $\mathbb{Z}_p$ has an inverse over $\mathbb{Z}_p$ if and only if its determinant and its inverse are in $\mathbb{Z}_p^\times$. Define

$$\mathrm{GL}_N \mathbb{Z}_p = \left\{A \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p \;\middle|\; \det A \in \mathbb{Z}_p^\times\right\}, \qquad \mathrm{SL}_N \mathbb{Z}_p = \{A \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p \mid \det A = 1\}.$$

Both are profinite groups.

Lemma 2.3.1. *For all $N \geq 1$ and p prime,*

$$\mathrm{GL}_N \mathbb{Z}_p = \varprojlim_n \mathrm{GL}_N \mathbb{Z}/p^n\mathbb{Z}, \qquad \mathrm{SL}_N \mathbb{Z}_p = \varprojlim_n \mathrm{SL}_N \mathbb{Z}/p^n\mathbb{Z}.$$

Proof. The diagrams

$$\begin{array}{ccc} \mathrm{Mat}_{N\times N} \mathbb{Z}_p & \longrightarrow & \mathrm{Mat}_{N\times N} \mathbb{Z}/p^n\mathbb{Z} \\ {\scriptstyle\det}\downarrow & & \downarrow{\scriptstyle\det} \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z} \end{array}$$

commute.

- $A \in \mathrm{GL}_N \mathbb{Z}_p$ if and only if $\det A \in \mathbb{Z}_p^\times$, if and only if $\det A_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ for all n, if and only if $A_n \in \mathrm{GL}_N \mathbb{Z}/p^n\mathbb{Z}$ for all n.
- $A \in \mathrm{SL}_N \mathbb{Z}_p$ if and only if $\det A = 1$, if and only if $\det A_n = 1$ for all n, if and only if $A_n \in \mathrm{SL}_N \mathbb{Z}/p^n\mathbb{Z}$ for all n.

□

Also have matrices over $\widehat{\mathbb{Z}}$. A warning is that $\widehat{\mathbb{Z}}$ is not an integral domain. Analogously,

$$\mathrm{GL}_N \widehat{\mathbb{Z}} = \left\{A \in \mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} \;\middle|\; \det A \in \widehat{\mathbb{Z}}^\times\right\} = \varprojlim_n \mathrm{GL}_N \mathbb{Z}/n\mathbb{Z} = \prod_p \mathrm{GL}_N \mathbb{Z}_p,$$

$$\mathrm{SL}_N \widehat{\mathbb{Z}} = \left\{A \in \mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} \;\middle|\; \det A = 1\right\} = \varprojlim_n \mathrm{SL}_N \mathbb{Z}/n\mathbb{Z} = \prod_p \mathrm{SL}_N \mathbb{Z}_p,$$

since $\mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} = \prod_p \mathrm{Mat}_{N\times N} \mathbb{Z}_p$, and

$$\mathrm{SL}_N \mathbb{Z} \leq \mathrm{SL}_N \mathbb{Z}_p, \qquad \mathrm{SL}_N \mathbb{Z} \leq \mathrm{SL}_N \widehat{\mathbb{Z}} = \varprojlim_n \mathrm{SL}_N \mathbb{Z}/n\mathbb{Z}$$

are dense. See problem sheet 2.

Example. $\left(\begin{smallmatrix} 7 & 9 \\ 4 & 9 \end{smallmatrix}\right) \in \mathrm{SL}_2 \mathbb{Z}/13\mathbb{Z}$ is in the image of $\mathrm{SL}_2 \mathbb{Z}$.

2.4 Subgroups, quotients, and homomorphisms

Proposition 2.4.1. *A closed subgroup of a profinite group is a profinite group.*

Proof. Let $G = \varprojlim_{j\in J} G_j$ be a profinite group for G_j finite. Take a closed subgroup $H \leq_{\mathrm{c}} G$ of G. Define $H_j = p_j(H) \leq G_j$. Then H_j, with transition maps $\phi_{ij}|_{H_i} : H_i \to H_j$, are an inverse system of finite groups. Define

$$H' = \varprojlim_j H_j = \left\{ (g_j) \in \prod_{j\in J} G_j \;\middle|\; \forall i \leq j,\ \phi_{ij}(g_i) = g_j,\ g_j \in H_j \right\}.$$

Show that $H = H'$. If $h = (h_j) \in H$, by definition $h_j = p_j(h) \in H_j$, so $H \leq H'$. Suppose $g = (g_j) \notin H$. Since H is closed, $G \setminus H$ is open, so there exists a basic open set containing g, which does not intersect H. There exists $j \in J$ such that $gU_j = p_j^{-1}(\{g_j\}) \leq G \setminus H$. Therefore for all $h \in H$, $p_j(h) \neq g_j$, since then $h \in H \cap p_j^{-1}(\{g_j\})$, so $g_j \notin H_j$, so $g \notin H'$. So $H = H'$. □

Remark.

- The two topologies on H agree by $\mathrm{id} : (H, \tau_{\text{profinite}}) \to (H, \tau_{\text{subspace}})$, which is continuous by Proposition 1.2.9.
- A better name for H' is $\overline{H}$, the closure. Actually proved that $H' = \overline{H} = H$.

Proposition 2.4.2. *Let $G = \varprojlim_j G_j$ and $H \leq G$. Set $H_j = p_j(H) \leq G_j$. Then the closure of H is $\overline{H} = \varprojlim_j H_j$.*

Lemma 2.4.3. *Let $f : G_1 \to G_2$ be a surjective homomorphism and $H \leq G_1$. Then $[G_1 : H] \geq [G_2 : f(H)]$.*

Proposition 2.4.4. *Let $G = \varprojlim_j G_j$ for (G_j) a surjective inverse system, so $G \twoheadrightarrow G_j$. Let $H \leq_{\mathrm{c}} G$ and set $H_j = p_j(H) \leq G_j$. Then H is finite index if and only if $[G_j : H_j]$ is constant on a cofinal subsystem, if and only if $[G_j : H_j]$ is bounded for all j. If this is true, then $[G : H] = [G_i : H_i]$ for $i \in I$.*

Proof. $p_j : G \to G_j$ are surjective, so $[G : H] \geq [G_j : H_j]$. Suppose $[G : H] \geq N$. There exist distinct cosets $g_1H, \ldots, g_NH$ of H in G, if and only if $g_n^{-1}g_m \notin H$ if $n \neq m$, so there exists $j_{n,m} \in J$ such that $p_{j_{n,m}}\left(g_n^{-1}g_m\right) \notin H_{j_{n,m}}$. Take $k \leq j_{n,m}$ for all n and m. Then $p_k\left(g_n^{-1}g_m\right) \notin H_k$ for all $n \neq m$, so $p_k(g_n)H_k$ are distinct cosets of H_k in G_k, so $[G_k : H_k] \geq N$. For any i in the cofinal subsystem $J_{\leq k}$, it follows $[G_i : H_i] \geq N$ for all $i \leq k$. If $[G : H] = N$ is finite, take k as above and $I = J_{\leq k}$. Then $[G : H] \geq [G_i : H_i] \geq N = [G : H]$ for all $i \in I$. If $[G : H]$ is infinite, assume I is cofinal and $[G_i : H_i] = N$ for all $i \in I$. Then there exists k such that $[G_k : H_k] \geq N + 1$. But there exists $i \in I$ such that $i \leq k$, then $[G_i : H_i] \geq [G_k : H_k] \geq N + 1 > N = [G_i : H_i]$, a contradiction. □

Proposition 2.4.5. *Let G be a profinite group and N a closed normal subgroup. Then G/N, with the quotient topology, is a profinite group.*

Proof. Take $G = \varprojlim_j G_j$ for (G_j) a surjective inverse system. Let $N_j = p_j(N) \triangleleft G_j = p_j(G)$. Recall $N = \varprojlim_j N_j$. Define $Q_j = G_j/N_j$. Since $\phi_{ij}(N_i) \leq N_j$, we get quotient homomorphisms $\psi_{ij} : Q_i \to Q_j$, which are transition maps for the Q_j. Set $Q = \varprojlim_j Q_j$. The map $\prod_h G_j \to \prod_j Q_j$ is continuous, so there is a continuous surjective group homomorphism $f : G \to Q$. The kernel of this map is N, since $f(g) = 1$ if and only if $q_j(f(g)) = 1$ for all j, if and only if $g_j \in N_j$ for all j, if and only if $g \in \varprojlim_j N_j = N$. By the first isomorphism theorem for groups,

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G/N & \xrightarrow[\overline{f}]{\sim} & Q \end{array}.$$

Since $G \to Q$ is continuous and $G \to G/N$ is the quotient map, $\overline{f}$ is continuous. Since G/N is compact and Q is Hausdorff, $\overline{f}$ is a homeomorphism. □

This is the first isomorphism theorem for profinite groups.

Lecture 7
Thursday
04/02/21

Definition. Let $(G_j)_{j\in J}$ and $(H_j)_{j\in J}$ be inverse systems of finite groups, over the same poset J. A **morphism of inverse systems** (f_j) is a family of homomorphisms $f_j : G_j \to H_j$, such that for all $i \leq j$,

$$\begin{array}{ccc} G_i & \xrightarrow{f_i} & H_i \\ {\scriptstyle \phi_{ij}^G}\downarrow & & \downarrow{\scriptstyle \phi_{ij}^H} \\ G_j & \xrightarrow[f_j]{} & H_j \end{array}$$

commutes, so $\phi_{ij}^H \circ f_i = f_j \circ \phi_{ij}^G$.

Proposition 2.4.6. *Let $(f_j) : (G_j) \to (H_j)$ be a morphism of inverse systems. Then there is a unique continuous homomorphism $f : G = \varprojlim_j G_j \to H = \varprojlim_j H_j$ such that*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle p_j^G}\downarrow & & \downarrow{\scriptstyle p_j^H} \\ G_j & \xrightarrow[f_j]{} & H_j \end{array},$$

so $p_j^H \circ f = f_j \circ p_j^G$ for all $j \in J$.

Proof. The maps $f_j \circ p_j^G : G \to H_j$ form a cone on the inverse system (H_j),

$$\begin{array}{ccccc} & & G & & \\ & {\scriptstyle f_i\circ p_i^G}\swarrow & & \searrow{\scriptstyle f_j\circ p_j^G} & \\ H_i & & \xrightarrow[\phi_{ij}^H]{} & & H_j \end{array},$$

since $\phi_{ij}^H \circ f_i \circ p_i^G = f_j \circ \phi_{ij}^G \circ p_i^G = f_j \circ p_j^G$. So by definition of limits, there exists a unique $f : G \to H = \varprojlim_j H_j$ such that $p_j^H \circ f = f_j \circ p_j^G$. □

Thus f is **induced** by the f_j by passing to an inverse limit.

Proposition 2.4.7. *Let $G = \varprojlim_{j\in J} G_j$ and $H = \varprojlim_{i\in I} H_i$ be inverse limits of finite groups, where I and J are countable inverse systems with no minimum element. Let $f : G \to H$ be a continuous homomorphism. Then there exist cofinal subsystems $J' \subseteq J$ and $I' \subseteq I$, an order-preserving bijection $J' \cong I'$, and a morphism of inverse systems $(f_j) : (G_j)_{j\in J'} \to (H_i)_{i\in I'}$ inducing f.*

Proof. Without loss of generality, use Proposition 1.3.4 to assume J and I are linearly ordered. Without loss of generality both are $\mathbb{N}$, with the wrong-way ordering. Construct an increasing sequence (k_n) of natural numbers as follows. Each map $p_n^H \circ f : G \to H \to H_n$ is a continuous homomorphism, so its kernel is open in G. By Proposition 1.2.12 there exists k_n such that $\ker(G \to G_{k_n}) \leq \ker(G \to H_n)$, which means there is a quotient homomorphism

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle p_{k_n}^G}\downarrow & \searrow & \downarrow{\scriptstyle p_n^H} \\ G_{k_n} & \xrightarrow[f_n]{} & H_n \end{array}.$$

Then $\ker(G \to G_{n+1}) \leq \ker(G \to G_n)$, so without loss of generality $k_n > k_{n-1}$. Now $J' = \{k_n\}_{n\in\mathbb{N}}$ give a cofinal subsystem of $J = \mathbb{N}$, and the f_n are the required morphisms of inverse systems. □

2.5 Generators of profinite groups

Definition. Let G be a topological group, and let S be a subset of G. Then S is a **topological generating set (TGS)** for G if the subgroup $\langle S \rangle$ is dense in G, and G is **topologically finitely generated (TFG)** if it has some finite TGS S.

Definition. Let G be a topological group and $S \subseteq G$. The **closed subgroup of G topologically generated by** S is the smallest closed subgroup of G which contains S. Denoted $\overline{\langle S \rangle}$.

Proposition 2.5.1. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G. Hence for $S \subseteq G$, the closed subgroup of G generated by S is equal to the closure of $\langle S \rangle$.*

Proof. Exercise. [2] □

Lemma 2.5.2. *A finite index subgroup of a finitely generated group is finitely generated.*

Proposition 2.5.3. *If a profinite group G is TFG and U is an open subgroup of G then U is TFG.*

Proof. Let S be a finite set such that $\langle S \rangle$ is dense in G. Then $\Gamma = U \cap \langle S \rangle$ is finite index in $\langle S \rangle$, hence Γ is finitely generated, so $\Gamma = \langle S' \rangle$ for S' finite. Since U is open, and $\langle S \rangle$ is dense, $\langle S' \rangle = U \cap \langle S \rangle$ is dense in U. So U is TFG. □

Proposition 2.5.4. *Let (G_j) be a surjective inverse system of finite groups with $G = \varprojlim_j G_j$. Let $S \subseteq G$. Then S is a TGS for G if and only if $p_j(S)$ generates G_j for all j.*

Proof. By Corollary 1.2.14, $\langle S \rangle$ is dense in G if and only if $G_j = p_j(\langle S \rangle) = \langle p_j(S) \rangle$ for all j. □

Lemma 2.5.5. *Let G be a TFG profinite group. Then G may be written as the inverse limit of a countable inverse system of finite groups.*

Proof. A continuous homomorphism from G to a finite group is determined by the image of a TGS S, since a function on S determines all of a homomorphism from $\langle S \rangle$ and continuity gives the behaviour on all of G. So there are only countably many continuous homomorphisms from G to $\mathcal{S}_n$ for $n \in \mathbb{N}$. Every open normal subgroup of G is the kernel of such a continuous homomorphism. So there are only countably many open normal subgroups of G. Then $\mathcal{U} = \{U \lhd_{\mathrm{o}} G\}$ is a neighbourhood base of the identity, so by Proposition 1.2.17, $G = \varprojlim_{U \in \mathcal{U}} G/U$. □

Example. Let G be a TFG profinite group. Then there are only finitely many open subgroups of G of index at most n. See Lemma 1.2.6. Define

$$G_n = \bigcap \{U \mid U \leq_{\mathrm{o}} G,\ [G : U] \leq n\}.$$

Then $G_n \lhd G$, and G_n is open in G. And $\{G_n\}$ is a neighbourhood base of the identity. So $G = \varprojlim_{n \in \mathbb{N}} G/G_n$.

Proposition 2.5.6. *Let $\mathbb{Z}_p^\times$ be the set of elements α of $\mathbb{Z}_p$ which topologically generate $\mathbb{Z}_p$. Then $\alpha \in \mathbb{Z}_p^\times$ if and only if $\alpha \not\equiv 0 \mod p$. Hence $\mathbb{Z}_p^\times$ is a closed uncountable subset of $\mathbb{Z}_p$. For every n, and every generator $a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ there is some $\alpha \in \mathbb{Z}_p^\times$ such that $\alpha \equiv a_n \mod p^n$.*

Proof. For the last part, a_n is the image of α, since it is a surjective inverse system, and if a_n generates $\mathbb{Z}/p^n\mathbb{Z}$, it is coprime to p. If $\alpha = (a_n)$ such that $a_1 \neq 0$, then $p \nmid a_n$ for any n. Hence a_n is coprime to p, and so generates $\mathbb{Z}/p^n\mathbb{Z}$ for all n. So $\langle \alpha \rangle$ is dense in $\mathbb{Z}_p$ by an earlier result. □

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Remark. $\mathbb{Z}_p^\times$ is the set of units in the ring $\mathbb{Z}_p$.

- $\Longleftarrow$ If α is a unit, then $\alpha \mod p^n$ is a unit in $\mathbb{Z}/p^n\mathbb{Z}$, so generates $\mathbb{Z}/p^n\mathbb{Z}$. Then α topologically generates $\mathbb{Z}_p$.
- $\Longrightarrow$ Consider the group homomorphism

$$\begin{array}{rccc} f & : & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p \\ & & x & \longmapsto & \alpha x \end{array},$$

  which is continuous as multiplication in a ring is continuous. So $\operatorname{im} f$ is a closed subgroup of $\mathbb{Z}_p$, containing α. Then α generates $\mathbb{Z}_p$, so the only closed subgroup containing α is $\mathbb{Z}_p$ itself. So $1 \in \operatorname{im} f$, so there exists β such that $\alpha\beta = 1$.

Thus α is a unit if and only if $\{\alpha\}$ is a TGS for $\mathbb{Z}_p$.

[2] Exercise

Example. If $p \neq 2$, then 2 is invertible in $\mathbb{Z}_p$, so 2^{-1} exists. If $p = 3$,

$$2^{-1} = (\ldots, 5, 2) \in \mathbb{Z}_3 \leq \prod_{n \in \mathbb{N}} \mathbb{Z}/3^n\mathbb{Z}.$$

Proposition 2.5.7. *$\alpha \in \widehat{\mathbb{Z}}^\times$ if and only if $\alpha \mod n \in (\mathbb{Z}/n\mathbb{Z})^\times$ for all n. For any n, and every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ there exists a generator $\alpha \in \widehat{\mathbb{Z}}^\times$ such that $\alpha \equiv k \mod n$.*

Proof. Follows from Proposition 2.5.6 via the CRT, since $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. □

Theorem 2.5.8 (Gaschutz's lemma for finite groups)**.** *Let $f : G \twoheadrightarrow H$ be a surjective homomorphism of finite groups. Suppose G has some generating set of size d. For any generating set $\{z_1, \ldots, z_d\} \subseteq H$, there exists a generating set $\{x_1, \ldots, x_d\} \subseteq G$ such that $f(x_i) = z_i$ for all i.*

Really, talking about generating vectors $\underline{x} = (x_1, \ldots, x_d) \in G^d$. Extend f to $f : G^d \to H^d$.

Proof. We will prove, by induction on $|G|$, for H fixed, the following statement. The number

$$N_G(\underline{y}) = \left|\left\{\text{generating vectors } \underline{x} \text{ of } G \;\middle|\; f(\underline{x}) = \underline{y}\right\}\right|,$$

where $\underline{y} \in H^d$ is a generating vector of H, is independent of $\underline{y}$. Want to show $N_G(\underline{z}) > 0$, and G has some generating vector $\underline{x'} \in G^d$ so $N_G(\underline{z}) = N_G(f(\underline{x'})) > 0$. Let $\underline{y} \in H^d$ be a generating vector. Let

$$\mathcal{C} = \{d\text{-generator proper subgroups of } G\}.$$

Every $\underline{x} \in G^d$ such that $f(\underline{x}) = \underline{y}$ either generates G or generates some $C \in \mathcal{C}$. Therefore

$$N_G(\underline{y}) + \sum_{C \in \mathcal{C}} N_C(\underline{y}) = \left|\left\{\underline{x} : f(\underline{x}) = \underline{y}\right\}\right| = |\ker f|^d.$$

Thus $N_G(\underline{y}) = |\ker f|^d - \sum_{C \in \mathcal{C}} N_C(\underline{y})$, which is independent of $\underline{y}$ by induction. □

Theorem 2.5.9 (Gaschutz's lemma for profinite groups)**.** *Let $f : G \to H$ be a continuous surjective homomorphism of profinite groups. Suppose G has a TGS of size d. Then for any TGS $\{z_1, \ldots, z_d\}$ of H, there is a TGS $\{x_1, \ldots, x_d\}$ of G such that $f(x_i) = z_i$ for all i.*

Proof. By Proposition 1.3.3 and Proposition 2.4.7 we may assume and write $G = \varprojlim_{j \in J} G_j$ and $H = \varprojlim_{j \in H} H_j$, surjective inverse systems of finite groups, with a morphism of inverse systems $(f_j) : (G_j) \to (H_j)$ such that $f = \varprojlim_j f_j$. It is forced that f_j is surjective, since

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle p_j^G}\downarrow & & \downarrow{\scriptstyle p_j^H} \\ G_j & \xrightarrow[f_j]{} & H_j \end{array}.$$

Let $\underline{z}$ be the given TGS of H. Set $\underline{z}_j$ for $j \in J$ to be the image of $\underline{z}$ in H_j. Then $\underline{z}_j = p_j^H(\underline{z})$ is a generating vector of H_j. Consider the finite sets

$$X_j = \left\{\text{generating vectors } \underline{x}_j \in G_j^d \;\middle|\; f_j(\underline{x}_j) = \underline{z}_j\right\} \neq \emptyset,$$

by Gaschutz. The X_j form an inverse system, so $\phi_{ij}(X_i) \subseteq X_j$, since

$$\begin{array}{ccc} G_i & \longrightarrow & H_i \\ \downarrow & & \downarrow \\ G_j & \longrightarrow & H_j \end{array}.$$

Therefore $\varprojlim_j X_j$ is non-empty. If $\underline{x} \in \varprojlim_j X_j \subseteq G^d$ such that $p_j^G(\underline{x}) \in X_j$, then $\underline{x}$ is a TGS of G and $p_j^H(f(\underline{x})) = \underline{z}_j$ for all j, so $f(\underline{x}) = \underline{z}$. □

3 Profinite completions

3.1 Residual finiteness

Notation. Discrete abstract groups will be Greek letters and profinite groups will be Roman letters.

Given an abstract group Γ and an inverse system $\mathcal{N} = \{N \lhd_{\mathrm{f}} \Gamma\}$, there is an inverse system of finite groups Γ/N. Then $\widehat{\Gamma} = \varprojlim_{N\in\mathcal{N}} \Gamma/N$, where $\Gamma/N_1 \to \Gamma/N_2$ if $N_1 \leq N_2$. Also had a canonical morphism $\iota_\Gamma = \iota : \Gamma \to \widehat{\Gamma}$. The image of ι is dense by Corollary 1.2.14. Also implies for any finite generating set $S \subseteq \Gamma$, $\iota(S)$ is a TGS of $\widehat{\Gamma}$, so if Γ is finitely generated, then $\widehat{\Gamma}$ is TFG.

Proposition 3.1.1. *Let $f : \Delta \to \Gamma$ be a group homomorphism. Then there exists a unique continuous group homomorphism $\widehat{f} : \widehat{\Delta} \to \widehat{\Gamma}$ such that $\widehat{f} \circ \iota_\Delta = \iota_\Gamma \circ f$, so*

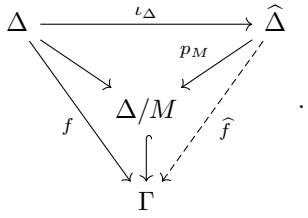
$$\begin{array}{ccc} \Delta & \xrightarrow{f} & \Gamma \\ {\scriptstyle \iota_\Delta}\downarrow & & \downarrow{\scriptstyle \iota_\Gamma} \\ \widehat{\Delta} & \xrightarrow[\widehat{f}]{} & \widehat{\Gamma} \end{array}.$$

Proof. Uniqueness will follow from the density of $\iota_\Delta(\Delta)$ in $\widehat{\Delta}$. Take two $\widehat{f_1}$ and $\widehat{f_2}$ satisfying Proposition 3.1.1. Consider

$$S = \left\{ \delta \in \widehat{\Delta} \;\middle|\; \widehat{f_1}(\delta) = \widehat{f_2}(\delta) \right\}.$$

Then S is closed, since it is the preimage of the diagonal in $\widehat{\Gamma}\times\widehat{\Gamma}$ under $\left(\widehat{f_1}, \widehat{f_2}\right) : \widehat{\Delta} \to \widehat{\Gamma}\times\widehat{\Gamma}$, and S contains $\iota_\Delta(\Delta)$, which is dense. So $S = \widehat{\Delta}$.

Case 1. Γ is finite, so $\Gamma = \widehat{\Gamma}$. Then $\ker f$ is a finite index normal subgroup M of Δ, so there exists a projection map $p_M : \widehat{\Delta} \to \Delta/M$. So we get a composition

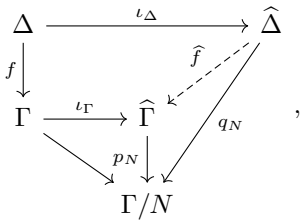


.

Case 2. General case. Take some $N \lhd_{\mathrm{f}} \Gamma$. There exists a unique $q_N : \widehat{\Delta} \to \Gamma/N$ such that $q_N \circ \iota_\Delta = p_N \circ \iota_\Gamma \circ f$. Then (q_N) form a cone on the inverse system, since

$$\phi^\Gamma_{N_1N_2} \circ q_{N_1} \circ \iota_\Delta = \phi^\Gamma_{N_1N_2} \circ p_{N_1} \circ \iota_\Gamma \circ f = p_{N_2} \circ \iota_\Gamma \circ f = q_{N_2} \circ \iota_\Delta.$$

Thus there exists a unique $\widehat{f} : \widehat{\Delta} \to \widehat{\Gamma}$ such that $p_N \circ \widehat{f} = q_N$ for all N, so



,

and

$$p_N \circ \widehat{f} \circ \iota_\Delta = q_N \circ \iota_\Delta = p_N \circ \iota_\Gamma \circ f.$$

□

Corollary 3.1.2. *$\widehat{\cdot}$ is a functor.*