Algebraic Geometry

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Syllabus

Contents

0 Brief review of classical algebraic geometry and motivation for scheme theory

Lecture 1
Friday
09/10/20

The following are the main references for the course.

- R Hartshorne, Algebraic geometry, 1977
- U Goertz and T Wedhorn, Algebraic geometry I, 2010
- R Vakil, The rising sea: foundations of algebraic geometry, 2017

0.1 Classical algebraic geometry

Throughout this discussion, we take the base field k to be algebraically closed. An **affine variety** $V \subseteq \mathbb{A}^n(k)$, where, once one has chosen coordinates, $\mathbb{A}^n(k) = k^n$, is given by the vanishing of polynomials $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. If $I = \langle f_1, \dots, f_r \rangle \subseteq k[x_1, \dots, x_n]$ is any ideal, we set

$$\mathbb{V}(I) = \{z \in \mathbb{A}^n \mid \forall f \in I, \ f(z) = 0\}.$$

First set $\mathbb{P}^n(k) = \left(k^{n+1} \setminus \{0\}\right)/k^*$ with **homogeneous coordinates** $(x_0 : \dots : x_n)$. A **projective variety** $V \subseteq \mathbb{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \dots, F_r \in k[x_0, \dots, x_n]$. If I is the ideal generated by the homogeneous ideals F_i, that is if $F \in I$ then so are all its homogeneous parts, we set

$$\mathbb{V}(I) = \{z \in \mathbb{P}^n \mid \forall F \in I \text{ homogeneous}, \ F(z) = 0\}.$$

If $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$, set

$$\mathbb{I}(V) = \{f \in k[x_1, \dots, x_n] \mid \forall x \in V, \ f(x) = 0\}.$$

Observe that $\mathbb{V}(\mathbb{I}(V)) = V$, by tautology, and $\mathbb{I}(\mathbb{V}(I)) \supseteq \sqrt{I}$, which is obvious. Recall that the **radical** $\sqrt{I}$ of the ideal I is defined by $f \in \sqrt{I}$ if and only if there exists $m > 0$ such that $f^m \in I$. **Hilbert's Nullstellensatz** states that, noting $k = \overline{k}$, $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. The **coordinate ring** is

$$k[V] = k[x_1, \dots, x_n] / \mathbb{I}(V).$$

This may be regarded as the ring of polynomial functions on V, and it is a finitely generated reduced k-algebra. Recall that a **k-algebra** is a commutative ring containing k as a subring. It is **finitely generated** if it is the quotient of a polynomial ring over k, and **reduced** if $a^m = 0$ implies that $a = 0$.

0.2 Why schemes?

A better question is what is wrong with varieties?

- With varieties, always work over algebraically closed fields. For example, let $I = \langle x^2 + y^2 + 1 \rangle \subseteq \mathbb{R}[x, y]$. Then $\mathbb{V}(I) = \emptyset$, but I is a prime ideal, hence radical, so $\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[x, y] \neq I$.
- Number theory? Diophantine equations. If $I \subseteq \mathbb{Z}[x_1, \dots, x_n]$ is an ideal, have $\mathbb{V}(I) \subseteq \mathbb{Z}^n$. For example, $x^n + y^n = z^n$.
- Why should we only consider radical, or prime, ideals? For example, a natural situation is

  $$X_1 = \mathbb{V}\left(x - y^2\right) \subseteq \mathbb{A}^2, \qquad X_2 = \mathbb{V}(x) \subseteq \mathbb{A}^2.$$

  Then $X_1 \cap X_2 = \mathbb{V}\left(x - y^2, x\right)$. Note $I = \left\langle x - y^2, x \right\rangle = \left\langle x, y^2 \right\rangle$ is not a radical ideal, because $y \notin I$ and $y^2 \in I$ so $y \notin \sqrt{I}$. Recall the coordinate ring of X_i is $k[X_i] = k[x, y]/I_i$. Then $k[X_1 \cap X_2] = k[x, y] / \left\langle x, y^2 \right\rangle \cong k[y] / \left\langle y^2 \right\rangle$. So thinking of the coordinate ring of $X_1 \cap X_2$ as functions on $X_1 \cap X_2$, we have a function y whose square is zero, but is not itself zero.

0.3 Categorical philosophy

What is a point? In the category of sets, objects are sets, and if A and B are sets, then morphisms are $\mathrm{Hom}(A, B)$, the set of maps $f : A \to B$. Let $*$ be a one-element set. Then the elements of any set X are in one-to-one correspondence with $\mathrm{Hom}(*, X)$. In the category of affine varieties, objects are affine varieties and morphisms are $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X])$. In this category, a point is a single point with coordinate ring k. Giving a morphism

$$\{\text{point}\} \to X = \mathbb{V}(I) \subseteq \mathbb{A}^n, \qquad I \subseteq k[x_1, \dots, x_n],$$

for I a radical ideal, is the same as giving a homomorphism

$$\begin{array}{cccccc} \phi & : & k[X] = k[x_1, \dots, x_n]/I & \longrightarrow & k \\ & & x_i & \longmapsto & a_i \end{array}.$$

Note that ϕ vanishes in I if and only if $f(a_1, \dots, a_n) = 0$ for all $f \in I$, which is if and only if $(a_1, \dots, a_n) \in \mathbb{V}(I) = X$. Note ϕ is surjective, and hence $\ker \phi$ is a maximal ideal. With k algebraically closed, the maximal ideals at $k[X]$ are all of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ for $(a_1, \dots, a_n) \in X$, a consequence of Hilbert's Nullstellensatz. That is, there exist one-to-one correspondences

$$\{\text{points of } X\} \quad \leftrightsquigarrow \quad \{k\text{-algebra homomorphisms } \phi : k[X] \to k\} \quad \leftrightsquigarrow \quad \{\text{maximal ideals of } k[X]\}.$$

What if k is not algebraically closed? We may want to consider solutions not just in $k^n = \mathbb{A}^n$ but $(k')^n$ for k' any field extension of k. That is, we may consider k-algebra homomorphisms

$$\begin{array}{cccccc} \phi & : & k[X] = k[x_1, \dots, x_r]/I & \longrightarrow & k' \\ & & x_i & \longmapsto & a_i \end{array}.$$

This gives a tuple $(a_1, \dots, a_n) \in (k')^n$ with $f(a_1, \dots, a_n) = 0$ for all $f \in I$. Then ϕ need not be surjective, so can only say the image of ϕ is a subring of a field, hence an integral domain. Thus $\ker \phi$ is a prime ideal, and maximal if and only if $\mathrm{im}\, \phi$ is a field.

Example. The $\mathbb{R}$-algebra homomorphism

$$\begin{array}{ccccc} \phi & : & \mathbb{R}[x, y] / \langle x^2 + y^2 + 1 \rangle & \longrightarrow & \mathbb{C} \\ & & x & \longmapsto & 0 \\ & & y & \longmapsto & i \end{array}$$

is surjective with kernel $\langle x, y^2 + 1 \rangle$, since $\mathbb{R}[y] / \langle y^2 + 1 \rangle \cong \mathbb{C}$. This is a maximal ideal but is not of the form $\langle x - a, y - b \rangle$ for $(a, b) \in \mathbb{R}^2$. If instead we considered the map

$$\begin{array}{ccc} \mathbb{R}[x, y] / \langle x^2 + y^2 + 1 \rangle & \longrightarrow & \mathbb{C} \\ x & \longmapsto & 0 \\ y & \longmapsto & -i \end{array},$$

we get the same kernel. That is, $(0, i)$ and $(0, -i)$ are solutions to $x^2 + y^2 + 1 = 0$, but they correspond to the same maximal ideal. In fact, this maximal ideal corresponds to a Galois orbit of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of solutions.

There are more exotic points by taking even bigger fields. Lecture 2 Monday 12/10/20

Example. Let $k(X)$ be the field of fractions of $k[X] = \mathbb{R}[x, y] / \langle x^2 + y^2 + 1 \rangle$. There is an inclusion

$$\begin{array}{ccc} k[X] & \longrightarrow & k(X) \\ f & \longmapsto & \dfrac{f}{1} \\ (x, y) & \longmapsto & (x, y) \end{array}.$$

The kernel of this map is zero. This gives a solution to the equation $x^2 + y^2 + 1 = 0$ with coordinates in the field $k(X)$. This solution is $(x, y) \in \mathbb{A}^2(k(X))$.

The moral is that once we start looking at solutions to equation over any field, then we get maps $k[X] \to k'$ with kernel not necessarily maximal. What about solutions over rings?

Example. Let $A = \mathbb{Z}[x_1, \dots, x_n]/I$, and let R be any commutative ring. We define an R-valued point of $\operatorname{Spec} A$ to be a ring homomorphism

$$\begin{array}{ccc} A & \longrightarrow & R \\ x_i & \longmapsto & r_i \end{array}.$$

Then $f(r_1, \dots, r_n) = 0$ for all $f \in I$. This gives a lot of flexibility. For example,

- $R = \mathbb{Z}$ gives diophantine equations,
- $R = \mathbb{F}_p$ gives solutions modulo p, and
- $R = \mathbb{Q}$ gives rational solutions.

Take this to its logical conclusion. Let A be a ring, where all rings are commutative in this course. Given A, we hope for some geometric object $\operatorname{Spec} A$, the **spectrum** of A. For a ring R, the set of **R-valued points** of X is

$$X(R) = \operatorname{Hom}_{\mathrm{ring}}(A, R).$$

A morphism $X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ should be the same thing as giving a morphism $\phi : B \to A$. Define the category of **affine schemes** to be the opposite category to the category of rings. Define a **scheme** to be something which is locally isomorphic to an affine scheme. By analogy, a **manifold** is a topological space with an open cover $\{U_i\}$ with each U_i homeomorphic to an open subset of $\mathbb{R}^n$. To make sense of the definition of schemes, we need a lot of language.

0.4 Spectrum of a ring

Definition. Let A be a ring. Then

$$\operatorname{Spec} A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ a prime ideal}\}.$$

For $I \subseteq A$ an ideal, define

$$\mathbb{V}(I) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime, } \mathfrak{p} \supseteq I\}.$$

Proposition 0.1. *The sets $\mathbb{V}(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the **Zariski topology**.*

Proof.

- $\mathbb{V}(A) = \emptyset$.
- $\mathbb{V}(0) = \operatorname{Spec} A$.
- If $\{I_i\}_{i \in J}$ is a collection of ideals, then

$$\mathbb{V}\left(\sum_{i \in J} I_i\right) = \bigcap_{i \in J} \mathbb{V}(I_i).$$

- Claim that

$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2).$$

  $\supseteq$ Obvious.

  $\subseteq$ If $\mathfrak{p} \supseteq I_1 \cap I_2$ is prime, then $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$. See Atiyah-Macdonald, Proposition 1.11.ii. [1]

□

Example. Let $A = k[x_1, \dots, x_n]$ with k algebraically closed and $I \subseteq A$ an ideal. Then the maximal ideals $\mathfrak{m}$ of A containing I are in one-to-one correspondence with the zero set of I in $\mathbb{A}^n(k)$, so

$$\left\{ \ \langle x_1 - a_1, \dots, x_n - a_n \rangle \supseteq I, \ a_i \in k \ \right\} \qquad \leftrightsquigarrow \qquad \left\{ \ (a_1, \dots, a_n) \in \mathbb{V}(I) \subseteq \mathbb{A}^n(k) \ \right\}.$$

The new $\mathbb{V}(I)$ now extends this notion of zero set by including possible other prime ideals.

Example. If k is a field, $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field.

We fix this by also thinking about what functions are on these spaces.

[1] Exercise: try to prove without looking up

1 Sheaves

Fix a topological space X.

1.1 Sheaves

Definition. A **presheaf** $\mathcal{F}$ on X consists of the following data.

- For every open set $U \subseteq X$ an abelian group $\mathcal{F}(U)$.
- Whenever given an inclusion $V \subseteq U \subseteq X$, a **restriction map** $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, a homomorphism, such that
  - $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and
  - if $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark. Can think of a presheaf as a contravariant functor from the category of open sets of X, the category whose objects are open subsets of X and whose morphisms are inclusions of open sets, to the category of abelian groups. Can replace the category of abelian groups with any desired category, such as commutative rings.

Definition. A **morphism of presheaves** $f : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V) \end{tikzcd}$$

is commutative.

Definition. A presheaf $\mathcal{F}$ is a **sheaf** if it satisfies the following additional axioms.

S1. If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = \rho_{UU_i}(s) = 0$ for all i, then $s = 0$.

S2. If U and $\{U_i\}$ are as in S1 and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all i.

Remark.

- If $\mathcal{F}$ is a sheaf, then $\emptyset \subseteq X$ is covered by the empty covering, and hence $\mathcal{F}(\emptyset) = 0$.
- S1 and S2 together can be described as saying, given U and $\{U_i\}_{i \in I}$,

  $$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

  is exact, where

  $$\alpha(s) = \left(s|_{U_i}\right)_{i \in I}, \qquad \beta_1\left((s_i)_{i \in I}\right) = \left(s_i|_{U_i \cap U_j}\right)_{i,j}, \qquad \beta_2\left((s_i)_{i \in I}\right) = \left(s_j|_{U_i \cap U_j}\right)_{i,j}.$$

  Exactness means
  - α is injective, which is S1,
  - $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and
  - for any $(s_i) \in \prod_{i \in I} \mathcal{F}(U_i)$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}(U)$ with $\alpha(s) = (s_i)$, which is S2.

1.2 Examples

Lecture 3 Wednesday 14/10/20

Example.

- Let X be any topological space, and let

$$\mathcal{F}(U) = \{\text{continuous functions } U \to \mathbb{R}\}.$$

This is a sheaf, by

$$\begin{array}{cccccc} \rho_{UV} & : & \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) \\ & & f & \longmapsto & f|_V \end{array}.$$

S1. A continuous function is zero if it is zero on every open set of a cover.

S2. Continuous functions can be glued.

- Let $X = \mathbb{C}$ with the Euclidean topology, and let

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ is a bounded analytic function}\}.$$

This is a presheaf. It satisfies S1, and does not satisfy S2. For example, consider the cover $\{U_i\}_{i\in\{1,2,\dots\}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$ and

$$\begin{array}{cccccc} s_i & : & U_i & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & z \end{array}.$$

Note if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. But if we glue we get the function $z : \mathbb{C} \to \mathbb{C}$, which is not bounded. Note $\mathcal{F}(\mathbb{C}) = \mathbb{C}$.

- Take any group G and set $\mathcal{F}(U) = G$ for any open set U. This is called the **constant presheaf**. This is not a sheaf. Let $U = U_1 \sqcup U_2$. If we wanted a sheaf,

$$\begin{tikzcd} \mathcal{F}(U_1) = G \arrow[dr] & & \mathcal{F}(U_2) = G \arrow[dl] \\ & \mathcal{F}(U_1 \cap U_2) = \mathcal{F}(\emptyset) = 0 & \end{tikzcd},$$

so if S2 is satisfied, would want $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ to glue. We would then want to have $\mathcal{F}(U) = G \times G$. Now give G the discrete topology, and define instead

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\},$$

that is f is locally constant. That is, if $x \in U$, there exists a neighbourhood $x \in V \subseteq U$ with $f|_V$ constant. This is called the **constant sheaf** and if U is non-empty and connected, then $\mathcal{F}(U) = G$.

- If X is an algebraic variety, and $U \subseteq X$ is a Zariski open subset, define

$$\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ regular function}\}.$$

Roughly f is **regular** means that every point of U has an open neighbourhood on which f is expressed as a ratio of polynomials g/h with h non-vanishing on the neighbourhood. Then $\mathcal{O}_X$ is a sheaf, called the **structure sheaf** of X.

1.3 Stalks

Definition. Let $\mathcal{F}$ be a presheaf on X. Let $p \in X$. Then the **stalk** of $\mathcal{F}$ at p is

$$\mathcal{F}_p = \{(U, s) \mid U \subseteq X \text{ is an open neighbourhood of } p,\ s \in \mathcal{F}(U)\} / \equiv,$$

where $(U, s) \equiv (V, s')$ if there exists $W \subseteq U \cap V$ also a neighbourhood of p such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a **germ**.

Remark. $\mathcal{F}_p = \varinjlim_{p\in U} \mathcal{F}(U)$.

Note that a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism

$$\begin{array}{rccc} f_p & : & \mathcal{F}_p & \longrightarrow & \mathcal{G}_p \\ & & (U, s) & \longmapsto & (U, f_U(s)) \end{array}.$$

Proposition 1.1. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$.*

Proof.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ Assume f_p is an isomorphism for all $p \in X$. Need to show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all $U \subseteq X$, as then we can define $\left(f^{-1}\right)_U = (f_U)^{-1}$. Check that with this definition, $\left(f^{-1}\right)_U$ is compatible with restriction maps, hence f^{-1} is a morphism of sheaves. [2]

- f_U is injective. Suppose $s \in \mathcal{F}(U)$, and $f_U(s) = 0$. Then for all $p \in U$, $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. Since f_p is injective, $(U, s) = 0$ in $\mathcal{F}_p$. That is, there exists a open neighbourhood V_p of p in U such that $s|_{V_p} = 0$. Since $\{V_p\}_{p \in U}$ cover U, we see by S1 that $s = 0$.
- f_U is surjective. Let $t \in \mathcal{G}(U)$ and write $t_p = (U, t) \in \mathcal{G}_p$. Since f_p is surjective, there exists $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. That is, there exists $V_p \subseteq U$ an open neighbourhood of p, and a germ (V_p, s_p) such that $\left(V_p, f_{V_p}(s_p)\right) \equiv (U, t)$. By shrinking V_p if necessary, we can assume that $t|_{V_p} = f_{V_p}(s_p)$. Now on $V_p \cap V_q$,

$$f_{V_p \cap V_q}\left(s_p|_{V_p \cap V_q} - s_q|_{V_p \cap V_q}\right) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0,$$

  and hence by injectivity of $f_{V_p \cap V_q}$ already proved, we have $s_p|_{V_p \cap V_q} = s_q|_{V_p \cap V_q}$. By S2 the s_p's glue to give an element $s \in \mathcal{F}(U)$ with $s|_{V_p} = s_p$, for all $p \in U$. Now

$$f_U(s)|_{V_p} = f_{V_p}\left(s|_{V_p}\right) = f_{V_p}(s_p) = t|_{V_p}.$$

  By S1, applied to $f_U(s) - t$, we get $f_U(s) = t$. Thus f_U is surjective.

□

1.4 Sheafification

Theorem 1.2 (Sheafification). *Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property. For any sheaf $\mathcal{G}$ and morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi^+ \circ \theta = \phi$, so*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dashed, "\phi^+"] \\ & \mathcal{G} \end{tikzcd}.$$

*The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism, and is called the **sheafification** of $\mathcal{F}$.*

Proof. See example sheet 1. The idea is to make $\mathcal{F}^+$ look like functions. Define

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall p \in U,\ s(p) \in \mathcal{F}_p, \\ \forall p \in U,\ \exists p \in V \subseteq U,\ \exists t \in \mathcal{F}(V),\ \forall q \in V,\ s(q) = (V, t) \in \mathcal{F}_q \end{array} \right\}.$$

Then

$$\begin{array}{rccc} \theta_U & : & \mathcal{F}(U) & \longrightarrow & \mathcal{F}^+(U) \\ & & s & \longmapsto & (p \mapsto (U, s) \in \mathcal{F}_p) \end{array}.$$

□

Exercise. A recommendation is to do all exercises in Chapter II.1 of Hartshorne.

[2] Exercise

1.5 Kernels, cokernels, and images

Lecture 4 Friday 16/10/20

Definition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a space X. We define the following.

- The **presheaf kernel** of f, $\ker f$, is the presheaf given by $(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$.
- The **presheaf cokernel** $\operatorname{coker} f$ is the presheaf given by $(\operatorname{coker} f)(U) = \operatorname{coker}(f_U) = \mathcal{G}(U) / \operatorname{im} f_U$.
- The **presheaf image** $\operatorname{im} f$ is the presheaf given by $(\operatorname{im} f)(U) = \operatorname{im} f_U$.

Exercise. Check that these are presheaves. That is, restrictions work.

Remark 1.3. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is also a sheaf.

Proof. S1 is certainly satisfied. If $s \in (\ker f)(U) \subseteq \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for all U_i in a cover of U so $s = 0$ by S1 for $\mathcal{F}$. Given $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ an open cover of U, and with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ by S2 for $\mathcal{F}$. But $f_U(s) = 0$ since $f_U(s)|_{U_i} = f_{U_i}(s|_{U_i}) = f_{U_i}(s_i) = 0$ so by S1, $f_U(s) = 0$. ☐

Example. Let $X = \mathbb{P}^1$, or think of the Riemann sphere. Let $P, Q \in X$ be distinct points. Let $\mathcal{G}$ be the sheaf of regular functions on X, or think of the sheaf of holomorphic functions. Let $\mathcal{F}$ be the sheaf of regular functions on X which vanish at P and Q. Note $\mathcal{F}(U) = \mathcal{G}(U)$ if $U \cap \{P, Q\} = \emptyset$. Let $U = \mathbb{P}^1 \setminus \{P\}$ and $V = \mathbb{P}^1 \setminus \{Q\}$. Note $\mathcal{F}(\mathbb{P}^1) = 0$ and $\mathcal{G}(\mathbb{P}^1) = k$, because regular functions on $\mathbb{P}^1$ are constants. Let $f : \mathcal{F} \to \mathcal{G}$ be the obvious inclusion. Then

$$(\operatorname{coker} f)(\mathbb{P}^1) = k, \qquad (\operatorname{coker} f)(U) = \mathcal{G}(U) / \mathcal{F}(U) = k[x] / \langle x \rangle = k,$$
$$(\operatorname{coker} f)(V) = k, \qquad (\operatorname{coker} f)(U \cap V) = \mathcal{G}(U \cap V) / \mathcal{F}(U \cap V) = 0.$$

If S2 holds, then we would need to have $(\operatorname{coker} f)(\mathbb{P}^1) = k \oplus k$. This is not a bug, but a feature.

Definition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

- The **sheaf kernel** $\ker f$ of f is just the presheaf kernel.
- The **sheaf cokernel** is the sheaf associated to the presheaf cokernel of f.
- The **sheaf image** is the sheaf associated to the presheaf image of f.

$\mathcal{F}$ is a **subsheaf** of $\mathcal{G}$ if we have inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ for all U compatible with restrictions.

Exercise. The sheaf image $\operatorname{im} f$ is a subsheaf of $\mathcal{G}$.

We say f is **injective** if $\ker f = 0$. We say f is **surjective** if $\operatorname{im} f = \mathcal{G}$. We say a sequence of morphisms of sheaves

$$\dots \to \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \to \dots$$

is **exact** if $\ker f^{i+1} = \operatorname{im} f^i$ for all i. If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, we write $\mathcal{F}/\mathcal{F}'$ for the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) / \mathcal{F}'(U)$. That is, this is the cokernel of the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$. A warning is if $f : \mathcal{F} \to \mathcal{G}$ is surjective, we do not necessarily have $\mathcal{F}(U) \to \mathcal{G}(U)$ surjective for all U.

Lemma 1.4. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then for all $p \in X$,*

$$(\ker f)_p = \ker(f_p : \mathcal{F}_p \to \mathcal{G}_p), \qquad (\operatorname{im} f)_p = \operatorname{im} f_p.$$

Proof. Have a map

$$\begin{array}{ccc} (\ker f)_p & \longrightarrow & \ker f_p \subseteq \mathcal{F}_p \\ (U, s) & \longmapsto & (U, s) \end{array}.$$

If $s \in (\ker f)(U) = \ker f_U$ represents a germ $(U, s) \in (\ker f)_p$, then $(U, s) \in \mathcal{F}_p$, and $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. So $(U, s) \in \ker f_p$.

- Injective. If $(U, s) = 0$ in $\mathcal{F}_p$, there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, s|_V) = (V, 0) = 0$ in $(\ker f)_p$.
- Surjective. If $(U, s) \in \ker f_p$, then $(U, f_U(s)) = 0$ in $\mathcal{G}_p$. That is, there exists a neighbourhood $V \subseteq U$ of p such that $0 = f_U(s)|_V = f_V(s|_V)$. Thus $s|_V \in (\ker f)(V)$, and $(V, s|_V) \in (\ker f)_p$, and $(V, s|_V)$ maps to the same element in $\ker f_p$ represented by (U, s).

Let $\mathrm{im}' f$ be the presheaf image. An easy fact is if $\mathcal{F}$ is a presheaf with associated sheaf $\mathcal{F}^+$, then $\mathcal{F}_p \cong \mathcal{F}_p^+$ for all $p \in X$. [3] Thus $(\mathrm{im}\, f)_p = \left(\mathrm{im}' f\right)_p$, so need to show $\left(\mathrm{im}' f\right)_p \cong \mathrm{im}\, f_p$. Define a map by

$$\begin{array}{rcl} \left(\mathrm{im}' f\right)_p & \longrightarrow & \mathrm{im}\, f_p \\ (U, s) & \longmapsto & (U, s) \end{array}.$$

- Injective. If $(U, s) = 0$ in $\mathcal{G}_p$ then there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, 0)$ in $\left(\mathrm{im}' f\right)_p$.
- Surjective. If $(U, s) \in \mathrm{im}\, f_p$, then there exists $(V, t) \in \mathcal{F}_p$ with $(U, s) = f_p(V, t) = (V, f_V(t))$, so after shrinking U and V if necessary, then we can take $U = V$ and $f_U(t) = s$. Then $(U, s) \in \left(\mathrm{im}' f\right)_p$.

□

Proposition 1.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then*

1. *f is injective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p, and*
2. *f is surjective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p.*

Proof.

1. f_p is injective for all p if and only if $\ker f_p = 0$ for all p, if and only if $(\ker f)_p = 0$ for all p, if and only if $\ker f = 0$, [4] which is if and only if f is injective.
2. f_p is surjective for all p if and only if $\mathrm{im}\, f_p = \mathcal{G}_p$ for all p, if and only if $(\mathrm{im}\, f)_p = \mathcal{G}_p$ for all p, if and only if $\mathrm{im}\, f = \mathcal{G}$, [5] which is if and only if f is surjective.

□

Remark. Given $f : \mathcal{F} \to \mathcal{G}$, in fact $\mathcal{G} / \mathrm{im}\, f \cong \mathrm{coker}\, f$. [6]

1.6 Passing between spaces

Lecture 5 Monday 19/10/20

Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Define $f_* \mathcal{F}$ by, for $U \subseteq Y$

$$(f_* \mathcal{F})(U) = \mathcal{F}\left(f^{-1}(U)\right).$$

Exercise. Check $f_* \mathcal{F}$ is a sheaf on Y.

Define $f^{-1} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \subseteq X \mapsto \{(V, s) \mid V \supseteq f(U),\ V \text{ open},\ s \in \mathcal{G}(V)\} / \sim,$$

where $(V, s) \sim (V', s')$ if there exists $W \subseteq V \cap V'$ such that $f(U) \subseteq W$, and $s|_W = s'|_W$.

Example. If $f : \{p\} \to X$ is an inclusion of a point, then $f^{-1} \mathcal{G} = \mathcal{G}_p$. This is a group but defines a sheaf on a one-point space. More generally, if $\iota : Z \hookrightarrow X$ is an inclusion of a subset with induced topology, we often write

$$\mathcal{F}|_Z = \iota^{-1} \mathcal{F}.$$

If Z is open in X, then this is easy, since if $U \subseteq Z$ then $\mathcal{F}|_Z(U) = \mathcal{F}(U)$.

Remark. If $s \in \mathcal{F}(U)$ we say s is a **section** of $\mathcal{F}$ over U. We often write

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}),$$

thinking of $\Gamma(U, \cdot)$ as a functor from the category of sheaves on a space X to the category of abelian groups.

[3] Exercise: check
[4] Exercise: check by S1
[5] Exercise: check using $\mathrm{im}\, f \subseteq \mathcal{G}$
[6] Exercise

2 Schemes

Want to construct a sheaf $\mathcal{O}$ on $\operatorname{Spec} A$, analogous to the sheaf of regular functions on a variety, and $\mathcal{O}$ will be a sheaf of rings. That is, $\mathcal{O}(U)$ will be a ring for each open set U and restriction maps will be ring homomorphisms.

2.1 Localisation of a ring

Importantly recall the following. Let A be a ring, where all rings are commutative with unity, and $S \subseteq A$ be a multiplicatively closed subset. That is, $1 \in S$, and if $s_1, s_2 \in S$ then $s_1 s_2 \in S$. We define a ring

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\} / \sim,$$

where $(a, s) \sim (a', s')$ if there exists $s'' \in S$ such that $s''(as' - a's) = 0$. Then $S^{-1}A$ is called the **localisation of A at S**. Note that we write a/s for the equivalence class of (a, s). The usual equivalence relation on fractions is $a/s = a'/s'$ if and only if $as' = a's$. We need the extra possibility of killing $as' - a's$ with s'' if A is not an integral domain.

Example.

- Take $f \in A$ and $S = \{1, f, \dots\} \subseteq A$. Then we write $A_f = S^{-1}A$. These will correspond to open subsets.
- If $\mathfrak{p} \subseteq A$ is a prime ideal and $S = A \setminus \mathfrak{p}$, then
  - $1 \in S$, and
  - $a, b \in S$ and $ab \in \mathfrak{p}$ is a contradiction by definition of prime ideals, so $ab \in S$.

  Then $A_{\mathfrak{p}} = S^{-1}A$ is the **localisation of A at $\mathfrak{p}$**. These will correspond to stalks.

2.2 Construction of the structure sheaf

Let

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in A_{\mathfrak{p}}, \\ \forall \mathfrak{p} \in U,\ \exists \mathfrak{p} \in V \subseteq U \text{ open},\ \exists a, f \in A,\ \forall \mathfrak{q} \in V,\ f \notin \mathfrak{q},\ s(\mathfrak{q}) = \dfrac{a}{f} \in A_{\mathfrak{q}} \end{array} \right\}.$$

Proposition 2.1. *For any $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$.*

Proof. Have a map

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}} \\ (U, s) & \longmapsto & s(\mathfrak{p}) \end{array}.$$

- Surjective. Any element of $A_{\mathfrak{p}}$ can be written as a/f for some $a \in A$ and $f \notin \mathfrak{p}$. Then $\mathbb{D}(f) = \operatorname{Spec} A \setminus \mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$, since $\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}$. Now a/f defines an element of $\mathcal{O}(\mathbb{D}(f))$ given by

  $$\begin{array}{cccc} s & : & \mathbb{D}(f) & \longrightarrow & A_{\mathfrak{q}} \\ & & \mathfrak{q} & \longmapsto & \dfrac{a}{f} \end{array},$$

  and in particular, $s(\mathfrak{p}) = a/f \in A_{\mathfrak{p}}$.
- Injective. Let $\mathfrak{p} \in U \subseteq \operatorname{Spec} A$ and $s \in \mathcal{O}(U)$ with $s(\mathfrak{p}) = 0$ in $A_{\mathfrak{p}}$. Want to show $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$. By shrinking U if necessary, we can assume that s is given by $a, f \in A$ with $s(\mathfrak{q}) = a/f$ for all $\mathfrak{q} \in U$. In particular $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$. Thus $a/f = 0/1$ in $A_{\mathfrak{p}}$ so there exists $h \in A \setminus \mathfrak{p}$ such that $0 = h \cdot (a \cdot 1 - f \cdot 0) = h \cdot a$ in A. Now let $V = U \cap \mathbb{D}(h)$. Then $(V, s|_V) = 0$, since for $\mathfrak{q} \in V$, $s|_V(\mathfrak{q}) = s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$ and $h \cdot a = 0$, and $h \in A \setminus \mathfrak{q}$ so $h \cdot a = 0$ implies $a/f = 0/1$ in $A_{\mathfrak{q}}$. Thus $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$.

∎

Proposition 2.2. *For any $f \in A$, $\mathcal{O}(\mathbb{D}(f)) = A_f$.*

In particular, as $\operatorname{Spec} A = \mathbb{D}(1)$, the **global sections** of $\mathcal{O}$ is $\mathcal{O}(\operatorname{Spec} A) = A_1 = A$.

Proof. Let

$$\begin{array}{cccc} \psi & : & A_f & \longrightarrow & \mathcal{O}(\mathbb{D}(f)) \\ & & \dfrac{a}{f^n} & \longmapsto & \left(\mathfrak{p} \in \mathbb{D}(f) \mapsto \dfrac{a}{f^n} \in A_{\mathfrak{p}}\right) \end{array},$$

since $f \notin \mathfrak{p}$ implies that $f^n \notin \mathfrak{p}$ for all $n \geq 0$.

- Injective. If $\psi(a/f^n) = 0$, then for all $\mathfrak{p} \in \mathbb{D}(f)$, $a/f^n = 0$ in $A_{\mathfrak{p}}$. That is, there exists $h \in A \setminus \mathfrak{p}$ such that $h \cdot a = 0$ in A. Let $I = \{g \in A \mid g \cdot a = 0\}$, the **annihilator** of a. So $h \in I$ and $h \notin \mathfrak{p}$, so $I \not\subseteq \mathfrak{p}$. This is true for all $\mathfrak{p} \in \mathbb{D}(f)$, so $\mathbb{V}(I) \cap \mathbb{D}(f) = \emptyset$. Thus $f \in \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p} = \sqrt{I}$, the radical, so $f^m \in I$ for some $m > 0$. Thus $f^m \cdot a = 0$, so $a/f^n = 0$ in A_f. Thus ψ is injective.

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- Surjective. Let $s \in \mathcal{O}(\mathbb{D}(f))$. Cover $\mathbb{D}(f)$ with open sets V_i on which s is represented as a_i/g_i with $a_i, g_i \in A$ such that $g_i \notin \mathfrak{p}$ whenever $\mathfrak{p} \in V_i$. Thus $V_i \subseteq \mathbb{D}(g_i)$. By question 1 on example sheet 1, the sets of the form $\mathbb{D}(h)$ form a base for the Zariski topology on $\operatorname{Spec} A$. Thus we can assume $V_i = \mathbb{D}(h_i)$ for some $h_i \in A$. Since $\mathbb{D}(h_i) \subseteq \mathbb{D}(g_i)$, we have $\mathbb{V}(h_i) \supseteq \mathbb{V}(g_i)$, so $\sqrt{\langle h_i \rangle} \subseteq \sqrt{\langle g_i \rangle}$, so $h_i^n \in \langle g_i \rangle$ for some n, say $h_i^n = c_i g_i$, so $a_i/g_i = c_i a_i / h_i^n$. Now replace h_i by h_i^n, since this does not change open sets because in general $\mathbb{D}(h_i) = \mathbb{D}(h_i^n)$, and replace a_i by $c_i a_i$. The situation so far is that we can assume $\mathbb{D}(f)$ is covered by sets $\mathbb{D}(h_i)$ such that s is represented by a_i/h_i on $\mathbb{D}(h_i)$. Claim that $\mathbb{D}(f)$ can be covered by a finite number of the $\mathbb{D}(h_i)$. That is, $\mathbb{D}(f)$ is quasi-compact. Since

$$\mathbb{D}(f) \subseteq \bigcup_i \mathbb{D}(h_i) \quad \iff \quad \mathbb{V}(f) \supseteq \bigcap_i \mathbb{V}(h_i) = \mathbb{V}\left(\sum_i \langle h_i \rangle\right) \quad \iff \quad f \in \bigcap_{\mathfrak{p} \in \mathbb{V}\left(\sum_i \langle h_i \rangle\right)} \mathfrak{p}$$

$$\iff \quad f \in \sqrt{\sum_i \langle h_i \rangle} \quad \iff \quad \exists n,\ f^n \in \sum_i \langle h_i \rangle,$$

we can write $f^n = \sum_{i \in I} b_i h_i$ for some finite index set I. Thus reversing this argument, $\mathbb{D}(f) \subseteq \bigcup_{i \in I} \mathbb{D}(h_i)$. We now pass to this finite subcover $\{\mathbb{D}(h_i)\}$. On $\mathbb{D}(h_i) \cap \mathbb{D}(h_j) = \mathbb{D}(h_i h_j)$, note a_i/h_i and a_j/h_j both represent s, so by injectivity shown in the last lecture, $a_i h_j / h_i h_j = a_i/h_i = a_j/h_j = a_j h_i / h_i h_j$ in $A_{h_i h_j}$. Thus for some n, $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A. We can pick an n sufficiently large to work for all pairs i and j. Rewriting, $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j a_j) = 0$. Replace each h_i by h_i^{n+1} and a_i by $h_i^n a_i$, since $a_i/h_i = a_i h_i^n / h_i^{n+1}$. Thus we can assume that s is still represented on $\mathbb{D}(h_i)$ by a_i/h_i but also for each i and j have $h_i a_j = h_j a_i$. Note $f^n = \sum_i b_i h_i$ for $b_i \in A$, since $\{\mathbb{D}(h_i)\}$ cover $\mathbb{D}(f)$. Let $a = \sum_i b_i a_i$. Then for any j, $h_j a = \sum_i b_i a_i h_j = \sum_i b_i a_j h_i = f^n a_j$. Thus $a/f^n = a_j/h_j$ on $\mathbb{D}(h_j)$. Thus $\psi(a/f^n) = s$, so ψ is surjective.

□

We now have a topological space $\operatorname{Spec} A$ equipped with a sheaf of rings $\mathcal{O}$.

2.3 Ringed spaces

Definition. A **ringed space** is a pair $(X, \mathcal{O}_X)$ where

- X is a topological space, and
- $\mathcal{O}_X$ is a sheaf of rings on X.

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the following data.

- $f : X \to Y$ a continuous map.
- $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ a morphism of sheaves of rings. That is, for each $U \subseteq Y$ open, we have a ring homomorphism $f_U^\# : \mathcal{O}_Y(U) \to (f_* \mathcal{O}_X)(U) = \mathcal{O}_X\left(f^{-1}(U)\right)$.

Example.

- Let X be a topological space, and let $\mathcal{O}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions. Then if $(Y, \mathcal{O}_Y)$ is similarly defined, given $f : X \to Y$, we get $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ defined by
$$\begin{array}{cccc} f_U^\# & : & \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X\left(f^{-1}(U)\right) \\ & & \phi & \longmapsto & \phi \circ f \end{array}.$$
- Let X be a variety, and let $\mathcal{O}_X$ be the sheaf of regular functions on X. A morphism of varieties $f : X \to Y$ is a continuous map inducing
$$\begin{array}{cccc} f_U^\# & : & \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X\left(f^{-1}(U)\right) \\ & & \phi & \longmapsto & \phi \circ f \end{array}.$$

A ring is **local** if it has a unique maximal ideal.

Definition. A **locally ringed space** $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$. A **morphism** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **of locally ringed spaces** is a morphism of ringed spaces such that the induced homomorphism $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a **local homomorphism** for all $p \in X$.

- The map is defined by [7]
$$\begin{array}{cccc} f_p^\# & : & \mathcal{O}_{Y,f(p)} & \longrightarrow & \mathcal{O}_{X,p} \\ & & (U, s) & \longmapsto & \left(f^{-1}(U), f_U^\#(s)\right) \end{array}.$$
- A ring homomorphism $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is **local** if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A. Note that $\phi(A \setminus \mathfrak{m}_A) = \phi(A^*) \subseteq B^* = B \setminus \mathfrak{m}_B$, where A^* is the set of invertible elements of A. Thus $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ always.

Example. In the case of varieties, $\mathcal{O}_{X,p}$ has a unique maximal ideal
$$\{(U, f) \in \mathcal{O}_X(U) \mid f(p) = 0\} / \sim.$$
If $f(p) \neq 0$, then f is nowhere vanishing on some neighbourhood of p, so after shrinking U, we can invert f. The local homomorphism condition just follows from the pull-back $\phi \circ f$ of a function ϕ vanishing at $f(p)$ vanishes at p.

2.4 Affine schemes

Lecture 7 Friday 23/10/20

The key example $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space, which we call an affine scheme.

Theorem 2.3. *The category of affine schemes with locally ringed morphisms is equivalent to the opposite category of rings.*

Need to show that

1. if $\phi : A \to B$ is a ring homomorphism, we obtain an induced morphism $\left(f, f^\#\right) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, and
2. any morphism of affine schemes as locally ringed spaces arises in this way.

Proof.

1. Given a ring homomorphism $\phi : A \to B$, define
$$\begin{array}{cccc} f & : & \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ & & \mathfrak{p} & \longmapsto & \phi^{-1}(\mathfrak{p}) \end{array}.$$
Note $\phi^{-1}(\mathfrak{p})$ is prime, since if $ab \in \phi^{-1}(\mathfrak{p})$, then $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, thus either $\phi(a) \in \mathfrak{p}$ or $\phi(b) \in \mathfrak{p}$, and hence either $a \in \phi^{-1}(\mathfrak{p})$ or $b \in \phi^{-1}(\mathfrak{p})$. Then f is continuous, since
$$\begin{aligned} f^{-1}(\mathbb{V}(I)) &= f^{-1}(\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq I\}) = \{\mathfrak{q} \in \operatorname{Spec} B \mid f(\mathfrak{q}) \supseteq I\} \\ &= \left\{\mathfrak{q} \in \operatorname{Spec} B \;\middle|\; \phi^{-1}(\mathfrak{q}) \supseteq I\right\} = \{\mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \supseteq \phi(I)\} = \mathbb{V}(\phi(I)). \end{aligned}$$

[7] Exercise: check well-defined

We need to construct $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$. For $\mathfrak{p} \in \operatorname{Spec} B$, we obtain a natural homomorphism

$$\begin{array}{cccc} \phi_{\mathfrak{p}} & : & A_{\phi^{-1}(\mathfrak{p})} & \longrightarrow & B_{\mathfrak{p}} \\ & & \dfrac{a}{s} & \longmapsto & \dfrac{\phi(a)}{\phi(s)} \end{array}.$$

Note $\phi_{\mathfrak{p}}$ is a local homomorphism, since the maximal ideal $\mathfrak{p} B_{\mathfrak{p}}$ of $B_{\mathfrak{p}}$ is generated by the image of $\mathfrak{p}$ under the map

$$\begin{array}{ccc} B & \longrightarrow & B_{\mathfrak{p}} \\ b & \longmapsto & \dfrac{b}{1} \end{array},$$

and the maximal ideal $\phi^{-1}(\mathfrak{p}) A_{\phi^{-1}(\mathfrak{p})}$ of $A_{\phi^{-1}(\mathfrak{p})}$ is generated by the image of $\phi^{-1}(\mathfrak{p})$ under the map

$$\begin{array}{ccc} A & \longrightarrow & A_{\phi^{-1}(\mathfrak{p})} \\ a & \longmapsto & \dfrac{a}{1} \end{array},$$

so have a commutative diagram

$$\begin{tikzcd} \phi^{-1}(\mathfrak{p}) & \subset & A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \supset & \mathfrak{p} \\ f(\mathfrak{p}) A_{f(\mathfrak{p})} & \subset & A_{\phi^{-1}(\mathfrak{p})} \arrow[r, "\phi_{\mathfrak{p}}"'] & B_{\mathfrak{p}} & \supset & \mathfrak{p} B_{\mathfrak{p}} \end{tikzcd},$$

thus $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p} B_{\mathfrak{p}}) = \phi^{-1}(\mathfrak{p}) A_{\phi^{-1}(\mathfrak{p})}$. Given $V \subseteq \operatorname{Spec} A$ open, we may define

$$\begin{array}{cccc} f_V^{\#} & : & \mathcal{O}_{\operatorname{Spec} A}(V) & \longrightarrow & \mathcal{O}_{\operatorname{Spec} B}\left(f^{-1}(V)\right) \\ & & (\mathfrak{p} \in V \mapsto s(\mathfrak{p}) \in A_{\mathfrak{p}}) & \longmapsto & \left(\mathfrak{q} \in f^{-1}(V) \mapsto \phi_{\mathfrak{q}}(s(f(\mathfrak{q}))) \in B_{\mathfrak{q}}\right) \end{array}.$$

Note that we need to check the local coherence part of the definition of $\mathcal{O}$. That is, if s is locally given by a/h, then $f_V^{\#}(s)$ is locally given by $\phi(a)/\phi(h)$. This gives the desired map $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$, and the induced map on stalks $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}$ agrees with $\phi_{\mathfrak{p}} : A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$, by construction. Hence $\left(f, f^{\#}\right)$ is a morphism of locally ringed spaces.

2. Now suppose given a morphism $\left(f, f^{\#}\right) : \operatorname{Spec} B \to \operatorname{Spec} A$ of locally ringed spaces. Take

$$\phi = f_{\operatorname{Spec} A}^{\#} : \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) = B.$$

We need to show ϕ gives rise to $\left(f, f^{\#}\right)$. We have $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} = A_{f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} = B_{\mathfrak{p}}$ a local homomorphism. This is compatible with the corresponding map on global sections. That is,

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, "f_{\operatorname{Spec} A}^{\#}"] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\ \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"'] & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} \end{tikzcd}$$

is commutative. That is, we have a commutative diagram

$$\begin{tikzcd} f(\mathfrak{p}) & \subset & A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \supset & \mathfrak{p} \\ f(\mathfrak{p}) A_{f(\mathfrak{p})} & \subset & A_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"'] & B_{\mathfrak{p}} & \supset & \mathfrak{p} B_{\mathfrak{p}} \end{tikzcd}.$$

Then $\left(f_{\mathfrak{p}}^{\#}\right)^{-1}(\mathfrak{p} B_{\mathfrak{p}}) = f(\mathfrak{p}) A_{f(\mathfrak{p})}$ since $f_{\mathfrak{p}}^{\#}$ is a local homomorphism, and by commutativity of the diagram, $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Thus f is induced by ϕ, and $f_{\mathfrak{p}}^{\#} = \phi_{\mathfrak{p}}$. So $f^{\#}$ is as constructed previously.

$\square$

Remark. Demanding $\left(f, f^{\#}\right)$ was a morphism of locally ringed spaces was crucial to make the proof work.

Definition. An **affine scheme** is a locally ringed space isomorphic, in the category of locally ringed spaces, to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\left\{\left(U_i, \mathcal{O}_X|_{U_i}\right)\right\}$ with each $\left(U_i, \mathcal{O}_X|_{U_i}\right)$ an affine scheme, where $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U_i$ open. A **morphism of schemes** is a morphism of locally ringed spaces.

Example. Let k be a field. Then $\operatorname{Spec} k = (\{0\}, k)$.

- What does giving a morphism $f : \operatorname{Spec} k \to X$ to a scheme mean? First, this selects a point $x \in X$, the image of f. Second, we get a local ring homomorphism $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} k,\{0\}} = k$. That is, $\left(f_x^\#\right)^{-1}(0) = \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$. Thus we get a factorisation $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x \to k$, where $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a field, written as $\kappa(x)$, called the **residue field** of X at x. Thus f induces an inclusion $\kappa(x) \hookrightarrow k$. Conversely, given such an inclusion $\iota : \kappa(x) \hookrightarrow k$ of fields, we get a scheme morphism by defining $f(0) = x$, and
$$f^\# \quad : \quad \begin{array}{ccc} \mathcal{O}_X & \longrightarrow & f_* k \\ s & \longmapsto & \iota(s(x)) \end{array}, \qquad s(x) \in \mathcal{O}_{X,x}.$$
The moral is that giving a morphism $f : \operatorname{Spec} k \to X$ is equivalent to giving a point $x \in X$ and an inclusion $\iota : \kappa(x) \to k$. Note that if $X = \operatorname{Spec} A$, giving $\operatorname{Spec} k \to \operatorname{Spec} A$ is equivalent to giving a homomorphism $A \to k$, which we viewed at the beginning of the course as a k-valued point on $\operatorname{Spec} A$. Lecture 8 Monday 26/10/20
- What does giving $f : X \to \operatorname{Spec} k$ mean? No information in the continuous map, but need also a map $f^\# : k \to f_*\mathcal{O}_X$, that is a map $k \to \Gamma(\operatorname{Spec} k, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. That is, $\Gamma(X, \mathcal{O}_X)$ carries a k-algebra structure. Note this induces k-algebra structures on $\mathcal{O}_X(U)$ for all U via the composition $k \to \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ and similarly all stalks $\mathcal{O}_{X,p}$ are also k-algebras. We say X is a **scheme defined over** k. For example, in affine varieties, consider $A = k[x_1, \dots, x_n]/I$ with $I = \sqrt{I}$. Then $\operatorname{Spec} A$ is our replacement for $\mathbb{V}(I) \subseteq \mathbb{A}_k^n$, viewing $\operatorname{Spec} A$ as a scheme over k. If $k \subseteq k'$ is a field extension, a k'**-valued point** of X/k is a commutative diagram
$$\begin{tikzcd} \operatorname{Spec} k' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}.$$
We write $X(k')$ for the set of such morphisms.

Remark. It is rare in algebraic geometry to work with schemes alone, but rather always working over a base scheme.

Fix a base scheme S. Define $\mathbf{Sch}/S$ to be the category whose objects are morphisms $T \to S$ and morphisms are commutative diagrams
$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & T' \arrow[dl] \\ & S & \end{tikzcd}.$$
We will frequently work with $\mathbf{Sch}/k = \mathbf{Sch}/\operatorname{Spec} k$. Given $T \to S$ and $X \to S$ objects in $\mathbf{Sch}/S$, a T**-valued point** of $X \to S$ is a morphism $T \to X$ over S, so
$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd},$$
and we write $X(T)$ for the set of T-valued points. The **Yoneda philosophy** is that $X(T)$ for all T determines X.

Example. Fix a field k, and let $D = \operatorname{Spec} k[t] / \langle t^2 \rangle = (\{\langle t \rangle\}, k[t] / \langle t^2 \rangle)$. Then t does not make sense as k-valued function anymore, as $t^2 = 0$. Let X be any scheme over k. What is $X(D)$? Given $f : D \to X$ a morphism of schemes over k, we get a point $x \in X$ as the image of f and a local homomorphism

$$\begin{array}{cccc} f_x^{\#} & : & \mathcal{O}_{X,x} & \longrightarrow & k[t] / \langle t^2 \rangle \\ & & \mathfrak{m}_x & \longmapsto & \langle t \rangle \end{array}.$$

Note that $\mathfrak{m}_x^2$ maps to zero, hence we get a k-linear map $\mathfrak{m}_x / \mathfrak{m}_x^2 \to \langle t \rangle \cong k$ as a k-vector space. We also have a composed surjective k-algebra homomorphism $\mathcal{O}_{X,x} \to k[t] / \langle t \rangle \cong k$ with kernel $\mathfrak{m}_x$, and hence we have $\kappa(x) = \mathcal{O}_{X,x} / \mathfrak{m}_x \cong k$. So we get

- a k-valued point x with residue field k, and
- a k-vector space map $\mathfrak{m}_x / \mathfrak{m}_x^2 \to k$, that is an element of $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$, the dual vector space.

Then $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$ is called the **Zariski tangent space** to X at x. Think of D as a point plus an arrow.

Example. **Glued schemes** are a special case of a question on example sheet 1. Suppose given two schemes X_1 and X_2 and open subsets $U_i \subseteq X_i$. Recall U_i is also a locally ringed space $(U_i, \mathcal{O}_{X_i}|_{U_i})$, and in fact U_i is then a scheme. Given an isomorphism $f : U_1 \xrightarrow{\sim} U_2$, can glue X_1 and X_2 along U_1 and U_2 to get a scheme X with an open cover $\{X_1, X_2\}$, so $X = X_1 \sqcup X_2 / \sim$ such that $x_1 \in U_1 \sim x_2 \in U_2$ if $f(x_1) = x_2$, and need to define $\mathcal{O}_X$. Now take $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$, so $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$. Take $X_1 = X_2 = \mathbb{A}_k^1$.

- Glue $U_1 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(x) \subseteq X_1$ and $U_2 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(x) \subseteq X_2$ via the identity map. This is the affine line with doubled origin.
- Could instead glue U_1 and U_2 via the map given by $x \mapsto x^{-1}$, where $U_1 = \operatorname{Spec} k[x]_x = U_2$ and

  $$\begin{array}{ccc} k[x]_x & \longrightarrow & k[x]_x \\ x & \longmapsto & x^{-1} \end{array}$$

  induces an isomorphism $U_1 \to U_2$. When we glue, we get the projective line over k, $\mathbb{P}_k^1$.

2.5 Projective schemes

Let S be a graded ring. That is,

$$S = \bigoplus_{d \geq 0} S_d,$$

with S_d an abelian group, and product law satisfies $S_d \cdot S_{d'} \subseteq S_{d+d'}$.

Example. $S = k[x_0, \dots, x_n]$, and S_d is the space of polynomials which are **homogeneous** of degree d. That is, spanned by monomials of degree d.

We write

$$S_+ = \bigoplus_{d \geq 1} S_d,$$

which we call the **irrelevant ideal**.

Definition. $I \subseteq S$ is a **homogeneous ideal** if I is generated by its homogeneous elements. That is, elements in S_d for various d.

Definition. Let

$$\operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \text{ is homogeneous}, \ \mathfrak{p} \not\supseteq S_+\}.$$

For $I \subseteq S$ a homogeneous ideal, set

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}.$$

Exercise. Check the $\mathbb{V}(I)$ form the closed sets of a topology on $\operatorname{Proj} S$.

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Notation. For $\mathfrak{p} \in \operatorname{Proj} S$, let
$$T = \{f \in S \setminus \mathfrak{p} \mid f \text{ is homogeneous}\}.$$
Then T is a multiplicatively closed subset of S, and let $S_{(\mathfrak{p})} \subseteq T^{-1}S$ be the subring of elements of degree zero. That is, written in the form s/s' with $s \in S$ homogeneous and $s' \in T$ with $\deg s = \deg s'$. For $f \in S$ homogeneous, we write $S_{(f)} \subseteq S_f$ for the subset of elements of degree zero.

Can now define a sheaf $\mathcal{O}$ on $\operatorname{Proj} S$. For $U \subseteq \operatorname{Proj} S$ open, set
$$\mathcal{O}(U) = \left\{ s : U \to \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in S_{(\mathfrak{p})}, \\ \forall \mathfrak{p} \in U,\ \exists \mathfrak{p} \in V \subseteq U \text{ open},\ \exists a, f \in S,\ \forall \mathfrak{q} \in V,\ f \notin \mathfrak{q},\ s(\mathfrak{q}) = \dfrac{a}{f} \in S_{(\mathfrak{q})} \end{array} \right\},$$
where a and f are homogeneous of the same degree. As before, $\mathcal{O}_{\mathfrak{p}} = S_{(\mathfrak{p})}$. [8] Is the locally ringed space $(\operatorname{Proj} S, \mathcal{O})$ a scheme?

Notation. If $f \in S$ is homogeneous, then we write
$$\mathbb{D}_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\},$$
which is an open set and $\mathbb{D}_+(f) = \operatorname{Proj} S \setminus \mathbb{V}(f)$.

Proposition 2.4. *$\left(\mathbb{D}_+(f), \mathcal{O}|_{\mathbb{D}_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$ as locally ringed spaces. Further, the open sets $\mathbb{D}_+(f)$ for $f \in S_+$ cover $\operatorname{Proj} S$. Hence $(\operatorname{Proj} S, \mathcal{O})$ is a scheme.*

Proof. Will be on example sheet 2. ☐

Definition. If A is a ring, define
$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n].$$

Example. If k is an algebraically closed field, consider $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. The **closed points**, that is points $\mathfrak{p}$ such that $\{\mathfrak{p}\}$ is closed, correspond to maximal elements of $\operatorname{Proj} S$. [9] These maximal elements are ideals of the form $\langle a x_0 - b x_1 \rangle$. The only maximal homogeneous ideal of $k[x_0, x_1]$ is $\langle x_0, x_1 \rangle = S_+$, since any maximal ideal is of the form $\langle x_0 - a_0, x_1 - a_1 \rangle$. The other prime ideals of $k[x_0, x_1]$ are principal. That is, of the form $\langle f \rangle$ with f irreducible or $f = 0$. For $\langle f \rangle$ to be homogeneous, f must be homogeneous. Any such polynomial splits into linear factors, all homogeneous, so in order for f to be irreducible it must be linear. Note we have a one-to-one correspondence between
$$\begin{array}{rcl} \{\langle a x_0 - b x_1 \rangle \mid a, b \in k \text{ not both zero}\} & \longrightarrow & \left(k^2 \setminus \{(0,0)\}\right)/k^* \\ \langle a x_0 - b x_1 \rangle & \longmapsto & (b : a) \end{array},$$
where k^* acts by $(a, b) \mapsto (\lambda a, \lambda b)$ for $\lambda \in k^*$. The conclusion is that the closed points of $\mathbb{P}^1_k$ are in one-to-one correspondence with points of $\left(k^2 \setminus \{(0,0)\}\right)/k^*$. More generally, the closed points of $\mathbb{P}^n_k$ are in one-to-one correspondence with points of $\left(k^{n+1} \setminus \{0\}\right)/k^*$. Can see this by making use of the open cover $\{\mathbb{D}_+(x_i) \mid 0 \le i \le n\}$, [10] which is an open cover since $\mathfrak{p} \notin \mathbb{D}_+(x_i)$ for any i implies that $x_i \in \mathfrak{p}$ for all i, so $S_+ \subseteq \mathfrak{p}$ and so $\mathfrak{p} \notin \operatorname{Proj} S$.

Example. Let $S = k[x_0, \dots, x_n]$, but grade by $\deg x_i = w_i$, where $w_0, \dots, w_n$ are positive integers. Define $\mathrm{W}\mathbb{P}^n(w_0, \dots, w_n) = \operatorname{Proj} S$, the **weighted projective space**. For example, $\mathrm{W}\mathbb{P}^2(1,1,2)$ has an open cover $\{\mathbb{D}_+(x_i) \mid 0 \le i \le 2\}$. Consider $\mathbb{D}_+(x_2) = \operatorname{Spec} S_{(x_2)}$. Note
$$S_{(x_2)} = k\left[\frac{x_0^2}{x_2}, \frac{x_0 x_1}{x_2}, \frac{x_1^2}{x_2}\right] \cong k[u, v, w] / \langle uw - v^2 \rangle \subseteq S_{x_2},$$
so $\operatorname{Spec} S_{(x_2)}$ is a quadric cone with a singular point. Similarly, $\mathbb{D}_+(x_0)$ and $\mathbb{D}_+(x_1)$ are both isomorphic to $\mathbb{A}^2_k$.

[8] Exercise: check
[9] Exercise: check
[10] Exercise: good exercise

Example. Let $M = \mathbb{Z}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$. Let $\Delta \subseteq M_{\mathbb{R}}$ be a compact convex lattice polytope. That is, there exists a finite set $V \subseteq M$ such that Δ is the convex hull of V, that is the smallest convex set containing V. Let

$$\mathrm{C}(\Delta) = \{(m,r) \in M_{\mathbb{R}} \oplus \mathbb{R} \mid m \in r\Delta,\ r \geq 0\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

Here $r\Delta = \{rm \mid m \in \Delta\}$. This is the **cone over** Δ. Let

$$S = k[\mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})] = \bigoplus_{P \in \mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})} kz^P,$$

with multiplication given by $z^P z^{P'} = z^{P+P'}$, since $\mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})$ is a monoid. That is, it is closed under addition and contains zero. This makes S into a ring, and it is graded by $\deg Z^{(m,r)} = r$. Define $\mathbb{P}_\Delta = \operatorname{Proj} S$. This is called a **projective toric variety**.

- Let Δ be the convex hull of $\{0, e_1, \dots, e_n\}$ with $e_1, \dots, e_n$ the standard basis of $M = \mathbb{Z}^n$. Check that $S = k[x_0, \dots, x_n]$ with standard grading $x_0 = z^{(0,1)}$ and $x_i = z^{(e_i,1)}$. [11] So $\mathbb{P}_\Delta = \mathbb{P}_k^n$.
- Let $n = 2$, and let Δ be the convex hull of $\{(0,0), (1,0), (0,1), (1,1)\}$. In S, the degree d monomials are $\left\{z^{(a,b,d)} \mid 0 \leq a \leq d,\ 0 \leq b \leq d\right\}$. Any of these can be written as a product of monomials of degree one. That is, the monomials $x = z^{(0,0,1)}$, $y = z^{(1,0,1)}$, $w = z^{(0,1,1)}$, and $t = z^{(1,1,1)}$. Thus $S = k[x,y,w,t] / \langle xt - yw \rangle$. So $\operatorname{Proj} S$ can be thought of as a quadric surface in $\mathbb{P}_k^3$.

2.6 Open and closed subschemes

Lecture 10 Friday 30/10/20

Definition. An **open subscheme** of a scheme X is a scheme $(U, \mathcal{O}_X|_U)$ for $U \subseteq X$ an open subset. Note that this is a scheme because from question 1 and question 11 on the first example sheet, open affine subsets of X form a basis for the topology on X. An **open immersion** is a morphism $f : X \to Y$ which induces a isomorphism of X with an open subscheme of Y. A **closed immersion** $f : X \to Y$ is a morphism which is a homeomorphism onto a closed subset of Y, and the induced morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. A **closed subscheme** of Y is an equivalence class of closed immersions, where

$$\begin{tikzcd} X \arrow[rr, dashed, "i"] \arrow[dr] & & X' \arrow[dl] \\ & Y & \end{tikzcd}$$

are equivalent if there exists an isomorphism i making the diagram commute.

Example.

- Let $Y = \operatorname{Spec} A$, let $I \subseteq A$ be an ideal, and let $X = \operatorname{Spec} A/I$. Note the map of schemes induced by the quotient map $A \to A/I$ identifies $\operatorname{Spec} A/I$ with $\mathbb{V}(I) \subseteq \operatorname{Spec} A$. Thus $f : X \to Y$, induced by $A \to A/I$, satisfies the first condition of being a closed immersion. Note that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective on stalks. For $\mathfrak{p} \in \mathbb{V}(I)$, $\mathcal{O}_{Y,\mathfrak{p}} = A_{\mathfrak{p}}$ and $(f_*\mathcal{O}_X)_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$ since all open sets in X are of the form $U \cap X$ for U an open set of Y and $\mathcal{O}_{X,\mathfrak{p}} = (A/I)_{\mathfrak{p}/I}$. Certainly $A_{\mathfrak{p}} \to (A/I)_{\mathfrak{p}/I}$ is surjective.
- Let $\operatorname{Spec} k[x,y] / \langle x \rangle \to \operatorname{Spec} k[x,y] = \mathbb{A}^2$. This gives a closed subscheme structure to the set $\mathbb{V}(x)$. Note $\mathbb{V}(x^2, xy) = \mathbb{V}(x)$. This gives a closed immersion $\operatorname{Spec} k[x,y] / \langle x^2, xy \rangle \to \mathbb{A}^2$. This gives a different closed subscheme structure on $\mathbb{V}(x)$. Note these two subschemes are isomorphic away from the origin, which we can see by looking at $\mathbb{D}(y) \subseteq \operatorname{Spec} k[x,y] / \langle x \rangle$, where

$$\mathbb{D}(y) \cong \operatorname{Spec} (k[x,y] / \langle x \rangle)_y = \operatorname{Spec} k[y]_y.$$

Looking at $\mathbb{D}(y) \subseteq \operatorname{Spec} k[x,y] / \langle x^2, xy \rangle$,

$$\mathbb{D}(y) \cong \operatorname{Spec} \left(k[x,y] / \langle x^2, xy \rangle\right)_y \cong \operatorname{Spec} k[x,y]_y / \langle x \rangle \cong \operatorname{Spec} k[y]_y.$$

[11] Exercise

3 Properties of schemes and morphisms of schemes

3.1 Fibre products

Let $\mathcal{C}$ be a category and

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a diagram in $\mathcal{C}$. Then the **fibre product**, if it exists, is an object W equipped with morphisms $p : W \to X$ and $q : W \to Y$ such that $f \circ p = g \circ q$ satisfying the following universal property. For any W' equipped with maps $p' : W' \to X$ and $q' : W' \to Y$ such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : W' \to W$ making the diagram

$$\begin{tikzcd} W' \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\ & W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

commute. That is, $p \circ h = p'$ and $q \circ h = q'$. Note that if the fibre product exists, it is unique up to unique isomorphism.

Example. Let $\mathcal{C}$ be the category of sets. Then

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

It will be helpful to think about the fibre product, and more generally other universal properties, via the Yoneda lemma.

Definition. Let $\mathcal{C}$ be a category. Write h_X for the contravariant functor

$$\begin{array}{cccll} \mathrm{h}_X & : & \mathcal{C} & \longrightarrow & \mathbf{Set} \\ & & Y & \longmapsto & \mathrm{Hom}(Y, X) \\ & & f : Y \to Z & \longmapsto & (\phi \in \mathrm{Hom}(Z, X) \mapsto \phi \circ f \in \mathrm{Hom}(Y, X)) \end{array}.$$

Recall that a **natural transformation** between contravariant functors $F, G : \mathcal{C} \to \mathcal{D}$, written as $T : \mathcal{C} \to \mathcal{D}$, consists of the data $T(X) : F(X) \to G(X)$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ such that for all $f : X \to Y$ in $\mathcal{C}$

$$\begin{tikzcd} F(X) \arrow[d, "T(X)"'] & F(Y) \arrow[l, "F(f)"'] \arrow[d, "T(Y)"] \\ G(X) & G(Y) \arrow[l, "G(f)"] \end{tikzcd}$$

is commutative.

Lemma 3.1 (Yoneda's lemma)**.** *The set of natural transformations between $\mathrm{h}_X : \mathcal{C} \to$ **Set** and $G : \mathcal{C} \to$ **Set** is $G(X)$.*

Proof. Given $\eta \in G(X)$, we need to define a map

$$\begin{array}{rcl} \mathrm{h}_X(Y) = \mathrm{Hom}(Y, X) & \longrightarrow & G(Y) \\ f & \longmapsto & G(f)(\eta) \end{array},$$

for all objects $Y \in \mathcal{C}$. Check that this defines a natural transformation $\mathrm{h}_X \to G$. [12] Conversely, given $T : \mathrm{h}_X \to G$ a natural transformation, take $\eta = T(X)(\mathrm{id}_X)$. Check that these two maps are inverse to each other. [13] $\square$

[12] Exercise
[13] Exercise

Corollary 3.2. *The set of natural transformations* $\mathrm{h}_X \to \mathrm{h}_Y$ *is* $\mathrm{h}_Y(X) = \mathrm{Hom}(X, Y)$.

Definition. A contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be **representable** if $F \cong \mathrm{h}_X$ for some $X \in \mathrm{Ob}\,\mathcal{C}$.

Lots of questions in algebraic geometry are about representability of functors. Redefining, the fibre product in a category $\mathcal{C}$ is an object which represents the functor

$$T \mapsto \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}(T, Y),$$

since an element of the set $\mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}(T, Y)$ is a commutative diagram

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}.$$

The advantage of using Yoneda is that we can check identities using fibre products using identities of fibre products of sets.

Example. In **Set**,

$$\begin{array}{rcl} (A \times_B C) \times_C D & \longleftrightarrow & A \times_B D \\ ((a, c), d) & \longmapsto & (a, d) \\ ((a, f(d)), d) & \longleftarrow & (a, d) \end{array}, \qquad f : D \to C.$$

Then we have two functors

$$\begin{tikzcd}
T \arrow[r] \arrow[dr] & \left(\mathrm{h}_A(T) \times_{\mathrm{h}_B(T)} \mathrm{h}_C(T)\right) \times_{\mathrm{h}_C(T)} \mathrm{h}_D(T) \arrow[d, "\sim"] \\
& \mathrm{h}_A(T) \times_{\mathrm{h}_B(T)} \mathrm{h}_D(T)
\end{tikzcd},$$

and natural transformations showing those functors are isomorphic, and hence represent isomorphic objects.

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Theorem 3.3. *Fibre products exist in the category of schemes.*

Proof. Will construct $X \times_S Y$ for various cases, bootstrapping up to the general case.

Step 1. Let $X = \mathrm{Spec}\, A$, let $Y = \mathrm{Spec}\, B$, and let $S = \mathrm{Spec}\, R$, so

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd} \qquad \Longleftrightarrow \qquad \begin{tikzcd} & B \\ A & R \arrow[l] \arrow[u] \end{tikzcd}.$$

Push-outs exist in the category of rings, so

$$\begin{tikzcd}
C & & \\
& A \otimes_R B \arrow[ul, dashed, "\exists! h"] & B \arrow[l, "p_2"'] \arrow[ull, bend right, "p_2'"'] \\
& A \arrow[u, "p_1"] \arrow[uul, bend left, "p_1'"] & R \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd},$$

where $p_1(a) = a \otimes 1$ and $p_2(b) = 1 \otimes b$. Here h is defined by $h(a \otimes b) = p_1'(a)\, p_2'(b)$. Check well-defined. [14] Thus $\mathrm{Spec}\, A \otimes_R B$ is $\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, B$ in the category of affine schemes.

[14] Exercise

If T is an arbitrary scheme, then giving a morphism $T \to \operatorname{Spec} A$ is the same as giving a morphism $A \to \Gamma(T, \mathcal{O}_T)$, by question 12, example sheet 1. Thus giving a commutative diagram

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

is equivalent to

$$\begin{tikzcd}
\Gamma(T, \mathcal{O}_T) & & \\
& A \otimes_R B \arrow[ul, dashed, "\exists! h"'] & B \arrow[l] \arrow[ull, bend right] \\
& A \arrow[u] \arrow[uul, bend left] & R \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd},$$

and $h : A \otimes_R B \to \Gamma(T, \mathcal{O}_T)$ induces a map $T \to \operatorname{Spec} A \otimes_R B$. Thus $\operatorname{Spec} A \otimes_R B$ is the fibre product $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ in the category of schemes.

Step 2. Will construct more general fibre products by gluing of schemes using question 14 on example sheet 1. We also glue morphisms, so if X and Y are schemes, $\{U_i\}$ an open cover of X, and we are given morphisms $f_i : U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then we obtain $f : X \to Y$ such that $f|_{U_i} = f_i$. The argument is given in the examples class.

Step 3. If $X, Y \to S$ are given and $U \subseteq X$ is open, suppose that $X \times_S Y$ exists, with projections $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$. Then $p_1^{-1}(U)$ is $U \times_S Y$. By commutativity of the diagram

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists! h"'] \arrow[drr, "\exists! h'"] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& p_1^{-1}(U) \arrow[r, hook] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& U \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd},$$

the image of h' must be contained in $p_1^{-1}(U)$. Thus h' factors through $p_1^{-1}(U) \hookrightarrow X \times_S Y$ giving the unique map h, so the universal property holds for $p_1^{-1}(U)$.

Step 4. Suppose $\{X_i\}$ is an open cover of X and $X_i \times_S Y$ exists for each i. Then $X \times_S Y$ exists. Let $X_{ij} = X_i \cap X_j$, and let $U_{ij} = p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y$. By step 3, $U_{ij} = X_{ij} \times_S Y$. By the universal property of fibre products there exists a unique isomorphism $\phi_{ij} : U_{ij} \to U_{ji}$. Check these gluing maps ϕ_{ij} satisfy the requirements of question 14 on example sheet 1. [15] Thus we can glue the $X_i \times_S Y$ via ϕ_{ij}'s to get a scheme $X \times_S Y$, but need to check it satisfies the fibre product axioms. So suppose given

$$\begin{tikzcd}
T \arrow[drr, bend left, "p_2'"] \arrow[ddr, bend right, "p_1'"'] & & \\
& & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}.$$

[15] Exercise: check

Let $T_i = (p_1')^{-1}(X_i)$, so get a morphism $\theta_i : T_i \to X_i \times_S Y \hookrightarrow X \times_S Y$, where $X_i \times_S Y \hookrightarrow X \times_S Y$ is an open immersion by construction. On $T_i \cap T_j$ these maps agree since they factor through $X_{ij} \times_S Y \subseteq X_i \times_S Y$ and $X_{ji} \times_S Y \subseteq X_j \times_S Y$ and by the universal property they agree. Thus using step 2, we can glue the θ_i's to get $\theta : T \to X \times_S Y$.

Step 5. Using step 4 and 1 we may construct $X \times_S Y$ when S and Y are affine. Repeating for Y, we obtain $X \times_S Y$ when S is affine, and X and Y are arbitrary.

Step 6. Let X, Y, S be arbitrary, take an open affine cover $\{S_i\}$ of S, let $f : X \to S$ and $g : Y \to S$, and let $X_i = f^{-1}(S_i)$ and $Y_i = g^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists and $X_i \times_{S_i} Y_i = X_i \times_S Y_i$. [16] Use the same gluing argument as before, to get $X \times_S Y$.

$\square$

3.2 Fibres of morphisms

The philosophy in **Set** is

$$\begin{tikzcd} f^{-1}(y) = \{y\} \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{y\} \arrow[r] & Y \end{tikzcd}.$$

Given $f : X \to Y$ a morphism and $y \in Y$, let $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of y, so we get a morphism $\operatorname{Spec} \kappa(y) \to Y$ with image y. Then we define

$$X_y = \operatorname{Spec} \kappa(y) \times_Y X$$

to be the **scheme-theoretic fibre** of f at y.

Example. Let $f : X = \operatorname{Spec} k[x] \to Y = \operatorname{Spec} k[t]$ be induced by

$$\begin{array}{ccc} k[t] & \longrightarrow & k[x] \\ t & \longmapsto & x^2 \end{array}.$$

For $y = \langle t - a \rangle \subseteq k[t]$ and $a \in k$, $\kappa(y) = k[t] / \langle t - a \rangle \cong k$. If B is an A-algebra then $A/I \otimes_A B = B/IB$, so

$$X_y = \operatorname{Spec} \kappa(y) \otimes_{k[t]} k[x] = \operatorname{Spec} k[x] / \left\langle x^2 - a \right\rangle.$$

If $a \neq 0$ and $\operatorname{ch} k \neq 2$, we obtain either X_y consists of two distinct points, if $\sqrt{a} \in k$, or a single point if $\sqrt{a} \notin k$. If $a = 0$, we get $\operatorname{Spec} k[x] / \left\langle x^2 \right\rangle$.

Remark.

- In general, it is hard to calculate fibre products, since $X \times_S Y$ is not the set-theoretic fibre product in general. For example, $\mathbb{A}_k^1 \times_{\operatorname{Spec} k} \mathbb{A}_k^1 = \operatorname{Spec} k[x] \otimes_k k[y] = \operatorname{Spec} k[x, y] = \mathbb{A}_k^2$.
- If we are interested only in varieties, such as schemes over a field k, the usual product of varieties $X \times Y$ corresponds to $X \times_{\operatorname{Spec} k} Y$. More generally, if we are working in the category $\mathbf{Sch}/S$, the natural product is $X \times_S Y$.
- Given schemes S and T with a morphism $T \to S$, we get a functor

$$\begin{array}{ccc} \mathbf{Sch}/S & \longrightarrow & \mathbf{Sch}/T \\ (X \to S) & \longmapsto & (X \times_S T \to T) \end{array}.$$

This functor is called **base-change**.

[16] Exercise: check, immediate from universal property

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Example. Consider a scheme X over $\operatorname{Spec} \mathbb{Z}$, such as $X = \operatorname{Proj} \mathbb{Z}[x, y, z] / \langle x^n + y^n - z^n \rangle \to \operatorname{Spec} \mathbb{Z}$. May consider base-changes

- $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p = \operatorname{Proj} \mathbb{F}_p[x, y, z] / I$,
- $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \to \mathbb{Q}$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q} = \operatorname{Proj} \mathbb{Q}[x, y, z] / I$, or
- $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$, induced by $\mathbb{Z} \to \mathbb{C}$, which gives $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C} = \operatorname{Proj} \mathbb{C}[x, y, z] / I \subseteq \mathbb{P}^2_{\mathbb{C}}$,

where $I = \langle x^n - y^n - z^n \rangle$.

3.3 Brief discussion of other properties

See example sheet 2 for more details or your favourite algebraic geometry text.

Definition. A scheme X is **integral** if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

Definition. A scheme X is **reduced** if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

Definition. A scheme X is **irreducible** if the underlying topological space X is irreducible. That is, if $X = X_1 \cup X_2$ with $X_1, X_2 \subseteq X$ closed, then either $X_1 = X$ or $X_2 = X$.

Example. Let $X = \operatorname{Spec} k[x, y] / \langle xy \rangle$.

- X is not integral because $\Gamma(X, \mathcal{O}_X) = k[x, y] / \langle xy \rangle$ is not an integral domain, since $xy = 0$.
- X is reduced.
- X is not irreducible, since $X = \mathbb{V}(x) \cup \mathbb{V}(y)$.

Theorem 3.4. *X is integral if and only if X is reduced and irreducible.*

Definition. Let X be a scheme. It is **locally Noetherian** if there exists a cover $\{U_i\}$ of X with $U_i = \operatorname{Spec} A_i$ affine and A_i Noetherian. Then X is **Noetherian** if the cover may be taken to be finite.

Example. $\operatorname{Spec} k[x_1, x_2, \dots]$ with a countable number of variables is not locally Noetherian.

Not obvious, but can show that X is locally Noetherian if and only if, if $U \subseteq X$ is affine and $U = \operatorname{Spec} A$, then A is Noetherian.

Definition. A morphism $f : X \to Y$ of schemes is **locally of finite type** if there is a covering of Y by affine open sets $\{V_i = \operatorname{Spec} B_i\}$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open sets $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, where each A_{ij} is a finitely generated B_i-algebra. We say f is of **finite type** if for each i, the cover $\{U_{ij}\}$ may be taken to be finite.

Definition. Let k be an algebraically closed field. A **variety over** k is a scheme X over $\operatorname{Spec} k$ which is integral and $X \to \operatorname{Spec} k$ is of finite type. That is, X can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$ with A_i a finitely generated k-algebra. The A_i must be integral domains, so $A_i = k[x_1, \dots, x_n] / I$ where I is a prime ideal.

Note that this still allows a non-Hausdorff scheme $\mathbb{A}^1 \cup \mathbb{A}^1$ obtained by gluing $\mathbb{D}(x) \subseteq \mathbb{A}^1$ to $\mathbb{D}(x) \subseteq \mathbb{A}^1$.

Example. Let $X_i = \operatorname{Spec} k[x_i, y_i] / \langle x_i y_i \rangle$ for $i \in \mathbb{Z}$. Glue X_i to X_{i+1} along open subsets $U_{i,i+1} \subseteq X_i$ given by $\mathbb{D}(x_i)$ and $U_{i+1,i} \subseteq X_{i+1}$ given by $\mathbb{D}(y_{i+1})$ via the map

$$\begin{array}{ccc} k[y_{i+1}]_{y_{i+1}} & \longrightarrow & k[x_i]_{x_i} \\ y_{i+1} & \longmapsto & x_i^{-1} \end{array}.$$

Doing this for all i, we get an infinite chain of $\mathbb{P}^1$'s. Note $\{X_i\}$ forms an open cover of X but has no finite subcover. Not quasi-compact, only locally of finite type over $\operatorname{Spec} k$.

3.4 Separated and proper morphisms

Remark. A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed.

Example. Let X be $\mathbb{R}$ with doubled origin in the usual Euclidean topology. Then $X \times X$ is $\mathbb{R}^2$ with doubled axes and four origins. Then Δ only contains two origins but other origins are in the closure of Δ.

Definition. Let $f : X \to Y$ be a morphism of schemes, and $\Delta : X \to X \times_Y X$ be the morphism induced by the diagram

$$\begin{tikzcd} X \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd} .$$

We say f is **separated** if Δ is a closed immersion.

Theorem 3.5 (Valuative criterion for separatedness). *Let $f : X \to Y$ be a morphism and X Noetherian. Then f is separated if and only if the following condition holds. For any field k and any valuation ring $R \subseteq k$, that is for any $x \in k$ such that $x \neq 0$ either $x \in R$ or $x^{-1} \in R$, let $T = \operatorname{Spec} R$ and $U = \operatorname{Spec} k$, and $\iota : U \to T$ be the morphism induced by the inclusion $R \hookrightarrow k$. Given a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\iota'"] & Y \end{tikzcd} ,$$

then there exists at most one morphism $\iota' : T \to X$ making the diagram commute.

The intuition is if R is a valuation ring, it has a zero prime ideal and a unique maximal ideal, such that $\overline{\{0\}} = \mathbb{V}(0) = \operatorname{Spec} R = T$ and the maximal ideal is a closed point. Lecture 13 Friday 06/11/20

Remark. We may now define a variety over a field k as a scheme X which is integral, and finite type and separated over $\operatorname{Spec} k$.

Definition. A morphism $f : X \to Y$ is **proper** if it is separated, of finite type, and **universally closed**. That is, for any morphism $Y' \to Y$ the induced projection $X \times_Y Y' \to Y'$ is a closed map, that is the image of a closed set is closed.

Example.

- $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n] \to \operatorname{Spec} k$ is proper.
- $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper. Consider the base-change by $\mathbb{A}^1_k \to \operatorname{Spec} k$. Let

$$\begin{array}{cccccc} p_2 & : & \mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k = \mathbb{A}^2_k = \operatorname{Spec} k[x] \otimes_k k[y] = \operatorname{Spec} k[x, y] & \longrightarrow & \mathbb{A}^1_k = \operatorname{Spec} k[t] \\ & & (x, y) & \longmapsto & y \end{array} .$$

  This is not a closed map. For example, $p_2(\mathbb{V}(xy - 1)) = \mathbb{D}(t)$, which is open and not closed.

Theorem 3.6 (Valuative criterion for properness). *Let $f : X \to Y$ be a finite type morphism with X Noetherian. Then f is proper if as in the criterion for separatedness, whenever given a diagram*

$$\begin{tikzcd} \operatorname{Spec} k = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R = T \arrow[r] \arrow[ur, dashed, "\exists! g"] & Y \end{tikzcd} ,$$

there exists a unique morphism $g : T \to X$ making the diagram commute.

Example. Projective varieties, that is closed subvarieties in $\mathbb{P}^n_k$, are proper over $\operatorname{Spec} k$.

4 Sheaves of $\mathcal{O}_X$-modules

The idea is to go from the notion of an A-module M to the notion of an $\mathcal{O}_X$-module $\mathcal{F}$.

4.1 Sheaves of modules

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf of abelian groups $\mathcal{F}$ on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module, compatible with restriction. That is, if $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, then $s|_V \cdot m|_V = (s \cdot m)|_V$ for $V \subseteq U$. A **morphism of sheaves of $\mathcal{O}_X$-modules** $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups such that for all $U \subseteq X$, $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

- Kernels, cokernels, and images of morphisms of sheaves of $\mathcal{O}_X$-modules are sheaves of $\mathcal{O}_X$-modules.
- $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the group of $\mathcal{O}_X$-module homomorphisms $\{\phi : \mathcal{F} \to \mathcal{G}\}$. This is an $\mathcal{O}_X(X)$-module. Then $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, which is an $\mathcal{O}_X(U)$-module, is a sheaf of $\mathcal{O}_X$-modules, written $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the **sheaf hom**.
- If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, we denote by $F \otimes_{\mathcal{O}_X} \mathcal{G}$ the sheaf associated to the presheaf
$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$
- Push-forwards and pull-backs. For modules, let $\phi : A \to B$ be a homomorphism of rings, let M be a B-module, and let N be an A-module. Then M is also an A-module such that
$$a \cdot m = \phi(a) \cdot m, \qquad a \in A, \qquad m \in M,$$
and $B \otimes_A N$ is a B-module via
$$b \cdot (b' \otimes n) = bb' \otimes n, \qquad b \in B, \qquad b' \otimes n \in B \otimes_A N.$$
Given $f : X \to Y$ a morphism of ringed spaces, so $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then the following holds.
  - $f_* \mathcal{F}$ is naturally a sheaf of $f_* \mathcal{O}_X$-modules, since $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ is an $(f_* \mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$-module, and hence $f_* \mathcal{F}$ is an $\mathcal{O}_Y$-module via $f^\#$.
  - $f^{-1} \mathcal{G}$ is naturally a sheaf of $f^{-1} \mathcal{O}_Y$-modules. But $f^\#$ induces the adjoint map $f^\# : f^{-1} \mathcal{O}_Y \to \mathcal{O}_X$, by question 10 on example sheet 1. Define
$$f^* \mathcal{G} = f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X.$$
This is a sheaf of $\mathcal{O}_X$-modules.

If $S \subseteq A$ is a multiplicatively closed subset, then
$$S^{-1} M = \left\{ \frac{m}{a} \;\middle|\; a \in S, \ m \in M \right\} / \sim,$$
where $m/a \sim m/a'$ if and only if there exists $b \in S$ such that $b(ma' - m'a) = 0$. Also, $S^{-1} M = M \otimes_A S^{-1} A$.

Example. Let $X = \mathrm{Spec}\, A$ be an affine scheme, and let M be an A-module. For $\mathfrak{p} \in \mathrm{Spec}\, A$, we have the localisation $M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$. Define a sheaf $\widetilde{M}$ on $\mathrm{Spec}\, A$ by
$$\widetilde{M}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U, \ s(\mathfrak{p}) \in M_\mathfrak{p}, \\ \forall \mathfrak{p} \in U, \ \exists \mathfrak{p} \in V \subseteq U \text{ open}, \ \exists m \in M, \ \exists s \in A, \ \forall \mathfrak{q} \in V, \ s \notin \mathfrak{q}, \ s(\mathfrak{q}) = \dfrac{m}{s} \end{array} \right\}.$$

Example. $\widetilde{A} = \mathcal{O}_{\mathrm{Spec}\, A}$.

Proposition 4.1.

- $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$.
- $\widetilde{M}(\mathbb{D}(f)) = M_f$.
- $\Gamma\left(\operatorname{Spec} A, \widetilde{M}\right) = M$.

Proof. Exactly as the corresponding statements for $\mathcal{O}_{\operatorname{Spec} A}$. □

4.2 Locally free and coherent modules

Definition. A sheaf of $\mathcal{O}_X$-modules is **free** if it is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ for some index set I. If $\# I = r < \infty$, then we say $\mathcal{F}$ has **rank** r. A sheaf $\mathcal{F}$ is **locally free** of rank r if there exists an open cover $\{U_i\}$ on X such that $\mathcal{F}|_{U_i}$ is free of rank r for each i. Then $\mathcal{F}$ is a **line bundle** if it is rank one. Often more generally, one might refer to a rank r locally free sheaf as a rank r **vector bundle**.

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Remark. One way to define the notion of a vector bundle over a k-scheme X as another scheme E with a morphism $\pi : E \to X$ whose fibres are $\mathbb{A}^r$, and there exists an open cover $\{U_i\}$ such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^r$, and other conditions. We get a sheaf

$$\mathcal{E}(U) = \left\{ s : U \to \pi^{-1}(U) \;\middle|\; \pi \circ s = \mathrm{id}_U \right\}.$$

This gives a locally free sheaf on X. See somewhere in Hartshorne Section II.5 exercises.

Example. Let $E = X \times \mathbb{A}^1$. Then $\mathcal{E}(U) = \mathcal{O}_X(U)$. Giving a morphism $s : U \to U \times_{\operatorname{Spec} k} \mathbb{A}^1_k$ whose composition with $\pi_1 : U \times_{\operatorname{Spec} k} \mathbb{A}^1_k \to U$ is the identity is the same as giving a morphism $f : U \to \mathbb{A}^1_k$, since

$$\begin{tikzcd}
U \arrow[dr, dashed, "s"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_U"'] & & \\
& U \times_{\operatorname{Spec} k} \mathbb{A}^1_k \arrow[r] \arrow[d, "\pi_1"'] & \mathbb{A}^1_k \arrow[d] \\
& U \arrow[r] & \operatorname{Spec} k
\end{tikzcd}.$$

Giving $U \to \mathbb{A}^1_k$ is the same thing as giving a k-algebra homomorphism

$$\begin{array}{ccc} k[x] & \longrightarrow & \mathcal{O}_X(U) \\ x & \longmapsto & \phi \end{array}.$$

The set of such homomorphisms is $\mathcal{O}_X(U)$.

Definition. Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X. We say $\mathcal{F}$ is **quasi-coherent** if X can be covered with affines $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i. We say $\mathcal{F}$ is **coherent** if each M_i can be taken to be finitely generated.

Example. A locally free sheaf is always quasi-coherent and coherent if of finite rank. If $U \subseteq X$ satisfies $\mathcal{F}|_U = \bigoplus_{i \in I} \mathcal{O}_U$, then $\mathcal{F}|_U = \widetilde{\bigoplus_{i \in I} A}$ for $U = \operatorname{Spec} A$.

Kernels, cokernels, images, tensor products, and hom sheaves of quasi-coherent sheaves of $\mathcal{O}_X$-modules are quasi-coherent. This follows since those operations commute with $\widetilde{\cdot}$, such as

$$\ker\left(\widetilde{M_1} \to \widetilde{M_2}\right) = \ker\widetilde{(M_1 \to M_2)}, \quad \widetilde{M_1} \otimes_{\mathcal{O}_X} \widetilde{M_2} = \widetilde{M_1 \otimes_A M_2}, \quad \mathcal{H}om_{\mathcal{O}_X}\left(\widetilde{M_1}, \widetilde{M_2}\right) = \widetilde{\operatorname{Hom}_A(M_1, M_2)}.$$

4.3 Line bundles and the Picard group

Remark. Note that if $\mathcal{L}$ is a line bundle, say with trivialising cover $\{U_i\}$, then we have on $U_i \cap U_j$

$$\phi_{ij} : \mathcal{O}_{U_i}|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{L}|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{O}_{U_j}|_{U_i \cap U_j},$$

using trivialisations on U_i and U_j. Then ϕ_{ij} is an automorphism of $\mathcal{O}_{U_i \cap U_j}$ as an $\mathcal{O}_{U_i \cap U_j}$-module, and as such is given by multiplication by $g_{ij} \in \mathcal{O}_X^* (U_i \cap U_j)$, where $\mathcal{O}_X^*$ is the subsheaf of $\mathcal{O}_X$ consisting of invertible sections of $\mathcal{O}_X$. Note on $U_i \cap U_j \cap U_k$, we have $g_{ij} g_{jk} = g_{ik}$.

Now suppose given $f : Y \to X$ a morphism. How do we think about $f^* \mathcal{L}$? Let $Y_i = f^{-1}(U_i)$ and $f_i : Y_i \to U_i$. Then

$$f_i^* \left(\mathcal{L}|_{U_i} \right) \cong f_i^* \mathcal{O}_{U_i} \cong f_i^{-1} \mathcal{O}_{U_i} \otimes_{f_i^{-1} \mathcal{O}_{U_i}} \mathcal{O}_{Y_i} \cong \mathcal{O}_{Y_i},$$

since $A \otimes_A M \cong M$. Now $(f^* \mathcal{L})|_{Y_i} \cong \mathcal{O}_{Y_i}$. So $\{U_i\}$ pulls back to a trivialising cover for $f^* \mathcal{L}$, so pull-back of a line bundle is a line bundle. Further the transition maps are given by $f^\# (g_{ij})$.

Remark. Push-forward is not as well-behaved. For example, $f_* \mathcal{L}'$ for $\mathcal{L}'$ a line bundle on Y need not be a line bundle. In fact, it will always be quasi-coherent but not necessarily coherent.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on X, with a common trivialising cover $\{U_i\}$ and with transition functions g_{ij} and h_{ij} respectively, then the following holds.

- The transition functions of $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ are $g_{ij} h_{ij}$. Note if $\cdot g : A \to A$ and $\cdot h : A \to A$ are given, then these two homomorphisms induce the homomorphism $\cdot g \otimes \cdot h : A \otimes_A A \to A \otimes_A A$, which is $\cdot gh : A \to A$.
- Set $\mathcal{L}_1^\vee \cong \mathcal{H}om_{\mathcal{O}_X} (\mathcal{L}_1, \mathcal{O}_X)$. This is also a line bundle because on U_i, $\mathcal{L}_1|_{U_i} \cong \mathcal{O}_{U_i}$, and since $\mathrm{Hom}_A (A, A) = A$, $\mathcal{H}om_{\mathcal{O}_{U_i}} (\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) = \mathcal{O}_{U_i}$. The transition maps are given by g_{ij}^{-1}, since $\cdot g_{ij} : \mathcal{O}_{U_i}|_{U_i \cap U_j} \to \mathcal{O}_{U_j}|_{U_i \cap U_j}$ has dual $\cdot \left(g_{ij}^\intercal \right)^{-1} = \cdot g_{ij}^{-1} : \mathcal{O}_{U_i}|_{U_i \cap U_j} \to \mathcal{O}_{U_j}|_{U_i \cap U_j}$.

Note that $\mathcal{L}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_1$ has transition maps $g_{ij}^{-1} g_{ij} = 1$. Thus

$$\mathcal{L}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_1 \cong \mathcal{O}_X.$$

Definition. Let X be a scheme. Define $\mathrm{Pic}\, X$, the **Picard group** of X, to be the set of isomorphism classes of line bundles on X. This is a group with product law

$$\mathcal{L}_1 \cdot \mathcal{L}_2 = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2, \qquad \mathcal{L}^{-1} = \mathcal{L}^\vee = \mathcal{H}om (\mathcal{L}, \mathcal{O}_X).$$

4.4 Morphisms to projective space

Why are line bundles important? Fix a base scheme $\mathrm{Spec}\, k$. Let $\mathbb{P}_k^n = \mathrm{Proj}\, k [x_1, \dots, x_n]$. Denote by **Sch**/$k$ the category of schemes over k. Let F be the functor

$$\begin{array}{rcl} \mathbf{Sch}/k & \longrightarrow & \mathbf{Set} \\ T & \longmapsto & \left\{ \text{surjections } \mathcal{O}_T^{\oplus(n+1)} \twoheadrightarrow \mathcal{L} \text{ for } \mathcal{L} \text{ a line bundle on } T \right\} / \cong \end{array},$$

where $\phi_1 : \mathcal{O}_T^{\oplus(n+1)} \to \mathcal{L}$, and $\phi_2 : \mathcal{O}_T^{\oplus(n+1)} \to \mathcal{L}_2$ are isomorphic if there exists an isomorphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ of $\mathcal{O}_X$-modules making

$$\begin{tikzcd} \mathcal{L}_1 \arrow[rr, "f"] & & \mathcal{L}_2 \\ & \mathcal{O}_T^{\oplus(n+1)} \arrow[ul, "\phi_1"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

commute. Given $f : T_1 \to T_2$ a morphism in **Sch**/k, we get a map of **Set**

$$\begin{array}{ccc} F(T_2) & \longrightarrow & F(T_1) \\ \left(\phi : \mathcal{O}_{T_1}^{\oplus(n+1)} \twoheadrightarrow \mathcal{L} \right) & \longmapsto & \left(f^* \phi : f^* \mathcal{O}_{T_2}^{\oplus(n+1)} = \mathcal{O}_{T_1}^{\oplus(n+1)} \twoheadrightarrow f^* \mathcal{L} \right) \end{array}.$$

This is a surjection by right exactness of tensor products.

Theorem 4.2. *F is represented by $\mathbb{P}^n_k$. That is, $F \cong \mathrm{h}_{\mathbb{P}^n_k}$.*

Remark. This is an example of a **Quot scheme**, which is a scheme which represents a functor of the form $T \mapsto \left\{\mathcal{O}_T^{\oplus k} \twoheadrightarrow \mathcal{E}\right\}$, where $\mathcal{E}$ is a coherent sheaf satisfying some properties.

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Proof. If the statement holds, then there is a **universal object**. That is, an element of $F(\mathbb{P}^n)$ corresponding to the identity $\mathrm{id}_{\mathbb{P}^n} \in \mathrm{h}_{\mathbb{P}^n}(\mathbb{P}^n)$, that is a surjective map $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$. Further, following the proof of Yoneda's lemma, given $f : X \to \mathbb{P}^n$ and $T : \mathrm{h}_{\mathbb{P}^n} \to F$ the natural transformation giving the natural isomorphism of functors, we get a commutative diagram

$$\begin{tikzcd} \mathrm{id}_{\mathbb{P}^n} \in \mathrm{h}_{\mathbb{P}^n}(\mathbb{P}^n) \arrow[r, "T(\mathbb{P}^n)"] \arrow[d, "\mathrm{h}_{\mathbb{P}^n}(f)"'] & F(\mathbb{P}^n) \ni \left(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{L}\right) \arrow[d, "F(f)"] \\ f \in \mathrm{h}_{\mathbb{P}^n}(X) \arrow[r, "T(X)"'] & F(X) \ni \left(\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{f^*\phi} f^*\mathcal{L}\right) \end{tikzcd}.$$

That is, the element $T(X)(f)$ is precisely $f^*\phi : \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{L}$. So the representing scheme $\mathbb{P}^n$ comes with the universal object $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$. So we will construct the universal object. The line bundle we construct has a name, $\mathcal{O}_{\mathbb{P}^n}(1)$.

- If $S = k[x_0, \dots, x_n]$, then $\mathbb{P}^n = \operatorname{Proj} S$ has an open cover
$$\mathcal{U} = \{\mathbb{D}_+(x_i) \mid 0 \le i \le n\}, \qquad \mathbb{D}_+(x_i) = \{\mathfrak{p} \in \operatorname{Proj} S \mid x_i \in \mathfrak{p}\}.$$
We will take $\mathcal{U}$ to be a trivialising cover for $\mathcal{O}_{\mathbb{P}^n}(1)$, with the transition map given by
$$g_{ij} = \frac{x_i}{x_j} = \frac{x_i^2}{x_i x_j} \in \mathcal{O}_{\mathbb{P}^n}^*(\mathbb{D}_+(x_i) \cap \mathbb{D}_+(x_j)) = \mathcal{O}_{\mathbb{P}^n}^*(\mathbb{D}_+(x_i x_j)) = S_{(x_i x_j)}^*,$$
so $g_{ji} = x_j/x_i = x_j^2/x_i x_j$ and $g_{ij} g_{jk} = (x_i/x_j)(x_j/x_k) = x_i/x_k = g_{ik}$. Have a morphism defined in $\mathbb{D}_+(x_i)$ by
$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1) \\ e_j & \longmapsto & \dfrac{x_j}{x_i} \end{array}, \qquad e_j = (0, \dots, 0, 1, 0, \dots, 0),$$
using the trivialisation of $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{D}_+(x_i)$. That is, we have an isomorphism $\mathcal{O}_{\mathbb{P}^n}(1)|_{\mathbb{D}_+(x_i)} \cong \mathcal{O}_{\mathbb{D}_+(x_i)} \ni x_j/x_i$. Well-defined globally, since
$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}\big|_{\mathbb{D}_+(x_i x_k)} \arrow[dl, "e_j \mapsto \frac{x_j}{x_i}"'] \arrow[dr, "e_j \mapsto \frac{x_j}{x_k}"] & \\ \mathcal{O}_{\mathbb{D}_+(x_i)}\big|_{\mathbb{D}_+(x_i x_k)} \arrow[rr, "\cdot g_{ik}"'] & & \mathcal{O}_{\mathbb{D}_+(x_k)}\big|_{\mathbb{D}_+(x_i x_k)} \end{tikzcd},$$
but $g_{ik}(x_j/x_i) = (x_i/x_k)(x_j/x_i) = x_j/x_k$. Note in particular each e_j maps to a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$. We now have a morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)$, and need to check surjective. On $\mathbb{D}_+(x_i)$, $e_i \mapsto x_i/x_i = 1 \in \Gamma(\mathbb{D}_+(x_i), \mathcal{O}_{\mathbb{P}^n}) = S_{(x_i)}$ so in particular, looking at sections over $\mathbb{D}_+(x_i)$, we get a homomorphism of $S_{(x_i)}$-modules
$$\begin{array}{ccc} S_{(x_i)}^{\oplus(n+1)} & \longrightarrow & S_{(x_i)} \\ e_i & \longmapsto & 1 \end{array},$$
so clearly a surjective map of modules. Thus $\left(\psi : \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}(1)\right) \in F(\mathbb{P}^n)$.

- It remains to show that given X and $\left(\phi : \mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}\right) \in F(X)$, we need that there exists a unique morphism $f : X \to \mathbb{P}^n$ such that
$$\left(\phi : \mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}\right) \cong \left(f^*\psi : \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{O}_{\mathbb{P}^n}(1)\right).$$
Indeed, this will give the natural transformation $F \to \mathrm{h}_{\mathbb{P}^n}$, and the inverse natural transformation $\mathrm{h}_{\mathbb{P}^n} \to F$ is given by pull-back. That is, $f : X \to \mathbb{P}^n$ gives $f^*\psi : \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{O}_{\mathbb{P}^n}(1)$.
  - Let $\phi(e_i) = s_i \in \Gamma(X, \mathcal{L})$. Define
$$Z_i = \{x \in X \mid (s_i)_x \in \mathfrak{m}_x\mathcal{L}_x\}, \qquad \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}.$$
Claim that this is a closed set. This can be checked on an open cover $\{U_i\}$, since $Z \subseteq X$ is closed if and only if $Z \cap U_i$ is closed in U_i for all i. Thus we may use a trivialising affine cover $\{U_i\}$ of X. So we reduce to the case that $X = \mathrm{Spec}\, A$ and $\mathcal{L} \cong \mathcal{O}_{\mathrm{Spec}\, A}$, so $\Gamma(X, \mathcal{L}) \cong A$ so $s_i \in A$ induces $(s_i)_{\mathfrak{p}} = s_i/1 \in A_{\mathfrak{p}}$. Now $s_i/1 \in \mathfrak{m}_{\mathfrak{p}}A_{\mathfrak{p}}$ if and only if s_i lies in the inverse image $\mathfrak{p}$ of $\mathfrak{m}_{\mathfrak{p}}A_{\mathfrak{p}}$ under the localisation map $A \to A_{\mathfrak{p}}$. Thus $Z_i = \mathbb{V}(s_i)$, a closed set. Let
$$U_i = X \setminus Z_i.$$
Then there is an isomorphism [17]
$$\begin{array}{ccc} \mathcal{O}_{U_i} & \longleftrightarrow & \mathcal{L}|_{U_i} \\ 1 & \longmapsto & s_i \\ \dfrac{s}{s_i} & \longleftarrow & s \end{array}.$$
Interpret s/s_i as the element of $\mathcal{O}_{U_i}$ such that $(s/s_i)\, s_i = s$.
  - We may now define a morphism $f_i : U_i = X \setminus Z_i \to \mathbb{D}_+(x_i) = \mathrm{Spec}\, S_{(x_i)}$ by giving a homomorphism by Lecture 16 Friday 13/11/20
$$\begin{array}{cccc} f_i^\# & : & S_{(x_i)} = k\left[\dfrac{x_0}{x_i}, \dots, \dfrac{x_n}{x_i}\right] & \longrightarrow & \Gamma(U_i, \mathcal{O}_X) \\ & & \dfrac{x_j}{x_i} & \longmapsto & \dfrac{s_j}{s_i} \end{array},$$
defining $f_i^\#$ as a k-algebra homomorphism. To get a morphism $f : X \to \mathbb{P}^n$ such that $f|_{U_i} = f_i$, we need to check $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Check that
$$\begin{array}{ccccc} f_i^\#\Big|_{U_i \cap U_j} & : & \Gamma(\mathbb{D}_+(x_i) \cap \mathbb{D}_+(x_j), \mathcal{O}_{\mathbb{P}^n}) = S_{(x_ix_j)} & \longrightarrow & \Gamma(U_i \cap U_j, \mathcal{O}_X) \\ & & \dfrac{x_k}{x_i} & \longmapsto & \dfrac{s_k}{s_i} \\ & & \dfrac{x_k}{x_j} = \dfrac{\frac{x_k}{x_i}}{\frac{x_j}{x_i}} & \longmapsto & \dfrac{\frac{s_k}{s_i}}{\frac{s_j}{s_i}} = \dfrac{s_k}{s_j} \end{array},$$
$$\begin{array}{ccccc} f_j^\#\Big|_{U_i \cap U_j} & : & \Gamma(\mathbb{D}_+(x_i) \cap \mathbb{D}_+(x_j), \mathcal{O}_{\mathbb{P}^n}) = S_{(x_ix_j)} & \longrightarrow & \Gamma(U_i \cap U_j, \mathcal{O}_X) \\ & & \dfrac{x_k}{x_j} & \longmapsto & \dfrac{s_k}{s_j} \\ & & \dfrac{x_k}{x_i} = \dfrac{\frac{x_k}{x_j}}{\frac{x_i}{x_j}} & \longmapsto & \dfrac{\frac{s_k}{s_j}}{\frac{s_i}{s_j}} = \dfrac{s_k}{s_i} \end{array}.$$
So $f_i^\#\Big|_{U_i \cap U_j} = f_j^\#\Big|_{U_i \cap U_j}$, so $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so the morphisms glue to give $f : X \to \mathbb{P}^n$. Further, $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}$, because the transition maps $g_{ij} = x_i/x_j$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ pull back under $f^\#$ to s_i/s_j, which are the transition maps for $\mathcal{L}$ using trivialisations for $\mathcal{L}|_{U_i}$ which we used above.

[17] Exercise: check on stalks

– For uniqueness, suppose given a surjection $\mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$ and a morphism $g : X \to \mathbb{P}^n$ such that

$$g^* \left(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)\right) \cong \left(\phi : \mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}\right).$$

We may think of ϕ as given by $n+1$ sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ with $s_i = \phi(e_i)$. Similarly the universal object on $\mathbb{P}^n$ is given by sections $x_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Note by the construction of the universal object, the section x_j is given on $\mathbb{D}_+(x_i)$ by $x_j/x_i \in S_{(x_i)}$. If $f : X \to Y$ and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, then $s \in \Gamma(Y, \mathcal{F})$ induces a section (Y, s) in $\Gamma\left(X, f^{-1}\mathcal{F}\right)$, and hence a section

$$f^* s = (Y, s) \otimes 1 \in \Gamma\left(X, f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_X\right) = \Gamma(X, f^*\mathcal{F}).$$

In particular, pull-back of the section $x_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is s_i. That is, $g^* x_i = s_i$. In particular, $(s_i)_x \in \mathfrak{m}_x \mathcal{L}_x$ for some $x \in X$ if and only if $(x_i)_{g(x)} \in \mathfrak{m}_{g(x)} \mathcal{O}_{\mathbb{P}^n}(1)_{g(x)}$. Thus $U_i = \{x \in X \mid (s_i)_x \notin \mathfrak{m}_x \mathcal{L}_x\}$ satisfies $U_i = g^{-1}(\mathbb{D}_+(x_i))$. So we have $g_i = g|_{U_i} : U_i \to \mathbb{D}_+(x_i)$ and it is enough to show $g_i = f_i$, where f_i was constructed previously from $\mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$. So it is enough to check $g_i^\# = f_i^\#$, and

$$g_i^\# \left(\frac{x_j}{x_i}\right) = \frac{g^* x_j}{g^* x_i} = \frac{s_j}{s_i} = f_i^\# \left(\frac{x_j}{x_i}\right).$$

Hence uniqueness.

□

Remark.

- If instead I had chosen $g_{ij} = x_j/x_i$, we would have obtained the line bundle

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n}(1)^\vee,$$

and $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$.

- If we were working in the world of varieties, locally the section s_i is viewed as a function and Z_i is the locus where s_i vanishes. On U_i, we define a morphism to projective space

$$\begin{array}{rcl} U_i & \longrightarrow & \mathbb{D}_+(x_i) \subseteq \mathbb{P}^n \\ p & \longmapsto & \left(\dfrac{s_0(p)}{s_i(p)}, \dots, \dfrac{s_n(p)}{s_i(p)}\right) \end{array}.$$

Equivalently, on X, we can view this function as

$$\begin{array}{rcl} X & \longrightarrow & \mathbb{P}^n \\ p & \longmapsto & (s_0(p), \dots, s_n(p)) \end{array}.$$

5 Divisors

Weil divisors are codimension one subvarieties and Cartier divisors are subschemes defined by a single equation.

5.1 Weil divisors

Recall the following.

Definition. The **dimension** of a topological space X is the length n of the longest chain $Z_0 \subsetneq \dots \subsetneq Z_n$ of irreducible closed subsets of X.

Example. $\dim \mathbb{A}_k^1 = 1$, since $\{\text{point}\} \subseteq \mathbb{A}_k^1$.

Definition. The **Krull dimension** of a ring A is $\dim A = \dim \operatorname{Spec} A$, which is the length of the longest chain $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ of prime ideals of A.

Definition. If $Z \subseteq X$ is an irreducible closed subset, then $\operatorname{codim}(Z, X)$ is the length n of the longest chain $Z = Z_0 \subsetneq \dots \subseteq Z_n$ of irreducible closed subsets.

Remark. Intuition on dimension may be faulty, even for Noetherian affine schemes. However, if B is a domain and a finitely generated k-algebra for k a field, then for any $\mathfrak{p} \subseteq B$,

$$\operatorname{Ht} \mathfrak{p} + \dim B/\mathfrak{p} = \dim B. \tag{1}$$

Here $\operatorname{Ht} \mathfrak{p}$ is the length n of the longest chain of primes $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. Write $\dim B/\mathfrak{p} = \dim \mathbb{V}(\mathfrak{p})$ and $\operatorname{Ht} \mathfrak{p} = \operatorname{codim}(\mathbb{V}(\mathfrak{p}), \operatorname{Spec} B)$, so we have from (1) that

$$\operatorname{codim}(\mathbb{V}(\mathfrak{p}), \operatorname{Spec} B) + \dim \mathbb{V}(\mathfrak{p}) = \dim \operatorname{Spec} B.$$

This implies that if X is a variety over k, so integral and finite type over k, and $Z \subseteq X$ an irreducible closed subset, that $\dim Z + \operatorname{codim}(Z, X) = \dim X$. Also if $\eta \in Z \subseteq X$ is the generic point of Z, then $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Z, X)$, by example sheet 3.

Proposition 5.1. *If X is a Noetherian scheme, then X is a Noetherian topological space, that is every decreasing sequence of closed sets is stationary, and every closed subset of X has a decomposition into a finite number of irreducible closed subsets.*

Proof. Exercise. [18] □

Assumption 5.2. X is a Noetherian integral scheme over $\operatorname{Spec} k$ which is **regular in codimension one**. That is, whenever a local ring $\mathcal{O}_{X,x}$ is of dimension one, it is **regular**, that is $\dim_{\mathcal{O}_{X,x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$. That is, the dimension of the Zariski tangent space to X at x coincides with $\dim \mathcal{O}_{X,x}$.

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Remark. Regularity measures non-singularity, so we tend to say a scheme X all of whose local rings are regular is **regular** or **non-singular**.

Example. If X is a non-singular curve then X is regular in codimension one, but $y^2 = x^2(x-1)$ is not regular at the origin since the Zariski tangent space at the origin is two-dimensional.

Remark. Standard commutative algebra fact in Atiyah-Macdonald. A regular Noetherian local domain A of dimension one is a **discrete valuation ring**. That is, if K is the field of fractions of A, then there is a group homomorphism $\nu : K^* \to \mathbb{Z}$, where K^* is the multiplicative group of K, such that

$$A = \{x \in K^* \mid \nu(x) \ge 0\} \cup \{0\},$$

and the maximal ideal of A is

$$\mathfrak{m} = \{x \in K^* \mid \nu(x) > 0\} \cup \{0\}.$$

Note that after rescaling ν so that $\nu(\mathfrak{m} \setminus \mathfrak{m}^2) = 1$, then $\nu(x) = k$ if $x \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$.

Definition. Assume Assumption 5.2 holds. Then a **prime divisor** on X is a closed subvariety, that is an irreducible and reduced, equivalently integral, closed subscheme of X, of codimension one. Let $\operatorname{Div} X$ be the free abelian group generated by prime divisors.

[18] Exercise

Let $\mathrm{K}(X)$ be the function field of X. See example sheet 2, question 7. Note $\mathrm{K}(X)$ is the field of fractions of A whenever $\operatorname{Spec} A \subseteq X$ is an open affine subset. For $Y \subseteq X$ a prime divisor, let $\eta \in Y$ be its generic point. Then $\dim \mathcal{O}_{X,\eta} = 1$, as follows from $\operatorname{codim}(Y, X) = 1$, and hence have valuation $\nu_Y : \mathrm{K}(X)^* \to \mathbb{Z}$, where $\mathrm{K}(X)$ is the field of fractions of $\mathcal{O}_{X,\eta}$, such that

$$\mathcal{O}_{X,\eta} = \left\{ f \in \mathrm{K}(X)^* \mid \nu_Y(f) \geq 0 \right\} \cup \{0\}.$$

May assume $\nu_Y\left(\mathfrak{m}_\eta \setminus \mathfrak{m}_\eta^2\right) = 1$.

Example. Let $X = \mathbb{A}_k^1 = \operatorname{Spec} k[x]$, and let $\mathfrak{p} = \langle x - a \rangle \subseteq k[x]$. Then $\mathcal{O}_{X,\mathfrak{p}} = k[x]_{\langle x-a \rangle}$ and $\mathrm{K}(X) = k(x)$. Given $f/g \in \mathrm{K}(X)$ non-zero, we may write $f/g = (p/q)(x-a)^k$ such that $\gcd(p, x-a) = \gcd(q, x-a) = 1$. Then the valuation $\nu_\mathfrak{p}(f/g) = k$ is the order of the zero or pole of f/g at zero, and

$$\mathcal{O}_{X,\mathfrak{p}} = \left\{ \frac{f}{g} \in \mathrm{K}(X)^* \;\middle|\; \nu_\mathfrak{p}\left(\frac{f}{g}\right) \geq 0 \right\} \cup \{0\}.$$

5.2 Class group of Weil divisors

Lemma 5.3. *With X satisfying Assumption 5.2, if $f \in \mathrm{K}(X)^*$, then $\nu_Y(f) = 0$ for all but a finite number of prime divisors Y.*

Proof. We can find an open affine subset $U = \operatorname{Spec} A$ of X such that $f \in \Gamma(U, \mathcal{O}_X)$. For example, first pass to an open affine $\operatorname{Spec} B$ on which we can write $f = a/s$ for $a \in B$ and $s \neq 0$, and then $f \in B_s$, so we may take $U = \mathbb{D}(s) \subseteq \operatorname{Spec} B$. Then $Z = X \setminus U$ is a proper closed subset of X. Since X is Noetherian, so is Z as a topological space and hence decomposes into a finite union of irreducible closed subsets. Thus Z contains only a finite number of prime divisors. So enough to check the statement on U, since any other prime divisor intersects U, and its generic point η is contained in U, since if $\eta \notin U$ then $\overline{\{\eta\}} \cap U = \emptyset$ as U is open. Thus we may assume $X = \operatorname{Spec} A$ is affine and $f \in A$. Thus $\nu_Y(f) \geq 0$ for all Y prime divisors in X and $\nu_Y(f) > 0$ if and only if $f/1 \in \mathfrak{m}_\eta \subseteq \mathcal{O}_{X,\eta}$ where η is the generic point of Y, if and only if $f \in \mathfrak{p}$ where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to η, if and only if $\mathfrak{p} \in \mathbb{V}(f)$, if and only if $Y \subseteq \mathbb{V}(f)$. Note $\mathbb{V}(f)$ is a proper closed subset of X since $f \neq 0$. Thus $\mathbb{V}(f)$ decomposes into a finite number of irreducible components, none of which are X, and hence at most a finite number of prime divisors contained in $\mathbb{V}(f)$. ☐

Definition. Let X satisfy Assumption 5.2, and $f \in \mathrm{K}(X)^*$. Then a **divisor of zeros and poles** of f, denoted as (f), is

$$(f) = \sum_{Y \subseteq X \text{ prime divisor}} \nu_Y(f) Y \in \operatorname{Div} X.$$

By Lemma 5.3, this makes sense. Note

$$\begin{array}{rcl} \mathrm{K}(X)^* & \longrightarrow & \operatorname{Div} X \\ f & \longmapsto & (f) \end{array}$$

is a group homomorphism as ν_Y is.

Definition. The **class group** of X, written as $\operatorname{Cl} X$, is the cokernel of the homomorphism $\mathrm{K}(X)^* \to \operatorname{Div} X$. Two divisors $D, D' \in \operatorname{Div} X$ are **linearly equivalent** if there exists $f \in \mathrm{K}(X)^*$ such that $(f) = D - D'$. We write $D \sim D'$. If $D \sim 0$, that is $D = (f)$ for some f, we say D is a **principal divisor**. So $\operatorname{Cl} X$ is the group of divisors modulo linear equivalence.

Remark. If $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of algebraic integers in a finite field extension $K/\mathbb{Q}$, then $\operatorname{Cl} \operatorname{Spec} \mathcal{O}_K = \operatorname{Cl} \mathcal{O}_K$ as defined in any algebraic number theory course.

Proposition 5.4. *If A is an integrally closed Noetherian domain, then*

$$A = \bigcap_{\operatorname{Ht} \mathfrak{p} = 1,\ \mathfrak{p} \subseteq A \text{ prime}} A_\mathfrak{p} \subseteq A_{\langle 0 \rangle}.$$

Proof. Matsumura, Commutative algebra, Theorem 38, Page 124. ☐

Theorem 5.5. *Let A be a Noetherian integral domain. Then A is a UFD if and only if $X = \operatorname{Spec} A$ is normal, that is A is integrally closed in its field of fractions, and $\operatorname{Cl} X = 0$.*

Proof. A UFD is integrally closed in its field of fractions. Also, A is a UFD if and only if every prime ideal of height one of A is principal. Thus we need to show that if A is an integrally closed domain, we have the equivalence that every height one prime of A is principal if and only if $\operatorname{Cl} \operatorname{Spec} A = 0$.

$\implies$ Given a prime divisor $Y \subseteq X$, Y corresponds to a height one prime $\mathfrak{p} \subseteq A$ and $\mathfrak{p} = \langle f \rangle$ for some $f \in A \setminus \{0\}$. Then $(f) = Y$, so every divisor is principal.

$\impliedby$ Suppose $\operatorname{Cl} X = 0$, $\mathfrak{p} \subseteq A$ is a prime of height one, and $Y = \mathbb{V}(\mathfrak{p})$. Then there exists $f \in \mathrm{K}(X)^* = A_{\langle 0 \rangle}^*$ such that $(f) = Y$. Since $\nu_Y(f) = 1$, $f \in A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$ and f generates the maximal ideal $\mathfrak{p} A_{\mathfrak{p}}$, since in a discrete valuation ring every element of $\mathfrak{m} \setminus \mathfrak{m}^2$ generates $\mathfrak{m}$. If $\mathfrak{p}' \subseteq A$ is any other height one prime, and $Y' = \mathbb{V}(\mathfrak{p}')$, then $\nu_{Y'}(f) = 0$, so $f \in A_{\mathfrak{p}'}$ is a unit. Now apply Proposition 5.4. Thus $f \in A$ and $f \in A \cap \mathfrak{p} A_{\mathfrak{p}} = \mathfrak{p}$. If we show f generates $\mathfrak{p}$, we will be done. Let g be any other element of $\mathfrak{p}$. Then $\nu_Y(g) \geq 1$ and $\nu_{Y'}(g) \geq 0$ for all $Y' \neq Y$ so $\nu_{Y'}(g/f) = \nu_{Y'}(g) - \nu_{Y'}(f) \geq 0$ for all Y'. Thus $g/f \in A$. Thus $g = (g/f) f \in \langle f \rangle$ so $\mathfrak{p} = \langle f \rangle$.

$\square$

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Proposition 5.6. *Let X satisfy Assumption 5.2, $Z \subseteq X$ a proper closed subset, and $U = X \setminus Z$ an open subscheme of X. Then*

1. *there exists a surjective homomorphism*

$$\begin{array}{ccc} \operatorname{Cl} X & \longrightarrow & \operatorname{Cl} U \\ \sum_i n_i Y_i & \longmapsto & \sum_i n_i (Y_i \cap U) \end{array},$$

interpreting as zero if $Y_i \cap U = \emptyset$,

2. *if $\operatorname{codim}(Z, X) \geq 2$, then this homomorphism is an isomorphism, and*
3. *if Z is irreducible of codimension one, then we have an exact sequence*

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \to \operatorname{Cl} U \to 0.$$

Proof.

1. Y being a prime divisor of X implies $Y \cap U$ is either a prime divisor of U or is empty. If $f \in \mathrm{K}(X)^*$, and $(f) = \sum_i n_i Y_i$, then the image of (f) is $\sum_i n_i (Y_i \cap U)$, and this coincides with $(f|_U)$. The main point is $\mathrm{K}(X) = \mathrm{K}(U)$. Thus $\operatorname{Cl} X \to \operatorname{Cl} U$ is well-defined. Surjective since if $Y \subseteq U$ is a prime divisor, then $\overline{Y} \subseteq X$ is a prime divisor of X with $Y = \overline{Y} \cap U$.
2. $\operatorname{Div} X$ and $\operatorname{Cl} X$ only depend on codimension one subvarieties, so obvious.
3. $\ker(\operatorname{Cl} X \to \operatorname{Cl} U)$ consists only of divisors supported on Z. If Z is irreducible of codimension one, there is precisely one such prime divisor, so $\ker(\operatorname{Cl} X \to \operatorname{Cl} U)$ is generated by $[Z]$.

$\square$

Proposition 5.7. $\operatorname{Cl} \mathbb{P}_k^n \cong \mathbb{Z}$, *generated by the class of a hyperplane $H = \mathbb{V}(x_i)$.*

Proof. As $\mathbb{P}^n \setminus H = \mathbb{D}_+(x_i) \cong \mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$ and $k[x_1, \dots, x_n]$ is a UFD, hence $\operatorname{Cl} \mathbb{A}^n = 0$. So we have an exact sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto [H]} \operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} \mathbb{A}^n = 0.$$

Thus $\operatorname{Cl} \mathbb{P}^n$ is generated by $[H]$. Now

$$\mathrm{K}(\mathbb{P}^n) = k[x_0, \dots, x_n]_{\langle 0 \rangle} = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[x_0, \dots, x_n] \text{ are homogeneous of the same degree, } g \neq 0 \right\} / \sim .$$

Thus if $dH \sim 0$, we would need a rational function f/g such that $(f/g) = dH$, and this is only possible if $d = 0$. More precisely, $(f/g) = Y_1 - Y_2$ where Y_1 and Y_2 are sums of hypersurfaces with the same total degree. ☐

Remark. If X is a projective non-singular curve, then $\mathrm{Cl}\, X$ was defined in Part II.

5.3 Cartier divisors and relation with Weil divisors

Definition. Let X be a scheme. We define the **sheaf of rational functions** on X, $\mathcal{K}_X$, to be the sheaf associated with the presheaf

$$U \mapsto \mathrm{S}(U)^{-1} \Gamma(U, \mathcal{O}_X),$$

where $\mathrm{S}(U) \subseteq \Gamma(U, \mathcal{O}_X)$ is the subset of elements whose stalks in $\mathcal{O}_{X,x}$ for each $x \in U$ are non-zero divisors.

Example. If X is integral, then $\mathrm{S}(U) \subseteq \Gamma(U, \mathcal{O}_X)$ consists of non-zero elements of $\Gamma(U, \mathcal{O}_X)$. Then $\mathcal{K}_X$ is the constant sheaf $U \mapsto \mathrm{K}(X)$.

Definition. Let $\mathcal{K}_X^* \subseteq \mathcal{K}_X$ be the sheaf of invertible elements of $\mathcal{K}_X$. Then there is an inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{K}_X^*$.[19] A **Cartier divisor** on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. A Cartier divisor is **principal** if it is in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Two divisors are **linearly equivalent** if their difference is principal. Note additive language for divisors. We write $\mathrm{Ca}\,\mathrm{Cl}\, X$, the **Cartier class group** of X, to be the Cartier divisors modulo principal divisors. That is, $\mathrm{Ca}\,\mathrm{Cl}\, X = \mathrm{coker}(\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*))$.

Remark. Note that an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ can be represented by $\{(U_i, f_i)\}$ where $\{U_i\}$ is some open cover of X and $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ and on $U_i \cap U_j$, we have $f_i|_{U_i \cap U_j} / f_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.

Proposition 5.8. *Let X satisfy Assumption 5.2. Then there exists a homomorphism $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \mathrm{Div}\, X$ descending to $\mathrm{Ca}\,\mathrm{Cl}\, X \to \mathrm{Cl}\, X$.*

Proof. Indeed, given $\{(U_i, f_i)\}$ as in the remark, and Y a prime divisor on X, associate a coefficient n_Y to Y by choosing some U_i such that $Y \cap U_i \neq \emptyset$, and setting $n_Y = \nu_Y(f_i)$. This is well-defined. If $Y \cap U_j \neq \emptyset$, then $Y \cap U_i \cap U_j \neq \emptyset$, as $U_i \cap Y$ is dense in Y, being irreducible. Then

$$\nu_Y(f_j) = \nu_Y\left(f_i\left(\frac{f_j}{f_i}\right)\right) = \nu_Y(f_i) + \nu_Y\left(\frac{f_j}{f_i}\right) = \nu_Y(f_i),$$

since f_j/f_i is invertible on $U_i \cap U_j$, hence has no zeros or poles. Now take the Cartier divisor $\{(U_i, f_i)\}$ to $\sum_Y n_Y Y$. You should check this is independent of the choice of representative $\{(U_i, f_i)\}$.[20] Note also we can always assume the cover $\{U_i\}$ is finite since X is Noetherian by Assumption 5.2 and hence is quasi-compact. Note also a principal divisor coming from $f \in \Gamma(X, \mathcal{K}_X^*)$ is represented by (X, f). Then this is mapped to (f) by construction. ☐

Proposition 5.9. *If X satisfies Assumption 5.2, and all local rings of X are UFD's, then the above map $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \mathrm{Div}\, X$ is an isomorphism.*

Remark. If X is a **non-singular variety**, that is all local rings of X are regular, then the hypotheses are satisfied as all regular local rings are UFD's, a non-trivial theorem in commutative algebra.

Definition. If all local rings of X are UFD's, we say X is **locally factorial**.

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Proof. Need to define the inverse map $\mathrm{Div}\, X \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Let $x \in X$ be any point. Then we get a morphism $\mathrm{Spec}\, \mathcal{O}_{X,x} \to X$. For example, if $x \in \mathrm{Spec}\, A \subseteq X$ is open affine, $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ where $\mathfrak{p}$ corresponds to x and then $A \to A_{\mathfrak{p}}$ induces the morphism $\mathrm{Spec}\, \mathcal{O}_{X,x} \to \mathrm{Spec}\, A \hookrightarrow X$. A prime divisor on X pulls back to a prime divisor on $\mathrm{Spec}\, \mathcal{O}_{X,x}$ by taking inverse images. More precisely, given $Y \subseteq X$ a prime divisor, if $x \notin Y$ then pull-back is empty, otherwise $\mathrm{Spec}\, A \cap Y$ is non-empty and is of the form $\mathbb{V}(\mathfrak{q})$ for $\mathfrak{q} \subseteq A$ a prime ideal with $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$, hence a prime divisor $\mathbb{V}(\mathfrak{q}A_{\mathfrak{p}})$ of $\mathrm{Spec}\, A_{\mathfrak{p}}$. This gives a map

$$\begin{array}{ccc} \mathrm{Div}\, X & \longrightarrow & \mathrm{Div}\,\mathrm{Spec}\, \mathcal{O}_{X,x} \\ D & \longmapsto & D_x \end{array}.$$

[19] Exercise: check at presheaf level, that is check $\Gamma(U, \mathcal{O}_X^*) \to \mathrm{S}(U)^{-1} \Gamma(U, \mathcal{O}_X)$ is injective

[20] Exercise

Since $\mathcal{O}_{X,x}$ is a UFD, D_x is a principal divisor on $\operatorname{Spec}\mathcal{O}_{X,x}$. That is, $D_x = (f_x)$ for $f_x \in \mathrm{K}(X)^*$, on $\operatorname{Spec}\mathcal{O}_{X,x}$. Thus D and (f_x) on X differ only in prime divisors which do not contain x. Thus if U_x is the complement of the union of prime divisors of X at which D and (f_x) have different coefficients, then $D|_{U_x} = (f_x)|_{U_x}$. Do this for every point x, and then represent a Cartier divisor by $\{(U_x, f_x)\}$. On $U_x \cap U_y$, (f_x) and (f_y) agree, as both agree with $D|_{U_x \cap U_y}$, so $(f_x/f_y) = 0$ on $U_x \cap U_y$, so f_x/f_y is invertible in $\mathcal{O}_{X,\mathfrak{p}}$ for all $\mathfrak{p} \in U_x \cap U_y$ points of height one. That is, generic points of prime divisors. If we cover $U_x \cap U_y$ with open affines $\operatorname{Spec} A$, this says that $f_x/f_y \in A_{\mathfrak{p}}^*$ for all $\mathfrak{p} \subseteq A$ primes of height one. Now since all $A_{\mathfrak{q}}$'s are UFD's, for all $\mathfrak{q} \subseteq A$ primes, $A_{\mathfrak{q}}$ is integrally closed. Thus A is integrally closed, see for example Atiyah-Macdonald, Proposition 5.13. Thus $A = \bigcap_{\mathfrak{p} \subseteq A,\ \mathrm{Ht}\,\mathfrak{p}=1} A_{\mathfrak{p}}$, so $f_x/f_y \in A^*$, so $f_x/f_y \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. Thus $\{(U_x, f_x)\}$ represents a section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. That is, a Cartier divisor. This gives the inverse map. ☐

5.4 Correspondence between Cartier divisors and line bundles

Definition. Let D be a Cartier divisor on X represented by $\{(U_i, f_i)\}$. Define $\mathcal{O}_X(D)$ to be the subsheaf of $\mathcal{O}_X$-modules of $\mathcal{K}_X$ generated by f_i^{-1} on U_i.

Note that as f_i/f_j is invertible on $U_i \cap U_j$, f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_{U_i \cap U_j}$-module. This is a line bundle.

Remark. The transition maps are

$$\begin{tikzcd} & \mathcal{O}_X(D)|_{U_i \cap U_j} & \\ \mathcal{O}_X|_{U_i \cap U_j} \arrow[ur, "1 \mapsto f_i^{-1}"] \arrow[rr, "1 \mapsto \frac{f_j}{f_i}"'] & & \mathcal{O}_X|_{U_i \cap U_j} \arrow[ul, "f_j^{-1} \leftarrow\!\shortmid 1"'] \end{tikzcd},$$

so $g_{ij} = f_j/f_i$ are the transition maps. Consequently, if D_1 and D_2 are Cartier divisors, represented by $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$, then $D_1 - D_2$ is represented by $\{(U_i, f_i/g_i)\}$ and the transition maps for $\mathcal{O}_X(D_1 - D_2)$ are $(f_j/g_j)/(f_i/g_i) = (f_j/f_i)/(g_j/g_i)$, which are also the transition maps for $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^\vee$. Thus

$$\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^\vee,$$

so we obtain a group homomorphism

$$\begin{array}{rcl} \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) & \longrightarrow & \operatorname{Pic} X \\ D & \longmapsto & \mathcal{O}_X(D) \end{array}.$$

Lemma 5.10. *$D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.*

Proof. It is enough to show D is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. If D is principal, then D is represented by (X, f) for $f \in \Gamma(X, \mathcal{K}_X^*)$. So $\mathcal{O}_X(D) = \mathcal{O}_X \cdot f^{-1} \cong \mathcal{O}_X$. Conversely, if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, let

$$\begin{array}{rcl} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(X, \mathcal{O}_X(D)) \subseteq \Gamma(X, \mathcal{K}_X) \\ 1 & \longmapsto & f \end{array}.$$

In fact $f \in \Gamma(X, \mathcal{K}_X^*)$. Then (X, f^{-1}) represents $D = \{(U_i, g_i)\}$ as f^{-1} and g_i only differ by a factor of an invertible function on U_i. Thus D is principal. ☐

Corollary 5.11. *On any scheme X, there is an injective homomorphism*

$$\begin{array}{rcl} \operatorname{Ca}\operatorname{Cl} X & \longrightarrow & \operatorname{Pic} X \\ D & \longmapsto & \mathcal{O}_X(D) \end{array}.$$

Proposition 5.12. *If X is integral, then this homomorphism is an isomorphism.*

Proof. Need to show every line bundle on X is isomorphic to a subsheaf of $\mathcal{K}_X$, which is in this case the constant sheaf $U \mapsto \mathrm{K}(X)$. Once this is shown, a trivialisation on a cover U_i leads to rational functions given by the isomorphism

$$\begin{array}{rcl} \mathcal{O}_{U_i} & \longrightarrow & \mathcal{L}|_{U_i} \subseteq \mathcal{K}_X|_{U_i} \\ 1 & \longmapsto & f_i \end{array},$$

and then $D = \left\{\left(U_i, f_i^{-1}\right)\right\}$ satisfies $\mathcal{L} \cong \mathcal{O}_X(D)$. So let $\mathcal{L}$ be a line bundle on X, and consider $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. On any open U with $\mathcal{L}|_U \cong \mathcal{O}_U$, we have $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)|_U \cong \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{K}_X|_U \cong \mathcal{K}_X|_U$. This is the constant sheaf $V \subseteq U \mapsto \mathrm{K}(X)$. Then $\mathcal{F} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is also the constant sheaf $V \mapsto \mathrm{K}(X)$. Indeed if V is any non-empty open subset and $\{U_i\}$ is a trivialising cover of $\mathcal{L}$, then $\mathcal{F}(V \cap U_i)$ can be identified with $\mathrm{K}(X)$ canonically, as we can identify $\mathcal{F}_\eta$ with $\mathrm{K}(X)$ where η is the generic point of X. Then the sheaf gluing axioms tell us that $\mathcal{F}(V) \cong \mathrm{K}(X)$. Thus $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \cong \mathcal{K}_X$ and we have a natural map

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \\ s & \longmapsto & s \otimes 1 \end{array},$$

thus exhibiting $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. □

5.5 Effective divisors

Definition. A Weil divisor $\sum_i a_i Y_i$ is **effective** if $a_i \geq 0$ for all i. A Cartier divisor $\{(U_i, f_i)\}$ is **effective** if $f_i \in \mathcal{O}_X(U_i)$ for all i. If $\mathcal{L}$ is a line bundle, $s \in \Gamma(X, \mathcal{L})$, and $\{U_i\}$ is a trivialising cover for $\mathcal{L}$, with trivialisations $\phi_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$, we obtain a Cartier divisor

$$(s)_0 = \{(U_i, \phi_i(s))\}, \qquad \phi_i(s) \in \mathcal{O}_X(U_i),$$

the **divisor of zeros** of s, necessarily effective.

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Theorem 5.13. *Let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Then $\Gamma(X, \mathcal{F})$ is a finite-dimensional k-vector space.*

Note that if $X = \mathbb{A}^1$ and $\mathcal{F} = \mathcal{O}_X$, then $\Gamma(X, \mathcal{F}) = k[x]$ is not a finite-dimensional k-vector space.

Proof. Hartshorne, Chapter II, Theorem 5.19. □

Theorem 5.14. *If $X \subseteq \mathbb{P}_k^n$ is an integral closed subscheme with k algebraically closed, then $\Gamma(X, \mathcal{O}_X) = k$.*

Proof. Hartshorne, Chapter I, Theorem 3.4. □

We need $k = \overline{k}$. [21]

Theorem 5.15. *Let X be an integral closed subscheme of $\mathbb{P}_k^n$ with k algebraically closed. Let D_0 be a Cartier divisor on X and $\mathcal{L} = \mathcal{O}_X(D_0)$. Then*

1. *for every $s \in \Gamma(X, \mathcal{L})$ such that $s \neq 0$, $(s)_0$ is an effective divisor linearly equivalent to D_0,*
2. *every effective divisor linearly equivalent to D_0 is $(s)_0$ for some section $s \in \Gamma(X, \mathcal{L})$, and*
3. *two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if there exists $\lambda \in k^*$ such that $s = \lambda s'$.*

Proof.

1. $\mathcal{O}_X(D_0) \subseteq \mathcal{K}_X$ so $s \in \Gamma(X, \mathcal{L})$ corresponds to a rational function $f \in \Gamma(X, \mathcal{K}_X) = \mathrm{K}(X)$. If D_0 is represented by $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated as an $\mathcal{O}_{U_i}$-module by f_i^{-1}, giving trivialisations
$$\begin{array}{ccccc} \phi_i & : & \mathcal{O}_X(D_0)|_{U_i} & \longrightarrow & \mathcal{O}_{U_i} \\ & & t & \longmapsto & t f_i \end{array},$$
so $D = (s)_0 = \{(U_i, f f_i)\} = D_0 + (f)$, since $(f) = \{(X, f)\}$. Thus $D \sim D_0$.
2. If D is effective and $D = D_0 + (f)$, then if we write $D = \{(U_i, g_i)\}$ and $D_0 = \{(U_i, f_i)\}$, then $g_i = f_i f$ and $g_i \in \mathcal{O}_X(U_i)$. Then $\phi_i^{-1}(g_i) = g_i f_i^{-1} = f_i f f_i^{-1} = f$. So f in fact is a section s of $\mathcal{O}_X(D_0) \cong \mathcal{L}$, and then $(s)_0 = D$.
3. If $(s)_0 = (s')_0$ then $(s)_0 = D_0 + (f)$ and $(s')_0 = D_0 + (f')$, and $(f/f') = 0$. That is, $f/f' \in \Gamma(X, \mathcal{O}_X^*)$. Now we use the fact that $\Gamma(X, \mathcal{O}_X) = k$, so $f/f' \in k^*$.

□

[21] Exercise: check

Example. $\mathbb{P}_k^n$ satisfies all the hypotheses of Theorem 5.15. We have isomorphisms $\mathbb{Z} \cong \mathrm{Cl}\,\mathbb{P}^n \cong \mathrm{Ca}\,\mathrm{Cl}\,\mathbb{P}^n \cong \mathrm{Pic}\,\mathbb{P}^n$, since $\mathbb{P}_k^n$ is non-singular, that is all local rings are regular. The generator of $\mathrm{Cl}\,\mathbb{P}^n$ is H, a hyperplane, and not so hard to see that $\mathcal{O}_{\mathbb{P}^n}(H) = \mathcal{O}_{\mathbb{P}^n}(1)$ constructed previously. [22] So $\mathrm{Pic}\,\mathbb{P}^n$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$. Define

$$\mathcal{O}_{\mathbb{P}^n}(d) = \begin{cases} \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d} & d > 0 \\ \mathcal{O}_{\mathbb{P}^n}(-d)^{\vee} & d < 0 \\ \mathcal{O}_{\mathbb{P}^n} & d = 0 \end{cases},$$

which is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(dH)$. We will see that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong S_d$ where $S = k[x_0, \dots, x_n] = \bigoplus_d S_d$ for S_d the degree d piece. Check that if $f \in S_d$ is a homogeneous polynomial of degree d and $f = \prod_{i=1}^n f_i^{d_i}$ its prime factorisation, then $(f)_0 = \sum_i d_i \mathbb{V}(f_i)$. [23]

5.6 Dictionary between line bundles and linear systems

Let X be an integral subscheme of $\mathbb{P}^n$ such that $k = \overline{k}$.

A line bundle $\mathcal{L}$.	A divisor $D \in \mathrm{Ca}\,\mathrm{Cl}\,X$ such that $\mathcal{L} \cong \mathcal{O}_X(D)$.
A section $s \in \Gamma(X, \mathcal{L})$ such that $s \neq 0$.	An effective divisor $(s)_0 \sim D$.
A projectivisation $\mathbb{P}(\Gamma(X, \mathcal{L})) = (\Gamma(X, \mathcal{L}) \setminus \{0\}) / k^*$.	A **complete linear system** $\lvert D\rvert = \{D' \text{ effective}, \ D' \sim D\}$.
Sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ define a morphism $\begin{array}{ccc} \mathcal{O}_X^{\oplus(n+1)} & \longrightarrow & \mathcal{L} \\ e_i & \longmapsto & s_i \end{array}.$ If this map is surjective, we say $\mathcal{L}$ is **generated by global sections** and we obtain a morphism $X \to \mathbb{P}_k^n$.	A linear subspace $\mathcal{D} \subseteq \lvert D\rvert$ is called a **linear system**. Think of this as the linear subspace of $\lvert D\rvert$ spanned by $(s_0)_0, \dots, (s_n)_0$. We say $\mathcal{D}$ is **base-point-free** if for all $x \in X$, there exists $D' \in \mathcal{D}$ such that $x \notin \mathrm{supp}\,D'$, where if $D' = \sum_i a_i Y_i$ with $a_i > 0$ then $\mathrm{supp}\,D' = \bigcup_i Y_i$. In this case $\mathcal{D}$ gives a morphism $\phi : X \to \mathbb{P}^n$. Note that if $\mathcal{D}$ is determined by $s_0, \dots, s_n$ then $\mathcal{D}$ is base-point-free if and only if $s_0, \dots, s_n$ generate $\mathcal{L} = \mathcal{O}_X(D)$. Also pull-backs of hyperplanes in $\mathbb{P}^n$ give elements of $\mathcal{D}$.
If sections of $\mathcal{L}$ induce a closed immersion in some $\mathbb{P}_k^n$, we say $\mathcal{L}$ is **very ample**.	If $\lvert D\rvert$ induces a closed immersion, we say D is **very ample**.
$\mathcal{L}$ is **ample** if $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$.	D is **ample** if nD is very ample for some $n > 0$.

Remark. There exists a good geometric criterion for very ampleness. See example sheets. There exist numerical criteria for ampleness. It is useful to control the size of $\Gamma(X, \mathcal{L})$.

[22] Exercise: check
[23] Exercise

6 Cohomology of sheaves

The problem is that given

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

a short exact sequence, we know

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$$

is exact. Can we extend this to a long exact sequence? The answer is the **right derived functors** of $\Gamma(X, -)$, which are written as $\mathrm{H}^i(X, -)$.

6.1 Injective resolutions

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An abelian group I is **injective** if given any diagram of abelian groups

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dashed] \end{tikzcd},$$

there exists a lifting making the diagram commutative.

Example. $\mathbb{Q}$ is injective.

Fact. Every abelian group A has an injection into an injective group.

This gives abelian groups

$$\begin{tikzcd} & & & & 0 \arrow[dr] & & 0 & \\ & & & & & C_1 \arrow[ur] \arrow[dr] & & \\ 0 \arrow[r] & A \arrow[r] & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[rr] \arrow[ur] & & I^2 \arrow[r] & \dots \\ & & & C_0 \arrow[ur] \arrow[dr] & & & & \\ & & 0 \arrow[ur] & & 0 & & & \end{tikzcd},$$

giving a long exact sequence

$$0 \to A \to I^\bullet,$$

an **injective resolution** of A.

6.2 Sheaf cohomology

We then get injective resolutions in the category of sheaves of abelian groups. If $\mathcal{F}$ is a sheaf on X, then have an inclusion

$$0 \to \mathcal{F}_x \to I_x, \qquad x \in X,$$

with I_x injective. Then define

$$\mathcal{I} = \prod_{x \in X} (\iota_x)_* I_x,$$

where $\iota_x : \{x\} \hookrightarrow X$. That is,

$$\mathcal{I}(U) = \prod_{x \in U} I_x.$$

Then we have an inclusion
$$\begin{array}{rcl} \mathcal{F}(U) & \longrightarrow & \mathcal{I}(U) \\ s & \longmapsto & (f_x(U,s))_{x\in U} \end{array},$$
and $\mathcal{I}$ is an injective object in the category of sheaves of abelian groups. This allows the construction of injective resolutions
$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \dots .$$
Then define
$$\mathrm{H}^i(X,\mathcal{F}) = \ker\left(d^1 : \Gamma\left(X,\mathcal{I}^i\right) \to \Gamma\left(X,\mathcal{I}^{i+1}\right)\right) / \operatorname{im}\left(d^{i-1} : \Gamma\left(X,\mathcal{I}^{i-1}\right) \to \Gamma\left(X,\mathcal{I}^i\right)\right).$$
That is, this is the cohomology of the chain complex
$$\Gamma\left(X,\mathcal{I}^0\right) \to \Gamma\left(X,\mathcal{I}^1\right) \to \dots .$$

Proposition 6.1.

- $\mathrm{H}^i(X,-)$ *is a well-defined covariant functor. That is, independent of the choice of resolution and* $f : \mathcal{F} \to \mathcal{G}$ *induces a map* $\mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^i(X,\mathcal{G})$.
- *Whenever*
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is exact, we obtain connecting homomorphisms $\delta : \mathrm{H}^i(X,\mathcal{F}'') \to \mathrm{H}^{i+1}(X,\mathcal{F}')$ *and a long exact sequence*
$$0 \to \mathrm{H}^0(X,\mathcal{F}') \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathcal{F}'') \xrightarrow{\delta} \mathrm{H}^1(X,\mathcal{F}') \to \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F}'') \xrightarrow{\delta} \dots .$$
- *Given a commutative diagram*
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0 \end{tikzcd},$$
with rows exact, we get a commutative square
$$\begin{tikzcd} \mathrm{H}^i(X,\mathcal{F}'') \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^{i+1}(X,\mathcal{F}') \arrow[d] \\ \mathrm{H}^i(X,\mathcal{G}'') \arrow[r, "\delta"'] & \mathrm{H}^{i+1}(X,\mathcal{G}') \end{tikzcd}.$$
- *Whenever* $\mathcal{F}$ *is* ***flasque****, or* ***flabby****, that is all restriction maps are surjective, then* $\mathrm{H}^i(X,\mathcal{F}) = 0$ *for all* $i > 0$.
- $\mathrm{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

Remark. May also work on a ringed space $(X,\mathcal{O}_X)$ and consider only sheaves of $\mathcal{O}_X$-modules. Injective resolutions of $\mathcal{O}_X$-modules by injective $\mathcal{O}_X$-modules exist, so could define cohomology using such resolutions, but in fact get the same answer as before.

Theorem 6.2 (Grothendieck)**.** *Let* X *be a Noetherian topological space of dimension* n *and* $\mathcal{F}$ *a sheaf of abelian groups on* X. *Then* $\mathrm{H}^i(X,\mathcal{F}) = 0$ *for all* $i > n$.

Proof. Hartshorne, Chapter III, Theorem 2.7. □

6.3 Čech cohomology

How do we calculate cohomology in practice? Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X. Choose a well-ordering on I, and write $U_{i_1 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. Define the group of **Čech p-cochains** to be

$$\check{\mathrm{C}}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}\left(U_{i_0 \dots i_p}\right).$$

Write $\alpha \in \check{\mathrm{C}}^p(\mathcal{U}, \mathcal{F})$ as $\alpha = \left(\alpha_{i_0 \dots i_p}\right)_{i_0 < \dots < i_p}$. Define the **Čech coboundary** by

$$\begin{array}{rcccl} \mathrm{d} & : & \check{\mathrm{C}}^p(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathrm{C}}^{p+1}(\mathcal{U}, \mathcal{F}) \\ & & \alpha & \longmapsto & \left(\sum_{k=0}^{p+1} (-1)^k \left.\alpha_{i_0 \dots \widehat{i_k} \dots i_{p+1}}\right|_{U_{i_0 \dots i_{p+1}}}\right)_{i_0 < \dots < i_{p+1}} \end{array}.$$

Exercise. $\mathrm{d}^2 = 0$.

Define

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = \mathrm{H}^p\left(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{F})\right) = \ker\left(\mathrm{d} : \check{\mathrm{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^{p+1}(\mathcal{U}, \mathcal{F})\right) / \operatorname{im}\left(\mathrm{d} : \check{\mathrm{C}}^{p-1}(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{C}}^p(\mathcal{U}, \mathcal{F})\right).$$

Example.

- Let $X = \mathrm{S}^1$ with the usual topology, and let $\mathcal{F} = \underline{\mathbb{Z}}$ be the constant sheaf. That is, the sheaf associated to the presheaf $U \mapsto \mathbb{Z}$, so

  $$\mathcal{F}(U) = \{\phi : U \to \mathbb{Z} \mid \phi \text{ locally constant}\}.$$

  Take as an open cover U and V connected such that $U \cap V$ has two connected components. Then

  $$\check{\mathrm{C}}^0(\mathcal{U}, \mathcal{F}) = \Gamma(U, \mathcal{F}) \times \Gamma(V, \mathcal{F}) = \mathbb{Z} \times \mathbb{Z}, \qquad \check{\mathrm{C}}^1(\mathcal{U}, \mathcal{F}) = \Gamma(U \cap V, \mathcal{F}) = \mathbb{Z}^2,$$

  and

  $$\begin{array}{rcccl} \mathrm{d} & : & \check{\mathrm{C}}^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{\mathrm{C}}^1(\mathcal{U}, \mathcal{F}) \\ & & (a, b) & \longmapsto & (b-a, b-a) \end{array},$$

  so $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \ker \mathrm{d} \cong \mathbb{Z}$ and $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) = \operatorname{coker} \mathrm{d} \cong \mathbb{Z}$. Note that this agrees with the regular cohomology of S^1. In this case, this also agrees with $\mathrm{H}^i\left(\mathrm{S}^1, \mathcal{F}\right)$.

- Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-2)$ for $\mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$. Then $\mathcal{O}_{\mathbb{P}^1}(1)$ had transition map from $U_0 = \mathbb{D}_+(x_0)$ to $U_1 = \mathbb{D}_+(x_1)$ given by x_0/x_1. Thus $\mathcal{O}_{\mathbb{P}^1}(-2)$ has transition map x_1^2/x_0^2. Taking $\mathcal{U} = \{U_0, U_1\}$, we get

  $$\check{\mathrm{C}}^0(\mathcal{U}, \mathcal{F}) = \Gamma\left(U_0, \mathcal{O}_{\mathbb{P}^1}(-2)\right) \times \Gamma\left(U_1, \mathcal{O}_{\mathbb{P}^1}(-2)\right) = k\left[\frac{x_1}{x_0}\right] \times k\left[\frac{x_0}{x_1}\right],$$

  and

  $$\check{\mathrm{C}}^1(\mathcal{U}, \mathcal{F}) = \Gamma\left(U_0 \cap U_1, \mathcal{O}_{\mathbb{P}^1}(-2)\right) = k\left[\frac{x_1}{x_0}\right]_{\frac{x_1}{x_0}} = k\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right],$$

  using the same trivialisation on $U_0 \cap U_1$ which we used on U_1. Then

  $$\mathrm{d}(f, g) = g - f\frac{x_1^2}{x_0^2}.$$

  Then $\ker \mathrm{d} = 0$ and $\operatorname{coker} \mathrm{d}$ is one-dimensional, generated by x_1/x_0. So $\check{\mathrm{H}}^0\left(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(-2)\right) = 0$ and $\check{\mathrm{H}}^1\left(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(-2)\right) = k$.

Theorem 6.3. *Let X be a Noetherian scheme with an open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ with the property that $U_{i_0 \dots i_n}$ are affine for all $i_0 < \dots < i_n$. Then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^i(X, \mathcal{F})$.*

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Remark. If $X \to S$ is a separated morphism with S affine, then any open affine cover of X has the desired property.

6.4 Calculation of cohomology of projective space

Fix a field k and $X = \mathbb{P}^r_k$. We saw every line bundle on $\mathbb{P}^r_k$ is of the form $\mathcal{O}_{\mathbb{P}^r}(m) = \mathcal{O}_X(m) = \mathcal{O}_X(mH)$ for some $m \in \mathbb{Z}$.

Definition. A **perfect pairing** is a bilinear map $\langle , \rangle : V \times W \to k$ with V and W two k-vector spaces such that the map

$$\begin{array}{ccc} V & \longrightarrow & W^* \\ v & \longmapsto & \langle v, \cdot \rangle \end{array}$$

is an isomorphism.

Theorem 6.4. *Let $S = k[x_0, \dots, x_r]$. Then*

1. *there is an isomorphism of graded S-modules*
$$S \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{O}_X(n)),$$
2. $\mathrm{H}^i(X, \mathcal{O}_X(n)) = 0$ *for* $0 < i < r$,
3. $\mathrm{H}^r(X, \mathcal{O}_X(-r-1)) \cong k$, *and*
4. *there is a perfect pairing*
$$\mathrm{H}^0(X, \mathcal{O}_X(n)) \times \mathrm{H}^r(X, \mathcal{O}_X(-n-r-1)) \to \mathrm{H}^r(X, \mathcal{O}_X(-r-1)) = k,$$
of finite-dimensional k-vector spaces for all $n \in \mathbb{Z}$.

Proof. Will calculate using Čech cohomology using the standard affine cover

$$\mathcal{U} = \{U_i = \mathbb{D}_+(x_i) \mid 0 \le i \le r\},$$

by calculating cohomology of $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$ as Čech cohomology respects direct sums. The key point is to recall the transition map for $\mathcal{O}_X(1)$ from U_i to U_j is x_i/x_j, and so the transition maps for $\mathcal{O}_X(m)$ are x_i^m/x_j^m. For $I \subseteq \{0, \dots, r\}$, we have $U_I = \bigcap_{i \in I} \mathbb{D}_+(x_i) = \mathbb{D}_+(x_I)$ where $x_I = \prod_{i \in I} x_i$. Thus $\Gamma(U_I, \mathcal{O}_{\mathbb{P}^r}) \cong S_{(x_I)}$. We will identify $\Gamma(U_I, \mathcal{O}_X(m))$ with the k-vector subspace of $S_{(x_I)}$ spanned by Laurent monomials of degree m. That is, monomials of the form $x_0^{a_0} \dots x_r^{a_r}$ with $\sum_i a_i = m$ and if $a_i < 0$ then $i \in I$. Given such a monomial M, then using the trivialisation on U_i, we will identify the section of $\mathcal{O}_X(m)$ defined by M with $M/x_i^m \in \Gamma(U_I, \mathcal{O}_{\mathbb{P}^r})$, with $i \in I$. If $i, j \in I$, then note $(M/x_i^m)(x_i^m/x_j^m) = M/x_j^m$. Thus we have a canonical identification of $\Gamma(U_I, \mathcal{O}_X(m))$ with the space spanned by Laurent monomials of degree m. Thus $\Gamma(U_I, \mathcal{F})$ can be identified with $S_{(x_I)}$. So now have a Čech complex $\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{F})$

$$\prod_{0 \le i_0 \le r} S_{(x_{i_0})} \xrightarrow{\mathrm{d}^0} \dots \xrightarrow{\mathrm{d}^{r-1}} S_{(x_0 \dots x_r)}.$$

1. Note $\mathrm{H}^0(X, \mathcal{F}) = \ker \mathrm{d}^0$. Note also all modules in the Čech complex are S-submodules of $S_{(x_0 \dots x_r)}$, and
$$\mathrm{d}^0\left((f_i)_{0 \le i \le r}\right) = (f_j - f_i)_{0 \le i < j \le r}.$$
Thus if $(f_i)_{0 \le i \le r} \in \ker \mathrm{d}^0$, we actually have $f_i = f_j$ for all i and j. Thus $f_i, f_j \in S$ since otherwise f_i involves a negative power of x_i, which cannot occur in f_j, or vice versa. Thus $f_i = f$ for all i with $f \in S$, so $\ker \mathrm{d}^0 \cong S$. Thus $\mathrm{H}^0(X, \mathcal{F}) \cong S$, preserving degrees. That is, $\mathrm{H}^0(X, \mathcal{O}_X(m)) = S_m$.

3. Now consider
$$\mathrm{d}^{r-1} : \prod_{0 \le k \le r} S_{(x_0 \dots \widehat{x_k} \dots x_r)} \to S_{(x_0 \dots x_r)}.$$
Note $S_{(x_0 \dots x_r)}$ is the k-vector space with basis $\prod_{i=0}^r x_i^{a_i}$ for $a_i \in \mathbb{Z}$ and $\mathrm{im}\,\mathrm{d}^{r-1}$ is spanned by monomials of the form $\prod_{i=0}^r x_i^{a_i}$ with at least one $a_i \ge 0$. Thus the basis for $\mathrm{coker}\,\mathrm{d}^{r-1}$ is
$$\left\{ \prod_{i=0}^r x_i^{a_i} \;\middle|\; \forall i,\ a_i \le -1 \right\}.$$
In particular, $\mathrm{H}^r(X, \mathcal{O}_X(-r-1))$ is generated by $x_0^{-1} \dots x_r^{-1}$. Thus $\mathrm{H}^r(X, \mathcal{O}_X(-r-1)) \cong k$.

4. Note $\mathrm{H}^0(X, \mathcal{O}_X(n)) = 0$ for $n < 0$ as $S_n = 0$ for $n < 0$, and $\mathrm{H}^r(X, \mathcal{O}_X(-n-r-1)) = 0$ for $n < 0$ as there are no monomials with only negative exponents of degree more than $-r-1$. Thus nothing to check in this case. If $n \geq 0$, we have a basis
$$\left\{ \prod_i x_i^{m_i} \;\middle|\; \sum_i m_i = n,\ m_i \geq 0 \right\}$$
for $\mathrm{H}^0(X, \mathcal{O}_X(n))$ and a basis
$$\left\{ \prod_i x_i^{l_i} \;\middle|\; \sum_i l_i = -n-r-1,\ l_i \leq -1 \right\}$$
for $\mathrm{H}^r(X, \mathcal{O}_X(-n-r-1))$. The perfect pairing is given by
$$\left(x_0^{m_0} \cdot \dots \cdot x_r^{m_r}\right) \cdot \left(x_0^{l_0} \cdot \dots \cdot x_r^{l_r}\right) = x_0^{m_0+l_0} \cdot \dots \cdot x_r^{m_r+l_r},$$
interpreting as zero if any $m_i + l_i \geq 0$. This gives a pairing
$$\mathrm{H}^0(X, \mathcal{O}_X(n)) \times \mathrm{H}^r(X, \mathcal{O}_X(-n-r-1)) \to \mathrm{H}^r(X, \mathcal{O}_X(-r-1)) = k \cdot (x_0 \dots x_r)^{-1}.$$
It is easy to check it is a perfect pairing. [24]

2. It remains to show $\mathrm{H}^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$, by induction on r. The base case $r = 1$ Lecture 23 Monday 30/11/20
has nothing to show. For the induction step, if we localise $\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{F})$ at x_r as graded S-modules, we get a Čech complex which calculates the cohomology groups $\mathrm{H}^i\left(U_r, \mathcal{F}|_{U_r}\right)$, by calculating using the Čech cover $\mathcal{U}' = \{U_i \cap U_r \mid 0 \leq i \leq r\}$. But $U_r \cong \mathbb{A}_k^r$, and Čech cohomology can also be calculated via the cover $\{U_r\}$, so $\mathrm{H}^i\left(U_r, \mathcal{F}|_{U_r}\right) = 0$ for all $i > 0$. Note that this implies that if $\mathcal{F}$ is in general a quasi-coherent sheaf on an affine scheme X, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$. Now localising at x_r is an exact functor, so $\mathrm{H}^i\left(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{F})_{x_r}\right) = \mathrm{H}^i\left(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{F})\right)_{x_r}$, so thus $\mathrm{H}^i(X, \mathcal{F})_{x_r} = \mathrm{H}^i\left(U_r, \mathcal{F}|_{U_r}\right) = 0$ for all $i > 0$. For this to be the case, every element of $\mathrm{H}^i(X, \mathcal{F})$ must be annihilated by some power of x_r. Now let $H = \mathbb{V}(x_r) \subseteq \mathbb{P}^r$. Thinking of this as a closed subscheme, $H = \operatorname{Proj} S / \langle x_r \rangle = \operatorname{Proj} k[x_0, \dots, x_{r-1}] = \mathbb{P}^{r-1}$. Have a surjective map $\mathcal{O}_{\mathbb{P}^r} \twoheadrightarrow \iota_* \mathcal{O}_H$ where $\iota : H \to \mathbb{P}^r$ is the inclusion. Because H is defined locally by a single equation, the kernel of $\mathcal{O}_{\mathbb{P}^r} \twoheadrightarrow \iota_* \mathcal{O}_H$ is a line bundle. Note this kernel is the ideal sheaf corresponding to H. On $U_i = \operatorname{Spec} S_{(x_i)}$, this kernel is generated by x_r / x_i and hence the transition maps for the ideal sheaf $\mathcal{I}_{H/X}$ are
$$\begin{tikzcd} \mathcal{O}_{U_i}|_{U_i \cap U_j} \arrow[r, "\frac{x_r}{x_i}"] \arrow[rr, bend right, "\frac{x_r}{x_i} \cdot \frac{x_j}{x_r} = \frac{x_j}{x_i}"'] & \mathcal{I}_{H/X}|_{U_i \cap U_j} & \mathcal{O}_{U_j}|_{U_i \cap U_j} \arrow[l, "\frac{x_r}{x_j}"'] \end{tikzcd}.$$
Thus $\mathcal{I}_{H/X} \cong \mathcal{O}_{\mathbb{P}^r}(-1) \cong \mathcal{O}_{\mathbb{P}^r}(-H)$. The upshot is that we have an exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r} \to \iota_* \mathcal{O}_H \to 0.$$
Multiplication by x_r makes sense, since on U_i, it means multiplying by x_r / x_i, recalling that x_r corresponds to the section x_r / x_i of $\mathcal{O}_{\mathbb{P}^r}(1)$ on U_i. We can tensor the exact sequence with $\mathcal{O}_{\mathbb{P}^r}(n)$. Still exact, so
$$0 \to \mathcal{O}_{\mathbb{P}^r}(n-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r}(n) \to \iota_*(\mathcal{O}_H(n)) \to 0.$$
Exactness on the left follows since $\mathcal{O}_{\mathbb{P}^r}(n)$ is locally free, hence flat, or more simply, on U_i, $\mathcal{O}_{\mathbb{P}^r}(n) \cong \mathcal{O}_{U_i}$, so tensoring with $\mathcal{O}_{U_i}$ does not do anything. Note also $\iota_* \mathcal{O}_H \otimes_{\mathcal{O}_{\mathbb{P}^r}} \mathcal{O}_{\mathbb{P}^r}(n) \cong \iota_*(\mathcal{O}_H(n))$. Frequently, we will drop the ι_* when dealing with sheaves on a closed subscheme. That is, if $\mathcal{F}$ is a sheaf on H, we often write $\mathcal{F}$ for $\iota_* \mathcal{F}$, where $(\iota_* \mathcal{F})(U) = \mathcal{F}(U \cap H)$. Thus we have an exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^r}(n-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r}(n) \to \mathcal{O}_H(n) \to 0.$$

[24] Exercise

Summing over all n,

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0, \qquad \mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(-1), \qquad \mathcal{F}_H = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_H(n).$$

Induction hypothesis implies that $\mathrm{H}^i(\mathbb{P}^r, \mathcal{F}_H) = 0$ for $0 < i < r-1$. Note $\mathrm{H}^i(\mathbb{P}^r, \mathcal{F}_H) = \mathrm{H}^i(H, \mathcal{F}_H)$, as the Čech complexes calculating them are the same. That is, if use $\mathcal{U} = \{U_i\}$ or $\mathcal{U}_H = \{U_i \cap H\}$. This is a general fact, where if $\iota : Y \to X$ is a closed immersion and $\mathcal{F}$ is a sheaf on Y, then $\mathrm{H}^p(X, \iota_* \mathcal{F}) = \mathrm{H}^p(Y, \mathcal{F})$.

– So if $1 < i < r-1$, get a piece of the long exact cohomology sequence

$$0 = \mathrm{H}^{i-1}(\mathbb{P}^r, \mathcal{F}_H) \to \mathrm{H}^i(\mathbb{P}^r, \mathcal{F}(-1)) \xrightarrow{\cdot x_r} \mathrm{H}^i(\mathbb{P}^r, \mathcal{F}) \to \mathrm{H}^i(\mathbb{P}^r, \mathcal{F}_H) = 0.$$

So $\cdot x_r : \mathrm{H}^i(\mathbb{P}^r, \mathcal{F}(-1)) \to \mathrm{H}^i(\mathbb{P}^r, \mathcal{F})$ is an isomorphism. But note $\mathrm{H}^i(\mathbb{P}^r, \mathcal{F}(-1)) = \mathrm{H}^i(\mathbb{P}^r, \mathcal{F})$ as non-graded S-modules. But we know every element of $\mathrm{H}^i(\mathbb{P}^r, \mathcal{F})$ is annihilated by some power of x_r. Thus $\mathrm{H}^i(\mathbb{P}^r, \mathcal{F}) = 0$ for $1 < i < r-1$.

– For $i = 1$, have

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(\mathbb{P}^r, \mathcal{F}(-1)) \arrow[r] \arrow[d, phantom, "\wr\parallel"] & \mathrm{H}^0(\mathbb{P}^r, \mathcal{F}) \arrow[r] \arrow[d, phantom, "\wr\parallel"] & \mathrm{H}^0(\mathbb{P}^r, \mathcal{F}_H) \arrow[r] \arrow[d, phantom, "\wr\parallel"] & \mathrm{H}^1(\mathbb{P}^r, \mathcal{F}(-1)) \arrow[r, "\cdot x_r"] & \mathrm{H}^1(\mathbb{P}^r, \mathcal{F}) \arrow[r] & 0 \\ & S(-1) \arrow[r, "\cdot x_r"'] & S \arrow[r, two heads] & S/\langle x_r \rangle & & & \end{tikzcd},$$

where $S(-1)$ is the S-module with $S(-1)_d = S_{d-1}$. Thus $\cdot x_r : \mathrm{H}^1(\mathbb{P}^r, \mathcal{F}(-1)) \to \mathrm{H}^1(\mathbb{P}^r, \mathcal{F})$ is injective, and hence as before, $\mathrm{H}^1(\mathbb{P}^r, \mathcal{F}) = 0$.

– For $i = r-1$, get

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}(-1)) \arrow[r, "\cdot x_r"] & \mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}) \arrow[r] & \mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}_H) \arrow[dll, out=0, in=180] \\ & \mathrm{H}^r(\mathbb{P}^r, \mathcal{F}(-1)) \arrow[r, "\cdot x_r"'] & \mathrm{H}^r(\mathbb{P}^r, \mathcal{F}) \arrow[r] & \mathrm{H}^r(\mathbb{P}^r, \mathcal{F}_H) = 0 \end{tikzcd}.$$

By our calculation, the kernel of $\cdot x_r : \mathrm{H}^r(\mathbb{P}^r, \mathcal{F}(-1)) \to \mathrm{H}^r(\mathbb{P}^r, \mathcal{F})$ is generated by

$$\left\{ x_0^{l_0} \dots x_r^{l_r} \;\middle|\; \forall i,\ l_i \le -1,\ l_r = -1 \right\}.$$

This is identified with $\mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}_H)$, so the connecting map is injective [25] and we conclude $\cdot x_r : \mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}(-1)) \to \mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F})$ is surjective. Thus $\cdot x_r$ is an isomorphism and we conclude as before that $\mathrm{H}^{r-1}(\mathbb{P}^r, \mathcal{F}) = 0$.

□

Remark. In general, given an effective Cartier divisor $D = \{(U_i, f_i)\}$ for $f_i \in \mathcal{O}_X(U_i)$, D defines a closed subscheme of X whose ideal on U_i is generated by f_i. This coincides with the line bundle $\mathcal{O}_X(-D)$.

[25] Exercise: check this by understanding of the Čech cohomology connecting maps

7 Differentials and Riemann-Roch

7.1 Normal and conormal bundles

Let X be a scheme and $\iota : Z \hookrightarrow X$ a closed immersion. Then have

$$\mathcal{I}_{Z/X} = \ker\left(\iota^* : \mathcal{O}_X \to \iota_* \mathcal{O}_Z\right).$$

We saw on the example sheet that $\mathcal{I}_Z$ is a coherent sheaf of $\mathcal{O}_X$-modules if X is Noetherian. We define the **conormal sheaf** of Z in X to be

$$\mathrm{N}_{Z/X}^\vee = \mathcal{I}_Z / \mathcal{I}_Z^2 \subseteq \mathcal{O}_X / \mathcal{I}_Z^2.$$

Here $\mathcal{I}_Z^2$ is the sheaf associated to the presheaf $U \mapsto \mathcal{I}_Z(U)^2 \subseteq \mathcal{O}_X(U)$.

Fact. Suppose X and Z are non-singular. That is, all local rings of X and Z are regular. Then $\mathrm{N}_{Z/X}^\vee$ is a locally free sheaf of rank $\operatorname{codim}(Z, X)$.

In this case we define the **normal bundle** of Z in X to be

$$\mathrm{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}\left(\mathrm{N}_{Z/X}^\vee, \mathcal{O}_Z\right).$$

Here we are using that $\mathrm{N}_{Z/X}^\vee$ is a sheaf of $\mathcal{O}_Z = \mathcal{O}_X / \mathcal{I}_Z$-modules.

7.2 Sheaf of differentials

Definition. Suppose $f : X \to Y$ is a separated morphism, so that $\Delta : X \to X \times_Y X$ is a closed immersion. Then the **sheaf of differentials** is the sheaf

$$\Omega_{X/Y} = \Delta^* \mathrm{N}_{X/X \times_Y X}^\vee.$$

Let B be an A-algebra, let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and M a B-module. An A**-derivation** $d : B \to M$ is a map such that Lecture 24 Wednesday 02/12/20

- $d(b + b') = d(b) + d(b')$ for all $b, b' \in B$,
- $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$, and
- $d(a) = 0$ for all $a \in A$.

The **module of relative differentials** $\Omega_{B/A}$ is a B-module satisfying the following universal property. There exists an A-derivation $d : B \to \Omega_{B/A}$ such that for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $g : \Omega_{B/A} \to M$ making the diagram

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "g"] \\ & M \end{tikzcd}$$

commute.

Example. Take $B = k[x_1, \dots, x_n]$ and $A = k$. Then

$$\Omega_{B/A} = \bigoplus_{i=1}^n B \mathrm{d}x_i, \qquad d(x_i) = \mathrm{d}x_i, \qquad d(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Given $d' : B \to M$, define

$$\begin{array}{cccc} g & : & \Omega_{B/A} & \longrightarrow & M \\ & & \mathrm{d}x_i & \longmapsto & d'(x_i) \end{array}.$$

Remark. In general, $\Omega_{B/A}$ can be constructed as follows. We have a homomorphism

$$\begin{array}{rcrcl} \phi & : & B \otimes_A B & \longrightarrow & B \\ & & b \otimes b' & \longmapsto & bb' \end{array}.$$

Take $I = \ker \phi$. Then I/I^2 is a B-module, and we may then define

$$\begin{array}{rcrcl} d & : & B & \longrightarrow & I/I^2 \\ & & b & \longmapsto & 1 \otimes b - b \otimes 1 \end{array}.$$

With this d, $I/I^2 = \Omega_{B/A}$ satisfies the universal property.

Note if $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, then ϕ induces the diagonal morphism $\Delta : X \to X \times_Y X$ and $\widetilde{I} = \mathcal{I}_{X/X \times_Y X}$. Then $\Delta^* \mathrm{N}^\vee_{X/X \times_Y X}$ coincides with $\widetilde{I/I^2}$, viewing I/I^2 as a B-module.

Example. If $Y = \operatorname{Spec} k$ and X is a non-singular connected variety, then so is $X \times_k X$ and $\dim X = \operatorname{codim}(\Delta(X), X \times_k X)$. So $\Omega_{X/\operatorname{Spec} k} = \Omega_X$ is a locally free sheaf of rank $\dim X$. For example, if $X = \mathbb{A}^n_k$, then

$$\Omega_X = \bigoplus_{i=1}^n \mathcal{O}_X \mathrm{d}x_i.$$

Think that Ω_X is the cotangent bundle and $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ is the tangent bundle.

Definition. If X is as above, we define the **canonical bundle** of X to be

$$\omega_X = \bigwedge^{\dim X} \Omega_X.$$

This is the sheaf associated to the presheaf $U \mapsto \bigwedge^{\dim X} \Omega_X(U)$ as an $\mathcal{O}_X(U)$-module. Alternatively if one takes a trivialising cover $\{U_i\}$ for Ω_X, with transition matrices $g_{ij} \in \mathrm{GL}_n(\Gamma(U_i \cap U_j, \mathcal{O}_X))$, then the transition functions for ω_X are $\det g_{ij}$. Then ω_X is a line bundle, and we call its corresponding Cartier divisor class as $\mathcal{K}_X$, the **canonical divisor** of X.

Theorem 7.1 (Serre duality)**.** *Let X be a non-singular projective variety over $\operatorname{Spec} k$ of dimension n. Then for any locally free sheaf $\mathcal{F}$ on X of finite rank, there is a natural isomorphism*

$$\mathrm{H}^i(X, \mathcal{F}^\vee \otimes \omega_X) \to \mathrm{H}^{n-i}(X, \mathcal{F})^\vee,$$

where $\mathcal{F}^\vee$ is the dual sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

The proof is mostly homological algebra, but ultimately reduces to the calculation of $\mathrm{H}^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$. In fact, for $\mathbb{P}^r$,

$$\omega_{\mathbb{P}^r} \cong \mathcal{O}_{\mathbb{P}^r}(-r-1),$$

so the perfect pairing we constructed is $\mathrm{H}^r(X, \mathcal{F}) \times \mathrm{H}^0(X, \mathcal{F}^\vee \otimes \omega_X) \to k$.

Definition. In general, if X is a projective scheme over k, then $\mathrm{H}^i(X, \mathcal{F})$ is a finite-dimensional k-vector space, for $\mathcal{F}$ a coherent sheaf on X. Then we may define the **Euler characteristic** of $\mathcal{F}$ to be

$$\chi(F) = \sum_{i=0}^{\dim X} (-1)^i \dim \mathrm{H}^i(X, \mathcal{F}).$$

This is additive on exact sequences. That is, if

$$\dots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \dots$$

is exact, then $\sum_{i=0}^{\dim X} \chi(\mathcal{F}_i) = 0$. In particular, for

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

exact, $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. These statements follow from the fact that if

$$\dots \to V_{i-1} \to V_i \to V_{i+1} \to \dots$$

is exact, then $\sum_i (-1)^i \dim V_i = 0$. Riemann-Roch states that $\chi(F)$ is a topological invariant.

7.3 Curves

First discuss for curves. For now, let X be a projective non-singular curve over a field k for k algebraically closed. If $P \in X$ is a closed point, we may think of it as a prime divisor defining a closed subscheme, and we have an exact sequence

$$0 \to \mathcal{I}_P \cong \mathcal{O}_X(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0,$$

where $\mathcal{O}_P$ is the structure sheaf of the point P. Now tensoring with a line bundle $\mathcal{L}$,

$$0 \to \mathcal{L}(-P) = \mathcal{L} \otimes \mathcal{O}_X(-P) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_P \cong \mathcal{O}_P \to 0.$$

Exactness on the left also holds since $\mathcal{L}$ is locally free. So $\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + \chi(\mathcal{O}_P) = \chi(\mathcal{L}(-P)) + 1$. Here we are using $k = \overline{k}$. So if $D \in \operatorname{Div} X$, then

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D,$$

where if $D = \sum_i a_i P_i$ then $\deg D = \sum_i a_i$.

Definition. The **genus** of X is

$$\mathrm{g} = \dim_k \mathrm{H}^1(X, \mathcal{O}_X).$$

Theorem 7.2 (Riemann-Roch for curves)**.** *For $D \in \operatorname{Div} X$,*

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(D)) - \dim \mathrm{H}^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = \deg D + 1 - \mathrm{g}.$$

Proof. By Serre duality,

$$\begin{aligned}
\dim \mathrm{H}^0(X, \mathcal{O}_X(D)) - \dim \mathrm{H}^0(X, \omega_X \otimes \mathcal{O}_X(-D)) &= \dim \mathrm{H}^0(X, \mathcal{O}_X(D)) - \dim \mathrm{H}^1(X, \mathcal{O}_X(D)) \\
&= \chi(\mathcal{O}_X(D)) \\
&= \chi(\mathcal{O}_X) + \deg D \\
&= \dim \mathrm{H}^0(X, \mathcal{O}_X) - \dim \mathrm{H}^1(X, \mathcal{O}_X) + \deg D \\
&= 1 - \mathrm{g} + \deg D.
\end{aligned}$$

□

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Remark.

- By Serre duality, $\chi(\omega_X) = \dim \mathrm{H}^0(X, \omega_X) - \dim \mathrm{H}^1(X, \omega_X) = \dim \mathrm{H}^1(X, \mathcal{O}_X) - \dim \mathrm{H}^0(X, \mathcal{O}_X) = \mathrm{g} - 1$. Riemann-Roch tells us that $\chi(\omega_X) = \deg \mathcal{K}_X + 1 - \mathrm{g}$. Thus
$$\deg \mathcal{K}_X = 2\mathrm{g} - 2.$$
- If $\deg D < 0$, then
$$\mathrm{H}^0(X, \mathcal{O}_X(D)) = 0.$$
Indeed linear equivalence must preserve degree. A silly way of seeing this is that the left hand side of Riemann-Roch is independent of the representative for D. Thus $|D|$ is empty, thus $\mathrm{H}^0(X, \mathcal{O}_X(D)) = 0$.
- Now if $\deg D > 2\mathrm{g} - 2$, then $\mathrm{H}^0(X, \mathcal{O}_X(-D) \otimes \omega_X) = 0$ since $\deg(\mathcal{K}_X - D) = 2\mathrm{g} - 2 - \deg D < 0$. Thus Riemann-Roch says
$$\dim \mathrm{H}^0(X, \mathcal{O}_X(D)) = \deg D + 1 - \mathrm{g}.$$
- A linear system $|D|$ on a curve is base-point-free if
$$\dim \mathrm{H}^0(X, \mathcal{O}_X(D - P)) = \dim \mathrm{H}^0(X, \mathcal{O}_X(D)) - 1,$$
as follows from the short exact sequence
$$0 \to \mathcal{O}_X(D - P) \to \mathcal{O}_X(D) \to \mathcal{O}_P = \mathcal{O}_X(D)_P / \mathfrak{m}_P \mathcal{O}_X(D)_P \to 0,$$
so
$$0 \to \mathrm{H}^0(X, \mathcal{O}_X(D - P)) \to \mathrm{H}^0(X, \mathcal{O}_X(D)) \to k.$$
There exists a section of $\mathcal{O}_X(D)$ not vanishing at P if and only if $\mathrm{H}^0(X, \mathcal{O}_X(D)) \to k$ is surjective, if and only if $\dim \mathrm{H}^0(X, \mathcal{O}_X(D - P)) = \dim \mathrm{H}^0(X, \mathcal{O}_X(D)) - 1$. In particular, if $\deg D > 2\mathrm{g} - 1$, then $|D|$ is base-point-free.

- It is easy to show from the very ampleness criterion on example sheet that D is very ample if and only if for all $P \in X$,

$$\dim \mathrm{H}^0\left(X, \mathcal{O}_X\left(D-P\right)\right) - 1 = \dim|D-P| = \dim|D| - 1 = \dim \mathrm{H}^0\left(X, \mathcal{O}_X\left(D\right)\right) - 2,$$

  the base-point-free condition, and for all $P, Q \in X$, not necessarily distinct,

$$\dim|D-P-Q| = \dim|D| - 2.$$

  Thus if $\deg D > 2\mathrm{g}$, then $|D|$ is very ample.

The most interesting range of divisors is $0 \le \deg D \le 2\mathrm{g} - 2$.

Example.

- Let $\mathrm{g} = 0$. Then if $\deg D = 1$, then D is very ample. For example, the linear system $|P|$ for $P \in X$ induces an embedding $f : X \to \mathbb{P}^1$, hence $X \cong \mathbb{P}^1$.
- Let $\mathrm{g} = 1$. Fix $P_0 \in X$. Then $|3P_0|$ is very ample and of dimension two, so we get an embedding $f : X \hookrightarrow \mathbb{P}^2$. This embeds X as a degree three plane curve. This comes from the fact that $f^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_X(3P_0)$, which is of degree three. Think about divisors of degree zero on X. Claim that if $D \in \operatorname{Div} X$ with $\deg D = 0$, then $D \sim P - P_0$ for some $P \in X$, which is unique. Consider $D + P_0$. We then have by Riemann-Roch $\dim \mathrm{H}^0\left(X, \mathcal{O}_X\left(D + P_0\right)\right) = \deg\left(D + P_0\right) + 1 - \mathrm{g} = 1 + 1 - 1 = 1$, so there exists a unique effective divisor P such that $D + P_0 \sim P$. Note $\deg P = 1$, so P is just a point. Thus $D \sim P - P_0$, which also shows P is unique. Hence we have an exact sequence

$$0 \to \mathrm{Cl}^0 X \to \mathrm{Cl}\, X \xrightarrow{\deg} \mathbb{Z} \to 0,$$

  where $\mathrm{Cl}^0 X$ is the linear equivalence classes of degree zero divisors. So there is a bijection between $\mathrm{Cl}^0 X$ and the closed points of X, since $k = \overline{k}$. So $\mathrm{Cl}^0 X$ acquires the structure of a variety. That is, it is the set of closed points of the scheme X. More generally, for X a curve of genus g, the group $\mathrm{Cl}^0 X$ forms the closed points of a g-dimensional variety called an **abelian variety** A. That is, it has a group structure compatible with the variety structure, that is morphisms $m : A \times A \to A$ for multiplication and $i : A \to A$ for inversion.

7.4 Surfaces*

Let X be a projective non-singular surface. Want to be able to count the number of intersection points of two curves $C, D \subseteq X$.

Theorem 7.3. *There exists a unique* ***intersection pairing*** *written as*

$$\begin{array}{rcl} \operatorname{Div} X \times \operatorname{Div} X & \longrightarrow & \mathbb{Z} \\ (C, D) & \longmapsto & C \cdot D \end{array},$$

satisfying

- *if C and D are non-singular curves meeting* ***transversally****, that is not tangent at any intersection point, then $C \cdot D = \#\left(C \cap D\right)$,*
- *$C \cdot D = D \cdot C$,*
- *$\left(C_1 + C_2\right) \cdot D = C_1 \cdot D + C_2 \cdot D$, and*
- *if $C_1 \sim C_2$, then $C_1 \cdot D = C_2 \cdot D$.*

Theorem 7.4 (Riemann-Roch for surfaces)**.**

$$\dim \mathrm{H}^0\left(X, \mathcal{O}_X\left(D\right)\right) - \dim \mathrm{H}^1\left(X, \mathcal{O}_X\left(D\right)\right) + \dim \mathrm{H}^0\left(X, \mathcal{O}_X\left(-D\right) \otimes \omega_X\right) = \frac{1}{2} D \cdot \left(D - \mathcal{K}_X\right) + 1 + \mathrm{P_a},$$

where $\mathrm{P_a}(X) = \chi\left(\mathcal{O}_X\right) - 1$ *is the* ***arithmetic genus*** *of* X*.*

The **blowup** of $\mathbb{A}^n$ at the origin is the variety X for $X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ with X defined by the equations $y_i x_j - x_i y_j = 0$ for all $1 \le i < j \le n$. If

$$\begin{array}{cccc} \phi & : & X & \longrightarrow & \mathbb{A}^n \\ & & \left(\left(x_1, \dots, x_n\right), \left(y_1 : \dots : y_n\right)\right) & \longmapsto & \left(x_1, \dots, x_n\right) \end{array},$$

then $\phi^{-1}\left(\mathbb{A}^n \setminus \{0\}\right) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism with $x_j / x_i = y_j / y_i$, and $\phi^{-1}(0) = \mathbb{P}^{n-1}$, so X is integral. Can globalise this operation. That is, if Y is a projective variety and $y \in Y$ is a non-singular point, we can blow up $y \in Y$ to get

$$\begin{array}{ccccc} \phi & : & \widetilde{Y} & \longrightarrow & Y \\ & & E & \longmapsto & y \end{array},$$

where $\phi^{-1}\left(Y \setminus \{y\}\right) \cong Y \setminus \{y\}$ and $\phi^{-1}\left(\{y\}\right) = E \cong \mathbb{P}^{n-1}$ if $\dim Y = n$.

Remark. There exists a more general notion of blowing up a sheaf of ideals. In this case we take ideal sheaf of $y \in Y$.

Let X be a non-singular projective surface and $\pi : \widetilde{X} \to X$ the blowup of a point $p \in X$. Then

$$\operatorname{Cl} \widetilde{X} = \operatorname{Cl} X \oplus \mathbb{Z}[E], \qquad E = \pi^{-1}\left(\{p\}\right),$$

since

$$0 \to \mathbb{Z}[E] \to \operatorname{Cl} \widetilde{X} \to \operatorname{Cl}\left(\widetilde{X} \setminus E\right) = \operatorname{Cl}\left(X \setminus \{p\}\right) = \operatorname{Cl} X \to 0.$$

Example. Let $p_1, \dots, p_6 \in \mathbb{P}^2$ be general points. That is, no three points contained in a line and not all six contained in a conic. Let $\pi : X \to \mathbb{P}^2$ be the blowup at $p_1, \dots, p_6$, so

$$\operatorname{Cl} X = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \dots \oplus \mathbb{Z}[E_6] = \mathbb{Z}^7.$$

Then $H^2 = H \cdot H = 1$, $H \cdot E_i = E_i \cdot E_j = 0$ for $i \neq j$, and $E_i^2 = E_i \cdot E_i = -1$. If $D = 3H - E_1 - \dots - E_6$, then $D \cdot D = 9 - 6 = 3$ and one can show that $|D|$ embeds X as a cubic surface in $\mathbb{P}^3$. Also, if C is any curve on X, then the degree of its image is $D \cdot C$. For example, there are six curves with $D \cdot E_i = 1$, there are fifteen curves with $\left(H - E_i - E_j\right) \cdot D = 1$ for $1 \le i < j \le 6$, and six curves with $\left(2H - E_1 - \dots - \widehat{E_i} - \dots - E_6\right) \cdot D = 1$. These are the twenty-seven straight lines on a cubic surface.