Algebraic Geometry

Lectured by Prof Mark Gross
Typed by David Kurniadi Angdinata

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Syllabus

Contents

0 Brief review of classical algebraic geometry and motivation for scheme theory

Lecture 1
Friday
09/10/20

The following are the main references for the course.

- R Hartshorne, Algebraic geometry, 1977
- U Goertz and T Wedhorn, Algebraic geometry I, 2010
- R Vakil, The rising sea: foundations of algebraic geometry, 2017

0.1 Classical algebraic geometry

Throughout this discussion, we take the base field k to be algebraically closed. An **affine variety** $V \subseteq \mathbb{A}^n(k)$, where, once one has chosen coordinates, $\mathbb{A}^n(k) = k^n$, is given by the vanishing of polynomials $f_1, \dots, f_r \in k[X_1, \dots, X_n]$. If $I = \langle f_1, \dots, f_r \rangle \lhd k[X_1, \dots, X_n]$ is any ideal, we set

$$\mathbb{V}(I) = \{z \in \mathbb{A}^n \mid \forall f \in I,\ f(z) = 0\}.$$

First set $\mathbb{P}^n(k) = \left(k^{n+1} \setminus \{0\}\right)/k^*$ with **homogeneous coordinates** $(x_0 : \dots : x_n)$. A **projective variety** $V \subseteq \mathbb{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \dots, F_r \in k[X_0, \dots, X_n]$. If I is the ideal generated by the homogeneous ideals F_i, that is if $F \in I$ then so are all its homogeneous parts, we set

$$\mathbb{V}(I) = \{z \in \mathbb{P}^n \mid \forall F \in I \text{ homogeneous},\ F(z) = 0\}.$$

If $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$, set

$$\mathbb{I}(V) = \{f \in k[X_1, \dots, X_n] \mid \forall x \in V,\ f(x) = 0\}.$$

Observe that $\mathbb{V}(\mathbb{I}(V)) = V$, by tautology, and $\mathbb{I}(\mathbb{V}(I)) \supseteq \sqrt{I}$, which is obvious. Recall that the **radical** $\sqrt{I}$ of the ideal I is defined by $f \in \sqrt{I}$ if and only if there exists $m > 0$ such that $f^m \in I$. **Hilbert's Nullstellensatz** states that, noting $k = \overline{k}$, $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. The **coordinate ring** is

$$k[V] = k[X_1, \dots, X_n]/\mathbb{I}(V).$$

This may be regarded as the ring of polynomial functions on V, and it is a finitely generated reduced k-algebra. Recall that a **k-algebra** is a commutative ring containing k as a subring. It is **finitely generated** if it is the quotient of a polynomial ring over k, and **reduced** if $a^m = 0$ implies that $a = 0$.

0.2 Why schemes?

A better question is what is wrong with varieties?

- With varieties, always work over algebraically closed fields. For example, let $I = \left\langle X^2 + Y^2 + 1 \right\rangle \subseteq \mathbb{R}[X, Y]$. Then $\mathbb{V}(I) = \emptyset$, but I is a prime ideal, hence radical, so $\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[X, Y] \ne I$.
- Number theory? Diophantine equations. If $I \subseteq \mathbb{Z}[X_1, \dots, X_n]$ is an ideal, have $\mathbb{V}(I) \subseteq \mathbb{Z}^n$. For example, $X^n + Y^n = Z^n$.
- Why should we only consider radical, or prime, ideals? For example, a natural situation is

  $$X_1 = \mathbb{V}\left(X - Y^2\right) \subseteq \mathbb{A}^2, \qquad X_2 = \mathbb{V}(X) \subseteq \mathbb{A}^2.$$

  Then $X_1 \cap X_2 = \mathbb{V}\left(X - Y^2, X\right)$. Note $I = \left\langle X - Y^2, X \right\rangle = \left\langle X, Y^2 \right\rangle$ is not a radical ideal, because $Y \notin I$ and $Y^2 \in I$ so $Y \notin \sqrt{I}$. Recall the coordinate ring of X_i is $k[X_i] = k[X, Y]/I_i$. Then $k[X_1 \cap X_2] = k[X, Y]/\left\langle X, Y^2 \right\rangle \cong k[Y]/\left\langle Y^2 \right\rangle$. So thinking of the coordinate ring of $X_1 \cap X_2$ as functions on $X_1 \cap X_2$, we have a function Y whose square is zero, but is not itself zero.

0.3 Categorical philosophy

What is a point? In the category of sets, objects are sets, and if A and B are sets, then morphisms are $\mathrm{Hom}(A, B)$, the set of maps $f : A \to B$. Let $*$ be a one-element set. Then the elements of any set X are in one-to-one correspondence with $\mathrm{Hom}(*, X)$. In the category of affine varieties, objects are affine varieties and morphisms are $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X])$. In this category, a point is a single point with coordinate ring k. Giving a morphism

$$\{\text{point}\} \to X = \mathbb{V}(I) \subseteq \mathbb{A}^n, \qquad I \subseteq k[X_1, \dots, X_n],$$

for I a radical ideal, is the same as giving a homomorphism

$$\begin{array}{cccc} \phi & : & k[X] = k[X_1, \dots, X_n]/I & \longrightarrow & k \\ & & X_i & \longmapsto & a_i \end{array}.$$

Note that ϕ vanishes in I if and only if $f(a_1, \dots, a_n) = 0$ for all $f \in I$, which is if and only if $(a_1, \dots, a_n) \in \mathbb{V}(I) = X$. Note ϕ is surjective, and hence $\ker \phi$ is a maximal ideal. With k algebraically closed, the maximal ideals at $k[X]$ are all of the form $\langle X_1 - a_1, \dots, X_n - a_n \rangle$ for $(a_1, \dots, a_n) \in X$, a consequence of Hilbert's Nullstellensatz. That is, there exists one-to-one correspondences

$$\{\text{points of } X\} \quad \leftrightsquigarrow \quad \{k\text{-algebra homomorphisms } \phi : k[X] \to k\} \quad \leftrightsquigarrow \quad \{\text{maximal ideals of } k[X]\}.$$

0.4 Solutions over non-algebraically closed fields

What if k is not algebraically closed? We may want to consider solutions not just in $k^n = \mathbb{A}^n$ but $(k')^n$ for k' any field extension of k. That is, we may consider k-algebra homomorphisms

$$\begin{array}{cccc} \phi & : & k[X] = k[X_1, \dots, X_r]/I & \longrightarrow & k' \\ & & X_i & \longmapsto & a_i \end{array}.$$

This gives a tuple $(a_1, \dots, a_n) \in (k')^n$ with $f(a_1, \dots, a_n) = 0$ for all $f \in I$. Then ϕ need not be surjective, so can only say the image of ϕ is a subring of a field, hence an integral domain. Thus $\ker \phi$ is a prime ideal, and maximal if and only if $\mathrm{im}\, \phi$ is a field.

Example. The $\mathbb{R}$-algebra homomorphism

$$\begin{array}{cccc} \phi & : & \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle & \longrightarrow & \mathbb{C} \\ & & X & \longmapsto & 0 \\ & & Y & \longmapsto & i \end{array}$$

is surjective with kernel $\langle X, Y^2 + 1 \rangle$, since $\mathbb{R}[Y] / \langle Y^2 + 1 \rangle \cong \mathbb{C}$. This is a maximal ideal but is not of the form $\langle X - a, Y - b \rangle$ for $(a, b) \in \mathbb{R}^2$. If instead we considered the map

$$\begin{array}{ccc} \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle & \longrightarrow & \mathbb{C} \\ X & \longmapsto & 0 \\ Y & \longmapsto & -i \end{array},$$

we get the same kernel. That is, $(0, i)$ and $(0, -i)$ are solutions to $X^2 + Y^2 + 1$, but they correspond to the same maximal ideal. In fact, this maximal ideal corresponds to a Galois orbit of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of solutions.

There are more exotic points by taking even bigger fields.

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Example. Let $k(X)$ be the field of fractions of $k[X] = \mathbb{R}[X, Y] / \langle X^2 + Y^2 + 1 \rangle$. There is an inclusion

$$\begin{array}{ccc} k[X] & \longrightarrow & k(X) \\ f & \longmapsto & \frac{f}{1} \\ (X, Y) & \longmapsto & (X, Y) \end{array}.$$

The kernel of this map is zero. This gives a solution to the equation $X^2 + Y^2 + 1 = 0$ with coordinates in the field $k(X)$. This solution is $(X, Y) \in \mathbb{A}^2(k(X))$.

The moral is that once we start looking at solutions to equation over any field, then we get maps $k[X] \to k'$ with kernel not necessarily maximal. What about solutions over rings?

Example. Let $A = \mathbb{Z}[X_1, \dots, X_n]/I$, and let R be any commutative ring. We define an R-valued point of $\operatorname{Spec} A$ to be a ring homomorphism

$$\begin{array}{ccc} A & \longrightarrow & R \\ X_i & \longmapsto & r_i \end{array}.$$

Then $f(r_1, \dots, r_n) = 0$ for all $f \in I$. This gives a lot of flexibility. For example,

- $R = \mathbb{Z}$ gives diophantine equations,
- $R = \mathbb{F}_p$ gives solutions modulo p, and
- $R = \mathbb{Q}$ gives rational solutions.

Take this to its logical conclusion. Let A be a ring, where all rings are commutative in this course. Given A, we hope for some geometric object $\operatorname{Spec} A$, the **spectrum** of A. For a ring R, the set of **R-valued points** of X is

$$X(R) = \operatorname{Hom}_{\mathrm{ring}}(A, R).$$

A morphism $X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ should be the same thing as giving a morphism $\phi : B \to A$. Define the category of **affine schemes** to be the opposite category to the category of rings. Define a **scheme** to be something which is locally isomorphic to an affine scheme. By analogy, a **manifold** is a topological space with an open cover $\{U_i\}$ with each U_i homeomorphic to an open subset of $\mathbb{R}^n$. To make sense of the definition of schemes, we need a lot of language.

0.5 Spectrum of a ring

Definition. Let A be a ring. Then

$$\operatorname{Spec} A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ a prime ideal}\}.$$

For $I \subseteq A$ an ideal, define

$$\mathbb{V}(I) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime}, \ \mathfrak{p} \supseteq I\}.$$

Proposition 0.1. *The sets $\mathbb{V}(I)$ form the closed sets of a topology on* $\operatorname{Spec} A$*, called the* ***Zariski topology****.*

Proof.

- $\mathbb{V}(A) = \emptyset$.
- $\mathbb{V}(0) = \operatorname{Spec} A$.
- If $\{I_i\}_{i \in J}$ is a collection of ideals, then
$$\mathbb{V}\left(\sum_{i \in J} I_i\right) = \bigcap_{i \in J} \mathbb{V}(I_i).$$
- Claim that
$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2).$$
  $\supseteq$ Obvious.
  $\subseteq$ If $\mathfrak{p} \supseteq I_1 \cap I_2$ is prime, then $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$. See Atiyah-Macdonald, Proposition 1.11.ii. [1]

$\square$

Example. Let $A = k[X_1, \dots, X_n]$ with k algebraically closed and $I \subseteq A$ an ideal. Then the maximal ideals $\mathfrak{m}$ of A containing I are in one-to-one correspondence with the zero set of I in $\mathbb{A}^n(k)$, so

$$\{ \ \langle X_1 - a_1, \dots, X_n - a_n \rangle \supseteq I, \ a_i \in k \ \} \quad \leftrightsquigarrow \quad \{ \ (a_1, \dots, a_n) \in \mathbb{V}(I) \subseteq \mathbb{A}^n(k) \ \}.$$

The new $\mathbb{V}(I)$ now extends this notion of zero set by including possible other prime ideals.

Example. If k is a field, $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field.

We fix this by also thinking about what functions are on these spaces.

[1] Exercise: try to prove without looking up

1 Sheaves

Fix a topological space X.

1.1 Sheaves

Definition. A **presheaf** $\mathcal{F}$ on X consists of the following data.

- For every open set $U \subseteq X$ an abelian group $\mathcal{F}(U)$.
- Whenever given an inclusion $V \subseteq U \subseteq X$, a **restriction map** $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, a homomorphism, such that
  - $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and
  - if $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark. Can think of a presheaf as a contravariant functor from the category of open sets of X, the category whose objects are open subsets of X and whose morphisms are inclusions of open sets, to the category of abelian groups. Can replace the category of abelian groups with any desired category, such as commutative rings.

Definition. A **morphism of presheaves** $f : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V) \end{tikzcd}$$

is commutative.

Definition. A presheaf $\mathcal{F}$ is a **sheaf** if it satisfies the following additional axioms.

S1. If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = \rho_{UU_i}(s) = 0$ for all i, then $s = 0$.

S2. If U and $\{U_i\}$ are as in S1 and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all i.

Remark.

- If $\mathcal{F}$ is a sheaf, then $\emptyset \subseteq X$ is covered by the empty covering, and hence $\mathcal{F}(\emptyset) = 0$.
- S1 and S2 together can be described as saying, given U and $\{U_i\}_{i \in I}$,

  $$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

  is exact, where

  $$\alpha(s) = \left(s|_{U_i}\right)_{i \in I}, \qquad \beta_1\left((s_i)_{i \in I}\right) = \left(s_i|_{U_i \cap U_j}\right)_{i,j}, \qquad \beta_2\left((s_i)_{i \in I}\right) = \left(s_j|_{U_i \cap U_j}\right)_{i,j}.$$

  Exactness means
  - α is injective, which is S1,
  - $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and
  - for any $(s_i) \in \prod_{i \in I} \mathcal{F}(U_i)$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}(U)$ with $\alpha(s) = (s_i)$, which is S2.

1.2 Examples

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Example.

- Let X be any topological space, and let

$$\mathcal{F}(U) = \{\text{continuous functions } U \to \mathbb{R}\}.$$

  This is a sheaf, by

$$\begin{array}{cccccc} \rho_{UV} & : & \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) \\ & & f & \longmapsto & f|_V \end{array}.$$

  S1. A continuous function is zero if it is zero on every open set of a cover.

  S2. Continuous functions can be glued.

- Let $X = \mathbb{C}$ with the Euclidean topology, and let

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ is a bounded analytic function}\}.$$

  This is a presheaf. It satisfies S1, and does not satisfy S2. For example, consider the cover $\{U_i\}_{i\in\{1,2,\dots\}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$ and

$$\begin{array}{cccccc} s_i & : & U_i & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & z \end{array}.$$

  Note if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. But if we glue we get the function $z : \mathbb{C} \to \mathbb{C}$, which is not bounded. Note $\mathcal{F}(\mathbb{C}) = \mathbb{C}$.

- Take any group G and set $\mathcal{F}(U) = G$ for any open set U. This is called the **constant presheaf**. This is not a sheaf. Let $U = U_1 \sqcup U_2$. If we wanted a sheaf,

$$\begin{tikzcd} \mathcal{F}(U_1) = G \arrow[dr] & & \mathcal{F}(U_2) = G \arrow[dl] \\ & \mathcal{F}(U_1 \cap U_2) = \mathcal{F}(\emptyset) = 0 & \end{tikzcd},$$

  so if S2 is satisfied, would want $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ to glue. We would then want to have $\mathcal{F}(U) = G \times G$. Now give G the discrete topology, and define instead

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\},$$

  that is f is locally constant. That is, if $x \in U$, there exists a neighbourhood $x \in V \subseteq U$ with $f|_V$ constant. This is called the **constant sheaf** and if U is non-empty and connected, then $\mathcal{F}(U) = G$.

- If X is an algebraic variety, and $U \subseteq X$ is a Zariski open subset, define

$$\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ regular function}\}.$$

  Roughly f is **regular** means that every point of U has an open neighbourhood on which f is expressed as a ratio of polynomials g/h with h non-vanishing on the neighbourhood. Then $\mathcal{O}_X$ is a sheaf, called the **structure sheaf** of X.

1.3 Stalks

Definition. Let $\mathcal{F}$ be a presheaf on X. Let $p \in X$. Then the **stalk** of $\mathcal{F}$ at p is

$$\mathcal{F}_p = \{(U, s) \mid U \subseteq X \text{ is an open neighbourhood of } p,\ s \in \mathcal{F}(U)\} / \equiv,$$

where $(U, s) \equiv (V, s')$ if there exists $W \subseteq U \cap V$ also a neighbourhood of p such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a **germ**.

Remark. $\mathcal{F}_p = \varinjlim_{p\in U} \mathcal{F}(U)$.

Note that a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism

$$\begin{array}{cccc} f_p & : & \mathcal{F}_p & \longrightarrow & \mathcal{G}_p \\ & & (U, s) & \longmapsto & (U, f_U(s)) \end{array}.$$

Proposition 1.1. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$.*

Proof.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ Assume f_p is an isomorphism for all $p \in X$. Need to show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all $U \subseteq X$, as then we can define $\left(f^{-1}\right)_U = (f_U)^{-1}$. Check that with this definition, $\left(f^{-1}\right)_U$ is compatible with restriction maps, hence f^{-1} is a morphism of sheaves. [2]

- f_U is injective. Suppose $s \in \mathcal{F}(U)$, and $f_U(s) = 0$. Then for all $p \in U$, $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. Since f_p is injective, $(U, s) = 0$ in $\mathcal{F}_p$. That is, there exists a open neighbourhood V_p of p in U such that $s|_{V_p} = 0$. Since $\{V_p\}_{p \in U}$ cover U, we see by S1 that $s = 0$.
- f_U is surjective. Let $t \in \mathcal{G}(U)$ and write $t_p = (U, t) \in \mathcal{G}_p$. Since f_p is surjective, there exists $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. That is, there exists $V_p \subseteq U$ an open neighbourhood of p, and a germ (V_p, s_p) such that $\left(V_p, f_{V_p}(s_p)\right) \equiv (U, t)$. By shrinking V_p if necessary, we can assume that $t|_{V_p} = f_{V_p}(s_p)$. Now on $V_p \cap V_q$,

$$f_{V_p \cap V_q}\left(s_p|_{V_p \cap V_q} - s_q|_{V_p \cap V_q}\right) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0,$$

  and hence by injectivity of $f_{V_p \cap V_q}$ already proved, we have $s_p|_{V_p \cap V_q} = s_q|_{V_p \cap V_q}$. By S2 the s_p's glue to give an element $s \in \mathcal{F}(U)$ with $s|_{V_p} = s_p$, for all $p \in U$. Now

$$f_U(s)|_{V_p} = f_{V_p}\left(s|_{V_p}\right) = f_{V_p}(s_p) = t|_{V_p}.$$

  By S1, applied to $f_U(s) - t$, we get $f_U(s) = t$. Thus f_U is surjective.

□

1.4 Sheafification

Theorem 1.2 (Sheafification). *Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property. For any sheaf $\mathcal{G}$ and morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi^+ \circ \theta = \phi$, so*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dashed, "\phi^+"] \\ & \mathcal{G} \end{tikzcd}.$$

*The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism, and is called the **sheafification** of $\mathcal{F}$.*

Proof. See example sheet 1. The idea is to make $\mathcal{F}^+$ look like functions. Define

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall p \in U,\ s(p) \in \mathcal{F}_p, \\ \forall p \in U,\ \exists p \in V \subseteq U,\ \exists t \in \mathcal{F}(V),\ \forall q \in V,\ s(q) = (V, t) \in \mathcal{F}_q \end{array} \right\}.$$

Then

$$\begin{array}{cccc} \theta_U & : & \mathcal{F}(U) & \longrightarrow & \mathcal{F}^+(U) \\ & & s & \longmapsto & (p \mapsto (U, s) \in \mathcal{F}_p) \end{array}.$$

□

Exercise. A recommendation is to do all exercises in chapter II.1 of Hartshorne.

[2] Exercise

1.5 Kernels, cokernels, and images

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Definition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a space X. We define the following.

- The **presheaf kernel** of f, $\ker f$, is the presheaf given by $(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$.
- The **presheaf cokernel** $\operatorname{coker} f$ is the presheaf given by $(\operatorname{coker} f)(U) = \operatorname{coker}(f_U) = \mathcal{G}(U) / \operatorname{im} f_U$.
- The **presheaf image** $\operatorname{im} f$ is the presheaf given by $(\operatorname{im} f)(U) = \operatorname{im} f_U$.

Exercise. Check that these are presheaves, that is restrictions work.

Remark 1.3. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is also a sheaf.

Proof. S1 is certainly satisfied. If $s \in (\ker f)(U) \subseteq \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for all U_i in a cover of U so $s = 0$ by S1 for $\mathcal{F}$. Given $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ an open cover of U, and with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ by S2 for $\mathcal{F}$. But $f_U(s) = 0$ since $f_U(s)|_{U_i} = f_{U_i}\left(s|_{U_i}\right) = f_{U_i}(s_i) = 0$ so by S1, $f_U(s) = 0$. ☐

Example. Let $X = \mathbb{P}^1$, or think of the Riemann sphere. Let $P, Q \in X$ be distinct points. Let $\mathcal{G}$ be the sheaf of regular functions on X, or think of the sheaf of holomorphic functions. Let $\mathcal{F}$ be the sheaf of regular functions on X which vanish at P and Q. Note $\mathcal{F}(U) = \mathcal{G}(U)$ if $U \cap \{P, Q\} = \emptyset$. Let $U = \mathbb{P}^1 \setminus \{P\}$ and $V = \mathbb{P}^1 \setminus \{Q\}$. Note $\mathcal{F}\left(\mathbb{P}^1\right) = 0$ and $\mathcal{G}\left(\mathbb{P}^1\right) = k$, because regular functions on $\mathbb{P}^1$ are constants. Let $f : \mathcal{F} \to \mathcal{G}$ be the obvious inclusion. Then

$$(\operatorname{coker} f)\left(\mathbb{P}^1\right) = k, \qquad (\operatorname{coker} f)(U) = \mathcal{G}(U) / \mathcal{F}(U) = k[X] / \langle X \rangle = k,$$
$$(\operatorname{coker} f)(V) = k, \qquad (\operatorname{coker} f)(U \cap V) = \mathcal{G}(U \cap V) / \mathcal{F}(U \cap V) = 0.$$

If S2 holds, then we would need to have $(\operatorname{coker} f)\left(\mathbb{P}^1\right) = k \oplus k$. This is not a bug, but a feature.

Definition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

- The **sheaf kernel** $\ker f$ of f is just the presheaf kernel.
- The **sheaf cokernel** is the sheaf associated to the presheaf cokernel of f.
- The **sheaf image** is the sheaf associated to the presheaf image of f.

$\mathcal{F}$ is a **subsheaf** of $\mathcal{G}$ if we have inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ for all U compatible with restrictions.

Exercise. The sheaf image $\operatorname{im} f$ is a subsheaf of $\mathcal{G}$.

We say f is **injective** if $\ker f = 0$. We say f is **surjective** if $\operatorname{im} f = \mathcal{G}$. We say a sequence of morphisms of sheaves

$$\dots \to \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \to \dots$$

is **exact** if $\ker f^{i+1} = \operatorname{im} f^i$ for all i. If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, we write $\mathcal{F}/\mathcal{F}'$ for the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) / \mathcal{F}'(U)$. That is, this is the cokernel of the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$. A warning is if $f : \mathcal{F} \to \mathcal{G}$ is surjective, we do not necessarily have $\mathcal{F}(U) \to \mathcal{G}(U)$ surjective for all U.

Lemma 1.4. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then for all $p \in X$,*

$$(\ker f)_p = \ker(f_p : \mathcal{F}_p \to \mathcal{G}_p), \qquad (\operatorname{im} f)_p = \operatorname{im} f_p.$$

Proof. Have a map

$$\begin{array}{ccc} (\ker f)_p & \longrightarrow & \ker f_p \subseteq \mathcal{F}_p \\ (U, s) & \longmapsto & (U, s) \end{array}.$$

If $s \in (\ker f)(U) = \ker f_U$ represents a germ $(U, s) \in (\ker f)_p$, then $(U, s) \in \mathcal{F}_p$, and $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. So $(U, s) \in \ker f_p$.

- Injective. If $(U, s) = 0$ in $\mathcal{F}_p$, there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, s|_V) = (V, 0) = 0$ in $(\ker f)_p$.
- Surjective. If $(U, s) \in \ker f_p$, then $(U, f_U(s)) = 0$ in $\mathcal{G}_p$. That is, there exists a neighbourhood $V \subseteq U$ of p such that $0 = f_U(s)|_V = f_V(s|_V)$. Thus $s|_V \in (\ker f)(V)$, and $(V, s|_V) \in (\ker f)_p$, and $(V, s|_V)$ maps to the same element in $\ker f_p$ represented by (U, s).

Let $\mathrm{im}' f$ be the presheaf image. An easy fact is if $\mathcal{F}$ is a presheaf with associated sheaf $\mathcal{F}^+$, then $\mathcal{F}_p \cong \mathcal{F}_p^+$ for all $p \in X$. [3] Thus $(\mathrm{im}\, f)_p = (\mathrm{im}' f)_p$, so need to show $(\mathrm{im}' f)_p \cong \mathrm{im}\, f_p$. Define a map by

$$\begin{array}{ccc} (\mathrm{im}' f)_p & \longrightarrow & \mathrm{im}\, f_p \\ (U, s) & \longmapsto & (U, s) \end{array}.$$

- Injective. If $(U, s) = 0$ in $\mathcal{G}_p$ then there exists a neighbourhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) \sim (V, 0)$ in $(\mathrm{im}' f)_p$.
- Surjective. If $(U, s) \in \mathrm{im}\, f_p$, then there exists $(V, t) \in \mathcal{F}_p$ with $(U, s) = f_p(V, t) = (V, f_V(t))$, so after shrinking U and V if necessary, then we can take $U = V$ and $f_U(t) = s$. Then $(U, s) \in (\mathrm{im}' f)_p$.

□

Proposition 1.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then*

1. *f is injective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p, and*
2. *f is surjective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p.*

Proof.

1. f_p is injective for all p if and only if $\ker f_p = 0$ for all p, if and only if $(\ker f)_p = 0$ for all p, if and only if $\ker f = 0$, [4] which is if and only if f is injective.
2. f_p is surjective for all p if and only if $\mathrm{im}\, f_p = \mathcal{G}_p$ for all p, if and only if $(\mathrm{im}\, f)_p = \mathcal{G}_p$ for all p, if and only if $\mathrm{im}\, f = \mathcal{G}$, [5] which is if and only if f is surjective.

□

Remark. Given $f : \mathcal{F} \to \mathcal{G}$, in fact $\mathcal{G} / \mathrm{im}\, f \cong \mathrm{coker}\, f$. [6]

1.6 Passing between spaces

Lecture 5 Monday 19/10/20

Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Define $f_* \mathcal{F}$ by, for $U \subseteq Y$

$$(f_* \mathcal{F})(U) = \mathcal{F}\left(f^{-1}(U)\right).$$

Exercise. Check $f_* \mathcal{F}$ is a sheaf on Y.

Define $f^{-1} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \subseteq X \mapsto \{(V, s) \mid V \supseteq f(U),\ V \text{ open},\ s \in \mathcal{G}(V)\} / \sim,$$

where $(V, s) \sim (V', s')$ if there exists $W \subseteq V \cap V'$ such that $f(U) \subseteq W$, and $s|_W = s'|_W$.

Example. If $f : \{p\} \to X$ is an inclusion of a point, then $f^{-1} \mathcal{G} = \mathcal{G}_p$. This is a group but defines a sheaf on a one-point space. More generally, if $\iota : Z \hookrightarrow X$ is an inclusion of a subset with induced topology, we often write

$$\mathcal{F}|_Z = \iota^{-1} \mathcal{F}.$$

If Z is open in X, then this is easy, since if $U \subseteq Z$ then $\mathcal{F}|_Z(U) = \mathcal{F}(U)$.

Remark. If $s \in \mathcal{F}(U)$ we say s is a **section** of $\mathcal{F}$ over U. We often write

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}),$$

thinking of $\Gamma(U, \cdot)$ as a functor from the category of sheaves on a space X to the category of abelian groups.

[3] Exercise: check
[4] Exercise: check by S1
[5] Exercise: check using $\mathrm{im}\, f \subseteq \mathcal{G}$
[6] Exercise

2 Schemes

Want to construct a sheaf $\mathcal{O}$ on $\operatorname{Spec} A$, analogous to the sheaf of regular functions on a variety, and $\mathcal{O}$ will be a sheaf of rings. That is, $\mathcal{O}(U)$ will be a ring for each open set U and restriction maps will be ring homomorphisms.

2.1 Localisation of a ring

Importantly recall the following. Let A be a ring, where all rings are commutative with unity, and $S \subseteq A$ be a multiplicatively closed subset, that is $1 \in S$ and if $s_1, s_2 \in S$ then $s_1 s_2 \in S$. We define a ring

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\} / \sim,$$

where $(a, s) \sim (a', s')$ if there exists $s'' \in S$ such that $s''(as' - a's) = 0$. Then $S^{-1}A$ is called the **localisation of A at S**. Note that we write a/s for the equivalence class of (a, s). The usual equivalence relation on fractions is $a/s = a'/s'$ if and only if $as' = a's$. We need the extra possibility of killing $as' - a's$ with s'' if A is not an integral domain.

Example.

- Take $f \in A$ and $S = \{1, f, \dots\} \subseteq A$. Then we write $A_f = S^{-1}A$. These will correspond to open subsets.
- If $\mathfrak{p} \subseteq A$ is a prime ideal and $S = A \setminus \mathfrak{p}$, then
  - $1 \in S$, and
  - $a, b \in S$ and $ab \in \mathfrak{p}$ is a contradiction by definition of prime ideals, so $ab \in S$.

  Then $A_{\mathfrak{p}} = S^{-1}A$ is the **localisation of A at $\mathfrak{p}$**. These will correspond to stalks.

2.2 Construction of the structure sheaf

Let

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ \forall \mathfrak{p} \in U,\ \exists \mathfrak{p} \in V \subseteq U \text{ open},\ \exists a, f \in A,\ \forall \mathfrak{q} \in V,\ f \notin \mathfrak{q},\ s(\mathfrak{q}) = \dfrac{a}{f} \in A_{\mathfrak{q}} \end{array} \right\}.$$

Proposition 2.1. *For any $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$.*

Proof. Have a map

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}} \\ (U, s) & \longmapsto & s(\mathfrak{p}) \end{array}.$$

- Surjective. Any element of $A_{\mathfrak{p}}$ can be written as a/f for some $a \in A$ and $f \notin \mathfrak{p}$. Then $\mathbb{D}(f) = \operatorname{Spec} A \setminus \mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$, since $\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}$. Now a/f defines an element of $\mathcal{O}(\mathbb{D}(f))$ given by

$$\begin{array}{cccc} s : & \mathbb{D}(f) & \longrightarrow & A_{\mathfrak{q}} \\ & \mathfrak{q} & \longmapsto & \dfrac{a}{f} \end{array},$$

  and in particular, $s(\mathfrak{p}) = a/f \in A_{\mathfrak{p}}$.
- Injective. Let $\mathfrak{p} \in U \subseteq \operatorname{Spec} A$ and $s \in \mathcal{O}(U)$ with $s(\mathfrak{p}) = 0$ in $A_{\mathfrak{p}}$. Want to show $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$. By shrinking U if necessary, we can assume that s is given by $a, f \in A$ with $s(\mathfrak{q}) = a/f$ for all $\mathfrak{q} \in U$. In particular $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$. Thus $a/f = 0/1$ in $A_{\mathfrak{p}}$ so there exists $h \in A \setminus \mathfrak{p}$ such that $0 = h \cdot (a \cdot 1 - f \cdot 0) = h \cdot a$ in A. Now let $V = U \cap \mathbb{D}(h)$. Then $(V, s|_V) = 0$, since for $\mathfrak{q} \in V$, $s|_V(\mathfrak{q}) = s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$ and $h \cdot a = 0$, and $h \in A \setminus \mathfrak{q}$ so $h \cdot a = 0$ implies $a/f = 0/1$ in $A_{\mathfrak{q}}$. Thus $(U, s) = 0$ in $\mathcal{O}_{\mathfrak{p}}$.

$\square$

Proposition 2.2. *For any $f \in A$, $\mathcal{O}(\mathbb{D}(f)) = A_f$.*

In particular, as $\operatorname{Spec} A = \mathbb{D}(1)$, the **global sections** of $\mathcal{O}$ is $\mathcal{O}(\operatorname{Spec} A) = A_1 = A$.

Proof. Let

$$\begin{array}{cccc} \psi & : & A_f & \longrightarrow & \mathcal{O}(\mathbb{D}(f)) \\ & & \dfrac{a}{f^n} & \longmapsto & \left(\mathfrak{p} \in \mathbb{D}(f) \mapsto \dfrac{a}{f^n} \in A_{\mathfrak{p}}\right) \end{array},$$

since $f \notin \mathfrak{p}$ implies that $f^n \notin \mathfrak{p}$ for all $n \geq 0$.

- Injective. If $\psi(a/f^n) = 0$, then for all $\mathfrak{p} \in \mathbb{D}(f)$, $a/f^n = 0$ in $A_{\mathfrak{p}}$, that is there exists $h \in A \setminus \mathfrak{p}$ such that $h \cdot a = 0$ in A. Let $I = \{g \in A \mid g \cdot a = 0\}$, the **annihilator** of a. So $h \in I$ and $h \notin \mathfrak{p}$, so $I \not\subseteq \mathfrak{p}$. This is true for all $\mathfrak{p} \in \mathbb{D}(f)$, so $\mathbb{V}(I) \cap \mathbb{D}(f) = \emptyset$. Thus $f \in \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p} = \sqrt{I}$, the radical, so $f^m \in I$ for some $m > 0$. Thus $f^m \cdot a = 0$, so $a/f^n = 0$ in A_f. Thus ψ is injective.

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- Surjective. Let $s \in \mathcal{O}(\mathbb{D}(f))$. Cover $\mathbb{D}(f)$ with open sets V_i on which s is represented as a_i/g_i with $a_i, g_i \in A$ such that $g_i \notin \mathfrak{p}$ whenever $\mathfrak{p} \in V_i$. Thus $V_i \subseteq \mathbb{D}(g_i)$. By question 1 on example sheet 1, the sets of the form $\mathbb{D}(h)$ form a base for the Zariski topology on $\operatorname{Spec} A$. Thus we can assume $V_i = \mathbb{D}(h_i)$ for some $h_i \in A$. Since $\mathbb{D}(h_i) \subseteq \mathbb{D}(g_i)$, we have $\mathbb{V}(h_i) \supseteq \mathbb{V}(g_i)$, so $\sqrt{\langle h_i \rangle} \subseteq \sqrt{\langle g_i \rangle}$, so $h_i^n \in \langle g_i \rangle$ for some n, say $h_i^n = c_i g_i$, so $a_i/g_i = c_i a_i / h_i^n$. Now replace h_i by h_i^n, since this does not change open sets because in general $\mathbb{D}(h_i) = \mathbb{D}(h_i^n)$, and replace a_i by $c_i a_i$. The situation so far is that we can assume $\mathbb{D}(f)$ is covered by sets $\mathbb{D}(h_i)$ such that s is represented by a_i/h_i on $\mathbb{D}(h_i)$. Claim that $\mathbb{D}(f)$ can be covered by a finite number of the $\mathbb{D}(h_i)$, that is $\mathbb{D}(f)$ is quasi-compact. Since

$$\mathbb{D}(f) \subseteq \bigcup_i \mathbb{D}(h_i) \quad \iff \quad \mathbb{V}(f) \supseteq \bigcap_i \mathbb{V}(h_i) = \mathbb{V}\left(\sum_i \langle h_i \rangle\right) \quad \iff \quad f \in \bigcap_{\mathfrak{p} \in \mathbb{V}\left(\sum_i \langle h_i \rangle\right)} \mathfrak{p}$$

$$\iff \quad f \in \sqrt{\sum_i \langle h_i \rangle} \quad \iff \quad \exists n,\ f^n \in \sum_i \langle h_i \rangle,$$

we can write $f^n = \sum_{i \in I} b_i h_i$ for some finite index set I. Thus reversing this argument, $\mathbb{D}(f) \subseteq \bigcup_{i \in I} \mathbb{D}(h_i)$. We now pass to this finite subcover $\{\mathbb{D}(h_i)\}$. On $\mathbb{D}(h_i) \cap \mathbb{D}(h_j) = \mathbb{D}(h_i h_j)$, note a_i/h_i and a_j/h_j both represent s, so by injectivity shown in the last lecture, $a_i h_j / h_i h_j = a_i/h_i = a_j/h_j = a_j h_i / h_i h_j$ in $A_{h_i h_j}$. Thus for some n, $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A. We can pick an n sufficiently large to work for all pairs i and j. Rewriting, $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j a_j) = 0$. Replace each h_i by h_i^{n+1} and a_i by $h_i^n a_i$, since $a_i/h_i = a_i h_i^n / h_i^{n+1}$. Thus we can assume that s is still represented on $\mathbb{D}(h_i)$ by a_i/h_i but also for each i and j have $h_i a_j = h_j a_i$. Note $f^n = \sum_i b_i h_i$ for $b_i \in A$, since $\{\mathbb{D}(h_i)\}$ cover $\mathbb{D}(f)$. Let $a = \sum_i b_i a_i$. Then for any j, $h_j a = \sum_i b_i a_i h_j = \sum_i b_i a_j h_i = f^n a_j$. Thus $a/f^n = a_j/h_j$ on $\mathbb{D}(h_j)$. Thus $\psi(a/f^n) = s$, so ψ is surjective.

∎

We now have a topological space $\operatorname{Spec} A$ equipped with a sheaf of rings $\mathcal{O}$.

2.3 Ringed spaces

Definition. A **ringed space** is a pair $(X, \mathcal{O}_X)$ where

- X is a topological space, and
- $\mathcal{O}_X$ is a sheaf of rings on X.

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the following data.

- $f : X \to Y$ a continuous map.
- $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ a morphism of sheaves of rings, that is for each $U \subseteq Y$ open, we have a ring homomorphism $f_U^\# : \mathcal{O}_Y(U) \to (f_* \mathcal{O}_X)(U) = \mathcal{O}_X\left(f^{-1}(U)\right)$.

Example.

- Let X be a topological space, and let $\mathcal{O}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions. Then if $(Y, \mathcal{O}_Y)$ is similarly defined, given $f : X \to Y$, we get $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ defined by
$$\begin{array}{rcccl} f_U^\# & : & \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X\left(f^{-1}(U)\right) \\ & & \phi & \longmapsto & \phi \circ f \end{array}.$$
- Let X be a variety, and let $\mathcal{O}_X$ be the sheaf of regular functions on X. A morphism of varieties $f : X \to Y$ is a continuous map inducing
$$\begin{array}{rcccl} f_U^\# & : & \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X\left(f^{-1}(U)\right) \\ & & \phi & \longmapsto & \phi \circ f \end{array}.$$

A ring is **local** if it has a unique maximal ideal.

Definition. A **locally ringed space** $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$. A **morphism** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **of locally ringed spaces** is a morphism of ringed spaces such that the induced homomorphism $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a **local homomorphism** for all $p \in X$.

- The map is defined by [7]
$$\begin{array}{rcccl} f_p^\# & : & \mathcal{O}_{Y,f(p)} & \longrightarrow & \mathcal{O}_{X,p} \\ & & (U, s) & \longmapsto & \left(f^{-1}(U), f_U^\#(s)\right) \end{array}.$$
- A ring homomorphism $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is **local** if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A. Note that $\phi(A \setminus \mathfrak{m}_A) = \phi(A^*) \subseteq B^* = B \setminus \mathfrak{m}_B$, where A^* is the set of invertible elements of A. Thus $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ always.

Example. In the case of varieties, $\mathcal{O}_{X,p}$ has a unique maximal ideal
$$\{(U, f) \in \mathcal{O}_X(U) \mid f(p) = 0\} / \sim .$$
If $f(p) \neq 0$, f is nowhere vanishing on some neighbourhood of p, so after shrinking U, we can invert f. The local homomorphism condition just follows from the pull-back $\phi \circ f$ of a function ϕ vanishing at $f(p)$ vanishes at p.

2.4 Affine schemes

Lecture 7 Friday 23/10/20

The key example $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space, which we call an affine scheme.

Theorem 2.3. *The category of affine schemes with locally ringed morphisms is equivalent to the opposite category of rings.*

Need to show that

1. if $\phi : A \to B$ is a ring homomorphism, we obtain an induced morphism $\left(f, f^\#\right) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, and
2. any morphism of affine schemes as locally ringed spaces arises in this way.

Proof.

1. Given a ring homomorphism $\phi : A \to B$, define
$$\begin{array}{rcccl} f & : & \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ & & \mathfrak{p} & \longmapsto & \phi^{-1}(\mathfrak{p}) \end{array}.$$
Note $\phi^{-1}(\mathfrak{p})$ is prime, since if $ab \in \phi^{-1}(\mathfrak{p})$, then $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, thus either $\phi(a) \in \mathfrak{p}$ or $\phi(b) \in \mathfrak{p}$, and hence either $a \in \phi^{-1}(\mathfrak{p})$ or $b \in \phi^{-1}(\mathfrak{p})$. Then f is continuous, since
$$\begin{aligned} f^{-1}(\mathbb{V}(I)) &= f^{-1}(\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq I\}) = \{\mathfrak{q} \in \operatorname{Spec} B \mid f(\mathfrak{q}) \supseteq I\} \\ &= \left\{\mathfrak{q} \in \operatorname{Spec} B \;\middle|\; \phi^{-1}(\mathfrak{q}) \supseteq I\right\} = \{\mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \supseteq \phi(I)\} = \mathbb{V}(\phi(I)). \end{aligned}$$

[7] Exercise: check well-defined

We need to construct $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$. For $\mathfrak{p} \in \operatorname{Spec} B$, we obtain a natural homomorphism

$$\begin{array}{cccc} \phi_{\mathfrak{p}} & : & A_{\phi^{-1}(\mathfrak{p})} & \longrightarrow & B_{\mathfrak{p}} \\ & & \dfrac{a}{s} & \longmapsto & \dfrac{\phi(a)}{\phi(s)} \end{array}.$$

Note $\phi_{\mathfrak{p}}$ is a local homomorphism, since the maximal ideal $\mathfrak{p} B_{\mathfrak{p}}$ of $B_{\mathfrak{p}}$ is generated by the image of $\mathfrak{p}$ under the map

$$\begin{array}{ccc} B & \longrightarrow & B_{\mathfrak{p}} \\ b & \longmapsto & \dfrac{b}{1} \end{array},$$

and the maximal ideal $\phi^{-1}(\mathfrak{p}) A_{\phi^{-1}(\mathfrak{p})}$ of $A_{\phi^{-1}(\mathfrak{p})}$ is generated by the image of $\phi^{-1}(\mathfrak{p})$ under the map

$$\begin{array}{ccc} A & \longrightarrow & A_{\phi^{-1}(\mathfrak{p})} \\ a & \longmapsto & \dfrac{a}{1} \end{array},$$

so have a commutative diagram

$$\begin{tikzcd} \phi^{-1}(\mathfrak{p}) & \subset & A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \supset & \mathfrak{p} \\ f(\mathfrak{p}) A_{f(\mathfrak{p})} & \subset & A_{\phi^{-1}(\mathfrak{p})} \arrow[r, "\phi_{\mathfrak{p}}"'] & B_{\mathfrak{p}} & \supset & \mathfrak{p} B_{\mathfrak{p}} \end{tikzcd},$$

thus $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p} B_{\mathfrak{p}}) = \phi^{-1}(\mathfrak{p}) A_{\phi^{-1}(\mathfrak{p})}$. Given $V \subseteq \operatorname{Spec} A$ open, we may define

$$\begin{array}{cccc} f_V^{\#} & : & \mathcal{O}_{\operatorname{Spec} A}(V) & \longrightarrow & \mathcal{O}_{\operatorname{Spec} B}\left(f^{-1}(V)\right) \\ & & (\mathfrak{p} \in V \mapsto s(\mathfrak{p}) \in A_{\mathfrak{p}}) & \longmapsto & \left(\mathfrak{q} \in f^{-1}(V) \mapsto \phi_{\mathfrak{q}}(s(f(\mathfrak{q}))) \in B_{\mathfrak{q}}\right) \end{array}.$$

Note that we need to check the local coherence part of the definition of $\mathcal{O}$. That is, if s is locally given by a/h, then $f_V^{\#}(s)$ is locally given by $\phi(a)/\phi(h)$. This gives the desired map $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$, and the induced map on stalks $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}$ agrees with $\phi_{\mathfrak{p}} : A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$, by construction. Hence $(f, f^{\#})$ is a morphism of locally ringed spaces.

2. Now suppose given a morphism $(f, f^{\#}) : \operatorname{Spec} B \to \operatorname{Spec} A$ of locally ringed spaces. Take

$$\phi = f_{\operatorname{Spec} A}^{\#} : \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) = B.$$

We need to show ϕ gives rise to $(f, f^{\#})$. We have $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} = A_{f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} = B_{\mathfrak{p}}$ a local homomorphism. This is compatible with the corresponding map on global sections, that is

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, "f_{\operatorname{Spec} A}^{\#}"] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\ \mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"'] & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}} \end{tikzcd}$$

is commutative. That is, we have a commutative diagram

$$\begin{tikzcd} f(\mathfrak{p}) & \subset & A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \supset & \mathfrak{p} \\ f(\mathfrak{p}) A_{f(\mathfrak{p})} & \subset & A_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"'] & B_{\mathfrak{p}} & \supset & \mathfrak{p} B_{\mathfrak{p}} \end{tikzcd}.$$

Then $\left(f_{\mathfrak{p}}^{\#}\right)^{-1}(\mathfrak{p} B_{\mathfrak{p}}) = f(\mathfrak{p}) A_{f(\mathfrak{p})}$ since $f_{\mathfrak{p}}^{\#}$ is a local homomorphism, and by commutativity of the diagram, $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Thus f is induced by ϕ, and $f_{\mathfrak{p}}^{\#} = \phi_{\mathfrak{p}}$. So $f^{\#}$ is as constructed previously.

□

Remark. Demanding $(f, f^{\#})$ was a morphism of locally ringed spaces was crucial to make the proof work.

Definition. An **affine scheme** is a locally ringed space isomorphic, in the category of locally ringed spaces, to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\left\{\left(U_i, \mathcal{O}_X|_{U_i}\right)\right\}$ with each $\left(U_i, \mathcal{O}_X|_{U_i}\right)$ an affine scheme, where $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U_i$ open. A **morphism of schemes** is a morphism of locally ringed spaces.

Example. Let k be a field. Then $\operatorname{Spec} k = (\{0\}, k)$.

- What does giving a morphism $f : \operatorname{Spec} k \to X$ to a scheme mean? First, this selects a point $x \in X$, the image of f. Second, we get a local ring homomorphism $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} k, 0} = k$, that is $\left(f_x^\#\right)^{-1}(0) = \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$. Thus we get a factorisation $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x \to k$, where $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a field, written as $\kappa(x)$, called the **residue field** of X at x. Thus f induces an inclusion $\kappa(x) \hookrightarrow k$. Conversely, given such an inclusion $\iota : \kappa(x) \hookrightarrow k$ of fields, we get a scheme morphism by defining $f(0) = x$, and

$$f^\# \quad : \quad \begin{array}{ccc} \mathcal{O}_X & \longrightarrow & f_* k \\ s & \longmapsto & \iota(s(x)) \end{array}, \qquad s(x) \in \mathcal{O}_{X,x}.$$

  The moral is that giving a morphism $f : \operatorname{Spec} k \to X$ is equivalent to giving a point $x \in X$ and an inclusion $\iota : \kappa(x) \to k$. Note that if $X = \operatorname{Spec} A$, giving $\operatorname{Spec} k \to \operatorname{Spec} A$ is equivalent to giving a homomorphism $A \to k$, which we viewed at the beginning of the course as a k-valued point on $\operatorname{Spec} A$. Lecture 8 Monday 26/10/20

- What does giving $X \to \operatorname{Spec} k$ mean? No information in the continuous map, but need also a map $f^\# : k \to f_* \mathcal{O}_X$, that is a map $k \to \Gamma(\operatorname{Spec} k, f_* \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. That is, $\Gamma(X, \mathcal{O}_X)$ carries a k-algebra structure. Note this induces k-algebra structures on $\mathcal{O}_X(U)$ for all U via the composition $k \to \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ and similarly all stalks $\mathcal{O}_{X,p}$ are also k-algebras. We say X is a **scheme defined over** k. For example, in affine varieties, consider $A = k[X_1, \dots, X_n]/I$ with $I = \sqrt{I}$. Then $\operatorname{Spec} A$ is our replacement for $\mathbb{V}(I) \subseteq \mathbb{A}_k^n$, viewing $\operatorname{Spec} A$ as a scheme over k. If $k \subseteq k'$ is a field extension, a k'**-valued point** of X/k is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}.$$

  We write $X(k')$ for the set of such morphisms.

Remark. It is rare in algebraic geometry to work with schemes alone, but rather always working over a base scheme.

Fix a base scheme S. Define $\mathbf{Sch}/S$ to be the category whose objects are morphisms $T \to S$ and morphisms are commutative diagrams

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & T' \arrow[dl] \\ & S & \end{tikzcd}.$$

We will frequently work with $\mathbf{Sch}/k = \mathbf{Sch}/\operatorname{Spec} k$. Given $T \to S$ and $X \to S$ objects in $\mathbf{Sch}/S$, a T**-valued point** of $X \to S$ is a morphism $T \to X$ over S, so

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd},$$

and we write $X(T)$ for the set of T-valued points. The **Yoneda philosophy** is that $X(T)$ for all T determines X.

Example. Fix a field k, and let $D = \operatorname{Spec} k[t] / \langle t^2 \rangle = (\{\langle t \rangle\}, k[t] / \langle t^2 \rangle)$. Then t does not make sense as k-valued function anymore, as $t^2 = 0$. Let X be any scheme over k. What is $X(D)$? Given $f : D \to X$ a morphism of schemes over k, we get a point $x \in X$ as the image of f and a local homomorphism

$$\begin{array}{cccc} f_x^\# & : & \mathcal{O}_{X,x} & \longrightarrow & k[t] / \langle t^2 \rangle \\ & & \mathfrak{m}_x & \longmapsto & \langle t \rangle \end{array}.$$

Note that $\mathfrak{m}_x^2$ maps to zero, hence we get a k-linear map $\mathfrak{m}_x / \mathfrak{m}_x^2 \to \langle t \rangle \cong k$ as a k-vector space. We also have a composed surjective k-algebra homomorphism $\mathcal{O}_{X,x} \to k[t] / \langle t \rangle \cong k$ with kernel $\mathfrak{m}_x$, and hence we have $\kappa(x) = \mathcal{O}_{X,x} / \mathfrak{m}_x \cong k$. So we get

- a k-valued point x with residue field k, and
- a k-vector space map $\mathfrak{m}_x / \mathfrak{m}_x^2 \to k$, that is an element of $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$, the dual vector space.

Then $(\mathfrak{m}_x / \mathfrak{m}_x^2)^*$ is called the **Zariski tangent space** to X at x. Think of D as a point plus an arrow.

Example. **Glued schemes** are a special case of a question on example sheet 1. Suppose given two schemes X_1 and X_2 and open subsets $U_i \subseteq X_i$. Recall U_i is also a locally ringed space $(U_i, \mathcal{O}_{X_i}|_{U_i})$, and in fact U_i is then a scheme. Given an isomorphism $f : U_1 \xrightarrow{\sim} U_2$, can glue X_1 and X_2 along U_1 and U_2 to get a scheme X with an open cover $\{X_1, X_2\}$, so $X = X_1 \sqcup X_2 / \sim$ such that $x_1 \in U_1 \sim x_2 \in U_2$ if $f(x_1) = x_2$, and need to define $\mathcal{O}_X$. Now take $\mathbb{A}_k^n = \operatorname{Spec} k[X_1, \dots, X_n]$, so $\mathbb{A}_k^1 = \operatorname{Spec} k[X]$. Take $X_1 = X_2 = \mathbb{A}_k^1$.

- Glue $U_1 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(X) \subseteq X_1$ and $U_2 = \mathbb{A}^1 \setminus \{0\} = \mathbb{D}(X) \subseteq X_2$ via the identity map. This is the affine line with doubled origin.
- Could instead glue U_1 and U_2 via the map given by $X \mapsto X^{-1}$, where $U_1 = \operatorname{Spec} k[X]_X = U_2$ and

  $$\begin{array}{ccc} k[X]_X & \longrightarrow & k[X]_X \\ X & \longmapsto & X^{-1} \end{array}$$

  induces an isomorphism $U_1 \to U_2$. When we glue, we get the projective line over k, $\mathbb{P}_k^1$.

2.5 Projective schemes

Let S be a graded ring, that is

$$S = \bigoplus_{d \geq 0} S_d,$$

with S_d an abelian group, and product law satisfies $S_d \cdot S_{d'} \subseteq S_{d+d'}$.

Example. $S = k[X_0, \dots, X_n]$, and S_d is the space of polynomials which are homogeneous of degree d, that is spanned by monomials of degree d.

We write

$$S_+ = \bigoplus_{d \geq 1} S_d,$$

which we call the **irrelevant ideal**.

Definition. $I \subseteq S$ is a **homogeneous ideal** if I is generated by its homogeneous elements, that is elements in S_d for various d.

Definition. Let

$$\operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \text{ is homogeneous}, \ \mathfrak{p} \not\supseteq S_+\}.$$

For $I \subseteq S$ a homogeneous ideal, set

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}.$$

Exercise. Check the $\mathbb{V}(I)$ form the closed sets of a topology on $\operatorname{Proj} S$.

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Notation. For $\mathfrak{p} \in \operatorname{Proj} S$, let
$$T = \{f \in S \setminus \mathfrak{p} \mid f \text{ is homogeneous}\}.$$
Then T is a multiplicatively closed subset of S, and let $S_{(\mathfrak{p})} \subseteq T^{-1}S$ be the subring of elements of degree zero, that is written in the form s/s' with $s \in S$ homogeneous and $s' \in T$ with $\deg s = \deg s'$. For $f \in S$ homogeneous, we write $S_{(f)} \subseteq S_f$ for the subset of elements of degree zero.

Can now define a sheaf $\mathcal{O}$ on $\operatorname{Proj} S$. For $U \subseteq \operatorname{Proj} S$ open, set
$$\mathcal{O}(U) = \left\{ s : U \to \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in S_{(\mathfrak{p})} \\ \forall \mathfrak{p} \in U,\ \exists \mathfrak{p} \in V \subseteq U \text{ open},\ \exists a, f \in S,\ \forall \mathfrak{q} \in V,\ f \notin \mathfrak{q},\ s(\mathfrak{q}) = \dfrac{a}{f} \in S_{(\mathfrak{q})} \end{array} \right\},$$
where a and f are homogeneous of the same degree. As before, $\mathcal{O}_{\mathfrak{p}} = S_{(\mathfrak{p})}$. [8] Is the locally ringed space $(\operatorname{Proj} S, \mathcal{O})$ a scheme?

Notation. If $f \in S$ is homogeneous, then we write
$$\mathbb{D}_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\},$$
which is an open set and $\mathbb{D}_+(f) = \operatorname{Proj} S \setminus \mathbb{V}(f)$.

Proposition 2.4. $\left(\mathbb{D}_+(f), \mathcal{O}|_{\mathbb{D}_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$ *as locally ringed spaces. Further, the open sets* $\mathbb{D}_+(f)$ *for* $f \in S_+$ *cover* $\operatorname{Proj} S$. *Hence* $(\operatorname{Proj} S, \mathcal{O})$ *is a scheme.*

Proof. Will be on example sheet 2. □

Definition. If A is a ring, define
$$\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \dots, X_n].$$

Example. If k is an algebraically closed field, consider $\mathbb{P}^1_k = \operatorname{Proj} k[X_0, X_1]$. The closed points, that is points $\mathfrak{p}$ such that $\{\mathfrak{p}\}$ is closed, correspond to maximal elements of $\operatorname{Proj} S$. [9] These maximal elements are ideals of the form $\langle aX_0 - bX_1 \rangle$. The only maximal homogeneous ideal of $k[X_0, X_1]$ is $\langle X_0, X_1 \rangle = S_+$, since any maximal ideal is of the form $\langle X_0 - a_0, X_1 - a_1 \rangle$. The other prime ideals of $k[X_0, X_1]$ are principal, that is of the form $\langle f \rangle$ with f irreducible or $f = 0$. For $\langle f \rangle$ to be homogeneous, f must be homogeneous. Any such polynomial splits into linear factors, all homogeneous, so in order for f to be irreducible it must be linear. Note we have a one-to-one correspondence between
$$\begin{array}{rcl} \{\langle aX_0 - bX_1 \rangle \mid a, b \in K,\ a, b \text{ not both zero}\} & \longrightarrow & \left(k^2 \setminus \{(0,0)\}\right)/k^* \\ \langle aX_0 - bX_1 \rangle & \longmapsto & (b : a) \end{array},$$
where k^* acts by $(a, b) \mapsto (\lambda a, \lambda b)$ for $\lambda \in k^*$. The conclusion is that the closed points of $\mathbb{P}^1_k$ are in one-to-one correspondence with points of $\left(k^2 \setminus \{(0,0)\}\right)/k^*$. More generally, the closed points of $\mathbb{P}^n_k$ are in one-to-one correspondence with points of $\left(k^{n+1} \setminus \{0\}\right)/k^*$. Can see this by making use of the open cover $\{\mathbb{D}_+(X_i) \mid 0 \le i \le n\}$, [10] which is an open cover since $\mathfrak{p} \notin \mathbb{D}_+(X_i)$ for any i implies that $X_i \in \mathfrak{p}$ for all i, so $S_+ \subseteq \mathfrak{p}$ and so $\mathfrak{p} \notin \operatorname{Proj} S$.

Example. Let $S = k[X_0, \dots, X_n]$, but grade by $\deg X_i = w_i$, where $w_0, \dots, w_n$ are positive integers. Define $\mathrm{W}\mathbb{P}^n(w_0, \dots, w_n) = \operatorname{Proj} S$, the **weighted projective space**. For example, $\mathrm{W}\mathbb{P}^2(1,1,2)$ has an open cover $\{\mathbb{D}_+(X_i) \mid 0 \le i \le 2\}$. Consider $\mathbb{D}_+(X_2) = \operatorname{Spec} S_{(X_2)}$. Note
$$S_{(X_2)} = k\left[\frac{X_0^2}{X_2}, \frac{X_0X_1}{X_2}, \frac{X_1^2}{X_2}\right] \cong k[U, V, W] / \left\langle UW - V^2 \right\rangle \subseteq S_{X_2},$$
so $\operatorname{Spec} S_{(X_2)}$ is a quadric cone with a singular point. Similarly, $\mathbb{D}_+(X_0)$ and $\mathbb{D}_+(X_1)$ are both isomorphic to $\mathbb{A}^2_k$.

[8] Exercise: check
[9] Exercise: check
[10] Exercise: good exercise

Example. Let $M = \mathbb{Z}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$. Let $\Delta \subseteq M_{\mathbb{R}}$ be a compact convex lattice polytope. That is, there exists a finite set $V \subseteq M$ such that Δ is the convex hull of V, that is the smallest convex set containing V. Let

$$\mathrm{C}(\Delta) = \{(m, r) \in M_{\mathbb{R}} \oplus \mathbb{R} \mid m \in r\Delta,\ r \geq 0\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

Here $r\Delta = \{rm \mid m \in \Delta\}$. This is the **cone over** Δ. Let

$$S = k\left[\mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})\right] = \bigoplus_{P \in \mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})} kZ^P,$$

with multiplication given by $Z^P Z^{P'} = Z^{P+P'}$, since $\mathrm{C}(\Delta) \cap (M \oplus \mathbb{Z})$ is a monoid, that is it is closed under addition and contains zero. This makes S into a ring, and it is graded by $\deg Z^{(m,r)} = r$. Define $\mathbb{P}_\Delta = \operatorname{Proj} S$. This is called a **projective toric variety**.

- Let Δ be the convex hull of $\{0, e_1, \dots, e_n\}$ with $e_1, \dots, e_n$ the standard basis of $M = \mathbb{Z}^n$. Check that $S = k[X_0, \dots, X_n]$ with standard grading $X_0 = Z^{(0,1)}$ and $X_i = Z^{(e_i,1)}$. [11] So $\mathbb{P}_\Delta = \mathbb{P}_k^n$.
- Let $n = 2$, and let Δ be the convex hull of $\{(0,0), (1,0), (0,1), (1,1)\}$. In S, the degree d monomials are $\left\{Z^{(a,b,d)} \mid 0 \leq a \leq d,\ 0 \leq b \leq d\right\}$. Any of these can be written as a product of monomials of degree one, that is the monomials $X = Z^{(0,0,1)}$, $Y = Z^{(1,0,1)}$, $W = Z^{(0,1,1)}$, and $T = Z^{(1,1,1)}$. Thus $S = k[X, Y, W, T] / \langle XT - YW \rangle$. So $\operatorname{Proj} S$ can be thought of as a quadric surface in $\mathbb{P}_k^3$.

2.6 Open and closed subschemes

Lecture 10 Friday 30/10/20

Definition. An **open subscheme** of a scheme X is a scheme $(U, \mathcal{O}_X|_U)$ for $U \subseteq X$ an open subset. Note that this is a scheme because from question 1 and question 11 on the first example sheet, open affine subsets of X form a basis for the topology on X. An **open immersion** is a morphism $f : X \to Y$ which induces a isomorphism of X with an open subscheme of Y. A **closed immersion** $f : X \to Y$ is a morphism which is a homeomorphism onto a closed subset of Y, and the induced morphism $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective. A **closed subscheme** of Y is an equivalence class of closed immersions, where

$$\begin{tikzcd} X \arrow[rr, dashed, "i"] \arrow[dr] & & X' \arrow[dl] \\ & Y & \end{tikzcd}$$

are equivalent if there exists an isomorphism i making the diagram commute.

Example.

- Let $Y = \operatorname{Spec} A$, let $I \subseteq A$ be an ideal, and let $X = \operatorname{Spec} A/I$. Note the map of schemes induced by the quotient map $A \to A/I$ identifies $\operatorname{Spec} A/I$ with $\mathbb{V}(I) \subseteq \operatorname{Spec} A$. Thus $f : X \to Y$, induced by $A \to A/I$, satisfies the first condition of being a closed immersion. Note that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective on stalks. For $\mathfrak{p} \in \mathbb{V}(I)$, $\mathcal{O}_{Y,\mathfrak{p}} = A_{\mathfrak{p}}$ and $(f_* \mathcal{O}_X)_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$ since all open sets in X are of the form $U \cap X$ for U an open set of Y and $\mathcal{O}_{X,\mathfrak{p}} = (A/I)_{\mathfrak{p}/I}$. Certainly $A_{\mathfrak{p}} \to (A/I)_{\mathfrak{p}/I}$ is surjective.
- Let $\operatorname{Spec} k[X, Y] / \langle X \rangle \to \operatorname{Spec} k[X, Y] = \mathbb{A}^2$. This gives a closed subscheme structure to the set $\mathbb{V}(X)$. Note $\mathbb{V}(X^2, XY) = \mathbb{V}(X)$. This gives a closed immersion $\operatorname{Spec} k[X, Y] / \langle X^2, XY \rangle \to \mathbb{A}^2$. This gives a different closed subscheme structure on $\mathbb{V}(X)$. Note these two subschemes are isomorphic away from the origin, which we can see by looking at $\mathbb{D}(Y) \subseteq \operatorname{Spec} k[X, Y] / \langle X \rangle$, where

$$\mathbb{D}(Y) \cong \operatorname{Spec}\left(k[X, Y] / \langle X \rangle\right)_Y = \operatorname{Spec} k[Y]_Y.$$

  Looking at $\mathbb{D}(Y) \subseteq \operatorname{Spec} k[X, Y] / \langle X^2, XY \rangle$,

$$\mathbb{D}(Y) \cong \operatorname{Spec}\left(k[X, Y] / \langle X^2, XY \rangle\right)_Y \cong \operatorname{Spec}\left(k[X, Y]_Y / \langle X \rangle\right) \cong \operatorname{Spec} k[Y]_Y.$$

[11] Exercise

2.7 Fibre products

Let $\mathcal{C}$ be a category and

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a diagram in $\mathcal{C}$. Then the **fibre product**, if it exists, is an object W equipped with morphisms $p : W \to X$ and $q : W \to Y$ such that $f \circ p = g \circ q$ satisfying the following universal property. For any W' equipped with maps $p' : W' \to X$ and $q' : W' \to Y$ such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : W' \to W$ making the diagram

$$\begin{tikzcd} W' \arrow[dr, dashed, "\exists ! h"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\ & W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

commute, that is $p \circ h = p'$ and $q \circ h = q'$. Note that if the fibre product exists, it is unique up to unique isomorphism.

Example. Let $\mathcal{C}$ be the category of sets. Then

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

It will be helpful to think about the fibre product, and more generally other universal properties, via the Yoneda lemma.

Definition. Let $\mathcal{C}$ be a category. Write h_X for the contravariant functor

$$\begin{array}{rcrcl} \mathrm{h}_X & : & \mathcal{C} & \longrightarrow & \mathbf{Set} \\ & & Y & \longmapsto & \mathrm{Hom}(Y, X) \\ & & f : Y \to Z & \longmapsto & (\phi \in \mathrm{Hom}(Z, X) \mapsto \phi \circ f \in \mathrm{Hom}(Y, X)) \end{array}.$$

Recall that a **natural transformation** between contravariant functors $F, G : \mathcal{C} \to \mathcal{D}$, written as $T : \mathcal{C} \to \mathcal{D}$, consists of the data $T(X) : F(X) \to G(X)$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ such that for all $f : X \to Y$ in $\mathcal{C}$

$$\begin{tikzcd} F(X) \arrow[d, "T(X)"'] & F(Y) \arrow[l, "F(f)"'] \arrow[d, "T(Y)"] \\ G(X) & G(Y) \arrow[l, "G(f)"] \end{tikzcd}$$

is commutative.

Lemma 2.5 (Yoneda's lemma)**.** *The set of natural transformations between* $\mathrm{h}_X : \mathcal{C} \to \textbf{\textit{Set}}$ *and* $G : \mathcal{C} \to \textbf{\textit{Set}}$ *is* $G(X)$.

Proof. Given $\eta \in G(X)$, we need to define a map

$$\begin{array}{rcl} \mathrm{h}_X(Y) = \mathrm{Hom}(Y, X) & \longrightarrow & G(Y) \\ f & \longmapsto & G(f)(\eta) \end{array},$$

for all objects $Y \in \mathcal{C}$. Check that this defines a natural transformation $\mathrm{h}_X \to G$. [12] Conversely, given $T : \mathrm{h}_X \to G$ a natural transformation, take $\eta = T(X)(\mathrm{id}_X)$. Check that these two maps are inverse to each other. [13] $\square$

Corollary 2.6. *The set of natural transformations* $\mathrm{h}_X \to \mathrm{h}_Y$ *is* $\mathrm{h}_Y(X) = \mathrm{Hom}(X, Y)$.

[12] Exercise
[13] Exercise

Definition. A contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be **representable** if $F \cong \mathrm{h}_X$ for some $X \in \mathrm{Ob}\,\mathcal{C}$.

Lots of questions in algebraic geometry are about representability of functors. Redefining, the fibre product in a category $\mathcal{C}$ is an object which represents the functor

$$T \mapsto \mathrm{Hom}\,(T, X) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}\,(T, Y)\,,$$

since an element of the set $\mathrm{Hom}\,(T, X) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}\,(T, Y)$ is a commutative diagram

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\ & W \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd} .$$

The advantage of using Yoneda is that we can check identities using fibre products using identities of the products of sets.

Example. In **Set**,

$$\begin{array}{rcl} (A \times_B B) \times_C D & \longleftrightarrow & A \times_B D \\ ((a, c)\,, d) & \longmapsto & (a, d) \\ ((a, f\,(d))\,, d) & \longleftarrow & (a, d) \end{array} , \qquad f : D \to C.$$

Then we have two functors

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & \mathrm{h}_A\,(T) \times_{\mathrm{h}_B(T)} \mathrm{h}_C\,(T) \times_{\mathrm{h}_C(T)} \mathrm{h}_D\,(T) \arrow[d, "\sim"] \\ & \mathrm{h}_A\,(T) \times_{\mathrm{h}_B(T)} \mathrm{h}_D\,(T) \end{tikzcd} ,$$

and natural transformations showing those functors are isomorphic, and hence represent isomorphic objects.