Profinite Groups and Group Cohomology

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Lent 2021

Syllabus

Contents

0 Introduction

Lecture 1
Thursday
21/01/21

A question is, when are things different?

- $\mathbb{Z}$ is in bijection with $\mathbb{Q}$, by writing down a bijection.
- $\mathbb{Q}$ is not in bijection with $\mathbb{R}$, by diagonalisation.

A solution is to try to find an invariant, which is

- easier to compute,
- computable, and
- preserved under isomorphism.

Example 0.0.1.

- Cardinality of a set.
- Dimension and base field of a vector space, which is complete.
- For an algebraic field extension K over $\mathbb{Q}$, the degree $[K : \mathbb{Q}]$ and the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
- For a topological space X, compactness, connectedness, simplicial homology groups $\mathrm{H}_\bullet(X)$, and the fundamental group $\pi_1(X)$.

Theorem 0.0.2. *There is no algorithm that decides whether a finite presentation represents the trivial group.*

Finite groups we can decide.

- List all the finite quotients of a group.
- If you have two such lists, you can compare.
- If two groups have different sets of finite quotients, they are not isomorphic.

How often does this work?

- Combine all the finite quotients into one object to study, the **profinite completion**, which is a limit of the finite groups.
- More generally, a limit of finite groups is called a **profinite group**.

Example 0.0.3.

- In Galois theory, let $K = \bigcup_{N \in \mathbb{N}} K_N$ be the extension of $\mathbb{Q}$ adjoining all p^N-th roots of unity for p a fixed prime and $N \in \mathbb{N}$, which gives a short exact sequence of Galois groups
$$\mathrm{Gal}(K/K_N) \to \mathrm{Gal}(K/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(K_N/\mathbb{Q}).$$
Then $\mathrm{Gal}(K_N/\mathbb{Q}) = \left(\mathbb{Z}/p^N\mathbb{Z}\right)^\times$ and $\mathrm{Gal}(K/\mathbb{Q}) = \varprojlim_N \left(\mathbb{Z}/p^N\mathbb{Z}\right)^\times = \mathbb{Z}_p^\times$.
- In algebraic geometry, étale fundamental groups are profinite groups.

The second part of the course is **group cohomology**, which is another invariant, with the following applications.

- Can tell if a group is free for some profinite groups.
- Given a group G and an abelian group A, group cohomology tells us how many groups E exist such that $A \triangleleft E$ and $E/A = G$.

1 Inverse limits

1.1 Categories and limits

Let A and B be sets. How to combine into one thing? The disjoint union $A \sqcup B$ has inclusion maps $i_A : A \hookrightarrow A \sqcup B$ and $i_B : B \hookrightarrow A \sqcup B$, and for any other set Z, with functions $j_A : A \to Z$ and $j_B : B \to Z$ there is a unique function defined by

$$\begin{array}{rccl} f & : & A \sqcup B & \longrightarrow & Z \\ & & a & \longmapsto & j_A(a) \\ & & b & \longmapsto & j_B(b) \end{array},$$

such that $f \circ i_A = j_A$ and $f \circ i_B = j_B$, so

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[dr, "j_A"'] & A \sqcup B \arrow[d, dashed, "\exists! f"] & B \arrow[l, hook', "i_B"'] \arrow[dl, "j_B"] \\ & Z & \end{tikzcd}.$$

The product $A \times B$ comes with $p_A : A \times B \to A$ and $p_B : A \times B \to B$ such that

$$\begin{tikzcd} A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \\ & Z \arrow[ul, "q_A"] \arrow[u, dashed, "\exists! f"] \arrow[ur, "q_B"'] & \end{tikzcd},$$

where $f(z) = (q_A(z), q_B(z))$. Reversed all arrows, so there is a duality, and disjoint union is a coproduct. What about groups, and group homomorphisms? The product still works, but the disjoint union is not a group. The coproduct is the free product $A * B$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A * B \arrow[d, dashed] & B \arrow[l] \arrow[dl] \\ & Z & \end{tikzcd}.$$

More generally is the pushout. Given groups A, B, and C, and homomorphisms $\phi_A : C \to A$ and $\phi_B : C \to B$, the **pushout** $A \sqcup_C B$ is

$$\begin{tikzcd} C \arrow[r, "\phi_A"] \arrow[d, "\phi_B"'] & A \arrow[d, "i_A"] \arrow[ddr, bend left, "j_A"] & \\ B \arrow[r, "i_B"'] \arrow[drr, bend right, "j_B"'] & A \sqcup_C B \arrow[dr, dashed, "\exists! f"] & \\ & & Z \end{tikzcd}.$$

Definition 1.1.1. A **category** $\mathcal{C}$ consists of

- a collection of **objects** $\mathrm{Obj}\,\mathcal{C}$,
- a collection of **morphisms** or **arrows** $\mathrm{Mor}\,\mathcal{C}$, such that each $f \in \mathrm{Mor}\,\mathcal{C}$ has a **domain** $X \in \mathrm{Obj}\,\mathcal{C}$ and a **codomain** $Y \in \mathrm{Obj}\,\mathcal{C}$ written as $f : X \to Y$,
- for all objects $X \in \mathrm{Obj}\,\mathcal{C}$, you have $\mathrm{id}_X : X \to X$, and
- if $f : X \to Y$ and $g : Y \to Z$, we have a defined composition $g \circ f : X \to Z$,

such that

- if $f : X \to Y$, then $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$, and
- if $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Example 1.1.2.

- In **Set**, objects are sets and morphisms are functions.
- In **Grp**, objects are groups and morphisms are group homomorphisms.
- In $\mathbf{Grp}_{\mathrm{fin}}$, objects are finite groups.
- In $\mathbf{Grp}_{\mathrm{inj}}$, morphisms are injective group homomorphisms.

Definition 1.1.3. A **partial ordering** on a set J is a binary relation $\leq$ such that

- $i \leq i$,
- if $i \leq j$ and $j \leq i$, then $i = j$, and
- if $i \leq j$ and $j \leq k$, then $i \leq k$.

A **poset** is a pair $(J, \leq)$, which is a **total ordering** if for all $i, j \in J$ either $i \leq j$ or $j \leq i$. The **poset category** $\mathcal{J}$ has objects $\operatorname{Obj} \mathcal{J} = J$ and morphisms $\operatorname{Mor} \mathcal{J} = \{i \to j \mid i \leq j\}$.

Lecture 2 Saturday 23/01/21

Definition 1.1.4. Let $\mathcal{C}$ be a category. A **product** of $A, B \in \operatorname{Obj} \mathcal{C}$ is an object P, equipped with morphisms $p_A : P \to A$ and $p_B : P \to B$, such that for all $Z \in \operatorname{Obj} \mathcal{C}$ and for all $q_A : Z \to A$ and $q_B : Z \to B$, there exists a unique $f : Z \to P$ such that $p_A \circ f = q_A$ and $p_B \circ f = q_B$, so

$$\begin{tikzcd} & Z \arrow[dl, "q_A"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "q_B"] & \\ A & P \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}.$$

Definition 1.1.5. Objects A and B in a category $\mathcal{C}$ are **isomorphic** if there exist $f : A \to B$ and $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proposition 1.1.6. *If a product of A and B in $\mathcal{C}$ exists, then it is unique up to a unique isomorphism.*

Proof. Let (P, p_A, p_B) and (P', p'_A, p'_B) be products. Then

$$\begin{tikzcd} & P' \arrow[dl, "p'_A"'] \arrow[dr, "p'_B"] \arrow[dd, dashed, bend right, "\exists! f"'] & \\ A & & B \\ & P \arrow[ul, "p_A"] \arrow[ur, "p_B"'] \arrow[uu, dashed, bend right, "\exists! g"'] & \end{tikzcd}.$$

Consider $f \circ g : P \to P$. Then $p_A \circ f \circ g = p'_A \circ g = p_A$ and $p_B \circ f \circ g = p'_B \circ g = p_B$. By uniqueness, $f \circ g = \mathrm{id}_P$. Similarly, $g \circ f = \mathrm{id}_{P'}$. □

Notation 1.1.7. Define $P = A \times B$.

Definition 1.1.8. Let $\mathcal{C}$ be a category and $A, B \in \operatorname{Obj} \mathcal{C}$. Then a **coproduct** is an object $A \sqcup B$, together with maps $i_A : A \to A \sqcup B$ and $i_B : B \to A \sqcup B$, with the universal property

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "j_A"'] & A \sqcup B \arrow[d, dashed, "\exists! f"] & B \arrow[l, "i_B"'] \arrow[dl, "j_B"] \\ & Z & \end{tikzcd}.$$

Products are examples of limits and coproducts are examples of colimits.

Definition 1.1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ associates an object $F(X) \in \operatorname{Obj} \mathcal{D}$ to each $X \in \operatorname{Obj} \mathcal{C}$, and a morphism $F(f) : F(X) \to F(Y)$ for each $f : X \to Y$ in $\mathcal{C}$, such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
- $F(g \circ f) = F(g) \circ F(f)$.

Definition 1.1.10. Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A **diagram of shape $\mathcal{J}$ in $\mathcal{C}$** is a functor $X : \mathcal{J} \to \mathcal{C}$. Often write $X(j) = X_j$, for $j \in \operatorname{Obj} \mathcal{J}$.

Very often, $\mathcal{J}$ is a poset category. In that case, if $i \le j$, there exists a unique arrow $f : i \to j$ and then denote $X(f) = \phi_{ij}$.

Definition 1.1.11. A **cone** on a diagram $X : \mathcal{J} \to \mathcal{C}$ is an object $Z \in \operatorname{Obj} \mathcal{C}$, together with maps $p_j : Z \to X_j = X(j)$ for all $j \in \operatorname{Obj} \mathcal{J}$ such that for all $f : i \to j$, $X(f) \circ p_i = p_j$, so

$$\begin{tikzcd} & Z \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ X_i \arrow[rr, "X(f)"'] & & X_j \end{tikzcd} .$$

A **limit** of a diagram $X : \mathcal{J} \to \mathcal{C}$ is a cone L, with morphisms p_j, such that for any cone Z, with morphisms q_j, there is a unique $g : Z \to L$ such that $p_j \circ f = q_j$, for all $j \in \operatorname{Obj} \mathcal{J}$, so

$$\begin{tikzcd} & Z \arrow[d, dashed, "\exists! g"] \arrow[ddl, bend right, "q_i"'] \arrow[ddr, bend left, "q_j"] & \\ & L \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ X_i \arrow[rr, "X(f)"'] & & X_j \end{tikzcd} ,$$

for $f : i \to j$. **Colimits** are as limits, but arrows are reversed.

Example 1.1.12.

- If $\mathcal{J}$ is the category

  $$\bullet \qquad \bullet,$$

  then a diagram of shape $\mathcal{J}$ is a pair of objects. The limit is the product and the colimit is the coproduct.
- If $\mathcal{J}$ is the category

  $$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} ,$$

  then a diagram of shape $\mathcal{J}$ in **Grp** would be

  $$\begin{tikzcd} C \arrow[r, "\phi_{CA}"] \arrow[d, "\phi_{CB}"'] & A \\ B & \end{tikzcd} .$$

  The colimit is the pushout.

Proposition 1.1.13. *Limits and colimits are unique up to unique isomorphism.*

1.2 Inverse limits and profinite groups

Let G be a group. Let $\mathcal{N}$ be the poset category whose objects are $\{N \triangleleft_{\mathrm{f}} G\}$, where $N \triangleleft_{\mathrm{f}} G$ are finite index, with ordering $N_1 \le N_2$ if and only if $N_1 \subseteq N_2$. There is a diagram of shape $\mathcal{N}$ in **Grp**,

$$\begin{array}{cccc} X & : & \mathcal{N} & \longrightarrow & \mathbf{Grp} \\ & & N & \longmapsto & X_N = G/N \end{array}.$$

If $N_1 \le N_2$, then $X(N_1 \to N_2)$ is the quotient map $\phi_{N_1 N_2} : G/N_1 \to G/N_2$, the transition maps.

Definition 1.2.1. Let G be a group. The **profinite completion** of G is the limit of this diagram, denoted $\widehat{G}$. Then G comes with **projections** $p_N : \widehat{G} \to G/N$ for all $N \triangleleft_{\mathrm{f}} G$ such that

- if $N_1 \subseteq N_2$, then $\phi_{N_1 N_2} \circ p_{N_1} = p_{N_2}$, and
- if Z is a group, with $q_N : Z \to G/N$ such that $\phi_{N_1 N_2} \circ q_{N_1} = q_{N_2}$, there exists a unique $f : Z \to \widehat{G}$ such that $p_N \circ f = q_N$ for all N.

In particular, $Z = G$ works, so there is a unique morphism $\iota_G : G \to \widehat{G}$, the **canonical morphism**, such that the diagrams commute.

Definition 1.2.2. A poset $(J, \le)$ is an **inverse system** if for all $i, j \in J$ there exists $k \in J$ such that $k \le i$ and $k \le j$. An **inverse system of groups** consists of an inverse system $(J, \le)$ and a diagram of shape $\mathcal{J}$ in **Grp**, so $G : \mathcal{J} \to \mathbf{Grp}$. Thus an inverse system is a group G_j for all $j \in J$ and transition maps $\phi_{ij} : G_i \to G_j$ if $i \le j$ such that $\phi_{ii} = \mathrm{id}$ and $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all $i \le j \le k$. The **inverse limit** of this inverse system of groups G_j is the limit of this diagram, denoted $\varprojlim_j G_j$.

Definition 1.2.3. A **profinite group** is the inverse limit of an inverse system of groups, all of which are finite.

Proposition 1.2.4. *Let $(G_j)_{j \in J}$ be an inverse system of groups. Then the inverse limit exists, and is given by the explicit description*

$$\varprojlim_j G_j = \left\{ (g_j)_{j \in J} \in \prod_{j \in J} G_j \;\middle|\; \forall i \le j,\ \phi_{ij}(g_i) = g_j \right\}.$$

Proof. This is a group. We have $p_j : \varprojlim_j G_j \to G_j$, restricted from $\prod_{j \in J} G_j \to G_j$. Take a cone Z on the system. Define

$$\begin{array}{cccc} f & : & Z & \longrightarrow & \varprojlim_j G_j \\ & & z & \longmapsto & (q_j(z))_{j \in J} \end{array}.$$

Then $\phi_{ij}(q_i(z)) = q_j(z)$, so $p_j \circ f = q_j$. □

Lecture 3 Tuesday 26/01/21

Definition 1.2.5. Let $(G_j)_{j \in J}$ be an inverse system of finite groups. Give each G_j the discrete topology. Give $\prod_j G_j$ the product topology. Then $\varprojlim_j G_j \le \prod_j G_j$ gets the subspace topology.

Proposition 1.2.6. *$\varprojlim_j G_j$ is compact Hausdorff.*

Proof. $\prod_j G_j$ is Hausdorff and compact, by Tychonoff's theorem. Each condition $\phi_{ij}(g_i) = g_j$ is a closed condition, since $\prod_{j \in J} G_j \to G_i \times G_j$, so $\varprojlim_j G_j$ is closed in $\prod_j G_j$. □

Proposition 1.2.7. *Let $(X_j)_{j \in J}$ be an inverse system of non-empty finite sets. Then $\varprojlim_j X_j$ is non-empty.*

Proof. Use the finite intersection property. Let $I_1 \subseteq J$ be a finite subset. Define a closed subset

$$Y_{I_1} = \left\{ (x_j) \in \prod_j X_j \;\middle|\; \forall i, j \in I_1,\ \forall i \le j,\ \phi_{ij}(x_i) = x_j \right\} \subseteq \prod_j X_j.$$

Since J is an inverse system and I_1 is finite, there exists $k \in J$ such that $k \leq i$ for all $i \in I_1$. Choose $x_k \in X_k \neq \emptyset$. Define $x_j = \phi_{kj}(x_k)$ for all $j \geq k$. Choose x_j arbitrarily elsewhere. This gives $x = (x_j) \in \prod_{j \in J} X_j$, which lies in Y_{I_1}, since if $i, j \in I_1$ such that $i \leq j$ then

$$x_j = \phi_{kj}(x_k) = \phi_{ij}(\phi_{ki}(x_k)) = \phi_{ij}(x_i).$$

So Y_{I_1} is non-empty. Then $Y_{I_1} \cap \dots \cap Y_{I_n} \supseteq Y_{I_1 \cup \dots \cup I_n} \neq \emptyset$. By the finite intersection property, since $\prod_j X_j$ is compact, $\bigcap_{I_1} Y_{I_1} = \varprojlim_j X_j$ is non-empty. □

Proposition 1.2.8. *Let J be a countable set and let $(X_j)_{j \in J}$ be a family of finite sets. Then $X = \prod_{j \in J} X_j$ is **metrisable**, so the metric topology equals to the other topology.*

Proof. Without loss of generality $J = \mathbb{N}$. Give each X_n the discrete metric d_n, where

$$d_n(x_n, y_n) = \begin{cases} 0 & x_n = y_n \\ 1 & x_n \neq y_n \end{cases}, \qquad x_n, y_n \in X_n.$$

Define

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{3^n} d_n(x_n, y_n), \qquad (x_n), (y_n) \in \prod_n X_n.$$

We need to show this gives the product topology. Let $f : (X, \tau_{\text{product}}) \to (X, d)$ be the identity function. A basis for the metric topology are open balls $\mathrm{B}(x, 1/3^n)$ for $x \in X$ and $n \in \mathbb{N}$. Then $d((x_n), (y_n)) < 1/3^m$ if and only if $x_n = y_n$ for all $n \leq m$, and

$$f^{-1}\left(\mathrm{B}\left((x_n), \frac{1}{3^m}\right)\right) = \{(y_n) \mid \forall n \leq m,\ y_n = x_n\} = \bigcap_{n=1}^{m} p_n^{-1}(\{x_n\}), \qquad p_n : \prod_n X_n \to X_n$$

is open in the product topology. So f is continuous, so a homeomorphism. □

Proposition 1.2.9. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Lemma 1.2.10. *Let G be a finitely generated group. For each $n \in \mathbb{N}$, there are only finitely many subgroups of index n.*

Proof. For a subgroup $H \leq G$ of index n, we get a homomorphism $G \to \mathrm{Sym}\, n$, since by labelling cosets $H, \dots, g_n H$ by symbols $1, \dots, n$, G permutes these right cosets by $g \cdot g_i H = (g g_i) H$ and H is recovered from this as $\mathrm{Stab}\, 1$. So there are at most as many subgroups H as homomorphisms to $\mathrm{Sym}\, n$, and there are only finitely many. □

Corollary 1.2.11. *If G is finitely generated, the inverse system $\mathcal{N} = \{N \triangleleft_{\mathrm{f}} G\}$ is countable.*

Proposition 1.2.12. *Let G be a profinite group. Then G is a **topological group**, so*

$$\begin{array}{rccc} \mathrm{m} : & G \times G & \longrightarrow & G \\ & (g, h) & \longmapsto & gh \end{array}, \qquad \begin{array}{rccc} \mathrm{i} : & G & \longrightarrow & G \\ & g & \longmapsto & g^{-1} \end{array}$$

are continuous.

Definition 1.2.13. Let G and H be topological groups. We say G and H are **isomorphic as topological groups** if and only if there exists $f : G \to H$ which is both an isomorphism of groups and a homeomorphism.

Recall that if G and H are profinite, this is the same as there exists f a continuous isomorphism.

Proposition 1.2.14. *Let H be a topological group and $G = \varprojlim_j G_j$ be an inverse limit of finite groups. Let $p_j : G \to G_j$ be the projection maps. A homomorphism $f : H \to G$ is continuous if and only if each map $f_j = p_j \circ f$ is continuous.*

Proof. $f : H \to G \leq \prod_j G_j$. This is continuous if and only if all f_j are continuous, by definition of the product topology. □

Proposition 1.2.15. *Let $f : H \to G_j$ be a homomorphism from a topological group to a finite group, with the discrete topology. Then f is continuous if and only if $\ker f$ is open in H.*

Proof. If f is continuous then $\ker f = f^{-1}(\{1\})$ is open. Assume $f^{-1}(\{1\})$ is open. Then $f^{-1}(\{g\})$ is open for all $g \in G$, since multiplication is continuous and $f^{-1}(\{g\}) = hf^{-1}(\{1\})$ for some $h \in H$. Taking unions, the preimage of any set in G_j is open in H, so f is continuous. □

Proposition 1.2.16. *Let G be a compact topological group. A subgroup of G is open if and only if it is closed and of finite index.*

Proposition 1.2.17. *Let $(G_j)_{j\in J}$ be an inverse system of finite groups. If $G = \varprojlim_j G_j$, then the open subgroups $U_j = \ker(p_j : G \to G_j)$ form a **basis of open neighbourhoods** of the identity $1 \in G$, so if $V \subseteq G$ is any open set with $1 \in V$, then there exists j such that $U_j \subseteq V$.*

Proof. Let $V \ni 1$ be open. By definition of the product topology,

$$V \supseteq p_{j_1}^{-1}(X_{j_1}) \cap \dots \cap p_{j_n}^{-1}(X_{j_n}) \supseteq p_{j_1}^{-1}(\{1\}) \cap \dots \cap p_{j_n}^{-1}(\{1\}) = U_{j_1} \cap \dots \cap U_{j_n}.$$

for $X_{j_i} \subseteq G_{j_i}$. There exists k such that $k \le j_i$. Since $p_{j_i} = \phi_{kj_i} \circ p_k$, $\ker p_k = U_k \subseteq U_{p_{j_i}} = \ker p_{j_i}$ for all i. Thus $V \supseteq U_k$. □

Lecture 4 Thursday 28/01/21

Corollary 1.2.18. *If $g = (g_j)_{j\in J} \in G$, then the open cosets $gU_j = p_j^{-1}(\{g_j\})$ form a neighbourhood base at g, so for all open set $V \ni g$, there exists $j \in J$ such that $gU_j \subseteq V$.*

Proof. Continuity of multiplication. □

Corollary 1.2.19. *A subset $X \subseteq G$ is dense if and only if $p_j(X) = p_j(G)$ for all $j \in J$.*

Proof. Suppose X is not dense. There exists a non-empty open set V such that $V \cap X = \emptyset$. Pick $g \in V$. There exists $j \in J$ such that $p_j^{-1}(\{g_j\}) = gU_j \subseteq V$, where $g_j = p_j(g)$. Then $g_j \in p_j(G)$. But for any $x \in X$, $p_j(x) \neq g_j$, otherwise $x \in p_j^{-1}(\{g_j\}) = gU_j \subseteq V$, so $p_j(X) \neq p_j(G)$. Assume X is dense. Then $p_j(X) \subseteq p_j(G)$ is obvious. If $g_j \in p_j(G)$, then $p_j^{-1}(\{g_j\})$ is a non-empty open set, so there exists $x \in X \cap p_j^{-1}(\{g_j\})$, then $p_j(x) = g_j$. So $g_j \in p_j(X)$, so $p_j(X) = p_j(G)$. □

Corollary 1.2.20. *Let Y be a compact topological space and let $f : Y \to G$ be a continuous function. Then f is surjective if and only if $p_j(f(Y)) = p_j(G)$ for all $j \in J$.*

Proof. $p_j(f(Y)) = p_j(G)$ if and only if $f(Y)$ is dense, if and only if $f(Y) = G$, since $f(Y)$ is closed. □

Proposition 1.2.21. *Let G be a profinite group and $X \subseteq G$ be a subset. Then the closure of X is*

$$\overline{X} = \bigcap_{N \le_{\mathrm{o}} G} XN,$$

where $N \le_{\mathrm{o}} G$ are open subgroups.

Proof. XN is a union of cosets, hence it is open and closed in G. So $\overline{X} \subseteq XN$ for all $N \le_{\mathrm{o}} G$, so $\overline{X} \subseteq \bigcap_{N \le_{\mathrm{o}} G} XN$. Take $g \notin \overline{X}$. There exists an open $V \subseteq G$ such that $g \in V$ but $X \cap V = \emptyset$. Then there exists $j \in J$ such that $V \supseteq gU_j$ for $N = U_j = \ker p_j$. Then $g \notin XN$, since if $g = xn$ for $x \in X$ and $n \in N = U_j$ then $x = gn^{-1} \in gN = gU_j \subseteq V$, a contradiction. Thus $g \notin \bigcap_N XN$, so $\bigcap_N XN \subseteq \overline{X}$. □

Proposition 1.2.22. *Let G be a profinite group and let $\mathcal{U}$ be a collection of open normal subgroups which form a neighbourhood base at the identity. Then*

$$G \cong \varprojlim_{U \in \mathcal{U}} G/U,$$

as topological groups, where G/U are finite groups.

Proof. The quotient maps $G \twoheadrightarrow G/U$ are a cone on the inverse system, so we get a well-defined homomorphism $f : G \to \varprojlim_U G/U$. Then

- f is continuous, since compositions with projection maps are continuous,
- f is surjective, since $G \twoheadrightarrow G/U$ are surjective, and
- f is injective, since if $g \in G \setminus \{1\}$, there exists an open subset V such that $1 \in V$ and $g \notin V$ and there exists $U \in \mathcal{U}$ such that $1 \in U \subseteq V$, then $g \notin \ker(G \to G/U)$, so $g \notin \ker f$.

□

1.3 Change of inverse system

Definition 1.3.1. Let $(J, \leq)$ be an inverse system. A **cofinal subsystem** of J is a subset $I \subseteq J$ such that for all $j \in J$ there exists $i \in I$ such that $i \leq j$.

Then I is an inverse system.

Example 1.3.2. If $k \in J$, then the set

$$J_{\leq k} = \{j \in J \mid j \leq k\},$$

the **principal cofinal subsystem**, is cofinal in J.

Proposition 1.3.3. *Let $(G_j)_{j \in J}$ be an inverse system of finite groups, and let $I \subseteq J$ be cofinal. Then $H = \varprojlim_{i \in I} G_i$ is topologically isomorphic to $G = \varprojlim_{j \in J} G_j$.*

Proof. The projection map $\prod_{j \in J} G_j \to \prod_{i \in I} G_i$ is a continuous homomorphism, and it restricts to $f : G \to H$. Check that f is bijective.

- Injective. Take $g = (g_j)_{j \in J} \in G$. Assume $f(g) = 1$, so $g_i = p_i(f(g)) = 1$ for all $i \in I$. For any $j \in J$, there exists $i \in I$ such that $i \leq j$. Then $g_j = \phi_{ij}(g_i) = \phi_{ij}(1) = 1$. So $g = 1$.
- Surjective. Let $h = (h_i)_{i \in I} \in H$ for $h_i \in G_i$. Define $g = (g_j) \in \prod_{j \in J} G_j$ by setting $g_j = \phi_{ij}(h_i)$ for some $i \in I$ such that $i \leq j$. If $i_1 \leq j$ and $i_2 \leq j$, there exists $i_0 \in I$ such that $i_0 \leq i_1$ and $i_0 \leq i_2$, then

  $$\phi_{i_1 j}(h_{i_1}) = \phi_{i_1 j}(\phi_{i_0 i_1}(h_{i_0})) = \phi_{i_0 j}(h_{i_0}) = \phi_{i_2 j}(\phi_{i_0 i_2}(h_{i_0})) = \phi_{i_2 j}(h_{i_2}).$$

  It also follows that $g \in G$, since if $j_1 \leq j_2$, choose $i \in I$ such that $i \leq j_1$, then

  $$g_{j_2} = \phi_{i j_2}(h_i) = \phi_{j_1 j_2}(\phi_{i j_1}(h_i)) = \phi_{j_1 j_2}(g_{j_1}).$$

  Finally, $f(g) = h$, since $g_i = \phi_{ii}(h_i) = h_i$ for all $i \in I$.

□

Definition 1.3.4. An inverse system of groups is **surjective** if all transition maps are surjective.

Proposition 1.3.5. *Let $(X_j)_{j \in J}$ be an inverse system of finite sets where all transition maps are surjective. Then the projection maps $p_j : \varprojlim_j X_j \to X_j$ are surjective.*

Proposition 1.3.6. *Let $(G_j)_{j \in J}$ be an inverse system of finite groups. Then there exists an inverse system $\left(G'_j\right)_{j \in J}$ such that $G'_j \leq G_j$, with surjective transition maps, such that $\varprojlim_j G_j = \varprojlim_j G'_j$.*

Proof. Let $p_j : G = \varprojlim_j G_j \to G_j$ be the projection. Define $G'_j = p_j(G)$. Since $\phi_{ij} \circ p_i = p_j$, $\left(G'_j\right)$ is an inverse system with $\phi_{ij}|_{G'_i} : G'_i \to G'_j$, and $\phi_{ij}|_{G'_i}$ is surjective. If $g = (g_j) \in G$ then $g_j = p_j(g) \in G'_j$, so $g \in \varprojlim_j G'_j \leq G \leq \prod_j G_j$. Thus $\varprojlim_j G'_j = G$. □

Definition 1.3.7. An inverse system $(J, \leq)$ is **linearly ordered** if there exists a bijection $f : J \to \mathbb{N}$ such that $i \leq j$ if and only if $f(i) \geq f(j)$, the **wrong-way ordering** on $\mathbb{N}$.

Thus cofinal if and only if increasing subsequence.

Proposition 1.3.8. *If J is a countable inverse system, with no* ***global minimum****, so there does not exist $m \in J$ such that $m \leq j$ for all j, then J has a linearly ordered cofinal subsystem.*

2 Profinite groups

2.1 The p-adic integers

Lecture 5 Saturday 30/01/21

Let p be a prime. Consider

$$\cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

The **ring of p-adic integers** is

$$\mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}.$$

Thus $\alpha \in \mathbb{Z}_p$ is a sequence $(a_n)_{n\in\mathbb{N}}$ of integers modulo p^n for $a_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that $a_n \equiv a_m \mod p^m$ whenever $n \geq m$, since $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, and

$$\begin{array}{cccc} p_n & : & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}/p^n\mathbb{Z} \\ & & \alpha & \longmapsto & a_n = \alpha \mod p^n \end{array}.$$

Given $a \in \mathbb{Z}$, setting $a_n = a \mod p^n$ gives an element $\iota(a) \in \mathbb{Z}_p$ for $\iota : \mathbb{Z} \to \mathbb{Z}_p$. Then ι is injective, since if $a \in \mathbb{Z}$, and $p^n > |a|$ then $a \not\equiv 0 \mod p^n$, so $\iota(a) \neq 0$ in $\mathbb{Z}_p$. Often $\mathbb{Z} \leq \mathbb{Z}_p$.

Definition 2.1.1. Let $\alpha = (a_n), \beta = (b_n) \in \mathbb{Z}_p$. If $\alpha = \beta$ then $\mathrm{d}(\alpha,\beta) = 0$. If $\alpha \neq \beta$, take the smallest n such that $a_n \neq b_n$, and set $\mathrm{d}(\alpha,\beta) = p^{-n}$, the **$p$-adic metric on** $\mathbb{Z}_p$. The restriction of d to $\iota(\mathbb{Z})$ is the **p-adic metric on** $\mathbb{Z}$.

Thus α and β are close if (a_n) and (b_n) agree modulo p^n for all but large n. Since

$$\mathrm{B}(0,r) = \left\{\alpha = (a_n) \mid \forall n \leq -\log_p r,\ a_n = 0\right\} = \ker\left(\mathbb{Z}_p \to \mathbb{Z}/p^{\lfloor -\log_p r\rfloor}\mathbb{Z}\right),$$

open balls are the subgroups $p^n\mathbb{Z}_p \leq \mathbb{Z}_p$.

- $\iota(\mathbb{Z})$ is dense in this metric. Let $\alpha = (a_n) \in \mathbb{Z}_p$ and $\epsilon > 0$. Take $n > -\log_p \epsilon$, and choose $a \in \mathbb{Z}$ such that $a \equiv a_n \mod p^n$. Then $\mathrm{d}(\alpha, \iota(a)) \leq p^{-n} < \epsilon$.
- The p-adic metric on $\mathbb{Z}$ is not complete, since $a_n = 1 + \cdots + p^n$ does not converge in $\mathbb{Z}$, but does converge in $\mathbb{Z}_p$.
- The p-adic metric on $\mathbb{Z}_p$ is complete. Let $\alpha^{(k)} = \left(a_n^{(k)}\right)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Z}_p$. For all n there exists K_n such that for all $k, l \geq K_n$, we have $\mathrm{d}\left(\alpha^{(k)}, \alpha^{(l)}\right) \leq p^{-n}$, so $a_n^{(k)} = a_n^{(l)}$ for all $k, l \geq K_n$ so for fixed n, $a_n^{(k)}$ is eventually a constant b_n. Then $\beta = (b_n) \in \mathbb{Z}_p$, and $\alpha^{(k)} \to \beta$ in $\mathbb{Z}_p$.

Thus $\mathbb{Z}_p$ is a completion of $\mathbb{Z}$, but is not the profinite completion of $\mathbb{Z}$.

Definition 2.1.2. Let p be a prime. A **p-group** is a finite group of order p^n for $n \geq 0$. A **pro-p group** is an inverse limit of p-groups.

Definition 2.1.3. Let G be a group and p prime. The set of normal subgroups $N \lhd G$ such that $[G : N] = p^n$ for some n form an inverse system $\mathcal{N}_p$. Since $G/N_1 \times G/N_2$ are p-groups, $N_1 \cap N_2 = \ker(G \to G/N_1 \times G/N_2)$ is a p-group. The **pro-p completion** is

$$\widehat{G_{(p)}} = \varprojlim_{N\in\mathcal{N}_p} G/N,$$

where $G/N_1 \to G/N_2$ if $N_1 \leq N_2$.

Proposition 2.1.4. *The additive group $\mathbb{Z}_p$ is abelian and torsionfree.*

Proof. $\mathbb{Z}_p \leq \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is abelian. Let $\alpha = (a_n) \in \mathbb{Z}_p \setminus \{0\}$. Suppose $m\alpha = 0$ for $m \in \mathbb{Z}$. We want $m = 0$. Assume $m = p^r s$ for s coprime to p. Then $\alpha \neq 0$, so there exists n such that $a_n \neq 0$. Consider a_{n+r}. Then $0 \equiv m a_{n+r} \equiv p^r a_{n+r} s \mod p^{n+r}$, so $p^n \mid a_{n+r} s$. Thus $p^n \mid a_{n+r}$, so $a_n \equiv a_{n+r} \equiv 0 \mod p^n$, a contradiction. □

Proposition 2.1.5. *The ring $\mathbb{Z}_p$ has no zero-divisors.*

Proof. Exercise. [1] □

2.2 The profinite completion of the integers

The **profinite completion of the integers** is

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z},$$

where $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ whenever $n\mathbb{Z} \leq m\mathbb{Z}$, which is if and only if $m \mid n$, so $n = mr$.

Theorem 2.2.1 (Chinese remainder theorem)**.** *There is an isomorphism of topological rings*

$$\widehat{\mathbb{Z}} \cong \prod_{p \ prime} \mathbb{Z}_p.$$

Proof. Each natural number n is written as a product of prime powers $n = \prod_{p \text{ prime}} p^{\mathrm{e}_p(n)}$. The classical CRT gives natural isomorphisms

$$\begin{array}{cccc} f_n & : & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \prod_{p \text{ prime}} \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z} \\ & & 1 & \longmapsto & (1, \dots, 1) \end{array},$$

and commutative diagrams

$$\begin{tikzcd} \mathbb{Z}/mn\mathbb{Z} \arrow[r, "f_{mn}", "\sim"'] \arrow[d, two heads] & \prod_p \mathbb{Z}/p^{\mathrm{e}_p(mn)}\mathbb{Z} \arrow[d, two heads] \\ \mathbb{Z}/n\mathbb{Z} \arrow[r, "\sim", "f_n"'] & \prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z} \end{tikzcd}.$$

Passing to inverse limits,

$$\begin{array}{ccc} \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\sim} & \varprojlim_n \prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z} \\ \cap & & \cap \\ \prod_n \mathbb{Z}/n\mathbb{Z} & \xrightarrow[\sim]{} & \prod_n \prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z} \end{array}.$$

The natural continuous surjections

$$\prod_p \mathbb{Z}_p \twoheadrightarrow \prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z}$$

form a cone on the inverse system $\left\{\prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z}\right\}$, so there exists

$$f : \prod_p \mathbb{Z}_p \twoheadrightarrow \varprojlim_n \prod_p \mathbb{Z}/p^{\mathrm{e}_p(n)}\mathbb{Z},$$

which is continuous by Proposition 1.2.14, surjective by Corollary 1.2.20, and injective since every non-trivial element of $\prod_p \mathbb{Z}_p$ is non-trivial in some quotient $\mathbb{Z}/p^e\mathbb{Z}$. So f is a topological isomorphism as required. □

Corollary 2.2.2. *The abelian group $\widehat{\mathbb{Z}}$ is torsionfree abelian.*

Corollary 2.2.3. *The ring $\widehat{\mathbb{Z}}$ is not an integral domain.*

Proof. Any product of non-trivial rings $R_1 \times R_2$ has zero-divisors, since $(r_1, 0) \cdot (0, r_2) = (0, 0)$. An element of $\widehat{\mathbb{Z}}$ is a zero-divisor if and only if it is zero in some $\mathbb{Z}_p$-factor. □

Elements of $\iota(\mathbb{Z})$ are not zero divisors in $\widehat{\mathbb{Z}}$.

[1] Exercise

2.3 Profinite matrix groups

Lecture 6 Tuesday 02/02/21

For a commutative ring R, we have

$$\mathrm{Mat}_{N\times M} R = \{N \times M \text{ matrices with elements in } R\}.$$

If $N = M$, we have a ring structure, where addition and multiplication are given by the usual formula. There exists a determinant function $\det : \mathrm{Mat}_{N\times N} R \to R$. Then

$$\mathbb{Z}_p^{NM} \cong \mathrm{Mat}_{N\times M} \mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathrm{Mat}_{N\times M} (\mathbb{Z}/p^n\mathbb{Z}).$$

By continuity of ring operations on $\mathbb{Z}_p$, addition and multiplication on matrices are continuous, and $\det : \mathrm{Mat}_{N\times N} \mathbb{Z}_p \to \mathbb{Z}_p$ is continuous. Since $\mathbb{Z}_p$ is an integral domain, it has a field of fractions $\mathbb{Q}_p$, so you can do linear algebra over $\mathbb{Q}_p$. A matrix over $\mathbb{Q}_p$ has an inverse over $\mathbb{Q}_p$ if and only if its determinant is non-zero, and a matrix over $\mathbb{Z}_p$ has an inverse over $\mathbb{Z}_p$ if and only if its determinant and its inverse are in $\mathbb{Z}_p^\times$. Define

$$\mathrm{GL}_N \mathbb{Z}_p = \left\{A \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p \;\middle|\; \det A \in \mathbb{Z}_p^\times\right\}, \qquad \mathrm{SL}_N \mathbb{Z}_p = \{A \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p \mid \det A = 1\}.$$

Both are profinite groups.

Lemma 2.3.1. *For all $N \ge 1$ and p prime,*

$$\mathrm{GL}_N \mathbb{Z}_p = \varprojlim_n \mathrm{GL}_N (\mathbb{Z}/p^n\mathbb{Z}), \qquad \mathrm{SL}_N \mathbb{Z}_p = \varprojlim_n \mathrm{SL}_N (\mathbb{Z}/p^n\mathbb{Z}).$$

Proof. The diagrams

$$\begin{tikzcd} \mathrm{Mat}_{N\times N} \mathbb{Z}_p \arrow[r] \arrow[d, "\det"'] & \mathrm{Mat}_{N\times N} (\mathbb{Z}/p^n\mathbb{Z}) \arrow[d, "\det"] \\ \mathbb{Z}_p \arrow[r] & \mathbb{Z}/p^n\mathbb{Z} \end{tikzcd}$$

commute.

- $A \in \mathrm{GL}_N \mathbb{Z}_p$ if and only if $\det A \in \mathbb{Z}_p^\times$, if and only if $\det A_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ for all n, if and only if $A_n \in \mathrm{GL}_N (\mathbb{Z}/p^n\mathbb{Z})$ for all n.
- $A \in \mathrm{SL}_N \mathbb{Z}_p$ if and only if $\det A = 1$, if and only if $\det A_n = 1$ for all n, if and only if $A_n \in \mathrm{SL}_N (\mathbb{Z}/p^n\mathbb{Z})$ for all n.

□

Also have matrices over $\widehat{\mathbb{Z}}$. A warning is that $\widehat{\mathbb{Z}}$ is not an integral domain. Analogously,

$$\mathrm{GL}_N \widehat{\mathbb{Z}} = \left\{A \in \mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} \;\middle|\; \det A \in \widehat{\mathbb{Z}}^\times\right\} = \varprojlim_n \mathrm{GL}_N (\mathbb{Z}/n\mathbb{Z}) = \prod_p \mathrm{GL}_N \mathbb{Z}_p,$$

$$\mathrm{SL}_N \widehat{\mathbb{Z}} = \left\{A \in \mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} \;\middle|\; \det A = 1\right\} = \varprojlim_n \mathrm{SL}_N (\mathbb{Z}/n\mathbb{Z}) = \prod_p \mathrm{SL}_N \mathbb{Z}_p,$$

since $\mathrm{Mat}_{N\times N} \widehat{\mathbb{Z}} = \prod_p \mathrm{Mat}_{N\times N} \mathbb{Z}_p$, and

$$\mathrm{SL}_N \mathbb{Z} \le \mathrm{SL}_N \mathbb{Z}_p, \qquad \mathrm{SL}_N \mathbb{Z} \le \mathrm{SL}_N \widehat{\mathbb{Z}} = \varprojlim_n \mathrm{SL}_N (\mathbb{Z}/n\mathbb{Z})$$

are dense. See problem sheet 2.

Example 2.3.2. $\left(\begin{smallmatrix} 7 & 9 \\ 4 & 9 \end{smallmatrix}\right) \in \mathrm{SL}_2 (\mathbb{Z}/13\mathbb{Z})$ is in the image of $\mathrm{SL}_2 \mathbb{Z}$.

2.4 Subgroups, quotients, and homomorphisms

Proposition 2.4.1. *A closed subgroup of a profinite group is a profinite group.*

Proof. Let $G = \varprojlim_{j \in J} G_j$ be a profinite group for G_j finite. Take a closed subgroup $H \leq_{\mathrm{c}} G$ of G. Define $H_j = p_j(H) \leq G_j$. Then H_j, with transition maps $\phi_{ij}|_{H_i} : H_i \to H_j$, are an inverse system of finite groups. Define

$$H' = \varprojlim_j H_j = \left\{ (g_j) \in \prod_{j \in J} G_j \;\middle|\; \forall i \leq j,\ \phi_{ij}(g_i) = g_j,\ g_j \in H_j \right\}.$$

Show that $H = H'$. If $h = (h_j) \in H$, by definition $h_j = p_j(h) \in H_j$, so $H \leq H'$. Suppose $g = (g_j) \notin H$. Since H is closed, $G \setminus H$ is open, so there exists a basic open set containing g, which does not intersect H. There exists $j \in J$ such that $gU_j = p_j^{-1}(\{g_j\}) \leq G \setminus H$. Therefore for all $h \in H$, $p_j(h) \neq g_j$, since then $h \in H \cap p_j^{-1}(\{g_j\})$, so $g_j \notin H_j$, so $g \notin H'$. So $H = H'$. ☐

Remark 2.4.2.

- The two topologies on H agree by $\mathrm{id} : (H, \tau_{\mathrm{profinite}}) \to (H, \tau_{\mathrm{subspace}})$, which is continuous by Proposition 1.2.14.
- A better name for H' is $\overline{H}$, the closure. Actually proved that $H' = \overline{H} = H$.

Proposition 2.4.3. *Let $G = \varprojlim_j G_j$ and $H \leq G$. Set $H_j = p_j(H) \leq G_j$. Then the closure of H is*

$$\overline{H} = \varprojlim_j H_j.$$

Lemma 2.4.4. *Let $f : G_1 \to G_2$ be a surjective homomorphism and $H \leq G_1$. Then $[G_1 : H] \geq [G_2 : f(H)]$.*

Proposition 2.4.5. *Let $G = \varprojlim_j G_j$ for (G_j) a surjective inverse system, so $G \twoheadrightarrow G_j$. Let $H \leq_{\mathrm{c}} G$ and set $H_j = p_j(H) \leq G_j$. Then H is finite index if and only if $[G_j : H_j]$ is constant on a cofinal subsystem, if and only if $[G_j : H_j]$ is bounded for all j. If this is true, then $[G : H] = [G_i : H_i]$ for $i \in I$.*

Proof. $p_j : G \to G_j$ are surjective, so $[G : H] \geq [G_j : H_j]$. Suppose $[G : H] \geq N$. There exist distinct cosets $g_1 H, \dots, g_N H$ of H in G, if and only if $g_n^{-1} g_m \notin H$ if $n \neq m$, so there exists $j_{n,m} \in J$ such that $p_{j_{n,m}}\left(g_n^{-1} g_m\right) \notin H_{j_{n,m}}$. Take $k \leq j_{n,m}$ for all n and m. Then $p_k\left(g_n^{-1} g_m\right) \notin H_k$ for all $n \neq m$, so $p_k(g_n) H_k$ are distinct cosets of H_k in G_k, so $[G_k : H_k] \geq N$. For any i in the cofinal subsystem $J_{\leq k}$, it follows $[G_i : H_i] \geq N$ for all $i \leq k$. If $[G : H] = N$ is finite, take k as above and $I = J_{\leq k}$. Then $[G : H] \geq [G_i : H_i] \geq N = [G : H]$ for all $i \in I$. If $[G : H]$ is infinite, assume I is cofinal and $[G_i : H_i] = N$ for all $i \in I$. Then there exists k such that $[G_k : H_k] \geq N + 1$. But there exists $i \in I$ such that $i \leq k$, then $[G_i : H_i] \geq [G_k : H_k] \geq N + 1 > N = [G_i : H_i]$, a contradiction. ☐

Proposition 2.4.6. *Let G be a profinite group and N a closed normal subgroup. Then G/N, with the quotient topology, is a profinite group.*

Proof. Take $G = \varprojlim_j G_j$ for (G_j) a surjective inverse system. Let $N_j = p_j(N) \triangleleft G_j = p_j(G)$. Recall $N = \varprojlim_j N_j$. Define $Q_j = G_j / N_j$. Since $\phi_{ij}(N_i) \leq N_j$, we get quotient homomorphisms $\psi_{ij} : Q_i \to Q_j$, which are transition maps for the Q_j. Set $Q = \varprojlim_j Q_j$. The map $\prod_j G_j \to \prod_j Q_j$ is continuous, so there is a continuous surjective group homomorphism $f : G \to Q$. The kernel of this map is N, since $f(g) = 1$ if and only if $q_j(f(g)) = 1$ for all j, if and only if $g_j \in N_j$ for all j, if and only if $g \in \varprojlim_j N_j = N$. By the first isomorphism theorem for groups,

$$\begin{tikzcd} G \arrow[d] \arrow[dr] & \\ G/N \arrow[r, "\sim", "\overline{f}"'] & Q \end{tikzcd}.$$

Since $G \to Q$ is continuous and $G \to G/N$ is the quotient map, $\overline{f}$ is continuous. Since G/N is compact and Q is Hausdorff, $\overline{f}$ is a homeomorphism. ☐

This is the first isomorphism theorem for profinite groups.

Lecture 7 Thursday 04/02/21

Definition 2.4.7. Let $(G_j)_{j \in J}$ and $(H_j)_{j \in J}$ be inverse systems of finite groups, over the same poset J. A **morphism of inverse systems** (f_j) is a family of homomorphisms $f_j : G_j \to H_j$, such that for all $i \le j$,

$$\begin{tikzcd} G_i \arrow[r, "f_i"] \arrow[d, "\phi^G_{ij}"'] & H_i \arrow[d, "\phi^H_{ij}"] \\ G_j \arrow[r, "f_j"'] & H_j \end{tikzcd}$$

commutes, so $\phi^H_{ij} \circ f_i = f_j \circ \phi^G_{ij}$.

Proposition 2.4.8. *Let $(f_j) : (G_j) \to (H_j)$ be a morphism of inverse systems. Then there is a unique continuous homomorphism $f : G = \varprojlim_j G_j \to H = \varprojlim_j H_j$ such that*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p^G_j"'] & H \arrow[d, "p^H_j"] \\ G_j \arrow[r, "f_j"'] & H_j \end{tikzcd},$$

so $p^H_j \circ f = f_j \circ p^G_j$ for all $j \in J$.

Proof. The maps $f_j \circ p^G_j : G \to H_j$ form a cone on the inverse system (H_j),

$$\begin{tikzcd} & G \arrow[dl, "f_i \circ p^G_i"'] \arrow[dr, "f_j \circ p^G_j"] & \\ H_i \arrow[rr, "\phi^H_{ij}"'] & & H_j \end{tikzcd},$$

since

$$\phi^H_{ij} \circ f_i \circ p^G_i = f_j \circ \phi^G_{ij} \circ p^G_i = f_j \circ p^G_j.$$

So by definition of limits, there exists a unique $f : G \to H = \varprojlim_j H_j$ such that $p^H_j \circ f = f_j \circ p^G_j$. ☐

Thus f is **induced** by the f_j by passing to an inverse limit.

Proposition 2.4.9. *Let $G = \varprojlim_{j \in J} G_j$ and $H = \varprojlim_{i \in I} H_i$ be inverse limits of finite groups, where I and J are countable inverse systems with no minimal element. Let $f : G \to H$ be a continuous homomorphism. Then there exist cofinal subsystems $J' \subseteq J$ and $I' \subseteq I$, an order-preserving bijection $J' \cong I'$, and a morphism of inverse systems $(f_j) : (G_j)_{j \in J'} \to (H_i)_{i \in I'}$ inducing f.*

Proof. Without loss of generality, use Proposition 1.3.8 to assume J and I are linearly ordered. Without loss of generality both are $\mathbb{N}$, with the wrong-way ordering. Construct an increasing sequence (k_n) of natural numbers as follows. Each map $p^H_n \circ f : G \to H \to H_n$ is a continuous homomorphism, so its kernel is open in G. By Proposition 1.2.17 there exists k_n such that $\ker(G \to G_{k_n}) \le \ker(G \to H_n)$, which means there is a quotient homomorphism

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p^G_{k_n}"'] \arrow[dr] & H \arrow[d, "p^H_n"] \\ G_{k_n} \arrow[r, "f_n"'] & H_n \end{tikzcd}.$$

Then $\ker(G \to G_{n+1}) \le \ker(G \to G_n)$, so without loss of generality $k_n > k_{n-1}$. Now $J' = \{k_n\}_{n \in \mathbb{N}}$ give a cofinal subsystem of $J = \mathbb{N}$, and the f_n are the required morphisms of inverse systems. ☐

2.5 Generators of profinite groups

Definition 2.5.1. Let G be a topological group, and let S be a subset of G. Then S is a **topological generating set** for G if the subgroup $\langle S\rangle$ is dense in G, and G is **topologically finitely generated** if it has some finite topological generating set S.

Definition 2.5.2. Let G be a topological group and $S \subseteq G$. The **closed subgroup of G topologically generated by** S is the smallest closed subgroup of G which contains S. Denoted $\overline{\langle S\rangle}$.

Proposition 2.5.3. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G. Hence for $S \subseteq G$, the closed subgroup of G generated by S is equal to the closure of $\langle S\rangle$.*

Proof. Exercise. [2] ☐

Lemma 2.5.4. *A finite index subgroup of a finitely generated group is finitely generated.*

Proposition 2.5.5. *If a profinite group G is topologically finitely generated and U is an open subgroup of G then U is topologically finitely generated.*

Proof. Let S be a finite set such that $\langle S\rangle$ is dense in G. Then $\Gamma = U \cap \langle S\rangle$ is finite index in $\langle S\rangle$, hence Γ is finitely generated, so $\Gamma = \langle S'\rangle$ for S' finite. Since U is open, and $\langle S\rangle$ is dense, $\langle S'\rangle = U \cap \langle S\rangle$ is dense in U. So U is topologically finitely generated. ☐

Proposition 2.5.6. *Let (G_j) be a surjective inverse system of finite groups with $G = \varprojlim_j G_j$. Let $S \subseteq G$. Then S is a topological generating set for G if and only if $p_j(S)$ generates G_j for all j.*

Proof. By Corollary 1.2.19, $\langle S\rangle$ is dense in G if and only if $G_j = p_j(\langle S\rangle) = \langle p_j(S)\rangle$ for all j. ☐

Lemma 2.5.7. *Let G be a topologically finitely generated profinite group. Then G may be written as the inverse limit of a countable inverse system of finite groups.*

Proof. A continuous homomorphism from G to a finite group is determined by the image of a topological generating set S, since a function on S determines all of a homomorphism from $\langle S\rangle$ and continuity gives the behaviour on all of G. So there are only countably many continuous homomorphisms from G to $\mathrm{Sym}\, n$ for $n \in \mathbb{N}$. Every open normal subgroup of G is the kernel of such a continuous homomorphism. So there are only countably many open normal subgroups of G. Then $\mathcal{U} = \{U \lhd_{\mathrm{o}} G\}$ is a neighbourhood base of the identity, so by Proposition 1.2.22, $G = \varprojlim_{U\in\mathcal{U}} G/U$. ☐

Example 2.5.8. Let G be a topologically finitely generated profinite group. Then there are only finitely many open subgroups of G of index at most n. See Lemma 1.2.10. Define

$$G_n = \bigcap \{U \mid U \leq_{\mathrm{o}} G,\ [G:U] \leq n\}.$$

Then $G_n \lhd G$, and G_n is open in G. And $\{G_n\}$ is a neighbourhood base of the identity. So

$$G = \varprojlim_{n\in\mathbb{N}} G/G_n.$$

Proposition 2.5.9. *Let $\mathbb{Z}_p^\times$ be the set of elements α of $\mathbb{Z}_p$ which topologically generate $\mathbb{Z}_p$. Then $\alpha \in \mathbb{Z}_p^\times$ if and only if $\alpha \not\equiv 0 \mod p$. Hence $\mathbb{Z}_p^\times$ is a closed uncountable subset of $\mathbb{Z}_p$. For every n, and every generator $a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ there is some $\alpha \in \mathbb{Z}_p^\times$ such that $\alpha \equiv a_n \mod p^n$.*

Proof. For the last part, a_n is the image of α, since it is a surjective inverse system, and if a_n generates $\mathbb{Z}/p^n\mathbb{Z}$, it is coprime to p. If $\alpha = (a_n)$ such that $a_1 \neq 0$, then $p \nmid a_n$ for any n. Hence a_n is coprime to p, and so generates $\mathbb{Z}/p^n\mathbb{Z}$ for all n. So $\langle\alpha\rangle$ is dense in $\mathbb{Z}_p$ by an earlier result. ☐

Lecture 8 Saturday 06/02/21

Remark 2.5.10. $\mathbb{Z}_p^\times$ is the set of units in the ring $\mathbb{Z}_p$.

- $\Longleftarrow$ If α is a unit, then $\alpha \mod p^n$ is a unit in $\mathbb{Z}/p^n\mathbb{Z}$, so generates $\mathbb{Z}/p^n\mathbb{Z}$. Then α topologically generates $\mathbb{Z}_p$.

[2] Exercise

$\Longrightarrow$ Consider the group homomorphism

$$\begin{array}{cccc} f & : & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p \\ & & x & \longmapsto & \alpha x \end{array},$$

which is continuous as multiplication in a ring is continuous. So $\operatorname{im} f$ is a closed subgroup of $\mathbb{Z}_p$, containing α. Then α generates $\mathbb{Z}_p$, so the only closed subgroup containing α is $\mathbb{Z}_p$ itself. So $1 \in \operatorname{im} f$, so there exists β such that $\alpha\beta = 1$.

Thus α is a unit if and only if $\{\alpha\}$ is a topological generating set for $\mathbb{Z}_p$.

Example 2.5.11. If $p \neq 2$, then 2 is invertible in $\mathbb{Z}_p$, so 2^{-1} exists. If $p = 3$, then $2^{-1} = (\dots, 5, 2) \in \mathbb{Z}_3 \leq \prod_{n \in \mathbb{N}} \mathbb{Z}/3^n\mathbb{Z}$.

Proposition 2.5.12. *$\alpha \in \widehat{\mathbb{Z}}^\times$ if and only if $\alpha \mod n \in (\mathbb{Z}/n\mathbb{Z})^\times$ for all n. For any n, and every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ there exists a generator $\alpha \in \widehat{\mathbb{Z}}^\times$ such that $\alpha \equiv k \mod n$.*

Proof. Follows from Proposition 2.5.9 via the CRT, since $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. $\square$

Theorem 2.5.13 (Gaschutz's lemma for finite groups)**.** *Let $f : G \twoheadrightarrow H$ be a surjective homomorphism of finite groups. Suppose G has some generating set of size d. For any generating set $\{z_1, \dots, z_d\} \subseteq H$, there exists a generating set $\{x_1, \dots, x_d\} \subseteq G$ such that $f(x_i) = z_i$ for all i.*

Really, talking about generating vectors $\underline{x} = (x_1, \dots, x_d) \in G^d$. Extend f to $f : G^d \to H^d$.

Proof. We will prove, by induction on $|G|$, for H fixed, the following statement. The number

$$N_G(\underline{y}) = \left|\left\{\text{generating vectors } \underline{x} \text{ of } G \mid f(\underline{x}) = \underline{y}\right\}\right|,$$

where $\underline{y} \in H^d$ is a generating vector of H, is independent of $\underline{y}$. Want to show $N_G(\underline{z}) > 0$, and G has some generating vector $\underline{x'} \in G^d$ so $N_G(\underline{z}) = N_G(f(\underline{x'})) > 0$. Let $\underline{y} \in H^d$ be a generating vector. Let

$$\mathcal{C} = \{d\text{-generator proper subgroups of } G\}.$$

Every $\underline{x} \in G^d$ such that $f(\underline{x}) = \underline{y}$ either generates G or generates some $C \in \mathcal{C}$. Therefore

$$N_G(\underline{y}) + \sum_{C \in \mathcal{C}} N_C(\underline{y}) = \left|\left\{\underline{x} \mid f(\underline{x}) = \underline{y}\right\}\right| = |\ker f|^d.$$

Thus $N_G(\underline{y}) = |\ker f|^d - \sum_{C \in \mathcal{C}} N_C(\underline{y})$, which is independent of $\underline{y}$ by induction. $\square$

Theorem 2.5.14 (Gaschutz's lemma for profinite groups)**.** *Let $f : G \to H$ be a continuous surjective homomorphism of profinite groups. Suppose G has a topological generating set of size d. Then for any topological generating set $\{z_1, \dots, z_d\}$ of H, there is a topological generating set $\{x_1, \dots, x_d\}$ of G such that $f(x_i) = z_i$ for all i.*

Proof. By Proposition 1.3.6 and Proposition 2.4.9 we may assume and write $G = \varprojlim_{j \in J} G_j$ and $H = \varprojlim_{j \in J} H_j$, surjective inverse systems of finite groups, with a morphism of inverse systems $(f_j) : (G_j) \to (H_j)$ such that $f = \varprojlim_j f_j$. It is forced that f_j is surjective, since

$$\begin{tikzcd} G \arrow[r, two heads, "f"] \arrow[d, two heads, "p_j^G"'] & H \arrow[d, two heads, "p_j^H"] \\ G_j \arrow[r, two heads, "f_j"'] & H_j \end{tikzcd}.$$

Let $\underline{z}$ be the given topological generating set of H. Set $\underline{z}_j$ for $j \in J$ to be the image of $\underline{z}$ in H_j, so $\underline{z}_j = p_j^H(\underline{z})$ is a generating vector of H_j. Consider the finite sets

$$X_j = \left\{\text{generating vectors } \underline{x}_j \in G_j^d \mid f_j(\underline{x}_j) = \underline{z}_j\right\} \neq \emptyset,$$

by Gaschutz. The X_j form an inverse system, since $\phi_{ij}(X_i) \subseteq X_j$. Therefore $\varprojlim_j X_j$ is non-empty. If $\underline{x} \in \varprojlim_j X_j \subseteq G^d$ such that $p_j^G(\underline{x}) \in X_j$, then $\underline{x}$ is a topological generating set of G and $p_j^H(f(\underline{x})) = \underline{z}_j$ for all j, so $f(\underline{x}) = \underline{z}$. $\square$

3 Profinite completions

3.1 Residual finiteness

Notation 3.1.1. Discrete abstract groups will be Greek letters and profinite groups will be Roman letters.

Given an abstract group Γ and an inverse system $\mathcal{N} = \{N \triangleleft_{\mathrm{f}} \Gamma\}$, there is an inverse system of finite groups Γ/N. Then $\widehat{\Gamma} = \varprojlim_{N \in \mathcal{N}} \Gamma/N$, where $\Gamma/N_1 \to \Gamma/N_2$ if $N_1 \le N_2$. Also had a canonical morphism $\iota_\Gamma = \iota : \Gamma \to \widehat{\Gamma}$. The image of ι is dense by Corollary 1.2.19. Also implies for any finite generating set $S \subseteq \Gamma$, $\iota(S)$ is a topological generating set of $\widehat{\Gamma}$, so if Γ is finitely generated, then $\widehat{\Gamma}$ is topologically finitely generated.

Proposition 3.1.2. *Let $f : \Delta \to \Gamma$ be a group homomorphism. Then there exists a unique continuous group homomorphism $\widehat{f} : \widehat{\Delta} \to \widehat{\Gamma}$ such that $\widehat{f} \circ \iota_\Delta = \iota_\Gamma \circ f$, so*

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "\iota_\Delta"'] & \Gamma \arrow[d, "\iota_\Gamma"] \\ \widehat{\Delta} \arrow[r, dashed, "\widehat{f}"'] & \widehat{\Gamma} \end{tikzcd}.$$

Proof. Uniqueness will follow from the density of $\iota_\Delta(\Delta)$ in $\widehat{\Delta}$. Take two $\widehat{f_1}$ and $\widehat{f_2}$ satisfying Proposition 3.1.2. Consider

$$S = \left\{ \delta \in \widehat{\Delta} \;\middle|\; \widehat{f_1}(\delta) = \widehat{f_2}(\delta) \right\}.$$

Then S is closed, since it is the preimage of the diagonal in $\widehat{\Gamma} \times \widehat{\Gamma}$ under $\left(\widehat{f_1}, \widehat{f_2}\right) : \widehat{\Delta} \to \widehat{\Gamma} \times \widehat{\Gamma}$, and S contains $\iota_\Delta(\Delta)$, which is dense. So $S = \widehat{\Delta}$.

Case 1. Γ is finite, so $\Gamma = \widehat{\Gamma}$. Then $\ker f$ is a finite index normal subgroup M of Δ, so there exists a projection map $p_M : \widehat{\Delta} \to \Delta/M$. So we get a composition

$$\begin{tikzcd} \Delta \arrow[rr, "\iota_\Delta"] \arrow[dr] \arrow[ddr, "f"'] & & \widehat{\Delta} \arrow[dl, "p_M"'] \arrow[ddl, dashed, "\widehat{f}"] \\ & \Delta/M \arrow[d, hook] & \\ & \Gamma & \end{tikzcd}.$$

Case 2. General case. Take some $N \triangleleft_{\mathrm{f}} \Gamma$. There exists a unique $q_N : \widehat{\Delta} \to \Gamma/N$ such that $q_N \circ \iota_\Delta = p_N \circ \iota_\Gamma \circ f$. Then (q_N) form a cone on the inverse system, since

$$\phi^\Gamma_{N_1 N_2} \circ q_{N_1} \circ \iota_\Delta = \phi^\Gamma_{N_1 N_2} \circ p_{N_1} \circ \iota_\Gamma \circ f = p_{N_2} \circ \iota_\Gamma \circ f = q_{N_2} \circ \iota_\Delta.$$

Thus there exists a unique $\widehat{f} : \widehat{\Delta} \to \widehat{\Gamma}$ such that $p_N \circ \widehat{f} = q_N$ for all N, so

$$\begin{tikzcd} \Delta \arrow[rr, "\iota_\Delta"] \arrow[d, "f"'] & & \widehat{\Delta} \arrow[dl, dashed, "\widehat{f}"'] \arrow[ddl, "q_N"] \\ \Gamma \arrow[r, "\iota_\Gamma"] \arrow[dr] & \widehat{\Gamma} \arrow[d, "p_N"'] & \\ & \Gamma/N & \end{tikzcd},$$

and

$$p_N \circ \widehat{f} \circ \iota_\Delta = q_N \circ \iota_\Delta = p_N \circ \iota_\Gamma \circ f.$$

$\square$

Corollary 3.1.3. $\widehat{\cdot}$ *is a functor.*

Lecture 9 Tuesday 09/02/21

Definition 3.1.4. Let Γ be an abstract group. Then Γ is **residually finite** if for every $\gamma \in \Gamma \setminus \{1\}$, there exists $N \lhd_{\mathrm{f}} \Gamma$ such that $\gamma \notin N$, if and only if $\gamma N \neq 1$ in Γ/N, if and only if there exists $\phi : \Gamma \to G$ finite such that $\phi(\gamma) \neq 1$.

Proposition 3.1.5. *Γ is residually finite if and only if $\iota : \Gamma \to \widehat{\Gamma}$ is injective.*

Proof.

$$\begin{array}{cccl} \iota & : & \Gamma & \longrightarrow & \widehat{\Gamma} \le \prod_N \Gamma/N \\ & & \gamma & \longmapsto & (\gamma N) \end{array}.$$

□

Proposition 3.1.6. *Any subgroup of a residually finite group is residually finite.*

Proposition 3.1.7. *Let Γ be an abstract group, and let $\Delta \le \Gamma$ be finite index. If Δ is residually finite, then Γ is residually finite.*

Proof. Let $\gamma \in \Gamma \setminus \{1\}$.

Case 1. If $\gamma \notin \Delta$, consider

$$\gamma \notin N = \mathrm{Core}_\Gamma \Delta = \bigcap_{g \in \Gamma} g\Delta g^{-1} \lhd_{\mathrm{f}} \Gamma,$$

which has finitely many distinct terms, since if $g\Delta = g'\Delta$ then $g = g'\delta$ so $g\Delta g^{-1} = g'\delta\Delta\delta^{-1}g'^{-1} = g'\Delta g'^{-1}$.

Case 2. If $\gamma \in \Delta$, there exists $N \lhd_{\mathrm{f}} \Delta$ such that $\gamma \notin N$. Now $\gamma \notin \mathrm{Core}_\Gamma N \lhd_{\mathrm{f}} \Gamma$.

□

Proposition 3.1.8. *Finitely generated abelian groups are residually finite.*

Proof. Exercise. [3] □

Proposition 3.1.9. *The groups $\mathrm{SL}_N \mathbb{Z} \le_{\mathrm{f}} \mathrm{GL}_N \mathbb{Z}$ are residually finite for all N.*

Proof. For $A \in \mathrm{GL}_N \mathbb{Z} \setminus \{\mathrm{I}\}$. Take a prime p larger than the absolute value of all entries of A. Then we have the homomorphism

$$\begin{array}{ccc} \mathrm{GL}_N \mathbb{Z} & \longrightarrow & \mathrm{GL}_N(\mathbb{Z}/p\mathbb{Z}) \\ A & \longmapsto & A_p \neq \mathrm{I} \end{array}.$$

□

These linear groups have as subgroups many important groups, such as free groups in $\mathrm{SL}_2 \mathbb{Z}$.

Theorem 3.1.10 (Malcev's theorem)**.** *Let Γ be a finitely generated subgroup of $\mathrm{GL}_N K$ where K is a field. Then Γ is residually finite.*

Proof (non-examinable). The entries of a generating set of Γ generate a finitely generated subring R of K. Commutative algebra says that R has many maximal ideals $\mathfrak{p} \subseteq R$, such that $R/\mathfrak{p}$ is a finite field. Use maps $\mathrm{GL}_N R \to \mathrm{GL}_N(R/\mathfrak{p})$ to show residual finiteness. □

Proposition 3.1.11. *The fundamental group of a surface is residually finite.*

Proof. Surface groups, via geometry, are subgroups of $\mathrm{Isom}\,\mathbb{H}^2 \cong \mathrm{PSL}_2 \mathbb{R}$. □

[3] Exercise: classification of finitely generated abelian groups

Lemma 3.1.12. *Let Γ be an abstract group. The open subgroups of $\widehat{\Gamma}$ are exactly $\overline{\iota(\Delta)}$ for $\Delta \leq_{\mathrm{f}} \Gamma$.*

Proof. If $\Delta \leq_{\mathrm{f}} \Gamma$ is finite index, take a finite set of coset representatives $\{\gamma_i\}$ of Δ in Γ, so $\Gamma = \bigcup_i \gamma_i \Delta$. Then

$$\widehat{\Gamma} = \overline{\iota(\Gamma)} = \overline{\bigcup_i \iota(\gamma_i \Delta)} = \bigcup_i \iota(\gamma_i) \overline{\iota(\Delta)},$$

so $\overline{\iota(\Delta)}$ is closed, and finite index, if and only if open. If $U \leq_{\mathrm{o}} \widehat{\Gamma}$, then $\iota(\Gamma)$ is dense, so $U = \overline{\iota(\Gamma) \cap U}$. Set $\Delta = \iota^{-1}(U) \leq_{\mathrm{f}} \Gamma$, and $\iota(\Delta) = \iota(\Gamma) \cap U$. Thus $U = \overline{\iota(\Delta)}$. ☐

Theorem 3.1.13. *Let G and H be topologically finitely generated profinite groups. Suppose the sets of isomorphism types of continuous finite quotients of G and H are equal. Then G and H are isomorphic profinite groups.*

Topologically finitely generated is necessary since $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \not\cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{R}}$. Continuous is not actually necessary by a hard theorem by Nikolov and Segal.

Proof. Let G_n be the intersection of all open subgroups of G of index at most n. Similarly, H_n. By Example 2.5.8, $G = \varprojlim_n G/G_n$ and $H = \varprojlim_n H/H_n$. By assumption there exists $V \lhd_{\mathrm{o}} H$, such that $G/G_n \cong H/V$. The intersection of index at most n subgroups of G/G_n is trivial, so the intersection of index at most n subgroups of H/V is trivial. Taking preimages, there exist index at most n open subgroups of H whose intersection is contained in V. Then $H_n \leq V$, so $|G/G_n| = |H/V| \leq |H/H_n|$. By symmetry, $|G/G_n| \geq |H/H_n|$, so equality holds and $V = H_n$. So $G/G_n \cong H/H_n$ for all n. We want a morphism of inverse systems, so commuting diagrams

$$\begin{tikzcd} G/G_n \arrow[r] \arrow[d] & H/H_n \arrow[d] \\ G/G_{n-1} \arrow[r] & H/H_{n-1} \end{tikzcd}.$$

Let

$$S_n = \{\text{isomorphisms } f_n : G/G_n \to H/H_n\} \neq \emptyset.$$

If $f_n \in S_n$, then f_n takes an index at most $n-1$ subgroup of G/G_n to an index at most $n-1$ subgroup of H/H_n. The intersection of such subgroups is G_{n-1}/G_n. So f_n maps G_{n-1}/G_n to H_{n-1}/H_n. So there is a well-defined quotient map such that the diagram

$$\begin{tikzcd} G/G_{n-1} \arrow[r, "\phi_{n,n-1}(f_n)", "\sim"'] & H/H_{n-1} \\ G/G_n \arrow[u] \arrow[r, "\sim", "f_n"'] & H/H_n \arrow[u] \end{tikzcd}$$

commutes. The $\phi_{n,n-1} : S_n \to S_{n-1}$ make (S_n) into an inverse system. Then $\varprojlim_n S_n$ is non-empty, and an element of $\varprojlim_n S_n \leq \prod_n S_n$ is a sequence of f_n such that all diagrams commute. Thus there is an isomorphism of inverse systems, so $G \cong H$. ☐

Theorem 3.1.14. *Let Γ and Δ be finitely generated abstract groups. Suppose the sets of isomorphism types of finite quotients of Γ and Δ are equal. Then $\widehat{\Gamma} \cong \widehat{\Delta}$.*

Lecture 10 Thursday 11/02/21

Definition 3.1.15. A property $\mathcal{P}$ of groups is a **profinite invariant** if, whenever two finitely generated residually finite groups G and H have $\widehat{G} \cong \widehat{H}$, G has $\mathcal{P}$ if and only if H has $\mathcal{P}$.

Proposition 3.1.16. *Being abelian is a profinite invariant.*

Proof. Let G and H be finitely generated residually finite groups such that $\widehat{G} \cong \widehat{H}$, with H abelian. Every quotient group of H is abelian, so every finite quotient of G is abelian. Suppose G is not abelian. There exist $g_1, g_2 \in G$ such that $[g_1, g_2] \neq 1$. Since G is residually finite, there exists a finite quotient Q of G and $\phi : G \twoheadrightarrow Q$, such that $[\phi(g_1), \phi(g_2)] = \phi([g_1, g_2]) \neq 1$. But Q is abelian, a contradiction. ☐

Proposition 3.1.17. *Let G and H be finitely generated groups with $\widehat{G} \cong \widehat{H}$. Then the abelianisations $G_{\mathrm{ab}} = G/[G,G]$ and $H_{\mathrm{ab}} = H/[H,H]$ are isomorphic.*

Proof. Suppose $\widehat{G} \cong \widehat{H}$. We claim $\widehat{G_{\mathrm{ab}}} \cong \widehat{H_{\mathrm{ab}}}$. Since G and H have the same finite quotients they have the same abelian finite quotients, which are the finite quotients of G_{ab} and H_{ab}, since

$$\begin{tikzcd} G \arrow[rr, two heads] \arrow[dr, two heads] & & G/[G,G] \arrow[dl, two heads] \\ & A & \end{tikzcd}.$$

It remains to show, if A and A' are finitely generated abelian groups with $\widehat{A} \cong \widehat{A'}$ then $A \cong A'$. By the classification, $A = \mathbb{Z}^r \times T$ and $A' \cong \mathbb{Z}^s \times T'$ for $r, s \in \mathbb{N}$ and T and T' finite. We can see r and T from finite quotients, so

$$r = \max \left\{ k \;\middle|\; \forall n,\ A \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^k \right\} = \max \left\{ k \;\middle|\; \forall n,\ A' \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^k \right\} = s.$$

Having found r, T is the largest finite group such that $A \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^r \times T$ for all n, which is T'. ☐

Corollary 3.1.18. *If G is abelian, the property of being isomorphic to G is a profinite invariant.*

Example 3.1.19. Let

$$\begin{array}{cccc} \phi & : & \mathcal{C}_{25} & \longrightarrow & \mathcal{C}_{25} \\ & & t & \longmapsto & t^6 \end{array}$$

be an automorphism, where $\mathcal{C}_{25} = \mathbb{Z}/25\mathbb{Z} = \langle t \rangle$. Form semidirect products

$$G_1 = \mathcal{C}_{25} \rtimes_\phi \mathbb{Z}, \qquad \left(t^a, s^b\right) *_1 \left(t^c, s^d\right) = \left(t^a \phi^b\left(t^c\right), s^{b+d}\right),$$

$$G_2 = \mathcal{C}_{25} \rtimes_{\phi^2} \mathbb{Z}, \qquad \left(t^a, s^b\right) *_2 \left(t^c, s^d\right) = \left(t^a \phi^{2b}\left(t^c\right), s^{b+d}\right),$$

where $\mathbb{Z} = \langle s \rangle$. Note that ϕ is of order five, so $\phi^5 = \mathrm{id}$ and $\phi^k = \phi^l$ if and only if $k \equiv l \mod 5$.

- Claim that G_1 is not isomorphic to G_2. Suppose $\Phi : G_2 \to G_1$ is an isomorphism. Each G_i has a unique order 25 subgroup. So $\Phi(\mathcal{C}_{25}) = \mathcal{C}_{25}$ and $\Phi(t, 1) = (t^a, 1)$ for some a coprime to 25. Set $\Phi(1, s) = \left(t^b, s^c\right)$, and s^c generates $\mathbb{Z}$, so $c = \pm 1$. A contradiction comes from the computation of

$$\begin{aligned} \left(\phi^2(t)^a, 1\right) &= \Phi\left(\phi^2(t), 1\right) = \Phi\left((1, s) *_2 (t, 1) *_2 \left(1, s^{-1}\right)\right) = \Phi(1, s) *_1 \Phi(t, 1) *_1 \Phi\left(1, s^{-1}\right) \\ &= \left(t^b, s^c\right) *_1 (t^a, 1) *_1 \left(\phi^{-c}\left(t^{-b}\right), s^{-c}\right) = \left(\phi^c(t^a), 1\right), \end{aligned}$$

  and since $\phi^2(t^a) = \phi^c(t^a)$, $\phi^2 = \phi^c$, so $c \equiv 2 \mod 5$.

- Consider finite quotients of G_1. Let $f : G_1 \to Q$ be a finite quotient map. If $\mathrm{im}(\mathbb{Z} \to G_1 \to Q)$ has order m, then $\ker f \geq 5m\mathbb{Z}$. Then f factors through the quotient $\mathcal{C}_{25} \rtimes_\phi \mathbb{Z}/5m\mathbb{Z}$, which is cofinal, so

$$\widehat{G_1} = \varprojlim_m \mathcal{C}_{25} \rtimes_\phi \mathbb{Z}/5m\mathbb{Z} = \mathcal{C}_{25} \rtimes_\phi \widehat{\mathbb{Z}}.$$

  By Gaschutz lemma, there exists $\kappa \in \widehat{\mathbb{Z}}^\times$ such that $\kappa \equiv 2 \mod 5$. We may now build an isomorphism defined by

$$\begin{array}{cccc} \Omega & : & \widehat{G_2} & \longrightarrow & \widehat{G_1} \\ & & \left(t^b, s^\lambda\right) & \longmapsto & \left(t^b, s^{\lambda\kappa}\right) \end{array}.$$

  This is a continuous bijection, and can compute it is a group homomorphism.

Question 3.1.20 (Remeslennikov's question)**.** Let F be a finitely generated free group, and G a finitely generated residually finite group. Is it true that $\widehat{F} \cong \widehat{G}$ implies that $F \cong G$?

Question 3.1.21. Does there exist G a finitely generated residually finite group, other than a free group, and an integer n such that a finite group Q is a quotient of G if and only if Q has a generating set with n elements?

Proposition 3.1.22. *Let F and F' be finitely generated free groups. If $\widehat{F} \cong \widehat{F'}$ then $F \cong F'$.*

Proof. From earlier, if $\widehat{F} \cong \widehat{F'}$ then $\mathbb{Z}^{\operatorname{rk} F} = F_{\mathrm{ab}} \cong F'_{\mathrm{ab}} = \mathbb{Z}^{\operatorname{rk} F'}$. Thus $\operatorname{rk} F = \operatorname{rk} F'$, so $F \cong F'$. □

How about surface groups? If S_g is the fundamental group of an orientable surface of genus g, then

$$\mathrm{S}_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle .$$

Then the abelianisation of S_g is $\mathbb{Z}^{2g}$. Hence $\widehat{\mathrm{S}_g} \ncong \widehat{\mathrm{F}_r}$, unless possibly $r = 2g$.

Theorem 3.1.23 (Basic correspondence)**.** *Let G_1 and G_2 be finitely generated residually finite groups, and suppose $\phi : \widehat{G_1} \cong \widehat{G_2}$. Then there is a bijection*

$$\psi : \{\textit{finite index subgroups of } G_1\} \to \{\textit{finite index subgroups of } G_2\},$$

such that, if $K \le_{\mathrm{f}} H \le_{\mathrm{f}} G_1$, then

- *$\psi(K) \le \psi(H)$ and $[H : K] = [\psi(H) : \psi(K)]$,*
- *$K \lhd H$ if and only if $\psi(K) \lhd \psi(H)$,*
- *if $K \lhd H$, then $H/K \cong \psi(H)/\psi(K)$, and*
- *$\widehat{H} \cong \widehat{\psi(H)}$.*

By the Nielsen-Schreier theorem, F_{2g} has an index two subgroup, which is free of rank $4g - 1$, so has abelianisation odd rank. Any finite index subgroup of a surface group is a surface group, so it has even rank abelianisation, contradicting the basic correspondence, so $\widehat{\mathrm{F}_{2g}} \ncong \widehat{\mathrm{S}_g}$.

Lecture 11 Saturday 13/02/21

Remark 3.1.24.

- Residually finite is not actually necessary, by replacing G_1 by $G_1/\ker \iota_{G_1}$ for $\iota_{G_1} : G_1 \to \widehat{G_1}$.
- ϕ and ψ do not depend on any homomorphism $G_1 \to G_2$.

Proposition 3.1.25. *Let G be a finitely generated residually finite group. Let ψ be the function*

$$\begin{array}{cccc} \psi & : & \{\textit{finite index subgroups } H \le G\} & \longrightarrow & \left\{\textit{open subgroups of } \widehat{G}\right\} \\ & & H & \longmapsto & \overline{H} \end{array}.$$

Then, if $K \le_{\mathrm{f}} H \le_{\mathrm{f}} G$,

1. *ψ is a bijection,*
2. *$[H : K] = \left[\overline{H} : \overline{K}\right]$,*
3. *$K \lhd H$ if and only if $\overline{K} \lhd \overline{H}$,*
4. *if $K \lhd H$, then $H/K \cong \overline{H}/\overline{K}$, and*
5. *$\overline{H} \cong \widehat{H}$.*

Proof.

1. Let $H \le_{\mathrm{f}} G$ and take coset representatives $\{g_i\}$ of H in G. Since $\widehat{G} = \overline{\bigcup_i g_i H} = \bigcup_i g_i \overline{H}$, $\overline{H}$ is finite index, so open. Conversely, if $U \le_{\mathrm{o}} \widehat{G}$ then $U = \overline{G \cap U}$, since G is dense and U is open and closed, so let $H = G \cap U$. So ψ is surjective. To show ψ is injective, we show $\overline{H} \cap G = H$. Considering the action of G on G/H, gives a continuous homomorphism

$$\begin{tikzcd} G \arrow[r] \arrow[d, phantom, "\cap"] & \mathrm{Sym}(G/H) \\ \widehat{G} \arrow[ur] & \end{tikzcd}.$$

Then H fixes the coset $1H$. By continuity of the action, $\overline{H}$ fixes $1H$. But if $g \in G \setminus H$, then $g \cdot 1H = gH \neq 1H$, so $g \notin \overline{H}$. So $\overline{H} \cap G = H$.

2. Let $\{g_i\}$ be a set of coset representatives. We know that the $g_i\overline{H}$ cover $\widehat{G}$. They are distinct cosets, since if $g_i\overline{H} = g_j\overline{H}$, then $g_i^{-1}g_j \in \overline{H} \cap G = H$. So $g_iH = g_jH$, so $g_i = g_j$, so $\left[\widehat{G} : \overline{H}\right] = [G : H]$. Also, there is a natural bijection of coset spaces $G/H \to \widehat{G}/\overline{H}$.

3. If $\overline{K} \lhd \overline{H}$ then $K = \overline{K} \cap G \lhd \overline{H} \cap G = H$. Conversely, if $K \lhd H$, consider the action of $\overline{H}$ on $\mathrm{Sym}\left(\overline{H}/\overline{K}\right) = \mathrm{Sym}\left(H/K\right) \leq \mathrm{Sym}\left(G/K\right)$. Then $K \lhd H$ if and only if K acts trivially on H/K, since $k \cdot hK = hK$ if and only if $h^{-1}kh \in K$. By continuity of the action, $\overline{K}$ acts trivially, so $\overline{K} \lhd \overline{H}$.

4. If $K \lhd H$, we already have our bijection $H/K \to \overline{H}/\overline{K}$, and this is an isomorphism of groups.

5. $\overline{H}$ maps onto all finite quotients H/K in a natural way, so we get a continuous homomorphism $\overline{H} \to \widehat{H}$. This is surjective because H is dense in $\widehat{H}$. For injectivity, if $h \in \overline{H} \setminus \{1\}$, then there is $U \lhd_{\mathrm{o}} \widehat{H}$ such that $h \notin U$, and the map

$$\begin{tikzcd} \overline{H} \arrow[rr] \arrow[dr] & & H/\left(U \cap H\right) \\ & \widehat{H} \arrow[ur] & \end{tikzcd}$$

shows that $h \not\mapsto 1 \in \widehat{H}$.

□

Remark 3.1.26. $\overline{H} \cap G = H$ and $\overline{H} \cong \widehat{H}$ are not always true if H is not of finite index.

Definition 3.1.27. A topological group G is **Hopfian**, or **has the Hopf property**, if every continuous surjection from G to itself is an isomorphism of topological groups.

Example 3.1.28. Finite groups, by the pigeonhole principle.

Proposition 3.1.29. *Let G be a topologically finitely generated profinite group. Let $f : G \to G$ be a continuous surjection. Then f is an isomorphism.*

Proof. Let G_n be the intersection of open subgroups of G of index at most n. Then $G_n \lhd_{\mathrm{o}} G$, and $G \cong \varprojlim_n G/G_n$. Since f is a surjection, $\left[G : f^{-1}\left(U\right)\right] = [G : U]$ for all $U \leq_{\mathrm{o}} G$. If U has index at most n, then $f^{-1}\left(U\right)$ has index at most n, so $f^{-1}\left(U\right) \geq G_n$, so $f^{-1}\left(G_n\right) \geq G_n$, so $f\left(G_n\right) \leq G_n$. So we have a quotient map $f_n : G/G_n \twoheadrightarrow G/G_n$, which are surjections, hence isomorphisms. So $\left(f_n\right)$ are a morphism of inverse systems giving f, so $f = \varprojlim_n f_n$ is an isomorphism. Or, if $g \in G \setminus \{1\}$, then $g \notin G_n$ for some n and then $p_n\left(f\left(g\right)\right) = f_n\left(p_n\left(g\right)\right) \neq 1$ so $g \notin \ker f$. □

Corollary 3.1.30. *Finitely generated residually finite groups are Hopfian.*

Proof. Let $f : G \to G$ be a surjection where G is finitely generated residually finite. By Proposition 3.1.2, we get an induced map

$$\begin{tikzcd} \widehat{G} \arrow[r, "\widehat{f}"] & \widehat{G} \\ G \arrow[u, hook] \arrow[r, "f"'] & G \arrow[u, hook] \end{tikzcd}.$$

Then $\widehat{f}$ is surjective, so it is an isomorphism. Thus f is injective. □

Proposition 3.1.31. *Let G be a Hopfian topological group and let H be a topological group. Suppose there exist continuous surjections $f : G \to H$ and $f' : H \to G$. Then f and f' are isomorphisms of topological groups.*

Proof. $f' \circ f : G \to G$ is a surjection, hence an isomorphism, and a homeomorphism. So f is injective and f' is injective, because f is a surjection, so isomorphisms. Also $f^{-1} = \left(f' \circ f\right)^{-1} \circ f'$ and $f'^{-1} = f \circ \left(f' \circ f\right)^{-1}$ are continuous. □

Let d be the minimal size of a generating set.

Proposition 3.1.32. *Let G be a finitely generated residually finite group. Assume there is a finite quotient Q of G such that $\mathrm{d}(Q) = \mathrm{d}(G)$. If $\widehat{G}$ is isomorphic to $\widehat{F}$ for F a free group, then $G \cong F$.*

Proof. Assume $\widehat{G} \cong \widehat{F}$. Then Q is a quotient of F, so $\mathrm{d}(F) \ge \mathrm{d}(Q) = \mathrm{d}(G)$. So there is a surjection $f : F \to G$. This induces $\widehat{f} : \widehat{F} \to \widehat{G}$. Then $\widehat{f}$ is surjective, so by the Hopf property, since $\widehat{F} \cong \widehat{G}$, $\widehat{f}$ is an isomorphism. Thus f is an isomorphism, since

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, hook] & G \arrow[d, hook] \\ \widehat{F} \arrow[r, "\sim"'] & \widehat{G} \end{tikzcd}.$$

□

Corollary 3.1.33. $\widehat{\mathrm{S}_g} \not\cong \widehat{\mathrm{F}_{2g}}$.

Proof. S_g has rank $2g$, and maps onto $Q = (\mathbb{Z}/2\mathbb{Z})^{2g}$. □

Example 3.1.34. Let n and m be coprime integers such that $|n|, |m| > 1$. Define

$$\mathrm{BS}(n, m) = \langle a, t \mid ta^n t^{-1} = a^m \rangle,$$

a HNN extension. Define

$$\begin{array}{cccc} f & : & \mathrm{BS}(n,m) & \longrightarrow & \mathrm{BS}(n,m) \\ & & t & \longmapsto & t \\ & & a & \longmapsto & a^n \end{array}.$$

This is well-defined, since

$$f : ta^n t^{-1} a^{-m} \mapsto ta^{n^2} t^{-1} a^{mn} = \left(ta^n t^{-1}\right)^n a^{-mn} = a^{mn} a^{-mn} = 1.$$

- f is surjective. Since $\operatorname{im} f \ni a^n, t$, $\operatorname{im} f \ni ta^n t^{-1} = a^m$, and so $\operatorname{im} f \ni a$, since there exist r and s such that $nr + ms = 1$ so $a = (a^n)^r (a^m)^s$.
- But f is not injective. By Britton's lemma, tat^{-1} does not commute with a, so $\left[tat^{-1}, a\right] \ne 1$. But $f\left(\left[tat^{-1}, a\right]\right) = \left[ta^n t^{-1}, a^n\right] = [a^m, a^n] = 1$.

So $\mathrm{BS}(m, n)$ is not Hopfian, hence not residually finite.

3.2 Finite quotients of free groups

Lecture 12 Tuesday 16/02/21

Theorem 3.2.1. *Free groups are residually finite.*

Previously, $\mathrm{F}_2 \hookrightarrow \mathrm{SL}_2 \mathbb{Z} \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

Remark 3.2.2. This is true for infinitely generated free groups. If $F = \langle a_i \rangle_{i \in I}$, take some $g \in F \setminus \{1\}$. Then g can be written as a finite product of $a_i^{\pm 1}$, so you need only finitely many a_i. Factoring out the others gives $F \twoheadrightarrow F' \twoheadrightarrow Q$, where F' is a finitely generated free group in which g is mapped to a non-trivial element.

Residual finiteness if and only if $\iota : G \hookrightarrow \widehat{G}$. Residual p-finiteness, stronger than residual finiteness, is $\iota : G \hookrightarrow \widehat{G_{(p)}}$, if and only if for all $g \in G \setminus \{1\}$, there exists $\phi : G \to Q$ where $|Q| = p^m$ such that $\phi(g) \ne 1$.

Proof 1 (non-examinable). Let p be a prime. Let X be a wedge of k circles, and $F = \pi_1(X)$. Construct $F_n \lhd F$ inductively, by

$$F_1 = F, \qquad F_{n+1} = \bigcap \{\ker f \mid f : F_n \to \mathbb{Z}/p\mathbb{Z}\} = \ker\left(F_n \to \prod_f \mathbb{Z}/p\mathbb{Z}\right).$$

Then F_n are characteristic subgroups, so normal, and $[F : F_n]$ is a power of p, by induction. Let $X_n \to X$ be the cover corresponding to $F_n \triangleleft F$. Claim that girth $X_{n+1} >$ girth X_n, so girth $X_n \geq n$. Let l be any loop in X_n of minimal length, girth X_n. We show l does not lift to X_{n+1}. Because l is minimal length, there exists an edge e which it crosses once exactly. Collapsing everything except e,

$$\begin{array}{rcl} F_n = \pi_1(X_n) & \longrightarrow & \pi_1(\mathrm{S}^1) = \mathbb{Z} \\ [l] & \longmapsto & 1 \end{array}.$$

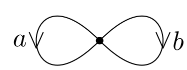
So we have a homomorphism

$$\begin{array}{rcl} F_n & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ [l] & \longmapsto & 1 \neq 0 \end{array},$$

so $[l] \notin F_{n+1}$, hence l does not lift to X_{n+1}. Let $g \in F \setminus \{1\}$. Write g as a loop in X. Let n be the number of edges of l. Then l cannot lift to X_{n+1}, with girth at least $n+1$. So $g \notin F_{n+1}$. ☐

Proof 2. Let $F = \langle a_1, \ldots, a_k \rangle$ be a free group. Let X be a bouquet of k circles with $\pi_1(X) = F$. Let $g \in F \setminus \{1\}$. Write g as a product $g = s_1 \ldots s_m$ where s_i is $a_j^{\pm 1}$. Let Y be a line segment labelled $s_1 \ldots s_m$. We add edges to Y to make it a covering space of X. This covering space $\widetilde{X}$ does not lift g, so $g \notin \pi_1\left(\widetilde{X}\right)$. ☐

Example 3.2.3. Let $F = \langle a, b \rangle$, and let X be



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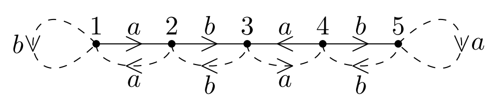
If $g = aba^{-1}b$, then $\widetilde{X}$ is



.

We get a homomorphism

$$\begin{array}{rrcl} \phi & : & F & \longrightarrow & \mathrm{Sym}\,5 \\ & & a & \longmapsto & (12)(34)(5) \\ & & b & \longmapsto & (1)(23)(45) \end{array},$$

acting on the right. Then

$$\phi(g): \qquad 1 \mapsto 5, \qquad 2 \mapsto 3, \qquad 3 \mapsto 4, \qquad 4 \mapsto 1, \qquad 5 \mapsto 2,$$

so $\phi(g) = (15234)$.

We can also answer stronger questions.

- Given $S \subseteq F$, does S generate F? Given $g \in F \setminus \{1\}$, does $g \in \langle S \rangle$?
- Does $\left\{abcb^2cb^{-1}c^{-1}b^{-1}a^{-1}, bc^{-1}b^{-1}abc, bcb^{-1}\right\}$ or $\left\{abcb^2cb^{-1}c^{-1}b^{-1}a^{-1}, bc^{-1}b^{-1}a^{-1}bc, bcb^{-1}\right\}$ generate $\langle a, b, c \rangle$?

Theorem 3.2.4 (Marshall Hall's theorem)**.** *Let S be a finite subset of a finitely generated free group F. Let $y \notin \langle S \rangle$. Then there exists a finite group Q and $f : F \to Q$ such that $f(y) \notin f(\langle S \rangle)$.*

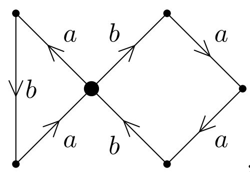
Corollary 3.2.5. *A finite subset $S \subset F$ generates F if and only if S topologically generates $\widehat{F}$.*

Proof. If S generates F, it generates $\widehat{F}$ topologically since $\langle S \rangle = F$ is dense in $\widehat{F}$. If $\langle S \rangle \neq F$, there exists $y \notin \langle S \rangle$. Take a finite group Q and $f : F \to Q$ as in Theorem 3.2.4. Then $f(y) \notin f(\langle S \rangle)$, so $f(\langle S \rangle) \neq f(F)$. Thus $\langle S \rangle$ is not dense in $\widehat{F}$. ☐

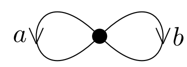
Marshall Hall's theorem says there exists $H \leq_{\mathrm{f}} F$ such that $H = \langle S \rangle * H'$.

Lecture 13
Thursday
18/02/21

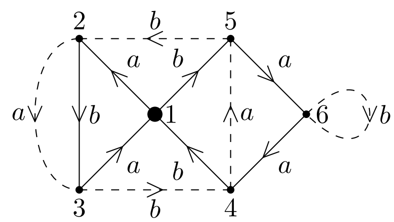
Example 3.2.6. Let $F = \langle a, b\rangle$, and let $S = \left\{aba, ba^2b\right\}$. We will show $\langle S\rangle \neq F$. Start by writing the elements of S as loops



and call it Y. We have a natural continuous map $Y \to X$, where X is



Then $\pi_1(Y) \to \langle S\rangle \leq \pi_1(X)$. Now add edges to make a covering space

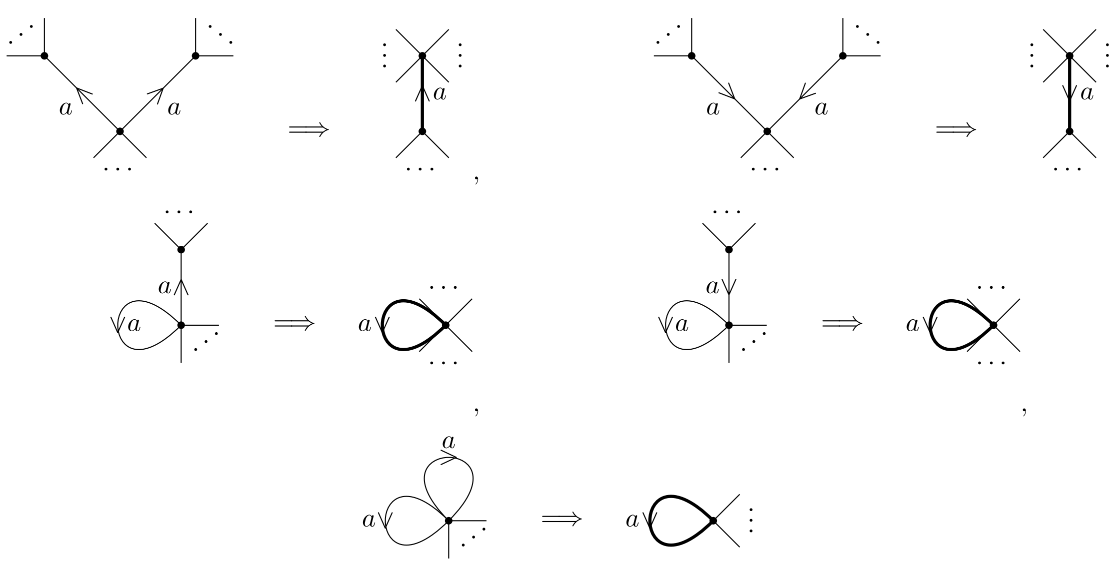


The explicit homomorphism to a finite group is

$$\begin{array}{rcrcl} \phi & : & F & \longrightarrow & \mathrm{Sym}\,6 \\ & & a & \longmapsto & (123)(456) \\ & & b & \longmapsto & (15234)(6) \end{array}.$$

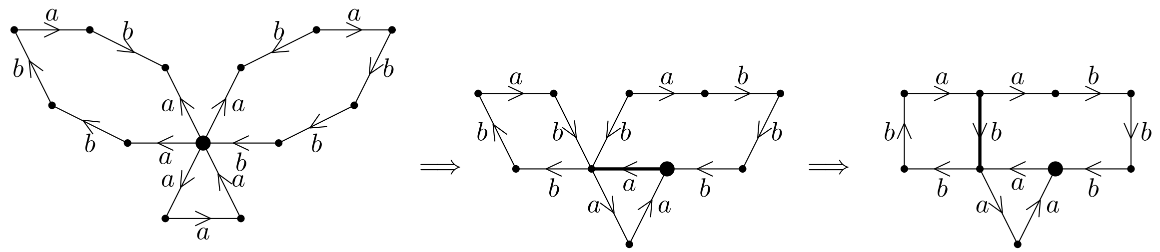
Note that $\phi(\langle S\rangle) \leq \mathrm{Stab}\,1$ and $\phi(a) \notin \mathrm{Stab}\,1$.

A **Stallings fold** is an operation on oriented, labelled graphs such that

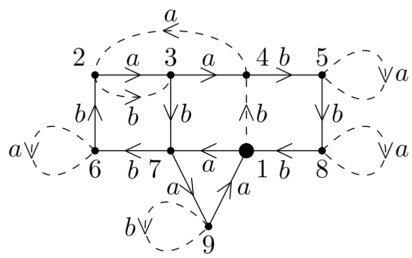


Fact 3.2.7. Folding Y gives a new graph Y' such that the image of $\pi_1(Y) \to \pi_1(Y') \to \pi_1(X)$ is still $\langle S\rangle$.

Example 3.2.8. Let $F = \langle a, b\rangle$, and let $S = \left\{a^3, ab^2aba^{-1}, ab^{-1}ab^3\right\}$. Folding,



Now can add edges to make a covering



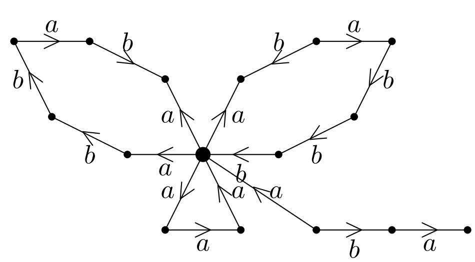
The homomorphism is

$$\begin{array}{rccl} \phi & : & F & \longrightarrow & \mathrm{Sym}\, 9 \\ & & a & \longmapsto & (179)(234)(5)(6)(8) \\ & & b & \longmapsto & (1458)(2376)(9) \end{array}.$$

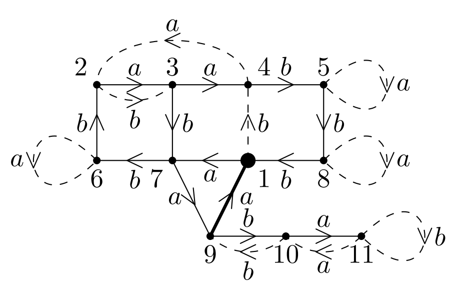
Then $\phi(\langle S\rangle) \le \mathrm{Stab}\, 1$ and $\phi(a) \notin \mathrm{Stab}\, 1$, so $\phi(\langle S\rangle) \neq \phi(F)$. Thus $\langle S\rangle \neq F$. The other case is that folding gives a one-vertex graph, then $\langle S\rangle$ is generated by some standard generators of F.

What if we want to know if a specific y lies in $\langle S\rangle$? Add y into the starting graph as a line.

Example 3.2.9. Let $y = a^{-1}ba$. Fold



and make a covering space



Thus $\phi(\langle S\rangle) \le \mathrm{Stab}\, 1$ and $\phi(y) = (1 \mapsto 11) \notin \mathrm{Stab}\, 1$. The other option is that y gets folded into being a loop, then $y \in \langle S\rangle$.

4 Pro-p groups

Recall that a pro-p group is an inverse limit of finite p-groups, groups of order p^n for p a fixed prime. For example, the pro-p completion of a group such as $\mathbb{Z}_p = \widehat{\mathbb{Z}_{(p)}}$.

4.1 Generators of pro-p groups

Definition 4.1.1. Let G be a finite group. The **Frattini subgroup** of G, denoted $\Phi(G)$, is

$$\Phi(G) = \bigcap \{M \mid M \text{ is a maximal proper subgroup of } G\},$$

such that if $M \le H \le G$ then $M = H$ or $H = G$.

Importantly, if G is finite, then every proper subgroup is contained in a maximal proper subgroup.

Proposition 4.1.2. *For G a finite group and $S \subseteq G$, the following are equivalent.*

1. *S generates G.*
2. *$S\Phi(G)$ generates G, so $\Phi(G)$ are non-generators.*
3. *The image of S in $G/\Phi(G)$ generates $G/\Phi(G)$.*

Proof.

$1 \implies 2$. Trivial.

$2 \implies 3$. Trivial.

$3 \implies 1$. Suppose S does not generate G. Then $\langle S\rangle$ is a proper subgroup, so, since G is finite, $\langle S\rangle$ is contained in a maximal proper subgroup M of G. Since $\Phi = \Phi(G) \le M$, $M/\Phi \ne G/\Phi$, so $S\Phi/\Phi \le M/\Phi \ne G/\Phi$, so $S\Phi/\Phi$ does not generate G/Φ.

□

Lecture 14 Saturday 20/02/21

Proposition 4.1.3. *Let $f : G \to H$ be a surjection of finite groups. Then $f(\Phi(G)) \le \Phi(H)$. Hence, $\Phi(G)$ is a characteristic subgroup of G.*

Remark 4.1.4. Surjection is necessary. For example, let $\mathbb{Z}/4\mathbb{Z} = \mathcal{C}_4 \hookrightarrow \operatorname{Sym} 5$. Then $\Phi(\mathbb{Z}/4\mathbb{Z}) = 2\mathbb{Z}/4\mathbb{Z} = \langle 2\rangle$ and $\Phi(\operatorname{Sym} 5) = 1$, since $\mathcal{A}_5$ is ruled out by $\operatorname{Stab} 1$, a maximal proper subgroup not containing $\mathcal{A}_5$.

Proof. Let M be a maximal proper subgroup of H. We claim $f^{-1}(M)$ is a maximal proper subgroup of G. Properness follows from surjectivity. If $\ker f \le f^{-1}(M) < G' \le G$, then $M < f(G') \le H = f(G)$. Since M is maximal, $f(G') = H$. Then $G' = G$, since if $g \in G$, then $f(g) = f(g') \in H$, for some $g' \in G'$, then $gg'^{-1} \in \ker f$, so $g \in g' \ker f \le G'$. Thus $\Phi(G) \le f^{-1}(M)$, so $f(\Phi(G)) \le M$, so $f(\Phi(G)) \le \Phi(H)$. □

Definition 4.1.5. Let G be a group and $H, K \le G$. Let m be an integer. Define

$$[H, K] = \langle\{[h, k] \mid h \in H,\ k \in K\}\rangle, \qquad H^m = \langle\{h^m \mid h \in H\}\rangle, \qquad HK = \{hk \mid h \in H,\ k \in K\}.$$

If $H \lhd G$ then HK is a subgroup and H^m is normal. If $H \lhd G$ and $K \lhd G$ then $HK \lhd G$ and $H \cap K \ge [H, K] \lhd G$.

Proposition 4.1.6. *Let G be a finite p-group. Then*

$$\Phi(G) = [G, G] G^p = \langle\{[g_1, g_2] g_3^p \mid g_1, g_2, g_3 \in G\}\rangle = \ker(G \to G_{\mathrm{ab}} \to G_{\mathrm{ab}}/pG_{\mathrm{ab}}),$$

where $\mathrm{H}_1(G, \mathbb{F}_p) = G_{\mathrm{ab}}/pG_{\mathrm{ab}}$ is a vector space $\mathbb{F}_p^{\mathrm{d}(G)}$ over $\mathbb{F}_p$.

Proof. On example sheet 2. □

Definition 4.1.7. Let G be a profinite group. Define the **Frattini subgroup**

$$\Phi(G) = \bigcap \{M \mid M \text{ is a maximal proper closed subgroup of } G\},$$

which is closed, where if $M \leq_{\mathrm{c}} H \leq_{\mathrm{c}} G$ then $H = M$ or $H = G$.

Proposition 4.1.8. *Any proper closed subgroup of a profinite group G is contained in a proper open subgroup. Hence a maximal proper closed subgroup is open, and any closed subgroup is contained in a maximal proper closed subgroup.*

Proof. Let $H \leq_{\mathrm{c}} G$ such that $H \neq G$. Then by Corollary 1.2.19, there exists $p : G \to Q$ for Q finite such that $p(H) \neq p(G)$. Then $p^{-1}(p(H))$ is open and proper, and contains H. Open subgroups have finite index, so maximal if and only if smallest index. ☐

Proposition 4.1.9. *Let $f : G \to H$ be a surjective continuous homomorphism of profinite groups. Then $f(\Phi(G)) \leq \Phi(H)$.*

Proposition 4.1.10. *Let G be profinite and $S \subseteq G$. Then the following are equivalent.*

- *S topologically generates G.*
- *$S\Phi(G)$ topologically generates G.*
- *$S\Phi(G)/\Phi(G)$ topologically generates $G/\Phi(G)$.*

Proposition 4.1.11. *Let $(G_j)_{j \in J}$ be a surjective inverse system of finite groups and $G = \varprojlim_j G_j$. Then*

$$\Phi(G) = \varprojlim_j \Phi(G_j).$$

Proof. $\Phi(G) = \varprojlim_j p_j(\Phi(G)) \leq \varprojlim_j \Phi(G_j)$. Let M be a maximal proper closed subgroup of G. Since M is open, there exists $i \in J$ such that $\ker p_i \leq M$. This implies $\ker p_j \leq M$ for $j \leq i$. Then $p_j(M)$ is a maximal proper subgroup of G_j for all $j \leq i$, so $\Phi(G_j) \leq p_j(M)$ for all $j \leq i$. Pass to the cofinal subsystem $\{j \leq i\}$. Now $\varprojlim_j \Phi(G_j) \leq \varprojlim_j p_j(M) = M$. So $\varprojlim_{j \in J} \Phi(G_j) \leq M$ for all M, so $\varprojlim_{j \in J} \Phi(G_j) \leq \Phi(G)$. ☐

Proposition 4.1.12. *Let G be a topologically finitely generated pro-p group. Then*

$$\Phi(G) = \overline{[G,G]\,G^p} = \mathrm{H}_1(G, \mathbb{F}_p), \qquad G/\Phi(G) \cong \mathbb{F}_p^d,$$

where $d = \mathrm{d}(G)$ is the minimal size of a topological generating set of G.

Proof. Write $G = \varprojlim_j G_j$ as a surjective inverse system of finite p-groups. We know $\Phi(G) = \varprojlim_j [G_j, G_j]\, G_j^p$. For any $[g_1, g_2]\, g_3^p$ for $g_1, g_2, g_3 \in G$ we have $p_j([g_1, g_2]\, g_3^p) = [p_j(g_1), p_j(g_2)]\, p_j(g_3)^p \in [G_j, G_j]\, G_j^p$, so $\overline{[G,G]\,G^p} \leq \varprojlim_j [G_j, G_j]\, G_j^p = \Phi(G)$. Since $G/\overline{[G,G]\,G^p}$ is topologically finitely generated, abelian, and every element has order p, it is finite and equal to $\mathbb{F}_p^d$ for some d. But $\Phi(\mathbb{F}_p^d) = \{0\}$, so $\Phi(G) \leq \overline{[G,G]\,G^p}$. ☐

Example 4.1.13. Generation of $\widehat{F_{(p)}}$ is easy. Let $F = \langle a, b \rangle$. Then

$$\begin{array}{rcl} \widehat{F_{(p)}} & \longrightarrow & \widehat{F_{(p)}}/\Phi = \mathbb{F}_p^2 \\ a & \longmapsto & (1,0) \\ b & \longmapsto & (0,1) \end{array}.$$

Corollary 4.1.14. *Let $f : G \to H$ be a continuous homomorphism of topologically finitely generated pro-p groups. Then $f(\Phi(G)) \leq \Phi(H)$. So f induces a map*

$$f_* : G/\Phi(G) \to H/\Phi(H),$$

and f is surjective if and only if f_* is surjective.

Proof. $f([g_1, g_2]\, g_3^p) = [f(g_1), f(g_2)]\, f(g_3)^p \in \Phi(H)$ for all $g_1, g_2, g_3 \in G$. Then $f([G,G]\,G^p) \leq \Phi(H)$, so $f(\Phi(G)) = f\left(\overline{[G,G]\,G^p}\right) \leq \Phi(H)$. If f_* is surjective, then the image of $f(G)$ in $H/\Phi(H)$ generates $H/\Phi(H)$, so $f(G)$ topologically generates H. So $f(G) = H$. ☐

4.2 Nilpotent groups

Lecture 15 Tuesday 23/02/21

Definition 4.2.1. The **lower central series** of a group G is the sequence $G_n = \gamma_n(G)$ defined by

$$\gamma_1(G) = G, \qquad \gamma_{n+1}(G) = [G, \gamma_n(G)], \qquad \gamma_{n+1}(G) \le \gamma_n(G).$$

Then G is **nilpotent of class** c if $\gamma_{c+1}(G) = 1$ but $\gamma_c(G) \neq 1$.

The following are properties.

Proposition 4.2.2. *$\gamma_n(G)$ is **fully characteristic**, so if $f : G \to H$ then $f(\gamma_n(G)) \le \gamma_n(H)$. If f is surjective, we have equality.*

Proposition 4.2.3. *Subgroups and quotients of nilpotent groups are nilpotent.*

Proposition 4.2.4. *Finite p-groups G are nilpotent.*

Proof. Proof by induction on $|G|$.

- $\gamma_2(\mathbb{F}_p) = 1$.
- There exists $z \in \mathrm{Z}(G) \setminus \{1\}$. Then $G/\langle z\rangle$ is nilpotent, so $\gamma_{c+1}(G/\langle z\rangle) = 1$ for some c. Thus $\gamma_{c+1}(G) \le \langle z\rangle$, so $\gamma_{c+2}(G) = [G, \gamma_{c+1}(G)] = 1$.

$\square$

The following is a variant. For pro-p groups, the **lower central p-series** is

$$\gamma_1^{(p)}(G) = G, \qquad \gamma_{n+1}^{(p)}(G) = \overline{\left[G, \gamma_n^{(p)}(G)\right]\left(\gamma_n^{(p)}(G)\right)^p},$$

so $\gamma_2^{(p)}(G) = \Phi(G)$. Then $\gamma_n^{(p)}(G)$ is open for topologically finitely generated pro-p groups, since by induction, $\gamma_{n+1}^{(p)}(G) \ge \Phi\left(\gamma_n^{(p)}(G)\right)$.

Proposition 4.2.5. *Let G be a p-group. Then $\gamma_n^{(p)}(G) = 1$ for some n.*

Proposition 4.2.6. *Let G be a topologically finitely generated pro-p group, then $\left\{\gamma_n^{(p)}(G)\right\}$ are a basis of open normal subgroups of G.*

Proof. If $N \lhd_{\mathrm{o}} G$, then G/N is a p-group, so $\gamma_n^{(p)}(G/N) = 1$. Thus $N \ge \gamma_n^{(p)}(G)$. $\square$

4.3 Invariance of topology

Theorem 4.3.1 (Serre)**.** *Let G be a topologically finitely generated pro-p group. Then all finite index subgroups are open.*

Thus

- every homomorphism to a finite group is continuous,
- by Proposition 1.2.14 every homomorphism to a profinite group is continuous, and
- no other topology on G makes it a profinite group, by applying Theorem 4.3.1 to $\mathrm{id} : G \to G$.

Proposition 4.3.2. *Let G be a pro-p group and let $K \le_{\mathrm{f}} G$. Then $[G : K]$ is a power of p.*

Proof. Without loss of generality K is normal. Let $[G : K] = m = p^r m'$ for m' coprime to p. Let

$$X = G^{\{m\}} = \{g^m \mid g \in G\} \subseteq K.$$

Then X is closed, since it is the image of G under $g \mapsto g^m$. Thus $X = \overline{X} = \bigcap_{N \lhd_{\mathrm{o}} G} XN$, by Proposition 1.2.21. Let $g \in G$. We will show $g^{p^r} \in K$ for all $g \in G$. This implies the result by Cauchy's theorem. Let $N \lhd_{\mathrm{o}} G$. Let $[G : N] = p^s$. Let $t = \max(r, s)$. Then $g^{p^t} \in N$ and $\gcd(p^t, m) = p^r$. So there exist $a, b \in \mathbb{Z}$ such that $p^r = ma + p^t b$. Then $g^{p^r} = (g^a)^m \left(g^{p^t}\right)^b \in XN$. $\square$

Lemma 4.3.3. *Let G be a nilpotent group with a finite generating set $a_1, \dots, a_d$. Then every $g \in [G, G]$ may be written*

$$g = [a_1, x_1] \dots [a_d, x_d], \qquad x_1, \dots, x_d \in G.$$

Proof. We induct on the nilpotency class c of G.

- If $c = 1$, then $1 = \gamma_2(G) = [G, G]$, so G is abelian, which is trivial.
- The result is true for $G/\gamma_c(G)$. So there exist $x_1, \dots, x_d \in G$ and $u \in \gamma_c(G) = [G, \gamma_{c-1}(G)]$ such that

  $$g = [a_1, x_1] \dots [a_d, x_d] u.$$

  Seek a nice form of u. There are commutator relations

  $$[xy, z] = [x, z]^y [y, z], \qquad [x, yz] = [x, z][x, y]^z.$$

  For any $v \in \gamma_{c-1}(G)$, these imply that

  $$[a_i a_j, v] = [a_i, v][a_j, v], \qquad [a_i, v]^2 = \left[a_i, v^2\right],$$

  $$\left[a_i^{-1}, v\right] = [a_i, v]^{-1} = \left[a_i, v^{-1}\right], \qquad [a_i, v][a_i, w] = [a_i, vw],$$

  since $[\cdot, v] \in \gamma_c(G)$ is central in G. We can write u in the form

  $$u = [a_1, v_1] \dots [a_d, v_d], \qquad v_i \in \gamma_{c-1}(G).$$

  Finally,

  $$g = [a_1, x_1] \dots [a_d, x_d][a_1, v_1] \dots [a_d, v_d] = [a_1, x_1 v_1] \dots [a_d, x_d v_d].$$

$\square$

Proposition 4.3.4. *If G is a topologically finitely generated pro-p group, then $[G, G] G^p$ is open and closed, and equals $\Phi(G)$.*

Proof. Let

$$G^{\{p\}} = \{g^p \mid g \in G\} \subseteq G^p.$$

Then $G/[G, G]$ is abelian, and in abelian groups we have $g^p h^p = (gh)^p$, so $g^p h^p (gh)^{-p} \in [G, G]$, so $[G, G] G^p = [G, G] G^{\{p\}}$. Claim that $[G, G]$ is closed. Let $a_1, \dots, a_d$ be a topological generating set of G. Let

$$X = \{[a_1, x_1] \dots [a_d, x_d] \mid x_1, \dots, x_d \in G\}.$$

Then X is closed, since it is the image of a continuous map $G^d \to G$. So $X = \overline{X} = \bigcap_{N \triangleleft_o G} XN$. We show $X = [G, G]$. Let $g \in [G, G]$. For any $N \triangleleft_o G$, $gN \in [G/N, G/N]$. Since G/N is nilpotent,

$$gN = [a_1 N, x_1 N] \dots [a_d N, x_d N], \qquad x_i N \in G/N.$$

Then $g \in XN$ for all $N \triangleleft_o G$, so $g \in \bigcap_N XN = \overline{X} = X$. Thus $[G, G] G^{\{p\}}$ is the image of $[G, G] \times G$ under the continuous function

$$\begin{array}{rcl} [G, G] \times G & \longrightarrow & G \\ (x, g) & \longmapsto & xg^p \end{array},$$

so $[G, G] G^{\{p\}}$ is closed. $\square$

Proof of Theorem 4.3.1. Proof by contradiction. Suppose G is topologically finitely generated pro-p and K is finite index but not open, such that $[G : K]$ is as small as possible. Without loss of generality K is normal. Consider

$$M = \Phi(G) K = [G, G] G^p K.$$

Then G/K is a non-trivial p-group. So the image of M is $\Phi(G/K) = [G/K, G/K](G/K)^p < G/K$. So M is proper in G, so $M = K$, otherwise $K <_o M <_o G$. Hence $\Phi(G) \leq K$ is open, so K is open. $\square$

4.4 Hensel's lemma and p-adic arithmetic

Lecture 16 Thursday 25/02/21

Previously, there exists x such that $\alpha x = 1$ if and only if $\alpha \not\equiv 0 \mod p$.

Lemma 4.4.1. *Let $f(X)$ be a polynomial with coefficients in $\mathbb{Z}_p$. Then f has a root in $\mathbb{Z}_p$ if and only if f has a root modulo p^k for all k.*

Example 4.4.2. Hensel lifting. Let $p = 7$. Then $3^2 = 9 \equiv 2 \mod 7$, so $X^2 - 2$ has a root modulo 7. To get a root modulo 49, consider $3 + 7a$ for $0 \le a \le p - 1 = 6$. Then

$$(3 + 7a)^2 = 3^2 + 7(6a) + 49a^2 \equiv 2 + 7(1 + 6a) \mod 49.$$

Choose the unique a such that $1 + 6a \equiv 0 \mod 7$, so $a = 1$. Then $3 + 7(1) = 10$ is a root of $X^2 - 2 \mod 49$, and $10^2 = 49(2) + 2$. Now we can repeat. To solve modulo 7^3,

$$(10 + 49a)^2 = 10^2 + 49(20a) + 49^2 a^2 \equiv 2 + 49(2 + 20a) \mod 7^3,$$

and solve for $a = 2$.

Proposition 4.4.3 (Hensel's lemma for square roots)**.** *Let $p \ne 2$ be prime. Suppose $\lambda \in \mathbb{Z}_p$ is congruent to a non-zero square $r_1^2 \mod p$, where $r_1 \in \mathbb{Z}$. Then there is a unique $\rho \in \mathbb{Z}_p$ such that $\rho^2 = \lambda$ and $\rho \equiv r_1 \mod p$.*

Proof. Construct elements $r_k \in \mathbb{Z}$, unique modulo p^k, such that $r_k^2 \equiv \lambda \mod p^k$ and $r_{k+1} \equiv r_k \mod p^k$. Then (r_k) is Cauchy, so there exists $\rho \in \mathbb{Z}_p$ such that $r_k \to \rho$ and $\rho^2 = \lambda$.

- r_1 is given.
- Suppose we have r_k. Consider $r_k + p^k a$ for $0 \le a \le p - 1$. We have $r_k^2 = \lambda + p^k b_k$ for $b_k \in \mathbb{Z}_p$. Then

  $$\left(r_k + p^k a\right)^2 = r_k^2 + 2r_k a p^k + p^{2k} a^2 \equiv \lambda + (b_k + 2r_k a) p^k \mod p^{k+1}.$$

  Now $2r_k \equiv 2r_1 \not\equiv 0 \mod p$, so we can solve $b_k + 2r_k a \equiv 0 \mod p$ and find a_k such that $\left(r_k + p^k a_k\right)^2 \equiv \lambda \mod p^{k+1}$. Set $r_{k+1} = r_k + p^k a_k$.

□

Proposition 4.4.4 (Hensel's lemma)**.** *Let $f(x)$ be a polynomial with coefficients in $\mathbb{Z}_p$. Let $r \in \mathbb{Z}_p$ such that $f(r) \equiv 0 \mod p^k$ for some k and $f'(r) \not\equiv 0 \mod p$, where $f' : \sum_n a_n x^n \mapsto \sum_n n a_n x^{n-1}$ is the formal derivative, and $f'(r)$ only depends on $r \mod p$. There there exists a unique $\rho \in \mathbb{Z}_p$ such that $f(\rho) = 0$ and $\rho \equiv r \mod p^k$.*

Lemma 4.4.5. *For $r, a \in \mathbb{Z}_p$ and $k \ge 1$ we have*

$$f\left(r + p^k a\right) \equiv f(r) + p^k a f'(r) \mod p^{k+1}.$$

Proof. It suffices to do $f(x) = x^n$. Then

$$\left(r + p^k a\right)^n = r^n + n r^{n-1} p^k a + \sum_{i=2}^{n} \binom{n}{i} p^{ki} a^i r^{n-i},$$

and $p^{k+1} \mid p^{2k} \mid p^{ki}$. □

Proof of Proposition 4.4.4. Construct r_k for $k \ge K$, such that $f(r_k) \equiv 0 \mod p^k$ and $r_{k+1} \equiv r_k \mod p^k$, and r_{k+1} will be unique modulo p^{k+1} with these properties. Then (r_k) is Cauchy and $r_k \to \rho$, so $f(\rho) = 0$.

- r_K is given.
- If r_k is constructed, consider $r_k + p^k a$ for $0 \le a \le p - 1$. We have $f(r_k) = b_k p^k$ for some $b_k \in \mathbb{Z}_p$. Now

  $$f\left(r_k + p^k a\right) \equiv f(r_k) + p^k a f'(r_k) \equiv p^k \left(b_k + a f'(r_k)\right) \mod p^{k+1}.$$

  Can solve $b_k + a f'(r_k) \equiv 0 \mod p$ since $f'(r_k) \equiv f'(r) \not\equiv 0 \mod p$ is invertible modulo p. So set a_k such that $b_k + a_k f'(r_k) \equiv 0 \mod p$ and set $r_{k+1} = r_k + p^k a_k$.

□

We can also do Hensel-type things in $\mathrm{GL}_N \mathbb{Z}_p$.

Definition 4.4.6. Let

$$\mathrm{GL}_N^{(k)} \mathbb{Z}_p = \ker\left(\mathrm{GL}_N \mathbb{Z}_p \to \mathrm{GL}_N\left(\mathbb{Z}/p^k\mathbb{Z}\right)\right) = \left\{\mathrm{I} + p^k A \mid A \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p\right\},$$
$$\mathrm{SL}_N^{(k)} \mathbb{Z}_p = \ker\left(\mathrm{SL}_N \mathbb{Z}_p \to \mathrm{SL}_N\left(\mathbb{Z}/p^k\mathbb{Z}\right)\right).$$

Proposition 4.4.7. *$\mathrm{GL}_N^{(1)} \mathbb{Z}_p$ and $\mathrm{SL}_N^{(1)} \mathbb{Z}_p$ are pro-p groups.*

Proof. $\left|\mathrm{GL}_N^{(1)}\left(\mathbb{Z}/p^m\mathbb{Z}\right)\right| = p^{(m-1)N^2}$, and

$$\mathrm{SL}_N^{(1)} \mathbb{Z}_p \le \mathrm{GL}_N^{(1)} \mathbb{Z}_p = \varprojlim_m \mathrm{GL}_N^{(1)}\left(\mathbb{Z}/p^m\mathbb{Z}\right).$$

□

Remark 4.4.8. $\mathrm{GL}_N \mathbb{Z}_p$ and $\mathrm{SL}_N \mathbb{Z}_p$ are not pro-p groups, since $\mathrm{SL}_N\left(\mathbb{Z}/p\mathbb{Z}\right)$ is not a p-group.

Proposition 4.4.9. *Let $p \ne 2$. The continuous function*

$$\begin{array}{ccc} \mathrm{GL}_N^{(k)} \mathbb{Z}_p & \longrightarrow & \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p \\ A & \longmapsto & A^p \end{array}$$

maps surjectively for $k \ge 1$. Also for $\mathrm{SL}_N^{(k)} \mathbb{Z}_p \twoheadrightarrow \mathrm{SL}_N^{(k+1)} \mathbb{Z}_p$.

Proof. For $r \ge 1$ and A a matrix over $\mathbb{Z}_p$, we have

$$(\mathrm{I} + p^r A)^p = \mathrm{I} + p^{r+1} A + \sum_{l=2}^{p} p^{rl} \binom{p}{p-l} A^l = \mathrm{I} + p^{r+1} A + p^{r+2} B,$$

for some B which commutes with A, unless $p = 2$, $l = 2$, and $r = 1$. Let $\mathrm{I} + p^{k+1} A \in \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p$. We show the following inductive statement for $n \ge 1$. There exist B_n and E_n, which are polynomials in A, hence commute with A and each other, such that

$$B_{n+1} \equiv B_n \mod p^n, \qquad \left(\mathrm{I} + p^k B_n\right)^p = \mathrm{I} + p^{k+1} A + p^{k+n+1} E_n.$$

Then (B_n) is Cauchy so $B_n \to B_\infty \in \mathrm{Mat}_{N\times N} \mathbb{Z}_p$, and $\left(\mathrm{I} + p^k B_\infty\right)^p = \mathrm{I} + p^{k+1} A$.

- Start with $B_1 = A$. Then
$$\left(\mathrm{I} + p^k A\right)^p = \mathrm{I} + p^{k+1} A + p^{k+2} E_1.$$
- Assume B_n and E_n are given. Set $B_{n+1} = B_n - p^n E_n$. Then
$$\begin{aligned} \left(\mathrm{I} + p^k B_{n+1}\right)^p &= \left(\mathrm{I} + p^k B_n - p^{k+n} E_n\right)^p = \left(\mathrm{I} + p^k B_n\right)^p - p\left(\mathrm{I} + p^k B_n\right)^{p-1} p^{k+n} E_n + \dots \\ &= \mathrm{I} + p^{k+1} A + p^{k+n+1} E_n - p^{k+n+1} E_n + \dots = \mathrm{I} + p^{k+1} A + p^{k+n+2} E_{n+1}. \end{aligned}$$

□

Proposition 4.4.10 (non-examinable)**.**

$$\Phi\left(\mathrm{GL}_N^{(k)} \mathbb{Z}_p\right) = \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p, \qquad \mathrm{GL}_N^{(k)} \mathbb{Z}_p / \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p \cong \mathbb{F}_p^{N^2},$$

*a **uniform pro-p group**, with isomorphisms*

$$\begin{array}{ccc} \mathrm{GL}_N^{(k)} \mathbb{Z}_p / \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p & \longrightarrow & \mathrm{GL}_N^{(k+1)} \mathbb{Z}_p / \mathrm{GL}_N^{(k+2)} \mathbb{Z}_p \\ x & \longmapsto & x^p \end{array}.$$

Theorem 4.4.11 (non-examinable)**.** *Let H be any closed subgroup of $\mathrm{GL}_N^{(1)} \mathbb{Z}_p$. Then $\mathrm{d}(H) \le N^2$.*

Compare to a free group as a subgroup of $\mathrm{SL}_2 \mathbb{Z}$.

Theorem 4.4.12 (non-examinable)**.** *If G is a pro-p group, such that $\mathrm{d}(H) \le R$ for all $H \le_{\mathrm{c}} G$, then*

$$G/\mathbb{Z}_p^a \hookrightarrow \mathrm{GL}_R \mathbb{Z}_p \times F,$$

where F is finite.

5 Cohomology of groups

Lecture 17 Saturday 27/02/21

In the homology of spaces, a simplicial complex X gives a family of abelian groups $\mathrm{H}_n(X)$ with $\mathbb{Z}$ coefficients. In the cohomology of groups, a group G gives a family of abelian groups $\mathrm{H}^n(G, M)$ with M coefficients.

5.1 Group rings and chain complexes

Let G be an abstract group.

Definition 5.1.1. The **group ring** of G is a ring $\mathbb{Z}G$ defined as follows. The additive group of $\mathbb{Z}G$ is the free abelian group with basis $\{g \mid g \in G\}$, so an element is a finite formal sum $\sum_{g\in G} n_g g$ for $n_g \in \mathbb{Z}$. The ring multiplication is defined on the basis by $g \cdot h = gh$ and extended bilinearly.

Thus $\mathbb{Z}G$ is non-commutative unless G is abelian, and has an identity e, the multiplicative identity in $\mathbb{Z}[G]$, usually called 1.

Example 5.1.2. If e is the identity of G, then $(e + g)(e - 2h) = e + g - 2h - 2gh$.

Definition 5.1.3. A **left** G**-module**, or $\mathbb{Z}G$**-module**, is an abelian group M equipped with a G-action, a function

$$\begin{array}{ccc} \mathbb{Z}G \times M & \longrightarrow & M \\ (r, m) & \longmapsto & r \cdot m \end{array},$$

such that for all $r_1, r_2 \in \mathbb{Z}G$ and for all $m_1, m_2 \in M$,

$$r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2, \qquad (r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1, \qquad (r_1 r_2) \cdot m_1 = r_1 \cdot (r_2 \cdot m).$$

A **trivial module**, or a **module with trivial** G**-action**, is a module such that $g \cdot m = m$ for all $g \in G$ and for all $m \in M$.

Definition 5.1.4. Let M_1 and M_2 be G-modules. A **morphism of** G**-modules**, or G**-linear map**, is an abelian group homomorphism $\alpha : M_1 \to M_2$ such that $\alpha(r \cdot m) = r \cdot \alpha(m)$ for all $r \in \mathbb{Z}G$ and $m \in M$.

Note that only need to check this for basis elements $r = g$.

Definition 5.1.5. Let M and N be G-modules. Define the **Hom-group**

$$\mathrm{Hom}_G(M, N) = \{G\text{-linear maps } \alpha : M \to N\},$$

with abelian group structure $(\alpha + \beta)(m) = \alpha(m) + \beta(m)$. If $\mathrm{Hom}(M, N)$, this means $\mathrm{Hom}_1(M, N)$, the abelian group homomorphisms.

Definition 5.1.6. If $f : M_1 \to M_2$ is a morphism of G-modules then we have a **dual map**

$$\begin{array}{cccc} f^* : & \mathrm{Hom}_G(M_2, N) & \longrightarrow & \mathrm{Hom}_G(M_1, N) \\ & \alpha & \longmapsto & \alpha \circ f \end{array}.$$

Also, we have an **induced map**

$$\begin{array}{cccc} f_* : & \mathrm{Hom}_G(N, M_1) & \longrightarrow & \mathrm{Hom}_G(N, M_2) \\ & \beta & \longmapsto & f \circ \beta \end{array}.$$

Thus

$$\begin{tikzcd} & N \arrow[dl, "\beta"'] \arrow[dr, dashed, "f_*\beta"] & \\ M_1 \arrow[rr, "f"] \arrow[dr, dashed, "f^*\alpha"'] & & M_2 \arrow[dl, "\alpha"] \\ & N & \end{tikzcd}.$$

Submodules and quotients are the obvious things.

Definition 5.1.7. Let M be a G-module. Then a G**-submodule** is a subgroup $N \leq M$ such that $g \cdot n \in N$ for all $g \in G$ and $n \in N$. If N is a submodule, we have a **quotient module** M/N, the abelian group M/N, with the G-action $g \cdot (m + N) = g \cdot m + N$.

Definition 5.1.8. A **chain complex of G-modules** $(M_n) = (M_n, d_n)_{t \le n \le s}$ is a sequence of G-modules

$$M_s \xrightarrow{d_s} M_{s-1} \to \dots \to M_{t+1} \xrightarrow{d_{t+1}} M_t,$$

where $s = \infty$ or $t = -\infty$ are possible, such that $d_n \circ d_{n+1} = 0$, so $\operatorname{im} d_{n+1} \le \ker d_n$. The complex is **exact at** M_n if $\operatorname{im} d_{n+1} = \ker d_n$. The complex is **exact** if it is exact at M_n for all $t < n < s$. The **homology** of the chain complex is the family of G-modules

$$\mathrm{H}_n (M_\bullet) = \begin{cases} \ker d_s & n = s \\ \ker d_n / \operatorname{im} d_{n+1} & t < n < s \\ M_t / \operatorname{im} d_{t+1} & n = t \end{cases}.$$

Example 5.1.9.

- The complex
$$0 \to M_1 \xrightarrow{\alpha} M_2$$
is exact if and only if α is injective.
- The complex
$$M_1 \xrightarrow{\alpha} M_2 \to 0$$
is exact if and only if α is surjective.
- A **short exact sequence** is an exact sequence
$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0,$$
that is α is injective, β is surjective, and $\ker \beta = \operatorname{im} \alpha$, such as
$$0 \to N \to M \to M/N \to 0.$$

Definition 5.1.10. Given a set X, the **free $\mathbb{Z}G$-module on** X, denoted $\mathbb{Z}G\{X\}$, is set of finite formal sums $\sum_{x \in X} r_x x$ for $r_x \in \mathbb{Z}G$. The G-action is the obvious one $g \cdot (\sum_x r_x x) = \sum_x (g r_x) x$.

If X is finite, $\mathbb{Z}G\{X\} \cong (\mathbb{Z}G)^{|X|}$.

Definition 5.1.11. A G-module P is **projective** if, for every surjective G-linear map $\alpha : M_1 \twoheadrightarrow M_2$ and every G-linear $\beta : P \to M_2$ there exists a G-linear $\overline{\beta} : P \to M_1$ such that $\alpha \circ \overline{\beta} = \beta$, so

$$\begin{tikzcd} & P \arrow[dl, dashed, "\overline{\beta}"'] \arrow[d, "\beta"] & \\ M_1 \arrow[r, two heads, "\alpha"'] & M_2 \arrow[r] & 0 \end{tikzcd}.$$

Proposition 5.1.12. *Free modules are projective.*

Proof. Let $\mathbb{Z}G\{X\}$ be a free module and take $\alpha : M_1 \twoheadrightarrow M_2$ and $\beta : \mathbb{Z}G\{X\} \to M_2$. For each $x \in X$ choose $m_x \in M_1$ such that $\alpha(m_x) = \beta(x)$, since α is surjective. Define

$$\begin{array}{rccc} \overline{\beta} & : & \mathbb{Z}G\{X\} & \longrightarrow & M_1 \\ & & x & \longmapsto & m_x \end{array},$$

and extend linearly, so $\overline{\beta}\left(\sum_{x \in X} r_x x\right) = \sum_{x \in X} r_x m_x$. $\square$

Definition 5.1.13. A **projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules** is an exact sequence

$$\dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{Z} \to 0,$$

where each F_n is projective, and $\mathbb{Z}$ has trivial G-action.

Definition 5.1.14. Take a projective resolution as above. Let M be a G-module. Apply the functor $\mathrm{Hom}_G(-, M)$ to get a sequence

$$\dots \xleftarrow{d^2 = d_2^*} \mathrm{Hom}_G(F_1, M) \xleftarrow{d^1 = d_1^*} \mathrm{Hom}_G(F_0, M),$$

where $d^n = d_n^*$ is the dual map, so $C_n = \mathrm{Hom}_G(F_{-n}, M)$ for $n \le 0$ is a chain complex. The ***n*-th cohomology group of G with coefficients in M** is

$$\mathrm{H}^n(G, M) = \begin{cases} \ker d^1 & n = 0 \\ \ker d^{n+1} / \operatorname{im} d^n & n > 0 \end{cases}.$$

Elements of $\ker d^{n+1}$ are called ***n*-cocycles**. Elements of $\operatorname{im} d^n$ are called ***n*-coboundaries**.

Lecture 18 Tuesday 02/03/21

Where do these come from? From topology. Consider a connected simplicial complex X whose universal cover $\widetilde{X}$ is contractible, with $\pi_1(X) = G$. Let X_n be the set of n-simplices of X. Then G acts on $\widetilde{X}$ with quotient X, and without fixing any points. Therefore the n-simplices of $\widetilde{X}$ are in bijection with $G \times X_n$. The simplicial chain complex of $\widetilde{X}$ is of the form

$$\dots \xrightarrow{d_2} \mathbb{Z}G\{X_1\} \xrightarrow{d_1} \mathbb{Z}G\{X_0\} \to \mathbb{Z} \to 0.$$

Since $\widetilde{X}$ is contractible, $\mathrm{H}_n\left(\widetilde{X}\right) = 0$ for $n > 0$, so exact at $\mathbb{Z}G\{X_n\}$ for $n > 0$, and $\mathrm{H}_0\left(\widetilde{X}\right) \cong \mathbb{Z}$. So we get a free resolution of $\mathbb{Z}$. Applying $\mathrm{Hom}_G(-, M)$ gives $\mathrm{Hom}_G(\mathbb{Z}G\{X_n\}, M)$. Take the case $M = \mathbb{Z}$. Then $\mathrm{Hom}_G(\mathbb{Z}G\{X_n\}, M) \cong \mathrm{Hom}(\mathbb{Z}\{X_n\}, \mathbb{Z})$, so

$$\dots \leftarrow \mathrm{Hom}(\mathbb{Z}\{X_1\}, \mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z}\{X_0\}, \mathbb{Z}),$$

which gives $\mathrm{H}^n(G, \mathbb{Z})$. The dual is

$$\dots \to \mathbb{Z}\{X_1\} \to \mathbb{Z}\{X_0\},$$

which gives $\mathrm{H}_n(X)$.

Example 5.1.15. Let $G = \mathbb{Z} = \langle t \rangle$. Consider the sequence

$$0 \to \mathbb{Z}G \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $d_1(x) = x(t-1)$ and

$$\epsilon\left(\sum_{g \in G} n_g g\right) = \sum_{g \in G} n_g$$

is the **augmentation map**. Claim that this is a free resolution of $\mathbb{Z}$.

- ϵ is obviously surjective.
- $\ker \epsilon \ge \operatorname{im} d_1$. If $x = \sum_g n_g g$, then

$$\epsilon(d_1(x)) = \epsilon(x(t-1)) = \epsilon(xt) - \epsilon(x) = \epsilon\left(\sum_g n_g(gt)\right) - \epsilon\left(\sum_g n_g g\right) = \sum_g n_g - \sum_g n_g = 0.$$

- $\ker \epsilon \le \operatorname{im} d_1$. Let $x = \sum_g n_g g$ such that $\sum_g n_g = 0$. Relabel each $g = t^k$ for some k, so rewriting,

$$\begin{aligned} x = \sum_k n_k t^k &= n_L t^L + \dots + n_K t^K \\ &= n_L t^{L-1}(t-1) + \dots + (n_L + \dots + n_{K-2}) t^{K-1}(t-1) + (n_L + \dots + n_K) t^K \\ &= \left(n_L t^{L-1} + \dots + (n_L + \dots + n_{K-2}) t^{K-1}\right)(t-1) \in \operatorname{im} d_1. \end{aligned}$$

- d_1 is injective. Let $x = \sum_k n_k t^k = n_L t^L + \dots$ for $n_L \ne 0$. Then $x(t-1)$ has highest coefficient $n_L t^{L+1} \ne 0$.

Let M be a G-module. Then

$$\begin{tikzcd} 0 & \mathrm{Hom}_G(\mathbb{Z}G, M) \arrow[l] \arrow[d, "\iota"', "\sim"] & \mathrm{Hom}_G(\mathbb{Z}G, M) \arrow[l, "d^1"'] \arrow[d, "\sim"', "\iota"] \\ 0 & M \arrow[l] & M \arrow[l] \end{tikzcd},$$

where $\iota(\phi) = \phi(1)$. Let $m \in M$, and let $\phi \in \mathrm{Hom}_G(\mathbb{Z}G, M)$ such that $\iota(\phi) = \phi(1) = m$. Then

$$\iota\left(d^1(\phi)\right) = d^1(\phi)(1) = \phi(d_1(1)) = \phi(t-1) = (t-1)\phi(1) = (t-1)m.$$

Thus

- $\mathrm{H}^0(G, M) = \{m \in M \mid tm = m\} = M^G$ are the **invariants**, the elements on which G acts trivially,
- $\mathrm{H}^1(G, M) = M/(t-1)M = M_G$ are the **coinvariants**, and
- $\mathrm{H}^n(G, M) = 0$ for $n \ge 2$.

Let $\alpha : \mathbb{Z}G\{X\} \to \mathbb{Z}G\{Y\}$ be G-linear for $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ finite, so $\mathbb{Z}G\{X\} \cong (\mathbb{Z}G)^n$ and $\mathbb{Z}G\{Y\} \cong (\mathbb{Z}G)^m$. Then α can be written as a matrix multiplication

$$\alpha(x_i) = \sum_j a_{ij} y_j, \qquad a_{ij} \in \mathbb{Z}G.$$

If $\begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}$ is a row vector corresponding to $\sum_i r_i x_i$, then

$$\alpha\left(\begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}\right) = \begin{pmatrix} \sum_i r_i a_{i1} & \dots & \sum_i r_i a_{im} \end{pmatrix} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}.$$

Now if M is a G-module, we have

$$\begin{array}{rccc} \iota_X & : & \mathrm{Hom}_G(\mathbb{Z}G\{X\}, M) & \longrightarrow & M^n \\ & & \psi & \longmapsto & \begin{pmatrix} \psi(x_1) \\ \vdots \\ \psi(x_n) \end{pmatrix} \end{array}, \qquad \begin{array}{rccc} \iota_Y & : & \mathrm{Hom}_G(\mathbb{Z}G\{Y\}, M) & \longrightarrow & M^m \\ & & \phi & \longmapsto & \begin{pmatrix} \phi(y_1) \\ \vdots \\ \phi(y_m) \end{pmatrix} \end{array}.$$

Then

$$\begin{tikzcd} \mathrm{Hom}_G(\mathbb{Z}G\{X\}, M) \arrow[d, "\iota_X"', "\sim"] & \mathrm{Hom}_G(\mathbb{Z}G\{Y\}, M) \arrow[l, "\alpha^*"'] \arrow[d, "\sim"', "\iota_Y"] \\ M^n & M^m \arrow[l, "\widetilde{\alpha}"] \end{tikzcd}.$$

Let $\begin{pmatrix} b_1 & \dots & b_m \end{pmatrix}^\intercal \in M^m$, and let $\phi \in \mathrm{Hom}_G(\mathbb{Z}G\{Y\}, M)$ such that $\iota_Y(\phi) = \begin{pmatrix} b_1 & \dots & b_m \end{pmatrix}^\intercal$, so $\phi(y_i) = b_i$. Then

$$\widetilde{\alpha}\left(\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\right) = \iota_X(\alpha^*(\phi)) = \begin{pmatrix} \alpha^*(\phi)(x_1) \\ \vdots \\ \alpha^*(\phi)(x_n) \end{pmatrix} = \begin{pmatrix} \phi(\alpha(x_1)) \\ \vdots \\ \phi(\alpha(x_n)) \end{pmatrix} = \begin{pmatrix} \phi\left(\sum_j a_{1j} y_j\right) \\ \vdots \\ \phi\left(\sum_j a_{nj} y_j\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_j a_{1j} \phi(y_j) \\ \vdots \\ \sum_j a_{nj} \phi(y_j) \end{pmatrix} = \begin{pmatrix} \sum_j a_{1j} b_j \\ \vdots \\ \sum_j a_{nj} b_j \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Proposition 5.1.16. *Let G be a finitely generated free group. If $n \geq 2$ then $\mathrm{H}^n(G, M) = 0$ for all G-modules M.*

Proof. Let X be a wedge of circles with $\pi_1(X) = G$. Then $\widetilde{X}$ is a tree, so contractible, which gives a free resolution of G

$$0 \to \mathbb{Z}G\{X_1\} \to \mathbb{Z}G\{X_0\} \to \mathbb{Z} \to 0,$$

where $|X_0| = 1$ and $|X_1| = \operatorname{rk} G$. Thus $\mathrm{H}^n(G, M) = 0$ for all $n \geq 2$. ☐

Definition 5.1.17. A group G has **cohomological dimension** $\operatorname{cd} G = n$ if $\mathrm{H}^m(G, M) = 0$ for all $m > n$ and all G-modules M but there exists M such that $\mathrm{H}^n(G, M) \neq 0$. If no n exists then $\operatorname{cd} G = \infty$.

Free groups have cohomological dimension one. By Stallings, groups with cohomological dimension one are free.

Definition 5.1.18. Let (A_n, α_n) and (B_n, β_n) be chain complexes. A **chain map** (f_n) is a sequence of G-linear maps $f_n : A_n \to B_n$ such that $f_{n-1} \circ \alpha_n = \beta_n \circ f_n$, so

$$\begin{tikzcd} A_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "\beta_n"'] & B_{n-1} \end{tikzcd}$$

commutes.

Proposition 5.1.19. *If (f_n) is a chain map, then (f_n) gives induced maps*

$$(f_*)_n : \mathrm{H}_n(A_\bullet) \to \mathrm{H}_n(B_\bullet).$$

These maps are functorial, so if $(g_n) : (B_n) \to (C_n)$ then $(g_*)_n \circ (f_*)_n = ((g \circ f)_*)_n : \mathrm{H}_n(A_\bullet) \to \mathrm{H}_n(C_\bullet)$.

Proof. Take $x \in \ker \alpha_n$. Define

$$(f_*)_n([x]) = [f_n(x)], \qquad [x] = x + \operatorname{im} \alpha_{n+1} \in \mathrm{H}_n(A_\bullet).$$

Then $\beta_n(f_n(x)) = f_{n-1}(\alpha_n(x)) = 0$, so $f_n(x) \in \ker \beta_n$. The choice of x does not matter, since if $x' = x + \alpha_{n+1}(y)$, then

$$f_n(x') + \operatorname{im} \beta_{n+1} = f_n(x) + f_n(\alpha_{n+1}(y)) + \operatorname{im} \beta_{n+1} = f_n(x) + \beta_{n+1}(f_{n+1}(y)) + \operatorname{im} \beta_{n+1} = f_n(x) + \operatorname{im} \beta_{n+1}.$$

☐

Corollary 5.1.20. *Let $f : M \to N$ be a map of G-modules. Then we get maps*

$$\begin{array}{cccc} (f_*)_n : & \mathrm{Hom}_G(F_n, M) & \longrightarrow & \mathrm{Hom}_G(F_n, N) \\ & \phi & \longmapsto & f \circ \phi \end{array}.$$

These are chain maps, so we have

$$(f_*)_n : \mathrm{H}^n(G, M) \to \mathrm{H}^n(G, N).$$

Lecture 19 Thursday 04/03/21

Lemma 5.1.21 (Snake lemma). *If*

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

is a short exact sequence of chain complexes, where $f_\bullet$ and $g_\bullet$ are chain maps and each

$$0 \to A_n \to B_n \to C_n \to 0$$

is exact. Then there exists $\delta_n : \mathrm{H}_{n+1}(C_\bullet) \to \mathrm{H}_n(A_\bullet)$ such that

$$\dots \to \mathrm{H}_{n+1}(C_\bullet) \xrightarrow{\delta_n} \mathrm{H}_n(A_\bullet) \xrightarrow{(f_*)_n} \mathrm{H}_n(B_\bullet) \xrightarrow{(g_*)_n} \mathrm{H}_n(C_\bullet) \to \dots.$$

Proof (non-examinable). See algebraic topology. ☐

Proposition 5.1.22. *Let*

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$$

be a short exact sequence of G-modules. There is a ***long exact sequence***

$$\dots \to \mathrm{H}^n(G, M_1) \xrightarrow{(\alpha_*)_n} \mathrm{H}^n(G, M_2) \xrightarrow{(\beta_*)_n} \mathrm{H}^n(G, M_3) \xrightarrow{\delta} \mathrm{H}^{n+1}(G, M_1) \to \dots,$$

where δ are ***connecting homomorphisms****.*

Proof. Apply the snake lemma to

$$0 \to \mathrm{Hom}_G(F_\bullet, M_1) \xrightarrow{(\alpha_*)_\bullet} \mathrm{Hom}_G(F_\bullet, M_2) \xrightarrow{(\beta_*)_\bullet} \mathrm{Hom}_G(F_\bullet, M_3) \to 0,$$

where $F_\bullet$ is a projective resolution of $\mathbb{Z}$ by G-modules. It remains to prove

$$0 \to \mathrm{Hom}_G(F_n, M_1) \xrightarrow{(\alpha_*)_n} \mathrm{Hom}_G(F_n, M_2) \xrightarrow{(\beta_*)_n} \mathrm{Hom}_G(F_n, M_3) \to 0$$

is exact for each n.

- $\ker(\alpha_*)_n = 0$. Let $\phi : F_n \to M_1$. If $(\alpha_*)_n(\phi) = 0$ then $0 = \alpha \circ \phi$, so for all $x \in F_n$, $0 = \alpha(\phi(x))$, so $0 = \phi(x)$ for all x, so $\phi = 0$.
- $\ker(\beta_*)_n = \mathrm{im}(\alpha_*)_n$. Let $\phi : F_n \to M_2$ be in the kernel of $(\beta_*)_n$. Then $\beta(\phi(x)) = 0$ for all $x \in F_n$, so $\phi(x) \in \ker\beta = \mathrm{im}\,\alpha$, so there exists a unique $y_x \in M_1$ such that $\alpha(y_x) = \phi(x)$. Declare

  $$\begin{array}{cccc} \psi & : & F_n & \longrightarrow & M_1 \\ & & x & \longmapsto & y_x \end{array}.$$

  Then $(\alpha_*)_n(\psi) = \phi$, and ψ is G-linear follows from uniqueness of y_x, since $\alpha(gy_x) = g\alpha(y_x) = g\phi(x) = \phi(gx)$ implies that $gy_x = y_{gx}$.
- $(\beta_*)_n$ is surjective. Exactly the definition of F_n projective.

□

5.2 Different projective resolutions

Theorem 5.2.1. *The definition of $\mathrm{H}^n(G, M)$ is independent of the choice of projective resolution.*

Proof (non-examinable). Take two projective resolutions (F_n, d_n) and (F'_n, d'_n) of $\mathbb{Z}$ by G-modules. Suppose we construct chain maps

- $f_n : F_n \to F'_n$ such that $f_{n-1} \circ d_n = d'_n \circ f_n$,
- $g_n : F'_n \to F_n$ such that $g_{n-1} \circ d'_n = d_n \circ g_n$,
- $s_n : F_n \to F_{n+1}$ such that $d_{n+1} \circ s_n + s_{n-1} \circ d_n = g_n \circ f_n - \mathrm{id}$, and
- $s'_n : F'_n \to F'_{n+1}$ such that $d'_{n+1} \circ s'_n + s'_{n-1} \circ d'_n = f_n \circ g_n - \mathrm{id}$.

These maps prove Theorem 5.2.1. Take a G-module M. Chain maps (f_n) and (g_n) give homomorphisms $f_n^* : \mathrm{Hom}_G(F'_n, M) \to \mathrm{Hom}_G(F_n, M)$, which give homomorphisms $f_n^* : \mathrm{H}^n_{F'_n}(G, M) \to \mathrm{H}^n_{F_n}(G, M)$. Take $\phi : F_n \to M$ such that $\phi \in \ker d^{n+1}$. Then

$$\begin{aligned} f_n^*(g_n^*(\phi)) &= \phi \circ g_n \circ f_n = \phi \circ (\mathrm{id} + d_{n+1} \circ s_n + s_{n-1} \circ d_n) = \phi + \phi \circ d_{n+1} \circ s_n + \phi \circ s_{n-1} \circ d_n \\ &= \phi + s_n^*\left(d^{n+1}(\phi)\right) + d^n\left(s_{n-1}^*(\phi)\right) = \phi + d^n\left(s_{n-1}^*(\phi)\right) \in \phi + \mathrm{im}\, d^n. \end{aligned}$$

So $f_n^*(g_n^*(\phi + \mathrm{im}\, d^n)) = \phi + \mathrm{im}\, d^n$, that is $f_n^* \circ g_n^* = \mathrm{id}$, on cohomology.

Construct f_n, inductively.

- Start with the identity $f_{-1} : \mathbb{Z} \to \mathbb{Z}$ and $f_{-2} : 0 \to 0$.
- Suppose we have f_n and f_{n-1}. Build f_{n+1}. Since $d'_n \circ f_n \circ d_{n+1} = f_{n-1} \circ d_n \circ d_{n+1} = 0$, there exists $f_{n+1} : F_{n+1} \to F'_{n+1}$ such that $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$, so

$$\begin{tikzcd}
 & F_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_n \circ d_{n+1}"] \arrow[dl, dashed, "f_{n+1}"'] & F_n \arrow[r, "d_n"] \arrow[d, "f_n"] & F_{n-1} \arrow[d, "f_{n-1}"] \\
F'_{n+1} \arrow[r, two heads, "d'_{n+1}"'] & \ker d'_n = \operatorname{im} d'_{n+1} \arrow[r, phantom, "\subset"] & F'_n \arrow[r, "d'_n"'] & F'_{n-1}
\end{tikzcd} .$$

Construct s_n such that $d_{n+1} \circ s_n + s_{n-1} \circ d_n = g_n \circ f_n - \mathrm{id} = h_n$.

- Start with the zero map $s_{-1} : \mathbb{Z} \to F_0$.
- Assume s_{n-1} and s_{n-2} are constructed. Define $t_n = h_n - s_{n-1} \circ d_n$. Since

$$d_n \circ t_n = d_n \circ h_n - d_n \circ s_{n-1} \circ d_n = h_{n-1} \circ d_n - (-s_{n-2} \circ d_{n-1} + h_{n-1}) \circ d_n = s_{n-2} \circ d_{n-1} \circ d_n = 0,$$

  there exists s_n such that $d_{n+1} \circ s_n = t_n = h_n - s_{n-1} \circ d_n$, so

$$\begin{tikzcd}
 & F_n \arrow[rr, "d_n"] \arrow[d, "t_n"] \arrow[dl, dashed, "s_n"'] \arrow[dr, "h_n"] & & F_{n-1} \arrow[r, "d_{n-1}"] \arrow[dl, "s_{n-1}"'] \arrow[d, "h_{n-1}"] & F_{n-2} \arrow[dl, "s_{n-2}"'] \arrow[d, "h_{n-2}"] \\
F_{n+1} \arrow[r, two heads, "d_{n+1}"'] & \ker d_n = \operatorname{im} d_{n+1} \arrow[r, phantom, "\subset"] & F_n \arrow[r, "d_n"'] & F_{n-1} \arrow[r, "d_{n-1}"'] & F_{n-2}
\end{tikzcd} .$$

$\square$

Definition 5.2.2. Let

$$G^{(n)} = \{[g_1 \mid \cdots \mid g_n] \mid g_1, \dots, g_n \in G\}, \qquad G^{(0)} = \{[]\}.$$

The **bar resolution** is $\mathrm{F}_n = \mathbb{Z}G\left\{G^{(n)}\right\}$, the free module with basis $G^{(n)}$, with

$$\begin{array}{rcrcl} \mathrm{d}_n & : & \mathrm{F}_n & \longrightarrow & \mathrm{F}_{n-1} \\ & & & & g_1 [g_2 \mid \cdots \mid g_n] \\ & & & & - \quad [g_1 g_2 \mid \cdots \mid g_n] \\ & & [g_1 \mid \cdots \mid g_n] & \longmapsto & + \quad \dots \\ & & & & + \quad (-1)^{n-1} [g_1 \mid \cdots \mid g_{n-1} g_n] \\ & & & & + \quad (-1)^n [g_1 \mid \cdots \mid g_{n-1}] \end{array},$$

and the augmentation map

$$\begin{array}{rcrcl} \epsilon & : & \mathrm{F}_0 = \mathbb{Z}G\{[]\} & \longrightarrow & \mathbb{Z} \\ & & g[] & \longmapsto & 1 \end{array}.$$

Fact 5.2.3. This is a chain complex, so $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$.

Proposition 5.2.4. *The bar resolution is exact.*

Proof (non-examinable). Forget the G-action. Then F_n is free abelian on the set $G \times G^{(n)} = \{g [g_1 \mid \cdots \mid g_n]\}$. Define abelian group homomorphisms

$$\begin{array}{rcrcl} s_n & : & \mathrm{F}_n & \longrightarrow & \mathrm{F}_{n+1} \\ & & g [g_1 \mid \cdots \mid g_n] & \longmapsto & [g \mid g_1 \mid \cdots \mid g_{n+1}] \end{array}.$$

By a calculation, $\mathrm{d}_{n+1} \circ s_n + s_{n-1} \circ \mathrm{d}_n = \mathrm{id}_{\mathrm{F}_n}$. If $x \in \ker \mathrm{d}_n$, then

$$x = \mathrm{id}(x) = \mathrm{d}_{n+1}(s_n(x)) + s_{n-1}(\mathrm{d}_n(x)) = \mathrm{d}_{n+1}(s_n(x)) \in \operatorname{im} \mathrm{d}_{n+1},$$

so $\ker \mathrm{d}_n = \operatorname{im} \mathrm{d}_{n+1}$. $\square$

Let M be a G-module. The **cochain group** is

$$\mathrm{C}^n(G,M)=\{\text{functions }\phi:G^n\to M\}\cong\mathrm{Hom}_G(\mathrm{F}_n,M).$$

The dual of the d_n in the bar resolution is

$$\begin{array}{rcll}\mathrm{d}^n & : & \mathrm{C}^{n-1}(G,M) & \longrightarrow & \mathrm{C}^n(G,M) \\ & & \phi & \longmapsto & \left(\begin{array}{rrl} & & g_1\phi(g_2,\dots,g_n) \\ & - & \phi(g_1g_2,\dots,g_n) \\ (g_1,\dots,g_n)\mapsto & + & \dots \\ & + & (-1)^{n-1}\phi(g_1,\dots,g_{n-1}g_n) \\ & + & (-1)^n\phi(g_1,\dots,g_{n-1})\end{array}\right)\end{array}.$$

The **group of n-cocycles** is

$$\mathrm{Z}^n(G,M)=\ker\mathrm{d}^{n+1}\le\mathrm{C}^n(G,M).$$

The **group of n-coboundaries** is

$$\mathrm{B}^n(G,M)=\mathrm{im}\,\mathrm{d}^n\le\mathrm{C}^n(G,M).$$

Then

$$\mathrm{H}^n(G,M)=\mathrm{Z}^n(G,M)/\mathrm{B}^n(G,M).$$

Lecture 20 Saturday 06/03/21

Corollary 5.2.5. *Let G be a group and M a G-module. Then*

$$\mathrm{H}^0(G,M)=\mathrm{Z}^0(G,M)=\ker\mathrm{d}^1=\{m\in M\mid\forall g,\ gm=m\}=M^G,$$

*the invariants. A function $\phi:G\to M$ is a **crossed homomorphism** if $\phi(gh)=g\phi(h)+\phi(g)$, and a **principal crossed homomorphism** if $\phi(g)=gm-m$ for some $m\in M$. Then*

$$\mathrm{H}^1(G,M)=\{\textit{crossed homomorphisms}\}/\{\textit{principal crossed homomorphisms}\},$$

which is $\mathrm{Hom}(G,M)$ *if M is trivial.*

Proof. Take $\phi\in\mathrm{C}^0(G,M)$ such that $\phi(1)=m$. Then

$$\mathrm{d}^1(\phi)(g)=g\phi(1)-\phi(1)=gm-m.$$

Let $\phi\in\mathrm{C}^1(G,M)$. Then

$$\mathrm{d}^2(\phi)(g,h)=g\phi(h)-\phi(gh)+\phi(g),$$

so $\phi\in\ker\mathrm{d}^2$ if and only if $\phi(gh)=g\phi(h)+\phi(g)$. If M has trivial G-action, ϕ is a homomorphism $G\to M$. ☐

Proposition 5.2.6. *Let $\alpha:G_1\to G_2$ be a group homomorphism. Let M be a G_2-module and make M into a G_1-module via*

$$g_1\cdot m=\alpha(g_1)m,\qquad g_1\in G_1,\qquad m\in M.$$

Then there is a natural homomorphism

$$\alpha^*:\mathrm{H}^n(G_2,M)\to\mathrm{H}^n(G_1,M).$$

If $\beta:G_0\to G_1$ then $(\alpha\circ\beta)^*=\beta^*\circ\alpha^*$.

Proof. Define maps by

$$\begin{array}{rcll}\alpha^* & : & \mathrm{C}^n(G_2,M) & \longrightarrow & \mathrm{C}^n(G_1,M) \\ & & \phi & \longmapsto & ((g_1,\dots,g_n)\mapsto\phi(\alpha(g_1),\dots,\alpha(g_2)))\end{array}.$$

Then $\mathrm{d}^n\circ\alpha^*=\alpha^*\circ\mathrm{d}^n$, so

$$\begin{tikzcd}\mathrm{C}^n(G_2,M) \arrow[r, "\alpha^*"] \arrow[d, "\mathrm{d}^n"'] & \mathrm{C}^n(G_1,M) \arrow[d, "\mathrm{d}^n"] \\ \mathrm{C}^{n+1}(G_2,M) \arrow[r, "\alpha^*"'] & \mathrm{C}^{n+1}(G_1,M)\end{tikzcd}.$$

Thus α^* induce maps on cohomology. ☐

We might like a sequence of groups

$$1 \to H \to G \to Q \to 1, \qquad H \triangleleft G, \qquad G/H = Q$$

to give a long exact sequence in cohomology. This is false.

Example 5.2.7. Let

$$1 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 1.$$

Then

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}^2(\mathbb{Z}, M) & \longrightarrow & \mathrm{H}^2\left(\mathbb{Z}^2, M\right) & \longrightarrow & \mathrm{H}^2(\mathbb{Z}, M) & \longrightarrow & \dots \\ & & \wr\wr & & & & \wr\wr & & \\ & & 0 & & & & 0 & & \end{array},$$

and there exists M such that $\mathrm{H}^2\left(\mathbb{Z}^2, M\right) \neq 0$.

Lemma 5.2.8. *Let $H \triangleleft G$. Let M be a G-module. Let G act on $\mathrm{C}^n(H, M)$ by*

$$g \cdot \phi = \left((h_1, \dots, h_n) \mapsto g\phi\left(g^{-1}h_1g, \dots, g^{-1}h_ng\right)\right).$$

Then this descends to an action of G on $\mathrm{H}^n(H, M)$, and H acts trivially, so this is an action of G/H.

Proof. We want $g \cdot \mathrm{d}^n(\phi) = \mathrm{d}^n(g \cdot \phi)$, which holds by direct computation. So the action of G is by chain maps, so gives an action on cohomology. For H acts trivially, we will just do $n = 1$. Take $\phi \in \mathrm{Z}^1(H, M)$ and let $\eta, h \in H$. Then

$$\begin{aligned}(\eta \cdot \phi)(h) - \phi(h) &= \eta\phi\left(\eta^{-1}h\eta\right) - \phi(h) = \eta\left(\eta^{-1}\phi(h\eta) + \phi\left(\eta^{-1}\right)\right) - \phi(h) = \phi(h\eta) + \eta\phi\left(\eta^{-1}\right) - \phi(h) \\ &= h\phi(\eta) + \phi(h) + \eta\phi\left(\eta^{-1}\right) - \phi(h) = h\phi(\eta) + \eta\phi\left(\eta^{-1}\right) = h\phi(\eta) - \phi(\eta) = \mathrm{d}^1(\phi(\eta))(h),\end{aligned}$$

since $\phi(1 \cdot 1) = 1\phi(1) + \phi(1)$ so $\phi(1) = 0$ and $0 = \phi(1) = \phi\left(\eta\eta^{-1}\right) = \eta\phi\left(\eta^{-1}\right) + \phi(\eta)$. ◻

The useful case is $n = 1$. If $\phi : H \to M$ is a crossed homomorphism $(g \cdot \phi)(h) = g\phi\left(g^{-1}hg\right)$. If M is trivial, this reads $(g \cdot \phi)(h) = \phi\left(g^{-1}hg\right)$ so the homomorphism ϕ is G**-invariant**.

Theorem 5.2.9 (Five-term inflation-restriction exact sequence)**.** *Let $H \triangleleft G$ and $Q = G/H$ and let M be a G-module. There is an exact sequence*

$$0 \to \mathrm{H}^1\left(Q, M^H\right) \to \mathrm{H}^1(G, M) \to \mathrm{H}^1(H, M)^Q \to \mathrm{H}^2\left(Q, M^H\right) \to \mathrm{H}^2(G, M).$$

Proof (non-examinable). Just define the maps.

- **Restriction maps**

$$\begin{array}{rccc} \mathrm{Res} & : & \mathrm{H}^k(G, M) & \longrightarrow & \mathrm{H}^k(H, M)^Q \\ & & \left(f : G^k \to M\right) & \longmapsto & \left(\mathrm{Res}\, f : H^k \leq G^k \xrightarrow{f} M\right) \end{array}.$$

- **Inflation maps**

$$\begin{array}{rccc} \mathrm{Inf} & : & \mathrm{H}^k\left(Q, M^H\right) & \longrightarrow & \mathrm{H}^k(G, M) \\ & & \left(f : Q^k \to M^H\right) & \longmapsto & \left(\mathrm{Inf}\, f : G^k \twoheadrightarrow Q^k \xrightarrow{f} M^H \leq M\right) \end{array}.$$

- **Transgression maps**. Let $s : Q \to G$ be a set-theoretic section, so $(Q \to G \to Q) = \mathrm{id}_Q$, with $s(1) = 1$. Define

$$\begin{array}{rcccc} \rho & : & G & \longrightarrow & H \\ & & g & \longmapsto & gs(gH)^{-1} \end{array}.$$

If $f : H \to M$ represents a Q-invariant cohomology class define

$$\begin{array}{rcccc} \mathrm{Tg} & : & \mathrm{H}^1(H, M)^Q & \longrightarrow & \mathrm{H}^2\left(Q, M^H\right) \\ & & f & \longmapsto & \left((g_1, g_2) \mapsto f(\rho(g_1)\rho(g_2)) - f(\rho(g_1g_2))\right) \end{array}.$$

◻

If G is a free group, $\mathrm{H}^2(G, M) = 0$ for all M.

Corollary 5.2.10 (Hopf's formula)**.** *Let F be a free group and $R \triangleleft F$ and $Q = F/R$. Let A be an abelian group, viewed as a trivial module. Then*

$$\mathrm{H}^2(Q, A) \cong \{F\text{-invariant homomorphisms } R \to A\} / \{\text{homomorphisms } F \to A\}.$$

Proof. There is an exact sequence

$$\mathrm{Hom}(F, A) \to \mathrm{Hom}(R, A)^F \to \mathrm{H}^2(Q, A) \to 0.$$

$\square$

If $Q = \langle x_1, \dots, x_d \mid r_1, \dots, r_m \rangle$ is a presentation, then $F = \langle x_1, \dots, x_d \rangle$ is free and $R = \langle\langle r_1, \dots, r_m \rangle\rangle$ is the normal subgroup generated by r_i. Then $\mathrm{d}\left(\mathrm{H}^1(Q, \mathbb{Z})\right) = \mathrm{d}(\mathrm{Hom}(Q, \mathbb{Z})) \le d$. An F-invariant homomorphism $R \to \mathbb{Z}$ is determined by images of r_i, so $\mathrm{d}\left(\mathrm{H}^2(Q, \mathbb{Z})\right) \le m$.

Example 5.2.11. Let $Q = \mathbb{Z}/3\mathbb{Z}$ and let Q act on $M = \mathbb{Z}^2$ via the order three matrix $A = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Consider the short exact sequence of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{\cdot 3} & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \wr\!\! \mid & & \wr\!\! \mid & & \wr\!\! \mid & & \\ & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/3\mathbb{Z} & & \end{array}.$$

Since H acts trivially on M, $\mathrm{H}^1(H, M) = \mathrm{Hom}(H, M) \cong \mathbb{Z}^2$ by $f \mapsto f(1)$. Then $f \in \mathrm{H}^1(H, M)^Q$ if and only if $f(1)$ is Q-invariant, if and only if $Af(1) = f(1)$. If $Ax = x$, then $x = 0$, so $\mathrm{H}^1(H, M)^Q = 0$ and $\mathrm{H}^2(G, M) = 0$. By the five-term exact sequence, $\mathrm{H}^2(Q, M) = 0$.

5.3 Cohomology and group extensions

Lecture 21 Tuesday 09/03/21

Take a group E, with an abelian normal subgroup M. Let $E/M = G$. Then E is an **extension of G by M**, and E and E' are **equivalent** extensions if there is a commuting diagram

$$\begin{tikzcd} & & E \arrow[dd, "\sim"] \arrow[dr, "\pi"] & & & \\ 1 \arrow[r] & M \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}.$$

The action of G on M is by conjugation in E. If $m \in M$ and $e \in E$, then $eme^{-1} \in M$, which gives an action of E on M, and M is abelian, so M acts trivially on itself, which gives an action of G on M. If $g \in G$, take e such that $\pi(e) = g$ and set

$$g \bullet m = eme^{-1}.$$

If this action is trivial, then E is a **central extension**. Given G and a G-module M, can construct the semidirect product $M \rtimes G$, where the set of elements is $M \times G$ with multiplication

$$(m_1, g_1) * (m_2, g_2) = (m_1 + g_1 \bullet m_2, g_1 g_2).$$

Then $M \triangleleft M \rtimes G$, with the correct action, and quotient G. So $M \rtimes G$ is an extension, called the **split extension**

$$1 \to M \to M \rtimes G \to G \to 1,$$

where the **splitting** is the group homomorphism

$$\begin{array}{ccc} G & \longrightarrow & M \rtimes G \\ g & \longmapsto & (0, g) \end{array}.$$

Notation 5.3.1. M is a G-module, written additively, and $M \le E$, which is non-abelian.

Proposition 5.3.2. *Let E be an extension of G by M. Assume there exists a splitting $s : G \to E$, a group homomorphism such that $\pi \circ s = \mathrm{id}_G$. Then E is equivalent to $M \rtimes G$.*

Proof. Write down the equivalence $(m, g) \mapsto ms(g)$, which is a group homomorphism. $\square$

Theorem 5.3.3. *We have a bijection*

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{extensions of } G \textit{ by } M \end{array} \right\} \qquad \leftrightsquigarrow \qquad \left\{ \ [\phi] \in \mathrm{H}^2(G, M) \ \right\}.$$

Proof. Let E be an extension of G by M. Let $\pi : E \to G$ be the quotient. Choose a set-theoretic section $s : G \to E$, a function such that $\pi \circ s = \mathrm{id}_G$, so $\pi(s(g)) = g$. Choose $s(1) = 1$. Consider the function

$$\begin{array}{cccc} \phi & : & G^2 & \longrightarrow & M \\ & & (g_1, g_2) & \longmapsto & s(g_1) s(g_2) s(g_1 g_2)^{-1} \end{array},$$

since $\pi\left(s(g_1) s(g_2) s(g_1 g_2)^{-1}\right) = g_1 g_2 (g_1 g_2)^{-1} = 1$, and $\phi \equiv 1$ if and only if s is a homomorphism.

- The 2-cochain $\phi \in \mathrm{C}^2(G, M)$ is a cocycle, that is $\mathrm{d}^3(\phi) = 0$. Since $s(g_1) s(g_2) = \phi(g_1, g_2) s(g_1 g_2)$,

$$\begin{aligned} \phi(g_1, g_2) \phi(g_1 g_2, g_3) s(g_1 g_2 g_3) &= \phi(g_1, g_2) s(g_1 g_2) s(g_3) = s(g_1) s(g_2) s(g_3) \\ &= s(g_1) \phi(g_2, g_3) s(g_2 g_3) = s(g_1) \phi(g_2, g_3) s(g_1)^{-1} s(g_1) s(g_2 g_3) \\ &= s(g_1) \phi(g_2, g_3) s(g_1)^{-1} \phi(g_1, g_2 g_3) s(g_1 g_2 g_3). \end{aligned}$$

  Rewriting in M,

$$\phi(g_1, g_2) + \phi(g_1 g_2, g_3) = g_1 \bullet \phi(g_2, g_3) + \phi(g_1, g_2 g_3),$$

  so

$$0 = g_1 \bullet \phi(g_2, g_3) - \phi(g_1 g_2, g_3) + \phi(g_1, g_2 g_3) - \phi(g_1, g_2) = \mathrm{d}^3(\phi)(g_1, g_2, g_3).$$

  Thus ϕ is a cocycle, and it is a **normalised cocycle**, since $\phi(1, g) = \phi(g, 1) = 0$ for all $g \in G$.

- Let $s' : G \to E$ be another section, with $s'(1) = 1$. Then $\pi\left(s'(g) s(g)^{-1}\right) = 1$ so define

$$\begin{array}{cccc} \psi & : & G & \longrightarrow & M \\ & & g & \longmapsto & s'(g) s(g)^{-1} \end{array}.$$

  Now let us compute the cocycle ϕ' coming from s'. Since $s'(g) = \psi(g) s(g)$,

$$\begin{aligned} s'(g_1) s'(g_2) &= \psi(g_1) s(g_1) \psi(g_2) s(g_2) = \psi(g_1) s(g_1) \psi(g_2) s(g_1)^{-1} s(g_1) s(g_2) \\ &= \psi(g_1) s(g_1) \psi(g_2) s(g_1)^{-1} \phi(g_1, g_2) s(g_1 g_2) \\ &= \psi(g_1) s(g_1) \psi(g_2) s(g_1)^{-1} \phi(g_1, g_2) \psi(g_1 g_2)^{-1} s'(g_1 g_2) \end{aligned}$$

  Back to M,

$$\phi'(g_1, g_2) = \psi(g_1) + g_1 \bullet \psi(g_2) + \phi(g_1, g_2) - \psi(g_1 g_2) = \phi(g_1, g_2) + \mathrm{d}^2(\psi)(g_1, g_2),$$

  so $[\phi] = \phi + \operatorname{im} \mathrm{d}^2 = \phi' + \operatorname{im} \mathrm{d}^2 = [\phi'] \in \mathrm{H}^2(G, M)$.

It remains to show that equivalent extensions give the same cohomology class, to construct an inverse, and show it is the inverse. Let $\zeta \in \mathrm{H}^2(G, M)$ and choose a normalised cocycle ϕ such that $\zeta = [\phi]$. Can always choose normalised. [4] Define E_ϕ, where the set is $M \times G$ with group operation

$$(m_1, g_1) *_\phi (m_2, g_2) = (m_1 + g_1 \bullet m_2 + \phi(g_1, g_2), g_1 g_2).$$

- If $s(g) = (0, g)$, then $s(g_1) *_\phi s(g_2) = (\phi(g_1, g_2), 1) *_\phi s(g_1 g_2)$.
  - Associativity since ϕ is a cocycle.
  - $(0, 1)$ is an identity since ϕ is normalised.
  - Existence of inverses since ϕ is a normalised cocycle.

  Then E_ϕ is an extension, since $M \triangleleft E_\phi$, with the right action, and $E_\phi / M = G$.

[4] Exercise: if $\phi \in \mathrm{Z}^2(G, M)$, set $\psi(g) = -\phi(1, g)$ and then $\phi + \mathrm{d}^2(\psi)$ turns out to be normalised

- If $[\phi] = [\phi']$, where ϕ' is also normalised, then $\phi - \phi' = \mathrm{d}^2(\psi)$ for some ψ. Define

$$\begin{array}{ccc} E_\phi & \longrightarrow & E_{\phi'} \\ (m, g) & \longmapsto & (m + \psi(g), g) \end{array},$$

which is an equivalence of extensions.

$\square$

Back to Hopf's formula. Suppose $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$. Take an abelian group A, a trivial G-module. Let E be a central extension of G by A. Then E is generated by A, and some choices $\overline{x_1}, \dots, \overline{x_n}$ of preimages of the x_i. Define $\overline{r_i}$ as r_i with bars on top. Then in E, $\overline{r_i} = a_i \in A$. Writing a presentation of E,

$$\overline{E} = \left\langle \overline{x_1}, \dots, \overline{x_n}, A \;\middle|\; \overline{r_1} a_1^{-1}, \dots, \overline{r_m} a_m^{-1},\ A \text{ central},\ \text{relations of } A \right\rangle.$$

Then

$$\begin{tikzcd} & A \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \overline{E} \arrow[r, two heads] \arrow[d] & G \arrow[d, phantom, "\cong" rotate=90] & \\ 1 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \end{tikzcd},$$

so $A \hookrightarrow \overline{E}$, and $\overline{E} \cong E$. Set $F = \langle \overline{x_1}, \dots, \overline{x_n} \rangle$ and $R = \ker(F \to G) = \langle\langle \overline{r_1}, \dots, \overline{r_m} \rangle\rangle$. Define an F-invariant homomorphism

$$\begin{array}{ccc} R & \longrightarrow & A \\ \overline{r_i} & \longmapsto & a_i \end{array},$$

since E is an extension. If we choose $\overline{x_i}'$ instead of $\overline{x_i}$, then $\overline{x_i}' = \overline{x_i} b_i$ for some $b_i \in A$. Then

$$\begin{array}{ccc} F & \longrightarrow & A \\ \overline{x_i} & \longmapsto & b_i \end{array}$$

is a homomorphism, and the homomorphism $R \to A$ is changed by subtracting the restriction of $F \to A$. Thus

$$\mathrm{H}^2(G, A) = \{F\text{-invariant homomorphisms } R \to A\} / \{\text{restrictions of homomorphisms } F \to A\}.$$

Example 5.3.4. Let $G = \left\langle x_1, x_2 \;\middle|\; x_1 x_2 x_1^{-1} x_2^{-1} x_1 \right\rangle$. Any extension of G by $\mathbb{Z}$ has a presentation

$$E = \left\langle \overline{x_1}, \overline{x_2}, a \;\middle|\; \overline{x_1 x_2 x_1}^{-1} \overline{x_2}^{-1} \overline{x_1} = a^k,\ a \text{ central} \right\rangle.$$

Then E is equivalent to the direct product, since by $\overline{x_1} \mapsto \overline{x_1}' = \overline{x_1} a^k$,

$$E \cong \left\langle \overline{x_1}', \overline{x_2}, a \;\middle|\; \overline{x_1}' \overline{x_2} \overline{x_1}'^{-1} \overline{x_2}^{-1} \overline{x_1}' = 1,\ a \text{ central} \right\rangle \cong \mathbb{Z} \times G.$$

Thus $\mathrm{H}^2(G, \mathbb{Z}) = 0$.

5.3.1 Worked example: extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$

Lecture 22
Thursday
11/03/21

Let $T = \mathbb{Z}^2 = \langle a, b \rangle$. Take the free resolution of $\mathbb{Z}$ by $\mathbb{Z}T$-modules

$$0 \to \mathbb{Z}T \xrightarrow{\beta} \mathbb{Z}T^2 \xrightarrow{\alpha} \mathbb{Z}T \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation $\epsilon(g) = 1$ for all $g \in T$, and

$$\alpha(x, y) = x(a-1) + y(b-1) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}, \qquad \beta(z) = \begin{pmatrix} z(1-b) & z(a-1) \end{pmatrix} = z \begin{pmatrix} 1-b & a-1 \end{pmatrix}.$$

Exactness comes from topology, since it is the chain complex for the square tiling of the plane. Applying $\mathrm{Hom}_T(-, \mathbb{Z})$,

$$\begin{tikzcd} 0 & \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \arrow[l] \arrow[d, "\sim"', "\phi \mapsto \phi(1)"] & \mathrm{Hom}_T\left(\mathbb{Z}T^2, \mathbb{Z}\right) \arrow[l, "\beta^*"'] \arrow[d, "\sim"', "\phi \mapsto (\phi(1,0), \phi(0,1))"] & \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \arrow[l, "\alpha^*"'] \arrow[d, "\sim"', "\phi \mapsto \phi(1)"] \\ 0 & \mathbb{Z} \arrow[l] & \mathbb{Z}^2 \arrow[l, "\beta^*"] & \mathbb{Z} \arrow[l, "\alpha^*"] \end{tikzcd}.$$

Then

$$\alpha^*(z) = \begin{pmatrix} a-1 \\ b-1 \end{pmatrix} z = 0, \qquad \beta^*\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1-b & a-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

so

$$\mathrm{H}^n(T, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & n > 2 \end{cases}.$$

Comparing the bar resolution and the topological resolution,

$$\begin{tikzcd} \dots \arrow[r] & \mathbb{Z}T\left\{T^{(2)}\right\} \arrow[r, "\mathrm{d}_2"] \arrow[d, "f_2"] & \mathbb{Z}T\left\{T^{(1)}\right\} \arrow[r, "\mathrm{d}_1"] \arrow[d, "f_1"] & \mathbb{Z}T\left\{T^{(0)}\right\} \arrow[r, "\epsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathbb{Z}T \arrow[r, "\beta"'] & \mathbb{Z}T^2 \arrow[r, "\alpha"'] & \mathbb{Z}T \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}.$$

- $f_0([]) = 1$.
- Let $[a^r b^s] \in T^{(1)}$ for $r, s \in \mathbb{Z}$. We want to set $f_1([a^r b^s]) = (x_{r,s}, y_{r,s}) \in \mathbb{Z}T^2$ such that $\alpha(x_{r,s}, y_{r,s}) = f_0(\mathrm{d}_1([a^r b^s]))$, and

  $$x_{r,s}(a-1) + y_{r,s}(b-1) = f_0(a^r b^s [] - []) = a^r b^s - 1 = (a^r - 1) b^s + b^s - 1.$$

  Set

  $$\mathrm{S}(a, r) = \begin{cases} 1 + \dots + a^{r-1} & r > 0 \\ 0 & r = 0 \\ -a^{-1} - \dots - a^r & r < 0 \end{cases},$$

  so $\mathrm{S}(a, r)(a-1) = a^r - 1$. Then $x_{r,s} = \mathrm{S}(a, r) b^s$ and $y_{r,s} = \mathrm{S}(b, s)$, so

  $$f_1([a^r b^s]) = (\mathrm{S}(a, r) b^s, \mathrm{S}(b, s)).$$

- We want $f_2([a^r b^s \mid a^t b^u]) = z_{r,s,t,u}$ such that $\beta(z_{r,s,t,u}) = f_1(\mathrm{d}_2([a^r b^s \mid a^t b^u]))$, and

  $$\begin{aligned} &((1-b) z_{r,s,t,u}, (a-1) z_{r,s,t,u}) \\ = {} & f_1\left(a^r b^s \left[a^t b^u\right] - \left[a^{r+t} b^{s+u}\right] + [a^r b^s]\right) \\ = {} & \left(a^r b^s \mathrm{S}(a, t) b^u - \mathrm{S}(a, r+t) b^{s+u} + \mathrm{S}(a, r) b^s, a^r b^s \mathrm{S}(b, u) - \mathrm{S}(b, s+u) + \mathrm{S}(b, s)\right). \end{aligned}$$

  By computation, $z_{r,s,t,u} = \mathrm{S}(a, r) b^s \mathrm{S}(b, u)$, so

  $$f_2\left(\left[a^r b^s \mid a^t b^u\right]\right) = \mathrm{S}(a, r) b^s \mathrm{S}(b, u).$$

A useful fact is that ϵ is a ring homomorphism, and $\epsilon(\mathrm{S}(a, r)) = r$, so the cochain representing $p \in \mathbb{Z} \cong \mathrm{H}^2(T, \mathbb{Z})$ is

$$\begin{array}{ccccc} T^{(2)} \subseteq \mathbb{Z}T\left\{T^{(2)}\right\} & \xrightarrow{f_2} & \mathbb{Z}T & \xrightarrow{p\epsilon} & \mathbb{Z} \\ [a^r b^s \mid a^t b^u] & \longmapsto & \mathrm{S}(a, r) b^s \mathrm{S}(b, u) & \longmapsto & pru \end{array}.$$

The corresponding central extension of T by $\mathbb{Z}$ is the set $\mathbb{Z} \times T$ with group operation

$$(m, a^r b^s) * \left(n, a^t b^u\right) = \left(m + n + pru, a^{r+t} b^{s+u}\right),$$

which is

$$\begin{pmatrix} 1 & pr & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & pt & n \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p(r+t) & m+n+pru \\ 0 & 1 & s+u \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the extension is isomorphic to

$$\left\{ \begin{pmatrix} 1 & pr & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; r, m, s \in \mathbb{Z} \right\}.$$

5.4 Cohomology of profinite groups

Definition 5.4.1. Let G be a profinite group. A **finite G-module** is a finite abelian group with an action

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g, m) & \longmapsto & g \cdot m \end{array},$$

which is continuous.

Definition 5.4.2. Let

$$\mathrm{C}^n(G, M) = \{\text{continuous functions } G^n \to M\},$$

with $\mathrm{d}^n : \mathrm{C}^{n-1}(G, M) \to \mathrm{C}^n(G, M)$ by the same formula, and let

$$\mathrm{Z}^n(G, M) = \ker \mathrm{d}^{n+1}, \qquad \mathrm{B}^n(G, M) = \operatorname{im} \mathrm{d}^n, \qquad \mathrm{H}^n(G, M) = \mathrm{Z}^n(G, M) / \mathrm{B}^n(G, M).$$

Theorem 5.4.3 (Course convention)**.** *All general results from section* 5.1 *to section* 5.3 *and from example sheet* 4*, can be applied to profinite groups, with substitutions groups to profinite groups, functions to continuous functions, and modules to finite modules.*

If E and E' are profinite groups, then extensions

$$\begin{tikzcd} & & E \arrow[dd] \arrow[dr] & & \\ 1 \arrow[r] & M \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

correspond to $\mathrm{H}^2(G, M)$.

Remark 5.4.4. Let $G = \widehat{\mathbb{Z}}$. Consider the short exact sequence of modules

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

By the course convention, $\mathrm{H}^1(G, M) = \operatorname{Hom}(G, M)$ and there is a long exact sequence in cohomology. By common sense, $\mathrm{H}^2\left(\widehat{\mathbb{Z}}, M\right) = 0$. Then $\mathrm{H}^1\left(\widehat{\mathbb{Z}}, \mathbb{Z}\right) = \mathrm{H}^1\left(\widehat{\mathbb{Z}}, \mathbb{Q}\right) = 0$, since any continuous homomorphism $\widehat{\mathbb{Z}} \to \mathbb{Q}$ has compact image, so must be trivial. There is an isomorphism

$$\begin{array}{ccc} \mathrm{H}^1\left(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}\right) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ a + n\mathbb{Z} & \longmapsto & \dfrac{am}{n} + \mathbb{Z} \end{array}.$$

Thus

$$\begin{array}{ccccc} \mathrm{H}^1\left(\widehat{\mathbb{Z}}, \mathbb{Q}\right) & \longrightarrow & \mathrm{H}^1\left(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}\right) & \longrightarrow & \mathrm{H}^2\left(\widehat{\mathbb{Z}}, \mathbb{Z}\right) \\ \cong & & \cong & & \\ 0 & & \mathbb{Q}/\mathbb{Z} & & \end{array},$$

so $\mathrm{H}^2\left(\widehat{\mathbb{Z}}, \mathbb{Z}\right)$ would be infinite, which is silly.

5.4.1 Pro-p groups of cohomological dimension one

Lecture 23 Saturday 13/03/21

Definition 5.4.5. A profinite group G has **cohomological dimension** $\operatorname{cd} G = n$ if $\mathrm{H}^m(G, M) = 0$ for all $m > n$ and for all finite modules M but there exists M such that $\mathrm{H}^n(G, M) \neq 0$.

Theorem 5.4.6. *Let G be a pro-p group. Then*

$$\operatorname{cd} G = \max\{n \mid \mathrm{H}^n(G, \mathbb{F}_p) \neq 0\}.$$

Corollary 5.4.7. *If $G \neq 1$, then $\operatorname{cd} G = 1$ if and only if $\mathrm{H}^2(G, \mathbb{F}_p) = 0$.*

Proof. $\mathrm{H}^1(G, \mathbb{F}_p) = \operatorname{Hom}(G, \mathbb{F}_p) \neq 0$. ☐

Definition 5.4.8. A G-module M is **simple** if the only G-submodules of M are zero and M.

Proposition 5.4.9. *Let G be a profinite group. Suppose $\mathrm{H}^n(G,S)=0$ for all finite simple modules S. Then $\mathrm{H}^n(G,M)=0$ for all finite M.*

Proof. Suppose not, and let M be the smallest G-module such that $\mathrm{H}^n(G,M)\neq 0$. Since M is not simple, there exists a non-trivial submodule $N\subsetneq G$. Then we have a short exact sequence of G-modules

$$0\to N\to M\to M/N\to 0.$$

By Proposition 5.1.22, there is a long exact sequence, of which part is

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}^n(G,N) & \longrightarrow & \mathrm{H}^n(G,M) & \longrightarrow & \mathrm{H}^n(G,M/N) & \longrightarrow & \dots \\ & & \wr\Vert & & & & \wr\Vert & & \\ & & 0 & & & & 0 & & \end{array},$$

so $\mathrm{H}^n(G,M)=0$, a contradiction. □

Definition 5.4.10. Let M be a finite G-module. For a prime p, M is called ***p*-primary** if $|M|$ is a power of p.

If M is finite, then

$$M=\bigoplus_{p\text{ prime}}M_p,$$

where M_p are p-Sylow subgroups or p-primary components, in a unique way, so this decomposition is invariant under G, so all M_p are G-submodules of M.

Proposition 5.4.11. *Let G be a pro-p group and let M be a finite G-module. Then*

$$\mathrm{H}^n(G,M)=\mathrm{H}^n(G,M_p),\qquad n\geq 1.$$

Proof. Write $M=M_p\oplus M'$. Then

$$\mathrm{H}^n(G,M)=\mathrm{H}^n(G,M_p)\oplus\mathrm{H}^n(G,M').$$

So we need $\mathrm{H}^n(G,M')=0$.

Part 1. If G is finite, this is on exercise sheet 4, since $|G|$ and $|M'|$ are coprime.

Part 2. Let G be a pro-p group. Let $\phi:G^n\to M'$ be a continuous function. We will find $K\lhd_{\mathrm{o}}G$ such that K acts trivially on M', and such that ϕ factors through $(G/K)^n$, so $\phi(g_1,\dots,g_n)=\phi(g_1k_1,\dots,g_nk_n)$. Then

$$\begin{array}{rcl} 0=\mathrm{H}^n(G/K,M') & \longrightarrow & \mathrm{H}^n(G,M') \\ \overline{\phi} & \longmapsto & \phi \end{array}.$$

The preimages of $m\in M'$ are closed and open subsets of G^n, so they are unions of basic open subsets, cosets of open subgroups $K_{i,m}^n$ where $K_{i,m}\lhd G$. By compactness, for each m only finitely many $K_{i,m}$ are needed. Take the intersection of all of these to get the required K. All preimages of $m\in M'$ are now unions of cosets of K^n, so ϕ factors through $(G/K)^n$. Making K smaller, by intersecting with the kernel of $G\to\mathrm{Aut}\,M'$, makes K act trivially. Now

$$\phi:G^n\xrightarrow{\pi}(G/K)^n\xrightarrow{\overline{\phi}}M'.$$

If $\mathrm{d}^{n+1}(\phi)=0$, then $\overline{\phi}=\mathrm{d}^n(\psi)$ for some $\psi:(G/K)^{n-1}\to M'$, by part 1. It follows that $\phi=\mathrm{d}^n(\psi\circ\pi)$.

□

Remark 5.4.12. In proving that all cocycles factor through some $(G/K)^n$, we really show that

$$\mathrm{H}^n(G,M)=\mathrm{H}^n\left(\varprojlim_{K\lhd_{\mathrm{o}}G,\ K\text{ acts trivially}}G/K,M\right)=\varinjlim_{K}\mathrm{H}^n(G/K,M).$$

Proposition 5.4.13. *Let G be a pro-p group. The only finite simple p-primary G-module is $\mathbb{F}_p$.*

Proposition 5.4.14. *Let G be a pro-p group. If $\mathrm{H}^n(G, \mathbb{F}_p) = 0$ for some n, then $\mathrm{H}^n(G, M) = 0$ for all finite G-modules M.*

Proposition 5.4.15. *Suppose there exists $N \geq 1$ such that $\mathrm{H}^N(G, M) = 0$ for all finite G-modules M. Then $\operatorname{cd} G \leq N - 1$.*

Proof for abstract groups, then apply course convention.

Proof (non-examinable). We show $\mathrm{H}^{N+1}(G, M) = 0$ for all M, then induct. Consider the **coinduced module**

$$\operatorname{Coind}_G^K M = \operatorname{Hom}_K(\mathbb{Z}G, M), \qquad K \leq G.$$

By **Shapiro's lemma**,

$$\mathrm{H}^n\left(G, \operatorname{Coind}_G^K M\right) = \mathrm{H}^n(K, M),$$

since $\operatorname{Hom}_G\left(F, \operatorname{Coind}_G^K M\right) \cong \operatorname{Hom}_K(F, M)$ for F a G-module. Consider $K = 1$. Consider

$$\begin{array}{cccc} \alpha & : & M & \longrightarrow & \operatorname{Coind}_G^1 M \\ & & m & \longmapsto & (x \mapsto xm) \end{array},$$

which is injective and G-linear. So there exists a short exact sequence of modules

$$0 \to M \xrightarrow{\alpha} \operatorname{Coind}_G^1 M \to M' \to 0,$$

where M' is the quotient, which gives a long exact sequence

$$\begin{array}{ccccccc} \mathrm{H}^n\left(G, \operatorname{Coind}_G^1 M\right) & \longrightarrow & \mathrm{H}^n(G, M') & \xrightarrow{\sim} & \mathrm{H}^{n+1}(G, M) & \longrightarrow & \mathrm{H}^{n+1}\left(G, \operatorname{Coind}_G^1 M\right) \\ \wr\| & & & & & & \wr\| \\ \mathrm{H}^n(1, M) = 0 & & & & & & \mathrm{H}^{n+1}(1, M) = 0, \end{array}, \qquad n \geq 1.$$

Since $\mathrm{H}^N(G, M') = 0$ for all M' by hypothesis, $\mathrm{H}^{N+1}(G, M) = 0$. ☐

We have now proved the following.

Theorem 5.4.16. *Let G be a pro-p group. If $\mathrm{H}^n(G, \mathbb{F}_p) = 0$ then $\operatorname{cd} G \leq n - 1$.*

Corollary 5.4.17. *Topologically finitely generated free pro-p groups have cohomological dimension one.*

Proof. To show $\mathrm{H}^2(G, \mathbb{F}_p) = 0$, if and only if every extension is a split extension. Let

$$1 \to \mathbb{F}_p \to E \to G \to 1.$$

If $\{x_1, \ldots, x_d\}$ is a generating set of G, choose preimages e_i of x_i in E. Then the function $x_i \mapsto e_i$ extends to a continuous homomorphism $G \to E$ by freeness. So E is a split extension, and $\mathrm{H}^2(G, \mathbb{F}_p) = 0$. ☐

Theorem 5.4.18. *Let $f : G \to G'$ be a continuous homomorphism of topologically finitely generated pro-p groups. Assume that*

$$f_1^* : \mathrm{H}^1(G', \mathbb{F}_p) \to \mathrm{H}^1(G, \mathbb{F}_p)$$

is an isomorphism and

$$f_2^* : \mathrm{H}^2(G', \mathbb{F}_p) \to \mathrm{H}^2(G, \mathbb{F}_p)$$

is injective. Then f is an isomorphism.

Corollary 5.4.19. *Let $f : \Gamma \to \Gamma'$ be a homomorphism of finitely generated abstract groups. Assume that*

$$f_1^* : \mathrm{H}^1(\Gamma', \mathbb{F}_p) \to \mathrm{H}^1(\Gamma, \mathbb{F}_p)$$

is an isomorphism and

$$f_2^* : \mathrm{H}^2(\Gamma', \mathbb{F}_p) \to \mathrm{H}^2(\Gamma, \mathbb{F}_p)$$

is injective. Then f yields an isomorphism of pro-p completions $\widehat{\Gamma_{(p)}} \to \widehat{\Gamma'_{(p)}}$.

Corollary 5.4.20. *Let G be a topologically finitely generated pro-p group of cohomological dimension one. Then G is free.*

Proof. If $\mathrm{d}(G) = d$, there is a surjection $f : F \twoheadrightarrow G$ where F is free of rank d. Then F gives an isomorphism $f_* : F/\Phi(F) \xrightarrow{\sim} G/\Phi(G) \cong \mathbb{F}_p^d$. Since

$$\mathrm{H}^1(G, \mathbb{F}_p) = \mathrm{Hom}(G, \mathbb{F}_p) = \mathrm{Hom}(G/\Phi(G), \mathbb{F}_p),$$

the dual spaces are

$$f_1^* : \mathrm{Hom}(G/\Phi(G), \mathbb{F}_p) = \mathrm{H}^1(G, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Hom}(F/\Phi(F), \mathbb{F}_p) = \mathrm{H}^1(F, \mathbb{F}_p),$$

and

$$f_2^* : \mathrm{H}^2(G, \mathbb{F}_p) = 0 \to \mathrm{H}^2(F, \mathbb{F}_p)$$

is an injection. Thus f is an isomorphism. ☐

Lecture 24 Tuesday 16/03/21

Proof of Theorem 5.4.18 (non-examinable). Let $G_n = \gamma_n^{(p)}(G)$ be the lower central p-series, where $G_1 = G$ and $G_{n+1} = \overline{[G, G_n] G_n^p}$, which are open in G. Since $f(G_n) \le G'_n$, there are $f_n : G/G_n \to G'/G'_n$. We show all f_n are isomorphisms, so $f = \varprojlim_n f_n$ is an isomorphism, by induction.

- $n = 1$ is trivial.
- Consider G_n/G_{n+1}, which is a finite-dimensional vector space over $\mathbb{F}_p$. Then f induces an isomorphism $G_n/G_{n+1} \to G'_n/G'_{n+1}$ if and only if $f^* : \mathrm{Hom}\left(G'_n/G'_{n+1}, \mathbb{F}_p\right) \to \mathrm{Hom}(G_n/G_{n+1}, \mathbb{F}_p)$ is an isomorphism. A continuous homomorphism $\phi : G_n \to \mathbb{F}_p$ factors through G_{n+1} if and only if $\phi([g, g_n]) = 0$ for all $g \in G$ and $g_n \in G_n$, if and only if $\phi(g_n) = \phi\left(g^{-1} g_n g\right)$, that is ϕ is G-invariant, so

  $$\mathrm{Hom}(G_n/G_{n+1}, \mathbb{F}_p) \cong \mathrm{Hom}(G_n, \mathbb{F}_p)^G = \mathrm{H}^1(G_n, \mathbb{F}_p)^G.$$

  For the short exact sequence

  $$1 \to G_n \to G \to G/G_n \to 1,$$

  we have a five-term exact sequence

  $$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(G/G_n, \mathbb{F}_p) \arrow[r] & \mathrm{H}^1(G, \mathbb{F}_p) \arrow[r] & \mathrm{H}^1(G_n, \mathbb{F}_p)^G \arrow[r] & \mathrm{H}^2(G/G_n, \mathbb{F}_p) \arrow[r] & \mathrm{H}^2(G, \mathbb{F}_p) \\ 0 \arrow[r] & \mathrm{H}^1(G'/G'_n, \mathbb{F}_p) \arrow[r] \arrow[u, "\sim"] & \mathrm{H}^1(G', \mathbb{F}_p) \arrow[r] \arrow[u, "\sim"] & \mathrm{H}^1(G'_n, \mathbb{F}_p)^{G'} \arrow[r] \arrow[u] & \mathrm{H}^2(G'/G'_n, \mathbb{F}_p) \arrow[r] \arrow[u, "\sim"] & \mathrm{H}^2(G', \mathbb{F}_p) \arrow[u, hook] \end{tikzcd}.$$

  By the five lemma, the middle map is an isomorphism, so $G_n/G_{n+1} \xrightarrow{\sim} G'_n/G'_{n+1}$. Then

  $$\begin{tikzcd} 1 \arrow[r] & G_n/G_{n+1} \arrow[r] \arrow[d, "\sim"] & G/G_{n+1} \arrow[r] \arrow[d] & G/G_n \arrow[r] \arrow[d, "\sim"] & 1 \\ 1 \arrow[r] & G'_n/G'_{n+1} \arrow[r] & G'/G'_{n+1} \arrow[r] & G'/G'_n \arrow[r] & 1 \end{tikzcd}.$$

  By the five lemma, $G/G_{n+1} \xrightarrow{\sim} G'/G'_{n+1}$.

☐

Example 5.4.21. Let $\Gamma = \left\langle x_1, x_2 \mid x_1 x_2 x_1^{-1} x_2^{-1} x_1 \right\rangle$. By Example 5.3.4, $\mathrm{H}^2(\Gamma, \mathbb{F}_p) = 0$. Then $\Gamma_{\mathrm{ab}} \cong \mathbb{Z}$, generated by x_2, and $\mathrm{H}^1(\Gamma, \mathbb{F}_p) \cong \mathbb{F}_p$, generated by $(x_1, x_2) \mapsto (0, 1)$. Let

$$\begin{array}{cccc} f & : & \mathbb{Z} & \longrightarrow & \Gamma \\ & & 1 & \longmapsto & x_2 \end{array}.$$

Then $f_1^* : \mathrm{H}^1(\Gamma, \mathbb{F}_p) \to \mathrm{H}^1(\mathbb{Z}, \mathbb{F}_p)$ is an isomorphism and $f_2^* : \mathrm{H}^2(\Gamma, \mathbb{F}_p) \to \mathrm{H}^2(\mathbb{Z}, \mathbb{F}_p)$ is an injection, so $\widehat{\Gamma_{(p)}} \cong \mathbb{Z}_p$.

5.4.2 Presentations of pro-p groups

Definition 5.4.22. Let X be a finite set and F the free pro-p group on the set X. Let $R \subseteq F$. The pro-p group with presentation $G = \lfloor X \mid R \rfloor_p$ is defined to be $G = F / \overline{\langle\langle R \rangle\rangle}$.

Lemma 5.4.23. *Let $F' \subseteq \widehat{F_{(p)}} = F$ be the free abstract group on X, and $R \subseteq F'$. Let $\Gamma = \langle X \mid R \rangle$ be an abstract group, and let $G = \lfloor X \mid R \rfloor_p$. Then $G = \widehat{\Gamma_{(p)}}$.*

Proof. Since $F' \xrightarrow{\subseteq} F$ such that $\langle\langle R \rangle\rangle \to \overline{\langle\langle R \rangle\rangle}$, $F' / \langle\langle R \rangle\rangle \to \widehat{\Gamma_{(p)}} \twoheadrightarrow G$. There exists $F \to \widehat{\Gamma_{(p)}}$ sending $X \to X$ such that $R \to 1$, so $G \twoheadrightarrow \widehat{\Gamma_{(p)}}$. By the Hopf property, surjections are isomorphisms. ☐

How many generators does a pro-p group need? Since

$$\mathrm{H}^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p) \cong \mathrm{Hom}(G/\Phi(G), \mathbb{F}_p) \cong \mathrm{Hom}\left(\mathbb{F}_p^d, \mathbb{F}_p\right),$$

the minimal number of generators is $\dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$.

Theorem 5.4.24. *Let G be a topologically finitely generated pro-p group, and let X be a finite topological generating set for G, so $\mathrm{F}(X) \twoheadrightarrow G$. Let r_X be the minimal size of a set $R \subseteq \mathrm{F}(X)$ such that $G = \lfloor X \mid R \rfloor_p$. Then*

$$|X| - \mathrm{r}_X = \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p).$$

Remark 5.4.25.

- Not true for abstract groups, since $|X| - \rho_X$ is not a constant for abstract groups, where ρ_X is the minimal size of R such that $\Gamma = \langle X \mid R \rangle$, and different X's need different numbers of relators.
- If G is a finite p-group, then $\mathrm{r}_X \leq \rho_X$. Not known if equality holds.

Lemma 5.4.26 (Non-examinable)**.** *Let F be a pro-p group and $N \triangleleft_{\mathrm{c}} F$. Then there exists $R \subseteq F$ of size r such that $N = \overline{\langle\langle R \rangle\rangle}$ if and only if $\dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} \leq r$.*

Proof. If $N = \overline{\langle\langle R \rangle\rangle}$, then any ϕ is determined by $\phi(r)$ for $r \in R$. Then there is an injection $\mathrm{H}^1(N, \mathbb{F}_p)^{F/N} \hookrightarrow \mathbb{F}_p^{|R|}$, so $\dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} \leq |R|$. Suppose $\dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} = r$, so there exists $R \subseteq N$ such that an F-invariant $\phi : N \to \mathbb{F}_p$ vanishes if and only if ϕ vanishes on R. Let $N' = \overline{\langle\langle R \rangle\rangle} \subseteq N$. Suppose $N' \neq N$. Then $N'\Phi(N) \neq N$, so $M = N/N'\Phi(N)$ is a non-trivial abelian pro-p group, and F acts by conjugation. So [5] there exists an F-invariant $M \twoheadrightarrow \mathbb{F}_p$, since the only simple F-module is $\mathbb{F}_p$, a contradiction. ☐

Proof of Theorem 5.4.24. By Lemma 5.4.26, $\mathrm{r}_X = \dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N}$. By the five-term exact sequence,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(G, \mathbb{F}_p) & \longrightarrow & \mathrm{H}^1(F, \mathbb{F}_p) & \xrightarrow{\alpha} & \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} & \xrightarrow{\beta} & \mathrm{H}^2(G, \mathbb{F}_p) & \longrightarrow & \mathrm{H}^2(F, \mathbb{F}_p) \\ & & & & \cong & & & & & & \cong \\ & & & & \mathbb{F}_p^{|X|} & & & & & & 0 \end{array},$$

where $\dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} = \mathrm{r}_X$. Then

$$\dim_{\mathbb{F}_p} \operatorname{im} \alpha = \dim_{\mathbb{F}_p} \mathrm{H}^1(F, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p) = |X| - \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p),$$

and

$$\dim_{\mathbb{F}_p} \ker \beta = \dim_{\mathbb{F}_p} \mathrm{H}^1(N, \mathbb{F}_p)^{F/N} - \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p) = \mathrm{r}_X - \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p).$$

Thus

$$|X| - \mathrm{r}_X = \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p).$$

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[5] Exercise: pure topology