

# Equivalent Potential Temperature

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RAF Algorithm Review

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# What Is Equivalent Potential Temperature?

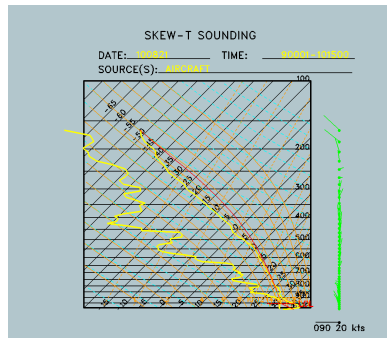
## Rossby Form

- $L_V$  and  $c_{pd}$  are kept constant.
- Basis for thermodynamic diagrams

$$\Theta_p^{[Rossby]} = \Theta_d \exp \left\{ \frac{L_V r}{c_{pd} T} \right\}$$

## Revised Forms Like Bolton

- Take into account the temperature dependence of  $L_V$  and  $c_{pd}$
- Often adjust  $L_V$  to minimize errors
- New: Davies-Jones, 2009



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$$\Theta_p^{[Rossby]} = \Theta_d \exp \left\{ \frac{L_v r}{c_{pd} T} \right\}$$

Davies-Jones (2009):

$$\Theta_E^* = \Theta e^{\left\{ \frac{[L_0^* - L_1^* (T_L - T_0)] r + K_2 r^2}{c_{pd} T_L} \right\}}$$

where  $L_0^*$ ,  $L_1^*$ , and  $K_2$  are coefficients that are adjusted to minimize errors.

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# Terminology

## Wet vs Dry

- wet-equivalent: carry all water with parcel ( $\Theta_q$ )
- pseudo-adiabatic equivalent: all water is removed as it condenses

## "Pseudo-adiabatic"

- Preferred terminology for all water removed when condensed
- Equivalent to neglecting the specific heat of liquid water

## AMS Glossary: Equivalent Potential Temperature

"A thermodynamic quantity, with its natural logarithm proportional to the entropy of moist air, that is conserved in a reversible moist adiabatic process. "

## Usage Here:

- pseudo-adiabatic equivalent
- $\Theta_p$  instead of  $\Theta_e$
- $\Theta_q$  for wet-equivalent

# Equations

$$\Theta_q = T \left( \frac{p_0}{p_d} \right)^{R_d/c_{pt}} \exp \left( \frac{L_v r}{c_{pt} T} \right) \quad (1)$$

$$\frac{dT}{dp_d} = \frac{TR_d + L_v r}{p_d} \left[ (c_{pd} + r_t c_w) + \frac{T \varepsilon}{p_d} \left( \frac{\partial \left( \frac{L_v e_s(T)}{T} \right)}{\partial T} \right)_{p_d} \right]^{-1} \quad (2)$$

- Quantities in red vary with temperature.
- Equation (1) is a straightforward definition if  $L_v$  and  $c_{pd}$  (entering  $c_{pt} = c_{pd} + r_t c_w$ ) are taken at the level of the LCL

# Equations

$$\Theta_q = T \left( \frac{p_0}{p_d} \right)^{R_d/c_{pt}} \exp \left( \frac{L_v r}{c_{pt} T} \right) \quad (1)$$

- Bolton: If  $\Theta_D$  is the dry-air potential temperature at the LCL,  $e$  the vapor pressure in mb,  $T_K$  the air temperature in kelvin,  $T_L$  the temperature at the LCL in kelvin and  $r$  the mixing ratio

$$T_L = \frac{2840}{3.5 \ln T_K - \ln e - 4.805} + 55$$

$$\Theta_p^{Bolton} = \Theta_D \exp \left\{ \left( \frac{3.376}{T_L} - 0.00254 \right) r (1 + 0.81 \times 10^{-3} r) \right\}$$

# Reasons For Proposing Changes

- 1 Bolton offered a second formula in his 1980 paper that is more accurate.
- 2 Davies-Jones repeated the analysis and obtained still better fit coefficients investigated over a wider numerical range.
- 3 If we change to the Davies-Jones formula, it may be useful to change the variable name to “pseudo-adiabatic equivalent potential temperature” at the same time to remove the conflict with the AMS definition.

# Three Associated Analyses

- ① Do revised vapor-pressure equations matter?
  - Davies-Jones (2009) still uses an approximate formula instead of a more accurate representation of  $e_s(T)$ .
  - Expect small effect because differences are at low temperature where vapor pressure is low.
- ② What is the effect of including the temperature dependence of the specific heats, especially for supercooled water?
  - Neglected in previous studies, although Murphy and Koop show that variation in  $c_w$  is particularly significant.
  - Applies to both  $\Theta_p$  and  $\Theta_q$ .
- ③ How accurate is the Bolton formula for the temperature at the LCL?
  - How much error is introduced?
  - Do we need to use a numerical solution to obtain better accuracy?



# The Approaches Used

- For all three questions, the approach was to compare solutions from equations to numerical solutions
  - Include new vapor pressure formulas
  - Allow specific heats and the latent heat of vaporization to vary with temperature
- Example: Adiabatic motion from in initial point with  $(T_1, p_1)$  to a new point with  $(T_2, p_2)$ : Given  $\{T_1, p_1, p_2\}$ , find  $T_2$  two ways:
  - 1 Integrate the exact equation for  $dT/dp_d$  from point 1 to point 2 to find  $T_2$ .
  - 2 Evaluate the equation for potential temperature at point 1, then invert it at point 2 to find  $T_2$ .

# Example: Check Davies-Jones Equation

## Method 1: Integration

- 1 Initialize: at  $\{p_1, T_1\}$
- 2 Each  $\Delta p$ , calculate  $\Delta T = \frac{dT}{dp_d} \Delta p_d$  from (3)
- 3 At  $\{p_2, T_2\}$ , find  $T_2$ .

$$\frac{dT}{dp_d} = \left( \frac{TR_d + L_v r}{p_d} \right) / \left[ (c_{pd} + r_t c_w) + \frac{T \epsilon}{P_d} \left( \frac{\partial \left( \frac{L_v e_s(T)}{T} \right)}{\partial T} \right)_{p_d} \right] \quad (3)$$

## Method 2: Inversion

- 1 Evaluate Davies-Jones  $\Theta_p$  at  $\{p_1, T_1\}$  from (4)
- 2 Set  $p$  in (4) to  $p_2$
- 3 With  $\{\Theta_p, p_2\}$  fixed, iterate to find  $T_2$

# Example: Check Davies-Jones Equation

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## Method 2: Inversion

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- 2 Set  $p$  in (4) to  $p_2$
- 3 With  $\{\Theta_p, p_2\}$  fixed, iterate to find  $T_2$

$$\Theta_p^{[DJ]} = \Theta_{DL} e^{\left\{ \frac{(L_0^* - L_1^*(T_L - T_0) + K_2 r)r}{c_{pd} T_L} \right\}} \quad (4)$$

$$\Theta_{DL} = T \left( \frac{p_0}{p - e} \right)^{2/7} \left( \frac{T}{T_L} \right)^{0.28 \times 10^{-3} r}$$

for iteration:

$$T_2' = \Theta_p^{[DJ]}(T_1, p_1) \left( \frac{T_2}{\Theta_p^{[DF]}(T_2, p_2)} \right)$$

# Results

Primed quantities result from numerical integration; unprimed from formula evaluation

T [°C]	p [hPa]	final $p_2$ [hPa]	$T_q$ [K]	$T'_q$ [K]	$T_p$ [K]	$T'_p$ [K]
25	850	100	206.50	207.69	200.73	200.76
15	750	100	189.34	189.73	185.39	185.42
10	750	100	178.69	178.71	175.66	175.69
0	700	100	167.05	166.90	165.39	165.41
-10	600	100	163.54	163.44	162.68	162.70
25	850	300	222.59	222.69	222.27	222.29
10	750	300	239.43	239.46	238.52	238.51
0	700	300	226.46	226.59	225.82	225.84

Example: 1.19 K error in  $\Theta_q$ , 0.03 K in  $\Theta_p$

# Results

Primed quantities result from numerical integration; unprimed from formula evaluation

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Largest error in  $\Theta_p$ : 0.03 K

# Results

Primed quantities result from numerical integration; unprimed from formula evaluation

T [°C]	p [hPa]	final $p_2$ [hPa]	$T_q$ [K]	$T'_q$ [K]	$T_p$ [K]	$T'_p$ [K]
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Error in  $\Theta_q$  larger when extended to low pressure

# Answers to First Two Questions

- 1 Do revised vapor-pressure equations matter?

NO: Have used Murphy-Koop for numerical integration but Davies-Jones equation based on an old formula for iterative solutions, and found good agreement for  $\Theta_p$ . For  $\Theta_q$ , differences seen were not caused by differences in vapor-pressure formulas used.

- 2 What is the effect of including the temperature dependence of the specific heats, especially for supercooled water?

NEGLIGIBLE FOR  $\Theta_p$ : Differences were typically smaller than 0.03 K for a range spanning typical conditions likely to be encountered in research flights.

POTENTIALLY WORTH ATTENTION FOR  $\Theta_q$ : Differences can exceed 1K but are more typically 0.1 K.

## Effect of LCL Formula?

- How accurate is the Bolton formula for the temperature at the LCL?
  - How much error is introduced?
  - Do we need to use a numerical solution to obtain better accuracy?

FORMULA IS GOOD: Numerical tests showed that results were within about 0.05 K and in most cases were better than this limit, suggesting that little error is introduced by using the Bolton formula.



# Recommendations

1. Change to (6.5) of Davies-Jones (2009), and change the variable name to “pseudo-adiabatic equivalent potential temperature”. Continue to use (21) of Bolton (1980) to determine the saturation temperature  $T_L$ .

$$\Theta_p^{[DJ]} = \Theta_{DL} \exp \left\{ \frac{(L_0^* - L_1^*(T_L - T_0) + K_2 r) r}{c_{pd} T_L} \right\}$$

$$\Theta_{DL} = T_k \left( \frac{1000}{p_d} \right)^{0.2854} \left( \frac{T_K}{T_L} \right)^{0.28 \times 10^{-3} r}$$

$$T_L = \frac{2840}{3.5 \ln T_K - \ln e - 4.805} + 55$$

## Recommendations

2. Add a new variable “wet-equivalent potential temperature” and use the standard equation for its evaluation. .

$$\Theta_q = T \left( \frac{p_0}{p_d} \right)^{R_d/c_{pt}} \exp \left( \frac{L_v r}{c_{pt} T} \right)$$

where  $c_{pt} = c_{pd} + r_t c_w$  and  $r_{tot}$  is the total water mixing ratio,  $r_{tot} = r + r_w$  where  $r_w = \chi/\rho_d$  with  $\chi$  the liquid water content and  $\rho_d$  the density of dry air:  $\rho_d = (p - e)/(R_d T)$ .