Proposta de resolução 2º teste

1

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x^2 + 2y^2 - 2x - 4y + 1$$

Pontos críticos

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - 2 = 0 \\ 4y - 4 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$(1,1)$ é o único ponto crítico

Matriz Hesseana: $H(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Os valores próprios da matriz Hessiana em $(1,1)$ são

2 e 4,

logo positivos. Assim sendo $(1,1)$ é ponto de máximo local de f e

$$f(1,1) = 1^2 + 2\times 1^2 - 2\times 1 - 4\times 1 + 1 = -2$$

é mínimo local de f.

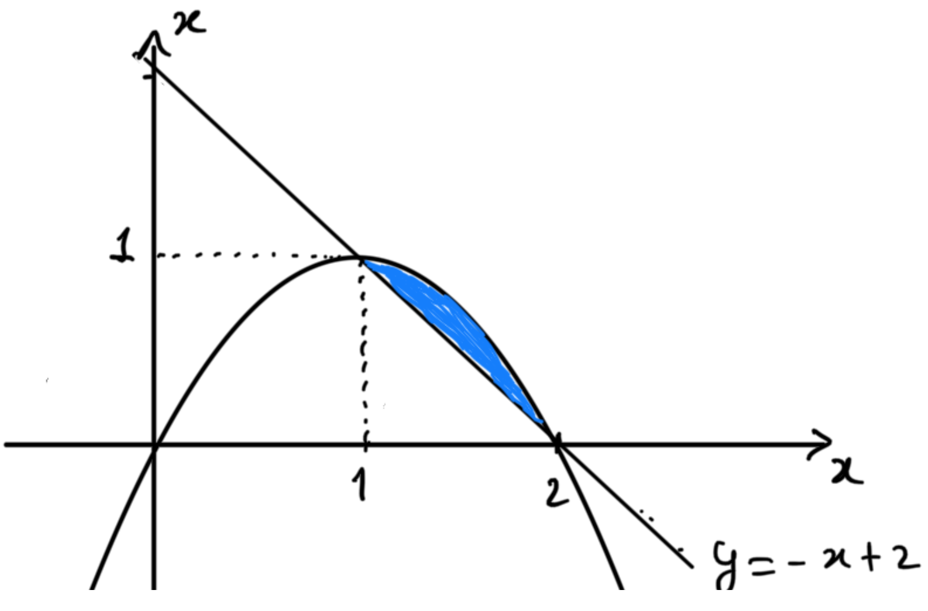
2

$$\int_1^2 \int_{-x+2}^{-x^2+2x} x \, dy \, dx$$

a) Região de integração:

$$R = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, -x+2 \leq y \leq -x^2+2x\}$$

Esboço da região:



$y=-x^2+2x$

$$\int_1^2 \int_{-x+2}^{-x^2+2x} x \, dy\, dx = \int_0^1 \int_{-y+2}^{1+\sqrt{1-y}} x \, dx\, dy.$$

$$y=-x^2+2x \Leftrightarrow x^2-2x+y=0 \Leftrightarrow x = \frac{2 \pm \sqrt{4-4y}}{2}$$

$$\Leftrightarrow x = 1 \pm \sqrt{1-y} \underset{x \geq 1}{\Leftrightarrow} x = 1+\sqrt{1-y}$$

e)
$$\int_1^2 \int_{-x+2}^{-x^2+2x} x\, dy\, dx = \int_1^2 \left[xy \right]_{y=-x+2}^{-x^2+2x} dx$$

$$= \int_1^2 x(-x^2+2x) - x(-x+2)\, dx = \int_1^2 -x^3+2x^2+x^2-2x\, dx$$

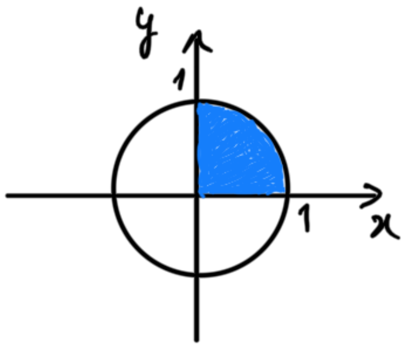
$$= \int_1^2 -x^3+3x^2-2x\, dx = \left[-\frac{x^4}{4}+x^3-x^2 \right]_1^2$$

$$= -\frac{2^4}{4}+2^3-2^2-\left(-\frac{1^4}{4}+1^3-1^2\right) = -4+8-4+\frac{1}{4} = \frac{1}{4}$$

$\boxed{3}$

$$\iint_R f(x,y)\, d(x,y)$$

$$R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1,\ x \geq 0, y \geq 0\} \text{ e } f(x,y) = \frac{1}{x^2+y^2}$$



$$\iint_R \frac{1}{x^2+y^2}\, d(x,y) = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{r^2} \cdot r\, dr\, d\theta$$

$\boxed{4}$

$$\int_0^1 \int_{x^2}^{x} \int_0^{x+y} x\, dz\, dy\, dx = \int_0^1 \int_{x^2}^{x} \left[xz \right]_{z=0}^{x+y} dy\, dz$$

$$\int_0^1 \int^{x} x(x+y)\, dy\, dz = \int_0^1 \left[x^2 y + x y^2 \right]^{x} dx$$

$$= \int_0 \int_{x^2} x(x+y)\, dy\, dx = \int_0 \left[x^2 y + \frac{x y^2}{2} \right]_{y=x^2}$$

$$= \int_0^1 x^3 + \frac{x^3}{2} - x^4 - \frac{x^5}{2}\, dx = \int_0^1 -\frac{x^5}{2} - x^4 + \frac{3}{2}x^3\, dx$$

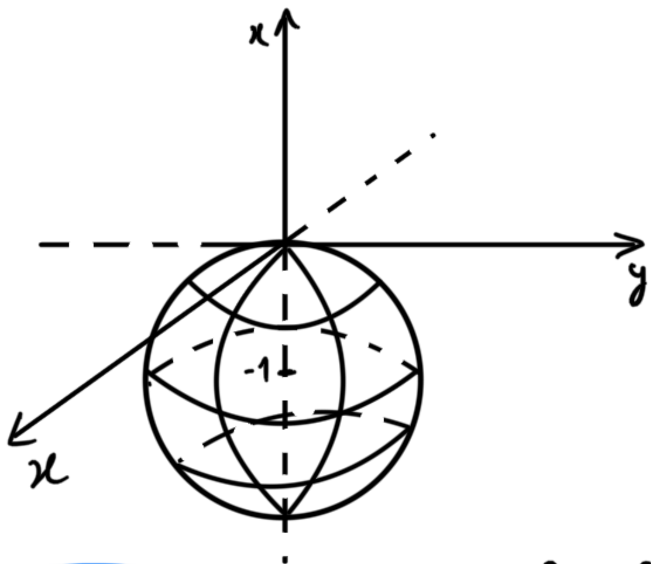
$$= \left[-\frac{x^6}{12} - \frac{x^5}{5} + \frac{3}{8}x^4 \right]_0^1 = -\frac{1}{12} - \frac{1}{5} + \frac{3}{8} = -\frac{2}{24} + \frac{9}{24} - \frac{1}{5}$$

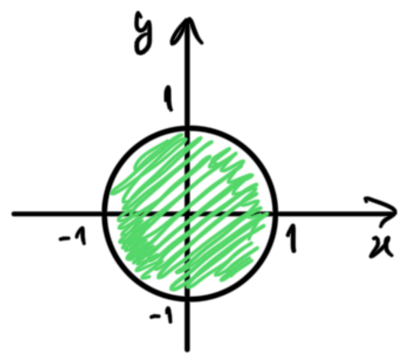
$$= \frac{7}{24} - \frac{1}{5} = \frac{35}{120} - \frac{24}{120} = \frac{11}{120}$$

$\boxed{5}$ $S = \{(x,y,z) \in \mathbb{R}^3 : z \leq x^2 + y^2,\ x^2 + y^2 \leq 1,\ z \geq 0\}$

$$\text{Volume}(S) = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r\, dz\, dr\, d\theta.$$

$\boxed{6}$ $Q = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z+1)^2 \leq 1\}$





$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$x^2 + y^2 + (z+1)^2 = 1 \Leftrightarrow x^2 + y^2 + z^2 + 2z + 1 = 1$$

$$\rho^2 + 2\rho\cos\varphi = 0 \Leftrightarrow \rho = 0 \text{ ou } \rho = -2\cos\varphi$$

$$\iiint_Q \frac{1}{x^2+y^2+z^2}\, d(x,y,z) = \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^{-2\cos\varphi} \frac{1}{\rho^2} \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\frac{\pi}{2}, 0$

$$= \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{-2\cos\varphi} \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_{\pi/2}^{\pi} \left[\rho \operatorname{sen}\varphi\right]_{\rho=0}^{-2\cos\varphi} d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/2}^{\pi} -2\operatorname{sen}\varphi\cos\varphi \, d\varphi \, d\theta = \int_0^{2\pi}\int_{\pi/2}^{\pi} -\operatorname{sen}(2\varphi) \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\cos(2\varphi)}{2}\right]_{\pi/2}^{\pi} d\theta = \int_0^{2\pi} \frac{\cos(2\pi)}{2} - \frac{\cos\pi}{2} \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$$