Universidade do Minho
Departamento de Matemática

Licenciatura em Ciências da Computação
Cálculo
2023/24

Primitivas

1. Comecemos por calcular $\displaystyle\int x^2 \operatorname{sen} x \, dx$.

   Sejam

   $$\begin{array}{ll} f'(x) = \operatorname{sen} x & f(x) = -\cos x \\ g(x) = x^2 & g'(x) = 2x \end{array}$$

   Aplicando o método de primitivação por partes, obtemos
   $$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x - \int -2x \cos x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$
   Vamos aplicar de novo o método de primitivação por partes para calcular $\int 2x \cos x \, dx$.
   Sejam

   $$\begin{array}{ll} f'(x) = \cos x & f(x) = \operatorname{sen} x \\ g(x) = 2x & g'(x) = 2 \end{array}$$

   Aplicando o método de primitivação por partes, obtemos
   $$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + 2\, x \operatorname{sen} x - \int 2 \operatorname{sen} x \, dx = -x^2 \cos x + 2\, x \operatorname{sen} x + 2\, \cos x + C, \qquad C \in \mathbb{R}.$$
   Vamos agora determinar o valor da constante C de modo a encontrar a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$. Tem-se que:

   $$-\frac{\pi^2}{4} \cos \frac{\pi}{2} + \pi \operatorname{sen} \frac{\pi}{2} + 2 \cos \frac{\pi}{2} + C = \pi \;\Leftrightarrow\; C = 0.$$

   Consequentemente, a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$ é

   $$F(x) = -x^2 \cos x + 2\, x \operatorname{sen} x + 2 \cos x.$$

2. (a) $f(x) = \dfrac{2x^3}{3} - \dfrac{x^2}{2} - 8x + \dfrac{65}{6}$.

   (b) $f(x) = \dfrac{5x}{4} - \dfrac{1}{8} \operatorname{sen}(2x)$.

3. [Primitivas imediatas]

(1) $$\int (\sqrt{x}+2)^2\,dx = \int (x+4\sqrt{x}+4)\,dx = \int x\,dx + \int 4\,x^{1/2}\,dx + \int 4\,dx$$
$$= \frac{x^2}{2} + 4\,\frac{x^{1/2+1}}{1/2+1} + 4x + C$$
$$= \frac{x^2}{2} + \frac{8}{3}\,x^{3/2} + 4x + C, \quad C \in \mathbb{R}$$

(2) $$\int (3x^2 - 2x^5)\,dx = x^3 - \frac{x^6}{3} + C, \quad C \in \mathbb{R}$$

(3) $$\int (2x+10)^{20}\,dx = \frac{1}{2}\int 2\,(2x+10)^{20}\,dx = \frac{1}{2}\frac{(2x+10)^{21}}{21} + C = \frac{(2x+10)^{21}}{42} + C, \quad C \in \mathbb{R}$$

(4) $$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int 3\,x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C, \quad C \in \mathbb{R}$$

(5) $$\int x^4(x^5+10)^9\,dx = \frac{1}{5}\int 5\,x^4(x^5+10)^9\,dx = \frac{1}{5}\frac{(x^5+10)^{10}}{10} + C = \frac{(x^5+10)^{10}}{50} + C, \quad C \in \mathbb{R}$$

(6) $$\int \frac{2x+1}{x^2+x+3}\,dx = \ln(x^2+x+3) + C, \quad C \in \mathbb{R}$$

(7) $$\int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx = \frac{1}{2}\int 2\,(2x+1)^{1/2}\,dx = \frac{1}{2}\frac{(2x+1)^{3/2}}{3/2} + C$$
$$= \frac{1}{3}(2x+1)^{3/2} + C, \quad C \in \mathbb{R}$$

(8) $$\int \frac{x}{3-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{3-x^2}\,dx = -\frac{1}{2}\,\ln|3-x^2| + C, \quad C \in \mathbb{R}$$

(9) $$\int \frac{1}{4-3\,x}\,dx = -\frac{1}{3}\int \frac{-3}{4-3x}\,dx = -\frac{1}{3}\,\ln|4-3x| + C, \quad C \in \mathbb{R}$$

$$(10)\quad \int \frac{1}{e^{3x}}\,dx = \int e^{-3x}\,dx = -\frac{1}{3}\int -3\,e^{-3x}\,dx = -\frac{1}{3}\,e^{-3x} + C, \quad C \in \mathbb{R}$$

$$(11)\quad \int \frac{-7}{\sqrt{1-5x}}\,dx = \int -7\,(1-5x)^{-1/2}\,dx = \frac{-7}{-5}\int -5\,(1-5x)^{-1/2}\,dx$$
$$= \frac{7}{5}\,\frac{(1-5x)^{1/2}}{1/2} + C = \frac{14}{5}\,(1-5x)^{1/2} + C, \quad C \in \mathbb{R}$$

$$(12)\quad \int \frac{\sqrt{1+3\,\ln x}}{x}\,dx = \int \frac{1}{x}(1+3\,\ln x)^{1/2}\,dx = \frac{1}{3}\int \frac{3}{x}(1+3\,\ln x)^{1/2}\,dx$$
$$= \frac{1}{3}\,\frac{(1+3\,\ln x)^{3/2}}{3/2} + C = \frac{2}{9}\,(1+3\,\ln x)^{3/2} + C, \quad C \in \mathbb{R}$$

$$(13)\quad \int x\,\mathrm{sen}\,(x^2)\,dx = \frac{1}{2}\int 2x\,\mathrm{sen}\,(x^2)\,dx = -\frac{1}{2}\,\cos(x^2) + C, \quad C \in \mathbb{R}$$

$$(14)\quad \int \frac{1}{x(\ln^2 x + 1)}\,dx = \int \frac{\frac{1}{x}}{1+\ln^2 x}\,dx = \mathrm{arctg}\,(\ln x) + C, \quad C \in \mathbb{R}$$

$$(15)\quad \int \left(\frac{2}{x}-3\right)^2 \frac{1}{x^2}\,dx = -\frac{1}{2}\int -\frac{2}{x^2}\left(\frac{2}{x}-3\right)^2\,dx$$
$$= -\frac{1}{2}\,\frac{\left(\frac{2}{x}-3\right)^3}{3} + C = -\frac{1}{6}\left(\frac{2}{x}-3\right)^3 + C, \quad C \in \mathbb{R}$$

$$(16)\quad \int \mathrm{sen}\,(\pi - 2\,x)\,dx = -\frac{1}{2}\int -2\,\mathrm{sen}\,(\pi - 2\,x)\,dx = \frac{1}{2}\,\cos(\pi - 2\,x) + C, \quad C \in \mathbb{R}$$

$$(17)\quad \int \mathrm{th}\,x\,dx = \int \frac{\mathrm{sh}\,x}{\mathrm{ch}\,x}\,dx = \ln(\mathrm{ch}\,x) + C, \quad C \in \mathbb{R}$$

$$(18)\quad \int \mathrm{sen}\,x\,\cos x\,dx = \frac{\mathrm{sen}^2 x}{2} + C, \quad C \in \mathbb{R}$$

$$(19) \quad \int \operatorname{sen}(2x)\, \cos x \, dx = \int 2 \operatorname{sen} x \cos^2 x \, dx = -2 \int -\operatorname{sen} x \cos^2 x \, dx = \frac{-2\cos^3 x}{3} + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}(20) \quad \int \operatorname{sen}^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) dx &= \int \left(\frac{1-\cos x}{2}\right)\left(\frac{1+\cos x}{2}\right) dx = \frac{1}{4}\int (1-\cos x)(1+\cos x)\, dx \\ &= \frac{1}{4}\int (1-\cos^2 x)\, dx = \frac{1}{4}\int \operatorname{sen}^2 dx = \frac{1}{4}\int \frac{1-\cos(2x)}{2}\, dx \\ &= \frac{1}{8}\int (1-\cos(2x))\, dx = \frac{1}{8}\int 1\, dx - \frac{1}{16}\int 2\, \cos(2x)\, dx \\ &= \frac{1}{8}\, x - \frac{1}{16}\operatorname{sen}(2x) + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}(21) \quad \int \operatorname{sen}^2 x \, dx &= \int \frac{1-\cos(2x)}{2}\, dx = \frac{1}{2}\int 1\, dx - \frac{1}{4}\int 2\, \cos(2x)\, dx \\ &= \frac{1}{2}\, x - \frac{1}{4}\operatorname{sen}(2x) + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}(22) \quad \int \cos^3 \, dx &= \int (1-\operatorname{sen}^2 x)\cos x \, dx = \int \cos x \, dx - \int \cos x \operatorname{sen}^2 x \, dx \\ &= \operatorname{sen} x - \frac{\operatorname{sen}^3 x}{3} + C, \quad C \in \mathbb{R}\end{aligned}$$

$$(23) \quad \int \frac{x}{x^2-1}\, dx = \frac{1}{2}\int \frac{2x}{x^2-1}\, dx = \frac{1}{2}\, \ln|x^2-1| + C, \quad C \in \mathbb{R}$$

$$(24) \quad \int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int 2x\,(x^2-1)^{-1/2}\, dx = \frac{1}{2}\, \frac{(x^2-1)^{1/2}}{1/2} + C = \sqrt{x^2-1} + C, \quad C \in \mathbb{R}$$

$$(25) \quad \int \frac{1}{x}\operatorname{sen}(\ln x)\, dx = -\cos(\ln x) + C, \quad C \in \mathbb{R}$$

$$(26) \quad \int \frac{-3}{x\,(\ln x)^3}\, dx = -3\int \frac{1}{x}\, (\ln x)^{-3}\, dx = -3\, \frac{(\ln x)^{-2}}{-2} + C = \frac{3}{2\,(\ln x)^2} + C, \quad C \in \mathbb{R}$$

$$(27) \quad \int \frac{e^x}{1+e^{2x}}\, dx = \int \frac{e^x}{1+(e^x)^2}\, dx = \operatorname{arctg}(e^x) + C, \quad C \in \mathbb{R}$$

(28) $$\int \frac{e^x}{1-2e^x}\,dx = -\frac{1}{2}\int \frac{-2\,e^x}{1-2e^x}\,dx = -\frac{1}{2}\ln|1-2e^x| + C, \quad C\in\mathbb{R}$$

(29) $$\int \frac{1}{\cos^2(7x)}\,dx = \frac{1}{7}\int \frac{7}{\cos^2(7x)}\,dx = \frac{1}{7}\,\mathrm{tg}\,(7x) + C, \quad C\in\mathbb{R}$$

(30) $$\begin{aligned}\int \left(\sqrt{2x-1}-\sqrt{1+3x}\right)\,dx &= \int (2x-1)^{1/2}\,dx - \int (1+3x)^{1/2}\,dx \\ &= \frac{1}{2}\int 2\,(2x-1)^{1/2}\,dx - \frac{1}{3}\int 3\,(1+3x)^{1/2}\,dx \\ &= \frac{1}{2}\,\frac{(2x-1)^{3/2}}{3/2} - \frac{1}{3}\,\frac{(1+3x)^{3/2}}{3/2} + C \\ &= \frac{1}{3}\,(2x-1)^{3/2} - \frac{2}{9}\,(1+3x)^{3/2} + C, \quad C\in\mathbb{R}\end{aligned}$$

(31) $$\begin{aligned}\int \frac{1}{x}\,(1+(\ln x)^2)\,dx &= \int \frac{1}{x}\,dx + \int \frac{1}{x}(\ln x)^2\,dx \\ &= \ln x + \frac{(\ln x)^3}{3} + C, \quad C\in\mathbb{R}\end{aligned}$$

(32) $$\begin{aligned}\int \frac{2\,+\,\sqrt{\mathrm{arctg}\,(2x)}}{1+4x^2}\,dx &= \int \frac{2}{1+(2x)^2}\,dx + \int \frac{1}{1+(2x)^2}(\mathrm{arctg}\,(2x))^{1/2}\,dx \\ &= \int \frac{2}{1+(2x)^2}\,dx + \frac{1}{2}\int \frac{2}{1+(2x)^2}(\mathrm{arctg}\,(2x))^{1/2}\,dx \\ &= \mathrm{arctg}\,(2x) + \frac{1}{2}\frac{(\mathrm{arctg}\,(2x))^{3/2}}{3/2} + C \\ &= \mathrm{arctg}\,(2x) + \frac{1}{3}(\mathrm{arctg}\,(2x))^{3/2} + C, \quad C\in\mathbb{R}\end{aligned}$$

(33) $$\int \frac{e^{\mathrm{arctg}\,x}}{1+x^2}\,dx = e^{\mathrm{arctg}\,x} + C, \quad C\in\mathbb{R}$$

(34) $$\begin{aligned}\int \frac{\mathrm{sen}\,x}{\sqrt{1+\cos x}}\,dx &= \int \mathrm{sen}\,x\,(1+\cos x)^{-1/2}\,dx = -\int -\mathrm{sen}\,x\,(1+\cos x)^{-1/2}\,dx \\ &= -\frac{(1+\cos x)^{1/2}}{1/2} + C = -2\sqrt{1+\cos x} + C, \quad C\in\mathbb{R}\end{aligned}$$

4. [Primitivação por partes]

(1) Calcule $\int \ln x \, dx$.

Sejam

$$\begin{array}{ll} f'(x) = 1 & f(x) = x \\ g(x) = \ln x & g'(x) = \dfrac{1}{x} \end{array}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln\, x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C, \quad C \in \mathbb{R}.$$

(2) Calcule $\int x \,\mathrm{sen}\,(2x) \, dx$.

Sejam

$$\begin{array}{ll} f'(x) = \mathrm{sen}\,(2x) & f(x) = -\frac{1}{2}\cos(2x) \\ g(x) = x & g'(x) = 1 \end{array}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int x \,\mathrm{sen}\,(2x) \, dx &= -\frac{1}{2} x \cos(2x) - \int -\frac{1}{2} \cos(2x).1 \, dx = -\frac{1}{2} x \cos(2x) + \frac{1}{2} \int \cos(2x) \, dx \\ &= -\frac{1}{2} x \cos(2x) + \frac{1}{4} \int 2 \cos(2x) \, dx = -\frac{1}{2} x \cos(2x) + \frac{1}{4} \mathrm{sen}\,(2x) + C, \quad C \in \mathbb{R}. \end{aligned}$$

(3) Calcule $\int \mathrm{arctg}\, x \, dx$.

Sejam

$$\begin{array}{ll} f'(x) = 1 & f(x) = x \\ g(x) = \mathrm{arctg}\, x & g'(x) = \dfrac{1}{1+x^2} \end{array}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int \ln\, x \, dx &= x \,\mathrm{arctg}\, x - \int \frac{x}{1+x^2} \, dx = x \,\mathrm{arctg}\, x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\ &= x \,\mathrm{arctg}\, x - \frac{1}{2} \ln(1+x^2) + C, \quad C \in \mathbb{R}. \end{aligned}$$

(4) Calcule $\int x \cos\, x \, dx$.

Sejam

$$\begin{array}{ll} f'(x) = \cos x & f(x) = \mathrm{sen}\, x \\ g(x) = x & g'(x) = 1 \end{array}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x \cos\, x \, dx = x \,\mathrm{sen}\, x - \int \mathrm{sen}\, x \, dx = x \,\mathrm{sen}\, x + \cos x + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\displaystyle\int \ln(1-x)\,dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$
$$g(x) = \ln(1-x) \qquad g'(x) = \frac{-1}{1-x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int \ln(1-x)\,dx &= x\,\ln(1-x) - \int \frac{-x}{1-x}\,dx = x\,\ln(1-x) - \int \frac{1-x-1}{1-x}\,dx \\
&= x\,\ln(1-x) - \int 1\,dx + \int \frac{1}{1-x}\,dx \\
&= x\,\ln(1-x) - \int 1\,dx - \int \frac{-1}{1-x}\,dx \\
&= x\,\ln(1-x) - x - \ln(1-x)\,dx + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(6) Calcule $\displaystyle\int x\,\ln\,x\,dx$.

Sejam

$$f'(x) = x \qquad f(x) = \tfrac{x^2}{2}$$
$$g(x) = \ln x \qquad g'(x) = \tfrac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int x\,\ln\,x\,dx &= \frac{x^2}{2}\,\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2}{2}\,\ln x - \int \frac{x}{2}\,dx \\
&= \frac{x^2}{2}\,\ln x - \frac{x^2}{4} + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(7) Calcule $\displaystyle\int x^2 \mathrm{sen}\,x\,dx$.

Sejam

$$f'(x) = \mathrm{sen}\,x \qquad f(x) = -\cos x$$
$$g(x) = x^2 \qquad g'(x) = 2x$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x^2 \mathrm{sen}\,x\,dx = -x^2\,\cos x - \int (-\cos x) 2x\,dx = -x^2\,\cos x + 2\int x\,\cos x\,dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\displaystyle\int x\,\cos x\,dx$.

Sejam

$$f'(x) = \cos x \qquad f(x) = \mathrm{sen}\,x$$
$$g(x) = x \qquad g'(x) = 1$$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int x^2 \operatorname{sen} x\, dx &= -x^2 \cos x + 2 \int x \cos x\, dx \\
&= -x^2 \cos x + 2 \left(x \operatorname{sen} x - \int \operatorname{sen} x\, dx \right) \\
&= -x^2 \cos x + 2\, x \operatorname{sen} x - 2 \int \operatorname{sen} x\, dx \\
&= -x^2 \cos x + 2\, x \operatorname{sen} x + 2 \cos x + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(8) Calcule $\displaystyle\int x \operatorname{sen} x \cos x\, dx$.

Sejam

$$f'(x) = \operatorname{sen} x \cos x \qquad f(x) = \frac{\operatorname{sen}^2 x}{2}$$

$$g(x) = x \qquad g'(x) = 1$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int x \operatorname{sen} x \cos x\, dx &= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{\operatorname{sen}^2 x}{2}\, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \frac{1}{2} \int \frac{1 - \cos(2x)}{2}\, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4}\, dx + \frac{1}{4} \int \cos(2x)\, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4}\, dx + \frac{1}{8} \int 2 \cos(2x)\, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \frac{x}{4} + \frac{1}{8} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(9) Calcule $\displaystyle\int \ln^2 x\, dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$

$$g(x) = \ln^2 x \qquad g'(x) = 2 \frac{\ln x}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln^2 x\, dx = x \ln^2 x - \int 2x \frac{\ln x}{x}\, dx = x \ln^2 x - 2 \int \ln x\, dx.$$

Pelo exercício 4.(1) temos que: $\displaystyle\int \ln x\, dx = x \ln x - x + C, \quad C \in \mathbb{R}$. Então, $\displaystyle\int \ln^2 x\, dx = x \ln^2 x - 2x \ln x + 2x + C, \quad C \in \mathbb{R}$.

(10) Calcule $\displaystyle\int e^x \cos\, x\, dx$. Este exercício é análogo a um exemplo resolvido na aula. A solução deste exercício é:

$$\int e^x \cos\, x\, dx = \frac{e^x \cos\, x + e^x \text{sen}\, x}{2} + C, \quad C \in \mathbb{R}.$$

(11) Calcule $\displaystyle\int \operatorname{arcsen} x\, dx$.

Sejam

$$f'(x) = 1 \qquad\qquad f(x) = x$$

$$g(x) = \operatorname{arcsen} x \qquad\qquad g'(x) = \frac{1}{\sqrt{1-x^2}}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \operatorname{arcsen} x\, dx &= x \operatorname{arcsen} x - \int \frac{x}{\sqrt{1-x^2}}\, dx = x \operatorname{arcsen} x - \int x\,(1-x^2)^{-1/2}\, dx \\ &= x \operatorname{arcsen} x + \frac{1}{2}\int -2x\,(1-x^2)^{-1/2}\, dx = x \operatorname{arcsen} x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} \\ &= x \operatorname{arcsen} x + (1-x^2)^{1/2} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(12) Calcule $\displaystyle\int e^{\text{sen}\, x}\, \text{sen}\, x \cos\, x\, dx$.

Sejam

$$f'(x) = e^{\text{sen}\, x} \cos x \qquad\qquad f(x) = e^{\text{sen}\, x}$$

$$g(x) = \text{sen}\, x \qquad\qquad g'(x) = \cos x$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int e^{\text{sen}\, x}\, \text{sen}\, x \cos\, x\, dx &= e^{\text{sen}\, x}\, \text{sen}\, x - \int e^{\text{sen}\, x} \cos x\, dx \\ &= e^{\text{sen}\, x}\, \text{sen}\, x - e^{\text{sen}\, x} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(13) Calcule $\displaystyle\int \frac{\operatorname{arcsen} \sqrt{x}}{\sqrt{x}}\, dx$.

Sejam

$$f'(x) = \tfrac{1}{\sqrt{x}} = x^{-1/2} \qquad\qquad f(x) = 2\sqrt{x}$$

$$g(x) = \operatorname{arcsen} \sqrt{x} \qquad\qquad g'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \frac{\operatorname{arcsen}\sqrt{x}}{\sqrt{x}}\,dx &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} - \int \frac{1}{\sqrt{1-x}}\,dx\\ &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} - \int (1-x)^{-1/2}\,dx\\ &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} + 2\sqrt{1-x}\ + C, \quad C\in\mathbb{R}.\end{aligned}$$

(14) Calcule $\displaystyle\int x\operatorname{arctg} x\,dx$.

Sejam

$$f'(x) = x \qquad f(x) = \tfrac{x^2}{2}$$

$$g(x) = \operatorname{arctg} x \qquad g'(x) = \frac{1}{1+x^2}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x\operatorname{arctg} x\,dx &= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx\\ &= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int 1\,dx + \frac{1}{2}\int\frac{1}{1+x^2}\,dx\\ &= \frac{x^2}{2}\operatorname{arctg} x - \frac{x}{2} + \frac{1}{2}\operatorname{arctg} x + C, \quad C\in\mathbb{R}.\end{aligned}$$

(15) Calcule $\displaystyle\int x^2\,\log x\,dx$.

Sejam

$$f'(x) = x^2 \qquad f(x) = \tfrac{x^3}{3}$$

$$g(x) = \log x \qquad g'(x) = \tfrac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x^2\,\log x\,dx &= \frac{x^3}{3}\log x - \int\frac{x^2}{3}\,dx\\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C, \quad C\in\mathbb{R}.\end{aligned}$$

(16) Calcule $\displaystyle\int \operatorname{sen}(\log\, x)\,dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$

$$g(x) = \operatorname{sen}(\log\, x) \qquad g'(x) = \tfrac{1}{x}\,\cos(\log\, x)$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \operatorname{sen}(\log x)\,dx = x\operatorname{sen}(\log x) - \int \cos(\log x)\,dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\displaystyle\int \cos(\log x)\,dx$.
Sejam

$f'(x) = 1 \qquad\qquad f(x) = x$

$g(x) = \cos(\log x) \qquad\qquad g'(x) = -\frac{1}{x}\operatorname{sen}(\log x)$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \operatorname{sen}(\log x)\,dx &= x\operatorname{sen}(\log x) - [x\,\cos(\log x) - \int -\operatorname{sen}(\log x)\,dx] \\ &= x\operatorname{sen}(\log x) - x\,\cos(\log x) - \int \operatorname{sen}(\log x)\,dx.\end{aligned}$$

Então,

$$\int \operatorname{sen}(\log x)\,dx = x\operatorname{sen}(\log x) - x\,\cos(\log x) - \int \operatorname{sen}\,(\log x)\,dx\,,$$

ou, de forma equivalente,

$$2\int \operatorname{sen}(\log x)\,dx = x\operatorname{sen}(\log x) - x\,\cos(\log x)\,.$$

Consequentemente,

$$\int \operatorname{sen}(\log x)\,dx = \frac{x\operatorname{sen}(\log x) - x\cos(\log x)}{2} + C, \quad C \in \mathbb{R}.$$

(17) Calcule $\displaystyle\int \operatorname{ch} x \operatorname{sen}(3x)\,dx$.
Sejam

$f'(x) = \operatorname{ch} x \qquad\qquad f(x) = \operatorname{sh} x$

$g(x) = \operatorname{sen}(3x) \qquad\qquad g'(x) = 3\cos(3x)$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \operatorname{ch} x \operatorname{sen}(3x)\,dx = \operatorname{sh} x \operatorname{sen}(3x) - 3\int \operatorname{sh} x \cos(3x)\,dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\displaystyle\int \operatorname{sh} x \cos(3x)\,dx$.
Sejam

$f'(x) = \operatorname{sh} x \qquad\qquad f(x) = \operatorname{ch} x$

$g(x) = \cos(3x) \qquad\qquad g'(x) = -3\operatorname{sen}(3x)$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \operatorname{ch} x \operatorname{sen}(3x)\, dx &= \operatorname{sh} x \operatorname{sen}(3x) - 3[\operatorname{ch} x\, \cos(3x) - \int -3 \operatorname{ch} x \operatorname{sen}(3x)\, dx] \\ &= \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x\, \cos(3x) - 9 \int \operatorname{ch} x \operatorname{sen}(3x)\, dx.\end{aligned}$$

Então,

$$\int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x\, \cos(3x) - 9 \int \operatorname{ch} x \operatorname{sen}(3x)\, dx\,,$$

ou, de forma equivalente,

$$10 \int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x\, \cos(3x)\,.$$

Consequentemente, $\displaystyle\int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \frac{\operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x\, \cos(3x)}{10} + C, \quad C \in \mathbb{R}.$

(18) Calcule $\displaystyle\int x^3\, e^{x^2}\, dx.$

Sejam

$f'(x) = x\, e^{x^2}$ $\qquad f(x) = \frac{1}{2}\, e^{x^2}$

$g(x) = x^2$ $\qquad g'(x) = 2x$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x^3\, e^{x^2}\, dx &= \frac{1}{2}\, x^2\, e^{x^2} - \int x\, e^{x^2}\, dx \\ &= \frac{1}{2}\, x^2\, e^{x^2} - \frac{1}{2}\, e^{x^2} + C, \quad C \in \mathbb{R}.\end{aligned}$$

5. [Primitivação por substituição]

(1) Resolvido na aula.

(2) Exercício análogo a um exercício resolvida na aula.

(3) Calcule $\displaystyle\int \sqrt{4 + x^2}\, dx$, efetuando a substituição $x = 2 \operatorname{sh} t, \quad t \geq 0.$

(i) Susbtituição:

Fazendo $x = 2 \operatorname{sh} t$, $t \geq 0$, tem-se que

$$\varphi(t) = 2 \operatorname{sh} t,\ \ \varphi'(t) = 2 \operatorname{ch} t,\ \ t = \operatorname{argsh}(x/2)\,.$$

(ii) Cálculo da nova primitiva:

$$
\begin{aligned}
\int \sqrt{4+x^2}\, dx &= \int \sqrt{4+4\,\mathrm{sh}^2 t}.\, \underbrace{2\,\mathrm{ch}\, t}_{\varphi'(t)}\, dt \\
&= \int 2\sqrt{1+\mathrm{sh}^2 t}.\, 2\,\mathrm{ch}\, t\, dt = \int 4\sqrt{\mathrm{ch}^2 t}.\mathrm{ch}\, t\, dt = \int 4\,\mathrm{ch}^2 t\, dt \\
&= 4\int \frac{1+\mathrm{ch}\,(2t)}{2}\, dt = 4\left(\frac{t}{2}+\frac{1}{4}\mathrm{sh}\,(2t)\right)+C, \quad C\in\mathbb{R} \\
&= 2t+\mathrm{sh}\,(2t)+C = 2t+2\,\mathrm{sh}\, t\,\mathrm{ch}\, t+C, \quad C\in\mathbb{R}.
\end{aligned}
$$

(iii) Regresso à variável inicial x:

Atendendo a que: $\mathrm{sh}\, t=\frac{x}{2}$, $t=\mathrm{argsh}\,(x/2)$ e a que

$$\mathrm{ch}^2 t-\mathrm{sh}\, t^2=1 \;\Rightarrow\; \mathrm{ch}\, t=\sqrt{1+\mathrm{sh}^2 t} \quad (\mathrm{ch}\, t\geq 1)$$

obtemos que

$$\int \sqrt{4+x^2}\, dx = 2\,\mathrm{argsh}\,(x/2)+x\sqrt{1+\frac{x^2}{4}}+C = 2\,\mathrm{argsh}\,(x/2)+\frac{x}{2}\sqrt{4+x^2}+C, \quad C\in\mathbb{R}.$$

(4) Resolvido na aula.

(5) Calcule $\displaystyle\int \frac{x}{\sqrt{2-3x}}\, dx$, efetuando a substituição $\sqrt{2-3x}=t$.

(i) Susbtituição:

Fazendo $\sqrt{2-3x}=t$, $t>0$, tem-se que $x=\frac{2}{3}-\frac{t^2}{3}$:

$$\varphi(t)=\frac{2}{3}-\frac{t^2}{3},\;\; \varphi'(t)=-\frac{2t}{3}.$$

(ii) Cálculo da nova primitiva:

$$
\begin{aligned}
\int \frac{x}{\sqrt{2-3x}}\, dx &= \int \frac{\frac{2}{3}-\frac{t^2}{3}}{t}.\left(-\frac{2t}{3}\right) dt = \int \left(-\frac{4}{9}+\frac{2t^2}{9}\right) dt \\
&= -\frac{4t}{9}+\frac{2t^3}{27}+C, \quad C\in\mathbb{R}.
\end{aligned}
$$

(iii) Regresso à variável inicial x:

Atendendo a que $t=\sqrt{2-3x}$, obtemos que

$$\int \frac{x}{\sqrt{2-3x}}\, dx = -\frac{4}{9}\sqrt{2-3x}+\frac{2}{27}\sqrt{(2-3x)^3}+C, \quad C\in\mathbb{R}.$$

(6) Resolvido na aula.

6. [Primitivação de funções racionais]

(1) Resolvido na aula.

(2) Resolvido na aula.

(3) Resolvido na aula.

(4) Resolvido na aula.

(5) $\displaystyle\int \frac{x^2 - x + 2}{x(x^2 - 2)}\, dx = -2\ln|x| + \ln|x-1| + 2\ln|x+1| + C, \quad C \in \mathbb{R}.$

(6) $\displaystyle\int \frac{27}{x^4 - 3\,x^3}\, dx = \frac{9}{2x^2} + \frac{3}{x} - \ln|x| + \log|x-3| + C, \quad C \in \mathbb{R}.$

(7) $\displaystyle\int \frac{x+3}{(x-2)\,(x^2 - 2\,x + 5)}\, dx = \ln|x-2| - \frac{1}{2}\log|x^2 - 2x + 5| + C, \quad C \in \mathbb{R}.$

(8) $\displaystyle\int \frac{x+1}{x(x^2+1)^2}\, dx = \ln|x| + \frac{1}{2(x^2+1)} - \frac{\ln(x^2+1)}{2} + \frac{\operatorname{arctg} x}{2} + \frac{x}{2(x^2+1)} + C, \ C \in \mathbb{R}.$

7. (1) Calcule $\displaystyle\int \frac{1}{(2+\sqrt{x})^7\,\sqrt{x}} dx.$

$$\begin{aligned}
\int \frac{1}{(2+\sqrt{x})^7\,\sqrt{x}}\, dx &= \int \frac{1}{\sqrt{x}}(2+\sqrt{x})^{-7}\, dx = 2\int \frac{1}{2\sqrt{x}}(2+\sqrt{x})^{-7}\, dx \\
&= 2\frac{(2+\sqrt{x})^{-6}}{-6} + C = -\frac{1}{3(2+\sqrt{x})^6} + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(2) Calcule $\displaystyle\int \operatorname{tg}^2 x\, dx.$

$$\begin{aligned}
\int \operatorname{tg}^2 x\, dx &= \int \frac{\operatorname{sen}^2 x}{\cos^2 x}\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}\, dx \\
&= \int \frac{1}{\cos^2 x}\, dx - \int 1\, dx = \operatorname{tg} x - x + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(3) Calcule $\displaystyle\int \frac{x + \left(\operatorname{arcsen}(3x)\right)^2}{\sqrt{1 - 9\,x^2}}\, dx.$

$$\begin{aligned}
\int \frac{x + \left(\operatorname{arcsen}(3x)\right)^2}{\sqrt{1 - 9\,x^2}}\, dx &= \int x(1-9x^2)^{-1/2}\, dx + \int \frac{1}{\sqrt{1-9\,x^2}}\left(\operatorname{arcsen}(3x)\right)^2\, dx \\
&= -\frac{1}{18}\int -18x(1-9x^2)^{-1/2}\, dx + \frac{1}{3}\int \frac{3}{\sqrt{1-(3x)^2}}\left(\operatorname{arcsen}(3x)\right)^2\, dx \\
&= -\frac{1}{18}\frac{(1-9x^2)^{1/2}}{1/2} + \frac{\left(\operatorname{arcsen}(3x)\right)^3}{9} + C \\
&= -\frac{1}{9}\sqrt{1-9x^2} + \frac{\left(\operatorname{arcsen}(3x)\right)^3}{9} + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(4) Calcule $\displaystyle\int \frac{1}{1+e^{\,x}}\,dx$.

$$\int \frac{1}{1+e^{\,x}}\,dx = \int \frac{1+e^x-e^x}{1+e^{\,x}}\,dx = \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx = x - \ln(1+e^x) + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\displaystyle\int \frac{1}{\cos^{\,2} x\ \mathrm{sen}^{\,2} x}\,dx$.

$$\begin{aligned}\int \frac{1}{\cos^2 x\ \mathrm{sen}^2}\,dx &= \int \frac{\cos^2 x + \mathrm{sen}^2 x}{\cos^{\,2} x\ \mathrm{sen}^2 x}\,dx\\ &= \int \frac{1}{\mathrm{sen}^2}\,dx + \int \frac{1}{\cos^2 x}\,dx = -\mathrm{cotg}\,x + \mathrm{tg}\,x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(6) Calcule $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$, efetuando a substituição $x = 2\,\mathrm{sen}\,t$.

(i) <u>Susbtituição</u>:

Fazendo $x = 2\,\mathrm{sen}\,t$, tem-se que

$$\varphi(t) = 2\,\mathrm{sen}\,t,\ \ \varphi'(t) = 2\ \cos t,\ \ t = \mathrm{arcsen}\,(x/2),\ \ t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

(ii) <u>Cálculo da nova primitiva</u>:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,dx &= \int \frac{1}{4\,\mathrm{sen}^2 t\sqrt{4-4\mathrm{sen}^2 t}}.\ \underbrace{2\cos t}_{\varphi'(t)}\,dt\\ &= \int \frac{1}{8\,\mathrm{sen}^2 t\ \cos t}.2\cos t\,dt\\ &= \frac{1}{4}\int \frac{1}{\mathrm{sen}^2 t}\,dt = -\frac{1}{4}\mathrm{cotg}\,t + C, \quad C \in \mathbb{R}\end{aligned}$$

(iii) <u>Regresso à variável inicial x</u>:

Atendendo a que:

$$x = 2\,\mathrm{sen}\,t \quad \text{e a que} \quad 1 + \mathrm{cotg}^{\,2} t = \frac{1}{\mathrm{sen}^2 t}$$

obtém-se que

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C, \quad C \in \mathbb{R}.$$