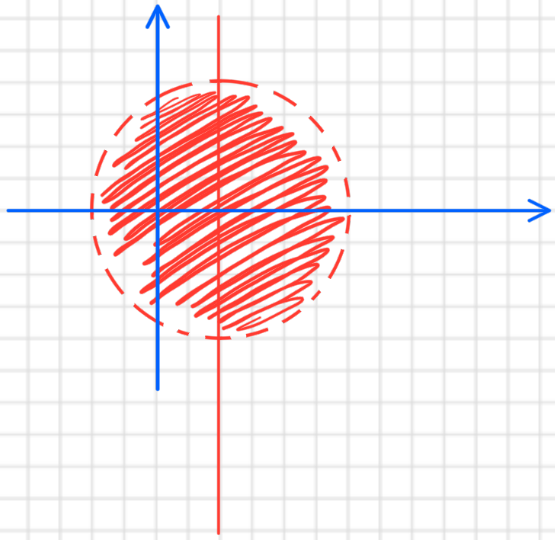
Proposta de resolução Teste 1 (Análise - LCC) 17-04-2024

$\boxed{1}$ $A = \{(x,y) \in \mathbb{R}^2 : x = 1 \text{ ou } (x-1)^2 + y^2 < 4\}$

a)



b) $\mathring{A} = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 4\}$

$\bar{A} = \{(x,y) \in \mathbb{R}^2 : x = 1 \text{ ou } (x-1)^2 + y^2 \leq 4\}$

$fr(A) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 4\} \cup \{(x,y) \in \mathbb{R}^2 : x = 1 \text{ e } |y| \geq 2\}$

$\boxed{2}$ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{y^3}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \underset{\substack{\downarrow \\ 0}}{y} \cdot \dfrac{y^2}{x^2+y^2} = 0 = f(0,0).$

$0 \leq \dfrac{y^2}{x^2+y^2} \leq 1$ (limitada)

logo f é contínua em $(0,0)$.

b) $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{0}{h^2+0} - 0}{h} = 0$

$\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{h^3}{0^2+h^2} - 0}{h} = \lim\limits_{h\to 0} \dfrac{h^3}{h^3} = 1$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - \left(f(0,0) + \frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y\right)}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{\frac{y^3}{x^2+y^2} - y}{\sqrt{x^2+y^2}}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{y^3 - y(x^2+y^2)}{(x^2+y^2)\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{-yx^2}{(x^2+y^2)\sqrt{x^2+y^2}}$

$\lim\limits_{\substack{(x,y)\to(0,0) \\ y=x}} \dfrac{-yx^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{-x^3}{2x^2\sqrt{2x^2}} = \lim\limits_{x\to 0} \dfrac{-x}{2\sqrt{2}\,|x|}$

$\lim\limits_{x\to 0^+} \dfrac{-x}{2\sqrt{2}\,|x|} = -\dfrac{1}{2\sqrt{2}}$

$\lim\limits_{x\to 0^-} \dfrac{-x}{2\sqrt{2}\,|x|} = \dfrac{1}{2\sqrt{2}}$

logo o limite não existe, donde f não é derivável em $(0,0)$.

$\boxed{3}$ $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\qquad f(x,y) = \frac{x}{y} + \cos(xy)$

a) $D = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$

b) $\frac{\partial f}{\partial x}(x,y) = \frac{1}{y} - \operatorname{sen}(xy) \cdot y$

$$\frac{\partial f}{\partial x} : D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{1}{y} - \operatorname{sen}(xy) \cdot y$$

$\frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2} - \operatorname{sen}(xy) \cdot x$

$$\frac{\partial f}{\partial y} : D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto -\frac{x}{y^2} - \operatorname{sen}(xy) \cdot x$$

c) Com $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são funções contínuas numa bola de centro em $(\pi, \frac{1}{2})$, então f é derivável em $(\pi, \frac{1}{2})$.

d)
$$f'(\pi, \tfrac{1}{2}) : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto \frac{\partial f}{\partial x}(\pi, \tfrac{1}{2})u + \frac{\partial f}{\partial y}(\pi, \tfrac{1}{2})v = \frac{3}{2}u - 5\pi v$$

$$\frac{\partial f}{\partial x}(\pi, \tfrac{1}{2}) = \frac{1}{1/2} - \operatorname{sen}\left(\frac{\pi}{2}\right) \cdot \frac{1}{2} = 2 - \frac{1}{2} = 3/2$$

$$\frac{\partial f}{\partial y}(\pi, \tfrac{1}{2}) = \frac{-\pi}{(1/2)^2} - \operatorname{sen}\left(\frac{\pi}{2}\right) \cdot \pi = -4\pi - \pi = -5\pi$$

e) A taxa de variação instantânea de f em $(\pi, \frac{1}{2})$ na direção do vector $\vec{u}\,(1,1)$ é

$$\frac{\partial f}{\partial (1,1)}(\pi, \tfrac{1}{2}) = \frac{1}{\|(1,1)\|} \cdot f'(\pi, \tfrac{1}{2})(1,1) = \frac{1}{\sqrt{2}}\left(\frac{3}{2} \cdot 1 - 5\pi \cdot 1\right) = \frac{3}{2\sqrt{2}} - \frac{5}{\sqrt{2}}\pi.$$

porque f é derivável em $(\pi, 1/2)$

$\boxed{4}$ $\ln(xy) + e^{xz} - z = 1 \iff \ln(xy) + e^{xz} - z - 1 = 0$

a) Considere-se $f(x,y,z) = \ln(xy) + e^{xz} - z - 1$.

Tem-se

- $f(2, \frac{1}{2}, 0) = \ln(2 \cdot \frac{1}{2}) + e^{2\times 0} - 0 - 1 = \ln(1) + 1 - 1 = 0$ ok!
- $\frac{\partial f}{\partial x}(x,y,z) = \frac{y}{xy} + z e^{xy} = \frac{1}{x} + z e^{xy}$
- $\frac{\partial f}{\partial y}(x,y,z) = \frac{x}{xy} = \frac{1}{y}$
- $\frac{\partial f}{\partial z}(x,y,z) = x e^{xz} - 1$

São contínuas no domínio de f, logo f é de classe C^1.

- $\frac{\partial f}{\partial z}(2, \frac{1}{2}, 0) = 2e^0 - 1 = 1 \neq 0$

Pelo Teorema da função implícita conclui-se que a equação dada define z como função de (x,y), i.e., $z = z(x,y)$, para (x,y,z) numa bola de centro $(2, \frac{1}{2}, 0)$

b) O teorema da função implícita garante que z é uma função de classe C^1, tendo-se

$$z'(2, \tfrac{1}{2}) : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(u,v) \longmapsto \frac{\partial z}{\partial x}(2, \tfrac{1}{2})\, u + \frac{\partial z}{\partial y}(2, \tfrac{1}{2})\, v = -\frac{1}{2}u - 2v$$

$$\frac{\partial z}{\partial x}(2, \tfrac{1}{2}) = -\frac{\frac{\partial f}{\partial x}(2, \frac{1}{2}, 0)}{\frac{\partial f}{\partial z}(2, \frac{1}{2}, 0)} = -\frac{\frac{1}{2} + 0}{1} = -\frac{1}{2}$$

$$\frac{\partial z}{\partial y}(2, \tfrac{1}{2}) = -\frac{\frac{\partial f}{\partial y}(2, \frac{1}{2}, 0)}{\frac{\partial f}{\partial z}(2, \frac{1}{2}, 0)} = -\frac{\frac{1}{1/2}}{1} = -2$$

c) Sendo $z(x,y)$ derivável em $(2, \frac{1}{2})$, então $\nabla z(2, \frac{1}{2})$ tem direcção normal à recta tangente à curva de nível $z(2, \frac{1}{2}) = 0$ no ponto $(2, \frac{1}{2})$. Assim

$$(x,y) = (2, \tfrac{1}{2}) + \lambda\,(2, -\tfrac{1}{2}), \quad \lambda \in \mathbb{R}$$

é uma equação da recta tangente pedida.

$\boxed{5}$ $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad \nabla f(2,3,0) = (-1,2,3)$

$(u,v,w) \longmapsto f(u,v,w)$

$g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ definida por $g(x,y) = f\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2} y\right)\right)$

a) Pela regra da cadeia tem-se

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial u}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) \cdot y + \frac{\partial f}{\partial v}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) \cdot 1 + \frac{\partial f}{\partial w}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) \cdot 0$$

$$\frac{\partial g}{\partial x}(1,2) = \frac{\partial f}{\partial u}(2,3,0) \times 2 + \frac{\partial f}{\partial v}(2,3,0) \times 1 + 0 = -1 \times 2 + 2 \times 1 = 0$$

b)

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) x + \frac{\partial f}{\partial v}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) \cdot 1 + \frac{\partial f}{\partial w}\left(xy, x+y, \operatorname{sen}\left(\frac{\pi}{2}y\right)\right) \cdot$$

$$\cdot \cos\left(\frac{\pi}{2}y\right) \cdot \frac{\pi}{2}$$

$$\frac{\partial g}{\partial y}(1,2) = \frac{\partial f}{\partial u}(2,3,0) \cdot 1 + \frac{\partial f}{\partial v}(2,3,0) \cdot 1 + \frac{\partial f}{\partial w}(2,3,0) \cos(\pi) \frac{\pi}{2}$$

$$= -1 + 2 - \frac{3\pi}{2} = 1 - \frac{3\pi}{2}$$