Ficha de trabalho

1. (a) $w = x^4 e^{x} + y\cos x$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(x^4 e^y + y\cos x\right)\right)$$

$$= \frac{\partial}{\partial x}\left(x^4 e^y + \cos x\right) = 4x^3 e^y - \operatorname{sen} x$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(x^4 e^y + y\cos x\right)\right)$$

$$= \frac{\partial w}{\partial y}\left(4w^3 e^y - y \operatorname{sen} x\right) = 4x^3 e^y - \operatorname{sen} x$$

Pelo que podemos concluir que $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$.

(b) $$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(x^2\cos(xy)\right)\right)$$

$$= \frac{\partial}{\partial x}\left(-x^3 \operatorname{sen}(xy)\right) = -3x^2 \operatorname{sen}(xy) - x^3 y \cos(xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(x^2\cos(xy)\right)\right)$$

$$= \frac{\partial}{\partial y}\left(2x\cos(xy) - x^2 y \operatorname{sen}(xy)\right)$$

$$= -2x^2 \operatorname{sen}(xy) - x^2 \operatorname{sen}(xy) - x^3 y \cos(xy)$$
$$= -3x^2 \operatorname{sen}(xy) - x^3 y \cos(xy)$$

Pelo que podemos concluir que $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$.

2. $w_{xyz} = \dfrac{\partial^3 w}{\partial z \partial y \partial x} = \dfrac{\partial^2}{\partial z \partial y}\left(\dfrac{\partial w}{\partial x}\right)$

$= \dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x}\left(x^3y^2z + xy^6z - yz\right)\right)\right)$

$= \dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(3x^2y^2z + y^6z\right)\right)$

$= \dfrac{\partial}{\partial z}\left(6x^2yz + 6y^5z\right) = 6x^2y + 6y^5$

3. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right)$

$= \dfrac{\partial}{\partial x}\left(-e^{-x}\cos y + e^{-y}\operatorname{sen} x\right)$

$= e^{-x}\cos y + e^{-y}\cos x$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right)$

$= \dfrac{\partial}{\partial y}\left(-e^{-x}\operatorname{sen} y + e^{-y}\cos x\right) = -e^{-x}\cos y - e^{-y}\cos x$

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = e^{-x}\cos y + \boxed{e^{-y}\cos x} - e^{-x}\cos y \boxed{- e^{-y}\cos x}$

$= 0$, logo é uma função harmónica.

4. (a) $z = x^3 + y^2$, $x = \cos t$ e $y = \dfrac{1}{t}$

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial t} = 3x^2 \cdot (-\operatorname{sen} t) + (2y) \cdot \left(-\dfrac{1}{t^2}\right)$

$= -3x^2 \operatorname{sen} t - \dfrac{2y}{t^2} = -3\cos^2 t \operatorname{sen} t - \dfrac{2}{t^3}$

(b) $z = \ln(x+y^2)$, $x = e^{t^2}$ e $y = t^3 + t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \left(\frac{1}{x+y^2}\right) 2te^{t^2} + \left(\frac{2y}{x+y^2}\right)(3t^2+1)$$

$$= \left(\frac{1}{x+y^2}\right)\left(2te^{t^2} + 2y(3t^2+1)\right)$$

$$= \left(\frac{1}{e^{t^2}+t^6+t^2}\right)\left(2te^{t^2} + 2(t^3+t)(3t^2+1)\right)$$

$$= \left(\frac{1}{e^{t^2}+t^6+t^2}\right)\left(2te^{t^2} + (2t^3+2t)(3t^2+1)\right)$$

$$= \frac{2te^{t^2} + 6t^5 + 2t^3 + 6t^3 + 2t}{e^{t^2}+t^6+t^2}$$

$$= \frac{2te^{t^2} + 6t^5 + 8t^3 + 2t}{e^{t^2}+t^6+t^2}$$

5. $w = u\cos v^2$, $u = x^3 + y$ e $v = x^2 y$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \cos(v^2)\,3x^2 - 2vu\,\text{sen}(v^2)\,2x$$

$$= 3x^2\cos(v^2) - 4vuxy\,\text{sen}(v^2)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \cos v^2 - 2vux^2\,\text{sen}(v^2)$$

6. $\frac{\partial z}{\partial r} = ?$, $z = u^2 + v^2 + w^2$, $u = re^{-s}$, $v = s^2 e^{-r}$, $w = re^{s}$
$r = 2$ e $s = 1$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial r} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial r}$$

$$= 2u(e^{-s}) - 2v(s^2 e^{-r}) + 2w(e^s)$$
$$= 2ue^{-s} - 2v(s^2 e^{-r}) + 2we^s$$
$$= 2ue^{-s} + 2we^s - 2vs^2 e^{-r}$$
$$= 2(re^{-s})e^{-s} + 2(re^s)e^s - 2(s^2 e^{-r})s^2 e^{-r}$$
$$= 2r(e^{-s})^2 + 2r(e^s)^2 - 2s^4(e^{-r})^2$$

Como sabemos qual o valor de r e de s, substituímos nesta expressão:

$$2 \times 2(e^{-1})^2 + 2 \times 2(e^1)^2 - 2 \times 1 \times (e^{-2})^2$$
$$= 4e^{-2} + 4e^2 - 2e^{-4}$$

7. $y = f(x)$, assim sendo $f(x,y) = x^3 + xy^2 + y^4 + x - 4$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-\frac{\partial}{\partial x}(x^3 + xy^2 + y^4 + x - 4)}{\frac{\partial}{\partial y}(x^3 + xy^2 + y^4 + x - 4)}$$

$$= \frac{-3x^2 + y^2 + 1}{2xy + 4y^3}$$, apenas possível para $(x,y) \neq 0$.

8. $z = f(x,y)$, logo $f(x,y,z) = z^2 y + 2xz^2 + xy^2 + 4z$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2z^2 + y^2}{2zy + 4xz + 4}$$, para $2zy + 4xz + 4 \neq 0$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{z^2 + 2xy}{2zy + 4xz + 4}$$, para $2zy + 4xz + 4 \neq 0$

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