

Essential Math Basics in Hydrodynamics

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1 Introduction

All phenomena can be found following [Lao Zi \[4th century BC\]](#), “The tendency of Dao (道) is to transform into its opposite” and “Anything that develops extreme qualities will invariably revert to the opposite qualities”. If one really needs to understand the most advanced or mysterious theory behind the physical process, the best start point for that person is the simplest or elemental step. To illustrate the opposite breakthrough, we will rename the traditional differential operator as “Dao” operator \mathfrak{D} since it can solve differentiation (or differential equations) by integration and solve (even difficult or impossible) integration by differentiation.

These topics become more and more important as it was found that wider and wider applications are manifesting the philosophy ubiquitous. And recent years, the emphasis and exploration of the [minesweeper](#) δ -function (can be either [Dirac delta](#) or [Kronecker delta](#)) definitely demonstrate that the evolutionary breakthrough in the various applications can happen in the near future. So for the theory in marine hydrodynamics, I would like to introduce the δ 's use as much as possible. On the other side as for the contents, I'm just selecting some interesting but discrete topics to discuss, therefore unfortunately, not a complete or systematic theory can be covered. Most of the contents are just smoothed learner notes or summary, but still there are several places, perhaps barely called new findings where I endeavored some derivations which you probably won't be able to find in any textbook or literature. The Chapters from 8th to the end are not well studies or organized, and are only kept for further start points. Some parts are abridged due to limited compiling time as a free user.

It will be claimed that all test functions to be discussed here are “smooth” functions for which we give the following definition.

Definition: A smooth function¹ is the function that is everywhere differentiable any number of times and such that it and all its derivatives are $\mathcal{O}(|x|^{-N})$ as $|x| \rightarrow \infty$ for all integer N .

This follows [Lighthill \[1959\]](#), where it is also proved that the Fourier transform of any smooth function is a smooth function.

¹Smooth functions are closely related to Schwartz space $\mathcal{S}(\mathbb{R}^n)$ which is the space of all smooth functions (as defined above are rapidly decreasing at infinity along with all partial derivatives). Smooth function is widely used as test function for instance in defining δ distribution as $\langle \delta, \phi \rangle = \langle \hat{\delta}, \phi \rangle$ for all Schwartz functions φ , where and $\hat{\phi}$ and $\hat{\delta}$ are the Fourier transforms of ϕ and δ , respectively. By the way, integration by parts is used here and for a general function $f(x)$ and the test function $\phi(a) = \phi(b) = 0$

$$\int_a^b \phi'(x)f(x) = \phi(x)f(x)|_a^b - \int_a^b f'(x)\phi(x)dx = - \int_a^b f'(x)\phi(x)dx. \quad (1)$$

As one will find later that several important definitions and theorems are fully restated from what I think classic resources. However, to avoid any unnecessary suspicion of plagiarism, here I claim that no theory is invented by myself and I have just learned and summarized all the essence of other distinguished publications mostly from [Strang \[2015\]](#), [Arnold \[1984\]](#), [Kempf et al. \[2014\]](#), [Kempf et al. \[2015\]](#) and [Jia et al. \[2017\]](#), etc. I just try my best to take notes and make it much easily to myself understandable and probably also for lots of non-mathematical background but interested people. I spent strenuous efforts to dig some coarse bricks which you might start with to polish and forge to gold.

Why I started to compose such a kind of lecture note about “Essential Math Basics”? This has to be traced back to my curved path in learning marine hydrodynamics, especially the period when I was invited to lecture and supervise master student in hydrodynamics. I started to ask myself, do I know some theory? What can I teach if I know nothing? I on and off turned to my books and read the prefaces or introduction. Most of the marine hydrodynamic textbook (for instance even the classic ones [Newman \[1977\]](#) and [Faltinsen \[1990\]](#)) claimed to students that if you have completed courses in some introductory mathematics, classical physics, and field theory, then you can start your carrier journey to become a hydrodynamicist or professor. Even some books stated that you have no difficulty in understanding the contents of complex analysis used in hydrodynamics since most introductory courses in mathematics at least have some components of complex theory. However from what I really heard, some of our lecturers or even professors themselves didn’t understand the theory completely. How could our students master the essences? Of course, there is one possible reason that I am the least knowledgeable person who needs to make extra strenuous effort to make the study progressive in life time. For example, one of my big puzzle is the frequency in the boundary conditions of surface wave. After many years investigation, it becomes clearer and clearer that the frequency of any mechanic oscillation depends on the energy distribution between kinetic and potential components and of course also the initial conditions. Note what mentioned here has nothing to do with the [Rayleigh–Ritz method](#). If the kinetic energy dominates, hence the water particles move faster relative to the simple harmonic oscillator and the time period decreases as the system’s energy increases, like slamming, springing, etc. On the other hand, the kinetic energy diminishes and the oscillation period increases with increasing energy. The typical case for long period estimation is so-called zero-frequency free surface boundary conditions of a fluid domain where we can utilize double body theory as in a infinite fluid. These kinds of knowledge are never be able to achieve in our recommended books during the whole university period. On the same time, I never underestimate to learn from books I can find but some aspects (or probably findings) that I had really Eureka moments and even experienced some sleepless nights. As a habit I suppressed or berried the exclamation and try to read more literature if they are already stated. One typical example is the differential operator which I found myself to express Laguerre polynomial and I verified

with the mathematical handbook [Spiegel and Liu \[1999\]](#) I have at hand for correctness and there was no such formula written in the book. I have wondered for some days whether it is possible that such a fancy formula has been neglected by so many mathematician. Then I searched online and then I realized that it is just one form of the Rodrigue's formula. Finally, as ever, all these would have been for naught had it not been further developed and applied, which naturally cast the light on my future life on the road ahead.

2 Linearization of Complicated Problem

A linear operator (can use symbol L or \mathfrak{D}) is such a function that satisfies the two properties

- Additivity: $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$,
- Homogeneity (scaling): $L(c\mathbf{u}) = cL(\mathbf{u})$

where \mathbf{u} and \mathbf{v} can represent vectors (of any type), but the linearity properties apply to matrices or functions as well. This is closely related to linear algebra. For instance, we need to find the derivative of a function $f(x) = x^5 + 3x^3 + 5x^2 + 7x + 11$. It becomes straight forward to express the differential operation in multiplication of the basis (power series) matrix and coordinate (coefficient) vector

$$\mathfrak{D}(x^5 + 3x^3 + 5x^2 + 7x + 11) \implies \mathbf{A}\mathbf{x} = \mathbf{b} \quad (2)$$

which means

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 5 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \begin{bmatrix} 11 \\ 7 \\ 5 \\ 3 \\ 0 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 9 \\ 0 \\ 5 \\ \vdots \end{bmatrix}. \quad (3)$$

There is a tremendous amount of beautiful features if the linearity can be allowed. Linear equations play a vital role in understanding our world. It is stated in [Arnold \[1984\]](#) that "Linear equations occupy a special place in the theory of differential equations because, by one of the basic principles of analysis, every smooth function is well approximated in a neighborhood of each point by a linear function. The operation of linearization that thereby arises leads to linear equations as a first approximation in the study of an arbitrary equation near any solution."

There are some basic ideas that are very essential to discuss earlier here. The first one is the power of the linearization for physical system. In marine hydrodynamics, the Green function is one of the most powerful and everyday tools, for instance, WAMIT, [Wamit \[2007\]](#). As a daily user, he or she seldom reminds oneself that Green function $G(x, y)$ is functioning as the inverse of differential operator due to the linearization which satisfies the equation

$$\mathfrak{D}G(x, y) = \delta(x - y) \quad (4)$$

with $\delta(x - y)$, the important delta function to be extensively surveyed below. This vastly states that the Green function is solving the differential equation with a single point source as the forcing and the solution to the same differential equation with an arbitrary forcing term can be obtained by simply integrating the Green function against the forcing term. All these shortcuts own thanks to the linearity of the differential operator. Explicitly, the general differential equation

$$\mathfrak{D}u(x) = f(x) \quad (5)$$

can be solved by

$$u(x) = \int G(x, y)f(y)dy \quad (6)$$

because we have

$$\mathfrak{D}u(x) = \int \mathfrak{D}G(x, y)f(y)dy = \int \delta(x - y)f(y)dy = f(x). \quad (7)$$

The other two topics are the delta (Dirac and Kronecker) function $\delta(x)$ and Heaviside function $H(x)$ and its expansion theorem (or [partial fraction expansion](#)). Some basic but

powerful techniques are introduced, to list some

$$\partial_x^{-n} \delta(x) = \frac{x^{n-1}}{(n-1)!} \quad (8)$$

$$\partial_x^{-n} H(x) = \frac{x^n}{n!} \quad (9)$$

$$\frac{\partial_x}{\partial_x - a} H(x) = \sum_{k=0}^{\infty} a^k \partial_x^{-k} H(x) = e^{ax} = \frac{\mathfrak{D}}{\mathfrak{D} - a} \quad (10)$$

$$\frac{\partial_x^2}{\partial_x^2 - a} H(x) = \cosh(\sqrt{a}x) \quad (11)$$

$$\frac{1}{\partial_x^2 - 1} H(x) = 2 \sinh^2\left(\frac{x}{2}\right) \quad (12)$$

$$\frac{\mathfrak{D}}{(\mathfrak{D} - a)^n} = \frac{x^{n-1}}{(n-1)!} e^{ax} \quad (13)$$

$$\frac{1}{(\mathfrak{D} - a)^n} = \sum_{k=1}^n \frac{(-1)^{k-1} x^{n-k}}{a^k (n-k)!} e^{ax} + (-1)^n \frac{1}{a^n} \quad (14)$$

$$\text{FT: } \mathcal{F}[f(t)](\omega) = 2\pi f(i\partial_\omega) \delta(\omega) \quad (15)$$

$$\text{Inverse FT: } \mathcal{F}^{-1}[\hat{f}(\omega)](t) = \hat{f}(-i\partial_t) \delta(t) \quad (16)$$

$$\text{LT: } \mathcal{L}[f(t)](s) = f(-\partial_s) \frac{1}{s} \quad (17)$$

$$\text{Inverse LT: } \mathcal{L}^{-1}[\hat{f}(s)](t) = \hat{f}(\partial_t) \delta(t) \quad (18)$$

where a is any number including complex or zero (note (14) is valid through limiting process $a \rightarrow 0$). The derivation and application will be given in the following sections. One significant declaration is that the differential operation function $\phi(\mathfrak{D})$ on a unit or Heaviside function can be expressed equivalently as the inverse Laplacian transform of $\phi(s)/s$, i.e.,

$$\phi(\mathfrak{D}) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\phi(s)}{s} e^{st} ds \quad (19)$$

where the integral is taken in the complex s -plane along a vertical line from $z = \sigma - i\infty$ to $z = \sigma + i\infty$ with σ fixed. The simplest verification can be seen if $\phi(\mathfrak{D}) = \mathfrak{D}/(\mathfrak{D} - a)$, then its r.h.s. of (19) is equal to e^{at} which is exactly what (13) stated when $n = 1$.

It might also be worthy mentioning that an interesting differential operator regarding the [generalized Rodrigue's formula](#) has been found enlightening to study a lot of potential differential equations. The generalized Rodrigue's formula is used to derive the polynomial eigenfunctions for polynomial differential operators. Their expansion and inversion can be very powerful to solve the differential equations. And the recursive form or inline feature is well suited to accelerate the computer computations. The most known applications of Rodrigues' type formulas are the formulas for Laguerre and Hermite polynomials:

For Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \partial_x^n [e^{-x} x^n] = \frac{1}{n!} (\partial_x - 1)^n x^n, \quad (20)$$

and the two kinds of Hermite polynomials are given by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \partial_x^n e^{-\frac{x^2}{2}}, \text{ or } = (-1)^n e^{x^2} \partial_x^n e^{-x^2}. \quad (21)$$

These two types can have the nice form of a Rodrigues' formula, respectively written as,

$$H_n(x) = (x - \partial_x)^n \cdot 1, \text{ or } = (2x - \partial_x)^n \cdot 1. \quad (22)$$

Note that these two definitions of Hermite polynomials differ just by scaling. To express in the binomial form of the operator expressions, the exponential containing formulas on the left hand side are actually the special forms of the following.

$$(-1)^n e^{ax^r} \partial_x^{\pm n} e^{-ax^r} = (a\partial_x x^r - \partial_x)^{\pm n} \cdot 1. \quad (23)$$

Note that the negative power of the operator ∂_x^{-n} stands for integration n times with respect to x . To be even radically generalizable, the exponential part can be conjectured to be a wave function $\phi(\mathbf{r})$.

$$(-1)^n e^{\phi(\mathbf{r})} \nabla^{\pm n} e^{-\phi(\mathbf{r})} = (\nabla\phi - \nabla)^{\pm n} \cdot \mathbf{r}. \quad (24)$$

The exponential function is probably the most important function in mathematics. But students are not often instilled enough. For instance, many people believe that Fourier and Laplace theories are most likely attached to telecommunication and control students. Rarely a marine student realizes that wave function is actually Fourier-Laplace transformation.

The main idea is the first order problem, but the techniques are easily extendable to higher order systems by breaking them up into a series of the first order equations. For instance, the most common 2nd order ODE

$$(\mathfrak{D}^2 + p\mathfrak{D} + q)y = r(x), \text{ with } y(0) = y_0 \text{ and } y'(0) = y_1 \quad (25)$$

can be completely solved by such a simple formula.

$$y = \frac{(\mathfrak{D}^2 + p\mathfrak{D})y_0 + \mathfrak{D}y_1 + r}{\mathfrak{D}^2 + p\mathfrak{D} + q}. \quad (26)$$

Note that the solution is definite with distinct eigenvalues and on the same time satisfies the Cauchy conditions (y_0 and y_1).

2.1 Linearization of the Fundamental Dynamic System

Nowadays within marine field, either in scientific publication or during the selecting a research topic for a master student, no one will be satisfied with the plain word “LINEAR

THEORY". If we are proposing some topic, for instance, "The linear hydrodynamic loads on a wind turbine structure". Most of students will just shout out the word "Nonlinear!" Yes, in real world, no system can be found exactly linear. "Linearity" is never exactly "true." (see Feynman et al. [1965], Georgi [2015]). Everyone knows that the linearity is extremely important and on the same time the nonlinearity is extremely difficult in an even very simply marine structure in waves. But in reality, almost all publications and master students are bravely taking the challenge to start with nonlinear problem.

In stead of suddenly berring one's head into the complex nonlinear problem, one should at least understand how to end up with a simpler solution from a linear system. First of all, one should have the assumption that any system will behave in the way that the system's energy can be expressed as a smooth function of the positions in space and the small displacement (compared to the object's dimension) from equilibrium will excite approximately linear restoring forces. To see the generic nature of linearity, we focus on potential theory, especially for marine structures where the ocean waves are the important loads.

Wait! Do I know how exactly the system's energy is related with its position?

Let's consider a dynamic system (can be simply a mass-spring system, but here to be general in 3D vectorial form) with the mass m moving to the position \mathbf{x} with the specific potential energy (potential energy per unit mass), $U(\mathbf{x})$. The generic conservative force $\mathbf{f}(\mathbf{x})$ per unit mass (here it is assumed that the force is conservative whose total work is path independent, for instance can be gravitational acceleration or the restoring force per unit mass) on the object at the position \mathbf{x} , can be proved to be minus the derivative of the potential energy $U(\mathbf{x})$, i.e., $\mathbf{f} = -\nabla U(\mathbf{x})$.

Let's assume that the conservation of energy holds which means the total energy

$$E = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} + mU(\mathbf{x}) \quad (27)$$

is conserved and its time-derivative vanishes.

Since the mass is just a scaling factor of the total energy from our simple model, it is convenient to study the specific energy, i.e., by dividing the mass from (27) we have

$$E_\rho = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + U(\mathbf{x}). \quad (28)$$

Note here the the potential energy $U(\mathbf{x})$ is a function of \mathbf{x} which itself is time dependent. Therefore when we take the time-derivative of the total energy, the so-called material differentiation shall be applied with respect to time

$$\begin{aligned} \frac{dE_\rho}{dt} &= \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{\partial U(\mathbf{x})}{\partial t} + \mathbf{v} \cdot \nabla U(\mathbf{x}) \\ &= \mathbf{v} \cdot [\mathbf{f} + \nabla U(\mathbf{x})] = 0. \end{aligned} \quad (29)$$

In (29), if the velocity \mathbf{v} is zero, the total energy is pure potential energy $U(\mathbf{x})$ which

indeed is time independent. When $\mathbf{v} \neq 0$, it leads that

$$\mathbf{f}(\mathbf{x}) = -\nabla U(\mathbf{x}). \quad (30)$$

Many textbooks or experts claimed that “the negative sign provides the convention that work done against a force field increases potential energy” or simply “The negative sign is just a convention”. But here the proof in (29) might indicate that the “strictly” mathematical derivation instead of just a convention².

For simplicity, we consider the unit mass moving along the x -axis with potential energy, $U(x)$. We assume that there is equilibrium at x_0 , where the spring force $f(x_0)$ vanishes, therefore, by (30), the derivative of the potential energy at x_0

$$f(x_0) = -U'(x_0) = 0. \quad (31)$$

If we give a small disturbance and let the mass move to the location x from its equilibrium $x = 0$, the resulting force can be obtained by Taylor expansion as

$$f(x) = -U'(x) = -U'(x_0) - xU''(x_0) - \frac{1}{2}x^2U'''(x_0) - \dots \quad (32)$$

The first term $-U'(x_0)$ in (32) vanishes because this system is in equilibrium at $x = 0$, from (31). The second term represents exactly what Hooke’s law states with

$$k = U''(0) \quad (33)$$

as the linear (spring) constant which to be shown has to be positive for a stable system. The equilibrium of a dynamic system is stable³ if the second derivative of the potential energy is positive, which implies that $x = 0$ is a local minimum of the potential energy.

Following discussion is best referred to [Georgi \[2015\]](#) in order to show to what extend the linearization can be thought accurate or acceptable.

The important point is that for sufficiently small x , the third term in (32), and all subsequent terms will be much smaller than the second. The third term is negligible if

$$|xU'''(0)| \ll U''(0). \quad (34)$$

Typically, each extra derivative will bring with it a factor of $1/L$, where L is the distance over which the potential energy changes by a large fraction. Then (34) becomes

$$x \ll L. \quad (35)$$

There are only two ways that a force derived from a potential energy can fail to be approximately linear for sufficiently small oscillations about stable equilibrium:

²This looseness in the scientific derivation has puzzled me for many years since the day when I was told to accept the convention during the period in proofreading the book [Faltinsen and Timokha \[2009\]](#).

³[Stability](#) can have different definitions, but here we assumes that the 2nd derivative is governing.

1. If the potential is not smooth so that the first or second derivative of the potential is not well defined at the equilibrium point, then we cannot do a Taylor expansion and the argument of (32) does not work. We will give an example of this kind at the end of this chapter.
2. Even if the derivatives exist at the equilibrium point, $x = 0$, it may happen that $U''(0) = 0$. In this case, to have a stable equilibrium, we must have $U'''(0) \neq 0$ as well, otherwise a small displacement in one direction or the other would grow with time. Then the next term in the Taylor expansion dominates at small x , giving a force proportional to x^3 .

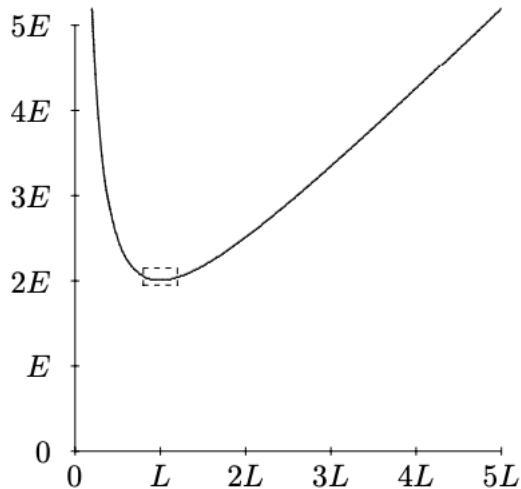


Fig. 1. The potential energy of (36)

Both of these exceptional cases are very rare in nature. Usually, the potential energy is a smooth function of the displacement and there is no reason for $U''(0)$ to vanish. The generic situation is that small oscillations about stable equilibrium are linear.

An example may be helpful. Almost any potential energy function with a point of stable equilibrium will do, so long as it is smooth. For example, consider the following potential energy

$$U(x) = E\left(\frac{L}{x} + \frac{x}{L}\right) \quad (36)$$

This is shown in Fig.(1). The minimum (at least for positive x) occurs at $x = L$, so we first redefine $x = X + L$, so that

$$U(X) = E\left(\frac{L}{X+L} + \frac{X+L}{L}\right) \quad (37)$$

The corresponding force is

$$F(X) = E \left(\frac{L}{(X+L)^2} - \frac{1}{L} \right) \quad (38)$$

we can look near $X = 0$ and expand in a Taylor Series:

$$F(X) = -2 \frac{E}{L} \left(\frac{X}{L} \right) + 3 \frac{E}{L} \left(\frac{X}{L} \right)^2 + \dots \quad (39)$$

Now, the ratio of the first nonlinear term to the linear term is

$$\frac{3X}{2L} \quad (40)$$

which is small if $X \ll L$.

In other words, the closer you are to the equilibrium point, the closer the actual potential energy is to the parabola that we would expect from the potential energy for a linear, Hooke's law force. You can see this graphically by blowing up a small region around the equilibrium point. In Fig.(1), the dotted rectangle in Fig.(2) has been blown up into a square. Note that it looks much more like a parabola than Fig.(1). If we repeated the procedure and again expanded a small region about the equilibrium point, you would not be able to detect the cubic term by eye.

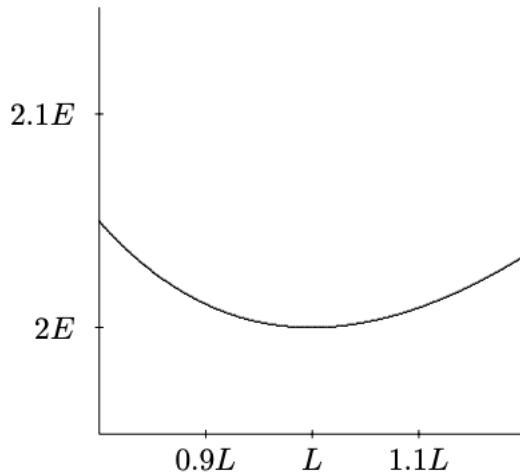


Fig. 2. The small dashed rectangle in Fig. 1 expanded

Often, the linear approximation is even better, because the term of order x^2 vanishes by symmetry. For example, when the system is symmetrical about $x = 0$, so that $U(x) = U(-x)$, the order x^3 term (and all x^n for n odd) in the potential energy vanishes, and then there is no order x^2 term in the force.

For a typical spring, linearity (Hooke's law) is an excellent approximation for small displacements. However, there are always nonlinear terms that become important if the

displacements are large enough. Usually, in this book we will simply stick to small oscillations and assume that our systems are linear. However, you should not conclude that the subject of nonlinear systems is not interesting. In fact, it is a very active area of current research in physics.

2.2 Delta Function

The delta function is a generalized function that can be defined as the limit of a class of delta sequences. It is not a function in the classical sense since it becomes infinity at a single point and vanishes otherwise along the real line. Its most important property is that its total contribution along the whole real line (either called summation or integral) is always a unit 1.

Solve an inhomogeneous linear equation we can introduce the delta-function and calculate

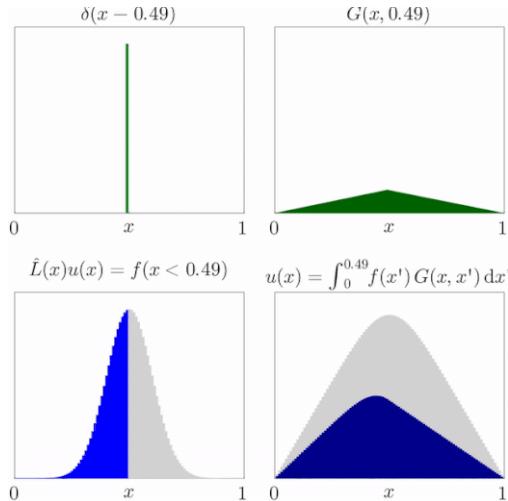


Fig. 3. If one knows the solution, [Green's function](#) $G(x, x')$, to a differential equation $L G(x, x') = \delta(x - x')$ subject to a point source, then one can superpose them to build the solution for a general source due to the linearity of differential operator.

the retarded Green's function which is the solution of (4). The solution of the generic initial-value problem (5) is expressed as the convolution of $G*$ through the superposition principle since any linear differential equation can be solved by summing the fundamental solution by splitting the forcing term into a sequence of delta functions. The procedure probably can be seen in Fig. 3.

2.3 Heaviside Expansion Theorem

The next is to introduce the very important techniques that are well underestimated in the applications of fundamental arithmetic, Fourier and Laplace transforms, calculus, operational methods, and various problems in mathematical physics, [Heaviside \[1892\]](#), [Heaviside](#)

[1893], Pennell [1929], etc.

Let nominator $N(s)$ and denominator $D(s)$ be two polynomials in s with $\deg N < \deg D$ (otherwise, $N(s)$ can be reduced so⁴). If $D(s)$ has n roots $\{s_i\}$, i.e., $D(s) = \prod_{i=1}^n (s - s_i) = 0$, but $N(s_i) \neq 0$ for all $1 \leq i \leq n$. We have

$$F(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=1}^n b_i s^{n-i}}{\sum_{i=0}^n c_i s^{n-i}} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{s^n + c_1 s^{n-1} + c_2 s^{n-2} + \dots + c_{n-1} s + c_n} \quad (42)$$

where the highest order of $D(s)$ set to $c_0 = 1$ to be monic. There are two cases for the n roots:

1. All s_i distinct and $D(s)$ to be factorized

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{(s - s_1)(s - s_2) \dots (s - s_n)} = \sum_{i=1}^n \frac{A_i}{s - s_i}. \quad (43)$$

Therefore,

$$A_j = \lim_{s \rightarrow s_j} \left[(s - s_j) \frac{N(s)}{D(s)} \right] = \frac{N(s_j)}{D'(s_j)} \quad (44)$$

which is the base of the well-known [cover-up method](#)⁵.

Note that (42) and (43) are closely related to the differential operator polynomial (92) for which I have fortuitously found the Vandermonde form and the connection between ODE's eigen-values and the solutions. The eigen-values of an ODE can form a Vandermonde matrix and the inversed matrix post-multiplied by the initial conditions gives directly the coefficients of the homogeneous ODE, and the source term of ODE can be processed through convolution in one matrix operation. This approach has similarity to Wronskian method, but is different. The one-step matrix operation might be mostly applicable for computer programming to solve the differential equations. It was later found that the partial fraction expansion using Vandermonde matrix was already discussed in publications, for instance, in Chen and Leung [1981] and Leyva-Ramos [1993], but none of them have directly applied the Vandermonde matrix to differential operators. Therefore, there will be no conflict to report some new approach for ODE applications shortly.

⁴For instance, we can detach a constant and have

$$F(s) = \frac{\sum_{i=0}^n b_i s^{n-i}}{\sum_{i=0}^n c_i s^{n-i}} = \frac{b_0}{c_0} + \frac{\sum_{i=1}^n b_i s^{n-i}}{\sum_{i=0}^n c_i s^{n-i}}. \quad (41)$$

⁵See also [brilliant.org: Partial Fractions - Cover Up Rule](#)

2. If one root repeats k times, WLOG⁶ say s_1 , then

$$F(s) = \frac{N(s)}{D(s)} = \sum_{j=0}^k \frac{A_j}{(s - s_1)^j} + \left\{ \sum_{j=k+1}^n \frac{A_j}{s - s_j} \right\} \quad (45)$$

$$= \sum_{j=0}^k \frac{A_j}{(s - s_1)^j} + \{Q(s)\} \quad (46)$$

if let $G(s) = (s - s_1)^k F(s)$, then $Q(s)$ terms will vanish for $s = s_1$, i.e.,

$$G(s) = A_1(s - s_1)^{k-1} + A_2(s - s_1)^{k-2} + \dots + \quad (47)$$

$$A_j(s - s_1)^{k-j} + \dots + A_{k-1}(s - s_1) + A_k \quad (48)$$

therefore

$$A_j = \frac{1}{(k-j)!} \frac{d^{k-j}}{ds^{k-j}} G(s_1) \quad (49)$$

i.e.,

$$A_k = G(s_1) \quad (50)$$

$$A_{k-1} = \frac{d}{ds} G(s_1) \quad (51)$$

$$A_{k-2} = \frac{1}{2!} \frac{d^2}{ds^2} G(s_1) \quad (52)$$

$$\vdots \quad (53)$$

$$A_1 = \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} G(s_1). \quad (54)$$

The inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)](t)$ is directly given by

$$f(t) = \sum_{k=1}^n \frac{N(s_k)}{D'(s_k)} e^{s_k t}. \quad (55)$$

Further, if in case $D(s) = sR(S)$, i.e.,

$$F(s) = \frac{N(s)}{sR(s)}, \quad (56)$$

the zeros of D now becomes $s_1 = 0, s_2, \dots, s_n$ and $D'(s) = R(s) + sR'(s)$, then

$$\sum_{k=1}^n \frac{N(s_k)}{D'(s_k)} e^{s_k t} = \frac{N(0)}{D'(0)} e^{s_1 t} + \sum_{k=2}^n \frac{N(s_k)}{D'(s_k)} e^{s_k t} = \frac{N(0)}{R(0)} + \sum_{k=2}^n \frac{N(s_k)}{s_k R'(s_k)} e^{s_k t} \quad (57)$$

assuming $R(s_1 = 0) \neq 0$. Here we will basically follow the formulation by [Chen and Leung \[1981\]](#). Actually, the coefficients denoted by column vector \mathbf{A} for (49) can be found equal to

⁶Abbreviation for Without Loss Of Generality, commonly used expression in mathematics.

$\mathbf{V}^{-1}\mathbf{b}$, the product of the inverse eigen-Vandermonde matrix \mathbf{V} and the coefficient vector \mathbf{b} of $N(s)$, in explicit

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ s_1^1 & s_2^2 & \dots & s_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{n-1} & s_2^{n-1} & \dots & s_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (58)$$

which can be utilized in the process of solving the ODEs. Note that the eigen-Vandermonde matrix \mathbf{V} will have simple determinant $|\mathbf{V}| = \prod_{0 \leq i < j \leq n} (s_j - s_i)$. After some basic algebra operations which are elaborated in details in Section 3.2.3, the final solution to one typical nth-order ODE (91) can be expressed by inverting the operator polynomial (92) as follows

$$y(x) = \begin{bmatrix} e^{s_1 x} & e^{s_2 x} & \dots & e^{s_n x} \end{bmatrix} \left[\begin{bmatrix} 1 & 1 & \dots & 1 \\ s_1^1 & s_2^2 & \dots & s_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{n-1} & s_2^{n-1} & \dots & s_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} + \frac{r(x)}{|\mathbf{V}|} \begin{bmatrix} \frac{1}{\mathfrak{D}-s_1} \\ \frac{1}{\mathfrak{D}-s_2} \\ \vdots \\ \frac{1}{\mathfrak{D}-s_n} \end{bmatrix} * \right] \quad (59)$$

where the $e^{s_i x}$, $i = 1 \dots n$ are the eigen-functions and the star $*$ means convolution in matrix multiplication. Here \mathbf{b} in (58) has been replaced by the Cauchy initial conditions $y_i = y^{(i)}(t = 0)$, $i = 0 \dots n - 1$ and $r(x)$ is the forcing function of the ODE. This compact formula looks fantastic and can work for any order of inhomogeneous ODEs, however until we face degenerate matrix, that is when any repeated eigen-value occurs. For instance, $D(s)$ appears repeated m -times roots, then only $n - m + 1$ roots are distinct⁷, in this situation, the Vandermonde matrix \mathbf{V} has to change and becomes

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 & \dots & 1 \\ s_1 & \frac{ds}{ds}|_{s_1} & 0 & \ddots & 0 & s_2^2 & \dots & s_{n-m}^2 \\ s_1^2 & \frac{ds^2}{ds^2}|_{s_1} & \frac{ds^2}{2!ds^2}|_{s_1} & \ddots & 0 & s_2^3 & \dots & s_{n-m}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_1^{n-1} & \frac{ds^{n-1}}{ds}|_{s_1} & \frac{ds^{n-1}}{2!ds^2}|_{s_1} & \dots & \frac{ds^{n-1}}{(m-1)!ds^{n-1}}|_{s_1} & s_2^{n-1} & \dots & s_{n-m}^{n-1} \end{bmatrix}. \quad (60)$$

2.4 Integral Representations

The extensive application for differential equations concerns complex function and its integral forms, i.e., Fourier and Laplace transformations. All functions to be worked on are of [exponential type](#) defined on the complex plane. If a function $f(z)$ is said to be of exponential type if there exist real-valued constants M and τ such that $|f(re^{i\theta})| \leq M e^{\tau r}$ in the limit of $r \rightarrow \infty$. Here, the complex variable z was written as $z = re^{i\theta}$ to emphasize that the limit must hold in all directions θ and sweep all the domain. Letting τ stand for the infimum of all such τ , one then says that the function f is of exponential type τ . For

⁷Here m is at most equal to $n - 1$, since otherwise the denominator $D(s)$ will be trivial to solve.

example, let $f(z) = \sin(\pi z)$. Then one says that $\sin(\pi z)$ is of exponential type π , since π is the smallest number that bounds the growth of $\sin(\pi z)$ along the imaginary axis (also called sharp since it vanishes on the integers but grows exponentially on the imaginary axis with a growth rate of $M = \pi$, and indeed it is not identically zero, i.e., or not a constant function).

The unilateral (one-sided) Laplace–Stieltjes transform is given by

$$\{\mathcal{L}^* g\}(s) = \lim_{\varepsilon \rightarrow 0^+} \int_{-\varepsilon}^{\infty} e^{-sx} dg(x). \quad (61)$$

The limit is necessary to ensure the transform captures a possible jump in $g(x)$ at $x = 0$, as is needed to make sense of the Laplace transform of the Dirac delta function. More general transforms can be considered by integrating over a contour in the complex plane. If a is the lower bound of the σ for which $|f(t)| = O(e^{\sigma t})$, $t \rightarrow \infty$, then $\sigma_a = a$; the number a is sometimes called the index of growth of $f(t)$. For example, if $f(t)$ has bounded variation in a neighbourhood of t_0 or if $f(t)$ is piece-wise smooth, then the inversion formula for the Laplace transform holds:

$$f^\circ(t_0) = \frac{f(t_0 + 0) + f(t_0 - 0)}{2} \quad (62)$$

$$= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\sigma-iR}^{\sigma+iR} F(p) e^{pt_0} dp, \quad \sigma > \sigma_a. \quad (63)$$

Since it is often to evaluate the Gaussian type integral, here we will use Laplace method to obtain a generalized integral $I = \int_0^\infty x^m e^{isx^k} dx$ on the condition that the proper parameters like $\Re\{s\} > 0$, $k > 1$, etc will guarantee its convergence.

Now let $u = -ix^k$, i.e., $x = (iu)^{\frac{1}{k}}$, so we have $dx = \frac{1}{k} i^{\frac{1}{k}} u^{\frac{1}{k}-1} du$, that is

$$\begin{aligned} I &= \int_0^\infty x^m e^{isx^k} dx = \frac{1}{k} i^{\frac{m+1}{k}} \int_0^\infty u^{\frac{m+1}{k}-1} e^{-su} du \\ &= \frac{1}{k} i^{\frac{m+1}{k}} \mathcal{L} \left[u^{\frac{m+1}{k}-1} \right] = \frac{1}{k} \left(\frac{i}{s} \right)^{\frac{m+1}{k}} \Gamma \left(\frac{m+1}{k} \right) \end{aligned} \quad (64)$$

where the last step is just the Laplace transform of $u^{\frac{m+1}{k}-1}$. Set $m = 0$ and $k = 2$ we will have

$$\int_0^\infty e^{icx^2} dx = \frac{1}{2} \left(\frac{i}{c} \right)^{\frac{1}{2}} \Gamma \left(\frac{1}{2} \right) = \frac{1}{2} \sqrt{\frac{i\pi}{c}}.$$

Some more powerful properties of the Laplace transform are introduced by [AlAhmad \[2020\]](#)⁸. One fancy property is

$$\mathcal{L}[f(x)h(x)] = \int_0^\infty (\mathcal{L}f)(\xi + s) (\mathcal{L}^{-1}h)(\xi) d\xi = \int_s^\infty (\mathcal{L}f)(\xi) (\mathcal{L}^{-1}h)(\xi - s) d\xi. \quad (65)$$

The following well-known integral technique is just one special form of (65) with $s = 0$

$$\int_0^\infty f(x)h(x) dx = \int_0^\infty (\mathcal{L}f)(\xi) (\mathcal{L}^{-1}h)(\xi) d\xi \quad (66)$$

⁸Note that unfortunately it contains misprints which must be corrected before use.

under suitable assumptions on the behaviour of f, h in a right neighbourhood of 0 and on the decay rate of f, h in a left neighbourhood of ∞ . The above formula is a variation of integration by parts, with the operators $\frac{d}{dx}$ and $\int dx$ being replaced by \mathcal{L} and \mathcal{L}^{-1} . This method can be used to compute integrals that would otherwise be difficult to compute using elementary methods of real calculus. The most typical one, for example,

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \mathcal{L}[1](x) \sin x dx = \int_0^\infty (1) \mathcal{L}[\sin](x) dx = \frac{\pi}{2} \quad (67)$$

which is a special case of (78). As an extended version of (65), the Laplace transform of the product of $h(t)$ and the convolution of $f * g$ is

$$\mathcal{L}[(f * g)(t)h(t)] = \int_0^\infty F(\xi + s)G(\xi + s)(\mathcal{L}^{-1}h)(\xi) d\xi \quad (68)$$

$$= \int_s^\infty F(\xi)G(\xi)(\mathcal{L}^{-1}h)(\xi - s) d\xi. \quad (69)$$

Another extended version of (65) can be used to evaluate complexed integral containing the product of $h(t)$ and the Laplace transform of product of $f(t)$ another Laplace transform of $g(t)$.

$$\int_0^\infty h(u) \mathcal{L}[f(t)(\mathcal{L}g)(t)](u) du = \int_0^\infty f(u) (\mathcal{L}_u[g(t)] \mathcal{L}[h(t)]) du \quad (70)$$

$$= \int_0^\infty f(u) \mathcal{L}[(g * h)(t)](u) du \quad (71)$$

$$= \int_0^\infty F(\xi)(g * h)(\xi) d\xi \quad (72)$$

The original author's intended derivation can be the following

$$\int_0^\infty h(u) \mathcal{L}[f(t)(\mathcal{L}g)(t)](u) du = \int_0^\infty h(u) \left\{ \int_u^\infty g(\xi - u) F(\xi) d\xi \right\} du \quad (73)$$

$$= \int_0^\infty h(u) \left\{ \int_0^\infty g(\xi) F(\xi + u) d\xi \right\} du \quad (74)$$

$$= \int_0^\infty F(\xi) \left\{ \int_0^\xi g(\xi - u) h(u) du \right\} d\xi \quad (75)$$

If $h(t) = e^{-st}$, above (70) and (73) will become

$$\mathcal{L}[f(t)(\mathcal{L}g)(t)] = \int_0^\infty F(\xi)(h(t) * e^{-st})(\xi) d\xi. \quad (76)$$

One application can be used to prove

$$\int_0^\infty \frac{\sin x}{x + x^2} dx = \int_0^\infty \frac{\pi - 2 \tan^{-1} x}{2e^x} dx \quad (77)$$

which⁹ is actually the representation of

$$\int_0^\infty e^x \mathcal{L} \left[\frac{\sin t}{t} \right] (x) dx = \int_0^\infty (\sin x) \mathcal{L} [e^{-t} * \mathbf{1}(t)](x) dx. \quad (79)$$

One more example can be shown that the consecutive usage of the above techniques can solve the difficult integrals. For example, we have

$$\mathcal{L} \left[\frac{\sin(at) - at \cos(at)}{at^3} \right] = \mathcal{L} \left[\frac{2 \sin(at) * \sin(at)}{at^3} \right] \quad (80)$$

$$= \int_s^\infty \frac{a^2}{(x^2 + a^2)^2} (x - s)^2 dx \quad (81)$$

$$= \frac{1}{2a} \left[(s^2 + a^2) \arctan \frac{a}{s} - as \right]. \quad (82)$$

Now, we will make a little change, and instead of integrating from s to ∞ , we evaluate the integral from 0

$$\int_0^\infty \frac{\sin(at) - at \cos(at)}{at^3} dt = \mathcal{L} \left[\frac{2 \sin(at) * \sin(at)}{at^3} \right]_{s=0} \quad (83)$$

$$= \int_0^\infty \frac{a^2}{(x^2 + a^2)^2} x^2 dx \quad (84)$$

$$= \int_0^\infty \mathcal{L}^{-1} \left[\frac{a^2 s}{(s^2 + a^2)^2} \right] (t) \mathcal{L}[x](t) dt \quad (85)$$

$$= \int_0^\infty \frac{1}{2} at \sin at \frac{1}{t^2} dt \quad (86)$$

$$= \frac{a}{2} \int_0^\infty \frac{\sin at}{at} d(at) \quad (87)$$

$$= \frac{\pi a}{4}. \quad (88)$$

3 Differentiation by Integration

A **Differential Operator** (can be called “Dao” operator \mathfrak{D} in a way to be explained later) is a functional operator which is defined as a function of the differentiation. It is helpful, as a simple notation first, to consider differentiation as an abstract operation that accepts a function and returns another function. Let’s start to show how the “Dao” operator \mathfrak{D} can be used to solve ordinary differential equations (ODE). The “Dao” differential operator is defined as $\mathfrak{D} = d/dx$ for single variable, and \mathfrak{D}_x or ∂_x for multi-index variables.

As in linear algebra, linear transformations can have eigenvalues and eigenvectors, which are called “eigenfunctions”. For example, the linear transformation $\mathfrak{D}^2 f(x)$ has eigenvalue -1 and corresponding eigenfunctions, $\sin(x)$ and $\cos(x)$, because $\mathfrak{D}^2 \sin(x) = -\sin(x)$ and

⁹ The power multiplication rule $\mathcal{L} [t^n f(t)](s) = (-1)^n \mathfrak{D}^n F(s)$ (note $n < 0$ stands for integral) can be applied here with $n = -1$

$$\mathcal{L} \left[\frac{\sin t}{t} \right] (s) = \int_x^\infty \mathcal{L} [\sin t](u) du = \cot^{-1} \frac{1}{x} = \frac{\pi}{2} - \tan^{-1} x. \quad (78)$$

$\mathfrak{D}^2 \cos(x) = -\cos(x)$. Sometimes one can use $\mathfrak{D}^2 + 1$ as an annihilator to suppress the source terms containing $\cos(x)$ or $\sin(x)$. One simple and typical eigenvalue equation is

$$\mathfrak{D}f(x) = \frac{d}{dx}f(x) = \lambda f(x). \quad (89)$$

This differential equation can be solved by the exponential function $f(x) = f_0 e^{\lambda x}$ which is called the eigenfunction of the differential operator \mathfrak{D} , with f_0 as the initial (or boundary) condition. The parameter λ is the associated eigenvalue. For the n -th order linear ODE

$$y^{(n)} + c_1 y^{(n-1)} + \dots + c_n y = q(x). \quad (90)$$

It can be written as

$$(\mathfrak{D}^n + c_1 \mathfrak{D}^{n-1} + \dots + c_n) y = q(x) \quad (91)$$

or in compact form, $p(\mathfrak{D})y = q(x)$, where we define

$$p(\mathfrak{D}) = \mathfrak{D}^n + c_1 \mathfrak{D}^{n-1} + c_2 \mathfrak{D}^{n-2} + \dots + c_{n-1} \mathfrak{D} + c_n = \sum_{j=0}^n c_j \mathfrak{D}^{n-j} \quad (92)$$

with constant coefficients c_j (setting $c_0 = 1$ to be monic) and the derivative operator \mathfrak{D}^{n-j} which becomes a unit when $j = n$. We call $p(\mathfrak{D})$ a polynomial differential operator with constant coefficients. We think of the formal polynomial $p(\mathfrak{D})$ as operating on a function $y(x)$, converting it into another function; it is like machine, in which the function $y(x)$ is fed in, and $p(\mathfrak{D})y$, i.e., the left side of (90) is produced.

3.1 Operator rules

Since the differential operator is a linear operator, it naturally implies the additivity $f(x+y) = f(x) + f(y)$ and homogeneity $f(\alpha x) = \alpha f(x)$, $\forall \alpha$. All the properties of differentiation rules in calculus will apply to the operators, but there are other peculiarities that need to summarize. In stating these rules, we will always assume that the functions involved are smooth (or sufficiently differentiable), so that the operators can be applied to them. We assume that the linear operators do not commute, i.e., the pre-/left-multiplication is different from post-/right-multiplication, except for constants. The first **four** rules (identity, sum, linearity and multiplication) are purely linearity requirements. The other **two** (substitution and translation) are manifesting the process of finding eigenfunctions. Note this part basically summarizes the lecture notes from ocw.mit.edu/courses/18-03.

3.1.1 Identity rule

Identity rule of \mathfrak{D} acting on any object (say function u) is just returning the same object as simply

$$\mathfrak{D}^0 u = u \quad (93)$$

3.1.2 Additive rule

If $p(\mathfrak{D})$ and $q(\mathfrak{D})$ are two polynomial operators, then for any (sufficiently differentiable) function u ,

$$(p(\mathfrak{D}) + q(\mathfrak{D}))u = p(\mathfrak{D})u + q(\mathfrak{D})u. \quad (94)$$

3.1.3 Linearity rule

If u_1 and u_2 are functions, and c_i constants,

$$p(\mathfrak{D})(c_1u_1 + c_2u_2) = c_1p(\mathfrak{D})u_1 + c_2p(\mathfrak{D})u_2. \quad (95)$$

The linearity rule is a familiar property of the operator $a \mathfrak{D}^k$. Together with the additive rule (94), thus (92) is true for operators which are polynomials in \mathfrak{D} . Note this rule is still valid even if the coefficients c_i in (92) are not constant, but functions of x , i.e., it also works when u_1 and u_2 are composite functions.)

3.1.4 Composition rule

If $p(\mathfrak{D}) = f(\mathfrak{D})g(\mathfrak{D})$, as polynomials in \mathfrak{D} , then

$$p(\mathfrak{D})u = f(\mathfrak{D})(g(\mathfrak{D})u). \quad (96)$$

This brackets illustrate the layers in the right side of (96). The property is obviously true when $f(\mathfrak{D})$ and $g(\mathfrak{D})$ are the simple operators like \mathfrak{D}^k , essentially because $\mathfrak{D}^m(\lambda\mathfrak{D}^k u) = \lambda\mathfrak{D}^{m+k}u$ which extends the linearity to the general polynomial operators $g(\mathfrak{D})$. Note that λ must be a constant; it's false otherwise. An important corollary of the multiplication property is that polynomial operators with constant coefficients (like the constant λ) commute; i.e., for every function $u(x)$,

$$p(\mathfrak{D})u(x) = f(\mathfrak{D})g(\mathfrak{D})u = g(\mathfrak{D})f(\mathfrak{D})u = f(\mathfrak{D})(g(\mathfrak{D})u) = g(\mathfrak{D})(f(\mathfrak{D})u). \quad (97)$$

Since \mathfrak{D} represents an operator, it naturally only applies with integral exponents and shall also include negative integers. But note that there will appear confusions if we do not have convention. Now if we have $\mathfrak{D}\mathfrak{D}^{-1}u = u$, then $\mathfrak{D}^{-1}\mathfrak{D}u = u + c$ where c is an integral constant. Therefore, for the composition rule to apply, i.e., $\mathfrak{D}^{-1}\mathfrak{D}u = \mathfrak{D}\mathfrak{D}^{-1}u$, we claim that the arbitrary constant is to be neglected when the inverse differential operator is performed. The remaining two rules are of a different type, and more concrete: they tell us how polynomial operators behave when applied to exponential functions and products involving exponential functions.

3.1.5 Substitution rule

$$p(\mathfrak{D})e^{\lambda x} = p(\lambda)e^{\lambda x} \quad (98)$$

Proof: We have, by repeated differentiation,

$$\mathfrak{D}e^{\lambda x} = \lambda e^{\lambda x}, \quad \mathfrak{D}^2 e^{\lambda x} = \lambda^2 e^{\lambda x}, \quad \dots, \quad \mathfrak{D}^k e^{\lambda x} = \lambda^k e^{\lambda x}; \quad (99)$$

therefore,

$$(\mathfrak{D}^n + c_1 \mathfrak{D}^{n-1} + \dots + c_n) e^{\lambda x} = (\lambda^n + c_1 \lambda^{n-1} + \dots + c_n) e^{\lambda x}, \quad (100)$$

which is the substitution rule (98).

3.1.6 Translation rule - I

This handles expressions such as $x^m e^{\lambda x}$ (therefore $x^m \cos \lambda x$ and $x^m \sin \lambda x$).

$$p(\mathfrak{D}) e^{\lambda x} u = e^{\lambda x} p(\mathfrak{D} + \lambda) u. \quad (101)$$

Proof It would be easier to breakdown the compact expression into individual terms and the linear combination will naturally hold. Successively, start with $p(\mathfrak{D}) = \mathfrak{D}$, this is true

$$\mathfrak{D} e^{\lambda x} u(x) = e^{\lambda x} \mathfrak{D} u(x) + \lambda e^{\lambda x} u(x) = e^{\lambda x} (\mathfrak{D} + \lambda) u(x) \quad (102)$$

by the product rule for differentiation, To show the rule is also true for \mathfrak{D}^k , we apply \mathfrak{D} to (102) repeatedly:

$$\mathfrak{D}^2 e^{\lambda x} u = e^{\lambda x} (\mathfrak{D} + \lambda) ((\mathfrak{D} + \lambda) u) = e^{\lambda x} (\mathfrak{D} + \lambda)^2 u \quad (103)$$

In the same way,

$$\mathfrak{D}^3 e^{\lambda x} u = \mathfrak{D} (\mathfrak{D}^2 e^{\lambda x} u) = \mathfrak{D} (e^{\lambda x} (\mathfrak{D} + \lambda)^2 u) \quad (104)$$

$$= e^{\lambda x} (\mathfrak{D} + \lambda) ((\mathfrak{D} + \lambda)^2 u) \quad (105)$$

$$= e^{\lambda x} (\mathfrak{D} + \lambda)^3 u \quad (106)$$

and so on. This shows that translation rule (101) is true for an operator of the form \mathfrak{D}^k . For the power function $u(x) = x^m$, it shows

$$\mathfrak{D}^k x^m e^{\lambda x} = e^{\lambda x} (\mathfrak{D} + \lambda)^k x^m. \quad (107)$$

Now we obtained an efficient differential operator by translation to reduce the power of the variable, or even to lead a annihilator which is to operated on something and can obliterate it (becomes zero). For instance, if we want to make x^2 vanish, we need to differentiate three times, but in order to obliterate $e^{\lambda x} x^2$, we have to use $(\mathfrak{D} - \lambda)^4$. Questions: if we need to annihilate

1. $x^m e^{\alpha x}$,
2. $x^m e^{\alpha x} \cos \beta x$

by differentiation, what operators should apply¹⁰?

A special case can be seen for (101) when $u(x)$ is a constant function, say y_0

$$p(\mathfrak{D})e^{\lambda x}u = e^{\lambda x}p(\mathfrak{D} + \lambda)y_0 = e^{\lambda x}p(\lambda)y_0. \quad (108)$$

The most common usage is to annihilate the source term at the right hand side of a ODE. For instance if $q(x) = y_0e^{-\lambda x}$ in (90) with y_0 a constant, then $\mathfrak{D}^2(\mathfrak{D} + \lambda)^{-1}q(x) = 0$.

3.1.7 Translation rule - II

This will handle the functions like $x^m u(x)$. We just need to show the case for $m = 1$ and the cases for $m > 1$ can be followed similarly.

$$p(\mathfrak{D})[xu(x)] = xp(\mathfrak{D})u(x) + p'(\mathfrak{D})u(x). \quad (109)$$

The proof can be simply elaborated by applying the same procedure as in Section and use

$$\mathfrak{D}^n[x^m u(x)] = \sum_{k=0}^n \binom{n}{k} \mathfrak{D}^k x^m \mathfrak{D}^{n-k} u(x). \quad (110)$$

Note that $p'(\mathfrak{D}) = n\mathfrak{D}^{n-1}$ if $p(\mathfrak{D}) = \mathfrak{D}^n$.

3.2 Some ODEs in Terms of Operators

3.2.1 General ODEs and Operators

With \mathfrak{D} operator

$$\mathfrak{D}^n y + c_1 \mathfrak{D}^{n-1} y + c_2 \mathfrak{D}^{n-2} y + \dots + c_{n-1} \mathfrak{D} y + c_n y = q(x) \quad (111)$$

so

$$y = \frac{1}{p(\mathfrak{D})} q(x) \quad (112)$$

where the monic differential operator polynomial $p(\mathfrak{D})$ is defined by (92). Note that the derivative operator \mathfrak{D}^{n-j} becomes a unit when $j = n$. In special case when the right side is of the form $xq(x)$ we can use following shortcut

$$y_p = \frac{1}{p(\mathfrak{D})} xq(x) = x \frac{1}{p(\mathfrak{D})} q(x) - \frac{p'(\mathfrak{D})}{p^2(\mathfrak{D})} q(x). \quad (113)$$

where $p'(\mathfrak{D})$ is the derivative with respect to its argument \mathfrak{D} . For instance $p'(\mathfrak{D}) = n\mathfrak{D}^{n-1}$ if $p(\mathfrak{D}) = \mathfrak{D}^n$. If we have a smooth function $f(x)$ (i.e., arbitrarily differentiable) then

$$p(\mathfrak{D}) [e^{\lambda x} f(x)] = e^{\lambda x} p(\mathfrak{D} + \lambda) f(x) \quad (114)$$

$$p(\mathfrak{D}) [xf(x)] = xp(\mathfrak{D})f(x) + p'(\mathfrak{D})f(x) \quad (115)$$

¹⁰ The answer to the first one is $(\mathfrak{D} - \alpha)^{m+2}$ or $\mathfrak{D}^2(\mathfrak{D} - \alpha)^m$ and the second can be solved by $[(\mathfrak{D} - \alpha)^2 + \beta^2]^{m+1}$.

If one complex function $z = f(\theta)e^{\pm i\theta}$, then

$$p(\mathfrak{D})z = e^{\mp i\theta} p(\mathfrak{D} \pm i)f(\theta). \quad (116)$$

The introduced methods can be applied to find the particular solutions since in general the standard approach can readily find the inhomogeneous solutions. We take one example to show the application of the operator algorithm rules. In this way, we need two steps to find the solution to the target ODE. The undetermined coefficients in the complete solution is to be solved by the initial conditions. However the operator techniques as stated earlier in (26) can be used to solve the ODE in one go. Details will be given later.

$$y'' - 2y' + 5y = e^x \cos(2x) \implies (D^2 - 2D + 5)y = e^x \cos(2x) \quad (117)$$

To find the particular solution, we can divide the operator polynomial

$$y_p = \frac{1}{D^2 - 2D + 5} e^x \cos(2x) \quad (118)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 5} \cos(2x) \quad (119)$$

$$= e^x \frac{1}{D^2 + 4} \cos(2x) \quad (120)$$

$$= e^x \Re \left\{ e^{2ix} \frac{1}{(D+2i)^2 + 4} \right\} \quad (121)$$

$$= e^x \Re \left\{ e^{2ix} \frac{1}{(D+4i)} \frac{1}{D} \right\} \quad (122)$$

$$= e^x \Re \left\{ e^{2ix} \frac{1}{4i(1 - \frac{Di}{4})} x \right\} \quad (123)$$

$$= e^x \Re \left\{ \frac{-ie^{2ix}}{4} \left(1 + \frac{Di}{4} \right) x \right\} \quad (124)$$

$$= e^x \Re \left\{ \frac{-ie^{2ix}}{4} \left(x + \frac{i}{4} \right) \right\} \quad (125)$$

$$= \frac{x}{4} e^x \sin(2x) + \frac{1}{16} e^x \cos(2x) \quad (126)$$

$$(127)$$

where the second addend in the last line is just a scaled source term, a by-product of particular solution. So the general solution is

$$y(x) = e^x [c_1 \sin(2x) + c_2 \cos(2x)] + \frac{x}{4} e^x \sin(2x), \quad c_1, c_2 \text{ are constants.} \quad (128)$$

If we use the last line of the following operator techniques, the final particular solution can be immediately shown from (120). There are complementary operator rules in addition to Section 3.1 which can be used solve the operator equations.

3.2.1.1 Resonance

The solution (112) will fail if $p(r) = 0$. There can be three approaches to deal with this scenario.

First, we can take limit process as we find the 2nd order ODE solution in Section 3.2.3.2. Second, we factorize $p(r) = (\mathfrak{D} - r)p(r)$ where $p(r) \neq 0$. Third, we can make use of (109).

3.2.1.2 Inverse Differential Operator Rules

- General rules (denominator is non-zero)

$$\frac{1}{D + a} p(x) = e^{ax} * p(x) = e^{-ax} \int e^{ax} p(x) \quad (129)$$

$$\frac{1}{f(D)} e^{ax} p(x) = e^{ax} \frac{1}{f(D + a)} p(x) \quad (130)$$

$$\frac{1}{f(D)} x^n \sin ax = \Im \left\{ e^{iax} \frac{1}{f(D + ia)} x^n \right\} \quad (131)$$

$$\frac{1}{f(D)} x^n \cos ax = \Re \left\{ e^{iax} \frac{1}{f(D + ia)} x^n \right\} \quad (132)$$

$$\frac{1}{f(D)} x^n (\cos ax + i \sin ax) = \frac{1}{f(D)} x^n e^{iax} = e^{iax} \frac{1}{f(D + ia)} x^n \quad (133)$$

- For trigonometric functions, the following rules applies if $f(D)$ can be expressed as $p(D^2)$ and $p(-a^2) \neq 0$, then

$$\frac{1}{f(D)} \sin ax = \frac{1}{p(D^2)} \sin ax = \frac{1}{p(-a^2)} \sin ax \quad (134)$$

$$\frac{1}{f(D)} \cos ax = \frac{1}{p(D^2)} \cos ax = \frac{1}{p(-a^2)} \cos ax \quad (135)$$

- If $f(D)$ can be expressed as $p(D^2) = D^2 + a^2$, then $p(-a^2) = 0$.

$$\frac{1}{f(D)} \sin ax = \frac{1}{p(D^2)} \sin ax = x \frac{1}{p'(D^2)} \sin ax = x \frac{1}{2D} \sin ax = -\frac{x}{2a} \cos ax \quad (136)$$

$$\frac{1}{f(D)} \cos ax = \frac{1}{p(D^2)} \cos ax = x \frac{1}{p'(D^2)} \cos ax = x \frac{1}{2D} \cos ax = \frac{x}{2a} \sin ax \quad (137)$$

3.2.2 First Order ODEs

The typical 1st-order ODE with constant coefficient

$$\frac{dy}{dx} \pm \lambda y = e^{\mp \lambda x} \frac{d}{dx} (e^{\pm \lambda x} y) \quad (138)$$

can be written as

$$\left(\frac{d}{dx} \pm \lambda \right) y = e^{\mp \lambda x} \frac{d}{dx} (e^{\pm \lambda x} y). \quad (139)$$

Therefore, we find

$$(\mathfrak{D} \pm \lambda)^n = (\partial_x \pm \lambda)^n = e^{\mp \lambda x} \partial_x^n e^{\pm \lambda x}. \quad (140)$$

Using the $\mathfrak{D}ao$ operators, the ODE

$$y' + \lambda y = (\mathfrak{D} + \lambda)y(x) = q(x), \text{ with } y(0) = y_0 \quad (141)$$

can be readily solved by relating the expression (139) with the forcing term $q(x)$, i.e.,

$$e^{-\lambda x} \frac{d}{dx} (e^{\lambda x} y) = q(x) \quad (142)$$

Note the positive sign on the left of (139) is taken to be consistent with (141).

For the both sides of (142) first multiply with $e^{\lambda x}$ and integral from x_0 to x , we have

$$[e^{\lambda s} y(s)]_{s=x_0}^{s=x} = e^{\lambda x} y(x) - y(0) = \int_{x_0}^x e^{\lambda s} q(s) ds \quad (143)$$

After rearrangement, we obtain

$$y(x) = e^{-\lambda x} \int_{x_0}^x e^{\lambda s} q(s) ds + y_0 e^{-\lambda x}. \quad (144)$$

Look at the difference between (141) and (144)! The bracketed stuff before $y(x)$ is gone and the leftover $y(x)$ is what we were going to sought after. Doesn't it resemble that we solve $Ay(x) = b$ by dividing or inverting the coefficient A ? So we can think of the above process in finding the solution $y(x)$ is the same as applying one inverse operator $(\mathfrak{D} + \lambda)^{-1}$ to $q(x)$ in (141)

$$(\mathfrak{D} + \lambda)^{-1}(\mathfrak{D} + \lambda)y(x) = y(x) = (\mathfrak{D} + \lambda)^{-1}q(x). \quad (145)$$

Since the factor $e^{-\lambda x}$ of the integral in (144) is independent of the integration variable s , it can be safely moved inside. Therefore one generic solution of the 1st-order ODE is achieved.

$$y = (\mathfrak{D} + \lambda)^{-1}q(x) = \underbrace{\int_{x_0}^x e^{-\lambda(x-s)} q(s) ds}_{y_p: \text{ steady-state}} + \underbrace{y_0 e^{-\lambda x}}_{y_h: \text{ transient}} \quad (146)$$

which is in fact the scalar version of (59).

Since all differential equations can be thought of being composed of first order ODEs, we need to dissect it and look into the solution internals. We see that the solution (146) consists of two parts and then we will investigate what they are and how they are functioning through the differential equation. The integral on the right of (146) represents the particular solution y_p which is the convolution of the source (input or forcing) term $q(x)$ and the growth term $e^{-\lambda x}$ (the convolution is commonly written as $y_p = q(x) * e^{-\lambda x}$). The second term on the right is the homogeneous solution y_h (sometimes also called null solution y_n). These two parts describe how the system is evolving as the independent variable x (it might be better to use an independent variable t representing time for explanation instead of x which is usually used for location variable, so we switch to t for the time being). We can see that the particular solution y_p does not involve any effect from the initial value y_0 . The first term y_p only starts to grow (or decrease dependent on the sign of the system's eigenvalue λ) after the instant (say time $t = t_0$) when the input (switch) is activated. This is why its exponential has the

shift $x - s$ in the dummy integration variable. If we are imagining the driving force $q(x)$ to be made up of an infinite succession of infinitesimally kicks $q_i(x)$, by the superposition principle, the response of the system can then be obtained by adding up (via integration) the responses of the system to each of these kicks. These are the great advantages of linear system. One part of the solution, y_h , deals with how the system behaves with only internal coordination, not on how it is being driven by external environments. Therefore, it is purely the self-trait of the system model and the solution y_h tells what the physical model by (141) will naturally do if it is disengaged from constraints or human interaction and left to its own disposal, but with the use of its initial conditions (for instance some energy stored) in the system at time $t = t_0$. And the other part y_p takes care of the external forcing $q(x)$ by applying convolution into the internal mechanism. With this and linearity advantage¹¹, the input forcing $q(x)$ can even contain discontinuities (for instance Dirac Delta $\delta(x)$) as impulses since such functions can always be integrated. Now the transient solution y_h takes care of the initial (boundary) condition y_0 . The initial (boundary) value y_0 will decay if $\lambda > 0$ or accumulate if $\lambda < 0$.

And finally, we readily write the solution to the ODE with non-constant coefficient, i.e., $p(x)$ and $q(x)$ are functions of the independent variable x .

$$\dot{y} + p(x)y = q(x), \text{ with } y(0) = y_0 \quad (147)$$

is solved by

$$y = \int_{x_0}^x e^{\int_x^s p(\tau)d\tau} q(s)ds + y_0 e^{-\int_{x_0}^x p(s)ds} \quad (148)$$

Further, since the negative power of the differential operator represents the integral, we can claim that (140) can be generalized to either positive or negative powers.

$$(\mathfrak{D} \pm \lambda)^{\pm n} = (\partial_x \pm \lambda)^{\pm n} = e^{\mp \lambda x} \partial_x^{\pm n} e^{\pm \lambda x} \quad (149)$$

3.2.3 Second Order ODEs

The second (or any higher) order ODEs can be split to a linear system of 1st order equations. For instance, if incurring the initial condition and use a new variable for the first derivative of y , we can write $z = \mathfrak{D}y = \mathfrak{D}(y - y_0)$. Then (25) can be rewritten as

$$(\mathfrak{D} + p)z + qy = \mathfrak{D}y_1 + r \quad (150)$$

which can be solved to obtain (26). In comparison with the standard method, the homogeneous (or null) solution is constructed by the eigenvalues, say α and β such that

$$\mathfrak{D}^2 + p\mathfrak{D} + q = (\mathfrak{D} - \alpha)(\mathfrak{D} - \beta). \quad (151)$$

¹¹Generally speaking, an arbitrary inhomogeneity $q(x)$ in (141) can be decomposed into a continuous linear combination of inhomogeneities “each concentrated at one point” and having the form of shifted δ -functions. According to the linearity, the principle of superposition can be used to construct a particular solution of (141) with an arbitrary inhomogeneity $q(x)$.

The complete solution will consist of the homogeneous solution and the particular solution where the particular solution involves the convolution of the source terms and eigenvalue terms.

The coefficients of the homogeneous solution $y_h = c_1 e^{\alpha x} + c_2 e^{\beta x} (+c_3 e^{\gamma x} + \dots)$ are determined from the initial conditions, i.e., y_0 and y_1 ($y_2 \dots$).

3.2.3.1 Solution Formulas to ODEs

By simple algebra and partial fractions of (152) and (153), the generic solution formulas can be derived for 2nd and 3rd order ODEs, respectively.

$$\frac{1}{(\mathfrak{D} - \alpha)(\mathfrak{D} - \beta)} = \frac{1}{\alpha - \beta} \left(\frac{1}{\mathfrak{D} - \alpha} - \frac{1}{\mathfrak{D} - \beta} \right) \quad (152)$$

$$\frac{1}{(\mathfrak{D} - \alpha)(\mathfrak{D} - \beta)(\mathfrak{D} - \gamma)} = \frac{1}{(\alpha - \beta)(\alpha - \gamma)(\mathfrak{D} - \alpha)} - \frac{1}{(\alpha - \beta)(\beta - \gamma)(\mathfrak{D} - \beta)} + \frac{1}{(\alpha - \gamma)(\beta - \gamma)(\mathfrak{D} - \gamma)}. \quad (153)$$

$$y = \begin{bmatrix} e^{\alpha x} & e^{\beta x} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} * r(x) \quad (154)$$

$$= \boxed{\frac{(y_1 - \beta y_0)e^{\alpha x} - (y_1 - \alpha y_0)e^{\beta x}}{\alpha - \beta} + \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} * r(x)}. \quad (155)$$

The 3rd order ODE can be solved by

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + \frac{r(x)}{(\alpha - \beta)(\alpha - \gamma)(\mathfrak{D} - \alpha)} * e^{\alpha x} + \frac{r(x)}{(\alpha - \beta)(\beta - \gamma)(\mathfrak{D} - \beta)} * e^{\beta x} \quad (156)$$

$$+ \frac{r(x)}{(\alpha - \gamma)(\beta - \gamma)(\mathfrak{D} - \gamma)} * e^{\gamma x} \quad (157)$$

where the coefficients are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{\beta\gamma}{(\alpha - \gamma)(\alpha - \beta)} & \frac{-(\beta + \gamma)}{(\alpha - \gamma)(\alpha - \beta)} & \frac{1}{(\alpha - \gamma)(\alpha - \beta)} \\ \frac{-\alpha\gamma}{(\beta - \gamma)(\alpha - \beta)} & \frac{\alpha + \gamma}{(\beta - \gamma)(\alpha - \beta)} & \frac{-1}{(\beta - \gamma)(\alpha - \beta)} \\ \frac{\alpha\beta}{(\alpha - \gamma)(\beta - \gamma)} & \frac{-(\alpha + \beta)}{(\alpha - \gamma)(\beta - \gamma)} & \frac{1}{(\alpha - \gamma)(\beta - \gamma)} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}. \quad (158)$$

3.2.3.2 Solution formula containing repeated eigenvalues

In case of the repeated eigenvalues, the denominator in (155) becomes zero, so we have to take limit. Then it turns out to be

$$y = \lim_{\beta \rightarrow \alpha} \left(\frac{(y_1 - \beta y_0)e^{\alpha x} - (y_1 - \alpha y_0)e^{\beta x}}{\alpha - \beta} + \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} * r(x) \right) \quad (159)$$

$$= e^{\alpha x} [x(y_1 - \alpha y_0) + y_0] + (x e^{\alpha x}) * r(x). \quad (160)$$

For repeated eigenvalues, the solution to the higher order ODE becomes complicated so other methods must be more effective.

4 Integration by differentiation

Various methods have been proposed for numerical damping of wave motions in diffraction analysis, and in computational fluid dynamics to simulate the viscous effects (see [Li \[2015\]](#) for a review). Popular commercial model introduces a damping term proportional to the free surface elevation inserted in the free surface boundary condition (scheme 6 of [Li \[2015\]](#)). The modified free-surface boundary condition becomes

$$\left[\partial_z \left(1 + \frac{i\omega}{g} \epsilon_w \right) - k \right] \phi = \delta(z)\delta(t) \quad (161)$$

$$\phi = \phi_0 e^{kz/(1+\frac{i\omega}{g}\epsilon_w)} \quad (162)$$

$$\text{for H*: } \phi = \phi_0 e^{(1-i\epsilon_h)kz} \quad (163)$$

$$\text{So: } \epsilon_w = -i \frac{g}{\omega} = \frac{g}{\omega} \epsilon_h \quad (164)$$

{..... to be complete

We probably can unify the Delta and Heaviside function by one general function. Let I denote the finite unit impulse function, (a Fig?):

$$I(h, \theta - \theta_0) = \begin{cases} \frac{1}{h} & \text{when } \theta_0 \leq \theta \leq \theta_0 + h, \\ 0 & \text{when } 0 \leq \theta < \theta_0 \text{ or } \theta_0 + h < \theta \leq 2\pi \end{cases} \quad (165)$$

where h is a positive number and $0 \leq \theta_0 < \theta_0 + h < 2\pi$. Note that

$$\int_{\theta_0}^{\theta_0+h} I(h, \theta - \theta_0) d\theta = 1. \quad (166)$$

Let's start with the power series as eigenvalues with the most important eigenfunction e^{ixt} (which actually forms orthogonal basis of Hilbert space [Strang \[2015\]](#)). It is trivial to know that

$$(-i\partial_t)^n e^{ixt} = x^n e^{ixt}. \quad (167)$$

If we have a “smooth” (as per Ch.[1](#)) function $f : \mathbb{R} \rightarrow \mathbb{R}$ is expanded in power series

$$f(x) = c_0 + c_1 x + \dots + c_n x^n = \sum_{k=0}^{\infty} c_k x^k. \quad (168)$$

then we can construct one special Dao operator $f(-i\mathfrak{D}_x)$, or probably better written as $f(-i\partial_x)$ to represent partial differentiation in general

$$f(-i\partial_t) e^{ixt} = \sum_{k=0}^{\infty} c_k (-i\partial_t)^k e^{ixt}. \quad (169)$$

In fact, for any complex number r , we can define one operator $f(r\partial_x)$ and it can impose on another smooth function $p(x)$ such that

$$f(r\partial_x) p(x) = \lim_{N \rightarrow \infty} \sum_{k=0}^N a_k (r\partial_x)^k p(x) = p(x+r) \quad (170)$$

4.1 Dirac Delta Function

In Wiki's definition, the mysterious Dirac delta ($\delta(x)$ distribution), also known as the unit impulse, is a generalized function or distribution¹² over the real numbers.

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad (171)$$

The Kronecker delta function (δ_{ij}), which is usually defined on a discrete domain and takes values 0 and 1, is the discrete analog of the Dirac delta function. A sequence of 2π -periodic $f_n(x)$ of function $f(x)$ can be called approximate identity if

$$1. f_n(x) = nf(nx) \geq 0 \text{ for } \forall n \in \mathbb{N} \quad (172)$$

$$2. \int_{-\pi}^{\pi} f_n(x) dx = 1 \text{ for } \forall n \in \mathbb{N} \quad (173)$$

$$3. \lim_{n \rightarrow \infty} \int_{\varepsilon \leq |x| \leq \pi} f_n(x) dx = 0 \text{ for } \forall \varepsilon > 0. \quad (174)$$

4.1.1 Fourier series and Dirac Delta

A periodic function $f(t) = f(t + T)$ of period T and circular frequency $\omega = 2\pi/T$ can be expanded into Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t} \quad (175)$$

where the coefficients c_k are given by the formula

$$c_k = \frac{1}{T} \int_0^T f(t) e^{-ik\omega t} dt. \quad (176)$$

Next we will prove one important property of δ -function by inserting $f(t) = \delta(t)$ in (176). The Fourier transform of $f(t)$ will be

$$F(\omega_s) = \int_{-\infty}^{\infty} f(t) e^{-ik\omega_s t} dt \quad (177)$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{ik(\omega - \omega_s)t} dt \quad (178)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_s). \quad (179)$$

Consider the simple case of the signal $f(t)$ being a cosine signal $f(t) = \cos(\omega_0 t)$ which is periodic with period $T = 2\pi/\omega_0$. In fact the most famous Euler's formula

$$\cos \omega_0 t = \frac{1}{2} (e^{-i\omega_0 t} + e^{i\omega_0 t}) \quad (180)$$

¹²Often called Schwartz distributions

can also be proved with the exponential Fourier series representation of $\cos \omega_0 t$, the coefficients $c_1 = c_{-1} = \frac{1}{2}$ ($c_k = 0$ for $|k| \neq 1$).

From (176), by $\omega = \omega_0 = 2\pi/T$ thus $\omega_0 T = 2\pi$,

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T \cos(\omega_0 t) e^{-ik\omega_0 t} dt \\ &= \frac{1}{2T} \int_0^T [e^{i\omega_0 t} + e^{-i\omega_0 t}] e^{-ik\omega_0 t} dt \\ &= \frac{1}{2T} \int_0^T [e^{i(1-k)\omega_0 t} + e^{-i(1+k)\omega_0 t}] dt \end{aligned} \quad (181)$$

$$\stackrel{?}{=} \frac{1}{2T} \left[\frac{e^{i(1-k)\omega_0 t}}{i(1-k)\omega_0} \Big|_0^T + \frac{e^{-i(1+k)\omega_0 t}}{-i(1+k)\omega_0} \Big|_0^T \right] \quad (182)$$

$$= \frac{1}{4\pi} \left[\frac{e^{i(1-k)2\pi} - 1}{i(1-k)} + \frac{e^{-i(1+k)2\pi}}{-i(1+k)} \right] \quad (183)$$

$$= \frac{1}{4\pi} \left[\frac{e^{i(1-k)2\pi} - 1}{i(1-k)} + \frac{e^{-i(1+k)2\pi}}{-i(1+k)} \right]. \quad (183)$$

Should we proceed the integral in (181) before we are sure that its convergence?

Since, j and k are integers so $e^{i(1-k)2\pi} = e^{-i(1+k)2\pi} = 1$ hence the above expression is zero for all k , except $k = \pm 1$ where the denominator terms above are also zero.

When $k = 1$, then returning to (181) indicates that the first integrand is equal to 1, and hence $c_{\pm 1} = \frac{1}{2}$.

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ft} dt \quad (184)$$

$$= \int_{-\infty}^{\infty} \cos(2\pi st) e^{-i2\pi ft} dt \quad (185)$$

$$= \int_{-\infty}^{\infty} \cos(2\pi st) [\cos(-2\pi ft) + i \sin(-2\pi ft)] dt \quad (186)$$

$$= \int_{-\infty}^{\infty} \cos(2\pi st) \cos(-2\pi ft) dt + i \int_{-\infty}^{\infty} \cos(2\pi st) \sin(-2\pi ft) dt \quad (187)$$

$$= \int_{-\infty}^{\infty} \cos(2\pi st) \cos(2\pi ft) dt \quad (188)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} \{ \cos[2\pi(f+s)t] + \cos[2\pi(f-s)t] \} dt \quad (189)$$

$$= \frac{1}{4\pi} \left\{ \frac{\sin[2\pi(f+s)t]}{f+s} + \frac{\sin[2\pi(f-s)t]}{f-s} \right\}_{t \rightarrow -\infty}^{t \rightarrow \infty} \quad (190)$$

$$= \begin{cases} \frac{1}{2}\delta(f+s) + \frac{1}{2}\delta(f-s) & \text{if } f = \pm s \\ 0 & \text{if } f \neq \pm s \end{cases} \quad (191)$$

4.1.2 Kronecker and Dirac Deltas

4.1.3 Dao operator and Delta

We begin by assuming the conventions for the Fourier transform:

$$\hat{g}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{itx} dx \quad (192)$$

its corresponding inversion formula, producing a symmetrical relationship

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(t) e^{-itx} dt. \quad (193)$$

To simplify the notation we denote integration over the real line by the absence of integration delimiters. Assume now that g is a smooth function (see Ch.1 for definition). Then

$$\delta(x) = \frac{1}{2\pi} \int e^{ixt} dt \quad (194)$$

$$= \frac{1}{2\pi} \int \frac{1}{g(t)} g(t) e^{ixt} dt \quad (195)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{g(-i\partial_x)} \left[\frac{1}{\sqrt{2\pi}} \int g(t) e^{ixt} dt \right] \quad (196)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{g(-i\partial_x)} \hat{g}(x). \quad (197)$$

Here, g must be sufficiently smooth such that $1/g$ can be expanded as a power series, which then gives meaning to $1/g(-i\partial_x)$ as a series in derivatives.

By this method, each suitable choice of g in (197) yields an exact representation of the Dirac delta. For example, if we choose g to be a Gaussian

$$g(x) = e^{-\frac{\sigma x^2}{2}} \quad (198)$$

$$\hat{g}(x) = \frac{1}{\sqrt{\sigma}} e^{-\frac{x^2}{2\sigma}} \quad (199)$$

with $\sigma > 0$, then

$$\delta(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\sigma\partial_x^2} e^{\frac{-x^2}{2\sigma}} \quad (200)$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}\sigma\partial_x^2)^k}{k!} e^{\frac{-x^2}{2\sigma}} \quad (201)$$

We obtain approximations of $\delta(x)$ by truncating the series of derivatives in equation (201). For example, when truncating the series after the first term, we obtain the standard Gaussian approximation $\delta(x) \approx \sqrt{2\pi\sigma} e^{-\frac{x^2}{2\sigma}}$ which converges to $\delta(x)$ as $\sigma \rightarrow 0$ in the weak limit.

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$f(-i\partial_t) e^{ixt} = \sum_{k=0}^{\infty} a_k (-i\partial_t)^k e^{ixt} \quad (202)$$

The normalized sinc function¹³ can be used as a representations of the delta function, i.e.,

$$\lim_{a \rightarrow 0} \frac{\sin(\frac{\pi x}{a})}{\pi x} = \lim_{a \rightarrow 0} \frac{1}{a} \operatorname{sinc}\left(\frac{x}{a}\right) = \delta(x). \quad (204)$$

Note that the limit in (204) is not an ordinary limiting process, since it does not converge. Rather, it converges in the sense of distributions or as a sequence of functions for all smooth test functions $f(x)$.

$$\lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{a} \operatorname{sinc}\left(\frac{x}{a}\right) f(x) dx = f(0) \quad (205)$$

5 Roll motion of a vessel - method of equivalent linearization - Regular waves and improvement

The equation of ship motions in the time domain and the coefficients are given by,

$$\sum_{i=1}^6 (M_{ij} + m_{ij}(\infty)) \ddot{x}_i(t) + \sum_{i=1}^6 \int_0^t L_{ij}(t-\tau) \dot{x}_i(\tau)(\tau) d\tau + \sum_{i=1}^6 (C_{ij} + G_{ij}) x_i(t) = F_j \quad (206)$$

¹³The normalized sinc function is commonly defined for $x \neq 0$ by

$$\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x} \quad (203)$$

where

$$L_{ij}(t) = \frac{2}{\pi} \int_0^\infty [A_{ij}(\omega) - A_{ij}(\infty)] \cos(\omega t) d\omega = \frac{2}{\pi} \int_0^\infty B_{ij}(\omega) \cos(\omega t) d\omega \quad (207)$$

$$m_{ij}(\infty) = A_{ij}(\omega) + \frac{1}{\omega} \int_0^\infty L_{ij}(t) \sin(\omega t) dt, \quad (208)$$

and

- M_{ij} : mass matrix (including moment of inertia)
- $m_{ij}(\infty)$: added mass (moment of inertia) matrix
- $x_i(t)$: displacement
- $L_{ij}(t)$: is the retardation function
- C_{ij} : hydrostatic restoring force coefficient
- G_{ij} : external (mooring) restoring force coefficient
- F_j : the external forcing
- t : time
- τ : integral variable
- $A_{ij}(\omega)$: added mass or added moment of inertia
- $B_{ij}(\omega)$: damping coefficient.

The retardation functions are concerning the integration to infinite frequency. The constant added masses is evaluated at the infinite frequency in (206). On the other hand, the retardation functions show the decaying oscillatory in all time which cause difficulty in the numerical integration. There are different approaches in dealing with this. For instance, [Wamit \[2007\]](#) has propose a truncation method and in [Kubo et al. \[1988\]](#) one alternative approximation is by,

$$L_{ij}(t) = r e^{-pt} \cos(qt + \epsilon) \quad (209)$$

where r , p , q and e are arbitrary constants. And the damping coefficients are given by applying the inverse Fourier transform to (207),

$$B_{ij}(\omega) = \int_0^\infty L_{ij}(t) \cos(\omega t) dt \quad (210)$$

Substituting (209) into (210) and carrying out the integration by parts in the right hand side, then

$$B_{ij}(\omega) = \frac{r}{2} \left[\frac{p \cos(\epsilon) - (q + \omega) \sin(\epsilon)}{p^2 + (q + \omega)^2} + \frac{p \cos(\epsilon) - (q - \omega) \sin(\epsilon)}{p^2 + (q - \omega)^2} \right] \quad (211)$$

where the constants r , p , q and e are decided by comparing the values obtained from (209) and (211) with ones from the numerical integral calculation. A new approach using Lobachevsky's formula ([Jolany \[2018\]](#)) is probably efficient and accurate.

5.1 Hilbert Transform for identification

The Hilbert transform (HT), as a kind of integral transformation, plays a significant role in signal processing. There are two common ways it can be used. First, the HT provides a direct examination of a vibration's instantaneous attributes: frequency, phase, and amplitude. It allows rather complex systems to be analyzed in the time domain. Second, the HT can find a system's real part of transfer function from the system's imaginary part and vice versa. This allows systems to be analyzed in the frequency domain. The HT can be used as an intermediate step in more elaborate system analysis. In addition to frequency response function analysis, it is useful for hysteretic damping characterizing and nonlinear system identification.

Let $\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$ be a complex function of the complex variable ω , where $\chi_1(\omega)$ and $\chi_2(\omega)$ are real. Suppose this function is Complex analytic function in the closed upper half-plane of ω and tends to 0 as $|\omega| \rightarrow \infty$. The Kramers–Kronig relations are given by

$$\chi_1(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega} d\omega' \quad (212)$$

and

$$\chi_2(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\chi_1(\omega')}{\omega' - \omega} d\omega', \quad (213)$$

where ω is real and where \mathcal{P} denotes the Cauchy principal value. The real and imaginary parts of such a function are not independent, allowing the full function to be reconstructed given just one of its parts.

The wave excitation contain the radiation and diffraction parts. The radiation loads are evaluated from damping $B_{ij}(\omega)$ and added mass $A_{ij}(\omega)$.

$$L_{ij}(j\omega) = B_{ij}(\omega) + j\omega[A_{ij}(\omega) - A_{ij}(\infty)] \quad (214)$$

The analytic representation of any real-valued response can be constructed from the original function and its Hilbert transform. For instance the roll decay motion signal can be converted into analytic one where its envelope (modulated) contains the damping information extracted from the original carrier signal. Using simple system identification, the damping coefficients can be obtained.

5.2 The method of equivalent linearization

The method of equivalent linearization is the most convenient way to solve a non-linear system with inevitable nonlinear and non-constant coefficient of the differential equation. The principle of this method is to express the non-linear equation of motion in a linearized by considering an equivalent amount of energy consumed during one cycle such that the work for each cycle is the same for the two equivalent systems. Thus we can utilized all the linearity properties by a equivalent linear damping term where the energy loss for the linear

equation shall be the same as the real energy loss per cycle. This approach of linearization process is called the method of equivalent linearization.

On one side, this approach solves the equation of motion efficiently, but on the other side, it assumes that the equivalent linearized coefficient holds constant during each cycle or the periods from one cycle to the other are invariant. However these are not true in reality. The typical case is when we do decay test which itself is used to determine the viscous damping coefficient. The standard roll motion equation in time domain can be written as

$$(M + A_\infty)\ddot{\phi} + B|\dot{\phi}|\dot{\phi} + C\phi + \int_0^t h(t - \tau)\dot{\phi}(\tau)d\tau = F_4(t). \quad (215)$$

To simulate the roll decay test, the forcing $F_4(t)$ vanishes. To illustrate the variation of the decaying effect of the solution, we simplify our system equation as

$$(\mathfrak{D}^2 + 1)\phi = \epsilon f(\phi, \dot{\phi}) \quad (216)$$

where $f(\phi, \dot{\phi})$ is the forcing term including all the damping terms. Let's consider a simpler case, a first-order nonlinear decay equation,

$$\dot{x} + |x|^{p-1}x = 0. \quad (217)$$

The solution has the form $x = \pm [(p-1)(t - t_0)]^{\frac{1}{1-p}}$ with certain conditions on t_0 . When $p < 1$, the solution drops to zero in finite time; when $p > 1$, it decays roughly as $t^{-1/(p-1)}$ which is much slower than exponential.

As a classical example, we look at $f(\phi, \dot{\phi}) = -\dot{\phi}^3$ in (216). A typical secular term will arise if a regular perturbation method is applied.

It means that we expand the solution $x(t)$ as a power series in a small parameter ϵ :

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots, \quad (218)$$

with initial conditions $x_0(0) = 1$, $\dot{x}_0(0) = 0$, and $x_n(0) = \dot{x}_n(0) = 0$ for $n \geq 1$. According to the powers of ϵ , a sequence of linear differential equations can be listed

$$(\mathfrak{D}^2 + 1)x_0 = 0 \quad (219)$$

$$(\mathfrak{D}^2 + 1)x_1 = -\dot{x}_0 \quad (220)$$

$$\dots \quad (221)$$

$$(\mathfrak{D}^2 + 1)x_n = -\dot{x}_{n-1}^3 \quad (222)$$

The solution to (219) with $x_0(0) = 1$ and $\dot{x}_0(0) = 0$ is simply $x_0(t) = \cos(t)$. Then the next order equation

$$(\mathfrak{D}^2 + 1)x_1 = -\dot{x}_0^3 = \sin^3(t) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t) \quad (223)$$

can be solved by

$$x_1 = \frac{1}{\mathfrak{D}^2 + 1} \left(\frac{3}{4} \sin(t) - \frac{1}{4} \sin(3t) \right). \quad (224)$$

We notice that $\sin(t)$ in the forcing term has the eigenvalue 1 as a root in left hand side, but $\sin(3t)$ is not. So let y_1 be the solution to

$$y_1 = \frac{1}{\mathfrak{D}^2 + 1} \left(-\frac{1}{4} \sin(3t) \right) \quad (225)$$

$$= \frac{1}{-3^2 + 1} \left(-\frac{1}{4} \sin(3t) \right) \quad (226)$$

$$= \frac{\sin(3t)}{32}. \quad (227)$$

and rest is to apply the formula (14) repeatedly

$$y_1 = \frac{1}{\mathfrak{D}^2 + 1} \left(\frac{3}{4} \sin(t) \right) \quad (228)$$

$$= \frac{3}{4} \Im \left(\frac{1}{\mathfrak{D}^2 + 1} e^{it} \right) \quad (229)$$

$$= \frac{3}{4} \Im \left(\frac{1}{(\mathfrak{D} - i)^2} \frac{\mathfrak{D}}{\mathfrak{D} + i} \right) \quad (230)$$

$$= \frac{3}{4} \Im \left(\frac{1}{(\mathfrak{D} - i)^2} e^{-it} \right) \quad (231)$$

$$= \frac{3}{4} \Im \left(e^{-it} \frac{1}{(\mathfrak{D} - 2i)^2} \right) \quad (232)$$

$$= \frac{3}{4} \Im \left(e^{-it} \left(\frac{te^{2it}}{2i} + e^{2it} - \frac{1}{4} \right) \right) \quad (233)$$

$$= \frac{-3t \cos(t)}{8} + \frac{3 \sin(t)}{8}. \quad (234)$$

Note we split the solutions which doesn't necessarily satisfy the initial conditions therefore we need to still to complete it with homogeneous solution $c_1 \cos(t) + c_2 \sin(t)$. Summarize to get the solution $x_1 = \frac{\sin(3t)}{32} - \frac{-3t \cos(t)}{8} + \frac{9 \sin(t)}{32} + \cos(t)$. However, the solution is found not realistic since the term $t \cos(t)$, whose amplitude grows with t , is known as a secular term. If we assume the solution to (216) can be written in the form

$$\phi(t) = a(t) \cos(t + b(t)) \quad (235)$$

Then we expect the estimated differential looks like

$$\mathfrak{D}\phi(t) = -a(t) \sin(t + b(t)) \quad (236)$$

The motivation for this ansatz is that when ϵ is zero, (216) has the form (235) with a and b constants. For small values of ϵ we expect the same form of the solution can be approximately valid, but with a and b be slowly varying functions of t . Differentiating (235) to represent (236), we need the following relation to hold

$$\dot{a} \cos(t + b(t)) - ab \sin(t + b(t)) = 0. \quad (237)$$

Solving these conditions, we can obtain the decaying solution

$$x(t) = \frac{1}{\sqrt{\frac{3}{4}\epsilon t + \frac{1}{a^2(0)}}} \cos(t) \quad (238)$$

where $a(0)$ is the initial amplitude at $t = 0$.

5.3 Dynamic Amplification Factor

For the 6Dof coupled equation system, the peak of RAO appears at the damped natural frequency, but in the hydrodynamic report, it is generally the undamped natural periods that are iteratively calculated based on the frequency dependent added mass (added moment of inertia). When forcing frequency of the incident wave is equal to the undamped natural frequency (i.e., frequency ratio $\Lambda = \omega/\omega_n = 1$), the RAO takes a local maximum as is called practical resonance with nonzero damping. These phenomena can be better seen from a parameter called dynamic amplification factor (or DAF which measures how many times a structural responses to a dynamic load magnifies compared to the amount when the load were applied statically).

5.3.1 Time Contant τ

Time Contant τ is the main characteristic unit of a first-order linear time-invariant (LTI) system, for instance (146). It characterizes the response to a step input of a first-order LTI system. In the frequency domain, the time constant τ also determines the bandwidth of a first-order LTI system, i.e., the frequency at which the output signal power drops to half the value it has at low frequencies. The homogeneous solution of (146) states the system response to an initial condition y_0 is $y_h(t) = y_0 e^{-t/\tau}$ where $\tau = 1/\lambda$. As the similar is elaborated following (146), if $\tau > 0$, the response of any system variable is an exponential decay from the initial value y_0 toward zero, and the system is stable. Otherwise the response grows for $\tau < 0$ exponentially from any finite value of y_0 , as shown in Fig.4. If we normalize time by the time constant τ and the response magnitude by the initial value as $y(t)/y_0$, and find the response levels after nondimensional time $\hat{t} = 1, 2, 3, 4$, the response levels will be decayed to $y(1\tau)/y_0 = e^{-1} \approx 36.8\%$, $y(2\tau)/y_0 = e^{-2} \approx 13.5\%$, $y(3\tau)/y_0 = e^{-3} \approx 0.5\%$ and $y(4\tau)/y_0 = e^{-4} \approx 0.2\%$. The first decayed level $t = 1\tau$ coincidentally has the same percentile as the MPM¹⁴ level as in extreme value distribution. Further, if $y_0 = \lambda$, the response solution y_h coincides with the probability density function (pdf) $f(x) = \lambda e^{-\lambda x}$ of an exponential failure distribution with the cumulative probability function $F(x) = 1 - e^{-\lambda x}$. So in this context,

¹⁴The MPM is the extreme value that is most likely to occur, which is equal to 36.8% for a linear Gaussian signal. For ocean engineering, the MPM is often considered as design value.

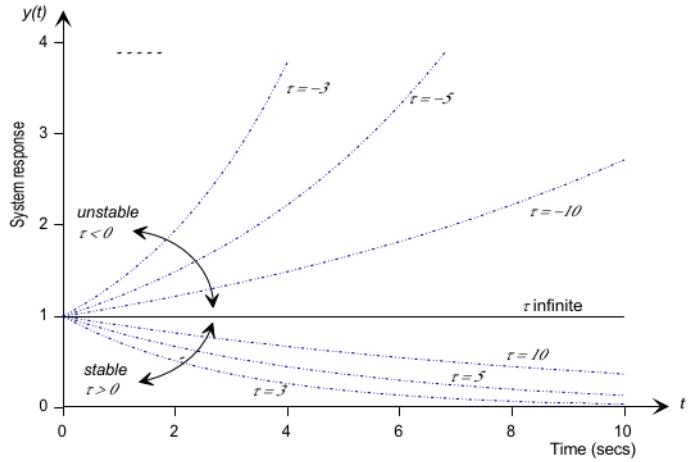


Fig. 4. Response of a first-order homogeneous solution of (146). The effect of the system time constant τ is shown for stable systems $\tau > 0$ and unstable systems $\tau < 0$. Courtesy of web.mit.edu/2.151.

$y_0 = \lambda$ is defined as failure rate, i.e., the frequency with which a system or component fails per unit of time and is often used in reliability engineering.

$$\lambda = \frac{f(t)}{1 - F(t)} = \frac{F'(t)}{1 - F(t)} \quad (239)$$

The bandwidth of this system is the frequency (here $\omega = \lambda$) where response amplitude drops to its half-value, corresponding to $\lambda\tau = 1$. Following signal processing convention, it is denoted by

$$f_{3\text{dB}} = \frac{1}{2\pi\tau} \quad (240)$$

This is so-called bandwidth defined as the frequency range where power drops by less than half. This notation $f_{3\text{dB}}$ stems from the expression of power in decibels and the observation that half-power corresponds to a drop in the value of response by a factor of $1/2$, since $(10 \log_{10} \frac{1}{2})\text{dB} \approx -3\text{dB}$. Thus, the time constant determines the bandwidth of this system.

5.3.2 Time domain solution

Let a linear system with mass m , damping b and stiffness k is excited by the external force $f(t)$ at initial conditions of displacement $x(0) = x_0$ and velocity v_0 . The equation of motion can be written as

$$m\ddot{x} + b\dot{x} + kx = f(t), \quad (241)$$

where the coefficients m , b and k are mass, damping and stiffness. They are independent of one another and time. As it can be seen later, the damping term b can have an important effect on the system's behavior. If the damping b reaches to the critical damping which

is $b_{cr} = 2\sqrt{mk}$, the system will lose the ability to oscillate, instead, the mass will quickly and smoothly returns to its equilibrium position x_0 . So how the damping is estimated will be critical for a system's response. The classical Rayleigh damping model can be used to construct a system damping matrix as a linear composition of the mass matrix and stiffness matrix. For our single degree-of-freedom, the damping coefficient b will be written as

$$b = \alpha k + \beta m, \quad (242)$$

where α is stiffness-proportional damping coefficient in unit [s] and β the mass-proportional damping with unit [s^{-1}]. The critical damping b_{cr} can be a valuable reference for the damping level of the system. So we can conveniently use a parameter ξ to represent the rate of the actual damping to the critical damping, i.e.,

$$\xi = b/b_{cr} \quad (243)$$

$$= \frac{1}{2m\omega_n}(\alpha k + \beta m) \quad (244)$$

$$= \frac{1}{2} \left(\alpha \omega_n + \frac{\beta}{\omega_n} \right), \quad (245)$$

where ω_n is another important parameter, the natural frequency of the dynamic system defined by $\omega_n = \sqrt{k/m}$. Thus, we note these

$$\frac{b}{m} = \frac{\xi b_{cr}}{m} = \xi \frac{2\sqrt{mk}}{m} = \sqrt{2\xi \frac{k}{m}} = 2\xi \omega_n \quad (246)$$

$$\frac{b}{k} = \frac{2\xi \sqrt{mk}}{k} = 2\xi \sqrt{\frac{m}{k}} = \frac{2\xi}{\omega_n} \quad (247)$$

and then (241) becomes

$$\begin{aligned} \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= \frac{f(t)}{m} \\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x &= \frac{f(t)}{m} \end{aligned} \quad (248)$$

What we obtained (248) is an inhomogeneous second-order ordinary differential equation which can be solved by different fundamental approaches, for instance through solving the characteristic equations to find the homogeneous solution plus any particular solution¹⁵. And our previous discussed formulas (59) or (155) can directly apply as well. But it is worth to show one alternative one, the powerful Laplace transforms. As we may always utilize, the “time” domain differential equation after Laplace transform turns into polynomial equations in terms of the “complex frequency”, i.e., $s = \sigma + i\omega$ domain. The roots of the polynomial equations in s domain correspond to eigenvalues in the time domain. Typically, the polynomials in the complex frequency domain (in the denominator) correspond to power series in the time domain, while time shifts in the complex frequency domain correspond to

¹⁵The complete solution of all linear equations can be found by the sum of the homogeneous (or null) solution with the initial conditions, and the particular solution with the source (forcing) term.

damping by decaying exponentials in the time domain. Let's take this roll equation of motion and solve it by Laplace approach.

After Laplace transform to both sides, (248) becomes

$$[(s^2 X(s) - sx_0 - v_0) + 2\xi\omega_n(sX(s) - x_0) + \omega_n^2 X(s)] = \frac{F(s)}{m} \quad (249)$$

where $X(s) = \mathcal{L}[x(t)]$ and $F(s) = \mathcal{L}[f(t)]$. Solving (249) obtains

$$\begin{aligned} X(s) &= \frac{F(s)}{m[(s + \xi\omega_n)^2 + (1 - \xi^2)\omega_n^2]} + \frac{x_0 s + 2\xi\omega_n x_0 + v_0}{(s + \xi\omega_n)^2 + (1 - \xi^2)\omega_n^2} \\ &= \frac{F(s)}{m[(s + \xi\omega_n)^2 + \omega^2]} + \frac{x_0(s + \xi\omega_n) + (\xi\omega_n x_0 + v_0)}{(s + \xi\omega_n)^2 + \omega^2} \end{aligned} \quad (250)$$

where we have used $\omega^2 = (1 - \xi^2)\omega_n^2$, implicitly including imaginary case. Actually ω can be one of the three cases, 1) ω is positive ($\xi < 1$), 2) $\omega = 0$ ($\xi = 1$) or 3) ω is imaginary ($\xi > 1$). The linear system (249) is then solved in s -domain by (250), which however is not the time-domain solution to (241). Therefore, we need to take inverse Laplace transform of (250) to obtain the time-domain solution. From Laplace transform table¹⁶, we know that the first term in (250) contains the product of $F(s)/m$ and a shifted Laplace transform of a sinusoidal function, i.e.,

$$\mathcal{L}^{-1}\left[\frac{1}{(s + \xi\omega_n)^2 + \omega^2}\right] = e^{-\xi\omega_n t} \frac{\sin(\omega t)}{\omega}. \quad (254)$$

The inverse Laplace transform of a product of two functions will result to a convolution integral. The two colored addends in the second term in (250) correspond to a cosine and sinusoidal Laplace transforms. Therefore, the full solution in time domain will be

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{F(s)}{m[(s + \xi\omega_n)^2 + \omega^2]} + \frac{x_0(s + \xi\omega_n) + (\xi\omega_n x_0 + v_0)}{(s + \xi\omega_n)^2 + \omega^2}\right] \\ &= e^{-\xi\omega_n t} \frac{\sin(\omega t)}{m\omega} * f(t) + x_0 e^{-\xi\omega_n t} \cos(\omega t) + (\xi\omega_n x_0 + v_0) e^{-\xi\omega_n t} \frac{\sin(\omega t)}{\omega} \\ &= \frac{1}{m\omega} \int_0^t f(\tau) e^{-\xi\omega_n(t-\tau)} \sin(\omega(t-\tau)) d\tau + \end{aligned} \quad (255)$$

$$+ e^{-\xi\omega_n t} \left[x_0 \cos(\omega t) + (\xi\omega_n x_0 + v_0) \frac{\sin(\omega t)}{\omega} \right] \quad (256)$$

1. If $\omega > 0$ ($\xi < 1$),

$$\begin{aligned} x(t) &= \frac{1}{m\omega} \int_0^t f(\tau) e^{-\xi\omega_n(t-\tau)} \sin(\omega(t-\tau)) d\tau \\ &\quad + e^{-\xi\omega_n t} \left[x_0 \cos(\omega t) + (\xi\omega_n x_0 + v_0) \frac{\sin(\omega t)}{\omega} \right] \end{aligned} \quad (257)$$

¹⁶Three formulas for Laplace transforms can be used.

$$\mathcal{L}^{-1}[f(s)g(s)] = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau) d\tau = \int_0^t f(t-\tau)g(\tau) d\tau \quad (251)$$

$$\mathcal{L}^{-1}\left[\frac{s+c}{(s+c)^2 + a^2}\right] = e^{-ct} \cos(at) \quad (252)$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+c)^2 + a^2}\right] = e^{-ct} \frac{\sin(at)}{a} \quad (253)$$

2. If $\omega = 0$ ($\xi = 1$)

$$\begin{aligned} x(t) &= \lim_{\omega \rightarrow 0} \left\{ \frac{1}{m\omega} \int_0^t f(\tau) e^{-\xi\omega_n(t-\tau)} \sin \omega(t-\tau) d\tau \right. \\ &\quad \left. + e^{-\xi\omega_n t} \left[x_0 \cos(\omega t) + (\xi\omega_n x_0 + v_0) \frac{\sin(\omega t)}{\omega} \right] \right\} \\ &= \frac{1}{m} \int_0^t (t-\tau) f(\tau) e^{-\omega_n(t-\tau)} d\tau + e^{-\xi\omega_n t} [x_0 + (\xi\omega_n x_0 + v_0) t] \end{aligned} \quad (258)$$

3. If ω is $\pm i\varpi = \pm i\omega_n \sqrt{\xi^2 - 1}$ ($\xi > 1$),

$$\begin{aligned} x(t) &= \frac{1}{m\varpi} \int_0^t f(\tau) e^{-\xi\omega_n(t-\tau)} \sinh \varpi(t-\tau) d\tau \\ &\quad + e^{-\xi\omega_n t} \left[x_0 \cosh(\varpi t) + (\xi\omega_n x_0 + v_0) \frac{\sinh(\varpi t)}{\varpi} \right]. \end{aligned} \quad (259)$$

5.3.3 Frequency domain solution

If a second order system (248) is driven by an exponential term, i.e., $f(t)/m = f_0 e^{i\omega t}$

$$\ddot{\phi} + 2\xi\omega_n \dot{\phi} + \omega_n^2 \phi = f_0 e^{i\omega t}. \quad (260)$$

It can be seen from ‘‘Exponential Response formula’’ that unless $\xi = 0$ and $\omega = \omega_n$ (in which case the system exhibits resonance, and has no periodic solutions), this has the particular solution

$$\phi_p = f_0 \frac{e^{i\omega t}}{p(i\omega)} \quad (261)$$

where $p(s) = s^2 + 2\xi s + \omega_n^2$ is the characteristic polynomial of the system. To put in a more familiar form by defining the transfer function $H(i\omega)$ (also called frequency response function or sometimes gain due to multiplication instead of ‘‘pain’’ by division), it becomes

$$\phi_p = f_0 H(i\omega) e^{i\omega t} \quad (262)$$

where $H(i\omega)$ can be written in polar form as,

$$W(i\omega) = \frac{1}{p(i\omega)} = g e^{-i\theta}. \quad (263)$$

so that the magnitude $g(\omega) = |H(i\omega)|$, and the phase $\theta(\omega) = \text{Arg}(H(i\omega))$.

The homogeneous solution

$$y_h(t) = y_0 e^{\frac{-\xi}{\sqrt{1-\xi^2}} \omega_n t} (c_1 \cos \omega_n t + c_2 \sin \omega_n t) \quad (264)$$

where damping ratio $\xi = b/b_{cr}$ and critical damping $b_{cr} = 2\sqrt{mc}$. We can express the particular solution by just recovering the dimensional mass m , damping b and stiffness k

$$y_p(t) = \Re \{ H(\omega) f_0 e^{i\omega t} \} \quad (265)$$

$$= \Re \left\{ \frac{m f_0 e^{i\omega t}}{(k - \omega^2 m) + ib\omega} \right\} \quad (266)$$

$$= \frac{m f_0}{m^2(\omega_n^2 - \omega^2)^2 + b^2\omega^2} [(k - \omega^2 m) \cos(\omega t) + \omega b \sin(\omega t)] \quad (267)$$

where $H(\omega)$ is independent of time t . We see that the two parts of the solution, y_h and y_p , oscillate with different frequencies ω_n and ω , respectively. This causes a “beating” effect in the time series until y_h is completely damped out. We will later be interested in the steady-state solution. This means the transient effects represented by y_h vanish and the solution becomes steady state and given by (267). The dynamic amplification ratio DAF¹⁷ is defined as the ratio between the amplitudes of y_p and the quasi-static response y_{st} which can be obtained as its name indicates by zeroize acceleration and velocity (damping) terms in (241). Let’s evaluate the DAF, which follows that

$$\begin{aligned} \text{DAF} &= \left| \frac{y_p}{y_{st}} \right| \\ &= \left| \frac{m f_0 e^{i\omega t}}{(k - \omega^2 m) + i b \omega} \right| \left| \frac{k}{m f_0} \right| \end{aligned} \quad (268)$$

$$= \frac{1}{\sqrt{(1 - \Lambda^2)^2 + (2\xi\Lambda)^2}} \quad (269)$$

where ξ is the damping ratio $b/(2\sqrt{mk})$ and Λ is the ratio of forcing frequency to natural frequency, i.e., $\Lambda = \omega/\omega_n$. Thus DAF is more clearly expressed in (269) being dependent on ξ and Λ which can be plotted in Fig.5. For low damping ratio $\xi < 0.2$, the decrement can be

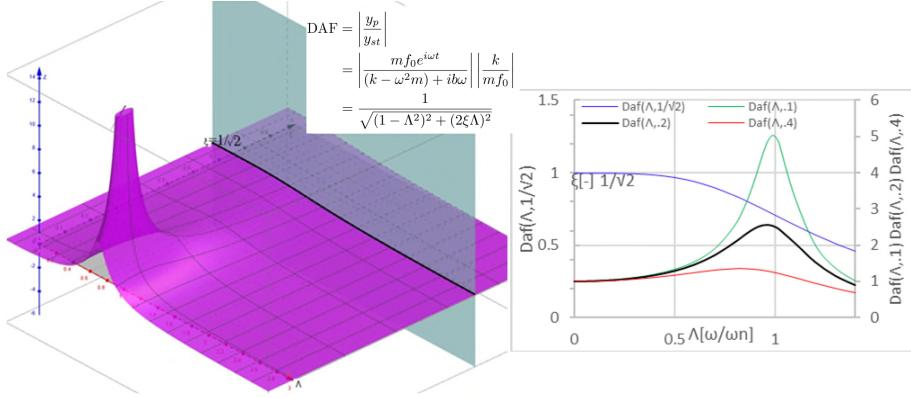


Fig. 5. Dynamic amplification factor. Left: DAF over damping and frequency ratio-plane (ξ, Λ), Right: DAF over frequency ratio for different damping ratios.

approximated to $\delta = \pi\xi$. The “Quality” factor of an oscillating system is something related to the number of cycles it takes for the motion to die down. It is defined as the ratio between the stored energy and the average energy loss during a time interval of length $1/\omega_d$.

$$\delta = \ln \frac{X_k}{X_{k+1}} = \frac{\pi\xi}{\sqrt{1 - \xi^2}}. \quad (270)$$

If energy time constant is $\tau = \Delta t$ we increase damping, the frequency range over which the dissipative characteristics dominate the behavior of the circuit increases. The Quality Factor Q is closely related to the sharpness of the peak since $Q = 1/(2\xi)$ and the bandwidth B_f of

¹⁷DAF can also be defined as the natural frequency times the transfer function, i.e., $\text{DAF} = \omega_n H(\omega)$

the peak resonance by $B_f = \omega_n/Q$.

The calibration of the nonlinear (viscous) damping itself is very challenging since the vessel motion in fluid (with or without environmental loads) involves both hysteretic damping and viscous damping. The latter is well-known for that the damping force is directly proportional to the relative velocity. The hysteretic damping also exists because the energy is dissipated more if the displacement is larger, i.e., larger roll amplitude will be damped more heavily and energy dissipation in a cycle can be velocity independent and is also independent on the frequency (do not confuse with damped natural frequency due to damping ratio), which often reflects reality better than viscous damping, in which the energy dissipation varies with frequency. When the frequency domain approach is applicable, it is often preferable to use the hysteretic damping model. However, when the time-domain approach is used, linear viscous damping is used for convenience. Both linear viscous damping and hysteretic damping require specification of the damping coefficient or ratio. Three common methods are used to estimate the nonlinear viscous damping level, i.e., logarithmic decrement from decay test, cross spectrum method from wave tests, and the half-power bandwidth. The 2nd and the 3rd methods can be called response curve method. The 1st and 2nd are mostly used but the 3rd one is simpler if the 2nd method is already used and can be a good reference. It is known from the frequency response functions that the damping not only reduces the peak amplitude, but it also broadens the shape of curve as a function of frequency. The half-power bandwidth method just utilizes this characteristic to estimate the damping level. The roll motion is excited harmonically by waves, an experimental response curve of peak roll motion versus frequency can be extracted here by cross-spectrum method. For small levels of damping, the frequency at peak response can be taken as undamped frequency f_n . Then the frequency f_L , to the left of the resonant frequency, can be found which corresponds to a level of the peak roll amplitude divided by $\sqrt{2}$. Similarly, the upper bound f_R , is the right of the resonant frequency corresponding to $1/\sqrt{2}$ of the peak roll amplitude. For small levels of damping, the damping can be estimated as $\xi = \frac{f_R - f_L}{2f_n}$.

6 Understanding the zero and infinite frequency of the free surface conditions

Various methods have been proposed for numerical damping of wave motions in diffraction analysis, and in computational fluid dynamics more generally (see Li [2015] for a review). A typical model introduces a damping term proportional to the free surface elevation, η . And free surface boundary condition (scheme 6 of Li [2015]) can be modified to be the following

$$g\phi Z - \omega^2 [1 - i\epsilon(\mathbf{X})] \phi = q_F(\mathbf{X}) \mathbf{X} \in S_F \quad (271)$$

where ϵ is a dimensionless damping coefficient. Setting $\epsilon = 0$ recovers the classical free surface boundary condition at the surface S_F , while assigning a positive value, $\epsilon > 0$, places

a damping lid on the free surface.

There is a significant body of empirical evidence in the fields of diffraction analysis and computational fluid dynamics that the boundary condition (271) has a desirable damping effect on resonant wave motions. However the theoretical motivation is rather weak, in the sense that (271) does not attempt to fully capture the viscous effects that are present in the Navier-Stokes equations but neglected in potential theory.

Typically the damping coefficient will be small and positive. For example, X-B [2004] recommends a value of $\epsilon = 0.016$ for a damping lid in the narrow gap between two barges, a value which gives good agreement with measurement data for that particular problem. Using a small value of ϵ also ensures that the impact of the damping lid is negligible at frequencies away from the resonance, as one would wish since potential theory is a good model of the physics at these frequencies.

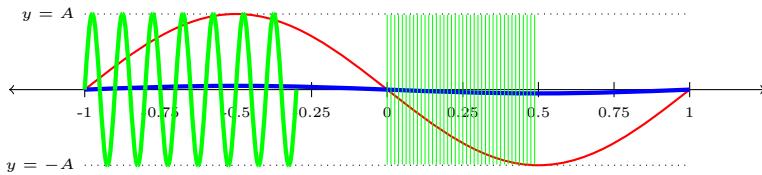


Fig. 6. All frequency waves - from zero to infinite frequency.

To understand piston mode, it will be better to investigate the zero-frequency surface “waves” which is just a flat horizontal line collapsed to the mean water line (MWL) and is the asymptotic of the blue curve shown in Fig.(6). In the zero-frequency case, all of the displacements (or elevations) are equal since the wave length is also infinite and is constant in time¹⁸. It might help to imagine a chain of masses connected by springs and free to move only horizontally. Now visualize the whole chain of oscillates rigidly as a whole up and down, which is actually impossible to move. So this is the zero frequency mode.

7 Estimation of high frequency radiation potential

7.1 Impulse Response Function and Response Amplitude Operator

The main idea was motivated by Benthien [2006]. In physical problems we have impulse response function $h(t) = 0$ for $t < 0$, so the Fourier transform of $h(t)$ becomes

$$H(f) = \int_0^\infty h(t)e^{-i2\pi ft} dt. \quad (272)$$

¹⁸Note that at this stage, the elevation is not necessarily zero, but asymptotically zero compared with the horizontal dimension. In addition, if we calculate the energy needed to have non-zero elevation, it can be proved to be zero

If $h(t)$ is sufficiently smooth and converges sufficiently rapidly to zero as $t \rightarrow \infty$, we can integrate (272) by parts k times to obtain

$$H(f) = \sum_{k=0}^n \frac{h^{(k)}(0)}{(i2\pi f)^{k+1}} + \frac{1}{(i2\pi f)^{n+1}} \int_0^\infty h^{(n+1)}(t)e^{-i2\pi ft} dt \quad (273)$$

$$= \frac{h(0)}{i2\pi f} + \frac{h'(0)}{(i2\pi f)^2} + \frac{h''(0)}{(i2\pi f)^3} + \frac{1}{(i2\pi f)^3} \int_0^\infty h'''(t)e^{-i2\pi ft} dt. \quad (274)$$

The expansion in terms of the derivatives of impulse response function shown in (274) tells us that the behavior of $H(f)$ for large f is related to the behavior of $h(t)$ for small t . Note that the large f terms are highly oscillatory functions, but since integral core $h(t)$ is not a function of f , the method of stationary phase approximation cannot be applied here. However, the (frequency domain) response function at high frequencies can be approximated by the series of derivatives of the (time domain) impulse response functions at their zeros.

7.2 Input-Output Relations

In many linear problems it is easier to solve the problem in the frequency domain than in the time domain. This is due largely to the fact that time derivatives become algebraic expressions in the frequency domain. The inverse Fourier transform can be used to relate the solution in the time domain to the solution in the frequency domain. In such problems we are often able to relate an output frequency function $G(f)$ to an input frequency function $H(f)$ by a linear relation

$$G(f) = R(f)H(f) \quad (275)$$

The function $R(f)$ is called a transfer function. Relations of this type are common in electric network problems. The Fourier transform of this relation is given by

$$g(t) = \int_{-\infty}^\infty K(t-\tau)f(\tau)d\tau \quad (276)$$

where

$$K(t) = \int_{-\infty}^\infty R(f)e^{i2\pi ft}dt. \quad (277)$$

The function $K(t)$ is called the **impulse response** since it is the output obtained from a delta function input. Since the output at time t can't depend on the input at times greater than t , it follows that

$$K(t) = 0 \text{ for } t < 0. \quad (278)$$

The above so-called causal condition thus leads

$$g(t) = \int_{-\infty}^t K(t-\tau)f(\tau)d\tau. \quad (279)$$

If the input $f(t)$ is zero for $t < 0$, then

$$g(t) = \int_0^t K(t-\tau)f(\tau)d\tau. \quad (280)$$

Taking the inverse transform of (277) and using the relation in (278), we obtain

$$R(f) = \int_0^\infty K(t)e^{-i2\pi ft}dt. \quad (281)$$

7.3 Negative added mass physical or not?

From [McIver and Evans \[1984\]](#), for a single oscillating body without an interior free surface, it appears that negative added mass occurs only when the body is submerged, and then only over a restricted range of frequencies.

This part is very important both in research and engineering since all computed results need to be verified representing the intended phenomena the design is following. But there are a series of papers for instance, [Falnes \[1980\]](#), [McIver and Linton \[1991\]](#), etc. to crack before this can be concluded.

7.4 Symmetric Decomposition of Quadratic Functions

The superscript $(*)$ operating on a complex number is used to denote its complex conjugate and the superscript (T) means transpose. Any square matrix can uniquely be written as sum of a symmetric ($A^T = A$) and a skew-symmetric matrix ($A^T = -A$). This decomposition is known as the Toeplitz decomposition, i.e.,

$$A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T) \quad (282)$$

We say that A is self-adjoint (physicists/engineers use the term **Hermitian**) if $A^H = A$. In linear algebra, a real symmetric matrix represents a self-adjoint operator over a real inner product space. The corresponding object for a complex inner product space is a Hermitian matrix with complex-valued entries, which is equal to its conjugate transpose. Thus, an arbitrary (complex) square matrix A can be written as the sum of a Hermitian matrix ($\iff a_{ij} = \bar{a}_{ji}$) and a skew-Hermitian matrix ($\iff a_{ij} = -\bar{a}_{ji}$).

$$A = \frac{1}{2} (A + A^H) + \frac{1}{2} (A - A^H) \quad (283)$$

In the notation to be followed the subscripts i, j, k are used to denote the frequencies of different linear solutions, and ℓ, m, n are used to denote the Fourier components of the same solutions, respectively. κ will be used for the wavenumber, to distinguish from the integer subscript k . First-order components such as the velocity potentials ϕ_j in [\(295\)](#) are distinguished either by the number of subscripts or superscripts (with bracketed order number, which shall not be confused by derivative) from second-order components ϕ_{ij} in [\(296\)](#). The velocity potential is expanded in the form

$$\Phi(\mathbf{x}, t) = \epsilon \Phi^0(\mathbf{x}, t) + \epsilon^2 \Phi^2(\mathbf{x}, t) + \dots \quad (284)$$

where \mathbf{x} is a fixed Cartesian coordinate system, and t denotes time. Assuming a discrete spectrum with frequency components $\omega_i > 0$ and $\omega_j > 0$, the product of two complex quantities can be expressed as

$$\Re\{z_1\}\Re\{z_2\} = \frac{1}{2}\Re\{z_1\}(z_2 + z_2^*) = \frac{1}{2}\Re\{z_1z_2\} + \frac{1}{2}\Re\{z_1z_2^*\} \quad (285)$$

Let's show the relationship (285) with any arbitrary (complex) quantities $z_1 = Ae^{i\omega_i t}$ and $z_2 = Be^{i\omega_j t}$, then

$$\begin{aligned} P = \Re\{Ae^{i\omega_i t}\}\Re\{Be^{i\omega_j t}\} &= \frac{1}{4}\{(Ae^{i\omega_i t} + A^*e^{-i\omega_i t})(Be^{i\omega_j t} + B^*e^{-i\omega_j t})\} \\ &= \frac{1}{4}\{ABe^{i\omega_{ij}^+ t} + A^*B^*e^{-i\omega_{ij}^+ t} + AB^*e^{i\omega_{ij}^- t} + A^*Be^{-i\omega_{ij}^- t}\} \\ &= \frac{1}{4}\{ABe^{i\omega_{ij}^+ t} + (ABe^{i\omega_{ij}^+ t})^* + AB^*e^{i\omega_{ij}^- t} + (AB^*e^{i\omega_{ij}^- t})^*\} \\ &= \frac{1}{2}\Re\{ABe^{i\omega_{ij}^+ t} + AB^*e^{i\omega_{ij}^- t}\} \\ &= \frac{1}{2}\Re\{(Ae^{i\omega_i})(Be^{i\omega_j t} + B^*e^{-i\omega_j t})\} \end{aligned} \quad (286)$$

$$= \frac{1}{2}\Re\{Q_{ij}^+e^{i\omega_{ij}^+ t}\} + \frac{1}{2}\Re\{Q_{ij}^-e^{i\omega_{ij}^- t}\} \quad (287)$$

where $\omega_{ij}^+ = \omega_i + \omega_j$ and $\omega_{ij}^- = \omega_i - \omega_j$. It can be seen that $AB = BA = Q_{ij}^+ = Q_{ji}^+$ and $AB^* = (BA^*)^* = Q_{ij}^- = Q_{ji}^-$.

It is often interesting to study the averaged value of the product of two quantities over a certain period of time T (finite). Since above-mentioned $\omega_i > 0$ and $\omega_j > 0$, this implies that $\omega_{ij}^+ > \omega_{ij}^-$. If let $T = 2k\pi/\omega_{ij}^+$ where k is the integer larger than zero. and on the same time let $T < 2\pi/\omega_{ij}^-$, then the short-period (high-frequency) part will vanish, only non-zero leftover will be the long-period part. Look at the behavior when $\omega_{ij}^- \rightarrow 0$, i.e., the case $\omega_i \rightarrow \omega_j$. The time averaged product of $z_1 = Ae^{i\omega_i t}$ and $z_2 = Be^{i\omega_j t}$ is denoted with a bar.

$$\bar{P} = \overline{\Re\{Ae^{i\omega_i t}\}\Re\{Be^{i\omega_j t}\}} \quad (288)$$

$$= \frac{1}{T} \int_0^T P dt \quad (289)$$

$$= \frac{1}{2T} \Re\left\{Q_{ij}^+ \int_0^T e^{i\omega_{ij}^+ t} dt\right\} + \frac{1}{2T} \Re\left\{Q_{ij}^- \int_0^T e^{i\omega_{ij}^- t} dt\right\} \quad (290)$$

$$= \frac{1}{2T} \Re\left\{Q_{ij}^+ \frac{e^{i\omega_{ij}^+ T} - 1}{i\omega_{ij}^+}\right\} + \frac{1}{2T} \Re\left\{Q_{ij}^- \int_0^T e^{i\omega_{ij}^- t} dt\right\} \quad (291)$$

$$= \frac{1}{2T} \Re\left\{Q_{ij}^+ \frac{e^{i2k\pi} - 1}{i\omega_{ij}^+}\right\} + \frac{1}{2} \Re\left\{Q_{ij}^- \frac{e^{i\omega_{ij}^- T} - 1}{i\omega_{ij}^- T}\right\} \quad (292)$$

$$= 0 + \frac{1}{2} \Re\left\{Q_{ij}^- \frac{i\omega_{ij}^- T + \frac{(i\omega_{ij}^- T)^2}{2!} + \frac{(i\omega_{ij}^- T)^3}{3!} + \dots}{i\omega_{ij}^- T}\right\} \quad (293)$$

$$= \frac{1}{2} \Re\{Q_{ij}^-\} + \frac{1}{2} \Re\left\{Q_{ij}^- \left(\frac{i\omega_{ij}^- T}{2!} + \frac{(i\omega_{ij}^- T)^2}{3!} + \dots\right)\right\} \quad (294)$$

where Taylor expansion has been used from step (292) to (293). It is obvious that the second term in the last line (294) will vanish if $\omega_{ij}^- \rightarrow 0$. But one must keep in mind that the additional terms can cause inaccuracy in the numerical computations of a random seas which contain tremendous frequency components (some singlet can be not so close to zero!).

Given a wave spectrum, it is customary to assume the spectrum is expressed as a linear superposition of the first-order incident waves of different frequencies. Thus the total potential for the wave-body interaction can be expressed by a sum of components circular frequencies of ω_j or $\omega_i \pm \omega_j$ for the first and second order solutions, respectively.

$$\Phi^{(1)}(\mathbf{x}, t) = \Re \sum_j \phi_j(\mathbf{x}) e^{i\omega_j t} \quad (295)$$

$$\Phi^{(2)}(\mathbf{x}, t) = \Re \sum_i \sum_j [\phi_{ij}^+(\mathbf{x}) e^{i(\omega_i + \omega_j)t} + \phi_{ij}^-(\mathbf{x}) e^{i(\omega_i - \omega_j)t}] \quad (296)$$

The second-order potentials ϕ_{ij}^\pm can be defined to satisfy the symmetry relations

$$\phi_{ij}^+ = \phi_{ji}^+ \quad \text{and} \quad \phi_{ij}^- = \phi_{ji}^{-*} \quad (297)$$

since anti-symmetric components will not contribute to (296). The free-surface boundary conditions satisfied by these potentials are

$$\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g \frac{\partial \Phi^{(1)}}{\partial z} = 0 \quad (298)$$

$$\frac{\partial^2 \Phi^{(2)}}{\partial t^2} + g \frac{\partial \Phi^{(2)}}{\partial z} = Q_F(x, y; t) \quad (299)$$

on $z = 0$. Here the inhomogeneous right-hand-side of the second-order free-surface condition (299) defines the quadratic forcing function [Newman \[1977\]](#)

$$Q_F = \frac{1}{g} \frac{\partial \Phi^{(1)}}{\partial t} \frac{\partial}{\partial z} \left(\frac{\partial^2 \Phi^{(1)}}{\partial t^2} + g \frac{\partial \Phi^{(1)}}{\partial z} \right) - \frac{\partial}{\partial t} (\nabla \Phi^{(1)} \cdot \nabla \Phi^{(1)}) \quad (300)$$

where the right-hand-side is to be evaluated on $z = 0$. Together with the body boundary condition

$$\frac{\partial \Phi^{(2)}}{\partial n} = Q_B(\mathbf{x}; t), \quad (301)$$

the forcing function on the body boundary is given by [Ogilvie \[1983\]](#)

$$\begin{aligned} Q_B = & - \frac{\partial \phi_I^2}{\partial n} + \mathbf{n} \cdot \frac{\partial \mathcal{H}}{\partial t} \mathbf{x} + (\boldsymbol{\xi}^{(1)} \times \mathbf{n}) \cdot \left(\frac{\partial (\boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(1)} \times \mathbf{x})}{\partial t} - \nabla \phi^{(1)} \right) \\ & - \mathbf{n} \cdot \left((\boldsymbol{\xi}^{(1)} + \boldsymbol{\xi}^{(1)} \times \mathbf{x}) \cdot \nabla \right) \nabla \phi^{(1)} + \sum_{k=1}^6 \frac{\partial \xi_k^{(2)}}{\partial t} n_k. \end{aligned} \quad (302)$$

where \mathcal{H} is expressed in (311).

In evaluation of the 2nd order quantities, the production of two 1st order oscillatory quantities can be expressed as

$$Q_F(\mathbf{x}, t) = \Re \sum_i \sum_j [Q_{ij}^+(\mathbf{x}) e^{i(\omega_i + \omega_j)t} + Q_{ij}^-(\mathbf{x}) e^{i(\omega_i - \omega_j)t}] \quad (303)$$

Adopting a form for Q analogous to (296) it is appropriate to symmetrize the functions Q_{ij}^\pm as,

$$Q_{ij}^+ = Q_{ji}^+ \quad \text{and} \quad Q_{ij}^- = Q_{ji}^{-*}. \quad (304)$$

The other second-order quantities such as the motion amplitude $\xi^{(2)} = \xi_{ij}^\pm$ are expressed in a form of (303) with the symmetry condition (304). With this in mind, we will omit the subscript ij hereafter. From combining (296) and (300), we have the expressions for the complex amplitudes of the free-surface forcing functions. Sum- and difference-frequency forcings on $z = 0$ are given by

$$Q_F^+ = \frac{i}{4g} \omega_i \phi_i \left(-\omega_j^2 \frac{\partial \phi_j}{\partial z} + g \frac{\partial^2 \phi_j}{\partial z^2} \right) + \frac{i}{4g} \omega_j \phi_j \left(-\omega_i^2 \frac{\partial \phi_i}{\partial z} + g \frac{\partial^2 \phi_i}{\partial z^2} \right) - \frac{i}{2} (\omega_i + \omega_j) \nabla \phi_i \cdot \nabla \phi_j \quad (305)$$

$$Q_F^- = \frac{i}{4g} \omega_i \phi_i \left(-\omega_j^2 \frac{\partial \phi_j^*}{\partial z} + g \frac{\partial^2 \phi_j^*}{\partial z^2} \right) - \frac{i}{4g} \omega_j \phi_j^* \left(-\omega_i^2 \frac{\partial \phi_i}{\partial z} + g \frac{\partial^2 \phi_i}{\partial z^2} \right) - \frac{i}{2} (\omega_i - \omega_j) \nabla \phi_i \cdot \nabla \phi_j^* \quad (306)$$

For future reference, we note that the first-order potential ϕ_i and ϕ_j in (305) and (306), consist of the incident wave potential (ϕ_I) and the body disturbances ($\phi_B = \phi_S + \phi_R$). Thus we may decompose Q_F^\pm into the quadratic interactions of the two incident wave potentials (Q_{II}^\pm), the incident and the body disturbance waves (Q_{IB}^\pm), and two body disturbance waves (Q_{BB}^\pm) such that

$$Q_F^\pm = Q_{II}^\pm + Q_{IB}^\pm + Q_{BB}^\pm \quad (307)$$

Next we consider the sum- and difference-frequency forcing on the body boundary in (302). They are given by

$$Q_B^+ = -\frac{\partial \phi_I^+}{\partial n} + \frac{i(\omega_i + \omega_j)}{2} \mathbf{n} \cdot H^+ \mathbf{x} + \frac{1}{4} [(\alpha_i \times \mathbf{n}) \cdot (i\omega_j (\boldsymbol{\xi}_j + \alpha_j \times \mathbf{x}) - \nabla \phi_j) + (\alpha_j \times \mathbf{n}) \cdot (i\omega_i (\boldsymbol{\xi}_i + \alpha_i \times \mathbf{x}) - \nabla \phi_i)] - \frac{1}{4} \mathbf{n} [((\boldsymbol{\xi}_i + \alpha_i \times \mathbf{x}) \cdot \nabla) \nabla \phi_j + ((\boldsymbol{\xi}_j + \alpha_j \times \mathbf{x}) \cdot \nabla) \nabla \phi_i] \quad (308)$$

$$Q_B^- = -\frac{\partial \phi_I^-}{\partial n} + \frac{i(\omega_i - \omega_j)}{2} \mathbf{n} \cdot H^- \mathbf{x} + \frac{1}{4} [(\alpha_i \times \mathbf{n}) \cdot (-i\omega_j (\boldsymbol{\xi}_j^* + \alpha_j^* \times \mathbf{x}) - \nabla \phi_j^*) + (\alpha_j^* \times \mathbf{n}) \cdot (i\omega_i (\boldsymbol{\xi}_i + \alpha_i \times \mathbf{x}) - \nabla \phi_i)] - \frac{1}{4} \vec{n} \cdot [((\boldsymbol{\xi}_i + \alpha_i \times \mathbf{x}) \cdot \nabla) \nabla \phi_j^* + ((\boldsymbol{\xi}_j^* + \alpha_j^* \times \mathbf{x}) \cdot \nabla) \nabla \phi_i] \quad (309)$$

The sum- and difference-frequency components of the last term of (302),

$$Q_B^\pm = i(\omega_i \pm \omega_j) \sum_k^6 \xi^\pm n_k. \quad (310)$$

are omitted from (308) and (309), since they are not a quadratic function of the first-order solution. These are proportional to the second-order motion and can be treated separately

from the rest of body forcing as is discussed below. The matrices H^\pm which account for the rotational motion of the body are given by (311) from [Wamit \[2007\]](#).

$$\begin{aligned} H^+ &= \mathcal{M}(\alpha_i, \alpha_j) \\ H^- &= \mathcal{M}(\alpha_i, \alpha_j^*) \\ \mathcal{M}(\alpha_i, \alpha_j) &= \frac{1}{2} \begin{bmatrix} -\alpha_{2i}\alpha_{2j} - \alpha_{3i}\alpha_{3j} & 0 & 0 \\ \alpha_{1i}\alpha_{2j} + \alpha_{2i}\alpha_{1j} & -\alpha_{1i}\alpha_{1j} - \alpha_{3i}\alpha_{3j} & 0 \\ \alpha_{1i}\alpha_{3j} + \alpha_{3i}\alpha_{1j} & \alpha_{2i}\alpha_{3j} + \alpha_{3i}\alpha_{2j} & -\alpha_{1i}\alpha_{1j} - \alpha_{2i}\alpha_{2j} \end{bmatrix} \end{aligned} \quad (311)$$

As in the first-order problem ($\phi^\pm = \phi_I^\pm + \phi_R^\pm + \phi_S^\pm$), it is convenient to decompose the total second-order potential into three components $\phi^\pm = \phi_I^\pm + \phi_R^\pm + \phi_S^\pm$. :the second order incident wave potential (ϕ_I^\pm), the second-order scattering wave potential (ϕ_S^\pm) and the second-order radiation potential (ϕ_R^\pm). We define ϕ_I^\pm as the potential that satisfies the second-order free surface condition in absence of the body. ϕ_R^\pm describes the disturbance due the second-order motion of the body and is linearly proportional to the motion amplitude. It can be decomposed into the mode dependent components

$$\phi_R^\pm = i(\omega_i \pm \omega_j) \sum_k^6 \xi^\pm \phi_k \quad (312)$$

where the unit-amplitude radiation velocity potential ϕ_k is defined in the same way as in first order problem. The rest of the second-order potential is defined to be ϕ_S^\pm , whether the body is fixed or moving. This decomposition is in accordance with the convention for the first-order problem where the wave exciting force is the pressure force due to the sum of the incident and scattering waves.

7.5 Volterra Functional Series and Quadratic Transfer Functions

7.5.1 Memory Effect

By Weierstrass theorem, any analytic functions¹⁹ function can be approximated nicely by a convergent power series.

For a discrete time system at the n -th sampling time, with input $x[n]$ and output $y[n]$, the power series with coefficients (all called weights a_i) for each term can be used to represent constant, linear and all nonlinear terms.

$$y[n] = a_0 + a_1 x[n] + a_2 x^2[n] + a_3 x^3[n] + \cdots + a_n x^n[n] + \cdots. \quad (313)$$

¹⁹A (real or complex) function $f(z)$ is called analytic at a point z_0 if it has a power series expansion that converges in some disk about this point (i.e., with $\rho \geq 0$) or if and only if its Taylor series about z_0 converges to the function in some neighborhood for every z_0 in its domain.

The first term a_0 is constant, i.e., independent of n , but the second term $a_1x[n]$ is linearly proportional to $x[n]$ and other terms are nonlinear ones.

The Taylor series can be used for approximating the response of a nonlinear system to a given input if the output of this system depends strictly on the input at that particular time. Whereas Volterra series can capture the output of the nonlinear system from the input to the system at all other times. This provides the ability to capture the “memory” effect of the dynamic system in the same way as devices of capacitors and inductors in a electric circuit.

7.5.1.1 1st Order System, linearity

- A typical linear system without memory effect

$$y(n) = a_1x(n) \quad (314)$$

where the output at n depends only on x at n .

- A typical linear system with memory effect can be written as

$$y(n) = \sum_{i=0}^n a(i)x(n-i) \quad (315)$$

which is actually the Cauchy product²⁰ of $a(i)$ and $x(i)$.

7.5.1.2 2nd Order System, nonlinearity

- A typical nonlinear system without memory effect is $y_2(n) = a_2x^2(n)$ where the output at n depends only on x at n .
- A typical nonlinear system with memory effect and uniformly weighted by a unit $a_2(n) \equiv 1$ can be written as

$$y(n) = \sum_{i=0}^n \sum_{j=0}^n x(i)x(j) \quad (316)$$

$$= x(n)x(n) + x(n)x(n-1) + \cdots + x(n)x(0) + \cdots + x(0)x(0). \quad (317)$$

In general, by adding proper weights $w(i, j)$,

$$y(n) = \sum_{i=0}^n \sum_{j=0}^n w(i, j)x(n-i)x(n-j). \quad (318)$$

²⁰Cauchy product is the discrete convolution of two infinite series, i.e.,

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n, \text{ where } c_n = \sum_{k=0}^n a_k b_{n-k}$$

7.5.1.3 Power series and Laplace Transform for Memoryless system

The Laplace Transform can be derived from the Power series. Let's consider the Power series

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} f(n) x^n = F(x) \quad (319)$$

where we use the notation $a_n = f(n)$ for the coefficients of the series. The function $f(n)$ with n as variable is transformed (\rightsquigarrow) into a function $F(x)$ with x as variable, by notation

$$f(n) \rightsquigarrow F(x). \quad (320)$$

We can illustrate this by a few examples. The simple function $f(n) = 1$ leads to a geometric series, which converges for $|x| < 1$, i.e.,

$$1 = f(n) \rightsquigarrow F(x) = \frac{1}{1-x} \quad \forall |x| < 1. \quad (321)$$

Another example is the exponential function,

$$\frac{1}{n!} = f(n) \rightsquigarrow F(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad (322)$$

by the Taylor expansion of e^x . If we now change the discrete variable $n = 1, 2, 3, \dots$ to the continuous variable t and replace the sum by an integral we have

$$\int_0^{\infty} f(t) x^t dt = F(x). \quad (323)$$

We are allowed to write

$$x = e^{\ln x} \quad (324)$$

and let $\ln x = -s$, so we get

$$x^t = e^{-s \cdot t} \quad (325)$$

which leads us to the definition of the Laplace transform

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad (326)$$

7.5.1.4 Complete System

To sum up the complete components, i.e., the zeroth, first, second and higher order terms and to make the time running continuously or take the Riemann sum, the system output without memory effect is

$$y(t) = h_0 + h_1 x(t) + h_2 x^2(t) + \dots \quad (327)$$

and the system including the memory effect is expressed as

$$y(t) = h_0 + \int_0^t h_1(\tau_1) x(t - \tau_1) d\tau_1 + \int_0^t h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_2 d\tau_2 + \dots \quad (328)$$

Note that h_0 is independent of the time t , which represent the mean component of the signal.

7.5.2 Volterra series and QTF

Volterra functional series representation of nonlinear random processes

$$y(t) = h[x(t)] = y_0^{21} + y_1(t) + y_2(t) + y_3(t) + \dots + y_n(t) \quad (329)$$

$$= h_0 + \int_0^t h_1(\tau_1)x(t - \tau_1)d\tau_1 + \int_0^t h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)d\tau_2 d\tau_1 + \dots \quad (330)$$

The functions $h_1(t_1)$ and $h_2(t_1, t_2)$ are respectively the linear and quadratic Volterra kernels or weighting functions. In terms of impulse responses, these functions represent the system's memory effect in the time domain affecting a linear response at time t due to an impulsive excitation at time $t - t_1$ and a quadratic response due to two impulsive excitations at times $t - t_1$ and $t - t_2$. By physical causality, only the excitation happened in the past can result in a response. It can be seen that $h_2(t_1, t_2)$ is symmetric with respect to t_1 and t_2 . To obtain the frequency domain responses (so-called transfer functions), we take the Fourier transform of the above impulse responses to the 2nd order:

$$Y(\omega) = \mathcal{F}[h_0] + \mathcal{F}[y_1(t)] + \mathcal{F}[y_2(t)] \quad (331)$$

$$= Y_0(\omega) + Y_1(\omega) + Y_2(\omega) \quad (332)$$

$$(333)$$

It can be shown that

$$Y_0(\omega) = \frac{h_0}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} dt = h_0\delta(\omega) \quad (334)$$

$$Y_1(\omega) = H_1(\omega)X_1(\omega) \quad (335)$$

$$Y_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_2(\omega_1, \omega - \omega_1)X(\omega_1)X(\omega - \omega_1)d\omega_1 \quad (336)$$

where $\omega_1 + \omega_2 = \omega$. It is straight-forward to see the Fourier transforms of a constant in (334) and of a first order convolution (335) in time domain, but (336) is a bit tedious to show. The uppercase letters denote the Fourier transformed functions (omitting the bounds of integration).

$$Y_2(\omega) = \iiint h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)e^{-i\omega t}dt d\tau_1 d\tau_2 \quad (337)$$

The issue here is that both x depend upon t , so I'm plopping in a δ to deal with that.

$$Y_2(\omega) = \iiint h_2(\tau_1, \tau_2)x(\theta - \tau_1)x(t - \tau_2)\delta(\theta - t)e^{-i\omega t}d\theta dt d\tau_1 d\tau_2 \quad (338)$$

Then I'm going to write the δ out in its integral form $\delta(\theta - t) = \frac{1}{2\pi} \int e^{-i\omega_1(\theta - t)}d\omega_1$.

²¹here y_0 is constant in time coming from the mean drift (diagonal terms of QFT)

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) x(\theta - \tau_1) x(t - \tau_2) e^{-i\omega_1(\theta-t)} e^{-i\omega t} d\theta dt d\tau_1 d\tau_2 d\omega_1 \quad (339)$$

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} x(\theta - \tau_1) e^{-i\omega_1(\theta-\tau_1)} x(t - \tau_2) e^{-i(\omega-\omega_1)t} d\theta dt d\tau_1 d\tau_2 d\omega_1 \quad (340)$$

With the substitution $\zeta_1 = \theta - \tau_1$ we can finally start getting rid of some of the mess.

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} x(\zeta_1) e^{-i\omega_1\zeta_1} x(t - \tau_2) e^{-i(\omega-\omega_1)t} d\zeta_1 dt d\tau_1 d\tau_2 d\omega_1 \quad (341)$$

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} X(\omega_1) x(t - \tau_2) e^{-i(\omega-\omega_1)t} dt d\tau_1 d\tau_2 d\omega_1 \quad (342)$$

Similarly I can transform the other x by taking $\zeta_2 = t - \tau_2$.

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} e^{-i(\omega-\omega_1)\tau_2} X(\omega_1) x(t - \tau_2) e^{-i(\omega-\omega_1)(t-\tau_2)} dt d\tau_1 d\tau_2 d\omega_1 \quad (343)$$

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} e^{-i(\omega-\omega_1)\tau_2} X(\omega_1) x(\zeta_2) e^{-i(\omega-\omega_1)\zeta_2} d\zeta_2 d\tau_1 d\tau_2 d\omega_1 \quad (344)$$

$$Y_2(\omega) = \frac{1}{2\pi} \iiint h_2(\tau_1, \tau_2) e^{-i\omega_1\tau_1} e^{-i(\omega-\omega_1)\tau_2} X(\omega_1) X(\omega - \omega_1) d\tau_1 d\tau_2 d\omega_1 \quad (345)$$

And finally we just have the Fourier transform of h_2 .

$$Y_2(\omega) = \frac{1}{2\pi} \int H_2(\omega_1, \omega - \omega_1) X(\omega_1) X(\omega - \omega_1) d\omega_1 \quad (346)$$

Therefore the total frequency domain response to the 2nd order are

$$Y(\omega) = h_0 \delta(\omega) + H_1(\omega) X_1(\omega) + \frac{1}{2\pi} \int_{-\infty}^{\infty} H_2(\omega_1, \omega - \omega_1) X(\omega_1) X(\omega - \omega_1) d\omega_1 \quad (347)$$

8 Extension of Stokes' Theorem and its Applications

8.1 Vector Algebra

No coordinates or basis vectors are actually needed for the laws of physics. The derivations are at most expressible without the aid of any coordinates or bases. That means that the components are secondary; they only exist after one has chosen a set of basis vectors. Components (we claim) are an impediment to a clear and deep understanding of the laws

of physics. The coordinate-free, component-free description is deeper, and once one becomes accustomed to it much more clear and understandable, see [GIBBS and WILSON, 1901].

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem, Pólya and Conway [2004]. If you could master one or two great skills in vector analysis, we will never be afraid of any complicated vector or tensor equations. One I can see is the Levi-Civita and the other is the splitting of magnitude and direction of any vector (but tensor can have multiple magnitude and direction pairs).

The following operations are satisfied for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and all real numbers λ , where $\mathbf{a} \cdot \mathbf{b}$ denotes the dot product (or scalar product), and $\mathbf{a} \times \mathbf{b}$ denotes the cross product (or vector product) between the vectors \mathbf{a} and \mathbf{b} .

$$\begin{aligned}
 \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} && \text{(commutative law)} \\
 (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}) && \text{(associative law)} \\
 \lambda \mathbf{a} &= \mathbf{a} \lambda && \text{(multiplication by a real)} \\
 \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} && \text{(commutative law)} \\
 \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} && \text{(distributive law)} \\
 \lambda(\mathbf{a} \cdot \mathbf{b}) &= (\lambda \mathbf{a}) \cdot \mathbf{b} && \text{(associative law)} \\
 \mathbf{a} \times \mathbf{b} &= -\mathbf{b} \times \mathbf{a} && \text{(non-commutativity)} \\
 \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} && \text{(distributive law)} \\
 \lambda(\mathbf{a} \times \mathbf{b}) &= (\lambda \mathbf{a}) \times \mathbf{b} && \text{(associative law)}
 \end{aligned}$$

\mathbb{E} is bivalent “metric unit identity” tensor (the one which is neutral to dot product operation), and that $\nabla \mathbf{r} = \mathbb{E}$.

General Rules for Directional Derivatives

1. Expand $\nabla = \hat{\mathbf{e}}_i \partial_i$, i.e., $\nabla \mathbf{a} = \hat{\mathbf{e}}_i \partial_i$ and $\nabla \bullet \mathbf{a} = \hat{\mathbf{e}}_i \partial_i \bullet = \hat{\mathbf{e}}_i \bullet \partial_i$
2. The commutativity of dot product of any two vectors
3. Dot product doesn't affect scalars (and coordinate derivative $\partial_i \triangleq \frac{\partial}{\partial q^i}$ is scalar), dot product affects only vectors and tensors of even bigger complexity
4. A scalar or magnitude multiplication is acting as a coefficient and can follow any vector factor
5. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for any two vectors \mathbf{a} and \mathbf{b}
6. Partial derivative of any vector with respect to scalar, like coordinate, is just a ordinary differentiation with its magnitude but keeping its direction. It results just a vector other than some more complex tensor.

7. The product rule: for differentiation of a product “ \circ ” of two multipliers, the famous **product rule** applies: $\partial_i(u \circ v) = (\partial_i u) \circ v + u \circ (\partial_i v)$

Here are some examples, for any constant vector \mathbf{c} ,

$$\mathbf{c} \cdot \int_C \mathbf{t} \times \mathbf{n} ds = \int_C (\mathbf{n} \times \mathbf{c}) \cdot \mathbf{t} ds = \int_S \nabla \times (\mathbf{n} \times \mathbf{c}) \cdot \mathbf{n} da \quad (348)$$

Using the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a} \quad (349)$$

and the fact that \mathbf{c} is constant, the above integral becomes

$$\int_S [\mathbf{c} \cdot \nabla \mathbf{n} - (\nabla \cdot \mathbf{n})\mathbf{c}] \cdot \mathbf{n} da = \mathbf{c} \cdot \int_S (\nabla \mathbf{n}) \cdot \mathbf{n} - (\nabla \cdot \mathbf{n})\mathbf{n} da \quad (350)$$

The first term in the integrand is zero, since it is just the gradient of the length of the unit normal. Also, since the vector \mathbf{c} was arbitrary, its dot product with the integral can be dropped:

$$\int_C \mathbf{t} \times \mathbf{n} ds = - \int_S (\nabla \cdot \mathbf{n}) \mathbf{n} da \quad (351)$$

8.2 Split Vector Operator into its Magnitude and Direction

For any vector functions \mathbf{u} , \mathbf{v} and \mathbf{n} defined on: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, i.e., $\mathbf{u} = (u_1, u_2, u_3)$ and same for \mathbf{v} and \mathbf{n} . If $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are the canonical basis vectors (that is, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$), it is often written as $\mathbf{n} = \sum_{j=1}^3 n_j \hat{\mathbf{e}}_j$. The partial derivative of a vector-valued function is computed by taking the partial derivative of each component function, i.e., one needs to identify the components of both independent (i.e., which x_i) and dependent variables (f_i). We deal with the pure **differential operator**, ∇ as a special vector, which therefore can also obtain magnitude and direction. And when we do vector differentiation, we can process the direction separately by splitting out its direction of the operator, while the rest for magnitude is taken over by the actual differentiation to the right vector(s) that should be kept in mind and followed. The most common differential operators can be expressed as follows.

$$\begin{aligned} \nabla \cdot &= \cdot \nabla = \sum_{i=1}^3 (\partial_i \hat{\mathbf{e}}_i) \cdot = \sum_{i=1}^3 \hat{\mathbf{e}}_i \cdot \partial_i, & \nabla \times &= \sum_{i=1}^3 (\partial_i \hat{\mathbf{e}}_i) \times = \sum_{i=1}^3 \hat{\mathbf{e}}_i \times \partial_i \\ (\dots \times \nabla) &= \sum_{i=1}^3 (\dots \times \hat{\mathbf{e}}_i) \partial_i & (\dots \times \nabla) \times &= \sum_{i=1}^3 (\dots \times \hat{\mathbf{e}}_i) \times \partial_i \end{aligned}$$

For instance, the following shows how the splitting is applied to general vector operations.

$$\begin{aligned}
 \nabla \cdot \mathbf{v} &= \sum_{i=1}^3 \hat{\mathbf{e}}_i \cdot \partial_i \mathbf{v}, & \nabla \times \mathbf{v} &= \sum_{i=1}^3 \hat{\mathbf{e}}_i \times \partial_i \mathbf{v} \\
 (\mathbf{v} \cdot \nabla) \mathbf{n} &= \sum_{i=1}^3 (\mathbf{v} \cdot \hat{\mathbf{e}}_i) \partial_i \mathbf{n}, & (\mathbf{v} \times \nabla) f &= \sum_{i=1}^3 (\mathbf{v} \times \hat{\mathbf{e}}_i) \partial_i \mathbf{n} \\
 (\mathbf{u} \times \nabla) \times \mathbf{v} &= \sum_{i=1}^3 (\mathbf{u} \times \hat{\mathbf{e}}_i) \times \partial_i \mathbf{v}, & [\nabla \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{n} &= \sum_{i=1}^3 (\mathbf{n} \times \hat{\mathbf{e}}_i) \cdot \partial_i (\mathbf{u} \times \mathbf{v}) \\
 &= \sum_{i=1}^3 [\hat{\mathbf{e}}_i \partial_i \mathbf{v} \cdot \mathbf{u} - (\hat{\mathbf{e}}_i \cdot \partial_i \mathbf{v}) \mathbf{u}], & &= \sum_{i=1}^3 [(\mathbf{n} \times \hat{\mathbf{e}}_i) \times \partial_i \mathbf{u}] \cdot \mathbf{v} \\
 &= \sum_{i=1}^3 [(\hat{\mathbf{e}}_i \partial_i \mathbf{v}) \cdot \mathbf{u} - (\hat{\mathbf{e}}_i \partial_i \cdot \mathbf{v}) \mathbf{u}], & &+ \sum_{i=1}^3 [(\mathbf{n} \times \hat{\mathbf{e}}_i) \times \mathbf{u}] \cdot \partial_i \mathbf{v}
 \end{aligned}$$

As a practice, let's familiarize with the concept of curvature in differential geometry from [do Carmo \[2016\]](#). For a surface defined in 3D space, the mean curvature is related to a unit surface normal of the surface: $2H = -\nabla \cdot \hat{\mathbf{n}}$ where the normal chosen affects the sign of the curvature. The sign of the curvature depends on the choice of normal: the curvature is positive if the surface curves "towards" the normal. The formula above holds for surfaces in 3D space defined in any manner, as long as the divergence of the unit normal may be calculated. Mean Curvature may also be calculated: $2H = \text{Tr}(I^{-1}II)$ where I and II denote first and second quadratic form matrices, respectively. Let the surface function defined by a vector $\sigma(x, y) = [x, y, \zeta(x, y)]$, then the derivatives can be found σ_{xy}

$$\sigma_x = [1, 0, \zeta_x] \tag{352}$$

$$\sigma_y = [0, 1, \zeta_y] \tag{353}$$

$$\sigma_{xx} = [0, 0, \zeta_{xx}] \tag{354}$$

$$\sigma_{xy} = [0, 0, \zeta_{xy}] \tag{355}$$

$$\sigma_{yy} = [0, 0, \zeta_{yy}]. \tag{356}$$

Immediately, the coefficients of the 1st fundamental form are determined $E = \sigma_x \cdot \sigma_x = 1 + \zeta_x^2$, $F = \sigma_x \cdot \sigma_y = \zeta_x \zeta_y$, $G = \sigma_y \cdot \sigma_y = 1 + \zeta_y^2$. So is the unit normal to the free surface:

$$\hat{\mathbf{n}} = \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} \tag{357}$$

With the normal $\hat{\mathbf{n}}$, we obtain the coefficients of the 2nd fundamental form:

$$L = \sigma_{xx} \cdot \hat{\mathbf{n}} = \frac{\zeta_{xx}}{1 + \zeta_x^2 + \zeta_y^2} \tag{358}$$

$$M = \sigma_{xy} \cdot \hat{\mathbf{n}} = \frac{\zeta_{xy}}{1 + \zeta_x^2 + \zeta_y^2} \tag{359}$$

$$N = \sigma_{yy} \cdot \hat{\mathbf{n}} = \frac{\zeta_{yy}}{1 + \zeta_x^2 + \zeta_y^2} \tag{360}$$

If $\zeta(x, y)$ is a free surface and \mathbf{u} and \mathbf{v} are two linearly independent vectors in parameter space then the mean curvature can be written in terms of the first (I) fundamental form and second (II) fundamental form as $\frac{LG - 2MF + NE}{2(EG - F^2)}$ where $E = I(\mathbf{u}, \mathbf{u})$, $F = I(\mathbf{u}, \mathbf{v})$, $G = I(\mathbf{v}, \mathbf{v})$, $L = II(\mathbf{u}, \mathbf{u})$, $M = II(\mathbf{u}, \mathbf{v})$ and $N = II(\mathbf{v}, \mathbf{v})$.

For the free surface function $F = z - \zeta(x, y) = 0$, and the (doubled) mean curvature expression is expressed as

$$2H = \nabla \cdot \frac{\nabla F}{|F|} \quad (361)$$

$$= \nabla \cdot \frac{[-\zeta_x, -\zeta_y, 1]}{\sqrt{1 + |\nabla \zeta|^2}} \quad (362)$$

$$= \frac{-\zeta_{xx} - \zeta_{yy}}{(1 + \zeta_x^2 + \zeta_y^2)^{1/2}} - \frac{[\zeta_x \zeta_{xx} + \zeta_y \zeta_{xy}, \zeta_x \zeta_{xy} + \zeta_y \zeta_{yy}, 0] \cdot [-\zeta_x, -\zeta_y, 1]}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}} \quad (363)$$

$$= \frac{-\zeta_{xx}(1 + \zeta_y^2) - \zeta_{yy}(1 + \zeta_x^2) + 2\zeta_x \zeta_y \zeta_{xy}}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}}. \quad (364)$$

In particular at a point where $\nabla \zeta = 0$, the mean curvature is half the trace of the Hessian matrix of ζ .

8.3 Levi-Civita Identities

The product of two Levi-Civita symbols can be expressed as a function of the Kronecker's symbol²² $\delta_{ij} = \delta_i^j = \delta_j^i = 1$ iff $i = j$, otherwise zero. The properties are easily observed by expressing ε_{ijk} in a [determinant form](#). This also can be generalized for $n > 3$. Let's look at the 3×3 identity matrix consisting of the three unit column vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$.

$$\begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (365)$$

Obviously, the unit columns are orthonormal,

$$\hat{\mathbf{e}}_i^T \hat{\mathbf{e}}_j = \delta_{ij} = \delta_i^j = \delta_j^i, \quad i, j = 1, 2, 3, \quad (366)$$

where δ_{ij} is the Kronecker delta.

Consider determinants consisting of three columns selected out of the three unit columns. Then by the properties of determinants:

$$\left| \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \right| = 0 \quad \text{if } i = j, i = k, \text{ or } j = k. \quad (367)$$

Further,

$$\left| \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 \right| = \left| \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_1 \right| = \left| \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \right| = 1 \quad (368)$$

$$\left| \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_3 \right| = \left| \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1 \right| = \left| \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_2 \right| = -1. \quad (369)$$

²²we will not make difference between upper or lower index notation

Hence

$$\varepsilon_{ijk} = \left| \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \right|. \quad (370)$$

Introduce two 3×3 unit matrices \mathbf{A} and \mathbf{B} as short-hand notations:

$$\det(\mathbf{A}) := \left| \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \right|, \quad \det(\mathbf{B}) := \left| \hat{\mathbf{e}}_\ell \hat{\mathbf{e}}_m \hat{\mathbf{e}}_n \right|. \quad (371)$$

Use $\varepsilon_{ijk}\varepsilon_{\ell mn} = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{A}^T)\det(\mathbf{B}) = \det(\mathbf{A}^T\mathbf{B})$. The determinant of the product of two matrices can be written as:

$$\varepsilon_{ijk}\varepsilon^{lmn} = \det \begin{pmatrix} \left[\begin{matrix} \hat{\mathbf{e}}_i^T \\ \hat{\mathbf{e}}_j^T \\ \hat{\mathbf{e}}_k^T \end{matrix} \right] & \left[\begin{matrix} \hat{\mathbf{e}}^\ell & \hat{\mathbf{e}}^m & \hat{\mathbf{e}}^n \end{matrix} \right] \end{pmatrix} = \begin{vmatrix} \delta_i^\ell & \delta_i^m & \delta_i^n \\ \delta_j^\ell & \delta_j^m & \delta_j^n \\ \delta_k^\ell & \delta_k^m & \delta_k^n \end{vmatrix} \quad (372)$$

$$= \delta_i^\ell (\delta_j^m \delta_k^n - \delta_j^n \delta_k^m) - \delta_i^m (\delta_j^\ell \delta_k^n - \delta_j^n \delta_k^\ell) + \delta_i^n (\delta_j^\ell \delta_k^m - \delta_j^m \delta_k^\ell) \quad (373)$$

If $i = \ell$ in (373), then

$$\begin{aligned} \varepsilon_{ijk}\varepsilon^{imn} &= \delta_i^i (\delta_j^m \delta_k^n - \delta_j^n \delta_k^m) - \delta_i^m (\delta_j^i \delta_k^n - \delta_j^n \delta_k^i) + \delta_i^n (\delta_j^i \delta_k^m - \delta_j^m \delta_k^i) \\ &= 3(\delta_j^m \delta_k^n - \delta_j^n \delta_k^m) - \delta_j^m \delta_k^n + \delta_k^m \delta_j^n + \delta_j^n \delta_k^m - \delta_j^m \delta_k^n \\ &= \delta_j^m \delta_k^n - \delta_j^n \delta_k^m \end{aligned} \quad (374)$$

It is seen that the two equal indices lead to the total index number (or dimension of the tensor) reduced by 2. If further let $j = m$ in (374), then

$$\begin{aligned} \varepsilon_{ijk}\varepsilon^{ijn} &= \delta_j^j \delta_k^n - \delta_j^n \delta_k^j \\ &= 3\delta_k^n - \delta_k^n \\ &= 2\delta_k^n \end{aligned} \quad (375)$$

From (375), the fact $\varepsilon_{ijk}\varepsilon^{ijk} = 2 \times 3 = 6$ is obtained²³.

In fact, repetition of any two indices will follow the same rule of contraction. To see the flexibility of the contraction, let $n = k$ in (373)

$$\mathbf{A}^T \mathbf{B} = \left[\begin{matrix} \hat{\mathbf{e}}_i^T \\ \hat{\mathbf{e}}_j^T \\ \hat{\mathbf{e}}_k^T \end{matrix} \right] \left[\begin{matrix} \hat{\mathbf{e}}_\ell & \hat{\mathbf{e}}_m & \hat{\mathbf{e}}_k \end{matrix} \right] = \begin{bmatrix} \delta_{i\ell} & \delta_{im} & \delta_{ik} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jk} \\ \delta_{k\ell} & \delta_{km} & \delta_{kk} \end{bmatrix} = \begin{bmatrix} \delta_{i\ell} & \delta_{im} & 0 \\ \delta_{j\ell} & \delta_{jm} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (376)$$

The zeros in the third row/column is due to the independence of unit basis $(\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k)$ with $i \neq j \neq k$ and $(\hat{\mathbf{e}}_\ell \hat{\mathbf{e}}_m \hat{\mathbf{e}}_k)$ with $\ell \neq m \neq k$, if otherwise, $\varepsilon_{ijk} = 0$. Therefore the determinant is

$$\det(\mathbf{A}^T \mathbf{B}) = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & 0 \\ \delta_{j\ell} & \delta_{jm} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \quad (377)$$

²³Einstein Summation Convention: If an index appears (exactly) twice, then it is summed over and appears only on one side of an equation. A single index (called a free index) appears once on each side of the equation.

A generalization of the property to arbitrary n is clear now:

$$\sum_{k=1}^n \varepsilon_{i_1 i_2 \dots i_{n-1} k} \varepsilon_{j_1 j_2 \dots j_{n-1} k} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \delta_{i_1 j_3} & \cdots & \delta_{i_1 j_{n-1}} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \cdots & \delta_{i_2 j_{n-1}} \\ \cdots & & & & \cdots \\ \delta_{i_{n-1} j_1} & \delta_{i_{n-1} j_2} & \cdots & \cdots & \delta_{i_{n-1} j_{n-1}} \end{vmatrix}. \quad (378)$$

The second property of the Levi-Civita symbol follows from

$$\begin{bmatrix} \hat{\mathbf{e}}_i^T \\ \hat{\mathbf{e}}_p^T \\ \hat{\mathbf{e}}_q^T \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_j & \hat{\mathbf{e}}_p & \hat{\mathbf{e}}_q \end{bmatrix} = \begin{bmatrix} \delta_{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (379)$$

The determinant of the last matrix is equal to δ_{ij} . The same holds for p and q interchanged. In the case of general n the sum is over $(n-1)!$ permutations [note that $(3-1)! = 2$]. The final property contains a summation over six ($3!$) non-zero terms; each term is the determinant of the identity matrix, which is unity. This section is extracted from a compendium note in [Kong \[2017\]](#).

8.4 General Stokes' Theorem

Stokes' theorem (full name: Kelvin–Stokes theorem after Lord Kelvin and George Stokes), is the fundamental theorem for curls or the curl theorem in vector calculus on \mathbb{R}^3 . Let Σ be a smooth oriented surface in \mathbb{R}^3 with boundary $\partial\Sigma$ and $\hat{\mathbf{n}}$ the unit outward normal. If a vector field $\mathbf{A} = (P(x, y, z), Q(x, y, z), R(x, y, z))$ is defined and has continuous first order partial derivatives in a region containing Σ , then

$$\iint_{\Sigma} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{\partial\Sigma} \mathbf{A} \cdot d\ell \quad (380)$$

More explicitly, the equality says that

$$\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) ds_x + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) ds_y + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds_z = \oint_{\partial\Sigma} P dx + Q dy + R dz \quad (381)$$

where the infinitesimal area components for the surface integral are $ds_x = d\mathbf{s} \cdot \hat{\mathbf{n}}_x = dy dz$, $ds_y = d\mathbf{s} \cdot \hat{\mathbf{n}}_y = dx dz$ and $ds_z = d\mathbf{s} \cdot \hat{\mathbf{n}}_z = dx dy$. By applying the Gauss' theorem,

$$\iiint_V (\nabla \cdot \nabla \times \mathbf{A}) dV = \iint_{\Sigma} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{\partial\Sigma} \mathbf{A} \cdot d\ell \quad (382)$$

A definition of $\nabla \times \mathbf{A}$ which is independent of the axes $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ may be obtained by applying Stokes's theorem to an infinitesimal plane area. Consider a point P . Pass a plane through P and draw in it, concentric with P , a small circle of area $d\mathbf{s}$.

$$(\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint \mathbf{A} \cdot d\ell \quad (383)$$

When $d\mathbf{s}$ has the same direction as $\nabla \times \mathbf{A}$ the value of the line integral will be a maximum, for the cosine of the angle between $\nabla \times \mathbf{A}$ and $d\mathbf{s}$ will be equal to unity. So $d\mathbf{s}/|d\mathbf{s}|$ can be

used to represent the direction of $\nabla \times \mathbf{A}$ but the magnitude $|ds|$ has to be divided from both sides in (383). For this value of ds goes to zero,

$$\nabla \times \mathbf{A} = \lim_{ds \rightarrow 0} \left[\frac{ds}{ds \cdot ds} \oint \mathbf{A} \cdot d\ell \right] \quad (384)$$

Hence the curl $\nabla \times \mathbf{A}$ of a vector function \mathbf{A} has at each point of space the direction of the normal to that plane in which the line integral of \mathbf{A} taken about a small circle concentric with the point in question is a maximum. The magnitude of the curl at the point is equal to the magnitude of that line integral of maximum value divided by the area of the circle about which it is taken.

8.5 The Different Forms of Stokes' Theorem and Variants

The following is to derive several different variants of the Stokes's Theory from the original (380) or (381). In the derivations, the notations are defined here unless specified otherwise. Let \mathbf{r} be the position vector $(x\hat{\mathbf{i}}, y\hat{\mathbf{j}}, z\hat{\mathbf{k}})$, \mathbf{c} an auxiliary constant vector and f, ϕ scalar functions. For compact expressions, some special notations are introduced, for instance \mathbf{r}_c is short for $\mathbf{r} \times \mathbf{c}$ and the up-down brackets around the three vectors $\llbracket \mathbf{a} \mathbf{b} \mathbf{c} \rrbracket$ stands for cyclic permutation equivalence of the scalar triple product.

$$\begin{aligned} \llbracket \mathbf{a} \mathbf{b} \mathbf{c} \rrbracket &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \\ &= -\llbracket \mathbf{a} \mathbf{c} \mathbf{b} \rrbracket &= -\llbracket \mathbf{b} \mathbf{a} \mathbf{c} \rrbracket = -\llbracket \mathbf{c} \mathbf{b} \mathbf{a} \rrbracket \end{aligned} \quad (385)$$

8.5.1 Original but in different equivalents

The original form (382) with $\mathbf{A} = \mathbf{b}$ can be written by the cyclic permutation notation as,

$$\oint_{\ell} \mathbf{b} \cdot d\ell = \iint_s (\nabla \times \mathbf{b}) \cdot ds = \iiint_V (\nabla \cdot \nabla \times \mathbf{b}) dV \quad (387)$$

$$= \iint_s [\mathbf{n} \cdot (\nabla \times \mathbf{b})] ds \quad (388)$$

$$= \iint_s \llbracket \mathbf{n} \nabla_{\mathbf{b}} \mathbf{b} \rrbracket ds \quad (389)$$

8.5.2 If $\mathbf{A} = f\mathbf{c}$, a scalar function multiplied by constant vector

In this case, the scalar function f can dependent on space (can also on time), but the vector \mathbf{c} is independent of space but no restriction on time.

$$\begin{aligned} \oint_{\ell} f\mathbf{c} \cdot d\ell &= \mathbf{c} \cdot \oint_{\ell} f d\ell = \iint_s (\nabla \times f\mathbf{c}) \cdot ds = \iint_s (\nabla f \times \mathbf{c}) \cdot ds = \mathbf{c} \cdot \iint_s \mathbf{n} \times \nabla f ds \\ \oint_{\ell} f d\ell &= \iint_s ds \times \nabla f = \iint_s \mathbf{n} \times \nabla f ds \end{aligned} \quad (390)$$

8.5.3 If $\mathbf{A} = \nabla(f\phi)$, a gradient of the product of two scalar functions

Then we have

$$\mathbf{A} = \nabla(f\phi) = f\nabla\phi + \phi\nabla f \quad (391)$$

$$\oint_{\ell} \nabla(f\phi) \cdot d\ell = \iint_s \nabla \times \nabla(f\phi) \cdot ds = 0 \quad (392)$$

$$\oint_{\ell} f\nabla\phi \cdot d\ell = \oint_{\ell} f d\phi = - \oint_{\ell} \phi df \quad (393)$$

$$= \iint_s \nabla f \times \nabla\phi \cdot ds \quad (394)$$

$$= \iint_s [\nabla f \cdot \nabla\phi] ds \quad (395)$$

8.5.4 If $\mathbf{A} = f\mathbf{a}$, a scalar function scaled vector

In this case, both the scalar function f and the vector \mathbf{a} are functions of space, as contrast to Section 8.5.2.

$$\oint_{\ell} f\mathbf{a} \cdot d\ell = \iint_s (\nabla \times f\mathbf{a}) \cdot ds = \iint_s (\nabla f \times \mathbf{a} + f\nabla \times \mathbf{a}) \cdot ds \quad (396)$$

$$= \iint_s [\nabla f \cdot \mathbf{a} \cdot \mathbf{n}] + [f \nabla \mathbf{a} \cdot \mathbf{n}] ds \quad (397)$$

8.5.5 If $\mathbf{A} = \mathbf{a} \times \mathbf{c}$, a cross product of scalar function scaled vector and a constant vector

Note \mathbf{c} denotes a constant vector, i.e., independent on space.

$$\oint_{\ell} (\mathbf{a} \times \mathbf{c}) \cdot d\ell = \mathbf{c} \cdot \oint_{\ell} d\ell \times \mathbf{a} \quad (398)$$

$$= \iint_s [\nabla \times (\mathbf{a} \times \mathbf{c})] \cdot \mathbf{n} ds \quad (399)$$

$$= \mathbf{c} \cdot \iint_s (\mathbf{n} \times \nabla) \times \mathbf{a} ds \quad (400)$$

$$= \mathbf{c} \cdot \iint_s \varepsilon_{ijk} (\mathbf{n} \times \nabla)_j a_k ds \quad (401)$$

$$= \mathbf{c} \cdot \iint_s \varepsilon_{ijk} \varepsilon_{jlm} n_l \partial_m a_k ds \quad (402)$$

$$= \mathbf{c} \cdot \iint_s n_k \partial_i a_k - n_i \partial_k a_k ds \quad (403)$$

$$\oint_{\ell} d\ell \times \mathbf{a} = \iint_s (\mathbf{n} \times \nabla) \times \mathbf{a} ds = \iint_s \mathbf{n} \cdot (\nabla \mathbf{a}) - (\nabla \mathbf{a}) \mathbf{n} ds \quad (404)$$

$$= \iint_s \mathbf{n} \times (\nabla \times \mathbf{a}) + (\mathbf{n} \cdot \nabla) \mathbf{a} - (\nabla \mathbf{a}) \mathbf{n} ds \quad (405)$$

8.5.6 If $\mathbf{A} = f\mathbf{a} \times \mathbf{c}$, a cross product of scalar function scaled vector and a constant vector

$$\oint_{\ell} (f\mathbf{a} \times \mathbf{c}) \cdot d\ell = \mathbf{c} \cdot \oint_{\ell} d\ell \times f\mathbf{a} \quad (406)$$

$$= \mathbf{c} \cdot \iint_s (\mathbf{n} \times \nabla) \times f\mathbf{a} ds \quad (407)$$

$$= \mathbf{c} \cdot \iint_s \varepsilon_{ijk} (\mathbf{n} \times \nabla)_j (fa_k) ds \quad (408)$$

$$= \mathbf{c} \cdot \iint_s \varepsilon_{ijk} \varepsilon_{jlm} n_l \partial_m (fa_k) ds \quad (409)$$

$$= \mathbf{c} \cdot \iint_s n_k f \partial_i a_k + n_k a_k \partial_i f - n_i f \partial_k a_k - n_i a_k \partial_k f ds \quad (410)$$

$$\oint_{\ell} d\ell \times f\mathbf{a} = \iint_s (\mathbf{n} \times \nabla) \times f\mathbf{a} ds \quad (411)$$

$$= \iint_s f \mathbf{n} \times (\nabla \times \mathbf{a}) + f (\mathbf{n} \cdot \nabla) \mathbf{a} + (\mathbf{n} \cdot \mathbf{a}) \nabla f - (\nabla f \cdot \mathbf{a}) \mathbf{n} - (\nabla \mathbf{a}) f \mathbf{n} ds \quad (412)$$

8.5.7 If $\mathbf{A} = f\nabla\phi \times \mathbf{c}$, a cross product of scalar function scaled gradient and a constant vector

Here, we introduce a potential function ϕ which satisfies Laplace equation, i.e. $\nabla^2\phi = 0$, therefore

$$\oint_{\ell} (f\nabla\phi \times \mathbf{c}) \cdot d\ell = \mathbf{c} \cdot \oint_{\ell} d\ell \times f\nabla\phi \quad (413)$$

neglecting the inner product by the constant vector \mathbf{c} , we obtain

$$\oint_{\ell} d\ell \times f\nabla\phi = \iint_s \left[f \frac{\partial}{\partial n} \nabla\phi + \frac{\partial\phi}{\partial n} \nabla f - (\nabla f \cdot \nabla\phi) \mathbf{n} \right] ds \quad (414)$$

8.5.8 If $\mathbf{A} = \mathbf{a} \times (\mathbf{r} \times \mathbf{c})$, a vector triple product

Note \mathbf{c} denotes a constant vector.

$$\oint_{\ell} \mathbf{a} \times (\mathbf{r} \times \mathbf{c}) \cdot d\ell = \oint_{\ell} (d\ell \times \mathbf{a}) \cdot (\mathbf{r} \times \mathbf{c}) = \oint_{\ell} [(d\ell \times \mathbf{a}) \times \mathbf{r}] \cdot \mathbf{c} \quad (415)$$

$$= \iint_s \{ \nabla \times [\mathbf{a} \times (\mathbf{r} \times \mathbf{c})] \} \cdot \mathbf{n} ds \xrightarrow{\mathbf{r}_c = \mathbf{r} \times \mathbf{c}} \iint_s \{ \nabla \times [\mathbf{a} \times \mathbf{r}_c] \} \cdot \mathbf{n} ds \quad (416)$$

$$= \iint_s \{ (\mathbf{r}_c \cdot \nabla) \mathbf{a} - \mathbf{r}_c \nabla \cdot \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{r}_c + \mathbf{a} \nabla \cdot \mathbf{r}_c \} \cdot \mathbf{n} ds \quad (417)$$

$$= \iint_s \left\{ \underbrace{(\mathbf{r}_c \cdot \nabla) \mathbf{a}}_{(1)} - \underbrace{(\mathbf{a} \cdot \nabla) \mathbf{r}_c}_{(2)} + \underbrace{\mathbf{a} \nabla \cdot \mathbf{r}_c}_{(3)} - \underbrace{\mathbf{r}_c \nabla \cdot \mathbf{a}}_{(4)} \right\} \cdot \mathbf{n} ds \quad (418)$$

$$(1) \cdot \mathbf{n} = \{[(\mathbf{r} \times \mathbf{c}) \cdot \nabla] \mathbf{a}\} \cdot \mathbf{n} \quad (419)$$

$$= \|\nabla \times \mathbf{a} \mathbf{r} \times \mathbf{c} \mathbf{n}\| + \|(\mathbf{n} \cdot \nabla) \mathbf{a} \mathbf{r} \mathbf{c}\| \quad (420)$$

$$= \|\mathbf{n} \times (\nabla \times \mathbf{a}) \mathbf{r} \mathbf{c}\| + \|(\mathbf{n} \cdot \nabla) \mathbf{a} \mathbf{r} \mathbf{c}\| \quad (421)$$

$$= \left\{ \underbrace{[\mathbf{n} \times (\nabla \times \mathbf{a})] \times \mathbf{r}}_{=} + [(\mathbf{n} \cdot \nabla) \mathbf{a}] \times \mathbf{r} \right\} \cdot \mathbf{c} \quad (422)$$

$$= \left\{ \underbrace{(\mathbf{r} \cdot \mathbf{n}) \nabla \times \mathbf{a} - [\mathbf{r} \cdot (\nabla \times \mathbf{a})] \mathbf{n}}_{=} + [(\mathbf{n} \cdot \nabla) \mathbf{a}] \times \mathbf{r} \right\} \cdot \mathbf{c} \quad (423)$$

$$(2) \cdot \mathbf{n} = -\{(\mathbf{a} \cdot \nabla)(\mathbf{r} \times \mathbf{c})\} \cdot \mathbf{n} \quad (424)$$

$$= -a_m (\partial_m \varepsilon_{ijk} r_j c_k) n_i = -\varepsilon_{ijk} a_m n_i (c_k \partial_m r_j + r_j \partial_m c_k) \quad (425)$$

$$= -\varepsilon_{ijk} a_j c_k n_i \quad (426)$$

$$= -\|\mathbf{a} \mathbf{c} \mathbf{n}\| = (\mathbf{a} \times \mathbf{n}) \cdot \mathbf{c} \quad (427)$$

$$(3) \cdot \mathbf{n} = \{\mathbf{a} \nabla \cdot (\mathbf{r} \times \mathbf{c})\} \cdot \mathbf{n} \quad (428)$$

$$= a_m [\varepsilon_{ijk} (c_k \partial_i r_j + r_j \partial_i c_k)] n_m \quad (429)$$

$$= \mathbf{a} [\mathbf{c} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{c})] \cdot \mathbf{n} \quad (430)$$

$$= 0 \quad (431)$$

$$(4) \cdot \mathbf{n} = (\nabla \cdot \mathbf{a})(\mathbf{r} \times \mathbf{c}) \cdot \mathbf{n} \quad (432)$$

$$= (\nabla \cdot \mathbf{a}) \|\mathbf{r} \mathbf{n} \mathbf{c}\| \quad (433)$$

$$= \{(\nabla \cdot \mathbf{a})(\mathbf{r} \times \mathbf{n})\} \cdot \mathbf{c} \quad (434)$$

$$\begin{aligned} \int_{\ell} (\mathbf{d} \ell \times \mathbf{a}) \times \mathbf{r} &= \iint_s \left\{ \underbrace{(\mathbf{r} \cdot \mathbf{n}) \nabla \times \mathbf{a} - [\mathbf{r} \cdot (\nabla \times \mathbf{a})] \mathbf{n}}_{\mathbf{n} \times (\nabla \times \mathbf{a}) \times \mathbf{r}} + [(\mathbf{n} \cdot \nabla) \mathbf{a}] \times \mathbf{r} + \right. \\ &\quad \left. \mathbf{a} \times \mathbf{n} + (\nabla \cdot \mathbf{a})(\mathbf{r} \times \mathbf{n}) \right\} ds \end{aligned} \quad (435)$$

Using above derivations, it is found that some literature contains incorrect use of Stokes formula. For instance the popularly cited [X-B \[2007\]](#) where the middle field reformulation for 2nd order wave loads used incorrectly Stokes's formulae, the corrected might be shown below.

$$\mathbf{r} \times [\mathbf{n} \cdot (\nabla \mathbf{a})] = \mathbf{r} \times [\mathbf{n} \times (\nabla \times \mathbf{a}) + (\mathbf{n} \cdot \nabla) \mathbf{a}] \quad (436)$$

$$\begin{aligned} \int_{\ell} (\mathbf{d} \ell \times \mathbf{a}) \times \mathbf{r} &= \iint_s \left\{ (\nabla \cdot \mathbf{a})(\mathbf{r} \times \mathbf{n}) + \mathbf{a} \times \mathbf{n} - \mathbf{r} \times [(\mathbf{n} \cdot \nabla) \mathbf{a}] \right. \\ &\quad \left. - [\mathbf{r} \cdot (\nabla \times \mathbf{a})] \mathbf{n} + (\mathbf{r} \cdot \mathbf{n}) \nabla \times \mathbf{a} \right\} ds \end{aligned} \quad (437)$$

$$\oint_c \mathbf{t} \times \mathbf{n} f ds = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{n} f ds \quad (438)$$

$$= \iint_S \varepsilon_{ijk} (\mathbf{n} \times \nabla)_j (f n_k) ds \quad (439)$$

$$= \iint_S \varepsilon_{ijk} \varepsilon_{jlm} n_\ell \partial_m (f n_k) ds \quad (440)$$

$$= \iint_S n_k f \partial_i n_k + n_k n_k \partial_i f - n_i f \partial_k n_k - n_i n_k \partial_k f ds \quad (441)$$

$$= \iint_S f \mathbf{n} \times (\nabla \times \mathbf{n}) + f (\mathbf{n} \cdot \nabla) \mathbf{n} + (\mathbf{n} \cdot \mathbf{n}) \nabla f - (\nabla f \cdot \mathbf{n}) \mathbf{n} - (\nabla \cdot \mathbf{n}) f \mathbf{n} ds \quad (442)$$

9 Reynolds theorem - Long story in short

When it comes to motions of fluid object, even a very small volume of fluid (here our object is called control volume, not quantum mechanics in the molecular sense), it is found horrible to apply the Newton's law to the tremendously large number of fluid particles and markers or taggers are in despair. Because fluid is continuous and deformable unlike isolated point masses, we have to find some new conceptual and mathematical techniques to apply these laws.

We call the traditional control volume, the Lagrangian type. This kind of control volume is a material volume (denoted $V_B^M(\xi, t)$, or more simply V_B^M). It consists of an infinitesimal mass of "tagged" fluid particles and always contains the same fluid particles. The boundaries of the control volume are defined such that there is zero flux of a specified extensive property B (the typical case is $B = m$, mass, so that the mass flow $b = 0$, i.e., there is zero flux of mass through the material volume's boundaries). The material volume is a function of time t , since the boundaries of the volume will change, with the fluid, in time. It is also a function of initial "marked" position ξ . The Reynolds theorem is used to describe the basic conservation laws of continuum quantities, i.e., that what was something changed inside the focused volume plus/minus what goes in/out of the surface is equal to the total change. But to be strict, one has to use Jacobian matrix to convert the Control value (CV) from Lagrangian frame to Eulerian frame. The derivation can take several pages and will use lots of differential geometry and calculus.

The former (Lagrangian CV) is particularly useful for deriving conservation laws, and the conservation of a quantity B can be expressed as $DB/Dt = 0$. The Eulerian CV control volume (denoted $V(\mathbf{x})$) and is a fixed and constant volume, whose location is fixed in the space coordinates. Therefore, it is not a function of time, only of location \mathbf{x} . It becomes more convenient for solving most problems in fluid mechanics because the field variables are related by a set of partial differential equations in which the independent variables are the coordinates x, y, z , and t . For instance, when we study the air flow dynamics, we just sit in a room and measure the air pressure, velocity, etc. to build up the equation system, or on a bridge over a river to study the hydraulic mechanics.

Here is provided I assume a simplest derivation which I have proposed as an answer in [Math.Exchange](#). I would use the $f(\vec{x}, t)$ for the physical quantity function and δV as the infinitesimal volume element. The product rule (or Leibniz rule) can well be used to express this:

$$D [f(\vec{x}, t) * \delta V] = \underbrace{\delta V}_{\text{hold constant}} * \partial f + \underbrace{f}_{\text{hold constant}} * \partial(\delta V) \quad (443)$$

The first differential term on the right is further divided²⁴ by δt and integral over the volume

$$\int_V \delta V \frac{\partial f}{\partial t} = \int_V \frac{\partial f}{\partial t} dV \quad (444)$$

and the last term in (443) is the differentiation of the volume over time which is nothing else than the volume change with the normal velocity U_n times the surface element dS , therefore, it reads

$$\int_{dV} f \frac{\partial \delta V}{\partial t} = \int_S f U_n dS \quad (445)$$

The full version can be readily obtained

$$\frac{D}{Dt} \int_V f dV = \int_V \frac{\partial f}{\partial t} dV + \int_S f U_n dS \quad (446)$$

10 Rigid Body Dynamics

10.1 Rigid body

If a mechanical system is constrained to move parallel to a fixed plane, then the rotation of a body in the system occurs around an axis $\hat{\mathbf{k}}$ perpendicular to this plane. In this case, the moment of inertia of the mass in this system is a scalar known as the polar moment of inertia. The definition of the polar moment of inertia can be obtained by considering momentum, kinetic energy and Newton's laws for the planar movement of a rigid system of particles.

If a system of N particles, P_i , $i = 1, \dots, n$, are assembled into a rigid body, then the momentum of the system can be written in terms of positions relative to a reference point \mathbf{R} , and absolute velocities \mathbf{v}_i

$$\delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R} \quad (447)$$

$$\mathbf{v}_i = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) + \mathbf{V} = \boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V}, \quad (448)$$

where $\delta \mathbf{r}_i$ (no confusion with Kroneck delta with subscripts) is the difference vector relative to the reference point \mathbf{R} , $\boldsymbol{\omega}$ is the angular velocity of the system and \mathbf{V} is the velocity of \mathbf{R} .

²⁴One might be imagining or be confused by applying chain rule again here. Note that the differentiation has been done already by applying Leibniz rule in (443).

For planar movement the angular velocity vector is directed along the unit vector $\hat{\mathbf{e}}_i$ which is perpendicular to the plane of movement. Introduce the unit vectors $\hat{\mathbf{e}}_i$ from the reference point \mathbf{R} to a point \mathbf{r}_i , and the unit vector $\hat{\mathbf{t}}_i = \hat{\mathbf{k}} \times \hat{\mathbf{e}}_i$ so

$$\hat{\mathbf{e}}_i = \frac{\delta \mathbf{r}_i}{\delta r_i}, \quad \hat{\mathbf{k}} = \frac{\boldsymbol{\omega}}{\omega}, \quad \hat{\mathbf{t}}_i = \hat{\mathbf{k}} \times \hat{\mathbf{e}}_i, \quad (449)$$

$$\mathbf{v}_i = \boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V} = \omega \hat{\mathbf{k}} \times \delta r_i \hat{\mathbf{e}}_i + \mathbf{V} = \omega \delta r_i \hat{\mathbf{t}}_i + \mathbf{V} \quad (450)$$

This defines the relative position vector and the velocity vector for the rigid system of the particles moving in a plane.

Note on the cross product: When a body moves parallel to a ground plane, the trajectories of all the points in the body lie in planes parallel to this ground plane. This means that any rotation that the body undergoes must be around an axis perpendicular to this plane. Planar movement is often presented as projected onto this ground plane so that the axis of rotation appears as a point. In this case, the angular velocity and angular acceleration of the body are scalars and the fact that they are vectors along the rotation axis is ignored. This is usually preferred for introductions to the topic. But in the case of moment of inertia, the combination of mass and geometry benefits from the geometric properties of the cross product. For this reason, in this section on planar movement the angular velocity and accelerations of the body are vectors perpendicular to the ground plane, and the cross product operations are the same as used for the study of spatial rigid body movement.

10.2 Angular momentum

Angular momentum is the rotational equivalent of linear momentum. The angular momentum vector for the planar movement of a rigid system of particles is the cross product of the particle's position vector \mathbf{r} (relative to some origin) and its momentum vector \mathbf{p} , given by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \mathbf{r} \times (\boldsymbol{\omega} \times (m\mathbf{r})) = mr^2\boldsymbol{\omega} = I\boldsymbol{\omega}$. The angular momentum is expressed in the discrete form

$$\mathbf{L} = \left(\sum_{i=1}^N m_i \delta \mathbf{r}_i \times \mathbf{v}_i \right) \quad (451)$$

$$= \left(\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{e}}_i \times (\omega \delta r_i \hat{\mathbf{t}}_i + \mathbf{V}) \right) \quad (452)$$

$$= \left(\sum_{i=1}^N m_i \delta r_i^2 \right) \omega \hat{\mathbf{k}} + \left(\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{e}}_i \right) \times \mathbf{V}. \quad (453)$$

Use the centre of mass \mathbf{C} as the reference point so

$$\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{e}}_i = \sum_{i=1}^N m_i (\mathbf{r}_i - \mathbf{C}) = 0, \quad (454)$$

and define the moment of inertia relative to the centre of mass $I_{\mathbf{C}}$ as

$$I_{\mathbf{C}} = \sum_i m_i \delta r_i^2 \quad (455)$$

then the equation for angular momentum simplifies to

$$\mathbf{L} = I_{\mathbf{C}} \omega \hat{\mathbf{k}} \quad (456)$$

The moment of inertia $I_{\mathbf{C}}$ about an axis perpendicular to the movement of the rigid system and through the centre of mass is known as the polar moment of inertia. Specifically, it is the second moment of mass with respect to the orthogonal distance from an axis (or pole).

For a given amount of angular momentum, a decrease in the moment of inertia results in an increase in the angular velocity.

10.3 Kinetic energy

The kinetic energy of a rigid system of particles moving in the plane is given by

$$E_K = \frac{1}{2} \left(\sum_{i=1}^N m_i \mathbf{v}_i \cdot \mathbf{v}_i \right), \quad (457)$$

$$= \frac{1}{2} \left(\sum_{i=1}^N m_i (\omega \delta r_i \hat{\mathbf{t}}_i + \mathbf{V}) \cdot (\omega \delta r_i \hat{\mathbf{t}}_i + \mathbf{V}) \right), \quad (458)$$

$$= \frac{1}{2} \omega^2 \left(\sum_{i=1}^N m_i \delta r_i^2 \hat{\mathbf{t}}_i \cdot \hat{\mathbf{t}}_i \right) + \omega \mathbf{V} \cdot \left(\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{t}}_i \right) + \frac{1}{2} \left(\sum_{i=1}^N m_i \right) \mathbf{V} \cdot \mathbf{V}. \quad (459)$$

Let the reference point be the centre of mass \mathbf{C} of the system so the second term becomes zero, and introduce the moment of inertia $I_{\mathbf{C}}$ so the kinetic energy is given by

$$E_K = \frac{1}{2} I_{\mathbf{C}} \omega^2 + \frac{1}{2} M \mathbf{V} \cdot \mathbf{V} \quad (460)$$

The moment of inertia $I_{\mathbf{C}}$ is the polar moment of inertia of the body.

10.3.1 Newton's laws

Newton's laws for a rigid system of N particles, P_i , $i = 1, \dots, N$, can be written in terms of a resultant force and torque at a reference point \mathbf{R} , to yield:

$$\mathbf{F} = \left(\sum_{i=1}^N m_i \mathbf{A}_i \right), \quad (461)$$

$$\boldsymbol{\tau} = \left(\sum_{i=1}^N \delta \mathbf{r}_i \times m_i \mathbf{A}_i \right), \quad (462)$$

where \mathbf{r}_i denotes the trajectory of each particle.

The kinematics of a rigid body yields the formula for the acceleration of the particle P_i in terms of the position \mathbf{R} and acceleration \mathbf{A} of the reference particle as well as the angular velocity vector $\boldsymbol{\omega}$ and angular acceleration vector $\boldsymbol{\alpha}$ of the rigid system of particles as,

$$\mathbf{A}_i = \boldsymbol{\alpha} \times \delta \mathbf{r}_i + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{A} \quad (463)$$

For systems that are constrained to planar movement, the angular velocity and angular acceleration vectors are directed along $\hat{\mathbf{k}}$ perpendicular to the plane of movement, which simplifies this acceleration equation. In this case, the acceleration vectors can be simplified by introducing the unit vectors $\hat{\mathbf{e}}_i$ from the reference point \mathbf{R} to a point \mathbf{r}_i and the unit vectors $\hat{\mathbf{t}}_i = \hat{\mathbf{k}} \times \hat{\mathbf{e}}_i$, so

$$\mathbf{A}_i = \alpha \hat{\mathbf{k}} \times \delta r_i \hat{\mathbf{e}}_i - \omega \hat{\mathbf{k}} \times \omega \hat{\mathbf{k}} \times \delta r_i \hat{\mathbf{e}}_i + \mathbf{A} \quad (464)$$

$$= \alpha \delta r_i \hat{\mathbf{t}}_i - \omega^2 \delta r_i \hat{\mathbf{e}}_i + \mathbf{A}. \quad (465)$$

This yields the resultant torque on the system as

$$\boldsymbol{\tau} = \left(\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{e}}_i \times (\alpha \delta r_i \hat{\mathbf{t}}_i - \omega^2 \delta r_i \hat{\mathbf{e}}_i + \mathbf{A}) \right) \quad (466)$$

$$= \left(\sum_{i=1}^N m_i \delta r_i^2 \right) \alpha \hat{\mathbf{k}} + \left(\sum_{i=1}^N m_i \delta r_i \hat{\mathbf{e}}_i \right) \times \mathbf{A}, \quad (467)$$

where $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_i = 0$, and $\hat{\mathbf{e}}_i \times \hat{\mathbf{t}}_i = \hat{\mathbf{k}}$ is the unit vector perpendicular to the plane for all of the particles P_i .

Use the centre of mass \mathbf{C} as the reference point and define the moment of inertia relative to the centre of mass $I_{\mathbf{C}}$, then the equation for the resultant torque simplifies to $\boldsymbol{\tau} = I_{\mathbf{C}} \alpha \hat{\mathbf{k}}$.

Motion in space of a rigid body, and the inertia matrix

The scalar moments of inertia appear as elements in a matrix when a system of particles is assembled into a rigid body that moves in three-dimensional space. This inertia matrix appears in the calculation of the angular momentum, kinetic energy and resultant torque of the rigid system of particles.

Let the system of particles $P_i, i = 1, \dots, N$ be located at the coordinates \mathbf{r}_i with velocities \mathbf{v}_i relative to a fixed reference frame. For a (possibly moving) reference point \mathbf{R} , the relative positions are

$$\delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R} \quad (468)$$

and the (absolute) velocities are

$$\mathbf{v}_i = \boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V}_{\mathbf{R}} \quad (469)$$

where $\boldsymbol{\omega}$ is the angular velocity of the system, and $\mathbf{V}_{\mathbf{R}}$ is the velocity of \mathbf{R} .

10.3.2 Angular momentum

Note that the cross product can be equivalently written as matrix multiplication by combining the first operand and the operator into a, skew-symmetric, matrix, $[\mathbf{b}]$, constructed

from the components of $\mathbf{b} = (b_x, b_y, b_z)$:

$$\mathbf{b} \times \mathbf{y} \equiv [\mathbf{b}]\mathbf{y} \quad (470)$$

$$[\mathbf{b}] \equiv \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}. \quad (471)$$

The inertia matrix is constructed by considering the angular momentum, with the reference point \mathbf{R} of the body chosen to be the centre of mass \mathbf{C} :

$$\mathbf{L} = \left(\sum_{i=1}^N m_i \delta \mathbf{r}_i \times \mathbf{v}_i \right) \quad (472)$$

$$= \left(\sum_{i=1}^N m_i \delta \mathbf{r}_i \times (\boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V}_{\mathbf{R}}) \right) \quad (473)$$

$$= \left(- \sum_{i=1}^N m_i \delta \mathbf{r}_i \times (\delta \mathbf{r}_i \times \boldsymbol{\omega}) \right) + \left(\sum_{i=1}^N m_i \delta \mathbf{r}_i \times \mathbf{V}_{\mathbf{R}} \right), \quad (474)$$

where the terms containing $\mathbf{V}_{\mathbf{R}} (= \mathbf{C})$ sum to zero by the definition of centre of mass.

Then, the skew-symmetric matrix $[\delta \mathbf{r}_i]$ obtained from the relative position vector $\delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{C}$, can be used to define,

$$\mathbf{L} = \left(- \sum_{i=1}^N m_i [\delta \mathbf{r}_i]^2 \right) \boldsymbol{\omega} = \mathbf{I}_{\mathbf{C}} \boldsymbol{\omega} \quad (475)$$

where $\mathbf{I}_{\mathbf{C}}$ defined by

$$\mathbf{I}_{\mathbf{C}} = - \sum_{i=1}^N m_i [\delta \mathbf{r}_i]^2 \quad (476)$$

is the symmetric inertia matrix of the rigid system of particles measured relative to the centre of mass \mathbf{C} .

10.3.3 Kinetic energy

The kinetic energy of a rigid system of particles can be formulated in terms of the centre of mass and a matrix of mass moments of inertia of the system. Let the system of particles $P_i, i = 1, \dots, N$ be located at the coordinates \mathbf{r}_i with velocities \mathbf{v}_i , then the kinetic energy is

$$E_K = \frac{1}{2} \left(\sum_{i=1}^N m_i \mathbf{v}_i \cdot \mathbf{v}_i \right) = \frac{1}{2} \left(\sum_{i=1}^N m_i (\boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V}_{\mathbf{C}}) \cdot (\boldsymbol{\omega} \times \delta \mathbf{r}_i + \mathbf{V}_{\mathbf{C}}) \right) \quad (477)$$

where $\delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{C}$ is the position vector of a particle relative to the centre of mass.

This equation expands to yield three terms

$$E_K = \frac{1}{2} \sum_{i=1}^N m_i (\boldsymbol{\omega} \times \delta \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \delta \mathbf{r}_i) + \sum_{i=1}^N m_i \mathbf{V}_{\mathbf{C}} \cdot (\boldsymbol{\omega} \times \delta \mathbf{r}_i) + \frac{1}{2} \sum_{i=1}^N m_i \mathbf{V}_{\mathbf{C}} \cdot \mathbf{V}_{\mathbf{C}} \quad (478)$$

The second term in this equation is zero because \mathbf{C} is the centre of mass. Introduce the skew-symmetric matrix $[\delta \mathbf{r}_i]$ so the kinetic energy becomes

$$E_K = \frac{1}{2} \left(\sum_{i=1}^N m_i ([\delta \mathbf{r}_i] \boldsymbol{\omega}) \cdot ([\delta \mathbf{r}_i] \boldsymbol{\omega}) \right) + \frac{1}{2} \left(\sum_{i=1}^N m_i \right) \mathbf{V}_C \cdot \mathbf{V}_C \quad (479)$$

$$= \frac{1}{2} \left(\sum_{i=1}^N m_i (\boldsymbol{\omega}^T [\delta \mathbf{r}_i]^T [\delta \mathbf{r}_i] \boldsymbol{\omega}) \right) + \frac{1}{2} \left(\sum_{i=1}^N m_i \right) \mathbf{V}_C \cdot \mathbf{V}_C \quad (480)$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \left(- \sum_{i=1}^N m_i [\delta \mathbf{r}_i]^2 \right) \boldsymbol{\omega} + \frac{1}{2} \left(\sum_{i=1}^N m_i \right) \mathbf{V}_C \cdot \mathbf{V}_C. \quad (481)$$

Thus, the kinetic energy of the rigid system of particles is given by

$$E_K = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_C \boldsymbol{\omega} + \frac{1}{2} M \mathbf{V}_C^2 \quad (482)$$

where \mathbf{I}_C is the inertia matrix relative to the centre of mass and M is the total mass.

10.3.4 Parallel axis theorem

The inertia matrix of a body depends on the choice of the reference point. There is a useful relationship between the inertia matrix relative to the centre of mass \mathbf{C} and the inertia matrix relative to another point \mathbf{R} . This relationship is called the parallel axis theorem.

Consider the inertia matrix \mathbf{I}_R obtained for a rigid system of particles measured relative to a reference point \mathbf{R} , given by

$$\mathbf{I}_R = - \sum_{i=1}^N m_i [\mathbf{r}_i - \mathbf{R}]^2 \quad (483)$$

Let \mathbf{C} be the centre of mass of the rigid system, then

$$\mathbf{R} = (\mathbf{R} - \mathbf{C}) + \mathbf{C} = \mathbf{d} + \mathbf{C} \quad (484)$$

where \mathbf{d} is the vector from the centre of mass \mathbf{C} to the reference point \mathbf{R} . Use this equation to compute the inertia matrix,

$$\mathbf{I}_R = - \sum_{i=1}^N m_i [\mathbf{r}_i - (\mathbf{C} + \mathbf{d})]^2 = - \sum_{i=1}^N m_i [(\mathbf{r}_i - \mathbf{C}) - \mathbf{d}]^2 \quad (485)$$

Distribute over the cross product to obtain

$$\mathbf{I}_R = - \left(\sum_{i=1}^N m_i [\mathbf{r}_i - \mathbf{C}]^2 \right) + \left(\sum_{i=1}^N m_i [\mathbf{r}_i - \mathbf{C}] \right) [\mathbf{d}] + [\mathbf{d}] \left(\sum_{i=1}^N m_i [\mathbf{r}_i - \mathbf{C}] \right) - \left(\sum_{i=1}^N m_i \right) [\mathbf{d}]^2 \quad (486)$$

The first term is the inertia matrix \mathbf{I}_C relative to the centre of mass. The second and third terms are zero by definition of the centre of mass \mathbf{C} . And the last term is the total mass of the system multiplied by the square of the skew-symmetric matrix $[\mathbf{d}]$ constructed from \mathbf{d} .

The result is the parallel axis theorem,

$$\mathbf{I}_R = \mathbf{I}_C - M[\mathbf{d}]^2 \quad (487)$$

where \mathbf{d} is the vector from the centre of mass \mathbf{C} to the reference point \mathbf{R} .

Note on the minus sign: By using the skew symmetric matrix of position vectors relative to the reference point, the inertia matrix of each particle has the form $-m[\mathbf{r}]^2$, which is similar to the mr^2 that appears in planar movement. However, to make this to work out correctly a minus sign is needed. This minus sign can be absorbed into the term $-m[\mathbf{r}]^T[\mathbf{r}]$, if desired, by using the skew-symmetry property of $[\mathbf{r}]$.

Scalar moment of inertia in a plane

The scalar moment of inertia, I_L , of a body about a specified axis whose direction is specified by the unit vector $\hat{\mathbf{k}}$ and passes through the body at a point \mathbf{R} is as follows:

$$I_L = \hat{\mathbf{k}} \cdot \left(- \sum_{i=1}^N m_i [\delta \mathbf{r}_i]^2 \right) \hat{\mathbf{k}} \quad (488)$$

$$= \hat{\mathbf{k}} \cdot \mathbf{I}_R \hat{\mathbf{k}} = \hat{\mathbf{k}}^T \mathbf{I}_R \hat{\mathbf{k}} \quad (489)$$

where \mathbf{I}_R is the moment of inertia matrix of the system relative to the reference point \mathbf{R} , and $[\delta \mathbf{r}_i]$ is the skew symmetric matrix obtained from the vector $\delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R}$.

This is derived as follows. Let a rigid assembly of N particles, P_i , $i = 1, \dots, N$, have coordinates \mathbf{r}_i . Choose \mathbf{R} as a reference point and compute the moment of inertia around a line \mathbf{L} defined by the unit vector $\hat{\mathbf{k}}$ through the reference point \mathbf{R} , $\mathbf{L}(t) = \mathbf{R} + t\hat{\mathbf{k}}$. The perpendicular vector from this line to the particle P_i is obtained from $\delta \mathbf{r}_i$ by removing the component that projects onto $\hat{\mathbf{k}}$.

$$\delta \mathbf{r}_i^\perp = \delta \mathbf{r}_i - (\hat{\mathbf{k}} \cdot \delta \mathbf{r}_i) \hat{\mathbf{k}} = (\mathbb{E} - \hat{\mathbf{k}} \hat{\mathbf{k}}^T) \delta \mathbf{r}_i \quad (490)$$

where \mathbb{E} is the identity matrix, so as to avoid confusion with the inertia matrix, and $\hat{\mathbf{k}} \hat{\mathbf{k}}^T$ is the outer product matrix formed from the unit vector $\hat{\mathbf{k}}$ along the line \mathbf{L} .

To relate this scalar moment of inertia to the inertia matrix of the body, introduce the skew-symmetric matrix $[\hat{\mathbf{k}}]$ such that $[\hat{\mathbf{k}}]\mathbf{y} = \hat{\mathbf{k}} \times \mathbf{y}$, then we have the identity

$$- [\hat{\mathbf{k}}]^2 \equiv |\hat{\mathbf{k}}|^2 (\mathbb{E} - \hat{\mathbf{k}} \hat{\mathbf{k}}^T) = \mathbb{E} - \hat{\mathbf{k}} \hat{\mathbf{k}}^T \quad (491)$$

noting that $\hat{\mathbf{k}}$ is a unit vector.

The magnitude squared of the perpendicular vector is

$$|\delta \mathbf{r}_i^\perp|^2 = \left(- [\hat{\mathbf{k}}]^2 \delta \mathbf{r}_i \right) \cdot \left(- [\hat{\mathbf{k}}]^2 \delta \mathbf{r}_i \right) \quad (492)$$

$$= (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \cdot (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \quad (493)$$

The simplification of this equation uses the triple scalar product identity

$$(\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \cdot (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \equiv ((\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \times \hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}} \times \delta \mathbf{r}_i) \quad (494)$$

where the dot and the cross products have been interchanged. Exchanging products, and simplifying by noting that $\delta \mathbf{r}_i$ and $\hat{\mathbf{k}}$ are orthogonal:

$$(\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \cdot (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \quad (495)$$

$$= ((\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \delta \mathbf{r}_i)) \times \hat{\mathbf{k}}) \cdot (\hat{\mathbf{k}} \times \delta \mathbf{r}_i) \quad (496)$$

$$= (\hat{\mathbf{k}} \times \delta \mathbf{r}_i) \cdot (-\delta \mathbf{r}_i \times \hat{\mathbf{k}}) \quad (497)$$

$$= -\hat{\mathbf{k}} \cdot (\delta \mathbf{r}_i \times \delta \mathbf{r}_i \times \hat{\mathbf{k}}) \quad (498)$$

$$= -\hat{\mathbf{k}} \cdot [\delta \mathbf{r}_i]^2 \hat{\mathbf{k}}. \quad (499)$$

Thus, the moment of inertia around the line L through \mathbf{R} in the direction $\hat{\mathbf{k}}$ is obtained from the calculation

$$I_L = \sum_{i=1}^N m_i |\delta \mathbf{r}_i^\perp|^2 \quad (500)$$

$$= \left(- \sum_{i=1}^N m_i \hat{\mathbf{k}} \cdot [\delta \mathbf{r}_i]^2 \hat{\mathbf{k}} \right) = \hat{\mathbf{k}} \cdot \left(- \sum_{i=1}^N m_i [\delta \mathbf{r}_i]^2 \right) \hat{\mathbf{k}} \quad (501)$$

$$= \hat{\mathbf{k}} \cdot \mathbf{I}_{\mathbf{R}} \hat{\mathbf{k}} = \hat{\mathbf{k}}^T \mathbf{I}_{\mathbf{R}} \hat{\mathbf{k}} \quad (502)$$

where $\mathbf{I}_{\mathbf{R}}$ is the moment of inertia matrix of the system relative to the reference point \mathbf{R} .

This shows that the inertia matrix can be used to calculate the moment of inertia of a body around any specified rotation axis in the body. Inertia tensor

The inertia matrix is often described as the inertia tensor, which consists of the same moments of inertia and products of inertia about the three coordinate axes. The inertia tensor is constructed from the nine component tensors, (the symbol \otimes is the tensor product) $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, where $i, j = 1, 2, 3$, where $\hat{\mathbf{e}}_i$, $i = 1, 2, 3$ are the three orthogonal unit vectors defining the inertial frame in which the body moves. Using this basis the inertia tensor is given by

$$\mathbf{I} = \sum_{i=1}^3 \sum_{j=1}^3 I_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad (503)$$

This tensor is of degree two because the component tensors are each constructed from two basis vectors. In this form the inertia tensor is also called the inertia binor.

For a rigid system of particles P_k , $k = 1, \dots, N$ each of mass m_k with position coordinates $\mathbf{r}_k = (x_k, y_k, z_k)$, the inertia tensor is given by

$$\mathbf{I} = \sum_{k=1}^N m_k [(\mathbf{r}_k \cdot \mathbf{r}_k) \mathbb{E} - \mathbf{r}_k \otimes \mathbf{r}_k] \quad (504)$$

where \mathbb{E} is the identity tensor: $\mathbb{E} = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$.

In this case, the components of the inertia tensor are given by

$$I_{11} = I_{xx} = \sum_{k=1}^N m_k (y_k^2 + z_k^2), \quad (505)$$

$$I_{22} = I_{yy} = \sum_{k=1}^N m_k (x_k^2 + z_k^2), \quad (506)$$

$$I_{33} = I_{zz} = \sum_{k=1}^N m_k (x_k^2 + y_k^2), \quad (507)$$

$$I_{12} = I_{21} = I_{xy} = - \sum_{k=1}^N m_k x_k y_k, \quad (508)$$

$$I_{13} = I_{31} = I_{xz} = - \sum_{k=1}^N m_k x_k z_k, \quad (509)$$

$$I_{23} = I_{32} = I_{yz} = - \sum_{k=1}^N m_k y_k z_k. \quad (510)$$

The inertia tensor for a continuous body is given by

$$\mathbf{I} = \iiint_Q \rho(\mathbf{r}) [(\mathbf{r} \cdot \mathbf{r}) \mathbb{E} - \mathbf{r} \otimes \mathbf{r}] dV \quad (511)$$

where \mathbf{r} defines the coordinates of a point in the body and $\rho(\mathbf{r})$ is the mass density function at that point. The integral is taken over the volume V of the body. The inertia tensor is symmetric because $I_{ij} = I_{ji}$.

Alternatively it can also be written in terms of the hat operator as:

$$\mathbf{I} = \iiint_Q \rho(\mathbf{r}) (\hat{\mathbf{r}})^2 dV \quad (512)$$

The inertia tensor can be used in the same way as the inertia matrix to compute the scalar moment of inertia about an arbitrary axis in the direction \mathbf{n} , $I_n = \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n}$, where the dot product is taken with the corresponding elements in the component tensors. A product of inertia term such as I_{12} is obtained by the computation $I_{12} = \hat{\mathbf{e}}_1 \cdot \mathbf{I} \cdot \hat{\mathbf{e}}_2$, and can be interpreted as the moment of inertia around the x-axis when the object rotates around the y-axis.

The components of tensors of degree two can be assembled into a matrix. For the inertia tensor this matrix is given by,

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad (513)$$

For mass matrix, the inertia tensor is placed on the modes $k = 4, 5, 6$. Let $I_{i+3,j+3}^R$, $I_{i+3,j+3}^G$ with $i, j = 1, 2, 3$ are the inertia tensors w.r.t. an arbitrary reference point \mathbf{R} and

center of mass G , then the [parallel-axis theorem](#) tells us

$$I_{i+3,j+3}^R = I_{i+3,j+3}^G + \sum_{n=1}^N m_n \left(\delta_{ij} \sum_l^3 r_l^2 - r_i r_j \right) \quad (514)$$

where $\sum_{n=1}^N m_n = M$ and $\sum_k^3 r_k^2 = r^2$, thus

$$I_{i+3,j+3}^R = I_{i+3,j+3}^G + M(r^2 \delta_{ij} - r_i r_j) \quad (515)$$

For example, the moments of inertia I_{44}^R around the X_1 axis induced by roll and pitch I_{45}^R , are respectively:

$$I_{44}^R = I_{44}^G + M[(r_1^2 + r_2^2 + r_3^2)\delta_{11} - r_1^2] = I_{44}^G + M(r_2^2 + r_3^2) \quad (516)$$

$$I_{45}^R = I_{45}^G + M[(r_1^2 + r_2^2 + r_3^2)\delta_{12} - r_1 r_2] = I_{45}^G - Mr_1 r_2 \quad (517)$$

As the Kronecker $\delta_{ij} = 0$ in (514) for $i \neq j$, the cross terms can be redefined as

$$I_{i+3,j+3}^R = I_{i+3,j+3}^G - Mr_i r_j = I_{i+3,j+3}^G + Mr_{ij} |r_{ij}| \quad (518)$$

As a generalization for diagonal radii of gyration, i.e., $r_{44} = \sqrt{r_2^2 + r_3^2}$, $r_{55} = \sqrt{r_1^2 + r_3^2}$ and $r_{66} = \sqrt{r_1^2 + r_2^2}$, r_{ij} with $i, j = 4, 5, 6$ can be defined as the cross radius of gyration

$$r_{ij} = (-1)^{\text{INT}(\frac{i}{j})+1} \text{sign}(r_{i-3} r_{j-3}) \sqrt{|r_{i-3} r_{j-3}|} \quad (519)$$

where $\text{INT}(n)$ is the integer function taking the argument $n \leq 1$, therefore it requires $i \leq j$ although r_{ij} is symmetric about i and j . This definition retains the sign from the product of r_{i-3} and r_{j-3} . The corresponding notations commonly used in rigid body mechanics can explicitly identify the rotation about x -, y -, and z - axes, such as I_{xx} and I_{xy} , for the components of the inertia tensor. Let the x_g , y_g and z_g be r_i with $i = 1, 2, 3$, then

$$\mathbf{I}^R = \begin{bmatrix} I_{44}^R & I_{45}^R & I_{46}^R \\ I_{54}^R & I_{55}^R & I_{56}^R \\ I_{64}^R & I_{65}^R & I_{66}^R \end{bmatrix} = \begin{bmatrix} I_{xx}^R & I_{xy}^R & I_{xz}^R \\ I_{yx}^R & I_{yy}^R & I_{yz}^R \\ I_{zx}^R & I_{zy}^R & I_{zz}^R \end{bmatrix} \quad (520)$$

To express the shifting formula explicitly in the components from the center of mass G to the reference \mathbf{R}

$$I_{44}^R = I_{xx}^G + M(y_g^2 + z_g^2), \quad (521)$$

$$I_{55}^R = I_{yy}^G + M(x_g^2 + z_g^2), \quad (522)$$

$$I_{66}^R = I_{zz}^G + M(x_g^2 + y_g^2), \quad (523)$$

$$I_{45}^R = I_{54}^R = I_{xy}^G - Mx_g y_g, \quad (524)$$

$$I_{46}^R = I_{64}^R = I_{xz}^G - Mx_g z_g, \quad (525)$$

$$I_{56}^R = I_{65}^R = I_{yz}^G - My_g z_g. \quad (526)$$

$$= \begin{bmatrix} M\mathbb{E}_{3 \times 3} & -M[\mathbf{S}]_{3 \times 3}(\mathbf{x}_g) \\ \vdots & \vdots \\ M[\mathbf{S}]_{3 \times 3}(\mathbf{x}_g) & [\mathbf{I}^G]_{3 \times 3} - M[\mathbf{S}]_{3 \times 3}^2(\mathbf{x}_g) \end{bmatrix} \quad (527)$$

where $[\mathbf{S}]_{3 \times 3}$ is the **skew-symmetric matrix** and the last term in the lower right is the squared skew-symmetric matrix $[\mathbf{S}]_{3 \times 3}^2$. It will be shown by using the Levi-Civita symbol.

$$[\mathbf{S}]^2(\mathbf{x}) = [\mathbf{S}](\mathbf{x})[\mathbf{S}](\mathbf{x}) = -[\mathbf{S}](\mathbf{x})[\mathbf{S}]^\top(\mathbf{x}) \quad (\text{due to } [\mathbf{S}]^\top = -[\mathbf{S}]) \quad (528)$$

$$= [-\epsilon_{ikm}x_m(-\epsilon_{kjn}x_n)] = [\epsilon_{ikm}\epsilon_{kjn}x_m x_n] \quad (529)$$

$$= [(\delta_{mj}\delta_{in} - \delta_{mn}\delta_{ij})x_m x_n] \quad (530)$$

$$= \left[x_i x_j - \delta_{ij} \sum_{n=1}^3 x_n^2 \right] = \mathbf{x} \mathbf{x}^\top - \mathbf{x}^\top \mathbf{x} [\mathbf{I}] \quad (531)$$

where \mathbf{x}^\top is the transpose of the vector \mathbf{x} and $\mathbb{E}_{3 \times 3}$ is the 3×3 identity matrix so as to avoid confusion with the inertia matrix. Note: for negative sign definition for cross product of inertia, one should not forget that the cross terms with respect to G must also flip the signs, i.e., different definition from the diagonal terms.

The principal axes of any rigid body are those for which the cross-products of inertia are all zero resulting in an diagonal inertia matrix. The principal axes are the eigenvectors of the tensor $[\mathbf{I}]$ and the principal moments are its eigenvalues. In component form, we can find these by solving a 3×3 eigenvalue problem. The eigenvectors satisfy the set of linear equations

$$\begin{bmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0 \quad (532)$$

In the principal basis the component equations become $[\mathbf{I}]\mathbf{x} = \lambda\mathbf{x}$ where λ is a constant of proportionality or can be called the slope of the of a mapping of $[\mathbf{I}]\mathbf{x}$ over \mathbf{x} .

Nontrivial solutions for ω exist only if the determinant vanishes,

$$\begin{bmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda \end{bmatrix} = 0 \quad (533)$$

11 Skew-symmetric matrix

In mathematics, particularly in linear algebra, a skew-symmetric (or antisymmetric or antimetric matrix is a square matrix whose transpose equals its additive inverse $\mathbf{A}^\top = -\mathbf{A}$.

- The sum of two skew-symmetric matrices is skew-symmetric.
- A scalar multiple of a skew-symmetric matrix is skew-symmetric.
- The elements on the diagonal of a skew-symmetric matrix are zero, and therefore its trace equals zero.
- If \mathbf{A} is a real skew-symmetric matrix and λ is a real eigenvalue, then $\lambda = 0$, i.e. the nonzero [eigenvalues](#) of a skew-symmetric matrix are non-real.
- If \mathbf{A} is a real skew-symmetric matrix, then $\mathbb{E} + \mathbf{A}$ is invertible, where \mathbb{E} is the identity matrix.
- If \mathbf{A} is a skew-symmetric matrix then \mathbf{A}^2 is a symmetric negative semi-definite matrix $\langle \mathbf{x}, \mathbf{A}^2 \mathbf{x} \rangle = \mathbf{x}^\top \mathbf{A}^2 \mathbf{x} = -\|\mathbf{Ax}\|_2^2 \leq 0$.
- Levi-Civita symbol ε_{ijk} is the completely skew-symmetric third-order tensor.
- The space of $n \times n$ skew-symmetric matrices has dimension of a vector space $\frac{1}{2}n(n-1)$.
- If \mathbf{A} is a skew-symmetric matrix then $[\mathbf{A}]^2 = \mathbf{AA}^\top - |\mathbf{A}|^2 \mathbb{E}$
- The real matrix \mathbf{A} is skew-symmetric iff $\langle \mathbf{Ax}, \mathbf{y} \rangle = (\mathbf{Ax})^\top \mathbf{y} = -\langle \mathbf{x}, \mathbf{Ay} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

To compute moment of inertia of a mass around an axis, the perpendicular vector from the mass to the axis is needed. If the axis L is defined by the unit vector $\hat{\mathbf{k}}$ through the reference point \mathbf{R} , then the perpendicular vector from the line L to the point \mathbf{r} is given by

$$\delta\mathbf{r}_i^\perp = \delta\mathbf{r}_i - (\hat{\mathbf{k}} \cdot \delta\mathbf{r}_i) \hat{\mathbf{k}} = (\mathbb{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}^\top) \delta\mathbf{r}_i \quad (534)$$

where \mathbb{E} is the 3×3 identity matrix, and $\hat{\mathbf{k}}\hat{\mathbf{k}}^\top$ is the outer product matrix formed from the unit vector $\hat{\mathbf{k}}$ along the axis line L . The matrix $\mathbb{E} - \hat{\mathbf{k}}\hat{\mathbf{k}}^\top$ in this equation subtracts the $\hat{\mathbf{k}}$ -parallel component from $\delta\mathbf{r}_i = \mathbf{r} - \mathbf{R}$ and then the perpendicular (to the line L) component is obtained.

The previous sections show that in computing the moment of inertia matrix this operator yields a similar operator using the components of the vector $\delta\mathbf{r}_i$ that is

$$[\mathbb{E} |\delta\mathbf{r}|^2 - \delta\mathbf{r}\delta\mathbf{r}^\top] \quad (535)$$

It is helpful to keep the following identities in mind to compare the equations that define the inertia tensor and the inertia matrix.

$$[(\mathbf{a} \times \mathbf{b}) \times \mathbb{E}] \cdot \mathbf{r} = [\mathbb{E} \times (\mathbf{a} \times \mathbf{b})] \cdot \mathbf{r} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{r} = (\mathbf{ba} - \mathbf{ab}) \cdot \mathbf{r} \quad (536)$$

where $\mathbb{E} = \hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}}$ is the idemfactor (versor), which is a complete dyadic, see [[GIBBS](#) and [WILSON, 1901](#)]. Therefore

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{r} = (\mathbf{ba} - \mathbf{ab}) \cdot \mathbf{r} \quad (537)$$

$$\mathbf{r} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{r} \cdot (\mathbf{ba} - \mathbf{ab}) \quad (538)$$

Let $[\mathbf{R}]$ be the skew symmetric matrix associated with the position vector $\mathbf{x} = (x, y, z)$ and a rotation vector $\boldsymbol{\omega}$, then the triple product, (also called Lagrange's formula) can be used to express unit moment of inertia matrix

$$-[\mathbf{R}]^2 \boldsymbol{\omega} = -[\mathbf{R}]([\mathbf{R}] \boldsymbol{\omega}) = -\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega}) = \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x}) \quad (539)$$

$$= - \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}^2 \boldsymbol{\omega} = \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} \boldsymbol{\omega} \quad (540)$$

The squared skew-symmetric matrix $[\mathbf{R}]^2$ can be proved by using the Levi-Civita symbol

$$[\mathbf{R}]^2(\mathbf{x}) = [\mathbf{S}](\mathbf{x})[\mathbf{S}](\mathbf{x}) = -[\mathbf{S}](\mathbf{x})[\mathbf{S}]^\top(\mathbf{x}) \quad (\text{due to } [\mathbf{S}]^\top = -[\mathbf{S}]) \quad (541)$$

$$= [-\epsilon_{ikm}x_m(-\epsilon_{kjn}x_n)] = [\epsilon_{ikm}\epsilon_{kjn}x_m x_n] \quad (542)$$

$$= [(\delta_{mj}\delta_{in} - \delta_{mn}\delta_{ij})x_m x_n] \quad (543)$$

$$= \left[x_i x_j - \delta_{ij} \sum_{n=1}^3 x_n^2 \right] = \mathbf{x} \mathbf{x}^\top - \mathbf{x}^\top \mathbf{x} [\mathbb{I}] \quad (544)$$

where \mathbf{x}^\top is the transpose of the vector \mathbf{x} and $[\mathbb{I}]$ is the 3×3 identity matrix so as to avoid confusion with the inertia matrix.

This can be viewed as another way of computing the perpendicular distance from an axis to a point, because the matrix formed by the outer product $[\mathbf{R} \mathbf{R}^\top]$ yields the identify

$$-[\mathbf{R}]^2 = |\mathbf{R}|^2 \hat{\mathbf{e}} - \mathbf{R} \mathbf{R}^\top \quad (545)$$

$$= \begin{bmatrix} x^2 + y^2 + z^2 & 0 & 0 \\ 0 & x^2 + y^2 + z^2 & 0 \\ 0 & 0 & x^2 + y^2 + z^2 \end{bmatrix} - \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} \quad (546)$$

Also notice, that

$$|\mathbf{R}|^2 = \mathbf{R} \cdot \mathbf{R} = \text{tr}(\mathbf{R} \mathbf{R}^\top) \quad (547)$$

where tr denotes the sum of the diagonal elements of the outer product matrix, known as its trace.

Inertia matrix in different reference frames

The use of the inertia matrix in Newton's second law assumes its components are computed relative to axes parallel to the inertial frame and not relative to a body-fixed reference frame. This means that as the body moves the components of the inertia matrix change with time. In contrast, the components of the inertia matrix measured in a body-fixed frame are constant.

12 Method of Equivalent Linearization - Irregular waves

Irregular waves are the sum of regular waves with a different period, amplitude, and phase, and is what we call a stochastic process. Since the incident waves are simplified to

be a random process, the excitation force is a random process, hence is difficult to obtain an exact solution of the motion $\eta(t)$. Since $\eta(t)$ is an approximation it is more relevant to look at the probability distribution of the amplitude of the response.

For irregular roll motion, we need to use another approach to linearize the roll damping term. We assume that the difference between the damping for the linearized term and the real value can be minimized by the least square method.

The quadratic roll-damping coefficient may be found in model-tests for given ship-hull types, independent of sea-state. For the actual sea-state in which the ship is to be analysed, stochastic linearization of the quadratic damping may be performed. The quadratic drag in the roll-motion gives a contribution to the moment about the x-axis on the form.

The error ε associated with the linearization is expressed below in motion equations

$$\varepsilon = B_1 \dot{\eta}_4 + B_2 |\dot{\eta}_4| \dot{\eta}_4 - B^{eq} \dot{\eta}_4 \quad (548)$$

We want determine the damping coefficient such that the error is as small as possible. One way to minimize ε is by the mean square value $E[\varepsilon^2]$. First of all we need to see if it is possible to minimize $E[\varepsilon^2]$ by application of (548)

$$\frac{\partial^2}{\partial (B^{eq})^2} E[\varepsilon^2] = 2E[\ddot{\eta}_4] > 0 \quad (549)$$

If the requirement are fulfilled, it is possible to minimize $E[\varepsilon^2]$

$$\frac{\partial}{\partial B^{eq}} E[\varepsilon^2] = 0 \quad (550)$$

Minimization of the mean squared value $E[\varepsilon^2]$ is expressed in (550) This is only valid if we assume a Gaussian random process, and that the damping coefficients B^{eq} , B_1 and B_2 remain constant.

$$\frac{\partial}{\partial B^{eq}} E[\varepsilon^2] = -2(B_1 - B^{eq})E[\dot{\eta}_4^2] - 2B_2 E[\dot{\eta}_4^2 |\dot{\eta}_4|] = 0 \quad (551)$$

Expressed by B^{eq} we get the formulation in

$$B^{eq} = B_1 + B_2 \frac{E[|\dot{\eta}_4|^3]}{E[\dot{\eta}_4^2]} \quad (552)$$

If it is assumed $\dot{\eta}_4$ is normally distributed, the half-normal distribution for $|\dot{\eta}_4|$ gives

$$E[\dot{\eta}_4^2] = \sigma_{\dot{\eta}_4}^2 \quad (553)$$

$$E[|\dot{\eta}_4|^3] = 2\sqrt{\frac{2}{\pi}} \sigma_{\dot{\eta}_4}^3 \quad (554)$$

Plugging into (552), we get

$$B^{eq} = B_1 + 2\sqrt{\frac{2}{\pi}} \sigma_{\dot{\eta}_4} B_2 \quad (555)$$

Where $\sigma_{\dot{\eta}_4}$ is the standard deviation of the response $\dot{\eta}_4$. Furthermore we express the variance, which is the standard deviation squared, as shown in Equation

$$\sigma_{\dot{\eta}_4}^2 = 2 \int_0^\infty S(\omega) |H(\omega)|^2 d\omega \quad (556)$$

Where $S(\omega)$ is the wave spectrum, and $|H(\omega)|^2$ is the squared transfer function known as the RAO. The RAO is defined as the motion amplitude divided by the wave elevation amplitude.

Based on a Rayleigh probability function we approximate the maxima, and the most probable largest value of the response, R_{max} is expressed by the following equation.

$$R_{max} = \sqrt{2\sigma_{\dot{\eta}_4}^2} \log \frac{t}{T} \quad (557)$$

$$\gamma_{xy}(f) = \frac{\hat{S}_{xy}(f)}{\sqrt{\hat{S}_{xx}(f)\hat{S}_{yy}(f)}} \quad (558)$$

Cross-spectral analyses with measured model responses. can extract the response RAO, the relative phase, and the coherence of the response with respect to the reference surface elevation, which is usually measured during wave calibration at the location of the model, without the model being present. During the cross-spectral analysis, it is assumed a linear relationship (i.e., RAO) in the frequency domain between the incident wave and output response at each specific frequency given.

For a pure linear system, if $S_{xy}(\omega)$ represents cross-spectral density between response and wave, and $S_{xx}(\omega)$ and $S_{yy}(\omega)$ for auto spectral density), respectively, then we have

$$S_{xy}(\omega) = H(\omega)S_{xx}(\omega), \quad S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \quad (559)$$

and the coherence $\gamma_{xy}(\omega)$ can be found to be unit.

$$|\gamma_{xy}(\omega)| \triangleq \left| \frac{S_{xy}(\omega)}{\sqrt{S_{xx}(\omega)S_{yy}(\omega)}} \right| \text{ by definition of coherence } \gamma_{xy}(\omega) \quad (560)$$

$$= \left| \frac{H(\omega)S_{xx}(\omega)}{\sqrt{S_{xx}(\omega)S_{yy}(\omega)}} \right| \quad (561)$$

$$= \frac{|H(\omega)|S_{xx}(\omega)}{\sqrt{S_{xx}(\omega)S_{yy}(\omega)}} \quad (562)$$

$$= \frac{|H(\omega)|S_{xx}(\omega)}{\sqrt{S_{xx}(\omega)|H(\omega)|^2 S_{xx}(\omega)}} \quad (563)$$

$$= 1. \quad (564)$$

13 Fun-to-know problems

13.1 Renewable Energy

13.1.1 Solar Energy

Although much hotter on the inside, we can closely approximate the surface of the sun, from which its emission occurs, as a black body at a temperature of about 5800 K. The Stefan-Boltzmann equation then gives the energy flux emitted at the sun's surface.

$$S_S = [5.67 \times 10^{-8} W/(m^2 K^4)](5800K)^4 = 63 \times 10^6 W/m^2 \quad (565)$$

If r_S is the radius of the Sun, the total energy it emits is $S_S 4\pi r_s^2$. When the energy emitted by the sun reaches the orbit of a planet, the large spherical surface over which the energy is spread has a radius, d_P , equal to the distance from the sun to the planet. The energy flux at any place on this surface, S_P , is less than what it was at the Sun's surface. But the total energy spread over this large surface is the same as the total energy that left the sun, so we can equate them:

$$S_S 4\pi r_s^2 = S_P 4\pi r_P^2 \quad (566)$$

$$S_P = S_S (r_s/r_P)^2 \quad (567)$$

Values for the average planetary distances, r_P , and the corresponding S_P , calculated using $r_s = 700,000$ km, are given in the table below in Table 13.1.1. The total amount of radiation incident on the planet (and atmosphere) is equal to the amount the planet intercepts to cast the imaginary shadow shown in the diagram below, i.e., $S_P \pi r_P^2$. If the average energy flux over the area of the planet is \bar{S} , the total energy for the planet is $\bar{S} 4\pi r_P^2$. These two total energies must be equal, so: $\bar{S} = S_P/4$.

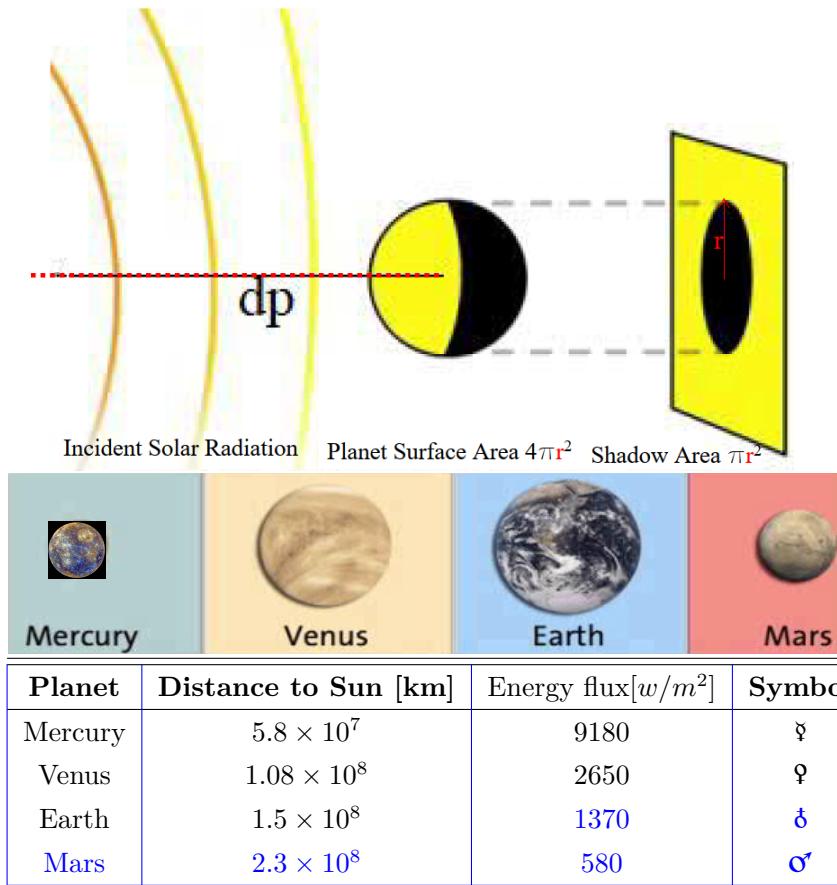
Solar Luminosity (L): the constant flux of energy put out by the sun is

$$L = 3.9 \times 10^{26} W \quad (568)$$

The solar constant (G_{SC}) is a flux density measuring mean solar electromagnetic radiation (total solar irradiance) per unit area. It is measured on a surface perpendicular to the rays, one astronomical unit (au) from the Sun (roughly the distance from the Sun to the Earth). Solar Constant (G_{SC}) is the solar energy density at the mean distance of Earth from the sun ($d = 1.5 \times 10^{11}$ m)

$$\begin{aligned} S &= L/(4\pi d^2) \\ &= (3.9 \times 10^{26} W)/[4\pi \times (1.5 \times 10^{11} m)^2] \\ &= 1370 W/m^2 \end{aligned} \quad (569)$$

The solar constant includes radiation over the entire electromagnetic spectrum. It is measured by satellite as being 1.361 kilowatts per square meter (kW/m^2) at solar minimum



Tab. 1. Energy flux over the area of the planets

(the time in the 11-year solar cycle when the number of sunspots is minimal) and approximately 0.1% greater (roughly 1.362 kW/m²) at solar maximum.

The standard solar panel has an input rate of around 1000 Watts per square meter, and the majority of solar panels available have around 15-20%. Therefore, if your solar panel was 1 square meter in size, then it would likely only produce around 150-200W in good sunlight.

13.1.2 Wind Energy

Betz' law estimates the maximum possible energy can be extracted from a wind turbine. It was developed in 1919 by German physicist Albert Betz. According to the rule, no turbine can capture more than 59.3% (16/27) of the potential energy in wind. The wind energy power can be estimated as

$$P_{Wind} = C_e \rho_{air} D_{WTblade}^2 v_{wind}^3 \quad (570)$$

where $\rho_{air}v^3$ is the energy flux.

13.2 Curl of unit normal vector on a surface

It really depends on how you define the vector field \vec{n} AWAY from the surface $\phi = \phi_{iso}$. On the surface \vec{n} is well-defined (up to choice of orientation).

1. Choice one: define \vec{n} , as you did, to be globally the normalized gradient of ϕ . That is, set $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$. In this case $\nabla \times \vec{n} = 0$, when evaluated at the surface $\{\phi = \phi_{iso}\}$, if and only if $|\nabla\phi|$ is constant along the surface.

2. Choice two: forget more or less about the function ϕ . Define the function

$$\psi = \frac{1}{|\nabla\phi|}(\phi - \phi_{iso})$$

Observe that the surface you are interested in is the surface $\{\psi = 0\}$. Computing the gradient $\nabla\psi$ you have that

$$\nabla\psi = \frac{\nabla(\phi - \phi_{iso})}{|\nabla\phi|} + (\phi - \phi_{iso})\nabla\frac{1}{|\nabla\phi|} \quad (571)$$

$$= \frac{\nabla(\phi)}{|\nabla\phi|} - \frac{(\phi - \phi_{iso})}{|\nabla\phi|^2}\nabla|\nabla\phi| \quad (572)$$

$$= \frac{\nabla\phi}{|\nabla\phi|} - \frac{(\phi - \phi_{iso})\nabla\phi \cdot \nabla(\nabla\phi)}{|\nabla\phi|^3} \quad (573)$$

The gradient of the absolute function from (572) to (573) is based on derivative of an absolute scalar function for each component.

$$|f(x)|' = \text{sign}[f(x)]f'(x) = \frac{f(x)}{|f(x)|}f'(x). \quad (574)$$

The key is that the second term vanishes on the surface, since there $\phi = \phi_{iso}$. So $\nabla\psi$ restricted to the surface is still the unit normal vector field. But $\nabla \times (\nabla\psi)$ is clearly zero. (Note, however, $\nabla\psi$ is not guaranteed to be a unit vector field away from the surface.)

More generally: given a compact smooth surface $\Sigma \subset \mathbb{R}^3$, there exists a radius $r > 0$ such that on the set $S = \{x \in \mathbb{R}^3 : \text{dist}(x, \Sigma) < r\}$ we can solve the [eikonal equation](#) ($|\nabla\Psi| = 1$) to get a function $\Psi : S \rightarrow \mathbb{R}$ such that $\Sigma = \Psi^{-1}(0)$ and $\nabla\Psi$ is the unit normal vector field for any level set $\Psi^{-1}(c)$. Then in this formulation we see that the unit normal vector field $\vec{n} = \nabla\Psi$ is curl-free everywhere in S . The number r , which is generically finite, is related to the radius of curvature of Σ .

13.2.1 The derivative of a unit normal vector is in the tangent plane

Let $\hat{\mathbf{n}}$ be the unit vector field AWAY from the surface and $d\hat{\mathbf{n}}$ the deferential or variation of the unit normal to the surface. The deferential $d\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{n}}$ because $\hat{\mathbf{n}}$ is a unit vector that can only change in its tangential direction at the vector end/head rather than its radial direction as shown in Fig.(7) and the algebraic proof can be readily done in (575).

$$\hat{\mathbf{n}} \cdot d\hat{\mathbf{n}} = \frac{1}{2}d(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = \frac{1}{2}d(1) = 0 \quad (575)$$

It is then shown that the deferential $d\hat{\mathbf{n}}$ is perpendicular to $\hat{\mathbf{n}}$.

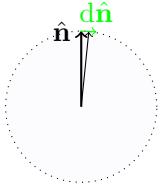


Fig. 7. A variation $d\hat{n}$ of a unit normal vector.

13.2.2 Perp Pass Unitary - a powerful dyadic

Here one dyadic that can be named as **Perp Pass Unitary**, will be introduced. The perp pass unitary is one operator that filters out the parallel component and pass the perpendicular component of its input and is constructed by the difference between a unit 2-order tensor and the juxtaposition of unit normal vectors, i.e., $\mathbb{E} - \hat{n}\hat{n}$. It is a complete dyadic and behaves as an idemfactor for all vectors perpendicular to \hat{n} and an annihilator for vectors parallel to \hat{n} .

Let one arbitrary vector denoted by \mathbf{c} . As we know, it can be decomposed into the perpendicular and parallel components with respect to the normal vector \hat{n} .

$$\mathbf{c} = \mathbf{c}^{\parallel} + \mathbf{c}^{\perp} \quad (576)$$

$$= (\mathbf{c} \cdot \hat{n})\hat{n} - \hat{n} \times (\hat{n} \times \mathbf{c}) \quad (577)$$

$$= (\mathbf{c} \cdot \hat{n})\hat{n} + \mathbf{c} \cdot (\mathbb{E} - \hat{n}\hat{n}) \quad (578)$$

If \mathbf{a} and \mathbf{b} represent these two components, respectively, i.e., $\mathbf{a} = \mathbf{c}^{\parallel}$ and $\mathbf{b} = \mathbf{c}^{\perp}$, then by definition, we get

$$\mathbf{a} = (\mathbf{c} \cdot \hat{n})\hat{n} = \lambda\hat{n} \quad \text{with } \lambda \text{ a scalar} \quad (579)$$

$$\mathbf{b} \cdot \hat{n} = b_i n_i = 0. \quad (580)$$

To prove rigorously,

$$\mathbf{c} \cdot (\mathbb{E} - \hat{n}\hat{n}) = (\mathbf{a} + \mathbf{b}) \cdot (\mathbb{E} - \hat{n}\hat{n}) \quad (581)$$

$$= \lambda n_i \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \delta_{ij} - n_i n_j \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) + b_i \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \delta_{ij} - n_i n_j \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) \quad (582)$$

$$= \lambda (n_j \hat{\mathbf{e}}_j - n_i n_j \hat{\mathbf{e}}_j) + b_i \hat{\mathbf{e}}_j \delta_{ij} - b_i n_i n_j \hat{\mathbf{e}}_j \quad (583)$$

$$= \lambda (\hat{n} - \hat{n}) + b_i \hat{\mathbf{e}}_i \quad (584)$$

$$= \mathbf{b} \quad (585)$$

The above process is acting like a filter which has the feature to pass the perpendicular component which is the effect of idemfactor or identity property of $\mathbb{E} - \hat{n}\hat{n}$. For the perpendicular vector \mathbf{b} , we have

$$\mathbf{b} \cdot (\mathbb{E} - \hat{n}\hat{n}) = b_i \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \delta_{ij} - n_i n_j \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) \quad (586)$$

$$= b_i \hat{\mathbf{e}}_j \delta_{ij} - b_i n_i n_j \hat{\mathbf{e}}_j = b_i \hat{\mathbf{e}}_i \quad (587)$$

$$= (\mathbb{E} - \hat{n}\hat{n}) \cdot \mathbf{b} = \mathbf{b} \quad (588)$$

And the blocking effect is acting like an annihilator for the parallel vector \mathbf{a} ,

$$\mathbf{a} \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) = \lambda n_i \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \delta_{ij} - n_i n_j \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j) \quad (589)$$

$$= \lambda(n_j \hat{\mathbf{e}}_j - n_i n_i n_j \hat{\mathbf{e}}_j) = \lambda(\hat{\mathbf{n}} - \hat{\mathbf{n}}) \quad (590)$$

$$= (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{a} = 0 \quad (591)$$

Let the surface geometry is expressed by an implicit equation $F(x, y, z) = 0$ of the variables x , y and z . Let \mathbf{r} be a radius vector drawn from an arbitrary fixed origin to a variable point of the surface. The increment²⁵ $d\mathbf{r}$ lies in the surface or in the tangent plane drawn to the surface at the terminus of \mathbf{r} . Thus we have

$$dF = d\mathbf{r} \cdot \nabla F = 0 \quad (592)$$

and the equation of the tangent plane to the surface at the point $\mathbf{r}_0 = [x_0, y_0, z_0]$ is given by

$$(\mathbf{r} - \mathbf{r}_0) \cdot \nabla F(\mathbf{r}_0) = 0. \quad (593)$$

Meanwhile, a normal line (orthogonal to the local surface) at a point \mathbf{r}_0 can be written in t -parametric form

$$\mathbf{r}_0 + t \nabla F(\mathbf{r}_0) = 0. \quad (594)$$

Hence, the derivative ∇F is collinear with the normal to the surface. If let $\mathbf{N} = \nabla F$ and the magnitude $N = |\nabla F|$ then the unit normal vector $\hat{\mathbf{n}}$ in the direction of ∇F

$$\hat{\mathbf{n}} = \frac{1}{N} \nabla F. \quad (595)$$

For example in ocean engineering, we study the surface waves and the free surface elevation $\eta(x, y)$ can be written implicitly as $F(x, y, z) = z - \eta(x, y)$, then the magnitude of the normal vector $N = \sqrt{1 + \eta_x^2 + \eta_y^2}$ and the unit normal vector will be

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|} = \frac{[-\eta_x \hat{\mathbf{i}}, -\eta_y \hat{\mathbf{j}}, \hat{\mathbf{k}}]}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \quad (596)$$

The derivative $d\hat{\mathbf{n}}$ in the tangential plane can be expressed as

$$d\hat{\mathbf{n}} = \frac{1}{N} d\nabla F - \frac{dN}{N^2} \nabla F \quad (597)$$

$$= \frac{1}{N} d\mathbf{r} \cdot \nabla \nabla F - \frac{1}{N^2} d\mathbf{r} \cdot \nabla N \nabla F \quad (598)$$

where $d\mathbf{r}$ is the infinitesimal difference of the radius vector. Use the $d\hat{\mathbf{n}}$ and ∇F to substitute \mathbf{a} and \mathbf{b} in (588) and (591), respectively, then we obtain

$$d\hat{\mathbf{n}} \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) = d\hat{\mathbf{n}}, \quad (599)$$

$$\nabla F \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) = 0. \quad (600)$$

²⁵Formally, $\delta\mathbf{r}$ defines a linear tangent differential form, and $\delta\mathbf{r}^\top \delta\mathbf{r}$ the associated quadratic tangent form (Arnold [1984]), both of which are differentiable with respect to time.

Therefore

$$d\hat{\mathbf{n}} = d\left(\frac{1}{N} \nabla F\right) \quad (601)$$

$$= \left[(\nabla F) d\frac{1}{N} + \frac{1}{N} d\nabla F \right] \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \quad (602)$$

$$= \frac{1}{N} d\mathbf{r} \cdot \nabla \nabla F \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \quad (603)$$

$$= d\mathbf{r} \cdot \frac{(\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \nabla \nabla F \cdot (\mathbb{E} - \hat{\mathbf{n}}\hat{\mathbf{n}})}{N} \quad (604)$$

13.2.3 differential Elements

The appropriate differential line element $d\ell$, surface area $d\mathbf{S}$, and volume d^3x can be defined in terms of any three differential line elements $d\ell(i)$, $i = 1, 2, 3$ that are linearly independent, i.e., $\|d\ell(1) d\ell(2) d\ell(3)\| = d\ell(1) \cdot d\ell(2) \times d\ell(3) \neq 0$ by

$$d\ell = d\ell(i), \quad \text{differential line element,} \quad (605)$$

$$d\mathbf{S} = d\ell(i) \times d\ell(j), \quad \text{differential surface area,} \quad (606)$$

$$d^3x = d\ell(1) \cdot d\ell(2) \times d\ell(3), \quad \text{scalar differential volume.} \quad (607)$$

In exploring properties of fluids and plasmas we often want to know how the differential line, surface and volume elements change as they move with the fluid flow velocity \mathbf{V} . In particular, when taking time derivatives of integrals, we need to know what the time derivatives of these differentials are as they are carried along with a fluid. To determine this, note first that if the flow is uniform then all points in the fluid would be carried along in the same direction at the same rate; hence, the time derivatives of the differentials would vanish. However, if the flow is nonuniform, the differential line elements and hence all the differentials would change in time. To calculate the time derivatives of the differentials, consider the motion of two initially close points \mathbf{x}_1 , \mathbf{x}_2 as they are carried along with a fluid flow velocity $\mathbf{V}(\mathbf{x}, t)$. Let $\delta\ell(t) = \mathbf{x}_2 - \mathbf{x}_1$ for the position difference at the time t and for this time instant, using the Taylor series expansion at the position \mathbf{x}_1 , we have $\mathbf{V}(\mathbf{x}_2, t) = \mathbf{V}(\mathbf{x}_1, t) + \delta\ell(t) \cdot \nabla \mathbf{V} + \dots$ and integrate the governing equation

$$\frac{d}{dt} \delta\ell(t) = \mathbf{V} \quad (608)$$

over time, we obtain

$$\delta\ell(t) - \delta\ell(t=0) = \mathbf{x}_2(t) - \mathbf{x}_1(t) - [\mathbf{x}_2(0) - \mathbf{x}_1(0)] \quad (609)$$

$$= [\mathbf{V}(\mathbf{x}_2, t) - \mathbf{V}(\mathbf{x}_1, t)] dt \quad (610)$$

$$= \int_0^t dt' \delta\ell(t') \cdot \nabla \mathbf{V} + \dots \quad (611)$$

in which $\mathbf{x}_2(0)$ and $\mathbf{x}_1(0)$ are the initial positions at $t = 0$. Taking the time derivative of this equation and identifying the differential line element $d\ell$ as $\mathbf{x}_2 - \mathbf{x}_1$ in the limit where the

points \mathbf{x}_2 and \mathbf{x}_1 become infinitesimally close and the zero variation of $\mathbf{x}_2(0) - \mathbf{x}_1(0)$ with respect to time, we find

$$d\dot{\ell} \equiv \frac{d}{dt}(d\ell) = d\ell \cdot \nabla \mathbf{V}. \quad (612)$$

The time derivative of the differential surface area $d\mathbf{S}$ can be calculated by taking the time derivative of (606) and using this last equation to obtain

$$d\dot{\mathbf{S}} \equiv \frac{d}{dt}(d\mathbf{S}) = d\dot{\ell}(1) \times d\ell(2) + d\ell(1) \times d\dot{\ell}(2) \quad (613)$$

$$= d\ell(1) \cdot \nabla \mathbf{V} \times d\ell(2) - d\ell(2) \cdot \nabla \mathbf{V} \times d\ell(1) \quad (614)$$

$$= (\nabla \cdot \mathbf{V})d\mathbf{S} - \nabla \mathbf{V} \cdot d\mathbf{S} \quad (615)$$

in which (D.93) and (D.33) have been used in obtaining the last form. Similarly, the time derivative of the differential volume element moving with the fluid is

$$\frac{d}{dt}(d^3x) = d\dot{\ell}(3) \cdot d\mathbf{S} + d\ell(3) \cdot d\dot{\mathbf{S}} \quad (616)$$

$$= d\ell(3) \cdot \nabla \mathbf{V} \cdot d\mathbf{S} + d\ell(3) \cdot (\nabla \cdot \mathbf{V})d\mathbf{S} - d\ell(3) \cdot \nabla \mathbf{V} \cdot d\mathbf{S} \quad (617)$$

$$= (\nabla \cdot \mathbf{V})d^3x, \quad (618)$$

which shows that the differential volume in a compressible fluid increases or decreases according to whether the fluid is rarefying ($\nabla \cdot \mathbf{V} > 0$) or compressing ($\nabla \cdot \mathbf{V} < 0$).

13.3 Permutation group

The product of two permutations QP is:

$$QP = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}_{(15)(24)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}_{(1234)} \quad (619)$$

$$= \begin{pmatrix} 2 & 4 & 1 & 3 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \quad (620)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \quad (621)$$

The composition of permutations, when they are written in cyclic form, is obtained by juxtaposing the two permutations (with the second one written on the left) and then simplifying to a disjoint cycle form if desired. Thus, in cyclic notation the above product would be given by $QP = (15)(24)(1234) = (1435)$

13.4 Recurrent Relation

Moreover, for the general first-order non-homogeneous linear recurrence relation with variable coefficients:

$$a_{n+1} = f_n a_n + g_n, \quad f_n \neq 0, \quad (622)$$

there is also a nice method to solve it:

$$a_{n+1} - f_n a_n = g_n \quad (623)$$

$$\frac{a_{n+1}}{\prod_{k=0}^n f_k} - \frac{f_n a_n}{\prod_{k=0}^n f_k} = \frac{g_n}{\prod_{k=0}^n f_k} \quad (624)$$

$$\frac{a_{n+1}}{\prod_{k=0}^n f_k} - \frac{a_n}{\prod_{k=0}^{n-1} f_k} = \frac{g_n}{\prod_{k=0}^n f_k} \quad (625)$$

Let

$$A_n = \frac{a_n}{\prod_{k=0}^{n-1} f_k}, \quad (626)$$

then

$$A_{n+1} - A_n = \frac{g_n}{\prod_{k=0}^n f_k} \quad (627)$$

$$\sum_{m=0}^{n-1} (A_{m+1} - A_m) = A_n - A_0 = \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^m f_k} \quad (628)$$

$$\frac{a_n}{\prod_{k=0}^{n-1} f_k} = A_0 + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^m f_k} \quad (629)$$

$$a_n = \left(\prod_{k=0}^{n-1} f_k \right) \left(A_0 + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^m f_k} \right) \quad (630)$$

Now solve:

$$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 0, a_1 = 1$$

A bare-hands method would be as follows:

$$a_n - 3a_{n-1} = 3(a_{n-1} - 3a_{n-2})$$

$$a_n - 3a_{n-1} = 3^{n-1}(a_1 - 3a_0)$$

$$\sum_{k=1}^n 3^{n-k} (a_k - 3a_{k-1}) = \sum_{k=1}^n 3^{n-k} (3^{k-1} (a_1 - 3a_0))$$

$$\sum_{k=1}^n 3^{n-k} a_k - 3^{n-(k-1)} a_{k-1} = \sum_{k=1}^n 3^{n-1} (a_1 - 3a_0)$$

$$a_n - 3^n a_0 = n 3^{n-1} (a_1 - 3a_0)$$

The constants were obtained from the factorization of the quadratic.

Note that the above method works regardless of whether the roots are repeated, although the last step will differ.

Here is a fun way to get to the answer:

Suppose we have a recursion $A_n = c_1 A_{n-1} + c_2 A_{n-2} + \dots + c_k A_{n-k}$. When you are doing the usual approach, you are looking for geometric sequences that satisfy this recursion, and you end up looking for nonzero roots r of:

$$x^n = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_k x^{n-k}$$

Now remember that when we have a double root of a polynomial, then it is also a root of the derivative of that polynomial. So if we are in that case and r is a double root of this let's differentiate the above equation, and then multiply by x to keep it looking nice:

$$nx^n = c_1(n-1)x^{n-1} + c_2(n-2)x^{n-2} + \dots + c_k(n-k)x^{n-k}$$

We know r must still be a root of this, so we plug in r and look carefully this says that the sequence $A_n = nr^n$ satisfies our original recursion. Hooray!

If r is a root of multiplicity more than 2, then we can differentiate the equation more times to get more sequences of the form $A_n = n^k r^n$ for k less than the multiplicity of r .

I know another way to get at this using linear algebra that I think is a bit more enlightening as to why these are the answers, and why you never get anything else, but in my opinion this differentiation trick is pretty slick.

13.5 Dice Probability

$$f_b(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Fig. 8. The binomial distribution function specifies the number of times x (total number of “successes” : pass or fail, win or lose, heads or tails etc.) that an event occurs in n independent trials where p is the probability of the event occurring in a single trial.

If you roll a fair six-sided dice three times, what's the probability that you get the same number three times?

If you want a specific number to appear three times in a row, you want it to appear on first, second and third roll. Which leads to $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$. If, however, you just want ANY number to appear three times in a row, then first $\frac{1}{6}$ disappears from previous formula. Why? It is because it does not really matter, what you roll first. You just want second and third roll to be the same as the first one. Hence first throw is not really important, since it only indicates which number has to be rolled twice from this moment. It's not a formal proof, but I guess you needed the simple reasoning behind the whole idea.

Ex 13.1 (Over Full Sampling) *Probability of seeing all numbers when rolling dice 7 times?*

Precisely one number must appear twice. Say, fix this to be 1. There are $\binom{7}{2}$ ways to pick where these appear, and then 5! options for the other five numbers. There 6 options for the fixed number, so the solution is

$$\frac{5! \cdot 6 \cdot \binom{7}{2}}{6^7} = \frac{6! \cdot \binom{7}{2}}{6^7} = 0.054 \quad (631)$$

Ex 13.2 (Get at least two) *If you roll 5 standard six-sided dice, what's the probability that you get at least two 4s?*

Let X the random variable that appears '4' in $n = 5$ trials where each has the probability of $\frac{1}{6}$ for a fair dice. Following the above binomial equation in Fig.(8), the probability of x times of trials that show face '4' is

$$f_b(X=x; n, p) = {}^n C_x p^x (1-p)^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}. \quad (632)$$

Therefore we need to find the probability that $X \geq 2$ in $n = 5$ with $p = \frac{1}{6}$ and can be calculated in the complimentary probability

$$f_b(X \geq 2; 5, \frac{1}{6}) = 1 - f_b(X < 2; 5, \frac{1}{6}) \quad (633)$$

Since the events are discrete, i.e., $X = 0$ and $X = 1$ are the independent and disjoint events.

$$\begin{aligned} f_b(X < 2; 5, \frac{1}{6}) &= f_b(0 \leq X \leq 1; 5, \frac{1}{6}) = f_b(X = 0; 5, \frac{1}{6}) + f_b(X = 1; 5, \frac{1}{6}) \\ &= \binom{5}{0} \left(\frac{1}{6}\right)^0 \left(1-\frac{1}{6}\right)^{5-0} + \binom{5}{1} \left(\frac{1}{6}\right)^1 \left(1-\frac{1}{6}\right)^{5-1} \end{aligned} \quad (634)$$

So the probability that you get at least two '4's is

$$f_b(X \geq 2; 5, \frac{1}{6}) = 1 - \left[\left(\frac{5}{6}\right)^5 + \frac{5}{6} \left(\frac{5}{6}\right)^4 \right] \quad (635)$$

$$= 1 - 2 \left(\frac{5}{6}\right)^5 \quad (636)$$

Ex 13.3 (Get results until) If you throw six, six-sided dice together repeatedly until I get three or more sixes in a single throw. What is the probability that this takes me no more than twelve throws of the six dice?

Let P denote the probability you're after. Then

$$P = 1 - q^{12} \quad (637)$$

where q is the probability that you do NOT get 3 or more sixes on a single roll of the 6 dice. To not get 3 or more sixes is to get either exactly 2 sixes, exactly 1 six, or none at all. Thus

$$q = \binom{6}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 + \binom{6}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 + \binom{6}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^6 \quad (638)$$

Ex 13.4 (Get at least) To get at least two 6's if throwing 3 fair dice in 2 Rolls?

The alternate approach would be to find the probability of the complementary event:

- 1) The probability of getting no 6's is given by $(\frac{5}{6})^3 \cdot (\frac{5}{6})^3 = (\frac{5}{6})^6$
- 2) The probability of getting exactly one 6 is given by

$$(\frac{5}{6})^3 \cdot \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 + \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 \cdot (\frac{5}{6})^2 = (\frac{1}{2}) \left(\frac{5}{6}\right)^5 + (\frac{1}{2}) \left(\frac{5}{6}\right)^4 \quad (639)$$

Therefore the probability of getting at least two 6's is given by

$$1 - (\frac{5}{6})^6 - (\frac{1}{2}) \left(\frac{5}{6}\right)^5 - (\frac{1}{2}) \left(\frac{5}{6}\right)^4 = \frac{5203}{23328} \approx .223 \quad (640)$$

Ex 13.5 (Pills-Drawing from FLP) Suppose you are taking one each of 5 different types of pills every day but you don't like having to open and close 5 different bottles, so at the beginning of each (30-day) month you put 30 of each type of pill into one big bottle. When it is time to take your pills, you draw them out of the big bottle one at a time until you have (at least) one of each type. On the last day of the month you will draw exactly 5 pills and they will all be different (because that's all that's left in the bottle), but on other days you will generally have to draw more than 5 pills in order to have (at least) one of each type. So, the question is: On the first day of each month (when there are 150 pills in the bottle), how many pills, on average, must you draw from the bottle in order to have (at least) one of each?. Take the 3 bottles and 2 pills as a start. **Note:** This is from the [discussion](#) with my friend Michael, FLP Webmaster who was given by Matt Sands (the last surviving FLP author).

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