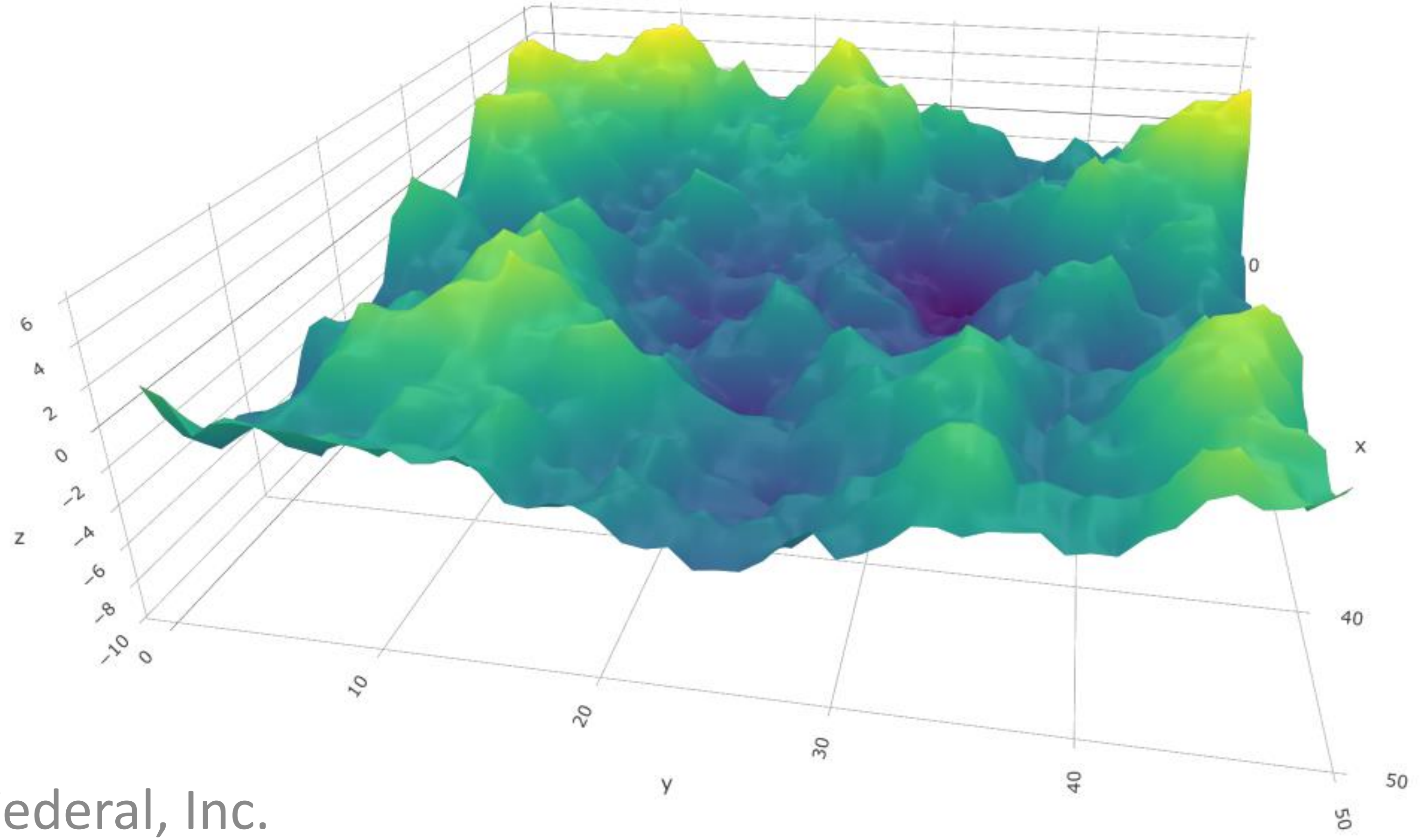
Spatial Modeling with Template Model Builder

Theory and Mechanics



Andrea Havron, PhD

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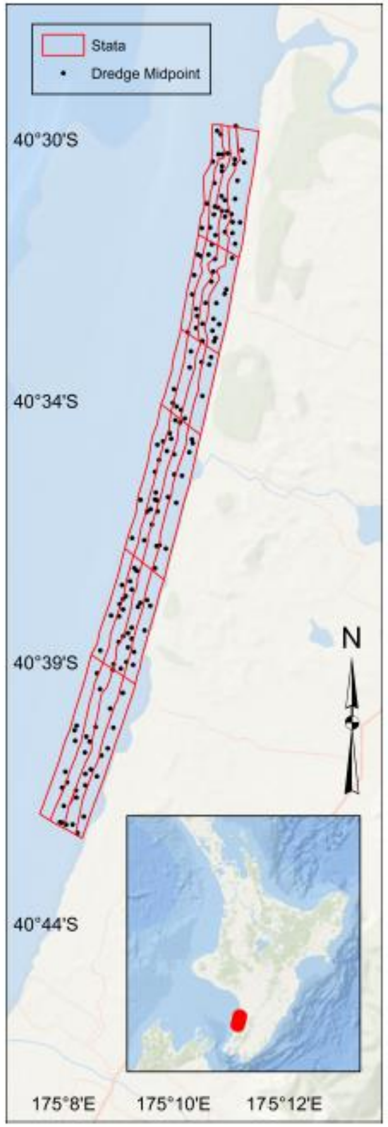
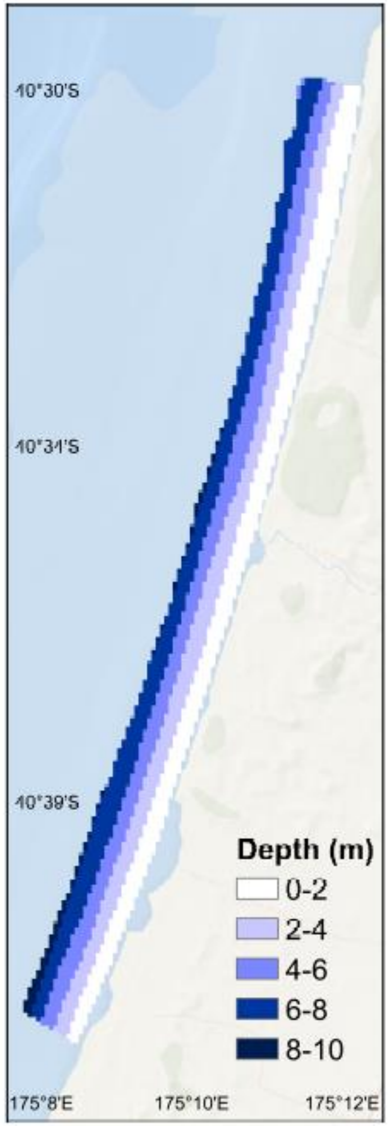
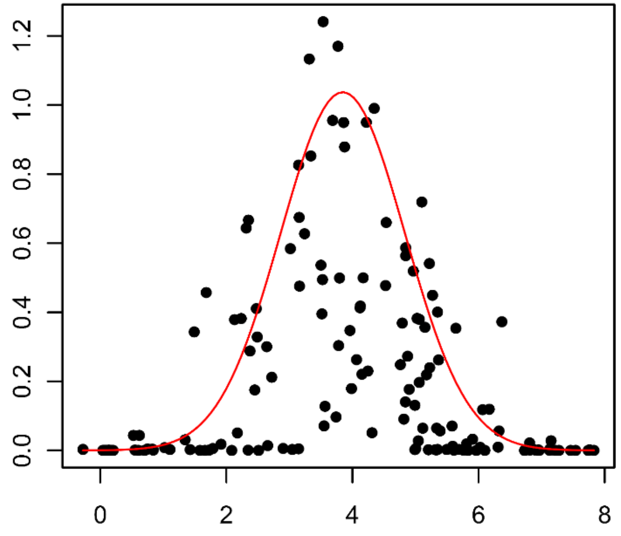
Species Distribution Modeling

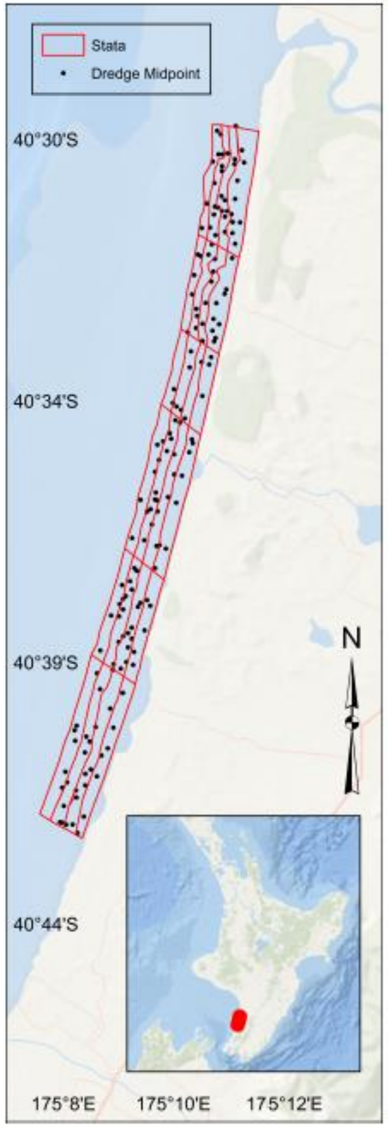


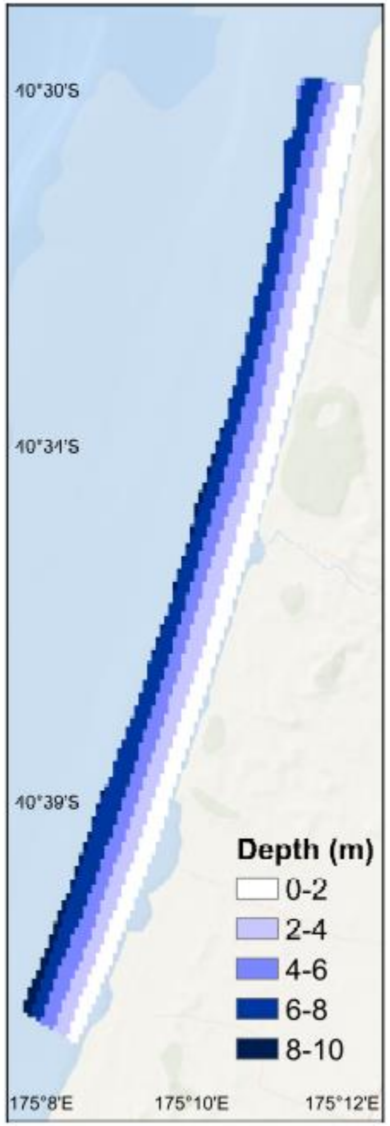


Triangle Shell Surf clam
Manawatu, NZ



Species Distribution Modeling





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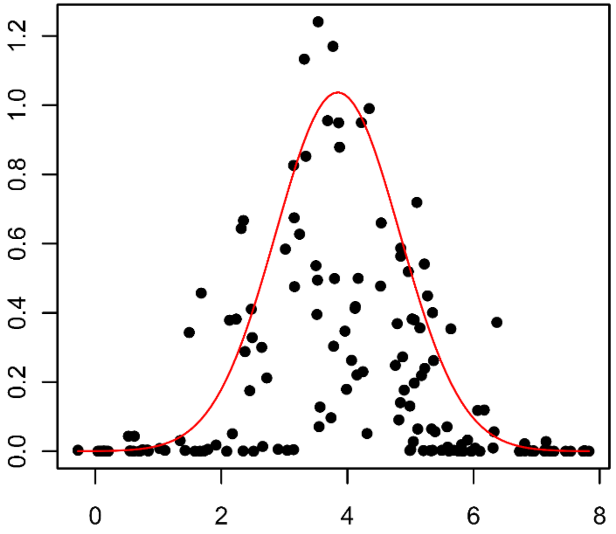
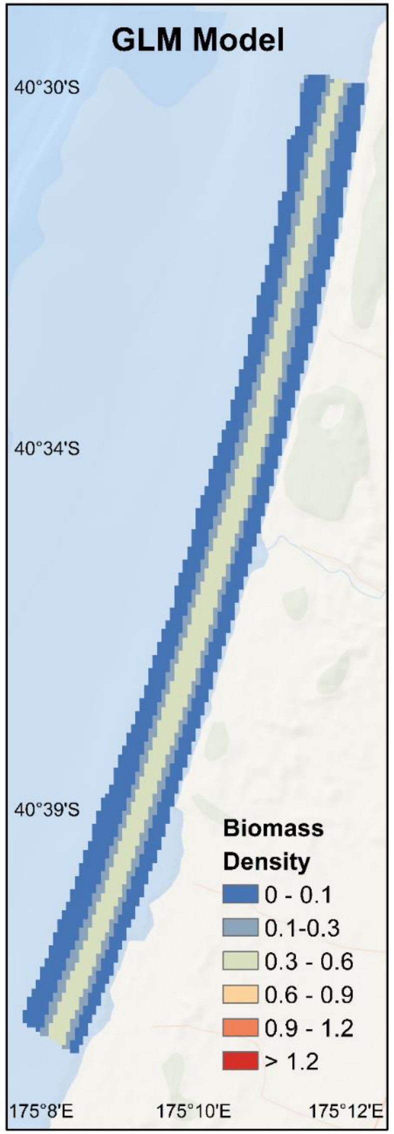
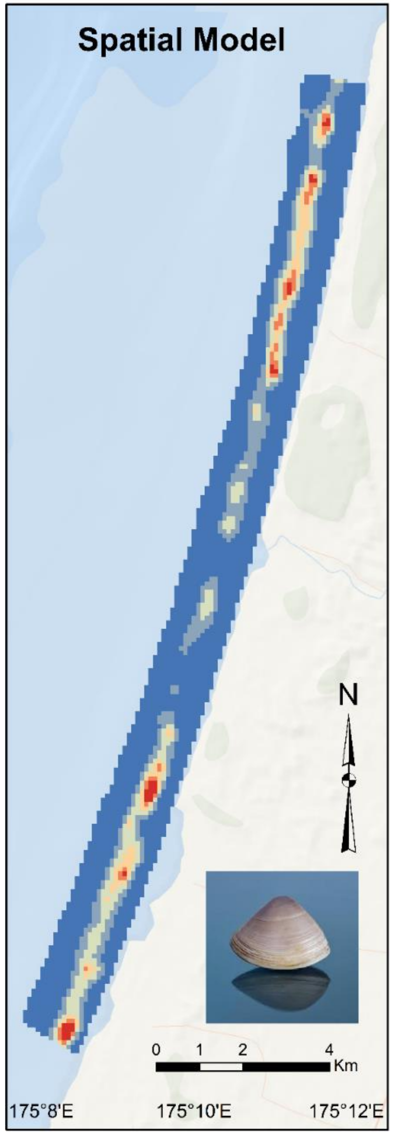






Benefits of modeling spatial correlation

- Failure to model spatial correlation violates independence assumptions
  - Narrower confidence intervals
  - Inflated Type I error
- We want to learn about the spatial structure
- More realistic predictions of species distributions

Hierarchical Spatial Model

$$\text{Y} \sim f_y\left(g^{-1}(\mu)\right)$$

$$\mu = \beta_0 + X\beta + \omega$$

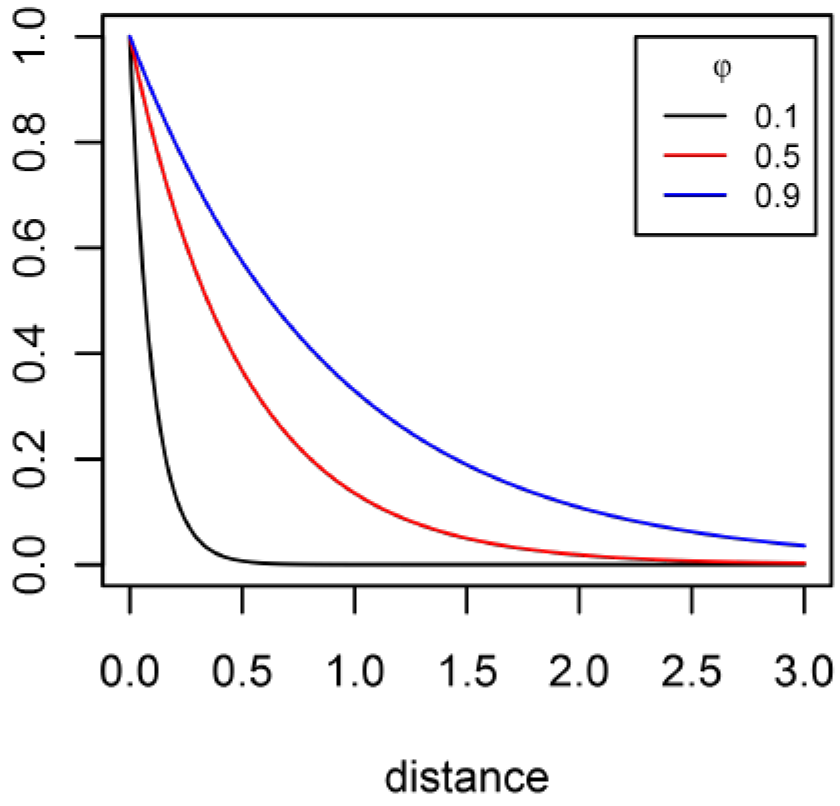
$$\omega \sim MVN(0, \Sigma)$$

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \Sigma^{-1} \boldsymbol{\omega})$$

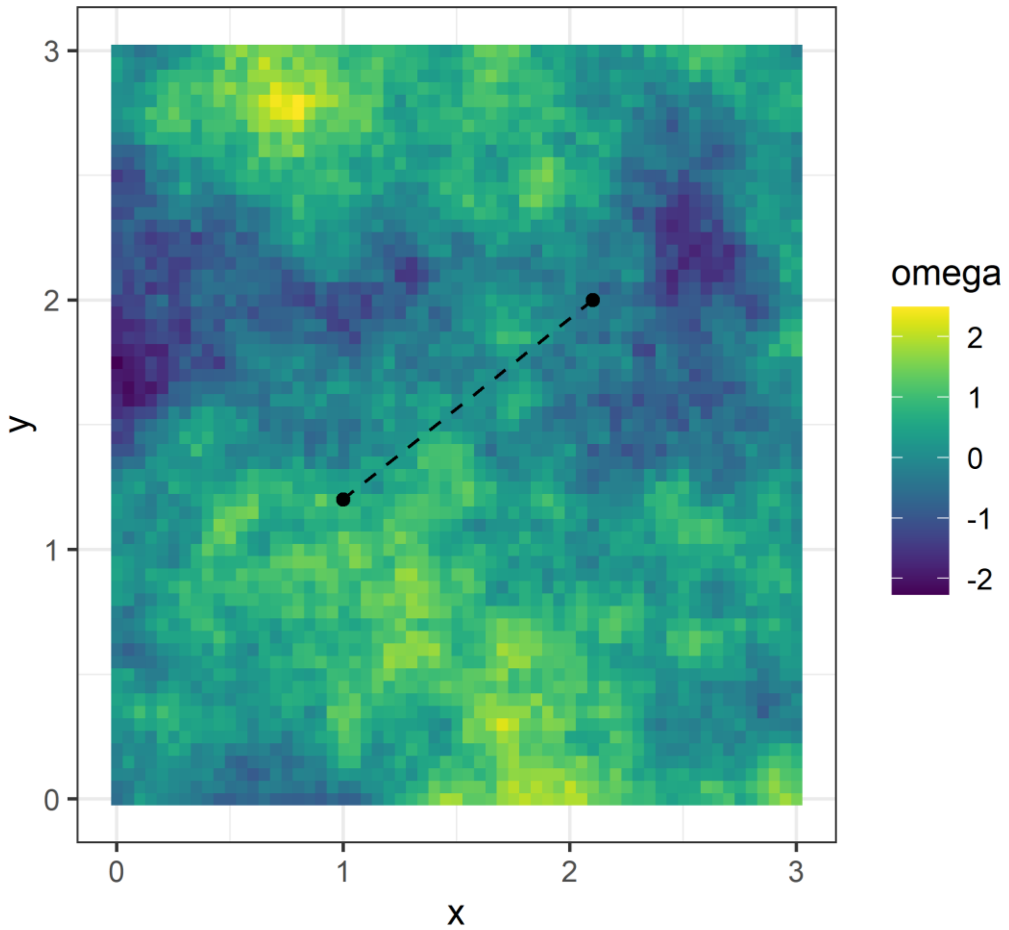
Covariance Matrix, Σ

Exponential Correlation

$$\Sigma_{i,j} = \exp(-\frac{d_{ij}}{\phi})$$



ϕ: distance spatial correlation ≈10%

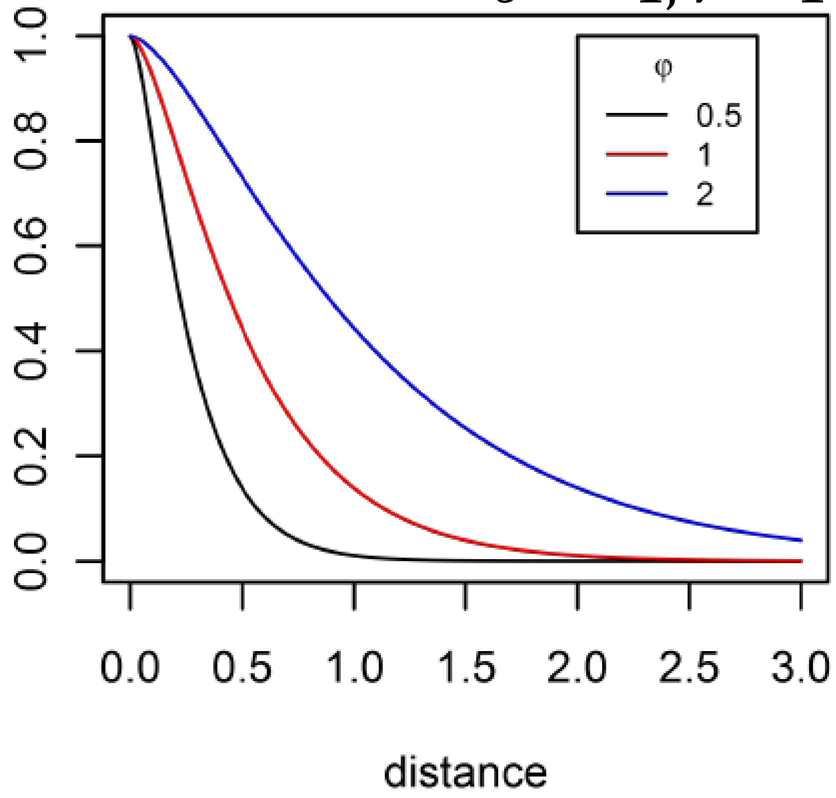


Covariance Matrix, Σ

Matérn Correlation

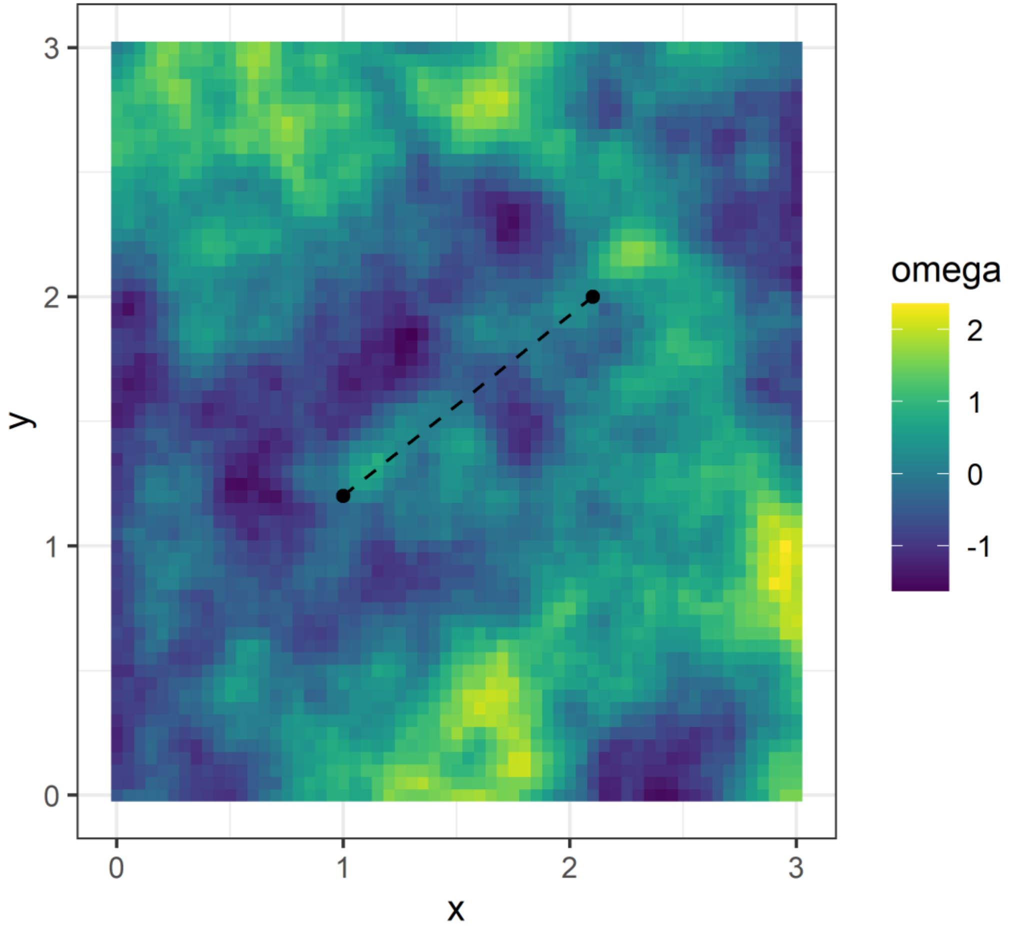
$$\Sigma_{i,j} = \frac{1}{2^{\nu-1}\Gamma(\nu)}(\kappa d)^{\nu} K_{\nu}(\kappa d)$$

$\sigma^2 = 1, \nu = 1$

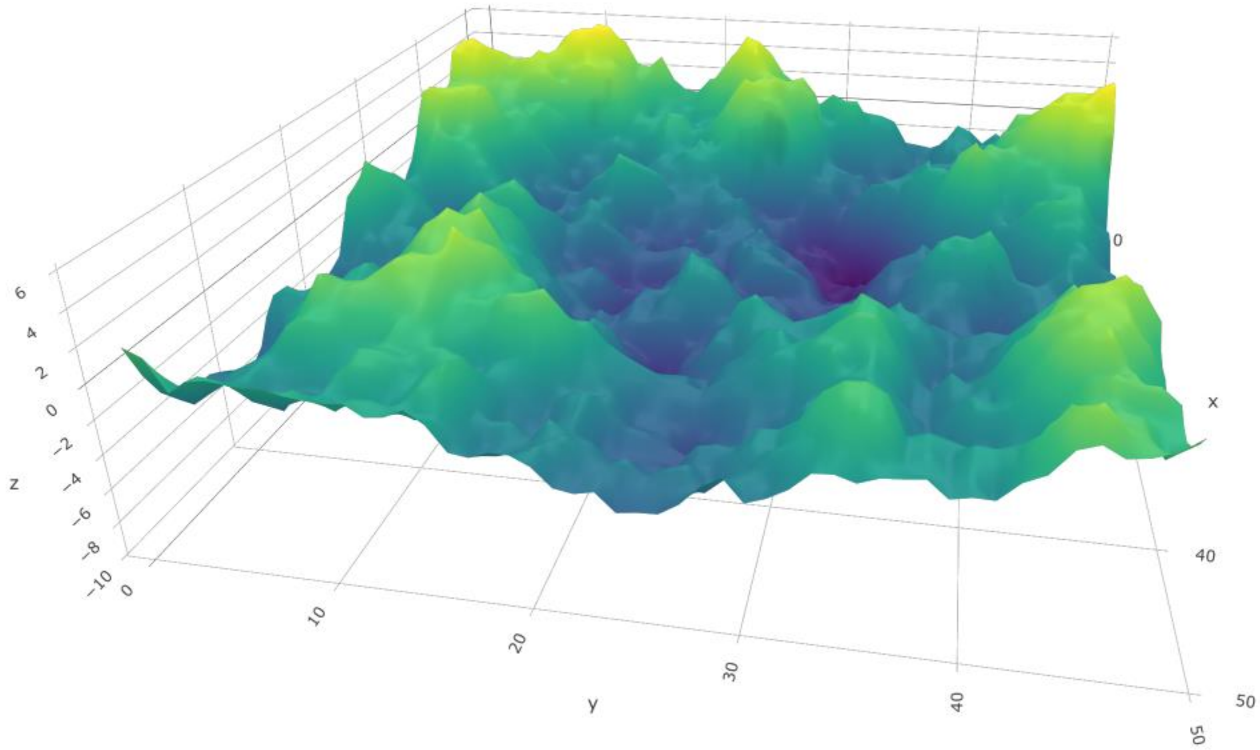


ϕ: distance spatial correlation ≈10%

$\kappa = \sqrt{8\nu}/\phi$



Gaussian Field



$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T\Sigma^{-1}\boldsymbol{\omega})$$

Likelihood Bottleneck

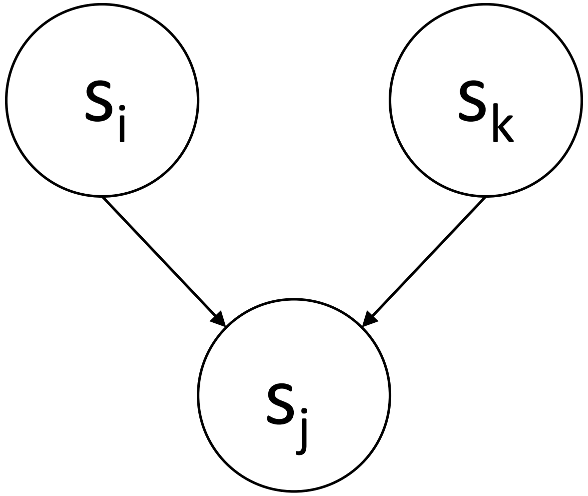
$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \Sigma^{-1} \boldsymbol{\omega})$$

- Operations are $\mathcal{O}(n^3)$

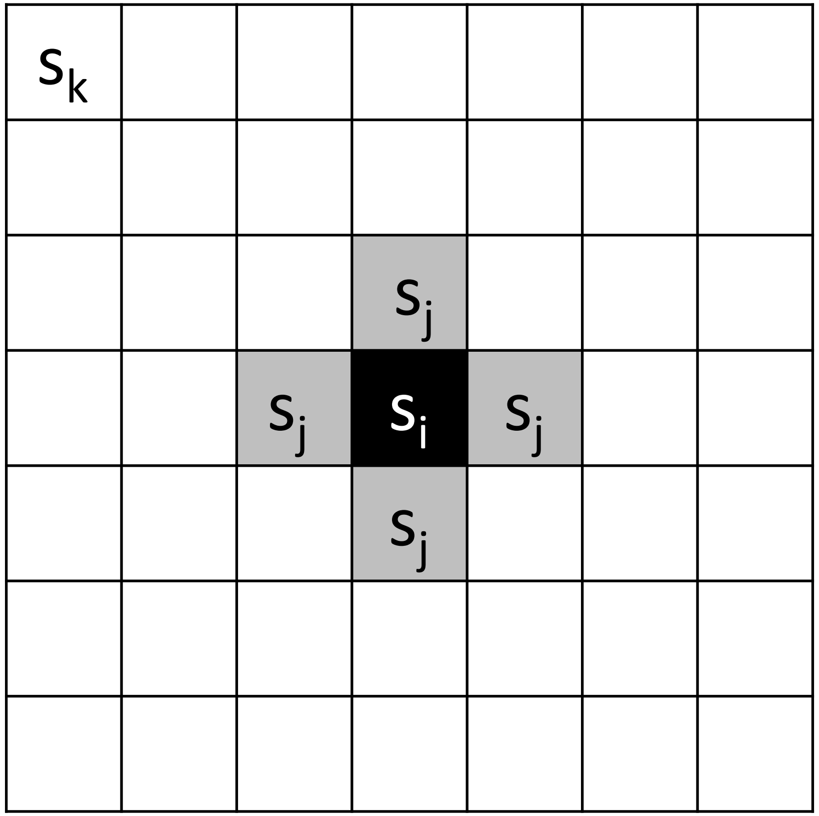
Gaussian Markov Random Field (Besag, 1974)

GMRF (Besag, 1974)

$$Y(s_i|\boldsymbol{s}_{-i}) = Y(s_i|\boldsymbol{s}_j : j \in Nb_i)$$

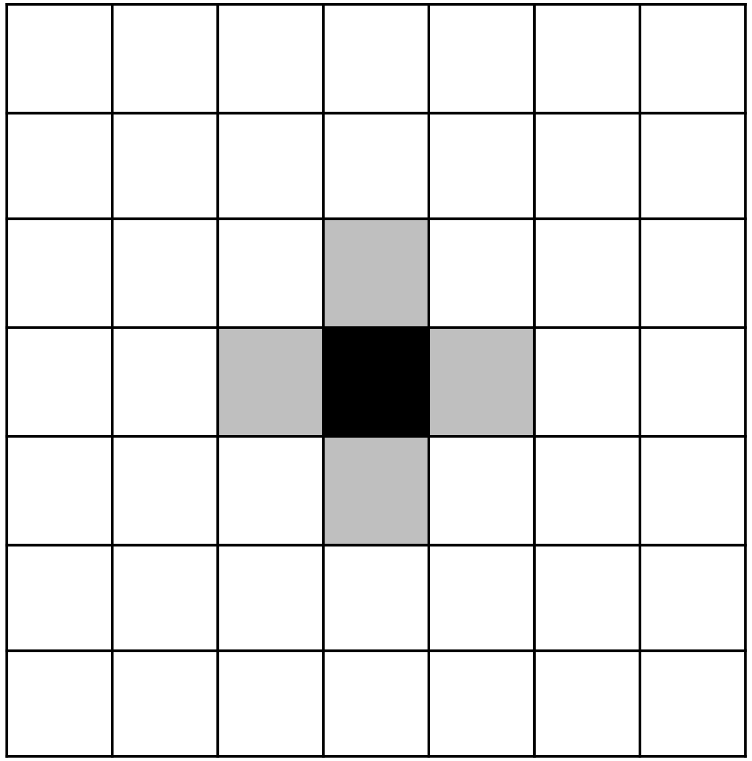


$$s_i \perp s_k \mid s_j$$

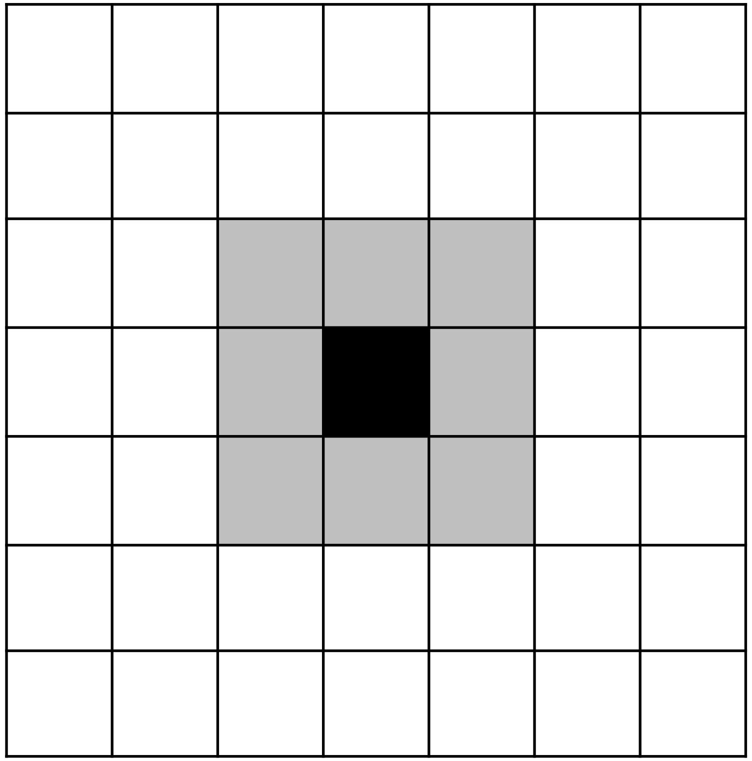


GMRF

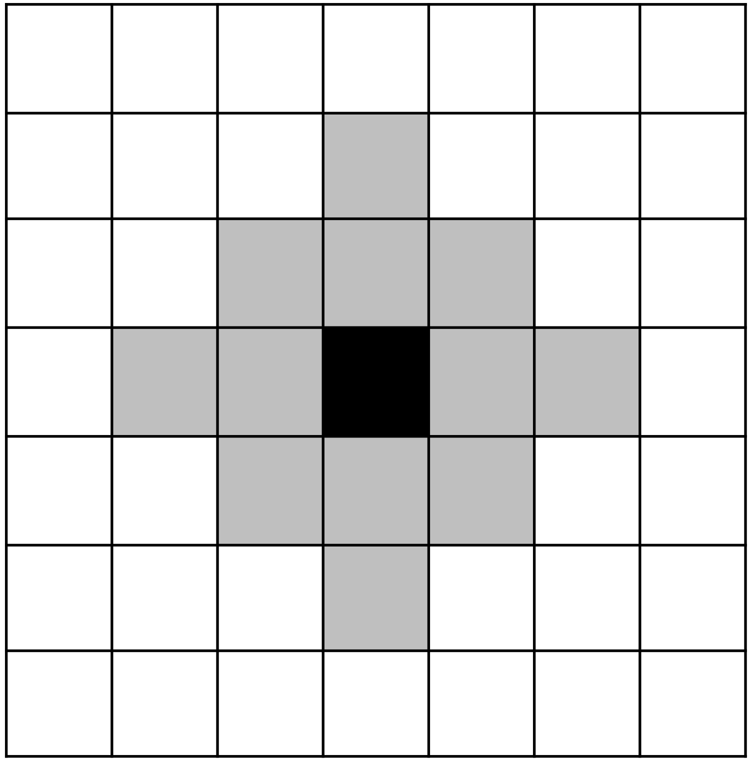
AR(1)



AR(2)



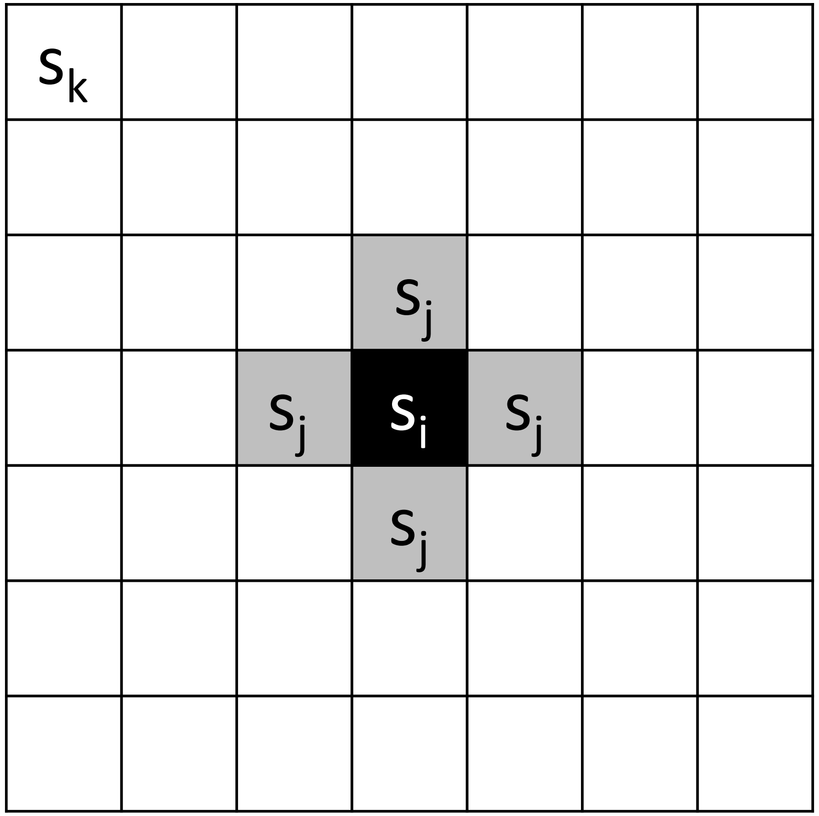
AR(3)



GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

AR(1), 1D

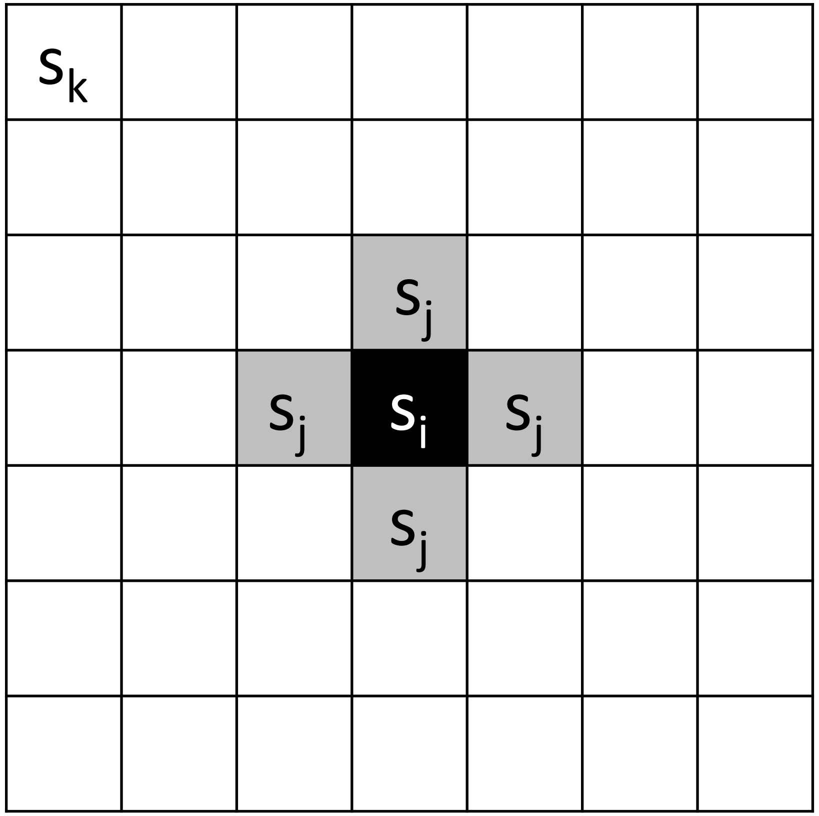


GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

$$Q = \Sigma^{-1}$$

AR(1), 1D



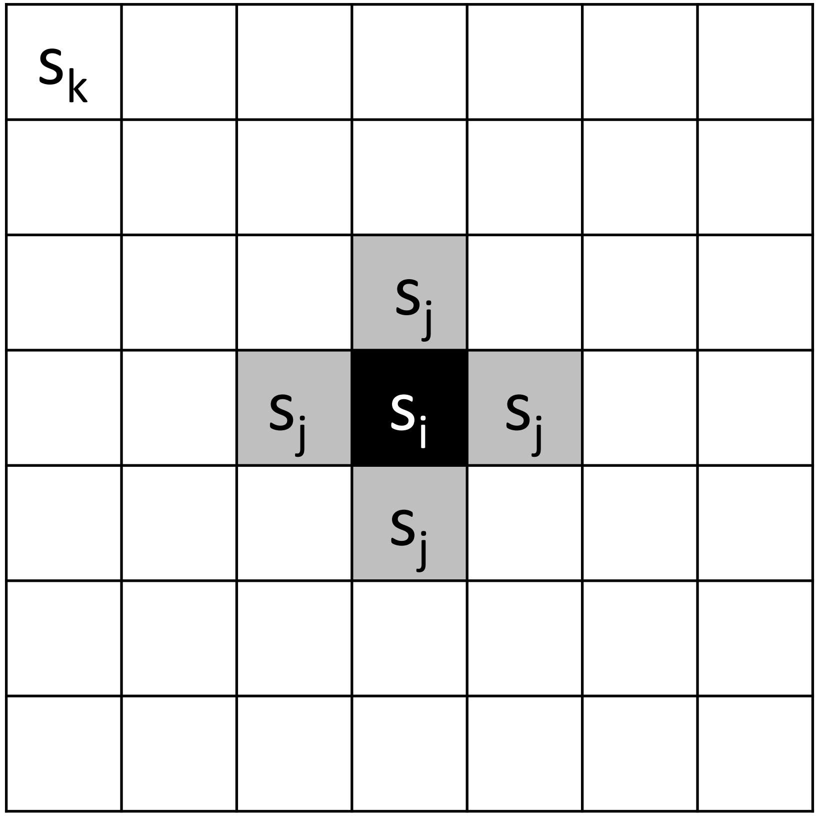
GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

$$Q = \Sigma^{-1}$$

$$\Sigma = \frac{\sigma^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}$$

AR(1), 1D



Likelihood Bottleneck

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{\Sigma}^{-\mathbf{1}} \boldsymbol{\omega})$$

- Operations are $\mathcal{O}(n^3)$

Resolving the Likelihood Bottleneck

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{Q} \boldsymbol{\omega})$$

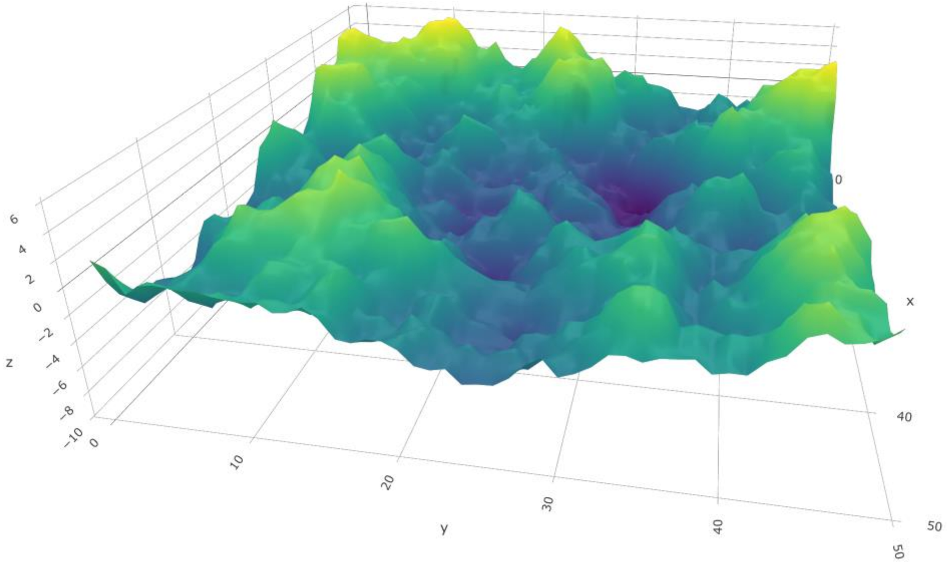
- Operations are $\mathcal{O}(n^{3/2})$

Stochastic Partial Differential Equation

SPDE (Whittle, 1963)

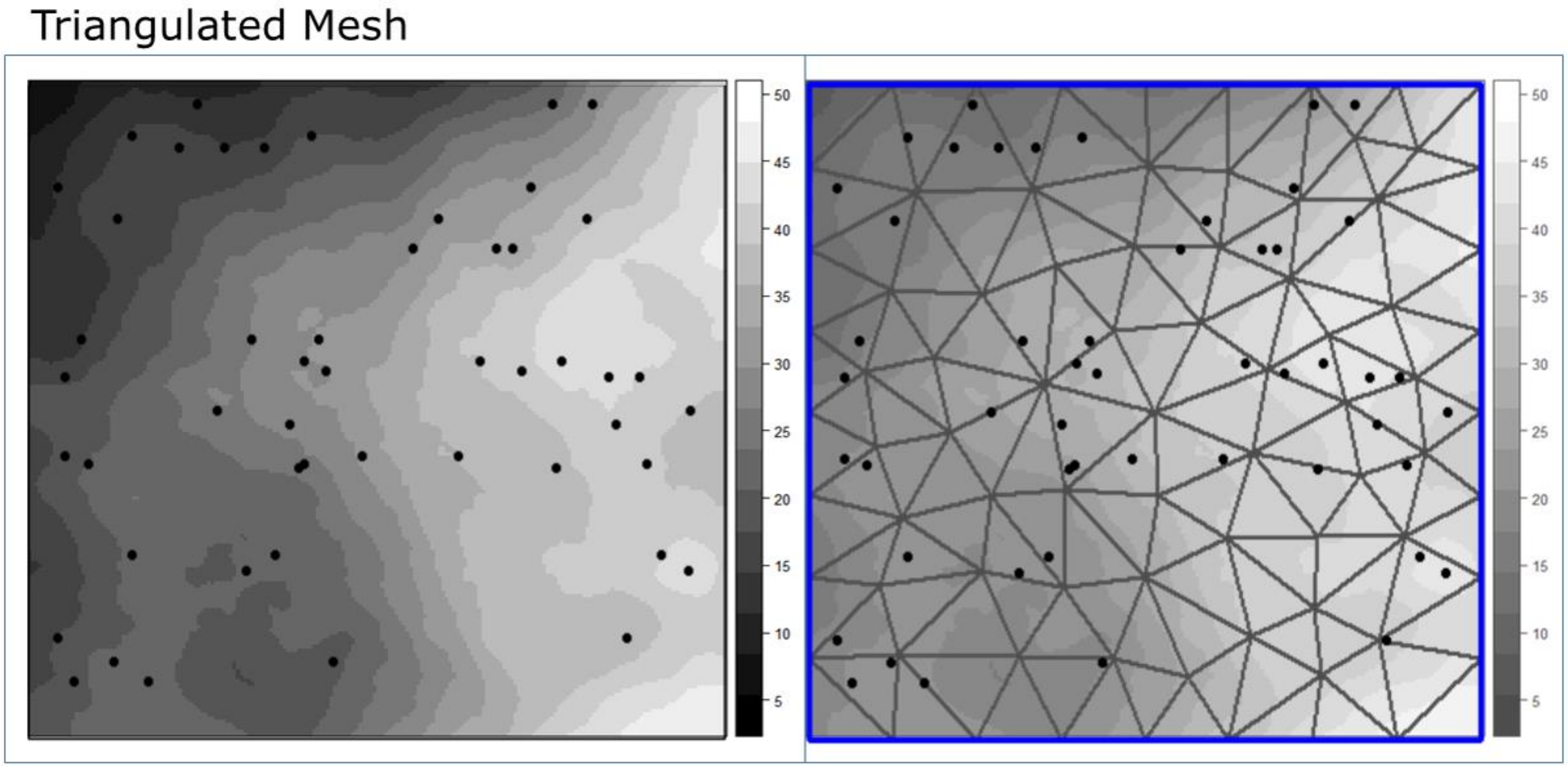
$$(\kappa^2 - \Delta)^{\alpha/2} x(s) = W(s)$$

Solution: Gaussian Field with a Matérn covariance matrix



Finite Element Method approach to the SPDE

FEM-SPDE (Lindgren et al., 2011)



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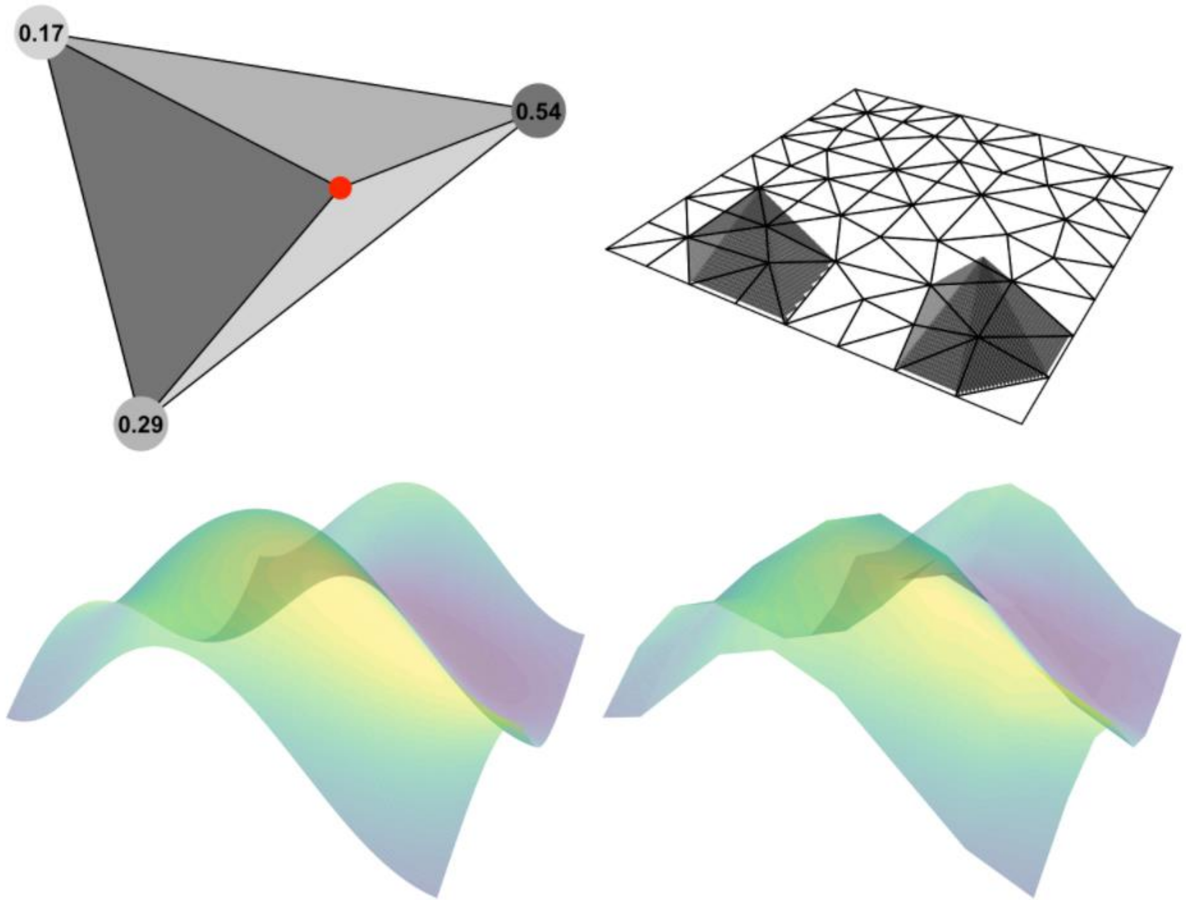
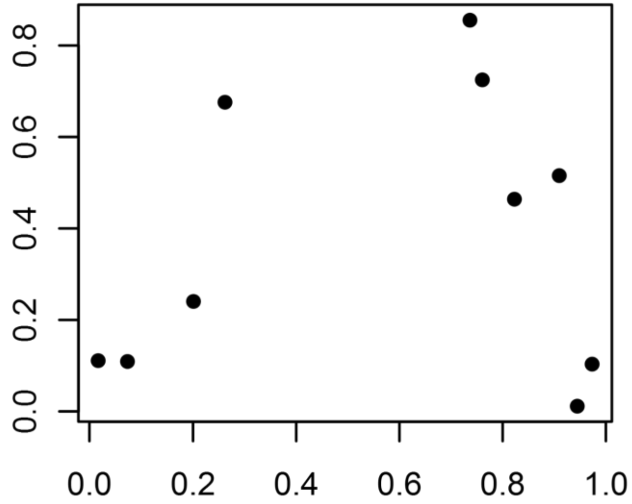
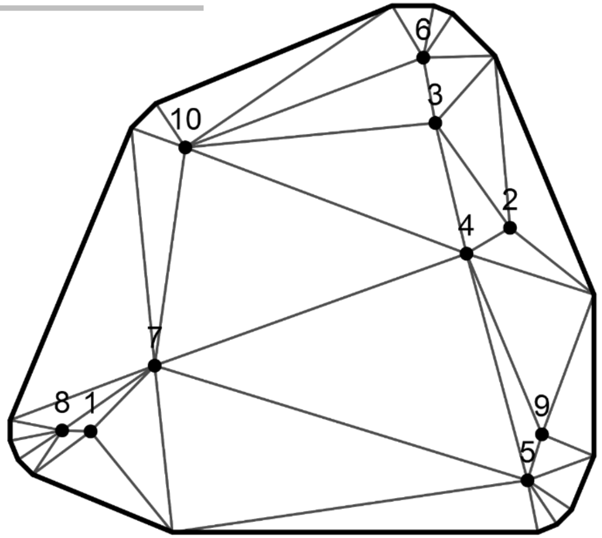


Image source: [Krainski et al. 2019]

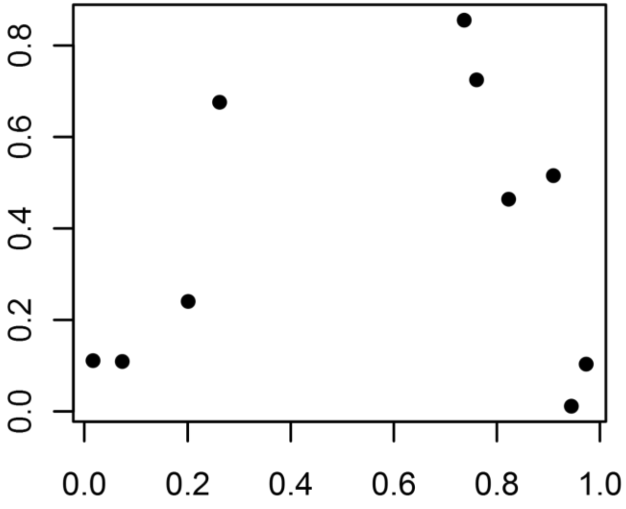
FEM-SPDE with R-INLA

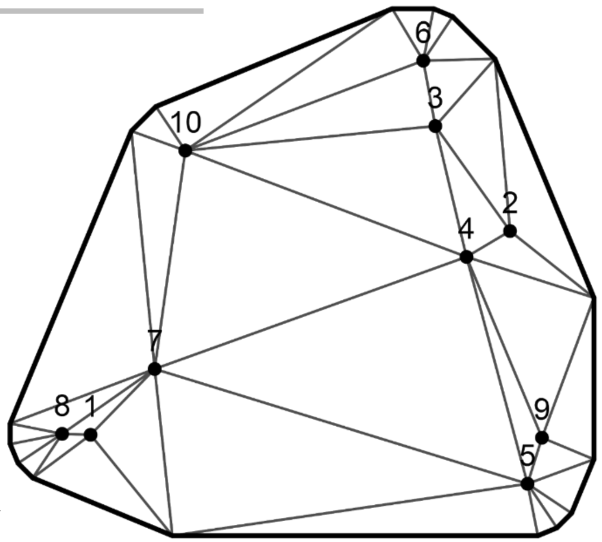



```
inla.mesh.2d()
inla.mesh.create()
```

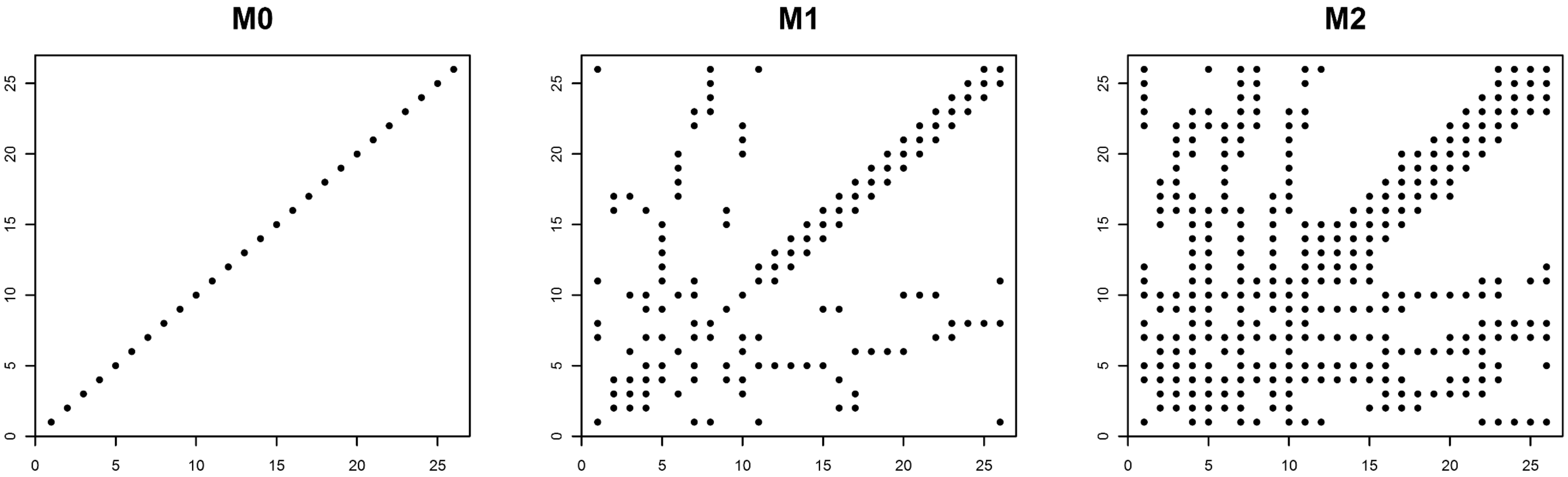


FEM-SPDE with R-INLA

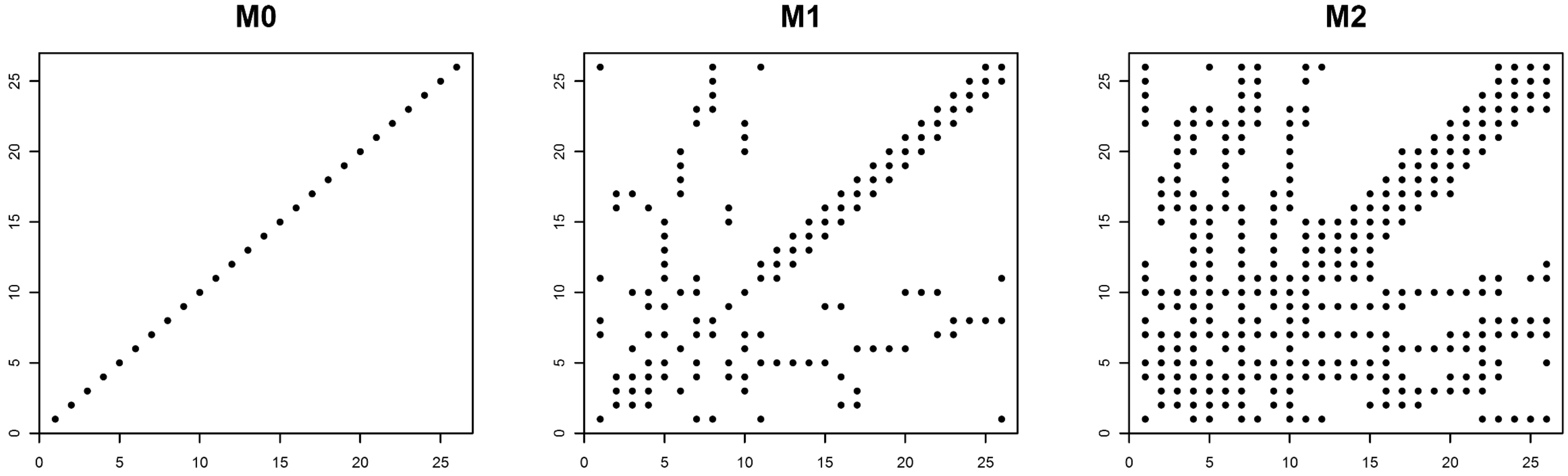




`inla.spde2.matern()`



Precision matrix from R-INLA output



$$Q = \tau^2(\kappa^4 M_1 + 2\kappa^2 M_1 + M_2)$$

$$\omega \sim GMRF(Q)$$

κ: rate of decay in spatial correlation

$$\tau^2 = \frac{1}{4\pi\kappa^2\sigma^2}, \nu = 1$$

TMB's Spatial Functionality

- Covariance Matrix, Σ:
  - matern()
  - MVNORM()
  - AR1()
- Precision Matrix, Q:
  - R_inla::Q_spde()
  - GMRF()
- Sparsity detection