Day 2: Notation and Logic

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Math Camp 2021

Miscellaneous Info

- Going to jump around a little today
- PS 490 – R workshop to help smooth the way with this course
- Will post today's slides by end of day

Now, to the good stuff!

Overview of Math Camp

- TeX or Rmd are great for homework–Michelle will walk you through soon
- http://detexify.kirelabs.org/classify.html
- Canvas for symbol cheat sheet
- OH: we'll build in time at the end of class sessions in addition to during sessions. Please take advantage of this!
- Email: jean.clipperton@northwestern.edu, MichelleBuenoVasquez2024@u.northwestern.edu
- Will LOOSELY follow recommended book – may jump around a bit
- Homeworks not graded but you will submit them

Math (P)refresher: Review Topics

- Broad Overview
- Sets
- Algebra review
- Inequalities
- Combinatorics
- Summation
- Derivatives
- Integrals

Broad Overview

From the Top: Variables and Constants

Theories

Theories are how we frame statements (hypotheses, propositions) about the world, using concepts.

Concepts

Concepts are the ideas, aspects to be measured

Example

Actors and Laws

Theory: More *political actors* involved in a process hampers *legislative productivity* because it's harder to reach an agreement

Need to define how you count an actor and measure productivity

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Abstract Thinking and Sets

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Set $\{3, 4, 0\}$.
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Set $\{3, 4, 0\}$.
Variable: actors in support of a bill.
Set $\{A, O, P\}$.

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Notes:
Subscript: Z_+ only the positive or Q_- negative elements of the set Superscript: N^2 dimensions of the space

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- Nominal (categorical): no mathematical relationship between the variables
- Ordinal: categorical variable with set relationship (can compare items to one another)
- Interval: distance between numerical values has meaning (e.g. 0,1, 2 – 2 is two greater than 0)
- Ratio (interval-ratio): distance between numerical values has meaning AND zero is meaningful

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How we measure concepts (and turn them into variables):

- Nominal (categorical): no mathematical relationship between the variables **Eye color**
- Ordinal: categorical variable with set relationship (can compare items to one another) **Age: above/under 18**
- Interval: distance between numerical values has meaning (e.g. 0,1, 2 – 2 is two greater than 0) **Approval from 1 to 5**
- Ratio (interval-ratio): distance between numerical values has meaning AND zero is meaningful **Number of years of grad school**

Variables and Sets: Putting Things Together

- Solution set: set of all solutions to an equation
- Sample space: set that contains all the values a variable can take
- Subsets: groupings that fall within other sets

Variables and Sets: Putting Things Together

Set operations

Can combine sets by looking at the *difference*, *complement*, *intersection*, *union*, and *partition* of sets.

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Example

Consider the set of all men and all women. The intersection is what is in both sets ('and') while the union ('or') is what is the collection of the two. We'll go into this vocabulary more tomorrow.

Notation Refresher

- There exists, $\exists$ for all, $\forall$
- Union $\cup$, Intersection $\cap$
- Excluding $\notin$, $\neg$
- Empty set $\emptyset$
- Element $\in$
- Equivalent $\equiv$
- Such that (s.t) or $|$, e.g. $\{x | x > 7\}$
- Subset $\subset, \subseteq$ (these function roughly like the less than/less than equal to, but for sets)

It's all Greek to me!

You'll want to have a basic familiarity with Greek letters as they'll come up from time to time.

α	θ	o	τ
β	ϑ	π	υ
γ	γ	ϖ	ϕ
δ	κ	ρ	φ
ϵ	λ	ϱ	χ
ε	μ	σ	ψ
ζ	ν	ς	ω
η	ξ		
Γ	Λ	Σ	Ψ
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Some commonly used letters include δ (integrals), Δ (difference/change), β (coefficients), μ (mean), σ (standard deviation), λ (eigenvalues (linear algebra)), ϵ (error)

Check in – all good?

Logic & Proofs

- Assumptions: taken to be true
- Proposition: statement thought to be true given the assumptions
- Theorem: proven proposition
- Lemma: theorem something of little interest
- Corollary: a type of proposition that follows directly from the proff of another proposition and does not require further proof

Necessary and Sufficient

Consider an outcome D with three possible input variables, A,B,C.

Sufficient

Something that occurs *also* when our outcome variable D occurs. Consider it as an if statement: D is true if A and B are true.

Necessary

Something that occurs *always* when our outcome variable occurs. Every time D is true, A and B are true. Consider it as an only if statement: D is true only if A and B are.

Necessary and Sufficient

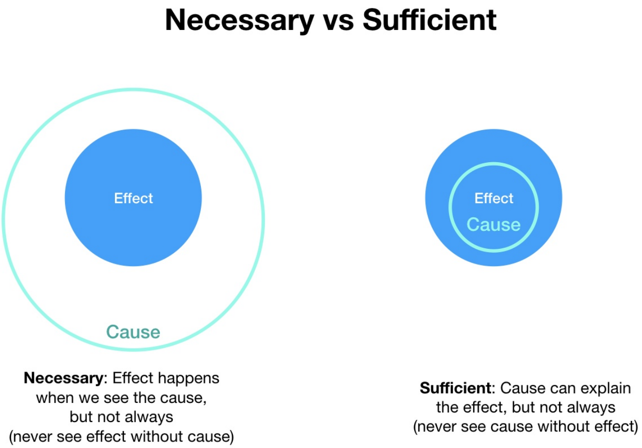

Figure 1:

Necessary and Sufficient

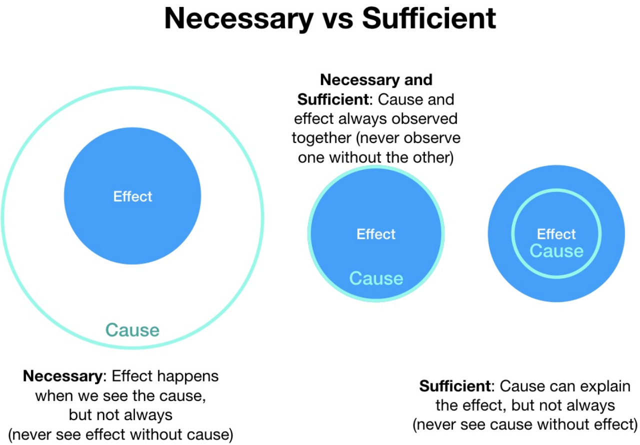

Figure 2:

Necessary and Sufficient

Example Suppose we're wondering if it will rain today. We notice the following:

Temperature	Pressure	Rain
L	H	N
L	M	N
H	M	N
H	L	Y

Recall that *necessary* means that something **ALWAYS** occurs when our outcome (rain) occurs while *sufficient* only doesn't happen when the outcome doesn't happen.

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Recall that *necessary* means that something **ALWAYS** occurs when our outcome (rain) occurs while *sufficient* only doesn't happen when the outcome doesn't happen. So, low pressure is necessary and sufficient for rain while a high temperature is only sufficient.

Logic

Consider the following elements: A,B,C that may be associated with an outcome, D. Suppose we're considering that A and B imply D ($A \wedge B \rightarrow D$).

Converse

The converse changes a necessary statement to a sufficient one, or vice versa. (not always logically true). Ex: Converse of $A \wedge B \rightarrow D$ is $D \rightarrow A \wedge B$ (trade places of elements; not always true).

Contrapositive

The contrapositive flips arrow *and* negates items. Ex: Contrapositive of $A \wedge B \rightarrow D$ is $\neg A \vee \neg B \leftarrow \neg D$. (Always true)

Fun Fact: The contrapositive is the negation of the converse of a statement.

Proofs: Direct proofs

We won't do a lot of proofs here, but it's helpful to understand how they work.

Direct proofs demonstrate the statement deductively by one of the following methods.

- General (deductive) proof: typically done using definitions, etc. Showing how the outcome logically follows building on rules and assumptions.
- Proof by exhaustion: Break up the outcome into sub cases and show for each case (done often in game theory for possible values)
- Proof by construction: These proofs demonstrate existence (is there a square that is the sum of two squares?).
- Proof by induction: Start small and show it is true for any number (e.g. start with a small n, n=1, then expand to n+1)

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Indirect proofs show that something must be true because there is no logically possible alternative. They are typically demonstrated through the following methods.

- Proof by counterexample: using a counterexample (x implies y, yet we observe y without x...x cannot imply y (aka x not *necessary* for y)).
- Proof by contradiction: assume that the statement is false and try to prove it wrong, eventually demonstrating that a contradiction emerges. Thus, the statement cannot be false.

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- Proof by contradiction: assume that the statement is false and try to prove it wrong, eventually demonstrating that a contradiction emerges. Thus, the statement cannot be false. (Essentially, hypothesis testing)

Questions?

Concluding CH 1 of Moore & Siegel – moving to Ch 2