Day 8: Calculus Fundamentals, Integrals

Jean Clipperton

Math Camp 2022

Agenda

- Intro to Integrals
- Indefinite integrals
- Definite integrals
- Advanced integrals: Double and Iterated integrals

Integrals

We can use integrals and, specifically, antiderivatives to 'undo' derivatives.

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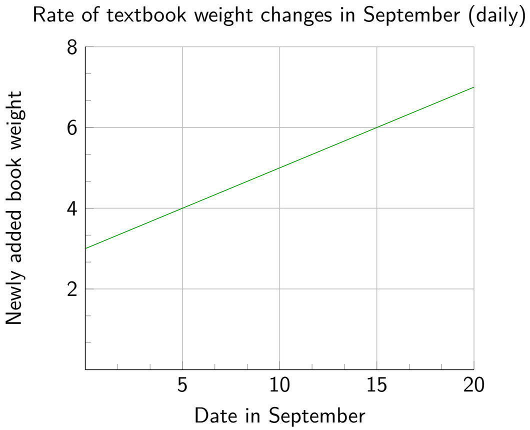
Essentially: integrals help us understand things like cumulative rates of change and add up things defined by a function. They're essentially sums and look that way: $\int f(x)dx$.

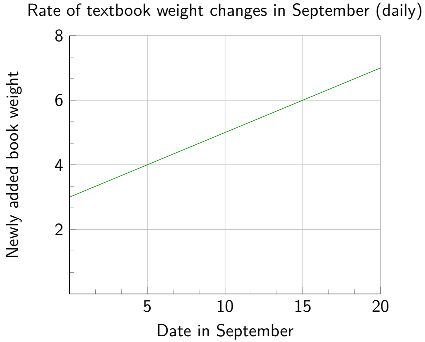
Antiderivatives

Just how they sound–know the rate of change but want to find the underlying function.



What questions could we answer with *this* graph?

What questions could we answer with the *original* function?



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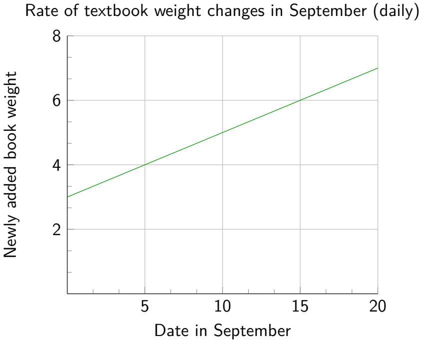

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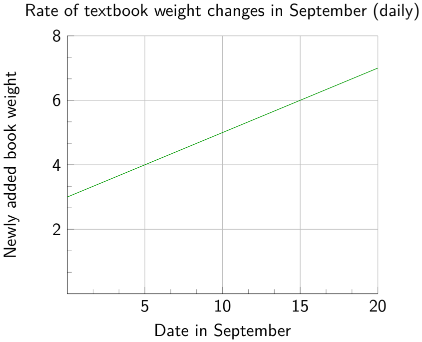

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- How many pounds new books have arrived so far?

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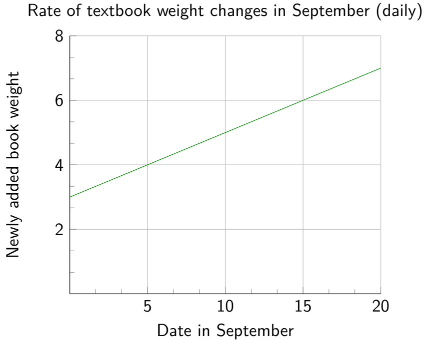
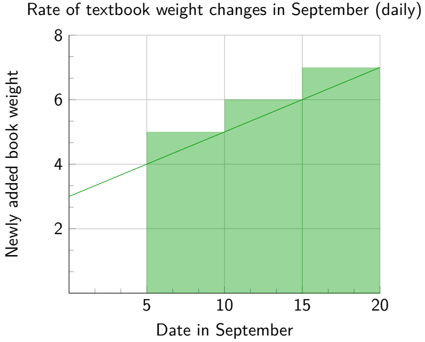


- What is the rate of change for newly arrived book weight on the 15th? 6 (units)
- How many pounds new books have arrived so far? To answer questions like these, we need to *undo* the derivative, in a sense.

Antiderivatives

How many pounds arrived between the 5^{th} and 15^{th}?



Integrals

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If the rate of new textbook arrivals is determined by ($0.2*x+3$), between the 5th and the 20th, then we need to take the integral of this function over the period. To take the integral, we will 'undo' the derivative.

Integral Rules

But first! The rules!

1. $f'(x) = 0$ has antiderivative $f(x) = c$

2. $f'(x) = x^n$ has antiderivative $f(x) = \frac{x^{n+1}}{n+1} + c$

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   - $f'(x) = 3x^2$, $f(x) = x^3 + c$

Fundamental Theorem of Calculus

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The area under the curve between a and b is equal to the difference between the antiderivative evaluated at these points.

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We get $f(x) = 0.1x^2 + 3x + c$.

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We get $f(x) = 0.1x^2 + 3x + c$. Now, we want to know how many pounds have arrived in TOTAL over a period. We can do that using the following integral:

$$\int_5^{20} 0.2*x+3 = 0.1x^2+3x\Big|_5^{20} = (0.1(400)+60)-(0.1(25)+15) = 82.5$$

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This tells us that approx 82 pounds of books arrived between Sept 5 through 20th. What we are doing is basically tallying the total number of books that arrived on each day using the 'book arriving' function when what we had was the *rate* of arrival function.

Excellent. Why do I care??

You'll be dealing with integrals indirectly in the fall and potentially directly in the winter/spring. Every time you're thinking about what happens 'overall', you are potentially dealing with an integral. Any time you're dealing with probability and statistics, you're almost certainly dealing with them indirectly (z tables are basically integral tables evaluating the function at different points). A basic understanding of integrals can prepare you for success later in your coursework!

Indefinite Integrals

The difference between definite and indefinite integrals is that indefinite integrals don't have numbers assigned – we want to know how we'd find the integral but aren't evaluating at a certain part of the function. We add a constant, c, to account for the lost constant (why lost?). For definite integrals, we want to know for particular values of our variable.

Integral Practice: Anti-Derivative for indefinite integrals

Practice

1. $f'(x) = 3$
2. $f'(x) = 0$
3. $f'(x) = 3x^2 - 2x + 4$
4. $f'(x) = e^x$

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1. $f'(x) = 3$, f(x)=c
2. $f'(x) = 0$, f(x)=c
3. $f'(x) = 3x^2 - 2x + 4$, f(x)=$x^3 - x^2 + 4x + c$
4. $f'(x) = e^x$, f(x)=$e^x + c$

Integral Practice: Definite

Practice

1. $\int_0^4 3dx$ f(x)$= 3x|_0^4 = 12 - 0 = 12$
2. $\int_0^4 0dx$
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3. $\int_0^2 (3x^2 - 2x + 4)dx$f(x)$= x^3 - x_4^2 x|_0^2 = 8 - 4 + 8 - 0 = 12$
4. $\int_2^3 e^x dx$

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4. $\int_2^3 e^x dx$ f(x)$e^x|_2^3 = e^3 - e^2 = e^2(e - 1)$

Integrals Gone Wild

We can get really sophisticated with integration, using complex techniques, such as integration by parts and substitution but these are beyond what we will need. Feel free to review these for fun (!).

Partial Derivatives

Similar to 'regular' derivative; treat additional variable(s) as constants (don't drop them, but don't deal with them). Written as ∂_x or $\frac{\partial f}{\partial x}(x, ...)$

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$f(x, z) = 7xz + 4x^2 + z$

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$f(x, z) = 7xz + 4x^2 + z$ $\partial_x = 7z + 8x$

$f(x, y) = x + 4y$ $\partial_x = 1$

Double Integrals & Iterated Integrals

Double integrals: two integrals that don't have definite values specified. Iterated integrals have values specified. You can take a partial derivative – how something changes as a function of two variables and ask about the overall total quantity that comes from the change in those two.

Suppose you have a function of x and y and want to know the volume of outcomes you'd observe as x and y move across their possible values.

Suppose you have the following function $f(x, y) = 7x + 8y$. You know that x can vary between 0 and 1 while y can vary between 1 and 3.

$$\int_0^1 \int_1^3 f(x, y) dy dx$$

Double Integrals & Iterated Integrals

Suppose you have the following function $f(x, y) = 7x + 8y$. You know that x can vary between 0 and 1 while y can vary between 1 and 3.

$$\int_0^1 \int_1^3 f(x, y) dydx = \int_0^1 \int_1^3 7x + 8ydydx$$

To do this, take the integral as you would for y (inner), then do it again (x).

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$\int_0^1 \int_1^3 7x + 8y \; dy \; dx$

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$= \int_0^1 (14x + 32)$

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$= \int_0^1 (14x + 32) = 7x^2 + 32x|_0^1$

Double Integrals & Iterated Integrals

$\int_0^1 \int_1^3 7x + 8y \ dy \ dx = \int_0^1 7xy + 4y^2|_1^3 = \int_0^1 21x + 36 - (7x + 4)$
$= \int_0^1 (14x + 32) = 7x^2 + 32x|_0^1 = 39$

BUT WAIT! There's MORE!!!

Double Integrals & Iterated Integrals

Recall we had $\int_0^1 \int_1^3 7x + 8y \quad dy\, dx$.

What if we reversed the order of the integrals? Meaning, what if it were $\int_1^3 \int_0^1 7x + 8y$ *dx dy* instead?

Well, sometimes, it will work out the same, and sometimes it will not. In this case, we're working with a pretty simple shape and so it works out to 39 as well (fun exercise to try on your own!). However, SOMETIMES, you're dealing with something a little more complex.

Double Integrals & Iterated Integrals

Generally, if you have a variable as one of your upper bounds, you can't move things around or else you're going to have a bad time. For example: $\int_{-1}^{3} \int_{0}^{x} x \; dy \; dx$. You can't switch the order here without trying to reconfigure what the x will end up being. This is honestly the hardest part of dealing with multiple integrals and generally won't be something you have to do (unless you go on to stats coursework). However, I want to flag this for you.

Basically, iterated integrals look hard and complicated, but really, they key is just to work your way through slow and steady!