Day 5: Matrix Algebra

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Math Camp 2021

Matrices: A basic introduction

- Not the most fun you'll ever have
- Not that scary once you get the hang of it
- A way of organizing things so you can do different types of operations on a large structure
- Can refer to a matrix as just the elements, or give it a name, like $[A]$ or $\mathbf{A}$

Matrices: overview

Filled with rows and columns – we refer to them by the number of rows and columns (e.g. 3 x 4 matrix). It can be a really efficient way to deal with data and to perform operations (like you would with a linear regression! woohoo!). It's also how your data will be organized any time you use a spreadsheet.

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$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The above matrix is a square matrix (same number of rows and columns) and each element is subscripted by its respective row and column number. Sometimes matrices are subscripted so you know their size. E.g $[A]_{2x2}$.

Matrices: Try it!

Suppose we have the following matrix. What are the dimensions of matrix [B] below, and what value is b_{23}?

$$\begin{bmatrix} 3 & 2 & 8 & 11 & 14 & 19 \\ 9 & 81 & 21 & 31 & 41 & 1 \\ 13 & 7 & 6 & 4 & 5 & 20 \end{bmatrix}$$

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Dimensions are 3 x 6; b_{23} is 21.

Elements of Matrices: the diagonal

The elements along the diagonal often are important in matrices. We typically focus upon the diagonal that starts in the upper left and goes down to the lower right.

$$[A] = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} \; [B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \; [C] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & 6 \end{bmatrix} \; [D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Types of Matrices

We won't get into the nitty gritty details of all sorts of different matrices but it might be helpful to know that there are 'special' matrices:

- Vector matrices (only one row (row vector) or column (column vector)
- Submatrix (subset of matrix)
- Triangular matrix: part of matrix is zeros – all bottom triangle zeros is upper triangular, all upper triangle zeros is lower triangular. (focus: where are the numbers?)
- Diagonal matrix: only the diagonal is non-zero
- Zero matrix: everything is zeros!
- Identity matrix: most important! all zeros except on diagonal AND diagonal is only ones...this is the matrix version of multiplying by 1
- Transpose (AKA transposition matrix): This is where you flip all the rows/columns. Meaning, if something was row 3, col 2, it will now be row 2, col 3. Denoted $[A]^T$. Done by 'reflecting' over the main diagonal (so the diagonal stays the same)

Identifying Matrices

What kinds are the matrices below? Also, notice their dimensions – they are all square. Square matrices tend to make the math nicer (it's all relative)

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A: Upper triangular, B: Zero matrix, C: Lower triangular, D: Identity matrix; Fun fact! A and C are the transpose of each other

Adding matrices

This is our last stop before things get too weird. Adding matrices (and subtracting) works exactly like you think it would: you need two matrices that have the same dimensions as each other. You then add the elements together (or subtract, as applicable). The final matrix has the same dimensions as the first two and everything is well and good.

$$\begin{bmatrix} 1 & 13 & 15 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 17 & 23 \\ 3 & 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 13 & 15 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

Multiplying Matrices

This might make your brain hurt a little, but this is the part where matrices get weird and complicated.

What you do is you take two matrices and to multiply the matrices you MULTIPLY things AND ADD THEM (!). To do this, the order really matters (you will not necessarily get the same thing if you do matrix A times matrix B if you were to do B times A, for example (although you might)). Thus, you need to have the matrices in the right order and the number of COLUMNS in the first matrix to be multiplied must equal the number of ROWS in the second matrix. (yes, that's right).

We'll start with two matrices, A is 1 x 2 and B is 2 x 3. Notice that it doesn't matter that they aren't exactly the same dimensions – only the middle two elements. The final matrix will have the outer numbers for the dimensions. Here, it will be 1 x 3.

Multiplying Matrices

$$[A] = \begin{bmatrix} 7 & 8 \end{bmatrix} [B] = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

Let's multiply! You take the row of the first matrix, multiply it by the COLUMN (hence the need to match) of the second matrix, ADD the sum of these products, and that goes into the first cell of the 'final' matrix. Then you do the same thing for the next column.

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$$[C] = \begin{bmatrix} 7*2+8*1 & 7*4+8*3 & 7*6+8*5 \end{bmatrix}$$

Note this is a row vector: $[C] = \begin{bmatrix} 22 & 52 & 82 \end{bmatrix}$

Multiplying Matrices: bigger matrices

If you have multiple rows in your initial matrix, you just do the same process over again, following the same procedure for each row. Your final matrix will have dimensions determined in the same way. For example, if you have a 2 x 3 and a 3 x 3, you'll have a 2 x 3 as your resulting matrix.

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$$[C] = \begin{bmatrix} 7*2+8*1 & 7*4+8*3 & 7*6+8*5 \\ 1*2+2*1 & 1*4+2*3 & 1*6+2*5 \end{bmatrix} = \begin{bmatrix} 22 & 52 & 82 \\ 4 & 10 & 16 \end{bmatrix}$$

Multiplying Matrices: Practice

$$[A] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 8 \end{bmatrix} [B] = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} [C] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} [D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Try the following:

- **A x B**
- **B x A**
- **A x C**
- **B x D**

Answers

- **A x B** $\begin{bmatrix} 1 & 9 & 41 \\ 2 & 10 & 46 \end{bmatrix}$
- **B x A** Not possible: 3 x 3 and 2 x 3 (middle numbers must match)
- **A x C** Not possible: 2 x 3 and 4 x 3 (middle numbers must match)
- **B x D** B (D is the identity matrix so you always get back whatever you multiplied it by)

Matrices: What you really need to know

- Identify a matrix (is it a matrix?)
- Determine dimensions of a matrix
- Add/Multiply simple matrices
- Understand that there's a whole rich world out there waiting (lurking?) for you re: matrices

Pre-Calculus Review: Summation and Limits

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$\sum_{n=1}^{5} 6$ Ans: $6 * 5 = 30$ $\sum_{n=1}^{4} 2n + 3$ Ans: $2 * (4 * 5)/2 + 3 * 4 = 32$

Limits

The sums on the previous page had *limits*: you can add elements and get an answer. We say that these series **converge** while those series that just keep getting bigger and bigger and bigger (or smaller and smaller and smaller) **diverge**.

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We can talk in the same way about sequences $lim_{i\to\infty} x_i = L$.

Limits

Limits are also useful in calculus – when we take a derivative, we are essentially asking: “What is the slope of the line at this infinitesimally tiny part of the line“?

A second way to ask this is to look at what happens to the slope as as the distance between points approaches zero.

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A second way to ask this is to look at what happens to the slope as as the distance between points approaches zero. However, to actually calculate the derivative, we need to first be sure that the point is differentiable. Again, we will use limits. (useful in PS 405 and Soc 401-1)

Behaving Badly

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Continuity

Continuous function: draw without picking up a pencil

REPLACE IMAGE HERE

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For us, you should be able to:

- Plug a value in to the limit and see what you get out (check for dividing by zero).
- Recognize an indeterminate limit: $\frac{0}{0}$ or $\frac{\infty}{\infty}$
- Compare powers in an indeterminate limit to find what happens: Same degree: limit goes to whatever fraction you get when you divide the top and bottom first terms

We don't go beyond here for our calculations, however the book has a nice explanation on limits and the bounds of limits and l'Hopital's rule. This is nifty, but more firepower than we need.

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Ans: $6/9$, simplifies to $2/3$ $\quad$ simplifies to x-2; limit is 2.