Day 6: Calculus II, Integrals

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Math Camp 2022

Day 6 Agenda

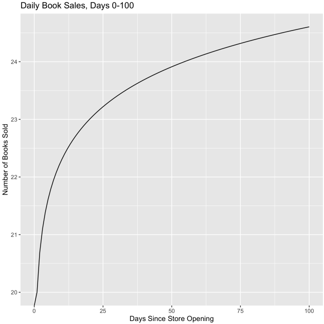
- Integrals: Concept, notation, and how-to
- Fundamental Theorem of Calculus
- Computing Integrals: Substitution and by Parts
- Advanced Rules of Integrals

Let's Take a Step Back...

- As a reminder, a derivative provides information about the *instantaneous rate of change* of a function.
- This means we can take a given value of x and find the rate of change of the function at that point, so long as the function is continuous at x. This also means we have an equation of the *marginal rate of change*.
- However, we are not always interested in just the *instantaneous rate of change*. Instead, we may be interested in the effect of a function's change over the *range of the function, or over a given interval.*

Motivating Example

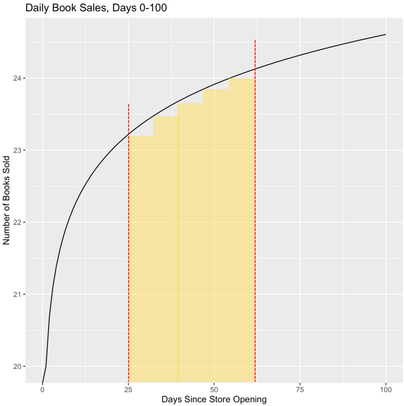
- Let's say a new book store opens and based on the first 100 days of sales we can model daily book sales given the function $f(x) = 20 + ln(x)$, where x is the number of days since the bookstore opened.
- As we know, solving for x will give us the amount of books sold on that particular day.



Ok, yes you can't sell 23.5 books over any of the days, but it's just a model for example's sake.

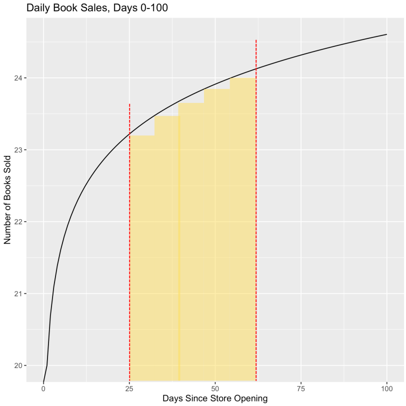
Cumulative Book Sales?

- Perhaps the bookstore is going through an audit, and instead wants to check for the *cumulative* number of books sold for the entire first 100 days.
- Or even more specifically, we want to know how many books were sold from Day 25 to Day 65.



Book Sales on the Interval?

- Remember our series and summation operations? This is essentially what we would be required to do to find this value!
- We *could* solve for x for each point in the interval and then add these all up. To make our estimate more accurate, we would have to break up the intervals between each x and sum those together– making the interval smaller and smaller to achieve higher precision.



Alternative?

- Instead, we can solve for the integral of this function to find the *total or cumulative* area under the curve within some specified bounds.
- This is much more precise and efficient.

Integrals

- Integrals are the inverse of derivatives. Where derivatives tell us about the marginal and instantaneous rate of change of a specific function, integrals are a generalization. The generalization of the function means that we can learn about the area that the function encompasses.

Indefinite Integrals and Anti-Derivatives

- **Indefinite Integrals** and **Anti-Derivatives** are terms that are essentially interchangeable for our purposes.
- Indefinite integrals assign an **integral** $F(x)$ for the **integrand** $f(x)$, wherein there are no limits on the integral. This means that we want to find the most general form of the integral.
- The indefinite integral is given by $\int f(x)dx = F(x) + c$, wherein $\int$ is the integration operator, dx refers to the variable to be integrated (x), and c is a constant.
- Indefinite integrals must always account for the constant c. Remember the rules of differentiation regarding constants–they go to 0.

Definite Integrals

- **Definite Integrals** are the same conceptually as indefinite integrals in that they give a function to be defined, however they place *bounds* on the interval , $[a, b]$, of the integral.
- *How does this look in practice?*

Definite Integrals

- **Definite Integrals** are the same conceptually as indefinite integrals in that they give a function to be defined, however they place *bounds* on the interval , $[a, b]$, of the integral.
- *How does this look in practice?* An indefinite integral solves for an *general equation* $f(x)$, whereas a definite integral solves for a *value*, or at least a function of difference between the upper and lower bounds.
- In the case of a definite integral, we instead notate the integral as $\int_a^b f(x)dx$. Notice that the bounds b and a notate the upper and lower bounds, respectively.

The *Anti*derivative

Ultimately the process of moving from derivatives to integrals should be possible both forward and backward. This is to say, these processes are inverse to each other and we *should* be able to perform differentation and anti-differentiation forward and backward.

This gives us the first fundamental theorem of calculus!!

$$f'(x) \longleftrightarrow f(x) \longleftrightarrow F(x)$$

Thus, $F'(x) = f(x)$!

Fundamental Theorem of Calculus

- We also want to use the definite integral to solve for the integral over a range of values, [a,b]. Mathematically this entails:

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

- This is the more common recitation of the **fundamental theorem of calculus**. Without getting into too much math, this theorem states that with this information that we solve for the *area under the function's curve over the stated interval*.
- With a bounded integral, we can ditch the constant, c, included in the indefinite integral because algebra will cancel it out.

Why does it matter?

As we will talk about more tomorrow, the "area under the curve" is important for calculating probability.

Perhaps you all are familiar with the Bell Curve distribution, wherein we can estimate the percentile of a given score based on its location on within the curve.

This Bell Curve model is possible because of the information that calculus reveals to us *about* a probability distribution.

OK... so now how do I do it?

Like derivatives, solving for integrals essentially comes down to a lot of rules. Solving for definite integrals also entails a bit of algebra.

Remember that the simple rule for calculating the derivative is $\frac{dx^n}{dx} = nx^{n-1}$; furthermore, remember that we are trying to achieve the 'original' function of f'(x) with the antiderivative.

Therefore, finding the integral requires that we obtain the inverse of the derivative. Finding the inverse implies that we will perform the opposite algebraic operation on the function. Furthermore, remember we must also account for a potential constant, c.

But really... how do I do it?

- When dealing with powers and basic integration:

$$\frac{dx^n}{dx} = nx^{n-1} \rightarrow \int \frac{x^{n+1}}{n+1} + c$$

- Note here that finding an integral in this form is not possible if $n \neq -1$.

Starting Out

Given this simple rule, let's practice with a couple of functions.

Let's try to integrate the following:

- $\int 3x + 3 \ \ dx$
- $\int 16x^2 + 2x + 7 \ \ dx$
- $\int \sqrt{x} + 13 \ \ dx$

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$$F(x) = \frac{2}{3}x^{\frac{3}{2}} + 13x + c$$

Integration as a Linear Operator

This has a few implications:

1. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

- This means that the sum/difference of two functions f and g integrated is equal to the sum/difference of their individual integration.

2. $\int_a^b Cf(x)dx = C \int_a^b f(x)dx$

- The integral of a function scaled by a constant is equal to the constant times the integral.

Practice

- $\int_0^2 4(x+2)dx$

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$4[6-0] = 24$

More on Integrals

There are more properties that will become somewhat self evident as long as you have more practice with them.

Beyond this though is continuing to address how to solve them. There are two ways of handling integrals, which are ultimately a bit more hard to deal with than derivatives.

- Integration by Substitution:
- Integration by Parts
- Integration given Trigonometric Identities

We can also have multiple integrals with multiple variables.

$$\int_c^d \int_a^b f(x, y) dx dy$$

The Fun Part Is. . . .

- You probably won't have to do any of these calculations by hand. Unless you go onto more advanced stats classes.
- So we won't cover solving for them extensively here.

Here's some more practice based on what we have learned so far

Solve for the following, be sure to pay attention whether the integral is definite or indefinite:

1. $\int (x^3 - x^2 + 2x) dx$

2. $\int_1^5 \frac{1}{x^3} dx$

3. $\int_1^5 x^3 + \frac{1}{x^3} dx$

4. $\int_0^1 e^x dx$ **hint: the integral of e^x is the same as its derivative.**

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$$F(x) = \frac{x^4}{4} - \frac{x^3}{3} + x^2 + c$$

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   $$\frac{1}{4}x^4\Big|_1^5 + \left(-\frac{1}{2x^2}\right)\Big|_1^5 = 156.48 \quad \text{or} \quad \frac{3912}{25}$$

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4. $\int_0^1 e^x dx$ **hint: the integral of e^x is the same as its derivative.**

   $$e^x\Big|_0^1 = (e - 1) \quad \text{or} \quad 1.7183$$