Day 5: Calculus I, Derivatives

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Math Camp 2022

Day 5 Agenda

- Derivatives: Concept, notation, and how-to
- Fundamental Derivative Rules
- Partial Derivatives
- Advanced Rules of Derivatives

Calculus

- This is often where we get a lot of nervous faces in the room. But, I can assure you that we are going to stick together in this and everyone will come out knowing some minimal calculus stuff to get you through these quant methods courses.

What is calculus?

- So basically, calculus is the study of *instantaneous change of a function*.

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Well, what does that mean?

- On the one hand, we could be interested in *discrete* change. Discrete change tells us the difference between two points on a graph, thus the difference between two observations modeled by a function.

- Finding discrete change means finding the slope of two points, or the *secant.* You all are probably familiar with $\frac{rise}{run}$ from algebra.

What is calculus?

- But, finding the secant has a limitation. Discrete change only tells about the functional behavior *over an interval*. Instead we might want to find the rate of change at a *very specific moment* in the function.

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- Calculus gives us some tools to calculate instantaneous change and other downstream quantities.

Limits, Secants, and Capturing Change

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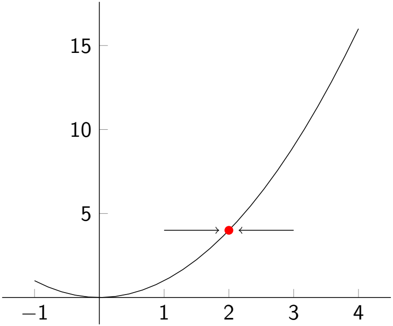
- Therefore, to capture discrete change we calculate the secant between two points by taking the difference of the functional limits, i.e. $\frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Limits, Secants, and Capturing Change

- To capture the instantaneous or continuous rate of change of a function, we *could* take the difference of the limits over iterations, making the interval between each x smaller and smaller.



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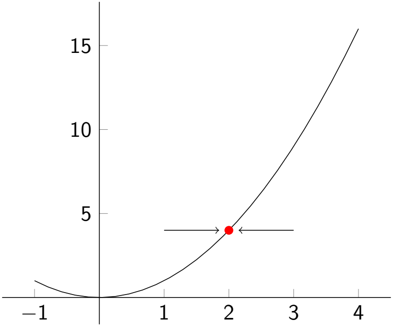

- However, the concept of **tangents**, and consequentially **derivatives**, makes this easier.

Tangents and Derivatives

- While a secant is a slope of a given line, a *tangent* is a line that touches the function at a given point. The tangent's slope tell us about the slope of the primary function, or the instantaneous rate of change, *at that particular point*.

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- Therefore, the problem becomes *how to find the tangent's slope*.

Notation

- There are a couple of ways to notate derivatives, all meaning the same thing.

$f'(x)$

$\frac{dy}{dx}$

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- Higher order derivatives will use the same types of notation, with details to indicate the order of the derivative

- $f''(x)$ or $\frac{d^2y}{dx^2}$ for a second derivative; $f'''(x)$ or $\frac{d^3y}{dx^3}$ for a third derivative

Calculating a Derivative

To calculate the derivative, begin with the secant formula. Use this formula to reduce the difference to some arbitrarily small value, *h*.

As h goes to zero, we go from discrete to instantaneous change.

Secant Formula:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

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$$= \lim_{h \to 0} 2x + 2h = 2x + 0 = 2x$$

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- Think back to what the slope of a linear equation tells us from the formula $y = mx + b$.
- If m is positive, the slope is increasing; if m is negative, the slope is decreasing.
- The derivative gives us information that we interpret similarly.

Derivative as information: Rate of change

- Positive Derivative

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- Positive Derivative Function is increasing
- Negative Derivative

Derivative as information: Rate of change

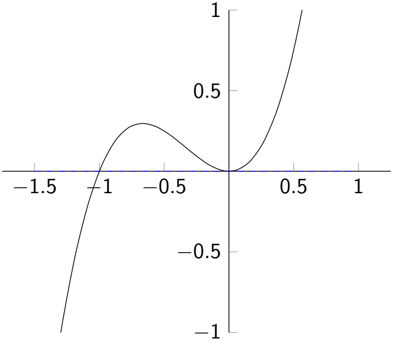
- Positive Derivative Function is increasing
- Negative Derivative Function is decreasing
- Zeroes?

Derivative as information: Rate of change

- Positive Derivative Function is increasing
- Negative Derivative Function is decreasing
- Zeroes? max or min

Derivatives and Extrema

Extrema: max or min of a function, i.e. where is the topmost or bottomost of the function?



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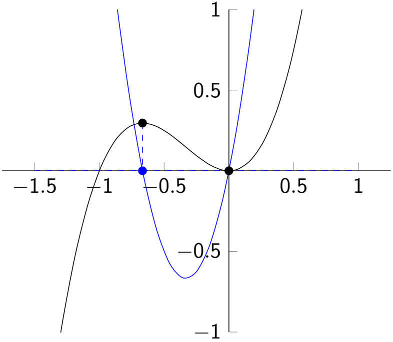


Worst Behavior

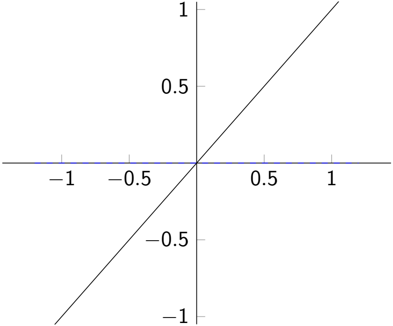
As we saw before: the function must be *continuous* on the *interval* to be differentiable.

Some functions are not differentiable *at all* or are not differentiable at *a certain point*. Need to determine the continuity of the function.

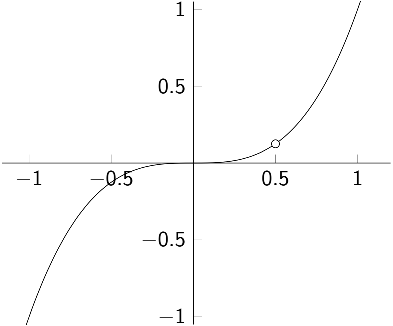
Continuity

Continuous function: draw without picking up a pencil

YES



NO



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Think through the work it would take to differentitate $4x^3 + 3x - 2$ using $\frac{f(x_0+h)-f(x_0)}{h}\dots$

Derivative Rules

Take the derivative $f(x)$ as $f'(x)$ below with constant k:

1. $f(k * x) = k * f(x), f'(k * x) = k * f'(x)$
2. $f(x) = k$ has derivative $f'(x) = 0$
3. $f(x) = x^n, f'(x) = n * x^{n-1}$
4. $[f(x) + g(x)]' = f'(x) + g'(x)$
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   - $f(x) = 4$,
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4. $[f(x)+g(x)]' = f'(x)+g'(x)$
   - $f(x) = 3x, g(x) = 7,$
5. $[f(x)-g(x)]' = f'(x)-g'(x)$

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   - $[3x - 7]',\ 3 - 0 = 3$

NOTE: $[f(x) * g(x)]! = f'(x) * g'(x)$ Ex: $(3x * 10x)'! = 30$

Derivatives Two Ways

We can check these handy formulas work as they should. Let's try. Find the derivative of $f(x) = \frac{1}{x}$

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$f'(x) = lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h}$

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We can check these handy formulas work as they should. Let's try. Find the derivative of $f(x) = \frac{1}{x}$

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Practice Problems

Find where functions are continuous and find derivatives

- $f(x) = 5$
- $f(x) = 3x - 7$
- $f(x) = 3x^2$
- $f(x) = \frac{x^2}{x}$
- $f(s) = s^{-2}$
- $f(y) = y(y + 7)(y - 3)$
- $f(z) = \frac{z^2 - 5z - 6}{z + 1}$

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- $f(x) = 3x - 7$,$f'(x) = 3$
- $f(x) = 3x^2$,$f'(x) = 6x$
- $f(x) = \frac{x^2}{x}$,$f'(x) = 1$
- $f(s) = s^{-2}$,$f'(s) = -2s^{-3}$ (not continuous at $s = 0$)
- $f(y) = y(y + 7)(y - 3)$,$f'(y) = 3y^2 + 8y - 21$
- $f(z) = \frac{z^2 - 5z - 6}{z + 1}$,$f'(z) = 1$ (not continuous at $z = -1$)

Higher Order Derivatives

Second derivatives (n^{th} derivatives): take a derivative a second (n^{th}) time

Rate of change of rate of change (velocity vs acceleration)

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(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: $f(x) * g(x)$
- Quotient Rule: $\frac{f(x)}{g(x)}$
- Chain Rule: $f(g(x))$
- Other: eg, exponentials: e^x, $ln(x)$

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- Quotient Rule: $\frac{f(x)}{g(x)}$, $\frac{x^4+3x}{x^2}$
- Chain Rule: $f(g(x))$, $(x^2 + 1)^3$ (composition!)
- Other: eg, exponentials: e^x, $ln(x)$

Product Rule

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- Simplify to get: $3x + 6 + 3x + 4 = 6x + 10$.
- To take the derivative using our previous approach, we first multiply: $3x^2 + 4x + 6x + 8 = 3x^2 + 10x + 8$. Then, just take the derivative of each term: $f'(x) = 6x + 10$.

Product Rule: A Motivating Example

Suppose that instead you had $(3x^2 + 3x + 4)(x^3 + 2x^2 + x + 2)$. Now the product rule is looking a little nicer!

$f(x) = (3x^2 + 3x + 4)$ and $g(x) = (x^3 + 2x^2 + x + 2)$

$f'(x) = 6x + 3$ and $g'(x) = 3x^2 + 4x + 1$.

We can substitute this into the formula:

$$f'(x) * g(x) + g'(x)f(x)$$

$$\rightarrow (6x + 3)(x^3 + 2x^2 + x + 2) + (3x^2 + 4x + 1)(3x^2 + 3x + 4)$$

.

Quotient Rule

Example: $\frac{3x^2}{x+2}$.

Formula is $\frac{f'(x)*g(x)-g'(x)f(x)}{(g(x))^2}$

So, we identify the following: $f(x) = 3x^2$ and $g(x) = x + 2$, therefore $f'(x) = 6x$ and $g'(x) = 1$.

Plug in to get:

$$\frac{6x(x+2) - 1(3x^2)}{(x+2)^2} = \frac{6x^2 + 12x - 3x^2}{(x+2)^2} = \frac{3x^2 + 12x}{(x+2)^2}$$

Practice: $\frac{x-4}{x+5}$ $\frac{3x^3}{x+2}$

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Practice: $\frac{x-4}{x+5}$ $\frac{3x^3}{x+2}$

Ans: $\frac{9}{(x+5)^2}$ $\frac{6x^3+18x^2}{(x+2)^2}$

Chain Rule

Sometimes, you have a function to a power: $f(g(x)) = (x+3)^3$. We can use the chain rule to evaluate this.

What we do is we take the derivative of the function and multiply it by the derivative of the inside: $f'(g(x)) * g'(x)$.

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More Chaing Rule

- **Try:** $f(x) = (2x^2 + 8x)^4$ $f(x) = (9x - x^2)^6$

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- **Try:** $f(x) = (2x^2 + 8x)^4$ $f(x) = (9x - x^2)^6$
- **Ans** $4(4x + 8)(2x^2 + 8x)^3$
  $6(9 - 2x)(9x - x^2)^5 = (54 - 12x)(9x - 2x^2)^5$

Exponentials: e and ln

You can take the derivative of continuous functions – including those with a log and/or e in them. The rules are a little hard, but once you learn them, it's not too bad:

$f(x) = e^x \quad f'(x) = e^x$ (a favorite of mine)

$f(x) = e^{g(x)} \quad f'(x) = e^{g(x)} * g'(x)$

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TRY: $ln(x^2)$, e^{2x} **ANS:** $\frac{2}{x}$ **and** $2e^{2x}$

Derivatives in Review

There are a few more handy rules and techniques that are important, perhaps even on your homework:

Common Derivatives

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$$

Figure 1: Source on Stack Exchange

And if you really want to explore more, check out all these techniques.