Day 3: Algebra Review

Sarah Moore and Jason Seawright

Math Camp 2022

Day 3 Agenda

- Algebraic Properties
- Inequalities and other relational operations
- Functions and Relations
- Combinatorics
- Exponents and Logarithms

Algebraic Properties of Numbers

1. **Associative property** $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$
2. **Commutative property** $a + b = b + a$ and $a \times b = b \times a$
3. **Distributive property** $a(b + c) = ab + ac$
4. **Identity property** $x + 0 = x$ and $x \times 1 = x$
5. **Inverse property** $-x + x = 0$.

Multiplicative inverse exists, but not for all numbers $x^{-1} \times x = 1$

Factoring

We may need to break down different functions.

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2. $m^2 + 3m + 2 =$
3. $x^2 + 5x + 6 =$

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Intervals and Inequalities

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Ranges of numbers can be expressed with either [] or () brackets.

$[a, b]$	a $\leq x \leq b$	Square brackets include end points (closed interval)
(a,b)	$a < x < b$	Parenthesis mean exclude end points (open interval)
$\{$a,b$\}$		Typically used for sets – not inequalities/intervals

Relational Expressions

You may remember how to graph relational expressions on a number line. Let's try to work through some of those here.

- $4 < x$
- $y > 12$
- $3 < z < 7$
- $(3, 9)$
- $[-7, 2)$

Solving Inequalities

We treat inequalities the same as any other equation when we wish to solve them, with the exception of when there are negative values.

When dividing or multiplying by a negative number in an inequality, flip the inequality sign.

Typically:

$8x + 2 < 3 \rightarrow 8x < 1 \rightarrow x < \frac{1}{8}$

Considering a negative value:

$-8x + 2 < 3 \rightarrow -8x < 1 \rightarrow x > -\frac{1}{8}$

Absolute Value

Magnitude of a numerical value indicated by $|x|$. Solved by typical arithmetic, but must solve for both the negative and positive equations.

For example:

$$|x + 4| < 5$$

- Must be solved as: $-x - 4 < 5$ AND $x + 4 < 5$

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- Try this one out: $|\frac{x+2}{x}| < 10$
- Answer: $-\frac{2}{11} > x > \frac{2}{9}$

Relations

Relations generally allow comparison of variables and expressions between a *range (y)* and *domain (x)*.

You may be familiar with the term *functions*, which are a type of relation. These are the most common relation you will encounter in statistical social science; while *correspondences* are a type of relation more likely found in game theory.

Functions

Mathematical functions have certain constitutive parameters.

- Specifically, functions assign *one* element of the *range (y)* to each element of the *domain (x)*.

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- Range: "the set of all values actually reached", a subset of the co-domain
- Thinking back to day 2: In social sciences, these sets will all typically be composed of $\mathbb{R}$, i.e. the set of all real numbers.

Mapping Functions

We might use some functional mapping to discern whether or not a relation constitutes a function:

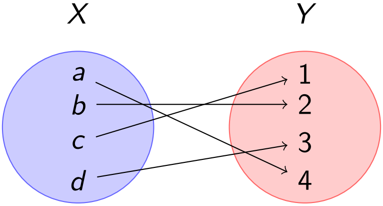

Figure 1: Mapping Relation S

Source Code

So if functions requiring that there is only one range element for each domain, does this mapping constitute a function?

More Function Mapping

Let's try out a couple more maps to see whether or not they are functions:

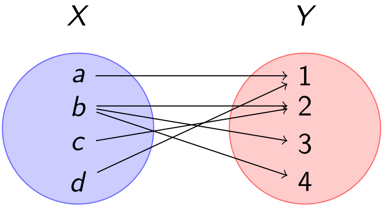

Figure 2: Mapping Relation H

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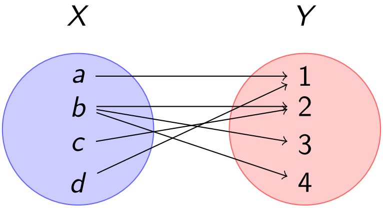


Figure 2: Mapping Relation H

Relation H is not a function.

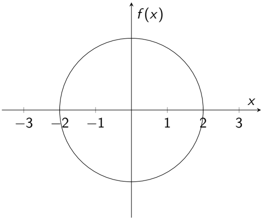
Functions

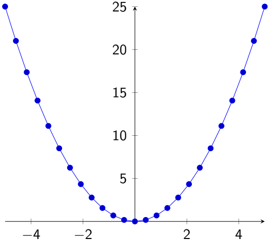
Vertical Line Test

Functions can only assign one x to one y. But sometimes our domain and range are not clearly defined and we may only have a graph immediately available to us. If any vertical line will pass through more than 1 point on the graph, then the relation is not a function.

Vertical Line Test

For example, based on the vertical line test which of the following graphs is not of a function:





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- Injective (one-to-one)– Each value in range comes from only one value in domain
- Bijective (invertible)– Both surjective and injective; function has an inverse

Monotonic Function Terms

Term	Meaning
Increasing	Function increases on subset of domain
Decreasing	Function decreases on subset of domain
Strictly increasing	Function always increases on subset of domain
Strictly decreasing	Function always decreases on subset of domain
Weakly increasing	Function does not decrease on subset of domain
Weakly decreasing	Function does not increase on subset of domain
(Strict) monotonicity	Order preservation; function (strictly) increasing or decreasing over domain

Table 3.2 from Moore and Siegel (pg 51)

Why is understanding functional behavior important?

Obviously useful to understand a function's behavior mathematically– i.e. to understand how the input relates to the output over the function.

BUT this is where we link some of the logic of math to theory construction. Ultimately you will choose what mathematical functions likely describe the relationship between x and y.

Therefore, we need to know how to model these things functionally.

Linear Equation versus Linear Function

- Thus far, we have addressed functions that are also linear equations – in that the highest power of the argument x is 1.
- *Linear functions* are a broader category of functions, to potentially include linear equations, that follow the rules of additivity and scaling:
  - Additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$
  - Scaling: $f(ax) = af(x)$ for all a
- Ultimately this is mathematical definition rarely followed by social scientists. But, for the purposes of further reading for some of you all the distinction will be important.

Nonlinear Functions

- More widely applicable is the differentiation between linear and non-linear functions.
- Generally a function is nonlinear when it is not linear and does not meet the properties of scalability or additivity.

Why does this matter?

- Up until now, the functions we've addressed assume a *linear* relationship between our x and y. But what if we have theoretical inclination to believe that the relationship is *nonlinear*– what if there is a ceiling effect to our x or diminishing returns?
- We need to be able to eventually account for these issues in our models, therefore, we must know how to model them mathematically.

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Exponents, Exponentials, Exponential functions

Exponents

Exponents are where you take a variable to some power – e.g. x^a where x is a variable and a is a constant. Typically, we focus on the numerical portion of the exponent–calling it 'the exponent'.

Exponential

An exponential is the reverse of the exponent – here it is a number to the power of a variable, e.g. a^x. To get the x 'down', we need to use logarithms (aka logs).

Exponential Function

The exponential function has a particular base, e, (where e is Euler's e and is approx 2.72.)

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- Quadratic polynomial: in the form $f(x) = ax^2 + bx + c$, wherein the highest power is 2
- Higher order polynomials: entail a degree higher than 2– can be 3 or greater (unlikely, but could be there!)

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Logarithms (typically base 10 ($log(x)$) or base e ($ln(e)$), but any base is possible, e.g. $log_{8675309}x$ (Bases aside from e and 10 will be specified).

- $y = log(z) \leftrightarrow 10^y = z$
- $y = ln(z) \leftrightarrow e^y = z$
- $log(1) = 0$

Exponents and Logs

Exponents in log are different from what you might expect:

- $log(x^2) = 2(log(x))$
- $log(x/y) = log(x) - log(y)$ provided $(x, y > 0)$

Logs help weigh smaller values more heavily; adding units not linear–less meaningful for larger values ($log(100) = 2, log(1000) = 3$).

Logs Practice Simplify the following

- $log(x^4)$
- $log(xy)$
- $ln(e^3)$
- $ln(1)$
- $log(3) + log(7)$

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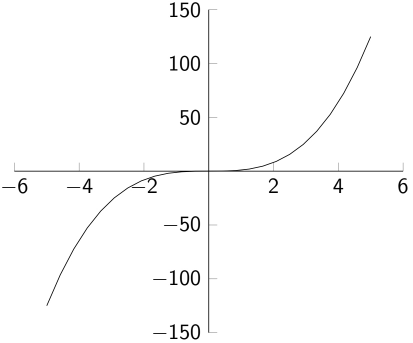
- $ln(e^3) = 3$
- $ln(1) = 0$
- $log(3) + log(7) = log(21)$

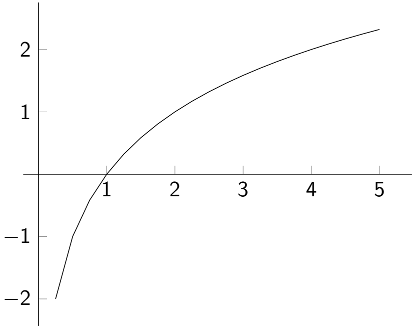
Graphical Behavior

Figure 3: Graph of x^3



Graphical Behavior

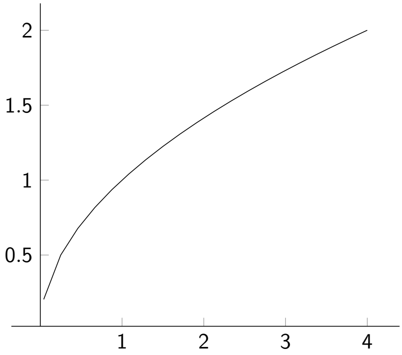
Figure 4: Graph of $\frac{ln(x)}{ln2}$



Source Code

Graphical Behavior

Figure 5: Graph of $\sqrt{x}$



Functional Continuity

As we will discuss in the coming days, and especially if you go further along in methods courses, functional continuity is also a source of concern.

- Some functions have certain points at which they *do not exist*.
- On one hand, there could be a certain reason that the function breaks at a certain x value.
- In other cases, there are limiting properties of certain function such that the function cannot exist over a certain interval.
- We won't go into more detail here, just know **continuity** and the interval we expect a function to exist over is important to understand conceptually.